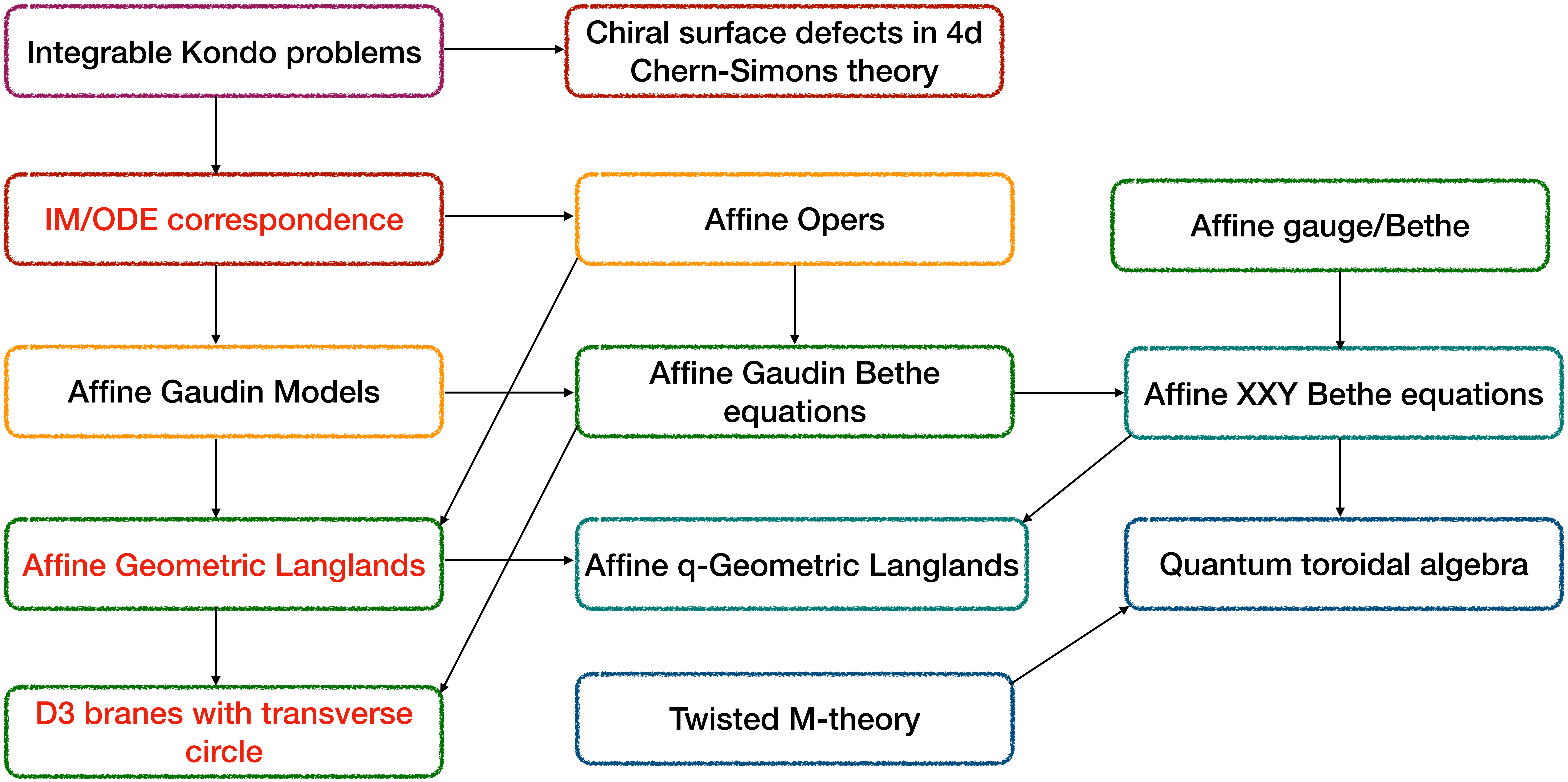
Integrable Kondo problems and affine Geometric Langlands

