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Integral transforms with generalized Legendre functions as kernels

by

B. L. J. Braaksma and B. Meulenbeld

Part I

0. Introduction

0.1. The functions $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$, two specified linearly independent solutions of the differential equation:

$$(0.1)\quad (1-z^2)\frac{d^2w}{dz^2}-2z\frac{dw}{dz}+\left\{k(k+1)-\frac{m^2}{2(1-z)}-\frac{n^2}{2(1+z)}\right\}w=0$$

have been introduced by Kuipers and Meulenbeld [1] as functions of z for all points of the z-plane, in which a cross-cut exists along the real x-axis from 1 to $-\infty$, and for complex values of the parameters k, m and n. On the segment $-1<x<1$ of the cross-cut these functions are defined in [2, (1) and (2)].

For the sake of brevity we put

$$\alpha = k+\tfrac{1}{2}(m+n),\quad \beta = k-\tfrac{1}{2}(m-n),$$
$$\gamma = k+\tfrac{1}{2}(m-n),\quad \delta = k-\tfrac{1}{2}(m+n).$$

In terms of hypergeometric functions we have:

(0.2)
$$P_k^{m,n}(z) = (z-1)^{-\frac{1}{2}m}(z+1)^{\frac{1}{2}n}\frac{1}{\Gamma(1-m)}F(\beta+1, -\gamma; 1-m; \tfrac{1}{2}(1-z))$$

if z is not lying on the cross-cut,

(0.3)
$$P_k^{m,n}(x) = (1-x)^{-\frac{1}{2}m}(1+x)^{\frac{1}{2}n}\frac{1}{\Gamma(1-m)}F(\beta+1, -\gamma; 1-m; \tfrac{1}{2}(1-x))$$

if $-1<x<1$, and

$$Q_k^{m,n}(z) = e^{\pi i m} 2^{\beta} \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(2k+2)} (z-1)^{-k-\frac{1}{2}n-1}(z+1)^{\frac{1}{2}n} F\left(\alpha+1, \beta+1; 2k+2; \frac{2}{1-z}\right) \tag{0.4}$$

if z is not lying on the cross-cut.

0.2. In the present paper we shall derive a number of inversion formulas for integral transforms, in which these functions occur in the kernel.

The main results are contained in the following four theorems.

THEOREM 1. *Let k_1 be a real number with*

$$k_1 > \tfrac{1}{2}\operatorname{Re} m + \tfrac{1}{2}|\operatorname{Re} n| - 1, \tag{0.5}$$

and $\varphi(t)$ a function such that for all $a > 1$

$$\begin{aligned} &\varphi(t)(t-1)^{-\frac{1}{2}-\frac{1}{2}|\operatorname{Re} m|} \in \mathfrak{L}(1, a) \quad \text{if } \operatorname{Re} m \neq 0,\\ &\varphi(t)(t-1)^{-\frac{1}{2}} \log (t-1) \in \mathfrak{L}(1, a) \quad \text{if } \operatorname{Re} m = 0,\\ &\varphi(t)t^{-1-k_1} \in \mathfrak{L}(a, \infty). \end{aligned} \tag{0.6}$$

Let further $\varphi(t)$ be of bounded variation in a neighborhood of $t = x$ $(x > 1)$. Then $\varphi(t)$ satisfies the relations:

$$\frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty} dk(2k+1)P_k^{m,n}(x)\int_1^{\infty} \varphi(t)e^{\pi i m}Q_k^{-m,-n}(t)dt = \frac{\varphi(x-0)+\varphi(x+0)}{2}, \tag{0.7}$$

and

$$\frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty} dk(2k+1)e^{\pi i m}Q_k^{-m,-n}(x)\int_1^{\infty} \varphi(t)P_k^{m,n}(t)dt = \frac{\varphi(x-0)+\varphi(x+0)}{2}. \tag{0.8}$$

THEOREM 2. *Let k_1 be a real number satisfying* (0.5), *k_0 a complex number with* $\operatorname{Re} k_0 \geqq -\frac{1}{2}$ *and*

$$\operatorname{Re} k_0 > \tfrac{1}{2}\operatorname{Re} m + \tfrac{1}{2}|\operatorname{Re} n| - 1. \tag{0.9}$$

Suppose $f(k)$ is a function continuous on $\operatorname{Re} k \geqq \min(k_1, \operatorname{Re} k_0)$, *and analytic on* $\operatorname{Re} k > \min(k_1, \operatorname{Re} k_0)$. *Let further*

$$f(k)k^{\frac{1}{2}+m} \in \mathfrak{L}(k_1-i\infty, k_1+i\infty), \tag{0.10}$$

$$
(0.11)\qquad
\begin{aligned}
f(k) &= o(k^{-1-|\mathrm{Re}\, m|}) \ \textit{if} \ \mathrm{Re}\, m < 1, m \neq 0;\\
f(k) &= \frac{o(1)}{k \log k} \ \textit{if} \ m = 0;\\
f(k) &= o(k^{1-3m}) \ \textit{if} \ \mathrm{Re}\, m > 1;\\
f(k) &= \frac{o(1)}{k^2 \log k} \ \textit{if} \ \mathrm{Re}\, m = 1
\end{aligned}
$$

as $k \to \infty$ *on* $\mathrm{Re}\, k \geqq \min(k_1, \mathrm{Re}\, k_0)$.

If $\mathrm{Re}\, k_0 = k_1 \geqq -\frac{1}{2}$, *then* $f(k)$ *has to satisfy a Hölder-condition in a right neighborhood of* $k = k_0$.

Then $f(k)$ *satisfies the relation*:

$$
(0.12)\qquad \frac{1}{2\pi i}\int_1^{\infty} dx e^{\pi i m} Q_{k_0}^{-m,-n}(x) \int_{k_1-i\infty}^{k_1+i\infty} (2k+1) P_k^{m,n}(x) f(k) dk = f(k_0).
$$

THEOREM 3. *Let* k_1 *be a real number satisfying* (0.5), *and* k_0 *a complex number satisfying*

$$|\mathrm{Re}\,(2k_0+1)| > \mathrm{Re}\, m + |\mathrm{Re}\, n| - 1.$$

Suppose $f(k)$ *is a function continuous on* $\mathrm{Re}\, k \geqq p$, *and analytic on* $\mathrm{Re}\, k > p$, *where* $p = \min\{k_1, |\mathrm{Re}\,(k_0+\frac{1}{2})|-\frac{1}{2}\}$. *Let further*

$$f(k)k^{\frac{1}{2}-m} \in \mathfrak{L}(k_1-i\infty, k_1+i\infty),$$

$$
(0.13)\qquad
\begin{aligned}
f(k) &= k^{-|\mathrm{Re}(m-1)|}\, o(1) \ \textit{if} \ \mathrm{Re}\, m \neq 1;\\
f(k) &= o\left(\frac{1}{\log k}\right) \ \textit{if} \ \mathrm{Re}\, m = 1
\end{aligned}
$$

as $k \to \infty$ *on* $\mathrm{Re}\, k \geqq p$.

Then we have:

$$
(0.14)\qquad \int_1^{\infty} dx P_{k_0}^{m,n}(x) \int_{k_1-i\infty}^{k_1+i\infty} (2k+1) Q_k^{-m,-n}(x) f(k) dk = 0.
$$

THEOREM 4. *Let* m *be a complex number with* $\mathrm{Re}\, m < 1$.

I. *Let* S *be the strip* $|\mathrm{Re}\, k| < a$ *in the* k*-plane, and* $\bar{S}$ *the strip* $|\mathrm{Re}\, k| \leqq a$, *where* a *is a positive number such that* $Q_{-\frac{1}{2}+k}^{-m,-n}(x)$ *has no poles in* $\bar{S}$. *Let* k_1 *be a real and* k_0 *a complex number in* $\bar{S}$.

Suppose $f(k)$ *is an even function, analytic on* S *and continous on* $\bar{S}$, *satisfying*

$$f(k)k^{\frac{1}{2}-m} \in \mathfrak{L}(k_1-i\infty, k_1+i\infty),$$

$$
(0.15)\qquad f(k) = o(k^{m-\frac{1}{2}}) \ \textit{as} \ k \to \infty \ \textit{in} \ \bar{S}.
$$

When $|\mathrm{Re}\, k_0| = a$, *then* $f(k)$ *has to satisfy a Hölder-condition in a* $\bar{S}$*-neighborhood of* k_0.

Then we have:

$$(0.16)\qquad \frac{1}{2\pi i}\int_1^\infty dx P^{m,n}_{-\frac{1}{2}+k_0}(x)\int_{k_1-i\infty}^{k_1+i\infty} k e^{\pi i m} Q^{-m,-n}_{-\frac{1}{2}+k}(x) f(k)dk = \tfrac{1}{2}f(k_0).$$

II. *This formula also holds when* $k_1 = \mathrm{Re}\, k_0 = 0$, *and* $f(k)$ *is an even function defined only on the line* $\mathrm{Re}\, k = 0$ *and satisfies the conditions:*

$$(0.17)\qquad \begin{aligned} &kf(k) \in \mathfrak{L}(0, i),\\ &f(k)k^{\frac{1}{2}-m} \in \mathfrak{L}(i, i\infty), \end{aligned}$$

$f(k)$ *is of bounded variation in a neighborhood of* $k = k_0$. *In the righthand side of* (0.16) $\frac{1}{2}f(k_0)$ *has to be replaced by* $\frac{1}{4}\{f(k_0+0i)+ +f(k_0-0i)\}$.

Formula (0.12) is a direct inversion of (0.7), and (0.16) a direct inversion of (0.8). If with the inversion of (0.8) similar strong conditions on $f(k)$ are required as in theorem 2, the righthand side turns out to be zero instead of $\frac{1}{2}f(k_0)$, as theorem 3 shows.

0.3. In section 1 we give some applications of these theorems. At first we will deduce some equivalent forms in which $P^{m,n}_k(x)$ and $Q^{m,n}_k(x)$ occur.

Special cases are obtained by choosing appropriate relations between the parameters. Among others we obtain generalizations of theorems of Vilenkin [3], Götze [4] and Mehler-Fock [5], [6].

Equivalent integral transforms with hypergeometric functions as kernels are deduced, of which one gives a result of Titchmarsh [7], and the other a result of Olevskiĭ [8].

In section 2 asymptotic approximations of hypergeometric functions are given, in section 3 of $P^{m,n}_k(x)$. In the sections 4—7 we give the proofs of theorems 1—4.

Examples of the inversion formulas may be obtained by a suitable choice of the occurring arbitrary functions. Since there are not many integrals known in which the integration is with respect to the parameters in $P^{m,n}_k(x)$ and $Q^{m,n}_k(x)$, it is obvious that it is easier to find examples of the inversion formulas in theorem 1 than in the other theorems. In section 8 we give a number of these examples with applications.

1. Applications

1.1. In order to obtain equivalent forms of the theorems 1—4, we make use of the following lemma.

Lemma. *If $x > -1$ and $x \neq 1$, we have:*

$$P_k^{m,n}(x) = 2^{k+\frac{1}{2}n+1}(1+x)^{-\frac{1}{2}} P^{m,-2k-1}_{-\frac{1}{2}(n+1)}\left(\frac{x-3}{-x-1}\right). \tag{1.1}$$

If $-1 < x < 1$, then

$$P_k^{m,n}(x) = \frac{2^{-k-\frac{1}{2}m+1}}{\Gamma(-\alpha)\Gamma(\delta+1)}(1-x)^{-\frac{1}{2}} e^{\pi i n} Q^{-n,2k+1}_{-\frac{1}{2}(m+1)}\left(\frac{-x-3}{x-1}\right). \tag{1.2}$$

If $x > 1$, then

(1.3)

$$e^{-\pi i m} Q_k^{m,n}(x) = 2^{-k+\frac{1}{2}n-1}\Gamma(\alpha+1)\Gamma(\gamma+1)(1+x)^{-\frac{1}{2}} P^{-2k-1,-m}_{\frac{1}{2}(n-1)}\left(\frac{x-3}{x+1}\right),$$

(1.4)

$$P_k^{m,n}(x) = \frac{2^{2-\frac{1}{2}m+k}}{\Gamma(\beta+1)\Gamma(\delta+1)}(x-1)^{-\frac{1}{2}} e^{-(2k+1)\pi i} Q^{2k+1,n}_{-\frac{1}{2}(m+1)}\left(\frac{x+3}{x-1}\right),$$

(1.5)

$$e^{-\pi i m} Q_k^{m,n}(x) = 2^{-k-\frac{1}{2}m-1}\Gamma(\alpha+1)\Gamma(\gamma+1)(x-1)^{-\frac{1}{2}} P^{-2k-1,n}_{\frac{1}{2}(m-1)}\left(\frac{x+3}{x-1}\right).$$

Proof. In [9] it is deduced for x not lying on the real axis [1]:

$$P_k^{m,n}(x) = e^{\mp\frac{1}{2}\pi i m} 2^{k+\frac{1}{2}n+1}(x+1)^{-\frac{1}{2}} P^{m,-2k-1}_{-\frac{1}{2}(n+1)}\left(\frac{x-3}{-x-1}\right),$$

where the upper or lower sign in the exponential has to be taken according as $\operatorname{Im} x$ is positive or negative. From this and the definition of $P_k^{m,n}(x)$ on the cross-cut it is easily seen that (1.1) holds.

Combining

$$Q_k^{m,n}(x) = e^{2\pi i m} 2^{n-m} \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\beta+1)\Gamma(\delta+1)} Q_k^{-m,-n}(x)$$

(see [10, (7)]),

and

$$Q_k^{n,m}(-x) = -2^{m-n} e^{\pi i(\pm k-m+n)} \frac{\Gamma(\beta+1)}{\Gamma(\gamma+1)} Q_k^{m,n}(x)$$

(see [10, (25)]),

we find:

$$Q_k^{m,n}(x) = -e^{\pi i(\mp k+m+n)} \frac{\Gamma(\alpha+1)}{\Gamma(\delta+1)} Q_k^{-n,-m}(-x),$$

[1] In the formula [9, p. 357, last formula] the signs $\mp$ are not correct, they have to be replaced by $\pm$.

and this substituted in

$$Q^{-2k-1,n}_{-\frac{1}{2}(m+1)}\left(\frac{x+3}{x-1}\right) = e^{-\pi i(2k+1)}2^{k+\frac{1}{2}m-1}\Gamma(-\alpha)\Gamma(-\gamma)(x-1)^{\frac{1}{2}}P^{m,n}_k(x)$$

(see [9, (1)],

yields:

$$P^{m,n}_k(x) = \frac{2^{-k-\frac{1}{2}m+1}}{\Gamma(-\alpha)\Gamma(\delta+1)}(x-1)^{-\frac{1}{2}}e^{\pi i(n+1\mp\frac{1}{2}(m+1))}Q^{-n,2k+1}_{-\frac{1}{2}(m+1)}\left(\frac{-x-3}{x-1}\right).$$

Passing to the limit gives (1.2) for x on the cross-cut $-1 < x < 1$. Combining (1.1) and (1.2) we get (1.4) for $x > 1$. (1.3) and (1.5) are the inverses of (1.2) and (1.4) respectively.

1.2. In the theorems 1—4 we have integrated over the lower parameter k of $P^{m,n}_k(x)$ and $Q^{m,n}_k(x)$. By means of the lemma it is possible to find equivalents of these theorems (theorems 1a—4a), in which integration is over the first upper parameter m.

Applying (1.4) and (1.5) in theorem 1, and replacing k by $\frac{1}{2}(m-1)$, k_1 by $\frac{1}{2}(m_1-1)$, m by $-2k-1$, t by $(t+3)/(t-1)$, x by $(x+3)/(x-1)$, and $\varphi(t)$ by $\varphi(t)(t-1)^{\frac{1}{2}}$, this theorem is transformed into

THEOREM 1a. *Let m_1 be a real number with*

$$m_1 > |\mathrm{Re}\, n|-2\,\mathrm{Re}\, k-2, \tag{1.6}$$

and $\varphi(t)$ a function such that for all $a > 1$

$$\begin{aligned} &\varphi(t)(t-1)^{\frac{1}{2}m_1-1} \in \mathfrak{L}(1, a),\\ &\varphi(t)t^{-\frac{5}{4}+|\mathrm{Re}(k+\frac{1}{2})|} \in \mathfrak{L}(a, \infty) \quad \text{if } \mathrm{Re}\, k \neq -\tfrac{1}{2},\\ &\varphi(t)t^{-\frac{5}{4}}\log t \in \mathfrak{L}(a, \infty) \qquad \text{if } \mathrm{Re}\, k = -\tfrac{1}{2}. \end{aligned} \tag{1.7}$$

Let further $\varphi(t)$ be of bounded variation in a neighborhood of $t = x$ $(x > 1)$.

Then $\varphi(t)$ satisfies the relations:

$$\frac{1}{2\pi i}\int_{m_1-i\infty}^{m_1+i\infty} dm\, me^{-\pi i m}Q^{m,n}_k(x)\int_1^\infty \varphi(t)P^{-m,-n}_k(t)\frac{dt}{t-1} = \frac{\varphi(x-0)+\varphi(x+0)}{2}, \tag{1.8}$$

and

$$\frac{1}{2\pi i}\int_{m_1-i\infty}^{m_1+i\infty} dm\, mP^{-m,-n}_k(x)\int_1^\infty \varphi(t)e^{-\pi i m}Q^{m,n}_k(t)\frac{dt}{t-1} = \frac{\varphi(x-0)+\varphi(x+0)}{2}. \tag{1.9}$$

Applying again (1.4) and (1.5) in theorem 2, and replacing k by $\frac{1}{2}(m-1)$, k_1 by $\frac{1}{2}(m_1-1)$, k_0 by $\frac{1}{2}(m_0-1)$, m by $-2k-1$, x by $(x+3)/(x-1)$, and $f(k)$ by $2^{-\frac{1}{2}(m-1)}\Gamma(\alpha+1)\Gamma(\gamma+1)f(m)$, this theorem is transformed into

THEOREM 2a. *Let m_1 be a real number satisfying* (1.6), *and m_0 a complex number with* $\operatorname{Re} m_0 \geqq 0$ *and*

$$\operatorname{Re} m_0 > |\operatorname{Re} n| - 2 \operatorname{Re} k - 2. \tag{1.10}$$

Suppose $f(m)$ is a function continuous on $\operatorname{Re} m \geqq \min(m_1, \operatorname{Re} m_0)$, *and analytic on* $\operatorname{Re} m > \min(m_1, \operatorname{Re} m_0)$. *Let further*

(1.11)

$$f(m)\Gamma(m+1) \in \mathfrak{L}(m_1-i\infty, m_1+i\infty);$$

$$f(m)\Gamma(m+1) = 2^{\frac{3}{2}m} m^{-\frac{3}{2}-2k-|\operatorname{Re}(2k+1)|} o(1) \ \textit{if} \ \operatorname{Re} k > -1,\ k \neq -\tfrac{1}{2};$$

$$f(m)\Gamma(m+1) = 2^{\frac{3}{2}m} m^{-\frac{1}{2}} \frac{o(1)}{\log m} \ \textit{if} \ k = -\tfrac{1}{2} \ \textit{and if} \ \operatorname{Re} k = -1;$$

$$f(m)\Gamma(m+1) = 2^{\frac{3}{2}m} m^{\frac{7}{2}+4k} o(1) \ \textit{if} \ \operatorname{Re} k < -1$$

as $m \to \infty$ on $\operatorname{Re} m \geqq \min(m_1, \operatorname{Re} m_0)$.

