Achieve, Inc.

High School Integrated Model Course Sequence

Integrated Mathematics Course 1

Integrated Mathematics Course 2

Integrated Mathematics Course 3

Achieve, Inc.
May 2008

Integrated Mathematics Course 1

Year One of a Model Three-Year Integrated High School Course Sequence

This first course in a three-course integrated mathematics sequence builds on a rigorous pre-algebra experience such as one indicated by the grade 8 expectations found in the National Assessment of Educational Progress (NAEP) guidelines and Achieve's Model Middle School Courses, as well as in many states' eighth-grade standards.

In particular, it is expected that students will come to this course with a strong conceptual foundation in ratios, rates, and proportional relationships and an understanding of simple linear and non-linear patterns of growth and their representation in the coordinate plane. In addition, students should have a thorough knowledge of the key characteristics of basic geometric shapes and objects. Students entering Integrated Mathematics Course 1 should be prepared to develop a more formal approach to similarity and congruence including, as the course develops, the proofs of key theorems about congruence and similarity in triangles. Proportional functions also follow from this foundation and offer an opportunity to reinforce students' experiences with linear relationships. An introduction to coordinate transformations of functions is applied here to linear, proportional, and simple reciprocal functions; this concept will be revisited again in Integrated Mathematics Course 3 when additional function types have been introduced. Following a discussion of propositional logic and geometric proof, the congruence and similarity theorems are verified, and basic geometric theorems are applied to geometric constructions and to the definitions of ratios in the trigonometry of right triangles. The course concludes with topics from algebra and discrete mathematics that involve reasoning about compound situations—that is, situations involving two or more events. The habits and tools of analysis and logical reasoning developed through geometric topics can and should be applied throughout mathematics. The closing unit in this course applies these tools to probability and probability distributions.

Appropriate use of technology is expected in all work. In Integrated Mathematics Course 1, this includes employing technological tools to assist students in forming and testing conjectures, creating graphs and data displays and determining and assessing lines of fit for data. Geometric constructions should be performed using geometric software as well as classical tools and technology should be used to aid three-dimensional visualization. Testing with and without technological tools is recommended.

How a particular subject is taught influences not only the depth and retention of the content of a course but also the development of skills in inquiry, problem solving, and critical thinking. Every opportunity should be taken to link the concepts of Integrated Mathematics Course 1 to those encountered in middle school mathematics as well as to other disciplines. Students should be encouraged to be creative and innovative in their approach to problems, to be productive and persistent in seeking solutions, and to use multiple means to communicate their insights and understanding.

The Major Concepts below provide the focus for Integrated Mathematics Course 1. They should be taught using a variety of methods and applications so that students attain a deep understanding of these concepts. Maintenance Concepts should have been taught previously and are important foundational concepts that will be applied in this course. Continued facility with and understanding of the Maintenance Concepts is essential for success in the Major Concepts defined for this course.

MAJOR CONCEPTS

- Proportion, Scale and Similarity
- Proportional Functions
- Fundamentals of Logic
- Geometric Relationships, Proof, and Constructions
- Linear Equations, Inequalities, and Systems
- Counting and Computing Probability for Compound Events

MAINTENANCE CONCEPTS

- Ratios, Rates, and Proportions
- Functions and Coordinate Graphs
- Linear Relationships
- Systems of Linear Equations
- Basic Geometric Shapes and Objects and Their Characteristics
- Rigid Motions in the Plane
- Basic Probability

A. Proportion, Scale, and Similarity

Rates, ratios and proportions are a major focus of a middle school curriculum. This course builds on that knowledge, extending proportions and scaling to arithmetic and geometric applications. The section following this one will extend these concepts to proportional functions in algebra.

Successful students will:

A1 Extend and apply understanding about rates and ratios, estimation, and measurement to derived measures, including weighted averages, using appropriate units and unit analysis to express and check solutions.

Derived measures are those achieved through calculations with measurement that can be taken directly.

a. Create and interpret scale drawings as a tool for solving problems.

b. Use unit analysis to clarify appropriate units in calculations.

Example: The calculation for converting 50 feet per second to miles per hour can be checked using the unit calculation.
$\frac{50 \text{ feet}}{1 \text{ second}} \bullet \frac{1 \text{ mile}}{5280 \text{ feet}} \bullet \frac{60 \text{ seconds}}{1 \text{ minute}} \bullet \frac{60 \text{ minute}}{1 \text{ hour}} = 34.\overline{09} \frac{\text{miles}}{\text{hour}}$ yields the correct units since the units feet, seconds and minutes all appear in both numerator and denominator.

c. Identify applications that can be expressed using derived measures or weighted averages; use and identify potential misuses of derived measures or weighted averages.

Examples: Percent change and density are examples of derived measures; grade averages, stock market indexes, the consumer price index, and unemployment rates are examples of weighted averages.

A2 Use ratios and proportional reasoning to apply a scale factor to a geometric object, a drawing, a three-dimensional space, or a model and analyze the effect.

A scale factor is a fixed positive real number, r, that multiplies the distances between any two points of a figure, resulting in a figure having the same shape.

a. Extend the concept of scale factor to relate the length, area, and volume of other figures and objects.

Example: Compare the metabolic rate of a man with that of someone twice his size, assuming that the metabolic rate of the human body is proportional to the body mass raised to the $\frac{3}{4}$ power.

A3 Identify and use relationships among volumes of common solids.

a. Identify and apply the 3:2:1 relationship between the volumes of circular cylinders, hemispheres, and cones of the same height and circular base.