In collaboration with K.Costello, J.H. Lee, B. Vicedo and J.Wu

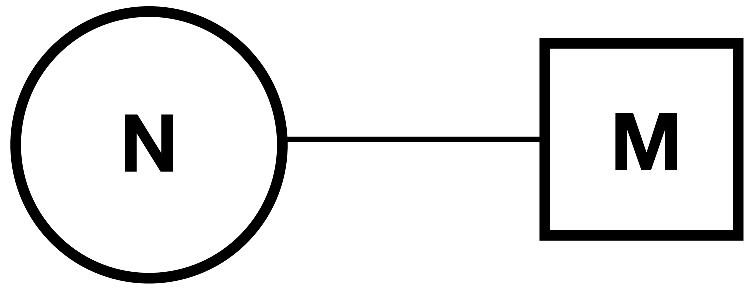
arXiv:2003.06694, to appear

Fragments of affine Geometric Langlands



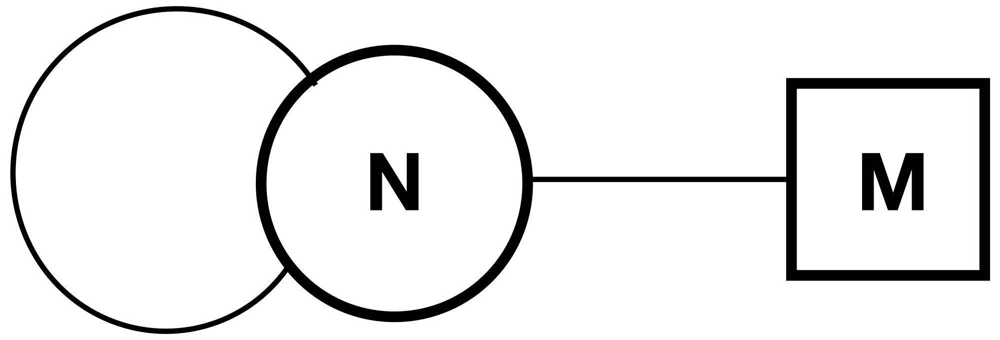
The Bethe/gauge perspective: the non-affine case

- Standard gauge/Bethe correspondence
  - 2d N=2* SUSY vacua of 3d N=4 ADE quiver gauge theories

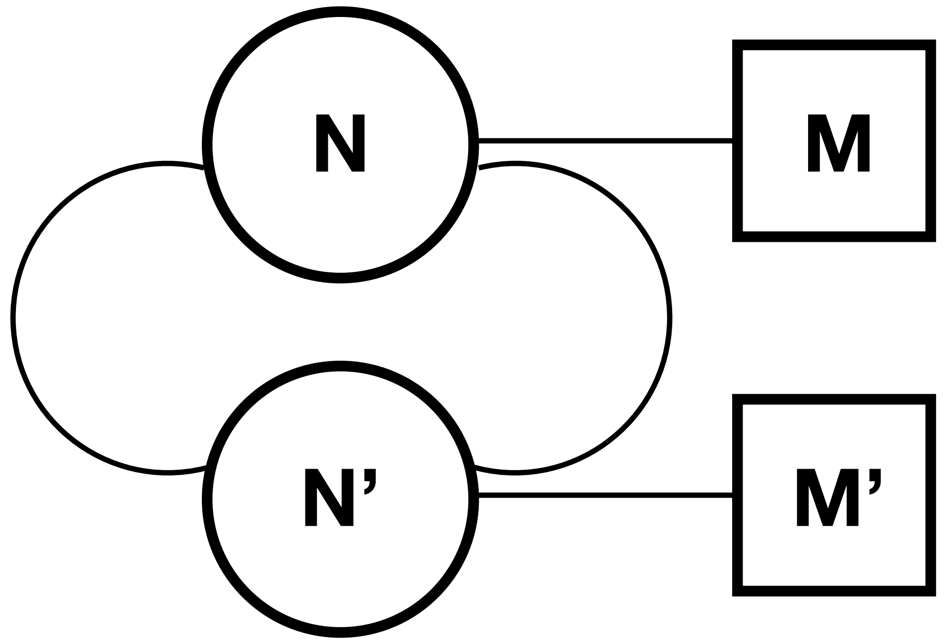


  - Bethe equations for XXZ spin chain, i.e. quantum group spin chain
- Rich dictionary, reduction to 2d (and lift 4d), mirror symmetry…
  - Possibly explained by 4d Chern-Simons, but intricate duality chain
- Scaling limit to Gaudin model relevant for Geometric Langlands

Affine Bethe/gauge correspondence



- Affine gauge/Bethe correspondence (new?)
    - 2d N=2* SUSY vacua of 3d affine ADE quiver gauge theories
    - Bethe equations for quantum toroidal algebras



- Much work to do. Spin chain studied for gl_n
    - Possibly explained by 5d Chern-Simons/twisted M-theory
- Scaling limit to affine Gaudin model