If $\operatorname{Re} m_0 = m_1 \geqq 0$, *then $f(m)$ has to satisfy a Hölder-condition in a right neighborhood of $m = m_0$.*

Then $f(m)$ satisfies the relation:

(1.12)

$$\frac{1}{2\pi i}\int_1^\infty \frac{dx}{x-1} P_k^{-m_0,-n}(x) \int_{m_1-i\infty}^{m_1+i\infty} m e^{-\pi i m} Q_k^{m,n}(x) f(m)\, dm = f(m_0).$$

Using all the substitutions of the preceding case, but now replacing

$$f(k) \text{ by } \frac{2^{\frac{1}{2}(m-1)}}{\Gamma(\alpha+1)\Gamma(\gamma+1)} f(m),$$

theorem 3 can be transformed into

THEOREM 3a. *Let m_1 be a real number satisfying* (1.6), *and m_0 a complex number satisfying*

$$|\operatorname{Re} m_0| > |\operatorname{Re} n| - 2 \operatorname{Re} k - 2. \tag{1.13}$$

Suppose $f(m)$ is a function continuous on

$$\operatorname{Re} m \geqq \min(m_1, |\operatorname{Re} m_0|),$$

and analytic on

$$\operatorname{Re} m > \min(m_1, |\operatorname{Re} m_0|).$$

Let further

$$f(m)\{\Gamma(m)\}^{-1} \in \mathfrak{L}(m_1 - i\infty, m_1 + i\infty),$$

$$(1.14)\quad f(m)\{\Gamma(m)\}^{-1} = 2^{-\frac{1}{2}m} m^{2k+\frac{1}{2}-2|\mathrm{Re}(k+1)|} o(1) \text{ if } \mathrm{Re}\, k \neq -1,$$
$$= 2^{-\frac{1}{2}m} m^{-\frac{3}{2}} o\left(\frac{1}{\log m}\right) \text{ if } \mathrm{Re}\, k = -1$$

as $m \to \infty$ *on* $\mathrm{Re}\, m \geqq \min(m_1, |\mathrm{Re}\, m_0|)$.

Then we have:

$$(1.15)\quad \int_1^\infty \frac{dx}{x-1} Q_k^{m_0, n}(x) \int_{m_1 - i\infty}^{m_1 + i\infty} m P_k^{-m, -n}(x) f(m) dm = 0.$$

Applying (1.4) and (1.5) in theorem 4, and replacing k by $\frac{1}{2}m$, k_1 by $\frac{1}{2}m_1$, k_0 by $\frac{1}{2}m_0$, m by $-2k-1$, x by $(x+3)/(x-1)$, and $f(k)$ by $2^{\frac{1}{2}m}/\Gamma(\alpha+1)\Gamma(\gamma+1) f(m)$, this theorem is transformed into

THEOREM 4a. *Let k be a complex number with* $\mathrm{Re}\, k > -1$.

I. *Let S be the strip* $|\mathrm{Re}\, m| < a$ *in the m-plane, and $\bar{S}$ the strip* $|\mathrm{Re}\, m| \leqq a$, *where a is a positive number. Let m_1 be a real and m_0 a complex number in $\bar{S}$. Suppose $f(m)$ is a function analytic on S and continuous on $\bar{S}$, such that*

$$(1.16)\quad \frac{2^{\frac{1}{2}m}}{\Gamma(\alpha+1)\Gamma(\gamma+1)} f(m) \text{ is an even function of } m \text{ in } \bar{S}.$$

Let further

$$(1.17)\quad \begin{aligned} &f(m)\{\Gamma(m)\}^{-1} \in \mathfrak{L}(m_1 - i\infty, m_1 + i\infty),\\ &f(m)\{\Gamma(m)\}^{-1} = o(1), \text{ as } m \to \infty \text{ in } \bar{S}. \end{aligned}$$

When $|\mathrm{Re}\, m_0| = a$, *then $f(m)$ has to satisfy a Hölder-condition in a $\bar{S}$-neighborhood of m_0.*

Then we have:

(1.18)

$$\frac{1}{2\pi i} \int_1^\infty \frac{dx}{x-1} e^{-\pi i m_0} Q_k^{m_0, n}(x) \int_{m_1 - i\infty}^{m_1 + i\infty} m P_k^{-m, -n}(x) f(m) dm = f(m_0).$$

II. *This formula also holds when* $m_1 = \mathrm{Re}\, m_0 = 0$, *and $f(m)$ is only defined on the line* $\mathrm{Re}\, m = 0$, *and satisfies the condition* (1.16),

$$(1.19)\quad \begin{aligned} &m f(m) \in \mathfrak{L}(0, i),\\ &f(m)\{\Gamma(m)\}^{-1} \in \mathfrak{L}(i, i\infty), \end{aligned}$$

$f(m)$ is of bounded variation in a neighborhood of $m = m_0$. *In the*

righthand side of (1.18) $f(m_0)$ *has to be replaced by*

$$\tfrac{1}{2}\{f(m_0-0i)+f(m_0+0i)\}.$$

1.3. In the formulas of the preceding theorems both functions P and Q occur. By means of the lemma it is also possible to derive theorems 1b—4b, equivalent to theorems 1—4, in which only the function $P_k^{m,n}$ occur. In this case the integration is carried out with respect to the upper parameters.

Instead of proving the theorems 1—4 we shall give the proofs of the theorems 1b—4b in sections 4—7.

Theorem 1 may be transformed by applying (1.1) and (1.2). Further, replacing k by $-\frac{1}{2}(n+1)$, k_1 by $-\frac{1}{2}(n_1+1)$, n by $-2k-1$, t by $-(t+3)/(t-1)$, x by $-(x+3)/(x-1)$, $\varphi(t)$ by $(1-t)^{\frac{1}{2}}\varphi(t)$, this theorem can be written in the form:

Theorem 1b. *Let n_1 be a real number with*

$$n_1 < \min\{\mathrm{Re}\,(2k+2-m),\ \mathrm{Re}\,(-2k-m)\}, \tag{1.20}$$

and $\varphi(t)$ a function such that for all a, $-1<a<1$

$$\begin{aligned}
&\varphi(t)(1+t)^{-\frac{1}{2}-\frac{1}{2}|\mathrm{Re}\,m|} \in \mathfrak{L}(-1, a) \qquad \textit{if } \mathrm{Re}\,m \neq 0,\\
&\varphi(t)(1+t)^{-\frac{1}{2}} \log(1+t) \in \mathfrak{L}(-1, a) \ \ \textit{if } \mathrm{Re}\,m = 0,\\
&\varphi(t)(1-t)^{-1-\frac{1}{2}n_1} \in \mathfrak{L}(a, 1).
\end{aligned} \tag{1.21}$$

Let further $\varphi(t)$ be of bounded variation in a neighborhood of $t=x(-1<x<1)$.

Then $\varphi(t)$ satisfies the relations:

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{n_1-i\infty}^{n_1+i\infty} dn\, n\Gamma(\delta+1)\Gamma(-\alpha)P_k^{m,n}(-x)\int_{-1}^{1}\varphi(t)P_k^{n,m}(t)\frac{dt}{1-t}\\
&\qquad\qquad = -\{\varphi(x-0)+\varphi(x+0)\},
\end{aligned} \tag{1.22}$$

and

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{n_1-i\infty}^{n_1+i\infty} dn\, n\Gamma(\delta+1)\Gamma(-\alpha)P_k^{n,m}(x)\int_{-1}^{1}\varphi(t)P_k^{m,n}(-t)\frac{dt}{1-t}\\
&\qquad\qquad = -\{\varphi(x-0)+\varphi(x+0)\}.
\end{aligned} \tag{1.23}$$

Using again (1.1) and (1.2) in theorem 2, and replacing k by $-\frac{1}{2}(n+1)$, k_1 by $-\frac{1}{2}(n_1+1)$, k_0 by $-\frac{1}{2}(n_0+1)$, n by $-2k-1$, x by $-(x+3)/(x-1)$, and $f(k)$ by $2^{\frac{1}{2}n}f(n)$, this theorem can be transformed into

Theorem 2b. *Let n_1 be a real number satisfying* (1.20), *n_0 a complex number with* $\mathrm{Re}\, n_0 \leqq 0$, *and*

(1.24) $\qquad \operatorname{Re} n_0 < \min \{\operatorname{Re}(2k+2-m), \operatorname{Re}(-2k-m)\}.$

Suppose $f(n)$ *is a function continuous on* $\operatorname{Re} n \leqq \max(n_1, \operatorname{Re} n_0)$, *and analytic on* $\operatorname{Re} n < \max \{n_1, \operatorname{Re} n_0\}$.

Let further

$$f(n)n^{m+\frac{1}{2}} \in \mathfrak{L}(n_1-i\infty, n_1+i\infty), \tag{1.25}$$

$$\begin{aligned} f(n) &= 2^{-\frac{1}{2}n}n^{-1-|\operatorname{Re} m|}o(1) && \text{if } \operatorname{Re} m < 1,\ m \neq 0; \\ f(n) &= 2^{-\frac{1}{2}n}\frac{o(1)}{n \log n} && \text{if } m = 0; \\ f(n) &= 2^{-\frac{1}{2}n}n^{1-3m}o(1) && \text{if } \operatorname{Re} m > 1; \\ f(n) &= 2^{-\frac{1}{2}n}\frac{o(1)}{n^2 \log n} && \text{if } \operatorname{Re} m = 1 \end{aligned} \tag{1.26}$$

as $n \to \infty$ *on* $\operatorname{Re} n \leqq \max \{n_1, \operatorname{Re} n_0\}$.

If $\operatorname{Re} n_0 = n_1 \leqq 0$, *then* $f(n)$ *has to satisfy a Hölder-condition in a left neighborhood of* $n = n_0$.

Then we have:

(1.27)

$$\frac{1}{2\pi i}\int_{-1}^{1} \frac{dx}{1-x} P_k^{n_0,m}(x) \int_{n_1-i\infty}^{n_1+i\infty} nP_k^{m,n}(-x)f(n)dn = \frac{-2f(n_0)}{\Gamma(\delta_0+1)\Gamma(-\alpha_0)},$$

where $\alpha_0 = k+\frac{1}{2}(m+n_0)$, $\delta_0 = k-\frac{1}{2}(m+n_0)$.

Using all the substitutions of the preceding case, but now replacing $f(k)$ by $2^{-\frac{1}{2}n}f(n)$, theorem 3 can be transformed into

THEOREM 3b. *Let* n_1 *be a real number satisfying* (1.20), *and* n_0 *a complex number with*

(1.28) $\qquad |\operatorname{Re} n_0| > \max \{\operatorname{Re}(m+2k), \operatorname{Re}(m-2k-2)\}.$

Suppose $f(n)$ *is a function continuous on*

$$\operatorname{Re} n \leqq \max \{n_1, -|\operatorname{Re} n_0|\},$$

and analytic on

$$\operatorname{Re} n < \max \{n_1, -|\operatorname{Re} n_0|\}.$$

Let further

$$f(n)n^{\frac{1}{2}-m} \in \mathfrak{L}(n_1-i\infty, n_1+i\infty), \tag{1.29}$$

$$\begin{aligned} f(n) &= 2^{\frac{1}{2}n}n^{-|\operatorname{Re}(m-1)|}o(1) && \text{if } \operatorname{Re} m \neq 1; \\ f(n) &= 2^{\frac{1}{2}n}\, o\left(\frac{1}{\log n}\right) && \text{if } \operatorname{Re} m = 1 \end{aligned} \tag{1.30}$$

as $n \to \infty$ *on* $\operatorname{Re} n \leqq \max \{n_1, -|\operatorname{Re} n_0|\}$.

Then we have:

(1.31)

$$\int_{-1}^{1}\frac{dx}{1-x}\,P_k^{m,\,n_0}(-x)\int_{n_1-i\infty}^{n_1+i\infty} n\Gamma(\delta+1)\Gamma(-\alpha)P_k^{n,\,m}(x)f(n)dn = 0.$$

Applying (1.1) and (1.2) in theorem 4, and replacing k by $-\frac{1}{2}n$, k_1 by $-\frac{1}{2}n_1$, k_0 by $-\frac{1}{2}n_0$, n by $-2k-1$, x by $(3+x)/(1-x)$, and $f(k)$ by $2^{-\frac{1}{2}n}f(n)$, this theorem is transformed into

THEOREM 4b. *Let m be a complex number with* $\operatorname{Re} m < 1$.

I. *Let S be the strip* $|\operatorname{Re} n| < a$ *in the n-plane, and $\bar{S}$ the strip* $|\operatorname{Re} n| \leqq a$, *where a is positive number such that $\Gamma(\delta+1)\Gamma(-\alpha)$ has no poles in $\bar{S}$. Let n_1 be a real and n_0 a complex number in $\bar{S}$.*

Suppose $f(n)$ is a function analytic in S, and continuous in $\bar{S}$, satisfying:

(1.32) $$f(n) = 2^n f(-n),$$

(1.33) $$f(n)n^{\frac{1}{2}-m} \in \mathfrak{L}(n_1-i\infty,\, n_1+i\infty),$$
$$f(n) = o(n^{m-\frac{1}{2}}) \text{ as } n \to \infty \text{ in } \bar{S}.$$

If $|\operatorname{Re} n_0| = a$, *then $f(n)$ has to satisfy a Hölder-condition in a $\bar{S}$-neighborhood of n_0.*

Then we have:

(1.34)

$$\frac{1}{2\pi i}\int_{-1}^{1}\frac{dx}{1-x}\,P_k^{m,\,n_0}(-x)\int_{n_1-i\infty}^{n_1+i\infty} n\Gamma(\delta+1)\Gamma(-\alpha)P_k^{n,\,m}(x)f(n)dn = -2f(n_0).$$

II. *This formula also holds when* $n_1 = \operatorname{Re} n_0 = 0$, *and $f(n)$ is only defined on the line* $\operatorname{Re} n = 0$, *and satisfies the conditions* (1.32),

(1.35) $$nf(n) \in \mathfrak{L}(0,\, i),$$
$$f(n)n^{\frac{1}{2}-m} \in \mathfrak{L}(i,\, i\infty),$$

$f(n)$ is of bounded variation in a neighborhood of $n = n_0$.

In the righthand side of (1.34) $-2f(n_0)$ *has to be replaced by* $-\{f(n_0-0i)+f(n_0+0i)\}$.

1.4. In this section we consider some special cases of the preceding theorems.

By choosing $k_1 = -\frac{1}{2}$ theorem 1 becomes

THEOREM 5. *Let m and n be complex numbers with*

$$(1.36) \qquad |\mathrm{Re}\, n| < 1 - \mathrm{Re}\, m,$$

and $\varphi(t)$ a function such that (0.6) *with $k_1 = -\frac{1}{2}$ holds. Let further this function be of bounded variation in a neighborhood of $t = x$ $(x > 1)$.*

Then $\varphi(t)$ satisfies the relation:

(1.37)

$$\int_0^\infty dr \frac{\Gamma\left(\frac{1-m+n}{2}+ir\right)\Gamma\left(\frac{1-m+n}{2}-ir\right)\Gamma\left(\frac{1-m-n}{2}+ir\right)\Gamma\left(\frac{1-m-n}{2}-ir\right)}{\Gamma(2ir)\Gamma(-2ir)}$$
$$P^{m,n}_{-\frac{1}{2}+ir}(x)\int_1^\infty P^{m,n}_{-\frac{1}{2}+ir}(t)\varphi(t)dt = \pi 2^{n-m+1}\{\varphi(x-0)+\varphi(x+0)\}.$$

Proof. If we substitute in (0.7) $k_1 = -\frac{1}{2}$, $k = -\frac{1}{2}+ir$, and use the relation $P^{m,n}_k(x) = P^{m,n}_{-1-k}(x)$, then we find for the lefthand side of (0.7):

$$(1.38) \qquad \frac{i}{\pi}\int_0^\infty dr\, r P^{m,n}_{-\frac{1}{2}+ir}(x)\int_1^\infty \varphi(t)e^{\pi i m}\{Q^{-m,-n}_{-\frac{1}{2}+ir}(t) - Q^{-m,-n}_{-\frac{1}{2}-ir}(t)\}dt.$$

In order to reduce the expression between the braces we combine the formulas:

$$Q^{-m,-n}_k(t) = e^{-2\pi i m}2^{m-n}\frac{\Gamma(\beta+1)\Gamma(\delta+1)}{\Gamma(\gamma+1)\Gamma(\alpha+1)}Q^{m,n}_k(t) \qquad \text{(see [10, (7)])}$$

and

$$\sin\alpha\pi\sin\gamma\pi Q^{m,n}_k(t) - \sin\delta\pi\sin\beta\pi Q^{m,n}_{-k-1}(t) = \frac{\pi}{2}e^{\pi i m}\sin 2k\pi P^{m,n}_k(t),$$

(see [10, (9)])

and obtain the formula:

$$(1.39) \qquad \begin{aligned} &e^{\pi i m}\{Q^{-m,-n}_k(t) - Q^{-m,-n}_{-k-1}(t)\} \\ &\quad = 2^{m-n-1}\Gamma(\beta+1)\Gamma(\delta+1)\Gamma(-\alpha)\Gamma(-\gamma)\frac{\sin 2k\pi}{\pi}P^{m,n}_k(t). \end{aligned}$$

Now (1.37) follows from (0.7), (1.38) and (1.39).

Remark 1. If in the proof we substitute in (0.8) instead of in (0.7), we obtain the same result (1.37).

To find the inverse of theorem 5 we transform case II of theorem 4. The result is

Theorem 6. *Let s be a positive number,* $\mathrm{Re}\, m < 1$; *$f(r)$ is defined for $r > 0$, and is of bounded variation in a neighborhood of $r = s$. Let further*

$$f(r) \in \mathfrak{L}(0, 1), \qquad f(r)r^{m-\frac{1}{2}} \in \mathfrak{L}(1, \infty). \tag{1.40}$$

Then we have:

(1.41)

$$\int_1^\infty dx\, P^{m,n}_{-\frac{1}{2}+is}(x) \int_0^\infty f(r) P^{m,n}_{-\frac{1}{2}+ir}(x) dr = \frac{\pi 2^{n-m+1}\Gamma(2is)\Gamma(-2is)\{f(s-0)+f(s+0)\}}{\Gamma\left(\frac{1-m+n}{2}+is\right)\Gamma\left(\frac{1-m+n}{2}-is\right)\Gamma\left(\frac{1-m-n}{2}+is\right)\Gamma\left(\frac{1-m-n}{2}-is\right)}.$$

PROOF. The proof is quite analogous to that of theorem 5. Now we substitute $k = ir$, $k_0 = is$, and replace $f(k)$ by

$$\frac{\Gamma(2ir)\Gamma(-2ir)f(r)}{\Gamma\left(\frac{1-m+n}{2}+ir\right)\Gamma\left(\frac{1-m+n}{2}-ir\right)\Gamma\left(\frac{1-m-n}{2}+ir\right)\Gamma\left(\frac{1-m-n}{2}-ir\right)}.$$

REMARK 2. Theorem 6 is an extension of theorem 1 of Götze [4, p. 402].

If in theorem 1—6 we choose $m = n$, then since $P^{m,m}_k(x) = P^m_k(x)$ and $Q^{m,m}_k(x) = Q^m_k(x)$, these theorems give inversion formulas for integral transformations with the associated Legendre functions $P^m_k(x)$ and $Q^m_k(x)$ as kernels. For $m = n = 0$ the ordinary Legendre functions occur. Theorem 5 simplifies for $m = n$ to

THEOREM 7. *Let m be a complex number with* $\operatorname{Re} m < \frac{1}{2}$, *and let $\varphi(t)$ be defined for $t > 1$ and be of bounded variation in a neighborhood of $t = x$, satisfying* (0.6) *with $k_1 = -\frac{1}{2}$.*

Then we have:

$$\int_0^\infty dr \frac{\Gamma(\frac{1}{2}-m+ir)\Gamma(\frac{1}{2}-m-ir)}{\Gamma(ir)\Gamma(-ir)} P^m_{-\frac{1}{2}+ir}(x) \int_1^\infty P^m_{-\frac{1}{2}+ir}(t)\varphi(t)\,dt = \frac{\varphi(x-0)+\varphi(x+0)}{2}. \tag{1.42}$$

If $m = 0$, (1.42) *becomes:*

$$\int_0^\infty dr\, r \tanh \pi r P_{-\frac{1}{2}+ir}(x) \int_1^\infty P_{-\frac{1}{2}+ir}(t)\varphi(t)dt = \frac{\varphi(x-0)+\varphi(x+0)}{2}. \tag{1.43}$$

Theorem 6 simplifies for $m = n$ to

THEOREM 8. *Suppose the conditions of theorem* 6 *are satisfied. Then we have*:

$$(1.44)\quad \begin{aligned}&\int_1^\infty dx P^m_{-\frac{1}{2}+is}(x)\int_0^\infty f(r)P^m_{-\frac{1}{2}+ir}(x)\,dr\\ &\qquad = \frac{\Gamma(is)\Gamma(-is)}{\Gamma(\frac{1}{2}-m+is)\Gamma(\frac{1}{2}-m-is)}\,\frac{f(s-0)+f(s+0)}{2}.\end{aligned}$$

If $m = 0$, (1.44) *becomes*:

$$(1.45)\quad \int_1^\infty dx P_{-\frac{1}{2}+is}(x)\int_0^\infty f(r)P_{-\frac{1}{2}+ir}(x)\,dr = \frac{f(s-0)+f(s+0)}{2s\tanh \pi s}.$$

REMARK 3. The formulas (1.42) and (1.44) have been found by Vilenkin [3]. The formulas (1.43) and (1.45) are the well-known Mehler-Fock transformation formulas (see [5] and [6]).