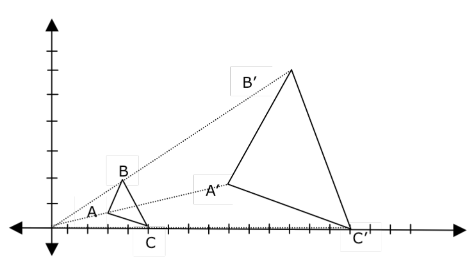
b. Recognize that the volume of a pyramid is one-third the volume of a prism of the same base area and height and use this to solve problems involving such measurements.

A4 Analyze, interpret, and represent origin-centered dilations and relate them to scaling and similarity.

An origin-centered dilation with scale factor r maps every point (x, y) in the coordinate plane to the point (rx, ry).

a. Interpret and represent origin-centered dilations of objects on the coordinate plane.

Example: In the following figure, triangle A′B′C′ with A′(9,3), B′(12,6), and C′(15,0) is the dilation of triangle ABC with A(3,1), B(4,2), and C(5,0). The scale factor for this dilation is 3.



b. Explain why the image under an origin-centered dilation is similar to the original figure.

c. Show that an origin-centered dilation maps a line to a line with the same slope, that dilations map parallel lines to parallel lines (lines passing through the origin remain unchanged and are parallel to themselves), and that a dilation maps a figure into a similar figure.

A5 Identify and apply conditions that are sufficient to guarantee similarity of triangles.

Informally, two geometric objects in the plane are similar if they have the same shape. More formally, having the same shape means that one figure can be mapped onto the other by means of rigid transformations and/or an origin-centered dilation.

a. Identify two triangles as similar if the ratios of the lengths of corresponding sides are equal (SSS criterion), if the ratios of the lengths of two pairs of corresponding sides and the measures of the corresponding angles between them are equal (SAS criterion), or if two pairs of corresponding angles are congruent (AA criterion).

b. Apply the SSS, SAS, and AA criteria to verify whether or not two triangles are similar.

c. Apply the SSS, SAS, and AA criteria to construct a triangle similar to a given triangle using straightedge and compass or geometric software.

d. Identify the constant of proportionality and determine the measures of corresponding sides and angles for similar triangles.

e. Use similar triangles to demonstrate that the rate of change (slope) associated with any two points on a line is a constant.

f Recognize, use, and explain why a line drawn inside a triangle parallel to one side forms a smaller triangle similar to the original one.

A6 Identify congruence as a special case of similarity; determine and apply conditions that guarantee congruence of triangles.

Informally, two figures in the plane are congruent if they have the same size and shape. More formally, having the same size and shape means that one figure can be mapped into the other by means of a sequence of rigid transformations.

a. Determine whether two plane figures are congruent by showing whether they coincide when superimposed by means of a sequence of rigid motions (translation, reflection, or rotation).

b. Identify two triangles as congruent if the lengths of corresponding sides are equal (SSS criterion), if the lengths of two pairs of corresponding sides and the measures of the corresponding angles between them are equal (SAS criterion), or if two pairs of corresponding angles are congruent and the lengths of the corresponding sides between them are equal (ASA criterion).

c. Apply the SSS, SAS, and ASA criteria to verify whether or not two triangles are congruent.

d. Apply the definition and characteristics of congruence to make constructions, solve problems, and verify basic properties of angles and triangles.

   Examples: Identify two triangles as congruent if two pairs of corresponding angles and their included sides are all equal (ASA criterion); verify that the bisector of the angle opposite the base of an isosceles triangle is the perpendicular bisector of the base; construct an isosceles triangle with a given base angle.

A7 Extend the concepts of similarity and congruence to other polygons in the plane.

a. Identify two polygons as similar if have the same number of sides and angles, if corresponding angles have the same measure, and if corresponding sides are proportional; identify two polygons as congruent if they are similar and their constant of proportionality equals one.

b. Determine whether or not two polygons are similar.

c. Use examples to show that analogues of the SSS, SAS, and AA criteria for similarity of triangles do not work for polygons with more than three sides.

B. Proportional Functions

Linear patterns of growth are a focus of the middle school curriculum. Description, analysis, and interpretation of lines should continue to be reinforced and extended, as students work in this course with functions that express direct proportions. The reciprocal functions introduced here should be linked back to student experience with proportions and with the simple exponential patterns of growth studied in middle school. These functions are strongly linked to the concepts of scaling and similarity addressed earlier in this course. Also included here is a first look at how changes in parameters affect the graph of a function.

Successful students will:

B1 Recognize, graph, and use direct proportional relationships.

A proportion is composed of two pairs of real numbers, (a, b) and (c, d), with at least one member of each pair non-zero, such that both pairs represent the same ratio. A linear function in which f(0) = 0 represents a direct proportional relationship. The function f(x) = kx, where k is constant, describes a direct proportional relationship.

a. Analyze the graph of a direct proportional relationship, *f(x) = kx* and identify its key characteristics.

   The graph of a direct proportional relationship is a line that passes through the origin (0, 0) whose slope is the constant of proportionality.

b. Compare and contrast the graphs of x = k, y = k and y = kx, where k is a constant.

c. Recognize and provide a logical argument that if *f(x)* is a linear function, *g(x) = f(x)* – *f*(0) represents a direct proportional relationship.

In this case, g(0) = 0, so g(x) = kx. The graph of f(x) = mx + b is the graph of the direct proportional relationship g(x) = mx shifted up (or down) by b units. Since the graph of g(x) is a straight line, so is the graph of f(x).

d. Recognize quantities that are directly proportional and express their relationship symbolically.