Finite dictionary

- Shape of the quiver => choice of ADE quantum group $U_q(\mathfrak{g})$
- Ranks => weight of state
- Flavour hypers => spin chain sites in minuscule irreps
- Mass parameters => spectral parameters of sites
- FI parameters => twist of spin chain boundary conditions
- N=2* mass => q
- Mirror symmetry => bispectral duality

Affine dictionary

- Affine A_n quiver => gl(n+1) quantum toroidal algebra $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_{n+1})$
- Flavour hypers => spin chain sites in (q-deformed Weyl?) Modules
- Mass parameters => spectral parameters of sites
- FI parameters => twist of spin chain boundary conditions
- N=2* mass, bifundamental/adjoint mass => q_1, q_2, $q_3 = 1/(q_1 q_2)$
- Mirror symmetry => bispectral duality

A simplification to $U_q(\hat{\mathfrak{g}})$

- Adjust bifundamental mass => simplified Bethe equations, fewer solutions
    - Only possibility for DE quivers
- Drop a gl_1 Fock space: sl_n version of the Weyl modules?
- Scaling limit: $$\prod \frac{u_i - qv_j}{u_i - qv_j} \to \sum \frac{\hbar}{u_i - v_j}$$
    - Bethe equations go to affine Gaudin Bethe equations

Gaudin refresher

- Gaudin Hamiltonians built from multiple copies of Lie algebra: $H_i = \sum_{j \neq i} \frac{J_i \cdot J_j}{z_i - z_j} + a \cdot J_i$
- Classical limit of XXX transfer matrices.
- Bethe equations, sl_2 example $c + \sum_i \frac{k_i}{w_a - z_i} - \sum_b \frac{2}{w_a - w_b} = 0$
- Bethe solutions build up tensor product of finite-dimensional irreps
- Kac-Moody at critical level: action of center on conformal blocks
- Gauge theory interpretation: vacua of 2d defects in 6d SCFT

Affine Gaudin

- Transfer matrices $Tr_R \mathrm{Pexp} \oint dw \sum_i \alpha(z, z_i) \frac{J_i(w) \cdot \mathfrak{t}}{z - z_i}$

- Complicated local Hamiltonians

- Affine Bethe equations, sl_2 example

$$c + \sum_i \frac{k_i}{w_a - z_i} + \sum_i \frac{2}{w_a - w'_b} - \sum_b \frac{2}{w_a - w_b} = 0 \quad c' + \sum_i \frac{k'_i}{w'_a - z_i} + \sum_i \frac{2}{w'_a - w_b} - \sum_b \frac{2}{w'_a - w'_b} = 0$$

- Bethe solutions build tensor product of Weyl modules

- Levels k+k', spins k/2

Gaudin and opers

- Bethe solutions of Gaudin model => opers with trivial monodromy
- sl_2 example: Schroedinger operator

$$\partial_x^2 \psi(x) = \left(a + \sum_i \frac{k_i(k_i+2)}{4(x-z_i)^2} + \sum_i \frac{c_i}{x-z_i} \right) \psi(x)$$

- Example of Geometric Langlands: Neumann to Nahm pole
- Boundary Wilson lines fo to Boundary 't Hooft lines
- Abelianization: w_a as positions of smooth monopoles

Affine Gaudin and affine opers

- Bethe solutions of affine Gaudin model => family of opers with trivial monodromy

- sl_2 example: Schroedinger operator

$$\partial_x^2 \psi(x) = \left(\frac{e^{2x} \prod_i (x - z_i)^{k_i + k_i'}}{\hbar^2} + \sum_i \frac{k_i(k_i+2)}{4(x-z_i)^2} + \sum_i \frac{c_i}{x - z_i} + \sum_a \frac{2}{(x - w_a')^2} + \sum_i \frac{d_i}{x - w_a} \right) \psi(x)$$

- Affine Geometric Langlands? D3's with a transverse circle?

- Abelianization: dynamical D1 segments going around a circle?