1.5. Since the functions $P^{m,n}_k(x)$ and $Q^{m,n}_k(x)$ can be expressed in terms of hypergeometric functions, the preceding theorems can be transformed into theorems on integral transforms with hypergeometric kernels. We shall not carry out all these transformations, but give two examples corresponding to the theorems 5 and 6. Theorem 5 gives rise to the following

THEOREM 9. *Let* a *and* c *be complex numbers with* $0 < \operatorname{Re} a < \operatorname{Re} c$. *Suppose* $\varphi(t)$ *is a function such that for all* $p > 0$

$$(1.46)\quad \begin{aligned}&\varphi(t)t^{-\frac{1}{4}-\frac{1}{2}|1-\operatorname{Re} c|} \in \mathfrak{L}(0, p) && \text{if } \operatorname{Re} c \neq 1,\\ &\varphi(t)t^{-\frac{1}{4}}\log t \in \mathfrak{L}(0, p) && \text{if } \operatorname{Re} c = 1,\\ &\varphi(t)t^{-\frac{1}{4}} \in \mathfrak{L}(p, \infty).\end{aligned}$$

Let further $\varphi(t)$ *be of bounded variation in a neighborhood of* $t = x$ $(x > 0)$. *Then we have*:

$$(1.47)\quad \begin{aligned}&\int_0^\infty db\, b \sinh 2b\pi\, \Gamma(a+ib)\Gamma(a-ib)\Gamma(c-a+ib)\Gamma(c-a-ib)\\ &x^{\frac{1}{2}(c-1)}(1+x)^{a-\frac{1}{2}c}F(a+ib, a-ib; c; -x)\int_0^\infty t^{\frac{1}{2}(c-1)}(1+t)^{a-\frac{1}{2}c}\\ &F(a+ib, a-ib; c; -t)\varphi(t)\,dt = \pi^2\{\Gamma(c)\}^2\cdot\frac{\varphi(x-0)+\varphi(x+0)}{2}.\end{aligned}$$

PROOF. Applying (0.2) to (1.37), and replacing n by $2a-c$, m by $1-c$, r by b, $\frac{1}{2}(1-x)$ by $-x$, $\frac{1}{2}(1-t)$ by $-t$, the theorem follows immediately.

REMARK 4. This theorem is an extension of a result of Titchmarsh [7, p. 93].

From theorem 6 we can derive

THEOREM 10. *Let a and c be complex numbers with* $\operatorname{Re} c > 0$. *Suppose $f(b)$ is a function such that*

$$
\begin{aligned}
&f(b) \in \mathfrak{L}(0, 1), \\
&f(b)b^{\frac{1}{2}-c} \in \mathfrak{L}(1, \infty),
\end{aligned} \tag{1.48}
$$

$f(b)$ is of bounded variation in a neighborhood of $b = b_0$ $(b_0 > 0)$. Then we have:

$$
\begin{aligned}
&\int_0^\infty dx\, x^{c-1}(1+x)^{2a-c} F(a+ib_0, a-ib_0; c; -x) \\
&\int_0^\infty f(b) F(a+ib, a-ib; c; -x) db \\
&= \frac{\pi^2\{\Gamma(c)\}^2}{b_0 \sinh 2\pi b_0 \Gamma(a+ib_0)\Gamma(a-ib_0)\Gamma(c-a+ib_0)\Gamma(c-a-ib_0)} \cdot \\
&\frac{f(b_0-0)+f(b_0+0)}{2}.
\end{aligned} \tag{1.49}
$$

The proof is analogous to that of theorem 9.

REMARK 5. The formula (1.49) has been found by Olevskiĭ [8].

Part II

2. Asymptotic approximations of a hypergeometric function

The object of this section is to obtain asymptotic approximations of the hypergeometric function $F(a+n, b+n; c+2n; x)$ as $n \to \infty$, uniformly on $0 \leqq x < 1$.

Let L_1 and L_2 be closed paths in the u-plane, starting from a point u_0 with $0 < u_0 < 1$, encircling the points $u = 0$, $u = 1$ respectively, both counter-clockwise. The point $u = x^{-1}$ is not encircled by these paths. It is easily seen that

$$
\begin{aligned}
F(a+n, b+n; c+2n; x) = \frac{\Gamma(c+2n)}{\Gamma(b+n)\Gamma(c-b+n)} \{&(-1+e^{2\pi i(n+b)})^{-1} I_1 \\
&+ (1-e^{2\pi i(n+c-b)})^{-1} I_2\},
\end{aligned} \tag{2.1}
$$

where

$$I_j = \int_{L_j} \left(\frac{u(1-u)}{1-ux}\right)^n u^{b-1}(1-u)^{c-b-1}(1-ux)^{-a}du \qquad (j = 1, 2),$$

if $c+2n \neq 0, -1, \ldots; b+n \neq 1, 2, \ldots; c-b+n \neq 1, 2, \ldots$.

The phases of u, $1-u$ and $1-ux$ will be taken to be zero at the starting point $u = u_0$. In order to apply the method of steepest descents we choose L_j in such a way that $|\{u(1-u)/1-ux\}^n|$ decreases or increases strongly along L_j. The derivative of $\log u(1-u)/1-ux$ is equal to $(xu^2-2u+1)/u(1-u)(1-ux)$, and vanishes for $u = (1\pm\sqrt{1-x})/x$.

Since $0 < (1-\sqrt{1-x})/x < 1$ we choose for the starting point $u_0 = (1-\sqrt{1-x})/x = 1/1+\sqrt{1-x}$.

Because of $u_0 \uparrow 1$ as $x \uparrow 1$, we change the variables x and u by means of $y = \sqrt{1-x}$ and $u = 1-yt$. I_j is then transformed into

$$I_j = -y^{c-b-a}\int_{C_j}\left(\frac{t(1-yt)}{t(1-y^2)+y}\right)^n (1-yt)^{b-1}t^{c-b-1}\{t(1-y^2)+y\}^{-a}dt$$
$$(j = 1, 2),$$

where C_1 and C_2 are closed paths with starting point $t_0 = 1/1+y$, encircling the points $t = y^{-1}$, $t = 0$ respectively, both in positive sense, whereas the point $t = -y/1-y^2$, corresponding to $u = x^{-1}$, is not encircled by C_j. In the starting point t_0 the powers in the integrand of I_j have the principal values. If we put

$$g(t) = \log \frac{t(1-yt)}{t(1-y^2)+y},$$

we get

$$g(t_0) = \log \frac{1}{(1+y)^2}.$$

Now we choose for C_1 and C_2 paths on which $\operatorname{Re} g(t) = g(t_0)$, so that

$$\left|\frac{t(1-yt)}{t(1-y^2)+y}\right| = \frac{1}{(1+y)^2}.$$

It is easily verified that these curves satisfy the conditions mentioned above.

If t describes the curves C_1 and C_2, the function $|e^{ng(t)}|$ decreases or increases monotonically according as $\operatorname{Im} n > 0$ or < 0. From

$$g^{\mathrm{I}}(t) = \frac{y\{(y^2-1)t^2-2yt+1\}}{t(1-yt)\{(1-y^2)t+y\}};$$

$$g^{\mathrm{II}}(t) = -\frac{1}{t^2}-\frac{y^2}{(1-yt)^2}+\frac{(1-y^2)^2}{\{(1-y^2)t+y\}^2};$$

$$g^{\mathrm{III}}(t) = \frac{2}{t^3}-\frac{2y^3}{(1-yt)^3}-\frac{2(1-y^2)^3}{\{(1-y^2)t+y\}^3};$$

$$g^{\mathrm{IV}}(t) = -\frac{6y^4}{(1-yt)^4}-\frac{6y\{4t^3(1-y^2)^3+6yt^2(1-y^2)^2+4y^2t(1-y^2)+y^3\}}{t^4\{(1-y^2)t+y\}^4},$$

we see that

$$g^{\mathrm{II}}(t_0) = -2y(1+y)^2, \quad g^{\mathrm{III}}(t_0) = 6y(1+y)^3(1-y),$$

and $y^{-1}g^{\mathrm{IV}}(t)$ is uniformly bounded on $|t-t_0| < \frac{1}{4}$, $0 < y \leqq 1$. Therefore

$$\begin{aligned} g(t) = &-2\log(1+y)-y(1+y)^2(t-t_0)^2 \\ &+y(1+y)^3(1-y)(t-t_0)^3+y(t-t_0)^4O(1). \end{aligned} \tag{2.2}$$

Furthermore we have

$$\begin{aligned} &(1-yt)^{b-1}t^{c-b-1}\{t(1-y^2)+y\}^{-a} \\ &\quad = (1+y)^{2-c}\{1+q(1+y)(t-t_0)+O(t-t_0)^2\} \end{aligned} \tag{2.3}$$

$(q = c-b-a-1+y(a-b+1))$ as $t \to t_0$, uniformly in y.

We first consider the case: $n = i\lambda e^{i\varphi}$, $\lambda > 0$, $|\varphi| \leqq \frac{1}{2}\pi-\eta$ $(0 < \eta < \frac{1}{2}\pi)$, $\lambda y \to \infty$.

In order to evaluate the contribution of the saddle point to the integral I_2, we consider the integral I_{21}, along the first part of C_2 from t_0 to t_1, lying inside the circle $|t-t_0| < (\lambda y)^{-\frac{1}{3}}$. Putting

$$t_1-t_0 = (\lambda y)^{-\frac{1}{3}}e^{\frac{3}{4}\pi i+i\varepsilon'}, \text{ where } \varepsilon' \to 0 \text{ as } \lambda y \to \infty, \tag{2.4}$$

and substituting $t-t_0 = (1+y)^{-1}(\lambda y)^{-\frac{1}{2}}e^{\frac{3}{4}\pi i}v^{\frac{1}{2}}$, one may verify, using (2.2) and (2.3), that

$$I_{21} = -y^{c-a-b}(1+y)^{1-c-2n}\frac{e^{\frac{3}{4}\pi i}}{2\sqrt{\lambda y}}\int_0^{v_1} e^{-e^{i\varphi}v}\cdot$$

$$\left(1+e^{(\frac{3}{4}\pi+\varphi)i}\frac{1-y}{\sqrt{\lambda y}}v^{\frac{3}{2}}+\frac{v^2}{\lambda y}O(1)+\frac{v^3}{\lambda y}O(1)\right)\left(1+e^{\frac{3}{4}\pi i}\frac{q}{\sqrt{\lambda y}}v^{\frac{1}{2}}+\frac{v}{\lambda y}O(1)\right)v^{-\frac{1}{2}}dv,$$

with $v_1 = (\lambda y)^{\frac{1}{3}}(1+y)^2e^{2i\varepsilon_1}$, as $\lambda y \to \infty$, uniformly on $0 < y \leqq 1$. From this one may deduce:

$$(2.5) \quad I_{21} = y^{c-a-b}(1+y)^{1-c-2n}\left\{\frac{\sqrt{\pi}}{2\sqrt{ny}} - \frac{1-y+q}{2ny} + \frac{O(1)}{(ny)^{\frac{3}{2}}}\right\}$$

as $ny \to \infty$, uniformly on $0 < y \leqq 1$.

In order to approximate the contribution I_{22} to the integral I_2 of the remaining part C_{22} of C_2, we remark that $|e^{ng(t)}|$ is monotonically decreasing, and that the arc length of C_2 is bounded for $0 < y \leqq 1$. Therefore from (2.2) and (2.4) we have

$$(2.6) \qquad |e^{ng(t)}| \leqq |e^{ng(t_1)}| \leqq K\,|(1+y)^{-2n}|\exp -c_0(\lambda y)^{\frac{1}{3}},$$

where c_0 and K are positive constants, independent of y and λ. Moreover, to estimate the other factors occurring in the integrand of I_2, we remark that on C_2 we have $|t| > \frac{1}{6}y$, and $|t+y/1-y^2| > \frac{1}{6}y$ for $0 < y \leqq 1$, as easily can be calculated, and find that these factors are bounded for $\delta \leqq y \leqq 1$, where δ is a positive constant < 1. Therefore

$$(2.7) \qquad |I_{22}| \leqq K'y^{\mathrm{Re}(c-a-b)}|(1+y)^{-2n}|\exp -c_0(\lambda y)^{\frac{1}{3}}$$

for $\delta \leqq y \leqq 1$.

In the case $0 < y \leqq \delta$ we split up C_{22} into three parts: C_{23} is the part where $|t| \geqq \delta_1$ (δ_1 will be suitably chosen $> \delta$), C_{24} where $y \leqq |t| \leqq \delta_1$ and C_{25} where $|t| \leqq y$. Let the points of the division be t_2, t_2'; t_3, t_3'; and t_4 the real point on C_{22} (see fig. 1). We denote the corresponding contributions to the integral I_2 by I_{23}, I_{24} and I_{25} respectively.

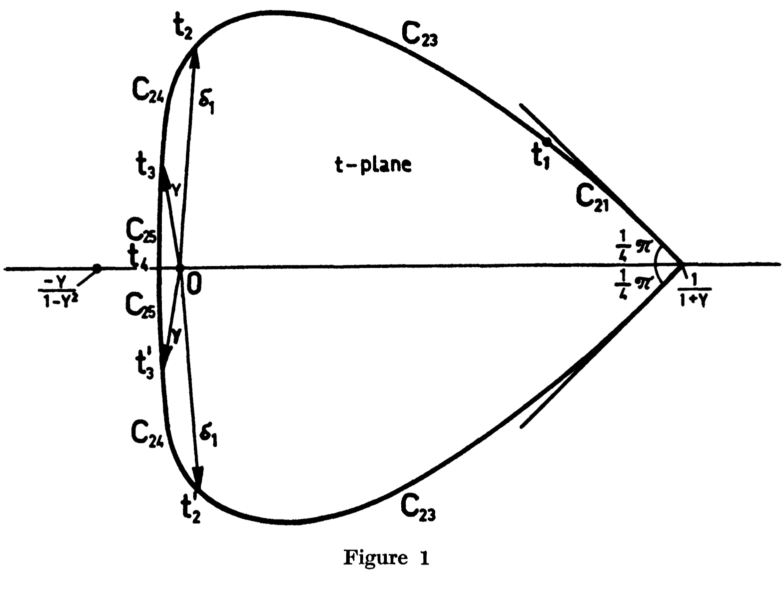

Figure 1

Since for sufficiently small δ we have $\frac{1}{6}y < |t| \leqq y$ and $\frac{1}{6}y < |t+y/1-y^2| < 2y$ on C_{25}, it follows from (2.6):

$$I_{25} = y^{2(c-a-b)}(1+y)^{-2n}e^{-c_0(\lambda y)^{\frac{1}{3}}}O(1) \tag{2.8}$$

uniformly in y and n for $0 < y \leqq \delta$.

We now choose δ and δ_1 so small that on C_{24} $|\mathrm{Im}\, t|$ increases with $\mathrm{Re}\, t$, and $\delta < \delta_1$. If we denote by s the arc length from t_4 to t on C_{22}, then we have $|t-t_4| \leqq s \leqq 2\,|t-t_4|$. Hence on C_{24} we have for sufficiently small δ:

$$(1-yt)^{b-1}t^{c-b-1}\{t(1-y^2)+y\}^{-a} = s^{c-b-a-1}O(1),$$

and from (2.6):

$$\begin{aligned} I_{24} &= y^{c-a-b}(1+y)^{-2n}e^{-c_0(\lambda y)^{\frac{1}{3}}}O(1)\int_{s_3}^{s_2} s^{\mathrm{Re}\,(c-a-b)-1}ds \\ &= y^{c-a-b}(1+y)^{-2n}e^{-c_0(\lambda y)^{\frac{1}{3}}}\{O(1)+O(y^{c-a-b})\} \end{aligned} \tag{2.9}$$

uniformly in y and n for $0 < y \leqq \delta$.

If $\mathrm{Re}\,(c-a-b) = 0$, then in this formula $O(y^{c-a-b})$ has to be replaced by $O\,(\log y)$.

Further it is evident that

$$I_{23} = y^{c-a-b}(1+y)^{-2n}e^{-c_0(\lambda y)^{\frac{1}{3}}}O(1) \tag{2.10}$$

uniformly in y and n for $0 < y \leqq \delta$. Hence from (2.5), (2.7), (2.8), (2.9) and (2.10) we obtain:

(2.11)
$$I_2 = y^{c-a-b}(1+y)^{1-c-2n}\left\{\frac{\sqrt{\pi}}{2\sqrt{ny}} - \frac{1-y+q}{2ny} + \frac{O(1)}{(ny)^{\frac{3}{2}}} + \frac{y^{c-a-b}}{(ny)^{\frac{3}{2}}}\,o(1)\right\}$$

uniformly as $ny \to \infty$ on $0 < y \leqq 1$.

In a similar way we may prove:

(2.12)
$$I_1 = y^{c-a-b}(1+y)^{1-c-2n}\left\{\frac{-\sqrt{\pi}}{2\sqrt{ny}} - \frac{1-y+q}{2ny} + \frac{O(1)}{(ny)^{\frac{3}{2}}} + \frac{y^{a+b-c}}{(ny)^{\frac{3}{2}}}\,o(1)\right\}$$

uniformly as $ny \to \infty$ on $0 < y \leqq 1$. If $\mathrm{Re}\,(c-a-b) = 0$, then the last terms of (2.11) and (2.12) have to be replaced by $\log y/(ny)^{\frac{3}{2}}\,o(1)$.

We now consider the case ny is bounded, $\mathrm{Im}\, n \geqq 0$. It is evident that $y \to 0$ and $n \to \infty$. Again $|\exp ng(t)|$ is monotonically decreasing along the contour C_2, and therefore absolutely less than $|\exp ng(t_0)| = |\exp -2n \log(1+y)| = O(1)$. Hence the formulas (2.8), (2.9), (2.10) hold for $\operatorname{Im} n \geqq 0$ if we omit the factor $\exp -c_0(\lambda y)^{\frac{1}{3}}$. Moreover it is clear that $I_{21} = O(y^{c-a-b})$. Consequently

$$I_2 = y^{c-a-b}\{O(1)+O(y^{c-a-b})\}, \tag{2.13}$$

and in a similar way

$$I_1 = O(y^{c-a-b})+O(1) \tag{2.14}$$

uniformly as ny is bounded, $\operatorname{Im} n \geqq 0$. If $\operatorname{Re}(c-a-b) = 0$ the last term in both formulas has to be replaced by $O(\log y)$. From the formulas (2.1), (2.11)—(2.14), $y = \sqrt{1-x}$ and Stirlings formula we obtain the final results:

(2.15)

$$\begin{aligned} F(a+n, b+n; c+2n; x) = {} & 2^{2n+c-1}(1-x)^{\frac{1}{2}(c-a-b-\frac{1}{2})}(1+\sqrt{1-x})^{1-c-2n} \\ & \{1+n^{-1}(1-x)^{-\frac{1}{2}}O(1)+n^{-1}(1-x)^{\frac{1}{2}(c-a-b-1)}o(1) \\ & +n^{-1}(1-x)^{\frac{1}{2}(a+b-c-1)}o(1)\} \end{aligned}$$

uniformly as $n \to \infty$, $n\sqrt{1-x} \to \infty$ on $0 \leqq x < 1$, $\eta \leqq \arg n \leqq \pi-\eta$, and

$$F(a+n, b+n; c+2n; x) = n^{\frac{1}{2}}2^{2n}\{O(1)+(1-x)^{c-a-b}O(1)\} \tag{2.16}$$

uniformly as $n \to \infty$ and $n\sqrt{1-x}$ is bounded, $0 \leqq x < 1$, $\operatorname{Im} n \geqq 0$, $|c+2n+g| \geqq \rho$, $|b+n-1-g| \geqq \rho$, $|c-b+n-1-g| \geqq \rho$ $(g=0, 1, 2, \ldots)$ where ρ is a positive number.