Example: The relationship between length of the side of a square and its perimeter is directly proportional.

B2 Recognize, graph, and use reciprocal relationships.

A function of the form f(x) = k/x where k is constant describes a reciprocal relationship. The term "inversely proportional" is sometimes used to identify such relationships, however, this term can be very confusing since the word "inverse" is also used in the term "inverse function" (the function $f^{-1}(x)$ *with the property that* $f \circ f^{-1}(x) = f^{-1} \circ f(x) = x$*, the identity function).*

a. Analyze the graph of reciprocal relationships, *f(x) = k/x* and identify its key characteristics.

The graph of f(x) = k/x is not a straight line and does not cross either the x– or the y–axis (i.e., there is no value of x for which f(x) = 0, nor is there any value for f(x) if x= 0).

b. Recognize quantities that are inversely proportional and express their relationship symbolically.

Example: The relationship between lengths of the base and side of a rectangle with fixed area is inversely proportional.

B3 Distinguish among and apply linear, direct proportional, and reciprocal relationships; identify and distinguish among applications that can be expressed using these relationships.

a. Identify whether a table, graph, formula, or context suggests a linear, direct proportional, or reciprocal relationship.

b. Create graphs of linear, direct proportional, and reciprocal functions by hand and using technology.

c. Distinguish practical situations that can be represented by linear, directly proportional, or inversely proportional relationships; analyze and use the characteristics of these relationships to answer questions about the situation.

B4 Create, interpret, and apply mathematical models to solve problems arising from contextual situations that involve linear relationships.

a. Distinguish relevant from irrelevant information, identify missing information, and find what is needed or make appropriate estimates.

b. Apply problem solving heuristics to practical problems: Represent and analyze the situation using symbols, graphs, tables, or diagrams; assess special cases; consider analogous situations; evaluate progress; check the reasonableness of results; and devise independent ways of verifying results.

B5 Explain and illustrate the effect of varying the parameters *m* and *b* in the family of linear functions and varying the parameter *k* in the families of directly proportional and reciprocal functions.

C. Fundamentals of Logic

This relatively short unit formalizes the vocabulary and methods of reasoning that form the foundation for logical arguments in mathematics. Examples should be taken from numeric and algebraic branches of mathematics as well as from everyday reasoning and argument. While this unit emphasizes the application of reasoning in a broad spectrum of contexts, the following unit will mainly apply logical thinking to geometric contexts.

Successful students will:

C1 Use mathematical notation, terminology, syntax, and logic; use and interpret the vocabulary of logic to describe statements and the relationship between statements.

a. Identify and give examples of definitions, conjectures, theorems, proofs, and counterexamples.

b. Describe logical statements using such terms as assumption, hypothesis, conclusion, converse, and contrapositive.

C2 Make, test, and confirm or refute conjectures using a variety of methods

a. Distinguish between inductive and deductive reasoning; explain and illustrate the importance of generalization in mathematics.

Inductive reasoning is based on observed patterns and can be used in mathematics to generate conjectures, after which deductive reasoning can be used to show that the conjectures are true in all circumstances. Inductive reasoning cannot prove propositions; valid conclusions and proof require deduction.

b. Construct simple logical arguments and proofs; determine simple counterexamples.

c. Demonstrate through example or explanation how indirect reasoning can be used to establish a claim.

d. Recognize syllogisms, tautologies, and circular reasoning and use them to assess the validity of an argument.

e. Recognize and avoid flawed reasoning; recognize flaws or gaps in the reasoning used to support an argument.

Example: The fact that A implies B does not imply that B implies A.

C3 Analyze and apply algorithms for searching, for sorting, and for solving optimization problems.

a. Identify and apply algorithms for searching, such as sequential and binary.

b. Describe and compare simple algorithms for sorting, such as bubble sort, quick sort, and bin sort.

   Example: Compare strategies for alphabetizing a long list of words; describe a process for systematically solving the Tower of Hanoi problem.

c. Know and apply simple optimization algorithms.

   Example: Use a vertex-edge graph (network diagram) to determine the shortest path needed to accomplish some task.

D. Geometric Relationships, Proof, and Constructions

Once students have gained experience with logic in multiple venues, geometry—partially because of its physical aspects—provides an excellent context in which to hone reasoning skills. This section identifies coordinate transformations as one example of generalization in mathematics. It applies generalization as well as inductive and deductive reasoning to establish similarity theorems (introduced earlier) and geometric constructions. This topic also offers the opportunity to reinforce the theorems about angles and triangles encountered in middle school.

Successful students will:

D1 Interpret, represent, and verify geometric relationships.

a. Use the Pythagorean theorem to determine slant height, surface area, and volume for pyramids and cones; justify the process through diagrams and logical reasoning.

b. Present and analyze geometric proofs using paragraphs or two-column or flow-chart formats.

   Example: Explain why, if two lines are intersected by a third line and the corresponding angles, alternate interior angles, or alternate exterior angles are congruent, then the two original lines must be parallel.

c. Use coordinates and algebraic techniques to interpret, represent, and verify geometric relationships in the plane.

   Examples: Given the coordinates of the vertices of a quadrilateral, determine whether it is a parallelogram; given a line segment in the coordinate plane whose endpoints are known, determine its length, midpoint, and slope*;* find an equation of a circle given its center and radius, and conversely, given an equation of a circle, find its center and radius.

D2 Analyze, execute, explain, and apply simple geometric constructions.

a. Perform and explain simple straightedge and compass constructions.

b. Apply properties of lines and angles to perform and justify basic geometric constructions.