Integrable Kondo problems

- Kondo problem: RG flow of line defects in a 2d chiral CFT
- Classic example: two-level system coupled to 2 complex chiral fermions in SU(2)-invariant way

$$g \int dt \left(\psi(t;0)^\dagger \vec{\sigma} \psi(t;0)\right) \cdot \vec{S}(t)$$

- Coupling only involves $SU(2)_1$ WZW currents

RG flow

- Coupling g marginal, marginally relevant for g>0

- Defines an UV complete line defect $L[\theta]$ depending on dimensionally transmuted scale, “Kondo temperature” at which dynamics becomes strong.

$$\mu = g^{\frac{1}{2}} e^{-\frac{1}{g}} \equiv e^{\theta}$$

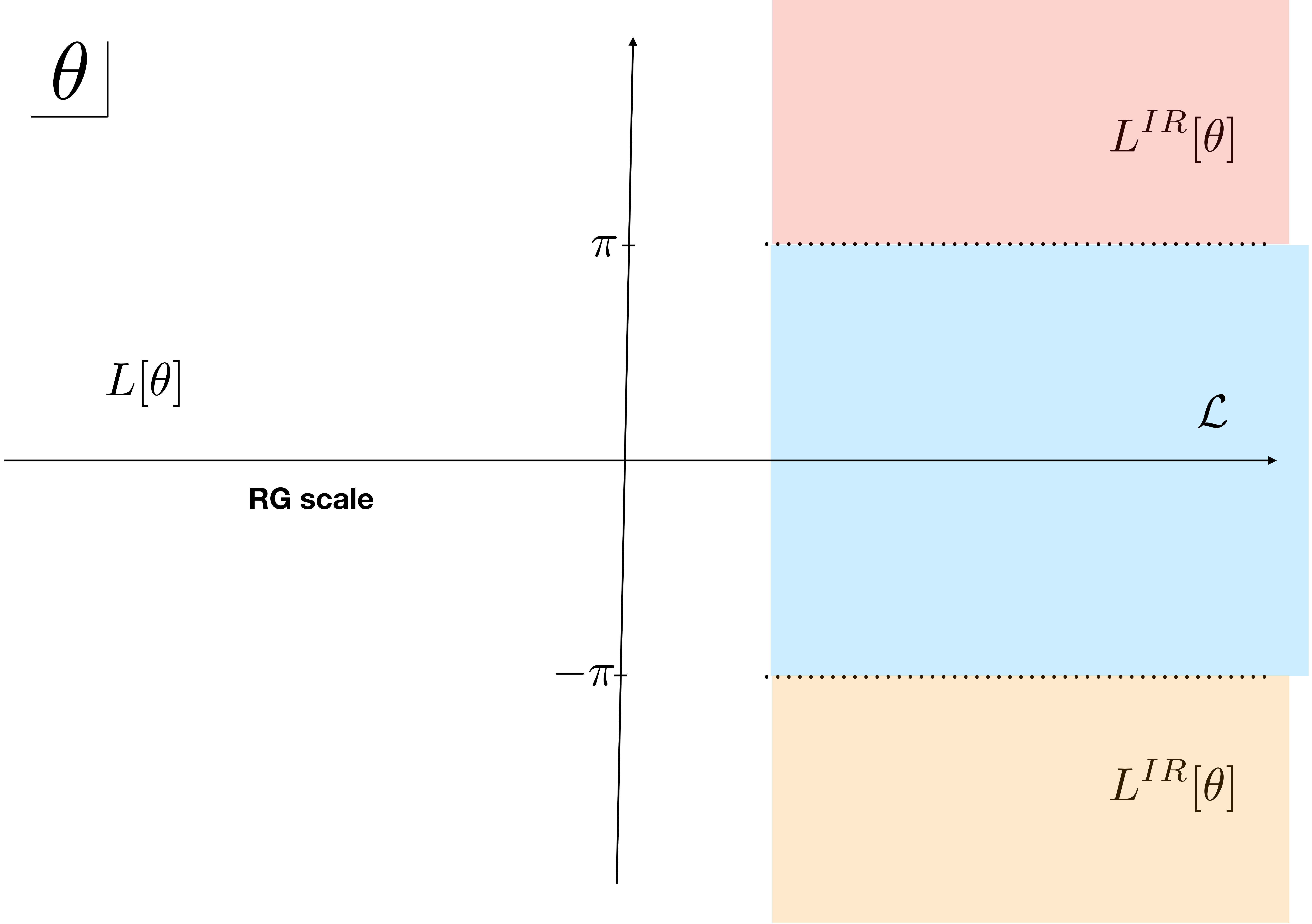
- After strong dynamics, flow conjecturally lands on topological line defect. $\mathcal{L}$, up to counterterm