If $\operatorname{Re}(c-a-b) = 0$, then we have to replace the last two terms in (2.15) by $n^{-1}(1-x)^{-\frac{1}{2}}\log(1-x)o(1)$, and the last term in (2.16) by $\log(1-x)O(1)$.

It can be shown by the same method that (2.15) also holds if $\eta \leqq -\arg n \leqq \pi-\eta$, and (2.16) for $\operatorname{Im} n \leqq 0$.

Remark 6. From the proof of (2.15) it follows that in (2.15) the terms $o(1)$ may be replaced by $(n\sqrt{1-x})^{\frac{2}{3}}\exp(-c_0|n\sqrt{1-x}|^{\frac{1}{3}})O(1)$.

Remark 7. If in (2.15) x is kept constant, then this formula is in accordance with the well known asymptotic expansion of the hypergeometric function given by Watson [11]. However, Watson's formula does not hold uniformly on the interval $0 \leqq x < 1$, but only on $0 \leqq x \leqq 1-\varepsilon$ with $0 < \varepsilon < 1$.

3. Asymptotic approximations of $P_k^{m,n}(x)$

In order to find an asymptotic approximation of $P_k^{m,n}(x)$ for large values of $|n|$, we use the relation:

(3.1)
$$P_k^{m,n}(x) = \frac{2^{-\gamma}}{\Gamma(1-m)}(1+x)^{k+\frac{1}{2}m}(1-x)^{-\frac{1}{2}m}F\left(-\gamma, -\alpha; 1-m; \frac{x-1}{x+1}\right),$$

$-1 < x < 1$ (see [12, (3)]), and the asymptotic expansion of the F in (3.1) given by Watson [11], and obtain after some calculation:

(3.2)
$$\begin{aligned} &P_k^{m,n}(1-2\tanh^2 u) \\ &\quad = \pi^{-\frac{1}{2}}2^{-\frac{1}{2}-\frac{3}{2}m+\frac{1}{2}n}(-n)^{m-\frac{1}{2}}(\tanh u)^{-\frac{1}{2}}e^{-nu}\left(1+O\left(\frac{1}{n}\right)\right)\end{aligned}$$

as $n \to \infty$ uniformly on $|\arg(-n)| \leqq \frac{1}{2}\pi-\eta$, $(0 < \eta < \frac{1}{2}\pi)$ and for fixed $u > 0$. Using the other asymptotic expansion of Watson we find:

(3.3)
$$P_k^{n,m}(2\tanh^2 u-1) = \frac{2^{\frac{1}{2}m-\frac{3}{2}n}}{\Gamma(1-n)}(\tanh u)^{-\frac{1}{2}}e^{nu}\left(1+O\left(\frac{1}{n}\right)\right)$$

as $n \to \infty$ and $|\arg(-n)| \leqq \pi-\eta$, for fixed $u > 0$.

The asymptotic behavior of $P_k^{n,m}(2\tanh^2 u-1)$ as $n \to \infty$ on $|\arg n| \leqq \frac{1}{2}\pi-\eta$ can be derived from those of $P_k^{-m,-n}(1-2\tanh^2 u)$ and $P_k^{m,-n}(1-2\tanh^2 u)$ by making use of the formulas:

$$P_k^{m,-n}(x) = 2^{-n}P_k^{m,n}(x)$$

(see [2, (13)]), and

(3.4)
$$\begin{aligned} P_k^{m,n}(x) &= \frac{-\pi 2^{-m}}{\Gamma(-\alpha)\Gamma(\delta+1)\sin n\pi}P_k^{-n,m}(-x) \\ &\quad + \frac{\pi 2^{n-m}}{\Gamma(\beta+1)\Gamma(-\gamma)\sin n\pi}P_k^{n,m}(-x).\end{aligned}$$

This last relation can be deduced from [2, (8) and (10)]. We obtain again (3.3) as $n \to \infty$ for fixed $u > 0$, uniformly on every subset of the sector $|\arg n| \leqq \frac{1}{2}\pi-\eta$ with a positive distance to the set of positive integers. Consequently (3.3) holds for such subsets of the sector $|\arg n| \leqq \pi-\eta$, and fixed u.

With aid of the results of section 2 we may deduce asymptotic approximations for the functions in the lefthand side of (3.3) and (3.2), valid uniformly in u for $u > 0$. Using (0.3) and (2.15) we obtain:

(3.5)
$$\begin{aligned} P_k^{n,m}(2\tanh^2 u-1) &= \frac{2^{\frac{1}{2}m-\frac{3}{2}n}}{\Gamma(1-n)}(\tanh u)^{-\frac{1}{2}}e^{nu}\{1+(n\tanh u)^{-1}O(1) \\ &\quad +(\tanh u)^{-m-1}n^{-1}o(1)+(\tanh u)^{m-1}n^{-1}o(1)\}\end{aligned}$$

as $n \to \infty$, $nu \to \infty$, $\eta \leqq |\arg n| \leqq \pi-\eta$. If $\operatorname{Re} m = 0$, then the last two terms have to be replaced by $(\log \tanh u)\,(n \tanh u)^{-1} o(1)$.

If $|nu|$ is bounded then we have from (2.16):

$$(3.6)\quad P_k^{n,m}(2\tanh^2 u-1) = \frac{1}{\Gamma(1-n)}\, 2^{-\frac{1}{2}n} n^{\frac{1}{2}} \{u^m O(1)+u^{-m} O(1)\}$$

as $n \to \infty$. If $\operatorname{Re} m = 0$, then the expression between braces has to be replaced by $\log(1-x)O(1)$.

Applying (3.4) and (3.5) we have:

(3.7)

$$\begin{aligned} P_k^{m,n}(1-2\tanh^2 u) &= \pi^{-\frac{1}{2}} 2^{-\frac{1}{2}-\frac{3}{2}m+\frac{1}{2}n}(-n)^{m-\frac{1}{2}}(\tanh u)^{-\frac{1}{2}} \\ &\cdot \left[e^{-nu}\{1+(n\tanh u)^{-1}O(1)+(\tanh u)^{-m-1}n^{-1}o(1)\right. \\ &+(\tanh u)^{m-1}n^{-1}o(1)\}+e^{nu\pm\pi i(m-\frac{1}{2})}\{1+(n\tanh u)^{-1}O(1) \\ &\left.+(\tanh u)^{-m-1}n^{-1}o(1)+(\tanh u)^{m-1}n^{-1}o(1)\}\right] \end{aligned}$$

as $n \to \infty$ and $nu \to \infty$, $\eta \leqq |\arg n| \leqq \pi-\eta$. The upper or the lower sign in the exponential is to be taken according as $\operatorname{Im} n \gtrless 0$. If $|nu|$ remains bounded we have from (3.4) and (3.6):

$$(3.8)\qquad P_k^{m,n}(1-2\tanh^2 u) = 2^{\frac{1}{2}n} n^m \{u^m O(1)+u^{-m}O(1)\}$$

as $n \to \infty$ in the entire n-plane. If $\operatorname{Re} m = 0$, in (3.7) and (3.8) similar corrections have to be made as in (3.5) and (3.6).

Remark 8. It is evident from (3.5) that (3.3) holds uniformly for $u \geqq a > 0$ and $\eta \leqq |\arg n| \leqq \pi-\eta$.

4. Proof of Theorem 1b

In order to apply the approximations deduced in sections 2 and 3 we may transform theorem 1b by means of the substitutions:

$$x = 2\tanh^2 u-1 \ (u > 0),\ t = 2\tanh^2 v-1 (v > 0),$$
$$\varphi(2\tanh^2 v-1) = f(v),$$

into the following

Theorem 1c. *Let n_1 be a real number satisfying* (1.20). *Suppose $f(v)$ is a function such that for all $a > 0$*

$$f(v)v^{\frac{1}{2}-|\operatorname{Re} m|} \in \mathfrak{L}(0, a) \ \textit{if}\ \operatorname{Re} m \neq 0,$$
$$f(v)v^{\frac{1}{2}} \log v \in L(0, a) \ \textit{if}\ \operatorname{Re} m = 0,$$
$$f(v)\, e^{n_1 v} \in \mathfrak{L}(a, \infty).$$

Let further $f(v)$ be of bounded variation in a neighborhood of $v = u$ with $u > 0$.

Then $f(v)$ satisfies the relations:

(4.1)

$$\frac{1}{2\pi i}\int_{n_1-i\infty}^{n_1+i\infty} dn\, n\Gamma(\delta+1)\Gamma(-\alpha)P_k^{m,n}(1-2\tanh^2 u)$$

$$\int_0^{\infty} f(v)P_k^{n,m}(2\tanh^2 v-1)\tanh v\,dv = -\tfrac{1}{2}\{f(u-0)+f(u+0)\},$$

and

(4.2)

$$\frac{1}{2\pi i}\int_{n_1-i\infty}^{n_1+i\infty} dn\, n\Gamma(\delta+1)\Gamma(-\alpha)P_k^{n,m}(2\tanh^2 u-1)$$

$$\int_0^{\infty} f(v)P_k^{m,n}(1-2\tanh^2 v)\tanh v\,dv = -\tfrac{1}{2}\{f(u-0)+f(u+0)\}.$$

PROOF OF THEOREM 1c.

Case A: First we shall prove this theorem in the case that $f(v) \equiv 0$ if $v < u$. Therefore we consider the integral

$$(4.3)\quad I(u, v, \lambda) = \int_{n_1-i\lambda}^{n_1+i\lambda} n\Gamma(\delta+1)\Gamma(-\alpha)P_k^{m,n}(1-2\tanh^2 u)P_k^{n,m}(2\tanh^2 v-1)dn,$$

where λ is a positive number. If we denote the integrand in (4.3) by $g(u, v, n)$, then $g(u, v, n)$ is an analytic function of n if $\operatorname{Re} n \leqq n_1$ on account of (1.20). Further from (3.2), (3.3) and the relation:

$$(4.4)\quad \frac{n\Gamma(\delta+1)\Gamma(-\alpha)}{\Gamma(1-n)} = -2^{n+m+\frac{1}{2}}\pi^{\frac{1}{2}}(-n)^{\frac{1}{2}-m}\left\{1+O\left(\frac{1}{n}\right)\right\}$$

as $n \to \infty$ on $|\arg(-n)| \leqq \pi-\eta$, we obtain for positive u and v:

$$g(u, v, n)\,(\tanh u \tanh v)^{\frac{1}{2}} = e^{n(v-u)}\left\{1+O\left(\frac{1}{n}\right)\right\}$$

as $n \to \infty$ on $|\arg(-n)| \leqq \frac{1}{2}\pi-\eta$.

From this and Cauchy's theorem we find for $v > u$:

$$(4.5)\quad I(u, v, \lambda) = \int_{n_1-i\lambda}^{\infty e^{-\frac{1}{2}\pi i}} g(u, v, n)dn - \int_{n_1+i\lambda}^{\infty e^{\frac{1}{2}\pi i}} g(u, v, n)dn.$$

To approximate $I(u, v, \lambda)$ for large values of λ and $v > u$, we need the behavior of $g(u, v, n)$ in the sector S:

$$\tfrac{1}{2}\pi-\eta \leqq |\arg n| \leqq \tfrac{1}{2}\pi+\eta.$$

From (4.4), (3.7) and (3.3) (see remark 8) we obtain:

$$(4.6)\quad \begin{aligned} g(u, v, n)(\tanh u \tanh v)^{\frac{1}{2}} = &-e^{n(v-u)}\{1+\psi_1(u, v, n)\} \\ &-e^{n(u+v)}\{e^{\pm\pi i(\frac{1}{2}-m)}+\psi_2(u, v, n)\}, \end{aligned}$$

with

$$(4.7)\qquad \psi_1(u, v, n) = O\left(\frac{1}{n}\right), \quad \psi_2(u, v, n) = O\left(\frac{1}{n}\right)$$

as $n \to \infty$, uniformly on $v \geqq u > 0$ (u fixed) and in the sector S. Defining

$$\varphi_1(u, v, \lambda) = \left(\int_{n_1+i\lambda}^{\infty e^{\frac{1}{2}\pi i}} - \int_{n_1-i\lambda}^{\infty e^{-\frac{1}{2}\pi i}}\right) e^{n(v-u)}\,dn$$
$$-\int_{n_1-i\lambda}^{-\infty-i\lambda} e^{n(u+v)+\pi i(\frac{1}{2}-m)}\,dn + \int_{n_1+i\lambda}^{-\infty+i\lambda} e^{n(u+v)-\pi i(\frac{1}{2}-m)}\,dn,$$

we obtain:

$$(4.8)\quad \begin{aligned} \varphi_1(u, v, \lambda) = &-\frac{e^{n_1(v-u)}}{v-u}\, 2i \sin \lambda(v-u) \\ &-\frac{e^{n_1(v+u)}}{v+u}\, 2i \sin\{\lambda(v+u)-\pi(\tfrac{1}{2}-m)\}. \end{aligned}$$

Similarly we define

$$(4.9)\quad \begin{aligned} \varphi_2(u, v, \lambda) = &\left(\int_{n_1+i\lambda}^{\infty e^{\frac{1}{2}\pi i}} - \int_{n_1-i\lambda}^{\infty e^{-\frac{1}{2}\pi i}}\right) \\ &\{e^{n(v-u)}\psi_1(u, v, n)+e^{n(v+u)}\psi_2(u, v, n)\}\,dn. \end{aligned}$$

From (4.5) and (4.6) we have:

$$(4.10)\quad I(u, v, \lambda) = (\tanh u \tanh v)^{-\frac{1}{2}}\{\varphi_1(u, v, \lambda)+\varphi_2(u, v, \lambda)\}.$$

Hence

(4.11)

$$\lim_{\lambda\to\infty}\int_u^\infty f(v)\varphi_1(u, v, \lambda)\left(\frac{\tanh v}{\tanh u}\right)^{\frac{1}{2}} dv = -\lim_{\lambda\to\infty} 2i\int_u^\infty f(v)e^{n_1(v-u)}\left(\frac{\tanh v}{\tanh u}\right)^{\frac{1}{2}}$$
$$\frac{\sin\lambda(v-u)}{v-u}\,dv - \lim_{\lambda\to\infty} 2i\int_u^\infty f(v)\frac{e^{n_1(v+u)}}{v+u}\left(\frac{\tanh v}{\tanh u}\right)^{\frac{1}{2}} \sin\{\lambda(v+u)-\pi(\tfrac{1}{2}-m)\}dv$$
$$= -\pi i f(u+0),$$

as follows from Dirichlet's formula and the Riemann-Lebesgue lemma.

To calculate

$$\lim_{\lambda\to\infty}\int_u^\infty f(v)\left(\frac{\tanh v}{\tanh u}\right)^{\frac{1}{2}}\varphi_2(u, v, \lambda)dv$$

we split the integral in two parts.

From (4.9) and (4.7) it follows that $e^{-n_1 v}\varphi_2(u, v, \lambda)$ tends to zero as $\lambda \to \infty$, uniformly on $v \geqq c > u$. On account of the conditions on $f(v)$ it is clear that

$$\lim_{\lambda\to\infty}\int_c^\infty f(v)\left(\frac{\tanh v}{\tanh u}\right)^{\frac{1}{2}}\varphi_2(u, v, \lambda)dv = 0. \tag{4.12}$$

From the assumptions on $f(v)$ it follows that we can choose c such that $f(v)$ is of bounded variation in the interval $u \leqq v \leqq c$. Therefore the real and imaginary part of $f(v)$ $(\tanh v)^{\frac{1}{2}}$ can be written as a difference of two monotonic increasing functions. Furthermore from (4.7) and (4.9) we have:

$$\int_u^d \varphi_2(u, v, \lambda)dv \to 0 \tag{4.13}$$

as $\lambda \to \infty$, uniformly on $u \leqq d \leqq c$. Applying Bonnet's mean value theorem and (4.13), it follows that

$$\lim_{\lambda\to\infty}\int_u^c f(v)\left(\frac{\tanh v}{\tanh u}\right)^{\frac{1}{2}}\varphi_2(u, v, \lambda)dv = 0. \tag{4.14}$$

From (4.12) and (4.14) we conclude:

$$\lim_{\lambda\to\infty}\int_u^\infty f(v)\left(\frac{\tanh v}{\tanh u}\right)^{\frac{1}{2}}\varphi_2(u, v, \lambda)dv = 0,$$

and from (4.11) and (4.10):

$$\lim_{\lambda\to\infty}\int_u^\infty f(v)\tanh v\, I(u, v, \lambda)dv = -\pi i f(u+0).$$

From (4.6), (4.7) and the conditions assumed on $f(v)$, it follows that

$$\int_u^\infty dv\, f(v)\tanh v\int_{n_1-i\lambda}^{n_1+i\lambda} g(u, v, n)dn$$
$$= \int_{n_1-i\lambda}^{n_1+i\lambda} dn\int_u^\infty f(v)\tanh v\, g(u, v, n)dv.$$

Passing to the limit $\lambda \to \infty$, we obtain (4.1) in the case that $f(v) \equiv 0$ if $v < u$.

Now we consider the lefthand side of (4.2) with $f(v) \equiv 0$ if $v < u$, and therefore we investigate the integral $I(v, u, \lambda)$, see (4.3). Applying (3.4) to $P_k^{m,n}(1-2\tanh^2 v)$ we find:

$$I(v, u, \lambda) = I_1(v, u, \lambda) + I_2(v, u, \lambda), \tag{4.15}$$

where

$$\begin{aligned} &I_1(v, u, \lambda) \\ &= \int_{n_1-i\lambda}^{n_1+i\lambda} -2^{-m}\pi \frac{n}{\sin n\pi} P_k^{n,m}(2\tanh^2 u-1) P_k^{-n,m}(2\tanh^2 v-1)\, dn, \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} I_2(v, u, \lambda) = \int_{n_1-i\lambda}^{n_1+i\lambda} & 2^{n-m}\pi \frac{n}{\sin n\pi} \frac{\Gamma(\delta+1)\Gamma(-\alpha)}{\Gamma(\beta+1)\Gamma(-\gamma)} \\ & P_k^{n,m}(2\tanh^2 u-1) P_k^{n,m}(2\tanh^2 v-1)\, dn. \end{aligned} \tag{4.17}$$

(In the case that n_1 is an integer we deform the path of integration around the point $n = n_1$). Denoting the integrands in (4.16) and (4.17) by $h_1(v, u, n)$ and $h_2(v, u, n)$ respectively, we see that the poles of $h_1(v, u, n)$ are $n = \pm 1, \pm 2, \ldots$, whereas $h_2(v, u, n)$ has poles at these points, and at the points where $k - \frac{1}{2}(m+n) = -1, -2, \ldots$ and $-k-\frac{1}{2}(m+n) = 0, -1, -2, \ldots$.

Applying (3.3) we find for the asymptotic behavior of $h_1(v, u, n)$:

$$h_1(v, u, n)(\tanh u \tanh v)^{\frac{1}{2}} = -e^{n(u-v)}\left\{1 + O\left(\frac{1}{n}\right)\right\}, \tag{4.18}$$

valid for $n \to \infty$ in every subset of the sector $|\arg n| \leqq \pi - \eta$ with a positive distance to the set of positive integers. In a similar way we find:

$$\begin{aligned} &h_2(v, u, n)(\tanh u \tanh v)^{\frac{1}{2}} \\ &\qquad = -\frac{2\sin\beta\pi \sin\gamma\pi}{\sin n\pi} e^{n(u+v)}\left\{1 + O\left(\frac{1}{n}\right)\right\}, \end{aligned} \tag{4.19}$$

valid for $n \to \infty$ on every subset of the sector $|\arg(-n)| \leqq \pi - \eta$ with a positive distance to the set of negative integers.