   Example: Use properties of alternate interior angles to construct a line parallel to a given line.

c. Use geometric computer or calculator packages to create and test conjectures about geometric properties or relationships.

D3 Show how similarity of right triangles allows the trigonometric functions sine, cosine, and tangent to be properly defined as ratios of sides.

a. Know the definitions of sine, cosine, and tangent as ratios of sides in a right triangle and use trigonometry to calculate the length of sides, measure of angles, and area of a triangle.

b. Derive, interpret and use the identity $\sin^2\theta + \cos^2\theta = 1$ for angles θ between 0° and 90°.

This identity is a special representation of the Pythagorean theorem.

E. Linear Equations, Inequalities and Systems

Considering what happens when two or more conditions exist is the theme that ties together the ideas found in the final two sections of the course. Understanding the language and meaning of mathematical terms lays the foundation for the solution of systems of equations and inequalities. Linear systems provide another opportunity to reinforce the basics of linear functions and offer a myriad of opportunities for contextual problem solving.

Successful students will:

E1 Know the concepts of sets, elements, empty set, relations (e.g., belong to), and subsets, and use them to represent relationships among objects and sets of objects.

a. Recognize and use different methods to define sets (lists, defining property).

b. Perform operations on sets: union, intersection, complement.

Example: Use Boolean search techniques to refine online bibliographic searches.

c. Create and interpret Venn diagrams to solve problems.

d. Identify whether a given set is finite or infinite; give examples of both finite and infinite sets.

E2 Use and interpret relational conjunctions ("and," "or," "not"), terms of causation ("if… then"), and equivalence ("if and only if").

a. Distinguish between the common uses of such terms in everyday language and their use in mathematics.

b. Relate and apply these operations to situations involving sets.

E3 Solve equations and inequalities involving the absolute value of a linear expression in one variable.

a. Use conjunctions and disjunctions to express equations and inequalities involving absolute value as compound sentences that do not involve absolute value.

Examples: Rewrite the absolute value inequality $|x-18| \geq 27$ as the disjunction $x-18 \geq 27$ or $x-18 \leq -27$.

b. Graph the solution of a single-variable inequality involving the absolute value of a linear expression as an open or closed interval on the number line or as a union of two of them.

E4 Solve and graph the solution of a linear inequality in two variables.

a. Know what it means to be a solution of a linear inequality in two variables, represent solutions algebraically and graphically, and provide examples of ordered pairs that lie in the solution set.

b. Graph a linear inequality in two variables and explain why the graph is always a half-plane (open or closed).

E5 Solve systems of two or more linear inequalities in two variables and graph the solution set.

Example: The set of points (x, y) that satisfy all three inequalities $5x - y \geq 3$, $3x + y \leq 10$, and $4x - 3y \leq 6$ is a triangle, the intersection of three half-planes whose points satisfies each inequality separately.

E6 Solve systems of linear equations in two and three variables using algebraic procedures; describe the possible arrangements of the graphs of three linear equations in three variables and relate these to the number of solutions of the corresponding system of equations.

E7 Recognize and solve problems that can be modeled using a linear inequality or a system of linear equations or inequalities; interpret the solution(s) in terms of the context of the problem.

Example: Optimization problems that can be approached through linear programming.

F. Counting and Computing Probability for Compound Events

The final topic addressed in this integrated course extends the compound thinking developed earlier from algebraic contexts to those involving discrete events. Counting the number of ways a series of events can occur and applying prior knowledge of probability encourages students to see linkages across mathematical content areas. As with linear equations, inequalities, and systems, these topics have important contextual applications.

Successful students will:

F1 Represent and calculate probabilities associated with compound events.

a. Distinguish between dependent and independent events.

b. Use Venn diagrams to summarize information about compound events.

c. Represent bivariate categorical data in a two-way frequency table; show how such a table can be used effectively to calculate and study relationships among probabilities for two events.

d. Recognize probability problems that can be represented by geometric diagrams, on the number line, or in the coordinate plane; represent such situations geometrically and apply geometric properties of length or area to calculate the probabilities.

e. Use probability to interpret odds and risks and recognize common misconceptions.

Examples: After a fair coin has come up heads four times in a row, explain why the probability of tails is still 50%; analyze the risks associated with a particular accident, illness, or course of treatment; assess the odds of winning the lottery or being selected in a random drawing.

F2 Construct and interpret discrete graphs and charts to represent contextual situations.

a. Construct and interpret network graphs and use them to diagram social and organizational networks.

A graph is a collection of points (nodes) and the lines (edges) that connect some subset of those points; a cycle on a graph is a closed loop created by a subset of edges. A directed graph is one with one-way arrows as edges.

Examples: Determine the shortest route for recycling trucks; schedule when contestants play each other in a tournament; illustrate all possible travel routes that include four cities; interpret a directed graph to determine the result of a tournament.

b. Construct and interpret decision trees to represent the possible outcomes of independent events.

A tree is a connected graph containing no closed loops (cycles).

Examples: Classification of quadrilaterals; repeated tossing of a coin; possible outcomes of moves in a game.

c. Construct and interpret flow charts.

F3 Determine the number of ways events can occur using permutations, combinations, and other systematic counting methods.

A permutation is a rearrangement of distinct items in which their order matters; a combination is a selection of a given number of distinct items from a larger number without regard to their arrangement (i.e., in which their order does not matter).

a. Know and apply organized counting techniques such as the Fundamental Counting Principle.