$$\psi \to -\psi$$

- The g<0 line defect IR free $L^{IR}[\theta]$, useful later

Generalities of defects in chiral CFTs

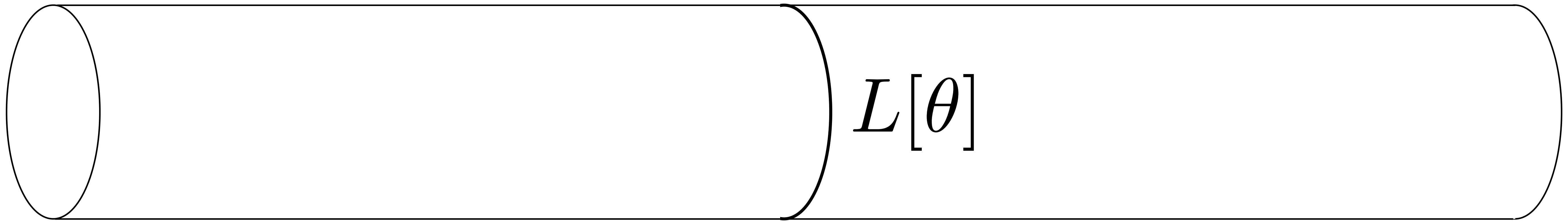
- Chiral interaction => invariance under rigid translations
- Local scale θ naturally complexified. Not unitary, but well defined and useful.
- Almost topological: changes of shape combined with imaginary shift of θ
- Phase transitions (level crossing) as a function of $\mathrm{Im}\,\theta$



θ
$L^{IR}[\theta]$
π
$L[\theta]$
$\mathcal{L}$
RG scale
$-\pi$
$L^{IR}[\theta]$

Line defects as transfer matrices

- Basic observable: line defect thermal free energy



- Wick rotate: $T[\theta; R] = \langle 0|\hat{T}[\theta; R]|0\rangle$
- transfer matrix $\hat{T}[\theta; R]$ commutes with Hamiltonian
- Only depends on $2\pi R e^{\theta}$, set $2\pi R = 1$

Hidden integrability

- We can compute in perturbation theory

$$\hat{T} = 2 + g^2 \hat{t}_2 + g^3 \hat{t}_3 + \cdots$$

- Surprise: $\left[\hat{T}[\theta], \hat{T}[\theta']\right] = 0$
- RG scale plays the role of spectral parameter
- Why? Affine Gaudin!

4d CS theory

- Holomorphic-topological theory

$$\int \omega(z) dz \wedge CS[A]$$

- Wilson lines W[z] along topological direction behave as transfer matrices
- Labelled by Yangian representations. Hirota fusion relations
- Useful local coordinate $d\theta = \omega(z) dz$
- Almost topological: changes of shape combined with imaginary shift of θ

Surface defects and Kondo lines

- 4d flat space: gauge theory has no dynamics, mediates integrable interactions which are local in topological plane
- Example: R-matrices R(z,z’) at “intersection" of W[z] and W[z’]
- Couple 4d CS to 2d chiral fermions/WZW model at $z=z_0$
- Gauge fields mediate no 2d couplings, but couple Wilson lines to WZW currents: Kondo line defects!

Predictions for basic Kondo

- Higher spin impurities: $L_j[\theta]$
- Conjectural flow to $L^{IR}_{j-\frac{1}{2}}[\theta] \otimes \mathcal{L}$
- Prediction: transfer matrices all commute
- Prediction: Hirota fusion
- $$T_j\left[\theta + \frac{i\pi}{2}\right] T_j\left[\theta - \frac{i\pi}{2}\right] = T_{j-\frac{1}{2}}[\theta] + T_{j+\frac{1}{2}}[\theta]$$

Generalizations

- Integrable Kondo lines in $\mathfrak{g}_{k_1} \times \cdots \times \mathfrak{g}_{k_n}$
- Labelled by Yangian irreps and positions $z_W; z_1, \cdots z_n$
- Commuting for different z_W
- Hirota-like fusion relations
- RG flow controlled by framing anomaly, beta function

$$\beta_{z_W} = \omega^{-1}(z_W) = \frac{1}{1 + \sum_i \frac{k_i}{2(z_W - z_i)}}$$

IM/ODE

- Kondo transfer matrix is transport coefficient of an ``affine oper with singularities of trivial monodromy"
- Example: g=su(2) vacuum vev from

$$\partial_x^2 \psi(x;\theta) = e^{2\theta+2x} \prod_i (1+g_i x)^{k_i} \psi(x;\theta)$$

- Excited states from extra trivial singularities
- IM/ODE as affine Geometric Langlands?