Now we have from Cauchy's theorem and (4.15)—(4.19) for $\lambda > \max\{|\mathrm{Im}\,(2k-m)|, |\mathrm{Im}\,(-2k-m)|\}$, and $v > u$:

(4.20)

$$I(v, u, \lambda) = \int_{n_1-i\lambda}^{\infty e^{-\frac{1}{4}\pi i}} h_1(v, u, n)dn - \int_{n_1+i\lambda}^{\infty e^{\frac{1}{4}\pi i}} h_1(v, u, n)dn$$

$-2\pi i \sum$ (residues of $h_1(v, u, n)$ at the poles to the right of $\operatorname{Re} n = n_1$)

$$+\int_{n_1-i\lambda}^{\infty e^{-\frac{1}{4}\pi i}} h_2(v, u, n)dn - \int_{n_1+i\lambda}^{\infty e^{\frac{1}{4}\pi i}} h_2(v, u, n)dn$$

$+2\pi i \sum$ (residues of $h_2(v, u, n)$ at the poles to the left of $\operatorname{Re} n = n_1$).

The residue of $h_1(v, u, n)$ at the point $n = g$ ($g = 1, 2, \ldots$) is equal to

$$(-1)^{g+1}2^{-m}gP_k^{g,m}(2\tanh^2 u-1)P_k^{-g,m}(2\tanh^2 v-1)$$
$$= 2^{-g-m}g\frac{\Gamma\left(k+\frac{g-m}{2}+1\right)\Gamma\left(k+\frac{g+m}{2}+1\right)}{\Gamma\left(k-\frac{g-m}{2}+1\right)\Gamma\left(k-\frac{g+m}{2}+1\right)}$$
$$P_k^{-g,m}(2\tanh^2 u-1)P_k^{-g,m}(2\tanh^2 v-1)$$

(see [2, (15)]). The residue of $h_2(v, u, n)$ at the point $n = -g$ ($g = 1, 2, \ldots$) is equal to

$$2^{-g-m}(-1)^{g+1}g\frac{\Gamma\left(k+\frac{g-m}{2}+1\right)\Gamma\left(-k+\frac{g-m}{2}\right)}{\Gamma\left(k-\frac{g+m}{2}+1\right)\Gamma\left(-k-\frac{g+m}{2}\right)}$$
$$P_k^{-g,m}(2\tanh^2 u-1)P_k^{-g,m}(2\tanh^2 v-1)$$

= the residue of $h_1(v, u, n)$ at the point $n = g$.

Since $h_1(v, u, n)+h_2(v, u, n)$ is equal to the integrand in $I(v, u, \lambda)$, it is regular at $n = g$ ($g = \pm 1, \pm 2, \ldots$). Therefore the residue of $h_1(v, u, n)$ at $n = g$ is equal to minus the residue of $h_2(v, u, n)$ at $n = g$. Hence

(Res. of $h_1(v, u, n)$ at $n = g$)+(Res. of $h_1(v, u, n)$ at $n = -g$) = 0;
(Res. of $h_2(v, u, n)$ at $n = g$)+(Res. of $h_2(v, u, n)$ at $n = -g$) = 0.

Consequently both sums of residues in (4.20) are equal. So

$$\begin{aligned} I(v, u, \lambda) &= \int_{n_1-i\lambda}^{\infty e^{-\frac{1}{4}\pi i}} h_1(v, u, n)dn - \int_{n_1+i\lambda}^{\infty e^{\frac{1}{4}\pi i}} h_1(v, u, n)dn \\ &+ \int_{n_1-i\lambda}^{\infty e^{-\frac{1}{4}\pi i}} h_2(v, u, n)dn - \int_{n_1+i\lambda}^{\infty e^{\frac{1}{4}\pi i}} h_2(v, u, n)dn. \end{aligned} \tag{4.21}$$

Now we may calculate

$$\lim_{\lambda\to\infty}\int_u^\infty\left\{\int_{n_1-i\lambda}^{\infty e^{-\frac{1}{4}\pi i}} h_1(v,u,n)dn-\int_{n_1+i\lambda}^{\infty e^{\frac{1}{4}\pi i}} h_1(v,u,n)dn\right\}f(v)\tanh v dv$$

in the same way as in the first part of this section, now using (4.18) instead of (4.6). The limit is equal to $-\pi i f(u+0)$.

Further, from

$$\frac{2\sin\beta\pi\sin\gamma\pi}{\sin n\pi}=e^{\pm\pi i(m-\frac{1}{2})}\left\{1+O\left(\frac{1}{\operatorname{Im} n}\right)\right\} \text{ as } \pm\operatorname{Im} n\to\infty,$$

and (4.19) we obtain:

$$h_2(v,u,n)(\tanh u\tanh v)^{\frac{1}{2}}=-e^{n(u+v)\pm\pi i(m-\frac{1}{2})}\left\{1+O\left(\frac{1}{\operatorname{Im} n}\right)\right\}$$
$$\text{as } \pm\operatorname{Im} n\to\infty.$$

Using this result and an analogous reasoning as in the first part of this section we find:

$$\lim_{\lambda\to\infty}\int_u^\infty\left\{\int_{n_1-i\lambda}^{\infty e^{-\frac{1}{4}\pi i}} h_2(v,u,n)dn-\int_{n_1+i\lambda}^{\infty e^{\frac{1}{4}\pi i}} h_2(v,u,n)dn\right\}f(v)\tanh v dv=0.$$

Combining this with (4.21) we obtain:

$$\lim_{\lambda\to\infty}\int_u^\infty f(v)\tanh v\, I(v,u,\lambda)dv=-\pi i f(u+0).$$

Changing the order of integration we have (4.2) in the case that $f(v)\equiv 0$ if $v<u$.

Case B: To prove (4.1) in the case that $f(v)\equiv 0$ if $v>u$, we have to investigate $I(u,v,\lambda)$ with $v<u$. Therefore we interchange the rôles of u and v, and apply the previous results on $I(v,u,\lambda)$ with $u<v$. Hence (4.21) holds. Applying (3.3) and (3.5) we find:

$$(4.22)\quad h_1(v,u,n)(\tanh u\tanh v)^{\frac{1}{2}}=-e^{n(u-v)}\{1+\psi_3(v,u,n)\},$$

where

$$(4.23)\qquad \psi_3(v,u,n)=\frac{1}{nu}\{O(1)+u^{-m}o(1)+u^m o(1)\}$$

as $n\to\infty$ and $nu\to\infty$ uniformly in the sector S and $0<u\leqq v$. Moreover from (3.3) and (3.6) we obtain:

$$(4.24)\qquad h_1(v,u,n)=n^{\frac{1}{2}}e^{-nv}\{u^m O(1)+u^{-m}O(1)\},$$

as $n\to\infty$ in S, and $|nu|$ is bounded.

If $\operatorname{Re} m = 0$ in (4.23) and (4.24) the expressions between the braces have to be replaced by $O(1)+o(\log u)$, $O(\log u)$ respectively. To calculate

(4.25)
$$\lim_{\lambda\to\infty}\int_0^v\left\{\int_{n_1-i\lambda}^{\infty e^{-\frac{1}{4}\pi i}}\{h_1(v,u,n)dn-\int_{n_1+i\lambda}^{\infty e^{\frac{1}{4}\pi i}}h_1(v,u,n)dn\right\}f(u)\tanh u du,$$

we use (4.22), and find:

$$(4.26)\qquad \begin{aligned}(\tanh u\tanh v)^{\frac{1}{2}}&\left(\int_{n_1-i\lambda}^{\infty e^{-\frac{1}{4}\pi i}}-\int_{n_1+i\lambda}^{\infty e^{\frac{1}{4}\pi i}}\right)h_1(v,u,n)dn\\ &=-2ie^{n_1(u-v)}\frac{\sin\lambda(u-v)}{u-v}+\varphi_3(v,u,\lambda),\end{aligned}$$

where

$$(4.27)\qquad \varphi_3(v,u,\lambda)=-\left(\int_{n_1-i\lambda}^{\infty e^{-\frac{1}{4}\pi i}}-\int_{n_1+i\lambda}^{\infty e^{\frac{1}{4}\pi i}}\right)e^{n(u-v)}\psi_3(v,u,n)dn.$$

From Dirichlet's formula we have:

(4.28)
$$\lim_{\lambda\to\infty}\int_d^v-2ie^{n_1(u-v)}\frac{\sin\lambda(u-v)}{u-v}\left(\frac{\tanh u}{\tanh v}\right)^{\frac{1}{2}}f(u)du=-\pi i f(v-0),$$

uniformly in d on $0\leqq d\leqq c<v$.

Now we choose c such that $f(u)$ is of bounded variation on the interval $c\leqq u\leqq v$. Applying Bonnet's mean value theorem as in the deduction of (4.14), we find:

$$(4.29)\qquad \lim_{\lambda\to\infty}\int_c^v f(u)\left(\frac{\tanh u}{\tanh v}\right)^{\frac{1}{2}}\varphi_3(v,u,\lambda)du=0.$$

From (4.27) and (4.23) we have for $\operatorname{Re} m\neq 0$:

$$\varphi_3(v,u,\lambda)=\frac{1}{\lambda u}\{O(1)+u^{-m}o(1)+u^m o(1)\}$$

as $\lambda\to\infty$, $\lambda u\to\infty$ uniformly on $0<u\leqq c$. From this we have:

$$\begin{aligned}\int_{K/\lambda}^c&\varphi_3(v,u,\lambda)\left(\frac{\tanh u}{\tanh v}\right)^{\frac{1}{2}}f(u)du\\ &=\frac{1}{K}\int_{K/\lambda}^c u^{\frac{1}{2}}f(u)\{O(1)+u^{-m}o(1)+u^m o(1)\}du,\end{aligned}$$

for positive K, $K\to\infty$ and $\lambda>K/c$. Hence on account of the conditions on $f(x)$ given $\varepsilon>0$, there exists a constant K_0 such that

$$\left|\int_{K_0/\lambda}^{c} \varphi_3(v, u, \lambda) \left(\frac{\tanh u}{\tanh v}\right)^{\frac{1}{2}} f(u)\,du\right| < \tfrac{1}{2}\varepsilon. \tag{4.30}$$

Using (4.28) with $d = K_0/\lambda$, (4.29) and (4.30), we obtain:

$$\begin{aligned}\Bigg|\int_{K_0/\lambda}^{v} \Bigg\{\int_{n_1-i\lambda}^{\infty e^{-\frac{1}{2}\pi i}} & h_1(v, u, n)\,dn \\ & - \int_{n_1+i\lambda}^{\infty e^{\frac{1}{2}\pi i}} h_1(v, u, n)\,dn\Bigg\} f(u)\tanh u\,du + \pi i f(v-0)\Bigg| < \varepsilon\end{aligned} \tag{4.31}$$

if $\lambda > \lambda_0(\varepsilon)$.

Finally we have to approximate:

(4.32)

$$\int_0^{K_0/\lambda} \left\{\int_{n_1-i\lambda}^{\infty e^{-\frac{1}{2}\pi i}} h_1(v, u, n)\,dn - \int_{n_1+i\lambda}^{\infty e^{\frac{1}{2}\pi i}} h_1(v, u, n)\,dn\right\} f(u)\tanh u\,du.$$

Since on the path of integration λu is bounded, we can use (4.24), and find for the expression between braces in (4.32): $\lambda^{\frac{1}{2}}\{u^m O(1) + u^{-m} O(1)\}$ if $\operatorname{Re} m \neq 0$. Hence the integral (4.32) is

$$\lambda^{\frac{1}{2}} O(1) \int_0^{K_0/\lambda} u^{m+1} f(u)\,du + \lambda^{\frac{1}{2}} O(1) \int_0^{K_0/\lambda} u^{-m+1} f(u)\,du,$$

and therefore from the conditions on $f(x)$ this expression tends to zero as $\lambda \to \infty$. Using this and (4.31) we see that the limit (4.25) is equal to $-\pi i f(v-0)$.

By an analogous reasoning we find:

$$\lim_{\lambda\to\infty} \int_0^{v} \left\{\int_{n_1-i\lambda}^{\infty e^{-\frac{1}{2}\pi i}} h_2(v, u, n)\,dn - \int_{n_1+i\lambda}^{\infty e^{\frac{1}{2}\pi i}} h_2(v, u, n)\,dn\right\} f(u)\tanh u\,du = 0.$$

From this, (4.3) and (4.21) we have, after interchanging u and v for $\operatorname{Re} m \neq 0$:

$$\lim_{\lambda\to\infty} \int_0^{u} f(v) \tanh v\, I(u, v, \lambda)\,dv = -\pi i f(u-0).$$

If $\operatorname{Re} m = 0$, we obtain the same result with the corresponding condition on $f(v)$.

Changing the order of integration we have (4.1) in the case that $f(v) \equiv 0$ if $v > u$.

In a similar way it may be derived that (4.2) holds in this case.

Combining the results obtained in A and B, theorem 1c is proved completely.

Part III

5. Proof of Theorem 2b

On account of (1.25) and (3.2), the integral in n occurring in the theorem is uniformly convergent for

$$a \leqq x \leqq b \text{ if } -1 < a < b < 1.$$

Hence

$$\begin{aligned}(5.1)\qquad &\int_a^b \frac{dx}{1-x} P_k^{n_0,m}(x) \int_{n_1-i\infty}^{n_1+i\infty} nP_k^{m,n}(-x)f(n)dn \\ &= \int_{n_1-i\infty}^{n_1+i\infty} dn\, nf(n) \int_a^b P_k^{n_0,m}(x)P_k^{m,n}(-x)\frac{dx}{1-x}.\end{aligned}$$

Putting $P_k^{n_0,m}(x) = w_1$ and $P_k^{m,n}(-x) = w_2$, we see that w_1 and w_2 satisfy the differential equations:

$$(1-x^2)w_1''-2xw_1'+\left\{k(k+1)-\frac{n_0^2}{2(1-x)}-\frac{m^2}{2(1+x)}\right\}w_1 = 0,$$

$$(1-x^2)w_2''-2xw_2'+\left\{k(k+1)-\frac{m^2}{2(1+x)}-\frac{n^2}{2(1-x)}\right\}w_2 = 0,$$

from which we derive:

$$(1-x^2)(w_2w_1''-w_1w_2'')-2x(w_2w_1'-w_1w_2')+\frac{n^2-n_0^2}{2(1-x)}w_1w_2 = 0.$$

Hence

$$(5.2)\qquad \int_a^b P_k^{n_0,m}(x)P_k^{m,n}(-x)\frac{dx}{1-x} = A(b)-A(a),$$

where

$$\begin{aligned}(5.3)\qquad A(x) = \frac{2}{n_0^2-n^2}(1-x^2)\Big\{&P_k^{m,n}(-x)\frac{d}{dx}P_k^{n_0,m}(x) \\ &-P_k^{n_0,m}(x)\frac{d}{dx}P_k^{m,n}(-x)\Big\}.\end{aligned}$$

Using [13, (24)]:

$$(5.4)\qquad \frac{dP_k^{m,n}(x)}{dx} = \frac{\alpha(\delta+1)}{(1-x^2)^{\frac{1}{2}}}P_k^{m-1,n-1}(x)+\frac{(m+n)x+m-n}{2(1-x^2)}P_k^{m,n}(x),$$

we find:

(5.5)
$$A(x) = \frac{x+1}{n_0+n} P_k^{m,n}(-x) P_k^{n_0,m}(x) + \frac{2}{n_0^2-n^2}(1-x^2)^{\frac{1}{2}}$$
$$[\alpha_0(\delta_0+1) P_k^{m,n}(-x) P_k^{n_0-1,m-1}(x) + \alpha(\delta+1) P_k^{m-1,n-1}(-x) P_k^{n_0,m}(x)].$$

From (5.1) and (5.2) we have:

$$\int_a^b \frac{dx}{1-x} P_k^{n_0,m}(x) \int_{n_1-i\infty}^{n_1+i\infty} n P_k^{m,n}(-x) f(n) dn = \int_{n_1-i\infty}^{n_1+i\infty} n f(n)\{A(b)-A(a)\} dn. \tag{5.6}$$

Case A: We now consider the case $\operatorname{Re} n_0 = n_1 < 0$.

The function $nf(n)\{A(b)-A(a)\}$ is analytic in n in the half plane $\operatorname{Re} n < n_1$, and continuous in the half plane $\operatorname{Re} n \leqq n_1$ (see (5.2)). Hence, from Cauchy's theorem we have:

$$\int_{n_0-i\varepsilon}^{n_0+i\varepsilon} nf(n)\{A(b)-A(a)\} dn = \int_{C_1} nf(n)\{A(b)-A(a)\} dn,$$

where $\varepsilon > 0$ and C_1 is the half circle $|n-n_0| = \varepsilon$, $\operatorname{Re} n \leqq n_1$, traversed from $n_0 - i\varepsilon$ to $n_0+i\varepsilon$. Therefore

$$\int_{n_1-i\infty}^{n_1+i\infty} nf(n)\{A(b)-A(a)\} dn = \left(\int_{n_0-i\infty}^{n_0+i\varepsilon} + \int_{C_1} + \int_{n_0+i\varepsilon}^{n_1+i\infty}\right) nf(n)\{A(b)-A(a)\} dn. \tag{5.7}$$

The function $nf(n)A(a)$ is analytic in n in the half plane $\operatorname{Re} n < n_1$, and is continuous on $\operatorname{Re} n \leqq n_1$, $n \neq n_0$. Applying again Cauchy's theorem we find that

$$\left(\int_{n_1-i\infty}^{n_0-i\varepsilon} + \int_{C_1} + \int_{n_0+i\varepsilon}^{n_1+i\infty}\right) nf(n)A(a) dn = \int_L nf(n)A(a) dn. \tag{5.8}$$

Herein L is defined as follows. Choose a positive number $R > |n_0|$. Let C_2 be the part of the circle $|n| = R$ to the left of the line $\operatorname{Re} n = n_1$. L is the straight line $\operatorname{Re} n = n_1$ with the part inside the circle $|n| = R$ replaced by C_2. L is traversed from $n_1 - i\infty$ to $n_1+i\infty$.

Putting $a = 2\tanh^2\eta - 1$ and choosing $R = \eta^{-1}$, we have to investigate the asymptotic behavior of $A(2\tanh^2\eta-1)$ on C_2 as $\eta \downarrow 0$. Since $|n\eta|$ is bounded on C_2, we may use (3.8) for $P_k^{m,n}(1-2\tanh^2\eta)$ and $P_k^{m-1,n-1}(1-2\tanh^2\eta)$ in (5.5). Moreover we apply (3.4) and (0.3) to $P_k^{n_0,m}(2\tanh^2\eta-1)$ and $P_k^{n_0-1,m-1}(2\tanh^2\eta-1)$ in (5.5). The result is

$$A(2\tanh^2\eta-1)$$
$$=\begin{cases} 2^{\frac{1}{2}n}n^{|\mathrm{Re}\,m|-1}O(1) \text{ if } \mathrm{Re}\,m<1,\ m\neq 0;\ 2^{\frac{1}{2}n}n^{-1}O(\log n) \text{ if } m=0;\\ 2^{\frac{1}{2}n}n^{3(m-1)}O(1) \text{ if } \mathrm{Re}\,m>1;\ 2^{\frac{1}{2}n}O(\log n) \text{ if } \mathrm{Re}\,m=1\end{cases}$$

as $\eta\downarrow 0$, $|n|=\eta^{-1}$, $\mathrm{Re}\,n\leqq n_1$. Hence from (1.26) we find:

$$\int_{C_2} nf(n)A(a)dn \to 0 \text{ as } a\downarrow -1.$$

For the straight parts of L we follow a similar argument; here we have to apply (3.7) besides (3.8), and find that the corresponding integral also tends to zero as $a\downarrow -1$. Hence by (5.8) the contribution of $A(a)$ to the integral in the righthand side of (5.7) tends to zero as $a\downarrow -1$.

To evaluate the contribution of $A(b)$ to the integral at the righthand side of (5.7) we use the asymptotic behavior of $A(b)$ as $n\to\infty$ uniformly for $b\uparrow 1$. Putting $b=2\tanh^2 N-1$ and using (5.5), (0.3) and (3.7), we obtain:

$$A(2\tanh^2 N-1)=(-n)^{m-\frac{3}{2}}O(1) \text{ as } n\to\infty,\ \mathrm{Re}\,n=n_1,$$

uniformly for $N\to\infty$. Hence from (1.25) it follows that

$$\text{(5.9)}\qquad \left(\int_{n_1+iM}^{n_1+i\infty}+\int_{n_1-i\infty}^{n_1-iM}\right) nf(n)A(b)dn=o(1)$$

as $M\to\infty$ uniformly for $b\uparrow 1$.