The Fundamental Counting Principal is a way of determining the number of ways a sequence of events can take place. If there are n ways of choosing one thing and m ways of choosing a second after the first has been chosen, then the Fundamental Counting Principal says that the total number of choice patterns is n • m.

Examples: How many different license plates can be formed with two letters and three numerals? If the letters had to come first, how many letters would be needed to create at least as many different license plate numbers? How many different subsets are possible for a set having six elements?

b. Distinguish between counting situations that do not permit replacement and situations that do permit replacement.

   Examples: How many different four-digit numbers can be formed if the first digit must be non-zero and each digit may be used only once? How many are possible if the first digit must be non-zero but digits can be used any number of times?

c. Distinguish between situations where order matters and situations where it does not; select and apply appropriate means of computing the number of possible arrangements of the items in each case.

d. Interpret and simplify expressions involving factorial notation; use factorial notation to express permutations and combinations.

   Examples: Interpret 6! as the product 6 • 5 • 4 • 3 • 2 • 1; recognize that $\frac{15!}{12!} = 15 \cdot 14 \cdot 13 = 2{,}730$.

Integrated Mathematics Course 2

Year Two of a Model Three-Year Integrated High School Course Sequence

The concept guiding the development of this model integrated course sequence is to provide rigorous high school mathematics organized around themes that cut across traditional mathematics sub-disciplines. In this second of three courses, statistical and algebraic topics are linked through the central and connecting concept of quadratic relationships.

Building on the work in Integrated Mathematics Course 1 with logic and its application in geometric, algebraic, and probabilistic arenas, this second course opens with a look at probability distributions and reasoning from data. The binomial expansion theorem, an example of an algebraic representation of a discrete distribution, provides a bridge to algebraic topics. Quadratic functions and equations follow, with the introduction of rational exponents, roots, and complex numbers opening the way for the solution of all quadratic equations. The concepts and techniques applied to quadratic functions are extended to the study of power and polynomial functions in the final unit of this course.

Throughout Integrated Mathematics Course 2, technology is an important tool for data analysis as well as for graphical visualization and deepening understanding of function relationships. If computer or calculator-based algebra manipulation is available, there are many opportunities in this course to use such technology for exploration. Care should be taken, however, to parallel any such use of technology with development of by-hand algebraic skills. Testing with and without technological tools is recommended.

How a particular subject is taught influences not only the depth and retention of the content of a course but also the development of skills in inquiry, problem solving, and critical thinking. Every opportunity should be taken to link the concepts of Integrated Mathematics Course 2 to those encountered in Integrated Mathematics Course 1 and middle school mathematics as well as to other disciplines. Students should be encouraged to be creative and innovative in their approach to problems, to be productive and persistent in seeking solutions, and to use multiple means to communicate their insights and understanding.

The **Major Concepts** below provide the focus for this second course in a three-course integrated mathematics sequence. They should be taught using a variety of methods and applications so that students attain a deep understanding of these concepts. **Maintenance Concepts** should have been taught previously and are important foundational concepts that will be applied in this course. Continued facility with and understanding of the Maintenance Concepts is essential for success in the Major Concepts defined for this course.

MAJOR CONCEPTS

- Reasoning from Data
- Applying Exponents
- Quadratic Functions and Equations with Real Zeros/Roots
- Quadratic Functions and Equations with Complex Zeros/Roots
- Power and Polynomial Functions and Expressions

MAINTENANCE CONCEPTS

- Elementary Data Analysis
- Integer Exponents and Roots
- Real Numbers
- Integer Exponents
- Linear and Proportional Relationships
- Fundamental Logic, Reasoning, and Proof

A. Reasoning from Data

Integrated Mathematics Course 1 ended with a look at probability and its applications, and this course begins by extending those concepts to probability distributions and the information they convey that leads to rational, reasoned decision-making. Since issues of precision and number comprehension often affect decisions, a short section on those topics is included as well.

Successful students will:

A1 Describe key characteristics of a distribution.

Key characteristics include measures of center and spread.

a. Identify and distinguish between discrete and continuous probability distributions.

b. Calculate and use the mean and standard deviation to describe the characteristics of a distribution.

c. Reason from empirical distributions of data to make assumptions about their underlying theoretical distributions.

A2 Know and use the chief characteristics of the normal distribution.

The normal (or Gaussian) distribution is actually a family of mathematical distributions that are symmetric in shape with scores more concentrated in the middle than in the tails. They are sometimes described as bell-shaped. Normal distributions may have differing centers (means) and scale (standard deviation). The standard normal distribution is the normal distribution with a mean of zero and a standard deviation of one. In normal distributions, approximately 68% of the data lie within one standard deviation of the mean and 95% within two.

a. Identify examples that demonstrate that the mean and standard deviation of a normal distribution can vary independently of each other (e.g., that two normal distributions with the same mean can have different standard deviations).

b. Identify common examples that fit the normal distribution (height, weight) and examples that do not (salaries, housing prices, size of cities) and explain the distinguishing characteristics of each.

A3 Apply probability to make and communicate informed decisions.

a. Apply probability to practical situations.

   Examples: Communicate an understanding of the inverse relation of risk and return; explain the benefits of diversifying risk.

Calculate the expected value of a random variable having a discrete probability distribution and interpret the results.

A4 Interpret and apply numbers used in practical situations.

a. Interpret and compare extreme numbers.

   Examples: Lottery odds, national debt, astronomical distances.

b. Determine a reasonable degree of precision in a given situation.

c. Assess the amount of error resulting from estimation and determine whether the error is within acceptable tolerance limits.

d. Choose appropriate techniques and tools to measure quantities in order to achieve specified degrees of precision, accuracy, and error (or tolerance) of measurements.