To determine

$$\text{(5.10)}\qquad \lim_{b\uparrow 1}\left(\int_{n_1-iM}^{n_0-i\varepsilon}+\int_{n_0+i\varepsilon}^{n_1+iM}\right) nf(n)A(b)dn$$

we use the asymptotic behavior of $A(b)$ as $b\uparrow 1$ uniformly in n on the path of integration. Using (0.3), (3.4) and (5.5) we may deduce:

(5.11)
$$A(2\tanh^2 N-1)=\frac{2^{\frac{1}{2}(n-n_0)+1}}{n_0-n}\frac{(\mathrm{sech}\,N)^{n-n_0}\Gamma(-n)}{\Gamma(1+\delta)\Gamma(-\alpha)\Gamma(1-n_0)}\left(1+O\left(\frac{1}{N}\right)\right)$$
$$+\frac{2^{\frac{1}{2}(n-n_0)+1}}{n_0+n}\frac{(\mathrm{sech}\,N)^{-n-n_0}\Gamma(n)}{\Gamma(1+\beta)\Gamma(-\gamma)\Gamma(1-n_0)}\left(1+O\left(\frac{1}{N}\right)\right)$$

as $N\to\infty$. Since $n-n_0$ is purely imaginary, we may apply the Riemann-Lebesgue lemma to

$$\left(\int_{n_1-iM}^{n_0-i\varepsilon}+\int_{n_0+i\varepsilon}^{n_1+iM}\right)\frac{n}{n_0-n}\,2^{\frac{1}{2}(n-n_0)+1}\,\frac{(\operatorname{sech} N)^{n-n_0}\Gamma(-n)}{\Gamma(1+\delta)\Gamma(-\alpha)\Gamma(1-n_0)}\,f(n)dn,$$

so that this integral vanishes as $N \to \infty$. Since Re $(-n-n_0) = -2n_1 > 0$, it is easily be seen that the other terms in (5.11) give contributions to the value of the integrals in (5.10) which tend to zero as $b \uparrow 1$. Hence the limit in (5.10) is equal to zero.

Furthermore

$$\text{(5.12)}\qquad \int_{C_1} nf(n)A(b)dn = \int_{C_1} n\{f(n)-f(n_0)\}A(b)dn + f(n_0)\int_{C_1} nA(b)dn.$$

From (5.5) and the Hölder-condition on $f(n)$ we see that

$$n\{f(n)-f(n_0)\}A(b) = (n-n_0)^{c-1}O(1) \quad \text{as } n \to n_0,\ \operatorname{Re} n \leqq n_1,\ 0 < c < 1.$$

Hence in the first term of the righthand side of (5.12) C_1 can be replaced by the straight line segment from $n_0-i\varepsilon$ to $n_0+i\varepsilon$. With the aid of the Riemann-Lebesgue lemma and (5.11) we see that the first term in the righthand side of (5.12) vanishes as $b \uparrow 1$.

Finally applying the theorem of residues we have:

$$\text{(5.13)}\qquad \int_{C_1} nA(b)dn = \int_{C_1'} nA(b)dn - 2\pi i \text{ (residue at } n = n_0),$$

where C_1' is the half circle $|n-n_0| = \varepsilon$, Re $n \geqq n_1$, traversed from $n_0-i\varepsilon$ to $n_0+i\varepsilon$. From (5.3) we find for the residue of $A(b)$ at the point $n = n_0$:

$$\frac{1}{n_0}(1-b^2)W\{P_k^{m,n_0}(b),\ P_k^{n_0,m}(-b)\},$$

and this is according to [14, (15)] equal to $2/n_0\Gamma(-\alpha_0)\Gamma(\delta_0+1)$. Hence the residue in (5.13) is equal to $2/\Gamma(-\alpha_0)\Gamma(\delta_0+1)$.

Choosing $\varepsilon < -2n_1$ it is easily seen from (5.11) that the last integral in (5.13) tends to zero as $b \uparrow 1$. So the first integral in (5.13) tends to $-4\pi i\{\Gamma(-\alpha_0)\Gamma(\delta_0+1)\}^{-1}$ as $b \uparrow 1$, and therefore the first integral in (5.12) to $-4\pi i f(n_0)\{\Gamma(-\alpha_0)\Gamma(\delta_0+1)\}^{-1}$. This completes the proof of theorem 2b in case A.

Case B: $n_1 = 0$, $\operatorname{Re} n_0 = 0$, $n_0 \neq 0$.

Without loss of generality we may assume $\operatorname{Im} n_0 > 0$.

In this case the righthand side of (5.6) is equal to

(5.14)
$$\left(\int_{-i\infty}^{-n_0-i\varepsilon} + \int_{C_-} + \int_{-n_0+i\varepsilon}^{n_0-i\varepsilon} + \int_{C_+} + \int_{n_0+i\varepsilon}^{i\infty}\right) nf(n)\{A(b)-A(a)\}dn,$$

where $C_\pm$ is the half circle $|n \mp n_0| = \varepsilon$, $\operatorname{Re} n \leqq 0$, traversed from $\pm n_0 - i\varepsilon$ to $\pm n_0 + i\varepsilon$. Here ε is a positive number with $\varepsilon < |n_0|$.

It follows from a similar argument as in case A that the contribution of $A(a)$ to the integrals in (5.14) tends to zero as $a \downarrow -1$, and that

$$\lim_{b \uparrow 1}\left(\int_{-i\infty}^{-n_0-i\varepsilon} + \int_{-n_0+i\varepsilon}^{n_0-i\varepsilon} + \int_{n_0+i\varepsilon}^{i\infty}\right) nf(n)A(b)dn = 0.$$

To evaluate $\int_{C_+} nf(n)A(b)dn$ we use (5.12) with C_1 replaced by C_+. Again the corresponding first term in the righthand side tends to zero as $b \uparrow 1$. To calculate the corresponding second term we apply (3.4) in (5.5), and write

$$A(b) = A_1(b) + A_2(b)$$

with

(5.15)
$$A_1(b) = \frac{-\pi 2^{-m}}{\Gamma(-\alpha)\Gamma(\delta+1)\sin n\pi}\left[\frac{b+1}{n_0+n} P_k^{n_0,m}(b)P_k^{-n,m}(b)\right.$$
$$\left. + \frac{2(1-b^2)^{\frac{1}{2}}}{n_0^2-n^2}\{\alpha_0(\delta_0+1)P_k^{n_0-1,m-1}(b)P_k^{-n,m}(b) + P_k^{n_0,m}(b)P_k^{1-n,m-1}(b)\}\right],$$

and

$$A_2(b) = \frac{\pi 2^{n-m}}{\Gamma(\beta+1)\Gamma(-\gamma)\sin n\pi}\left[\frac{b+1}{n_0+n} P_k^{n_0,m}(b)P_k^{n,m}(b)\right.$$
(5.16)
$$+ \frac{2(1-b^2)^{\frac{1}{2}}}{n_0^2-n^2}\{\alpha_0(\delta_0+1)P_k^{n_0-1,m-1}(b)P_k^{n,m}(b)$$
$$\left. - \alpha(\delta+1)P_k^{n_0,m}(b)P_k^{n-1,m-1}(b)\}\right].$$

$A_1(b)$ is an analytic function in n in a deleted neighborhood of $n = n_0$. So we may apply (5.13) with C_1 replaced by C_+ and $A(b)$ by $A_1(b)$. The residue of $A_2(b)$ at $n = n_0$ is equal to zero, so that the residue of $A_1(b)$ is the same as that of $A(b)$ and this is, as in case A, equal to $2/n_0\Gamma(-\alpha_0)\Gamma(\delta_0+1)$. Further $\int_{C'_+} nA_1(b)dn \to 0$

as $b \uparrow 1$, since $A_1(b) = (1-b)^{\frac{1}{2}(n-n_0)}O(1)$ as $b \uparrow 1$ (see (5.15) and (0.3)).

Using (5.16) and (0.3) we may deduce that

$$A_2(b) = (1-b)^{-\frac{1}{2}(n+n_0)}O(1) \text{ as } b \uparrow 1.$$

Hence

$$\int_{C_+} nf(n)A_2(b)dn \to 0 \text{ as } b \uparrow 1.$$

Consequently

$$\lim_{b \uparrow 1} \int_{C_+} nf(n)A(b)dn = \frac{-4\pi i f(n)}{\Gamma(-\alpha_0)\Gamma(\delta_0+1)}.$$

The above method is applicable to the integral over C_-. In this case the residue of $A_1(b)$ at $n = -n_0$ is equal to

$$\frac{\pi 2^{-m}}{n_0\Gamma(-\gamma_0)\Gamma(\beta_0+1)\sin n_0\pi} P_k^{n_0, m}(b)\{n_0(b+1)P_k^{n_0, m}(b)$$
$$+\alpha_0(\delta_0+1)(1-b^2)^{\frac{1}{2}}P_k^{n_0-1, m-1}(b)+2(1-b^2)^{\frac{1}{2}}P_k^{n_0+1, m-1}(b)\},$$

where

$$\beta_0 = k-\tfrac{1}{2}(m-n_0),\ \gamma_0 = k+\tfrac{1}{2}(m-n_0),$$

and using the recurrence relations between the $P_k^{n, m}(b)$ it can be shown that this expression vanishes.

Hence

$$\lim_{b \uparrow 1} \int_{C_-} nf(n)A(b)dn = 0.$$

Combining the results obtained above the theorem is proved in this case.

Case C: $n_1 = n_0 = 0$.

We use the same argument as in case B. The residue of $nA_1(b)$ at $n = 0$ is given by

$$\frac{1}{2\pi i}\int_C nA_1(b)dn,$$

where C is the circle $|n| = r$ traversed in positive sense ($r < 1$). To evaluate this integral we first consider the case $n_0 \neq 0$ and $|n_0| < r$. Then we have two poles $n = \pm n_0$ inside C, and the residues in these poles are according to case B equal to $2/\Gamma(-\alpha_0)\Gamma(\delta_0+1)$ and 0 respectively. Since the integrand is a continuous function of n on C, the residue at $n = 0$ remains $2/\Gamma(-\alpha_0)\Gamma(\delta_0+1)$ if $n_0 = 0$. Hence the result is the same as in the former cases.

Case D: $n_1 \neq \operatorname{Re} n_0$.

This case can be reduced to the preceding cases. Now we will prove that

$$(5.17) \qquad \left(\int_{n_1-i\infty}^{n_1+i\infty} - \int_{\operatorname{Re} n_0-i\infty}^{\operatorname{Re} n_0+i\infty}\right) nP_k^{m,n}(-x)f(n)dn = 0$$

if $-1 < x < 1$.

The integrand is analytic in n for $\operatorname{Re} n < \max(\operatorname{Re} n_0, n_1)$, and continuous for $\operatorname{Re} n \leqq \max(\operatorname{Re} n_0, n_1)$. Furthermore the asymptotic behavior of $P_k^{m,n}(-x)$ is given in (3.2). From (1.25) it follows that the integrand tends to zero as $|\operatorname{Im} n| \to \infty$ if $\min(\operatorname{Re} n_0, n_1) \leqq \operatorname{Re} n \leqq \max(\operatorname{Re} n_0, n_1)$. Cauchy's theorem yields (5.17) immediately.

From the cases A, B and C it follows that

$$\int_{-1}^{1} \frac{dx}{1-x} P_k^{n_0,m}(x) \int_{\operatorname{Re} n_0-i\infty}^{\operatorname{Re} n_0+i\infty} nP_k^{m,n}(-x)f(n)dn = \frac{-4\pi i f(n)}{\Gamma(-\alpha_0)\Gamma(\delta_0+1)}.$$

Combining this and (5.17) the theorem is proved in this case. Herewith theorem 2b is proved completely.

6. Proof of Theorem 3b

Since

$$P_k^{m,-n_0}(x) = 2^{-n_0} P_k^{m,n_0}(x),$$

it is sufficient to prove this theorem for $\operatorname{Re} n_0 \leqq 0$. The proof in that case may be given in the same way as that of theorem 2b. Now we have:

$$(6.1) \qquad \begin{aligned} &\int_a^b \frac{dx}{1-x} P_k^{m,n_0}(-x) \int_{n_1-i\infty}^{n_1+i\infty} n\Gamma(\delta+1)\Gamma(-\alpha)P_k^{n,m}(x)F(n)dn \\ &= \int_{n_1-i\infty}^{n_1+i\infty} n\Gamma(\delta+1)\Gamma(-\alpha)F(n)\{A^*(b, n)-A^*(a, n)\}dn, \end{aligned}$$

where (see (5.3))

$$(6.2) \qquad A^*(x, n) = \frac{2}{n^2-n_0^2}(1-x^2)\left\{P_k^{m,n_0}(-x)\frac{d}{dx}P_k^{n,m}(x) - P_k^{n,m}(x)\frac{d}{dx}P_k^{m,n_0}(-x)\right\}$$

and (see (5.4))

(6.3) $A^*(x, n)$

$$= \frac{x+1}{n+n_0} P_k^{m, n_0}(-x) P_k^{n, m}(x) + \frac{2}{n^2-n_0^2} (1-x^2)^{\frac{1}{2}} [\alpha(\delta+1) P_k^{m, n_0}(-x)$$
$$P_k^{n-1, m-1}(x) + \alpha_0(\delta_0+1) P_k^{n, m}(x) P_k^{m-1, n_0-1}(-x)].$$

$A^*(x, n)$ is obtained from $A(x)$ of the proof of theorem 2b by interchanging n and n_0. Moreover the analogues of (5.7) and (5.8) hold. From (0.3) and (3.6) it follows that

$$\begin{aligned} A^*(2 \tanh^2 \eta - 1, n) &= \frac{2^{-\frac{1}{2}n}}{\Gamma(1-n)} n^{-\frac{1}{2}} O(1) \quad \text{if } \operatorname{Re} m < 1; \\ &= \frac{2^{-\frac{1}{2}n}}{\Gamma(1-n)} n^{-\frac{1}{2}} O(\log n) \text{ if } \operatorname{Re} m = 1; \\ &= \frac{2^{-\frac{1}{2}n}}{\Gamma(1-n)} n^{2m-\frac{5}{2}} O(1) \quad \text{if } \operatorname{Re} m > 1 \end{aligned}$$

as $n \to \infty$, $\eta = 1/|n|$, $\operatorname{Re} n \leqq n_1$. Hence from (4.4) and (1.30) we find

$$\lim_{a \downarrow -1} \int_{C_2} n\Gamma(\delta+1)\Gamma(-\alpha) f(n) A^*(a, n) dn = 0.$$

Proceeding further as in the proof of theorem 2b, it is easily shown that the contribution of $A^*(a, n)$ to the integral in the right-hand side of the analogue of (5.7) tends to zero.

The contribution of $A^*(b, n)$ to this integral can also be determined in a similar way as in the proof of theorem 2b. However, the calculation of

$$\lim_{b \uparrow 1} \int_{C_1} n\Gamma(\delta+1)\Gamma(-\alpha) f(n) A^*(b, n) dn$$

is much easier, since $A^*(b, n)$ on C_1 tends to zero on account of the formula (5.11) with n_0 and n interchanged. Therefore the Hölder condition, the evaluation of the residue at $n = n_0$, and in the case $\operatorname{Re} n_0 = n_1 = 0$ the splitting up of $A^*(x, n)$ in two functions, is superfluous, and it follows that the lefthand side of (6.1) tends to zero as $a \downarrow -1$ and $b \uparrow 1$.

7. Proof of Theorem 4b

Without loss of generality we may assume that $\operatorname{Re} n_0 \leqq 0$ on account of

$$P_k^{m,-n_0}(x) = 2^{-n_0}P_k^{m,n_0}(x) \text{ and } f(-n_0) = 2^{-n_0}f(n_0).$$

Moreover, from the conditions on $f(n)$ and Cauchy's theorem, it follows that we only have to treat the case $\operatorname{Re} n_0 = n_1$. Further we may use again the formulas (6.1), (6.2) and (6.3).

Case A: Suppose $n_1 = \operatorname{Re} n_0 < 0$.

The integrand in the righthand side of (6.1) is analytic in n in the strip S and continuous in $\bar{S}$. Hence, by Cauchy's theorem, we may replace the part of the path of integration $[n_0-i\varepsilon, n_0+i\varepsilon]$ by the half circle $C_1 : |n-n_0| = \varepsilon$, $\operatorname{Re} n \geqq n_1$. Denoting the new path by L_1, we have to prove:

(7.1)
$$\lim_{\substack{b\uparrow 1\\ a\downarrow -1}} \frac{1}{2\pi i}\int_{L_1} n\Gamma(\delta+1)\Gamma(-\alpha)f(n)\{A^*(b,n)-A^*(a,n)\}dn = -2f(n_0).$$

By (1.33) and Cauchy's theorem we have:

(7.2)
$$\int_{L_1} n\Gamma(\delta+1)\Gamma(-\alpha)f(n)A^*(a,n)dn = \int_{-i\infty}^{i\infty} n\Gamma(\delta+1)\Gamma(-\alpha)f(n)A^*(a,n)dn$$
$$= \int_0^{i\infty} nf(n)\{\Gamma(\delta+1)\Gamma(-\alpha)A^*(a,n)-\Gamma(\beta+1)\Gamma(-\gamma)2^{-n}A^*(a,-n)\}dn.$$

From (6.2), (3.4) and (5.4) we may deduce:

(7.3)
$$n\Gamma(\delta+1)\Gamma(-\alpha)A^*(a,n)-n\Gamma(\beta+1)\Gamma(-\gamma)2^{-n}A^*(a,-n) = B(a,n),$$

where

(7.4)
$$B(x,n) = \Gamma(-\alpha)\Gamma(\beta+1)\Gamma(-\gamma)\Gamma(\delta+1)2^{m-n}\frac{n\sin n\pi}{\pi}$$
$$\left[\frac{x+1}{n+n_0}P_k^{m,n_0}(-x)P_k^{m,n}(-x)+\frac{2}{n^2-n_0^2}(1-x^2)^{\frac{1}{2}}\right.$$
$$\left.\{\alpha_0(\delta_0+1)P_k^{m-1,n_0-1}(-x)P_k^{m,n}(-x)-\alpha(\delta+1)P_k^{m,n_0}(-x)P_k^{m-1,n-1}(-x)\}\right].$$

Hence from (7.2):

$$\text{(7.5)}\quad \int_{L_1} n\Gamma(\delta+1)\Gamma(-\alpha)f(n)A^*(a,n)dn = \int_0^{i\infty} f(n)B(a,n)dn.$$

The function $B(a, n)$ is regular at the point $n = n_0$. We split up the last integral into three parts:

$$(7.6)\qquad \int_0^{i\infty} = \int_0^{iN_0} + \int_{iN_0}^{i/\eta} + \int_{i/\eta}^{i\infty},$$

where N_0 is sufficiently large and $2\tanh^2\eta - 1 = a$. From (7.4) and (0.3) we find:

$$(7.7)\qquad B(x, n) = (1+x)^{1-m}O(1) \text{ as } x \downarrow -1 \text{ on } \lfloor 0, iN_0].$$

Hence

$$(7.8)\qquad \int_0^{iN_0} f(n)B(a, n)dn \to 0 \text{ as } a \downarrow -1,$$

since $\operatorname{Re} m < 1$. In the second integral in the righthand side of (7.6), $|n\eta|$ is bounded, and we can use (3.8). Since

$$n\Gamma(-\alpha)\Gamma(\beta+1)\Gamma(-\gamma)\Gamma(\delta+1)2^{m-n}\frac{\sin n\pi}{\pi} = n^{1-2m}O(1)$$

as $n \to \infty$, $\operatorname{Im} n = 0$ (see (4.4)), we obtain:

$$(7.9)\qquad B(2\tanh^2\eta - 1, n) = n^{-m}O(1)$$

as $\eta \downarrow 0$, $n \to \infty$, $\operatorname{Im} n = 0$ and $|n\eta|$ bounded. Hence from (1.33):

$$\int_{iN_0}^{i/\eta} f(n)B(2\tanh^2\eta - 1, n)dn \to 0 \text{ as } N_0 \to \infty,\ \eta \downarrow 0.$$

To the third integral in (7.6) we apply (3.7) instead of (3.8), and we obtain again (7.9) as $n\eta \to \infty$. Using (1.33) we see that this integral tends to zero as $\eta \downarrow 0$. Consequently from (7.6) and (7.5) we get:

$$(7.10)\qquad \lim_{a\downarrow -1}\int_{L_1} n\Gamma(\delta+1)\Gamma(-\alpha)f(n)A^*(a, n)dn = 0.$$

The contribution of $A^*(b, n)$ to the integral in (7.1) may be evaluated in the same way as that of $A(b)$ to the integral in the righthand side of (5.7). We now use the asymptotic behavior:

$$A^*(2\tanh^2 N - 1, n) = \frac{1}{n\Gamma(1-n)}O(1)$$

as $n \to \infty$, $\operatorname{Re} n = n_1$ uniformly for $N \to \infty$, and (5.11) with A replaced by A^* and n and n_0 interchanged. Then we find:

(7.11) $$\lim_{b\uparrow 1}\int_{L_1} n\Gamma(\delta+1)\Gamma(-\alpha)f(n)A^*(b,n)\,dn = -4\pi i f(n_0).$$

From (7.10) and (7.11) we find (7.1).