   Example: Humans have a reaction time to visual stimuli of approximately 0.1 sec. Thus, it is reasonable to use hand-activated stopwatches that measure tenths of a second.

e. Apply significant figures, orders of magnitude, and scientific notation when making calculations or estimations.

B. Applying Exponents

Building on the understanding of whole number exponents, students in Integrated Mathematics Course 2 will develop an understanding of the impact of a negative exponent and generalize the properties of exponents to all rational exponents. Application of the laws of exponents to numerical and algebraic monomials and their use in operations with binomials forms a foundation for important algebraic skills. Basic factoring and multiplication enable algebraic expressions to be written in various forms that provide insight and clarify information. The binomial theorem is an example of the multiplication of a binomial. Its links to the binomial distribution and to probability studied in the previous unit provide an effective bridge from the study of reasoning with data and distributions of data to the study of algebraic expressions, equations, and functions.

Successful students will:

B1 Interpret negative integer and rational exponents; use them to rewrite numeric expressions in alternative forms.

a. Convert between expressions involving negative exponents and those involving only positive ones; apply the properties as necessary.

Examples: $3^{-2} = \frac{1}{9}$; $\frac{2^{-3}}{7^{-1}} = \frac{7}{2^3} = \frac{7}{8}$.

b. Convert between expressions involving rational exponents and those involving roots and integral powers; apply the properties of exponents as necessary.

Examples: $5^{\frac{3}{2}} = \sqrt{5^3} = 5\sqrt{5}$; $\sqrt[4]{27} = \sqrt[4]{3^3} = 3^{\frac{3}{4}}$; $3^{\frac{2}{3}} = \left(3^2\right)^{\frac{1}{3}} = 9^{\frac{1}{3}}$; $\left(2^3\right)^{\frac{1}{4}} = 2^{\frac{3}{4}}$; $\frac{5^{\frac{4}{3}}}{5^{\frac{1}{3}}} = 5$.

B2 Apply the properties of exponents to transform variable expressions involving integer exponents.

a. Know and apply the laws of exponents for integer exponents.

Examples: $a^m \cdot a^n = a^{m+n}$, for *m, n* real; $\frac{x^5}{x^7} = \frac{1}{x^2} = x^{-2}$; $9^x = 3^{2x}$; $64^{\frac{5}{6}} = 32$.

b. Factor out common factors in expressions involving integer exponents.

Factoring transforms an expression that was written as a sum or difference into one that is written as a product.

Examples: $6v^7 + 12v^5 - 8v^3 = 2v^3\left(3v^4 + 6v^2 - 4\right)$;

$27x^{-2} - 12x^{-4} + 45x = 3x^{-4}\left(9x^2 - 4 + 15x^5\right)$; $3x\left(x+1\right)^2 - 2\left(x+1\right)^2 = \left(x+1\right)^2\left(3x-2\right)$.

Chunking *is a term often used to describe treating an expression, such as the x + 1 above, as a single entity.*

B3 Make regular fluent use of basic algebraic identities such as $(a + b)^2 = a^2 + 2ab + b^2$; $(a - b)^2 = a^2 - 2ab + b^2$; and $(a + b)(a - b) = a^2 - b^2$.

a. Use the distributive law to derive each of these formulas.

Examples: $(a + b)(a - b) = (a + b)a - (a + b)b = (a^2 + ab) - (ab + b^2) = a^2 + ab - ab - b^2 = a^2 - b^2$; applying this to specific numbers, 37 • 43 = (40 – 3)(40 + 3) = 1,600 – 9 = 1,591.

b. Use geometric representations to illustrate these formulas.

Example: Use a partitioned square or tiles to provide a geometric representation of $(a + b)^2 = a^2 + 2ab + b^2$.

B4 Know and use the binomial expansion theorem.

a. Relate the expansion of $(a + b)^n$ to the possible outcomes of a binomial experiment and the n^{th} row of Pascal's triangle.

B5 Convert between forms of numerical expressions involving roots and perform operations on numbers expressed in radical form.

Example: Convert $\sqrt{8}$ to $2\sqrt{2}$ and use the understanding of this conversion to perform similar calculations and to compute with numbers in radical form.

B6 Solve linear and simple nonlinear equations involving several variables for one variable in terms of the others; use fractional exponents and roots as needed to express the solution.

Example: Solve $A = \pi r^2 h$ for h or for r.

C. Quadratic Functions and Equations with Real Zeros/Roots

The study of quadratic functions and equations builds on the work with algebraic identities and forms begun in the last unit. Early work with quadratic functions and equations should focus on those with real zeros/roots.

Successful students will:

C1 Identify quadratic functions expressed in multiple forms; identify the specific information each form clarifies.

a. Express a quadratic function as a polynomial, $f(x) = ax^2 + bx + c$, where a, b and c are constants with $a \neq 0$, and identify its graph as a parabola that opens up when a > 0 and down when $a < 0$; relate c to where the graph of the function crosses the y-axis.

b. Express a quadratic function in factored form, $f(x) = (x - r)(x - s)$, when r and s are integers; relate the factors to the solutions of the equation $(x - r)(x - s) = 0$ ($x = r$ and $x = s$) and to the points ((r, 0) and (s, 0)) where the graph of the function crosses the x-axis.

C2 Transform quadratic functions and relate their symbolic and graphical forms.

a. Write a quadratic function in polynomial or standard form, $f(x) = ax^2 + bx + c$, to identify the y-intercept of the function's parabolic graph or the x-coordinate of its vertex, $x = -\frac{b}{2a}$.

b. Write a quadratic function in factored form, $f(x) = a(x - r)(x - s)$, to identify the zeros of the function.

c. Write a quadratic function in vertex form, $f(x) = a(x - h)^2 + k$, to identify the vertex and axis of symmetry of the function's parabolic graph.

d. Describe the effect that changes in the leading coefficient or constant term of $f(x) = ax^2 + bx + c$ have on the shape, position, and characteristics of the graph of $f(x)$.

Examples: If a and c have opposite signs, then the roots of the quadratic must be real and have opposite signs; varying c varies the y-intercept of the graph of the parabola; if a is positive, the parabola opens up, if a is negative, it opens down; as $|a|$ increases, the graph of the parabola is stretched vertically, i.e., it looks narrower.

e. Determine domain and range, intercepts, axis of symmetry, and maximum or minimum.

C3 Solve and graph quadratic equations having real solutions using a variety of methods.

a. Solve quadratic equations having real solutions by factoring, by completing the square, and by using the quadratic formula.

b. Estimate the real zeros of a quadratic function from its graph; identify quadratic functions that do not have real zeros by the behavior of their graphs.

c. Use a calculator to approximate the roots of a quadratic equation and as an aid in graphing.

D. Quadratic Functions and Equations with Complex Zeros/Roots

This unit begins with the definition of complex numbers. Extension of the real number system to the complex number system permits solution of all quadratic equations. Students should be comfortable using a variety of solutions methods for quadratic equations and in identifying and interpreting their graphs. These techniques should then be applied to solving and graphing quadratic inequalities and transforming quadratic expressions and equations, including those that are not functions, to extract information.

Successful students will:

D1 Know that if *a* and *b* are real numbers, expressions of the form *a* + *bi* are called complex numbers and explain why every real number is a complex number.

Every real number, a, is a complex number because it can be expressed as a + 0i.

The imaginary unit, sometimes represented as $i = \sqrt{-1}$*, is a solution to the equation* $x^2 = -1$.

a. Explain why every real number is a complex number.

Every real number, a, is a complex number because it can be expressed as a + 0i.

b. Express the square root of a negative number in the form *bi,* where *b* is real.

Just as with square roots of positive numbers, there are two square roots for negative numbers; in $\sqrt{-4} = \pm 2i$*, 2i is taken to be the principal square root based on both the Cartesian and trigonometric representations of complex numbers.*

Examples: Determine the principal square root for each of the following: $\sqrt{-7} = i\sqrt{7}$; $\sqrt{-256} = 16i$.

c. Identify complex conjugates.

The conjugate of a complex number a + bi is the number a – bi.

D2 Solve and graph quadratic equations having complex roots and find those roots.

a. Use the quadratic formula to solve any quadratic equation and write it as a product of linear factors.

b. Use the discriminant $D = b^2 - 4ac$ to determine the nature of the roots of the equation $ax^2 + bx + c = 0$.

c. Know that complex solutions of quadratic equations with real coefficients occur in conjugate pairs and show that multiplying factors related to conjugate pairs results in a quadratic equation having real coefficients.

Example: The complex numbers $\left(3+i\sqrt{5}\right)$ and $\left(3-i\sqrt{5}\right)$ are the roots of the equation $\left(x-\left(3+i\sqrt{5}\right)\right)\left(x-\left(3-i\sqrt{5}\right)\right)=x^2-6x+14=0$ whose coefficients are real.

D3 Recognize and solve practical problems that can be expressed using simple quadratic equations; interpret their solutions in terms of the context of the situation.

Examples: Determine the height of an object above the ground t seconds after it has been thrown upward from a platform d feet above the ground at an initial velocity of v_0 feet per second; find the area of a rectangle with perimeter 120 in terms of the length, L, of one side.

a. Create, interpret, and apply mathematical models to solve problems arising from contextual situations that involve quadratic relationships; distinguish relevant from irrelevant information, identify missing information, and find what is needed or make appropriate estimates and apply problem solving heuristics.

b. Select and explain a method of solution (e.g., exact vs. approximate) that is effective and appropriate to a given problem.

D4 Solve and graph quadratic inequalities in one or two variables.

Example: Solve $(x - 5)(x + 1) > 0$ and relate the solution to the graph of $(x - 5)(x + 1) > y$.

D5 Manipulate quadratic equations to extract information.

Example: Use completing the square to determine the center and radius of a circle from its equation given in general form.

E. Power and Polynomial Functions and Expressions

Power and polynomial functions are natural extensions of the work done in this course with quadratic functions. The majority of work in this unit involves recognizing power and polynomial functions, identifying some of their characteristics, and applying them to contextual situations. Manipulation of polynomial and rational expressions completes the unit.

Successful students will:

E1 Analyze power functions and identify their key characteristics.

Power functions include positive integer power functions such as $f(x) = -3x^4$*, root functions such as* $f(x) = f\sqrt{x}$ *and* $f(x) = 4x^{\frac{1}{3}}$*, and reciprocal functions such as* $f(x) = kx^{-4}$

a. Recognize that the inverse proportional function $f(x) = k/x$ ($f(x) = kx^n$ for $n = -1$) and the direct proportional function $f(x) = kx$ ($f(x) = kx^n$ for $n = 1$) are special cases of power functions.

b. Distinguish between *odd* and *even* power functions.