Case B: Suppose Re $n_0 = n_1 = 0$, $n_1 \neq 0$.

Since the case I of theorem 4b for Re $n_0 = n_1 = 0$ is contained in case II, we may restrict ourselves to this last case. It is also clear that without loss of generality we may assume Im $n_0 > 0$.

In a similar way as in case A it may be shown that

(7.12) $$\int_{-i\infty}^{i\infty} n\Gamma(\delta+1)\Gamma(-\alpha)f(n)\{A^*(b,n)-A^*(a,n)\}dn = \int_0^{i\infty} f(n)\{B(b,n)-B(a,n)\}dn,$$

and that the contribution of $B(a, n)$ to the last integral tends to zero as $a \downarrow -1$.

To calculate

(7.13) $$\lim_{b\uparrow 1}\int_0^{i\infty} f(n)B(b,n)\,dn$$

we now split it into two integrals:

(7.14) $$\int_0^{i\infty} = \int_0^{iN_0} + \int_{iN_0}^{i\infty}.$$

To approximate $B(b, n)$ in the first integral we apply (3.4) and (0.3), and find:

$$\begin{aligned} B(x,n) = {} & (1-x)^{\frac{1}{2}(n+n_0)}g_1(n,x)+(1-x)^{-\frac{1}{2}(n+n_0)}g_2(n,x) \\ & + \frac{2}{n-n_0}\{(1-x)^{\frac{1}{2}(n-n_0)}-(1-x)^{\frac{1}{2}(n_0-n)}\}\{1+g_3(x)\} \\ & +(1-x)^{\frac{1}{2}(n-n_0)}g_4(n,x)+(1-x)^{\frac{1}{2}(n_0-n)}g_5(n,x), \end{aligned}$$

where the functions g_j are analytic in n and x in the neighborhood of $x = 1$, whereas $g_3(x) \to 0$ as $x \uparrow 1$.

Since

$$(1-x)^{\frac{1}{2}(n-n_0)}-(1-x)^{\frac{1}{2}(n_0-n)} = 2i \sin \frac{n-n_0}{2i} \log(1-x),$$

and $\log(1-x) \to -\infty$ as $x \uparrow 1$, we may apply the Riemann-Lebesgue lemma and Dirichlet's formula to the first integral in the righthand side of (7.14) as $b \uparrow 1$, and find:

$$\lim_{b\uparrow 1}\int_0^{iN_0} f(n)B(b,n)\,dn = -2\pi i\{f(n_0+oi)+f(n_0-oi)\}.$$

By applying (3.7) and (3.4) to the integrand in the last integral of (7.14), it is easily found that this integrand is $f(n)n^{-\frac{1}{2}-m}O(1)$ as $n\to\infty$, $b\uparrow 1$. Hence from (1.35) it follows that the integral tends to zero as $b\uparrow 1$, $N_0\to\infty$. This completes the proof in this case.

Case C: Suppose $n_0 = n_1 = 0$.

The proof in this case differs from that in B only in the approximation of $B(b, n)$. To estimate $P_k^{m,0}(-b)$ it is not possible to use (3.4). We now apply (0.3) and the formula:

$$F(a,b;a+b;x) = \frac{-\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\log(1-x)+O(1), \text{ as } x\uparrow 1$$

(see [15, p. 110, (14)]). The integrand in (7.13) can be written as

$$\begin{aligned}\frac{4f(n)}{n}&\{(1-b)^{\frac{1}{2}n}-(1-b)^{-\frac{1}{2}n}\}(1+h_1(b))\\ &+f(n)(1-b)^{\frac{1}{2}n}\{\log(1-b)h_2(n,b)+h_3(n,b)\}\\ &+f(n)(1-b)^{-\frac{1}{2}n}\{\log(1-b)h_4(n,b)+h_5(n,b)\},\end{aligned}$$

where the functions h_j are analytic in n and b in the neighborhood of $n=0$ and $b=1$. Further the proof is quite similar to that in case B. With this theorem 4b is now proved completely.

Part IV

8. Examples

8.1. LEMMA. *Let p and q be arbitrary complex numbers with* $\mathrm{Re}\,(p-\frac{1}{2}n) > -1$ *and* $\mathrm{Re}\,q+1 > \frac{1}{2}|\mathrm{Re}\,m|$. *Then we have*:

(8.1)

$$\int_{-1}^{1}(1-t)^p(1+t)^q P_k^{n,m}(t)\,dt$$
$$= \frac{\Gamma(q+\frac{1}{2}m+1)\Gamma(p-\frac{1}{2}n+1)}{\Gamma(1-n)\Gamma(p+q+\frac{1}{2}(m-n)+2)}$$
$$2^{p+q+\frac{1}{2}(m-n)+1}{}_3F_2(\gamma+1,-\beta,p-\tfrac{1}{2}n+1;1-n,p+q+\tfrac{1}{2}(m-n)+2;1).$$

PROOF: From (0.3) it follows that the lefthand side is equal to

$$\frac{1}{\Gamma(1-n)}\int_{-1}^{1}(1-t)^{p-\frac{1}{2}n}(1+t)^{q+\frac{1}{2}m}F\{\gamma+1,\ -\beta;\ 1-n;\ \tfrac{1}{2}(1-t)\}\,dt.$$

The integral is convergent on account of the conditions on p and q. Substituting $\frac{1}{2}(1-t) = x$ and applying [16, p. 399, (5)] we obtain (8.1).

REMARK 9. (8.1) is an extension of [17, (38)].

COROLLARY. *Choosing* $q = k-p-1$ *we obtain:*
If $\mathrm{Re}\,(p-\frac{1}{2}n) > -1$ *and* $\mathrm{Re}\,(k-p) > \frac{1}{2}\,|\mathrm{Re}\, m|$, *we have:*

$$\int_{-1}^{1}(1-t)^{p}(1+t)^{k-p-1}P_{k}^{n,m}(t)dt = 2^{\gamma}\,\frac{\Gamma(k-p+\frac{1}{2}m)\Gamma(k-p-\frac{1}{2}m)\Gamma(p-\frac{1}{2}n+1)}{\Gamma(\gamma+1)\Gamma(\delta+1)\Gamma(-p-\frac{1}{2}n)}. \tag{8.2}$$

THEOREM 11. *Let* p *be a complex number, and* n_1 *a real number with*

$$n_1 < \min\{2+2\,\mathrm{Re}\,p,\ -\mathrm{Re}\,(2k+m)\} \tag{8.3}$$

and

$$\mathrm{Re}\,(k-p) > \tfrac{1}{4}. \tag{8.4}$$

Then for $-1 < x < 1$ *we have:*

$$\frac{1}{2\pi i}\int_{n_1-i\infty}^{n_1+i\infty} n2^{-\frac{1}{2}n}\,\frac{\Gamma(-\alpha)\Gamma(p-\frac{1}{2}n+1)}{\Gamma(\gamma+1)\Gamma(-p-\frac{1}{2}n)}\,P_{k}^{m,n}(x)\,dn = -2^{1-k-\frac{1}{2}m}\,\frac{(1+x)^{p+1}(1-x)^{k-p-1}}{\Gamma(k-p+\frac{1}{2}m)\Gamma(k-p-\frac{1}{2}m)}. \tag{8.5}$$

PROOF. We apply theorem 1b, formula (1.22), with

$$\varphi(t) = (1-t)^{p+1}(1+t)^{k-p-1}$$

and (8.2). Then we obtain (8.5) with the conditions (8.3) and the condition:

$$|\mathrm{Re}\, m| < 2\,\mathrm{Re}\,(k-p)-\tfrac{1}{2}.$$

The asymptotic behavior as $n \to \infty$, $\mathrm{Re}\, n \to n_1$ of the integrand in (8.5) can be found from (3.7) and Stirling's formula. From this it is easily seen that the integral has a meaning and is an analytic function of m if (8.3) and (8.4) are satisfied. Hence (8.5) holds for these conditions.

(8.5) becomes for $p = -\frac{3}{2}$:

$$\frac{1}{2\pi i}\int_{n_1-i\infty}^{n_1+i\infty} \frac{n}{n^2-1}\, 2^{-\frac{1}{2}n}\, \frac{\Gamma(-\alpha)}{\Gamma(\gamma+1)}\, P_k^{m,n}(x)\, dn = -2^{-k-\frac{1}{2}m-1}\, \frac{(1+x)^{-\frac{1}{2}}(1-x)^{k+\frac{1}{2}}}{\Gamma(k+\frac{1}{2}m+\frac{3}{2})\Gamma(k-\frac{1}{2}m+\frac{3}{2})}$$

if $n_1 < \min\left(-1, -\operatorname{Re}(2k+m)\right)$, $\operatorname{Re} k > -\frac{5}{4}$, $-1 < x < 1$. For $k = -\frac{1}{2}m-\frac{1}{2}$ we have:

$$\frac{1}{2\pi i}\int_{n_1-i\infty}^{n_1+i\infty} \frac{n}{n^2-1}\, 2^{-\frac{1}{2}n} P_{-(m+1)/2}^{m,n}(x)\, dn = -2^{-\frac{1}{2}}\, \frac{(1+x)^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}m}}{\Gamma(1-m)}$$

if $n_1 < -1$, $\operatorname{Re} m < \frac{3}{2}$, $-1 < x < 1$.

(8.5) becomes for $p = -1$:

$$\frac{1}{2\pi i}\int_{n_1-i\infty}^{n_1+i\infty} 2^{-\frac{1}{2}n}\, \frac{\Gamma(-\alpha)}{\Gamma(\gamma+1)}\, P_k^{m,n}(x)\, dn = 2^{-k-\frac{1}{2}m}\, \frac{(1-x)^k}{\Gamma(k+\frac{1}{2}m+1)\Gamma(k-\frac{1}{2}m+1)}$$

if $n_1 < -\operatorname{Re}(2k+m)$, $\operatorname{Re} k > -\frac{3}{4}$, $-1 < x < 1$.
For $k = -\frac{1}{2}m-\frac{1}{2}$ we have:

$$\frac{1}{2\pi i}\int_{n_1-i\infty}^{n_1+i\infty} 2^{-\frac{1}{2}n} P_{-\frac{1}{2}(m+1)}^{m,n}(x)\, dn = \sqrt{\frac{2}{\pi}}\, \frac{(1-x)^{-\frac{1}{2}(m+1)}}{\Gamma(\frac{1}{2}-m)}$$

if $\operatorname{Re} m < \frac{1}{2}$, $-1 < x < 1$.

(8.5) becomes for $p = -\frac{1}{2}$:

$$\frac{1}{2\pi i}\int_{n_1-i\infty}^{n_1+i\infty} n 2^{-\frac{1}{2}n}\, \frac{\Gamma(-\alpha)}{\Gamma(\gamma+1)}\, P_k^{m,n}(x)\, dn = 2^{1-k-\frac{1}{2}m}\, \frac{(1+x)^{\frac{1}{2}}(1-x)^{k-\frac{1}{2}}}{\Gamma(k+\frac{1}{2}m+\frac{1}{2})\Gamma(k-\frac{1}{2}m+\frac{1}{2})}$$

if $n_1 < -\operatorname{Re}(2k+m)$, $\operatorname{Re} k > -\frac{1}{4}$, $-1 < x < 1$. For $k = -\frac{1}{2}m-\frac{1}{2}$ we have:

$$\frac{1}{2\pi i}\int_{n_1-i\infty}^{n_1+i\infty} n 2^{-\frac{1}{2}n} P_{-\frac{1}{2}(m+1)}^{m,n}(x)\, dn = 0 \quad \text{if } \operatorname{Re} m < -\tfrac{1}{2},\ -1 < x < 1,$$

a trivial result since the integrand is an odd function of n.

8.2. Theorem 11 may be transformed by applying (1.1) to $P_k^{m,n}(x)$ in (8.5), substituting $(x-3)/(-x-1)$ by x, and replacing k by $-\frac{1}{2}(n+1)$, n by $-2k-1$ and n_1 by $-2k_1-1$, into the following

THEOREM 12. *Let p be a complex number and k_1 a real number with*

(8.6) $k_1 > \max\{-\frac{3}{2}-\operatorname{Re} p, \frac{1}{2}\operatorname{Re}(m-n)-1\}$ and $\operatorname{Re}(n+2p) < -\frac{3}{2}$.

Then we have for $x > 1$:

$$\frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}(2k+1)\frac{\Gamma(\beta+1)\Gamma(p+k+\frac{3}{2})}{\Gamma(\gamma+1)\Gamma(-p+k+\frac{1}{2})}P_k^{m,n}(x)dk = 2^{p+\frac{1}{2}(n-m)+\frac{3}{2}}\frac{(x-1)^{-p-\frac{1}{2}n-\frac{3}{2}}(x+1)^{\frac{1}{2}n}}{\Gamma(-p+\frac{1}{2}m-\frac{1}{2}n-\frac{1}{2})\Gamma(-p-\frac{1}{2}m-\frac{1}{2}n-\frac{1}{2})}. \tag{8.7}$$

For $m = n$ we have:

$$\frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}(2k+1)\frac{\Gamma(p+k+\frac{3}{2})}{\Gamma(-p+k+\frac{1}{2})}P_k^{m}(x)dk = 2^{p+\frac{3}{2}}\frac{(x-1)^{-\frac{1}{2}m-p-\frac{3}{2}}(x+1)^{\frac{1}{2}m}}{\Gamma(-p-\frac{1}{2})\Gamma(-p-m-\frac{1}{2})} \tag{8.8}$$

if $k_1 > -\frac{3}{2}-\operatorname{Re} p$, $\operatorname{Re}(m+2p) < -\frac{3}{2}$, $x > 1$.

For $p = -\frac{3}{2}$ (8.7) becomes:

$$\frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}\frac{2k+1}{k(k+1)}\frac{\Gamma(\beta+1)}{\Gamma(\gamma+1)}P_k^{m,n}(x)dk = 2^{\frac{1}{2}(n-m)}\frac{(x-1)^{-\frac{1}{2}n}(x+1)^{\frac{1}{2}n}}{\Gamma(1+\frac{1}{2}m-\frac{1}{2}n)\Gamma(1-\frac{1}{2}m-\frac{1}{2}n)} \tag{8.9}$$

if $k_1 > \max\{0, \frac{1}{2}\operatorname{Re}(m-n)-1\}$, $\operatorname{Re} n < \frac{3}{2}$, $x > 1$.
Taking $m = n$ in (8.9) or $p = -\frac{3}{2}$ in (8.8) we find:

$$\frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}\frac{2k+1}{k(k+1)}P_k^{m}(x)dk = \frac{1}{\Gamma(1-m)}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}$$

if $k_1 > 0$, $\operatorname{Re} m < \frac{3}{2}$, $x > 1$.

For $p = -1$, we derive from (8.7):

$$\frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}\frac{\Gamma(\beta+1)}{\Gamma(\gamma+1)}P_k^{m,n}(x)dk = 2^{-\frac{1}{2}+\frac{1}{2}(n-m)}\frac{(x-1)^{-\frac{1}{2}-\frac{1}{2}n}(x+1)^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}+\frac{1}{2}m-\frac{1}{2}n)\Gamma(\frac{1}{2}-\frac{1}{2}m-\frac{1}{2}n)} \tag{8.10}$$

if $k_1 > \frac{1}{2}\operatorname{Re}(m-n)-1$, $\operatorname{Re} n < \frac{1}{2}$, $x > 1$.
For $m = n$ we have from this last formula:

$$\frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty} P_k^m(x)dk = \frac{1}{\sqrt{2\pi}\,\Gamma(\frac{1}{2}-m)}(x-1)^{-\frac{1}{2}-\frac{1}{2}m}(x+1)^{\frac{1}{2}m}$$

if $\operatorname{Re} m < \frac{1}{2}$, $x > 1$.

For $p = -\frac{1}{2}$ we obtain from (8.7):

$$\text{(8.11)}\quad \begin{aligned}\frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}&(2k+1)\frac{\Gamma(\beta+1)}{\Gamma(\gamma+1)}P_k^{m,n}(x)dk\\ &= 2^{\frac{1}{2}(n-m)+1}\frac{(x-1)^{-\frac{1}{2}n-1}(x+1)^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}m-\frac{1}{2}n)\Gamma(-\frac{1}{2}m-\frac{1}{2}n)}\end{aligned}$$

if $k_1 > \frac{1}{2}\operatorname{Re}(m-n)-1$, $\operatorname{Re} n < -\frac{1}{2}$, $x > 1$.

For $m = n$ the integral vanishes, a trivial result since the integrand is an odd function of $k-\frac{1}{2}$.

8.3. (8.7) can be transformed by means of (0.2) in an integral with a hypergeometric function. After replacing $\frac{1}{2}(n-m)$ by a, $1-m$ by m, $1-x$ by $-2x$ we obtain:

$$\begin{aligned}\frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}&\frac{(2k+1)\Gamma(k+a+1)\Gamma(p+k+\frac{3}{2})}{\Gamma(m)\Gamma(k-a+1)\Gamma(-p+k+\frac{1}{2})}\\ &\qquad F(-k+a,\, k+a+1;\, m;\, -x)dk\\ &= \frac{x^{-p-a-\frac{3}{2}}}{\Gamma(-p-a-\frac{1}{2})\Gamma(m-p-a-\frac{3}{2})}\end{aligned}$$

if $k_1 > \max(-\frac{3}{2}-\operatorname{Re} p, -\operatorname{Re} a-1)$, $\operatorname{Re}(2p+2a-m) < -\frac{5}{2}$, $x > 0$.

For $p = -1$ we obtain:

$$\begin{aligned}\frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}&\frac{\Gamma(k+a+1)}{\Gamma(k-a+1)}\frac{1}{\Gamma(m)}F(-k+a,\, k+a+1;\, m;\, -x)dk\\ &= \frac{1}{2}\frac{x^{-a-\frac{1}{2}}}{\Gamma(\frac{1}{2}-a)\Gamma(m-a-\frac{1}{2})}\end{aligned}$$

if $k_1 > -\operatorname{Re} a-1$, $\operatorname{Re}(2a-m) < -\frac{1}{2}$, $x > 0$.

For $a = 0$ this becomes:

$$\frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}\frac{1}{\Gamma(m)}F(-k,\, k+1;\, m;\, -x)dk = \frac{x^{-\frac{1}{2}}}{2\sqrt{\pi}\,\Gamma(m-\frac{1}{2})}$$

if $\operatorname{Re} m > \frac{1}{2}$, $x > 0$.

8.4. Now we return to (8.10) and deduce from this formula a summation formula for the Q.

We suppose $m-n \neq 1, 2, \ldots$, and deform the path of integration in (8.10) such that the points $k = \frac{1}{2}(m-n)-g$ $(g = 1, 2, \ldots)$ and $k = -\frac{1}{2}, -1, -1\frac{1}{2}, -2, \ldots$ are to the left, and the points $k = -\frac{1}{2}(m-n)+h$ $(h = 0, 1, 2, \ldots)$ and $k = 0, \frac{1}{2}, 1, 1\frac{1}{2}, \ldots$ are to the right of the new path L. From (1.39) and (8.10) we obtain:

$$\begin{aligned}(8.12)\quad &\frac{1}{2\pi i}\int_L \frac{-\sin\gamma\pi\, 2^{n-m+1}}{\Gamma(\delta+1)\Gamma(-\alpha)\sin 2k\pi} e^{\pi i m}\{Q_k^{-m,-n}(x)-Q_{-k-1}^{-m,-n}(x)\}dk \\ &= 2^{-\frac{1}{2}+\frac{1}{2}(n-m)} \frac{(x-1)^{-\frac{1}{2}-\frac{1}{2}n}(x+1)^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}+\frac{1}{2}m-\frac{1}{2}n)\Gamma(\frac{1}{2}-\frac{1}{2}m-\frac{1}{2}n)}.\end{aligned}$$

To find the asymptotic behavior of $Q_k^{-m,-n}(x)$ as $k \to \infty$ on $|\arg k| \leqq \pi-\eta$ with $0 < \eta < \pi$, we use the formula ([12, (9)]):

$$Q_k^{-m,-n}(x) = e^{-\pi i m}2^{\gamma} \frac{\Gamma(\delta+1)\Gamma(\beta+1)}{\Gamma(2k+2)} (x+1)^{-k-\frac{1}{2}m-1}(x-1)^{\frac{1}{2}m}$$
$$F\left(\alpha+1, \gamma+1; 2k+2; \frac{2}{1+x}\right).$$

Applying (2.15) we obtain:

$$F\left(\alpha+1, \gamma+1; 2k+2; \frac{2}{1+x}\right) = \left(\frac{1+\sqrt{\dfrac{x-1}{x+1}}}{2}\right)^{-2k} O(1),$$

so that, using Stirling's formula, we have:

$$(8.13)\qquad Q_k^{-m,-n}(x) = k^{-m-\frac{1}{2}}(x+\sqrt{x^2-1})^{-k}O(1)$$

as $k \to \infty$ on $|\arg k| \leqq \pi-\eta$, $x > 1$. Furthermore

$$\frac{-\sin\gamma\pi 2^{n-m+1}e^{\pi i m}}{\Gamma(\delta+1)\Gamma(-\alpha)\sin 2k\pi} = k^{m+n}O(1),$$

valid for $k \to \infty$ on every set with a positive distance to the set of integers.

From this formula and (8.13) the asymptotic behavior of the integrand in (8.12) can be found. Then we see that for $\operatorname{Re} n < -\frac{1}{2}$ we may split up the integral in (8.12) into the two parts:

$$(8.14)\qquad \frac{1}{2\pi i}\int_L \frac{-\sin\gamma\pi\cdot 2^{n-m+1}e^{\pi i m}}{\Gamma(\delta+1)\Gamma(-\alpha)\sin 2k\pi} Q_k^{-m,-n}(x)dk$$

and

$$(8.15)\qquad \frac{1}{2\pi i}\int_L \frac{\sin\gamma\pi\, 2^{n-m+1}e^{\pi i m}}{\Gamma(\delta+1)\Gamma(-\alpha)\sin 2k\pi} Q_{-k-1}^{-m,-n}(x)dk.$$

Moreover we see that (8.14) is equal to

$-\sum$ (residues at the poles of the integrand to the right of L),

and (8.15) is equal to

$\sum$ (residues at the poles of the integrand to the left of L).

Hence the righthand side of (8.12) is equal to

$$-\sum_{g=0}^{\infty} \frac{-\sin \frac{1}{2}(m-n)\pi\,(-1)^g\, 2^{n-m+1}}{\Gamma(g+1-\frac{1}{2}(m+n))\,\Gamma(-g-\frac{1}{2}(m+n))}\, \frac{e^{\pi i m}}{2\pi}\, Q_g^{-m,-n}(x)$$
$$-\sum_{g=0}^{\infty} \frac{-\cos \frac{1}{2}(m-n)\pi\,(-1)^g\, 2^{n-m+1}}{\Gamma(g+\frac{3}{2}-\frac{1}{2}(m+n))\,\Gamma(-g-\frac{1}{2}-\frac{1}{2}(m+n))}\, \frac{-e^{\pi i m}}{2\pi}\, Q_{g+\frac{1}{2}}^{-m,-n}(x)$$
$$-\sum_{g=1}^{\infty} \frac{-\sin \frac{1}{2}(m-n)\pi\,(-1)^g\, 2^{n-m+1}}{\Gamma(-g+1-\frac{1}{2}(m+n))\,\Gamma(g-\frac{1}{2}(m+n))}\, \frac{e^{\pi i m}}{2\pi}\, Q_{g-1}^{-m,-n}(x)$$
$$-\sum_{g=0}^{\infty} \frac{\cos \frac{1}{2}(m-n)\pi\,(-1)^g\, 2^{n-m+1}}{\Gamma(-g+\frac{1}{2}-\frac{1}{2}(m+n))\,\Gamma(g+\frac{1}{2}-\frac{1}{2}(m+n))}\, \frac{-e^{\pi i m}}{2\pi}\, Q_{g-\frac{1}{2}}^{-m,-n}(x)$$
$$= \frac{\cos \frac{1}{2}(m-n)\pi}{\Gamma(\frac{1}{2}-\frac{1}{2}(m+n))^2}\, \frac{2^{n-m}}{\pi}\, e^{\pi i m} Q_{-\frac{1}{2}}^{-m,-n}(x)$$
$$+2\sum_{g=1}^{\infty} \frac{\cos \frac{1}{2}(m-n)\pi\,(-1)^g\, 2^{n-m}}{\Gamma(g+\frac{1}{2}-\frac{1}{2}(m+n))\,\Gamma(-g+\frac{1}{2}-\frac{1}{2}(m+n))}\, \frac{e^{\pi i m}}{\pi}\, Q_{g-\frac{1}{2}}^{-m,-n}(x).$$

Replacing m by $-m$ we find after some simplifications the relation:

(8.16)
$$e^{\pi i m} 2^{\frac{1}{2}(n-m-1)} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}m+\frac{1}{2}n)}{\Gamma(\frac{1}{2}+\frac{1}{2}m-\frac{1}{2}n)} (x-1)^{-\frac{1}{2}n-\frac{1}{2}}(x+1)^{\frac{1}{2}n}$$
$$= \frac{1}{\Gamma(\frac{1}{2}+\frac{1}{2}m-\frac{1}{2}n)^2}\, Q_{-\frac{1}{2}}^{m,n}(x) + 2\sum_{g=1}^{\infty} \frac{(-1)^g Q_{g-\frac{1}{2}}^{m,n}(x)}{\Gamma(g+\frac{1}{2}+\frac{1}{2}m-\frac{1}{2}n)\Gamma(-g+\frac{1}{2}+\frac{1}{2}m-\frac{1}{2}n)}.$$

This formula has been proved under the conditions $x > 1$, $\operatorname{Re} n < -\frac{1}{2}$, $m+n \neq -1, -2, \ldots$. From (8.13) it follows that the series in (8.16) converges for every n. Hence it can be shown that (8.16) is valid for $x > 1$.

For $m = n$ (8.16) becomes:

$$e^{\pi i m} \sqrt{\frac{\pi}{2}}\, (x+1)^{\frac{1}{2}m}(x-1)^{-\frac{1}{2}m-\frac{1}{2}} = \frac{1}{\Gamma(m+\frac{1}{2})} \left\{ Q_{-\frac{1}{2}}^{m}(x) + 2\sum_{g=1}^{\infty} Q_{g-\frac{1}{2}}^{m}(x) \right\}.$$

This is a special case of [15, p. 166 (3)].

With the same method we can deduce from (8.11) the following relation:

$$e^{\pi i m}\, 2^{\frac{1}{2}(n-m)} \frac{\Gamma(1+\frac{1}{2}m-\frac{1}{2}n)}{\Gamma(\frac{1}{2}m+\frac{1}{2}n)} (x-1)^{\frac{1}{2}n-1}(x+1)^{-\frac{1}{2}n}$$
$$= \sum_{g=0}^{\infty} (-1)^g \frac{2g+1}{\Gamma(g+1+\frac{1}{2}m+\frac{1}{2}n)\Gamma(-g+\frac{1}{2}m+\frac{1}{2}n)} Q_g^{m,n}(x),$$

valid for $x > 1$, and for $m = n$:

$$\frac{e^{\pi i m}}{\Gamma(m)} (x-1)^{\frac{1}{2}m-1}(x+1)^{-\frac{1}{2}m} = \sum_{g=0}^{\infty} (-1)^g \frac{2g+1}{\Gamma(g+m+1)\Gamma(m-g)} Q_g^m(x).$$

It is evident that from (8.7) more similar relations can be deduced.

8.5. If we choose in (8.1) $p = q = -\frac{1}{2}$ we may apply Watson's theorem [15, p. 189 (6)] to the ${}_3F_2$ and obtain:

$$\int_{-1}^{1} (1-t^2)^{-\frac{1}{2}} P_k^{n,m}(t)dt$$
$$= \frac{\pi^2\, 2^{\frac{1}{2}m+\frac{1}{2}n}}{\cos \frac{1}{2}m\pi} \frac{1}{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha)\Gamma(\frac{1}{2}-\frac{1}{2}\beta)\Gamma(\frac{1}{2}\gamma+1)\Gamma(\frac{1}{2}\delta+1)}$$

if $\operatorname{Re} n < 1$, $|\operatorname{Re} m| < 1$. Applying theorem 1b, (1.22), with $\varphi(t) = (1-t)^{\frac{1}{2}}(1+t)^{-\frac{1}{2}}$ we find:

$$\frac{1}{2\pi i}\int_{n_1-i\infty}^{n_1+i\infty} n 2^{-\frac{1}{2}n} \frac{\Gamma(-\frac{1}{2}\alpha)\Gamma(\frac{1}{2}\delta+\frac{1}{2})}{\Gamma(-\frac{1}{2}\beta+\frac{1}{2})\Gamma(\frac{1}{2}\gamma+1)} P_k^{m,n}(x)dn$$
$$= -\frac{1}{\pi} \cos \tfrac{1}{2}m\pi\, 2^{2+\frac{1}{2}m} \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}, \tag{8.17}$$

valid for (1.20) and $-1 < x < 1$.

From (8.17) we may deduce with the aid of (1.1) and the usual substitutions:

$$\frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty} (2k+1) \frac{\Gamma(\frac{1}{2}\beta+\frac{1}{2})\Gamma(\frac{1}{2}\delta+\frac{1}{2})}{\Gamma(\frac{1}{2}\alpha+1)\Gamma(\frac{1}{2}\gamma+1)} P_k^{m,n}(x)dk$$
$$= \frac{1}{\pi} \cos \tfrac{1}{2}m\pi \cdot 2^{\frac{1}{2}m+\frac{1}{2}n+2}(x^2-1)^{-\frac{1}{2}}, \tag{8.18}$$

valid for (0.5) and $x > 1$. For $m = n$ this becomes:

$$\frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty} (2k+1) \frac{\Gamma(\frac{1}{2}k+\frac{1}{2})\Gamma(\frac{1}{2}k-\frac{1}{2}m+\frac{1}{2})}{\Gamma(\frac{1}{2}k+1)\Gamma(\frac{1}{2}k+\frac{1}{2}m+1)} P_k^m(x)dk$$
$$= \frac{1}{\pi} \cos \tfrac{1}{2}m\pi \cdot 2^{m+2}(x^2-1)^{-\frac{1}{2}}.$$

8.6. In order to apply (1.23) we transform (8.2) by interchanging m and n and replacing t by $-t$. We obtain:

$$\int_{-1}^{1}(1-t)^{k-p-1}(1+t)^{p}P_k^{m,n}(-t)\,dt = 2^{\beta}\,\frac{\Gamma(k-p+\frac{1}{2}n)\Gamma(p-\frac{1}{2}m+1)\Gamma(k-p-\frac{1}{2}n)}{\Gamma(\beta+1)\Gamma(\delta+1)\Gamma(-p-\frac{1}{2}m)} \tag{8.19}$$

if Re $(p-\frac{1}{2}m) > -1$ and $\frac{1}{2}\,|\text{Re}\, n| <$ Re $(k-p)$.

This leads to the result:

$$\frac{1}{2\pi i}\int_{n_1-i\infty}^{n_1+i\infty} n2^{\frac{1}{2}n}\Gamma(k-p+\tfrac{1}{2}n)\Gamma(k-p-\tfrac{1}{2}n)\,\frac{\Gamma(-\alpha)}{\Gamma(\beta+1)}\,P_k^{n,m}(x)\,dn = -2^{-k+\frac{1}{2}m+1}\,\frac{\Gamma(-p-\frac{1}{2}m)}{\Gamma(p-\frac{1}{2}m+1)}\,(1-x)^{k-p}(1+x)^{p}, \tag{8.20}$$

if Re $p > -\frac{1}{4}$, $|n_1| < 2$ Re $(k-p)$, $n_1 <$ −Re $(2k+m)$, $-1 < x < 1$.

Choosing $p = k-1$ we obtain:

$$\frac{1}{2\pi i}\int_{n_1-i\infty}^{n_1+i\infty} 2^{\frac{1}{2}n}\,\frac{n^2}{\sin\frac{1}{2}n\pi}\,\frac{\Gamma(-\alpha)}{\Gamma(\beta+1)}\,P_k^{n,m}(x)\,dn = -\frac{1}{\pi}\,2^{-k+\frac{1}{2}m+2}\,\frac{\Gamma(1-k-\frac{1}{2}m)}{\Gamma(k-\frac{1}{2}m)}\,(1-x)(1+x)^{k-1}$$

if Re $k > \frac{3}{4}$, $|n_1| < 2$, $n_1 <$ −Re $(2k+m)$.

Formula (8.20) may be transformed by (1.2) into

$$\frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}(2k+1)\,\frac{\Gamma(-k+\frac{1}{2}n-p-1)\Gamma(k+\frac{1}{2}n-p)}{\Gamma(-\delta)\Gamma(\alpha+1)}\,e^{-\pi i m}Q_k^{m,n}(x)\,dk = 2^{-\frac{1}{2}m+n-p-1}\,\frac{\Gamma(-p+\frac{1}{2}m)}{\Gamma(p+\frac{1}{2}m+1)}\,(x-1)^{p}(x+1)^{-\frac{1}{2}n}, \tag{8.21}$$

valid for $x > 1$, Re $p > -\frac{1}{4}$, $|2k_1+1| <$ Re $(n-1-2p)$, $k_1 > \frac{1}{2}$ Re $(n-m)-1$.

For $p = \frac{1}{2}n-1$ this formula becomes:

$$\frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}\frac{2k+1}{\sin k\pi}\,\frac{1}{\Gamma(-\delta)\Gamma(\alpha+1)}\,e^{-\pi i m}Q_k^{m,n}(x)\,dk = -\frac{1}{\pi}\,2^{\frac{1}{2}n-\frac{1}{2}m}\,\frac{\Gamma(1+\frac{1}{2}m-\frac{1}{2}n)}{\Gamma(\frac{1}{2}m+\frac{1}{2}n)}\,(x-1)^{\frac{1}{2}n-1}(x+1)^{-\frac{1}{2}n},$$

valid for $x > 1$, Re $n > \frac{3}{2}$, $-1 < k_1 < 0$, $k_1 > \frac{1}{2}$Re $(n-m)-1$.

8.7. From [16, p. 400, (10)], (0.2) and (0.3) the following formula is easily derived:

(8.22)

$$\int_1^\infty (t-1)^{-\frac{1}{2}m}(t+1)^{-\frac{1}{2}n}(t+x)^{-2l-1}P_k^{m,n}(t)\,dt$$
$$= (x+1)^{-l-\frac{1}{2}m}(1-x)^{-l-\frac{1}{2}n}\,\frac{\Gamma(\alpha+2l+1)\Gamma(-\delta+2l)}{\Gamma(2l+1)}\,P_k^{-n-2l,-m-2l}(x),$$

valid for $\operatorname{Re} m<1$, $|\operatorname{Re}(2k+1)|<\operatorname{Re}(m+n+4l+1)$, $-1<x<1$.

Applying (0.8) of theorem 1 we obtain:

(8.23)

$$\frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}(2k+1)\Gamma(\delta-2l+1)\Gamma(-\alpha-2l)e^{-\pi i m}Q_k^{m,n}(x)P_k^{n+2l,m+2l}(z)\,dk$$
$$= \Gamma(1-2l)(x-1)^{\frac{1}{2}m}(x+1)^{\frac{1}{2}n}(1-z)^{-l-\frac{1}{2}n}(z+1)^{-l-\frac{1}{2}m}(x+z)^{2l-1},$$

valid for $x>1$, $-1<z<1$, $\operatorname{Re} m>-\frac{3}{4}$, $k_1>-\frac{1}{2}\operatorname{Re} m+\frac{1}{2}|\operatorname{Re} n|-1$, $|2k_1+1|<1-\operatorname{Re}(m+n+4l)$.

For $m=n$ (8.23) becomes:

(8.24)

$$\frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}(2k+1)\Gamma(k-m-2l+1)\Gamma(-k-m-2l)e^{-\pi i m}Q_k^{m}(x)P_k^{m+2l}(z)\,dk$$
$$= \Gamma(1-2l)(x^2-1)^{\frac{1}{2}m}(1-z^2)^{-l-\frac{1}{2}m}(x+z)^{2l-1}.$$

Choosing in (8.23) $l=0$, and replacing m by $-m$, we find:

(8.25)
$$\frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}(2k+1)\Gamma(-\beta)\Gamma(\gamma+1)e^{\pi i m}Q_k^{-m,-n}(x)P_k^{n,m}(z)\,dk$$
$$= \frac{2^{m-n}}{x+z}\left(\frac{1-z}{x+1}\right)^{-\frac{1}{2}n}\left(\frac{1+z}{x-1}\right)^{\frac{1}{2}m},$$

valid for $x>1$, $-1<z<1$, $\operatorname{Re} m<\frac{3}{4}$, (0.5) and $|2k_1+1|<1+\operatorname{Re}(m-n)$. For $m=n$ this becomes:

(8.26)
$$\frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}\frac{2k+1}{\sin k\pi}\,e^{\pi i m}Q_k^{-m}(x)P_k^{m}(z)\,dk$$
$$= \frac{-1}{\pi(x+z)}\left(\frac{1+z}{1-z}\cdot\frac{x+1}{x-1}\right)^{\frac{1}{2}m}.$$

8.8. Other applications of theorem 1, (0.8), may be found by using the formulas:

$$\int_1^\infty e^{-at}(t^2-1)^{-\frac{1}{2}m}P_k^m(t)dt = \sqrt{\frac{2}{\pi}}\, a^{m-\frac{1}{2}}K_{k+\frac{1}{2}}(a),$$

where $\operatorname{Re} a > 0$, $\operatorname{Re} m < 1$, and $K_k(x)$ denotes the modified Bessel function (see [16, p. 323, (11)]), and

$$\int_1^\infty \left(\frac{t+1}{t-1}\right)^{\frac{1}{2}m} e^{-at}P_k^m(t)dt = a^{-1}W_{m,k+\frac{1}{2}}(2a),$$

where $\operatorname{Re} a > 0$, $\operatorname{Re} m < 1$, and $W_{k,m}(x)$ denotes the Whittaker function.

The results are:

$$\begin{aligned}\frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty} &(2k+1)e^{-\pi i m}Q_k^m(x)K_{k+\frac{1}{2}}(a)dk \\ &= \sqrt{\frac{\pi}{2}}\, a^{\frac{1}{2}+m}e^{-ax}(x^2-1)^{\frac{1}{2}m},\end{aligned} \tag{8.27}$$

valid for

$$x > 1,\ \operatorname{Re} a > 0,\ \operatorname{Re} m > -\tfrac{3}{4},\ k_1 > -\tfrac{1}{2}\operatorname{Re} m+\tfrac{1}{2}\,|\operatorname{Re} m| -1,$$

and

$$\frac{1}{2\pi i}\int_{k_1-i\infty}^{k_1+i\infty}(2k+1)e^{\pi i m}Q_k^{-m}(x)W_{m,k+\frac{1}{2}}(2a)dk = a\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}e^{-ax}, \tag{8.28}$$

valid for $x > 1$, $\operatorname{Re} a > 0$, $\operatorname{Re} m < \frac{3}{4}$, $k_1 > \frac{1}{2}\operatorname{Re} m+\frac{1}{2}\,|\operatorname{Re} m|-1$.

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