Examples: When the exponent of a power function is a positive integer, then even power functions have either a minimum or maximum value, while odd power functions have neither; even power functions have reflective symmetry over the *y*-axis, while odd power functions demonstrate rotational symmetry about the origin.

E2 Transform the algebraic expression of power functions using properties of exponents and roots.

Example: $f(x) = 3x^2\left(-2x^{-\frac{3}{2}}\right)$ can be more easily identified as a root function once it is rewritten as $f(x) = -6x^{\frac{1}{2}} = -6\sqrt{x}$.

a. Explain and illustrate the effect that a change in a parameter has on a power function (a change in *a* or *n* for $f(x) = ax^n$).

E3 Analyze polynomial functions and identify their key characteristics.

a. Know that polynomial functions of degree *n* have the general form $f(x) = ax^n + bx^{n-1} + \ldots + px^2 + qx + r$ for *n* an integer, $n \geq 0$ and $a \neq 0$.

The degree of the polynomial function is the largest power of its terms for which the coefficient is non-zero.

b. Know that a power function with an exponent that is a positive integer is a particular type of polynomial function, a *monomial* function, whose graph contains the origin.

c. Distinguish among polynomial functions of low degree, i.e., constant functions, linear functions, quadratic functions, or cubic functions.

d. Explain why every polynomial function of odd degree has at least one zero; identify any assumptions that contribute to your argument.

At this level students are expected to recognize that this result requires that polynomials are connected functions without "holes." They are not expected to give a formal proof of this result.

e. Communicate understanding of the concept of the multiplicity of a root of a polynomial equation and its relationship to the graph of the related polynomial function.

If a zero, r_1, of a polynomial function has multiplicity 3, $(x - r_1)^3$ is a factor of the polynomial. The graph of the polynomial touches the horizontal axis at $x = r_1$ but does not change sign (does not cross the axis) if the multiplicity of r_1 is even; it changes sign at $x = r_1$ (crosses over the axis) if the multiplicity is odd.

E4 Use key characteristics to identify the graphs of simple polynomial functions.

Simple polynomial functions include constant functions, linear functions, quadratic functions or cubic functions such as $f(x) = x^3$, $f(x) = x^3 - a$, or $f(x) = x(x - a)(x + b)$.

a. Decide if a given graph or table of values suggests a simple polynomial function.

b. Distinguish between the graphs of simple polynomial functions.

c. Where possible, determine the domain, range, intercepts and end behavior of polynomial functions.

It is not always possible to determine exact horizontal intercepts.

E5 Recognize and solve problems that can be modeled using power or polynomial functions; interpret the solution(s) in terms of the context of the problem.

a. Use power or polynomial functions to represent quantities arising from numeric or geometric contexts such as length, area, and volume.

Examples: The number of diagonals of a polygon as a function of the number of sides; the areas of simple plane figures as functions of their linear dimensions; the surface areas of simple three-dimensional solids as functions of their linear dimensions; the sum of the first n integers as a function of n.

b. Solve simple polynomial equations and use technology to approximate solutions for more complex polynomial equations.

E6 Perform operations on polynomial expressions.

a. Add, subtract, multiply, and factor polynomials.

b. Divide one polynomial by a lower-degree polynomial.

E7 Use factoring to reduce rational expressions that consist of the quotient of two simple polynomials.

E8 Perform operations on simple rational expressions.

Simple rational expressions are those whose denominators are linear or quadratic polynomial expressions.

a. Add, subtract, multiply, and divide rational expressions having monomial or binomial denominators.

b. Rewrite complex fractions composed of simple rational expressions as a simple fraction in lowest terms.

Example: $\dfrac{(a+b)}{\left(\dfrac{1}{a}+\dfrac{1}{b}\right)} = \dfrac{(a+b)}{\dfrac{(b+a)}{ab}} = (a+b)\bullet\dfrac{ab}{(b+a)} = ab$.

Integrated Mathematics Course 3

Year Three of a Model Three-Year Integrated High School Course Sequence

This final course in the model integrated mathematics sequence opens with a more sophisticated and formal look at reasoning and proof in mathematics. Such reasoning is applied in numeric, algebraic, and everyday situations as well as through geometric propositions, such as the theorems in this course relating circles and their associated lines and angles. An optional unit on spherical geometry connects well to the study of circles, if time permits. Following an introduction to iteration and recursion, iterative thinking can be applied through a second optional unit on sequences and series. In this final course of the model integrated sequence, piecewise-defined and exponential functions round out the toolkit of function families now available to students who have experienced the earlier integrated courses. With all basic function types introduced, the characteristics of different prototypical functions can be explored, compared, and applied. Students will perform transformations on the various functions, noting that the effect of specific transformations generalize across function types. The habits and tools of analysis and logical reasoning developed throughout the three integrated courses are applied in the closing unit on mathematical modeling. This topic provides an excellent opportunity for students to engage in extended projects involving research and analysis of bivariate data. If time permits, an optional unit on transforming data connects back to the function transformations unit and offers students an excellent opportunity to engage in the type of work actually done by statisticians and researchers.

Appropriate use of technology is expected in all work. In Integrated Mathematics Course 3, this includes employing technological tools to assist students in the formation and testing of conjectures, creating graphs and data displays, and determining and assessing lines of fit for data. Geometric constructions should be performed using geometric software as well as classical tools, and technology should be used to aid three-dimensional visualization. Testing with and without technological tools is recommended.

How a particular subject is taught influences not only the depth and retention of the content of a course but also the development of skills in inquiry, problem solving, and critical thinking. Every opportunity should be taken to link the concepts in Integrated Mathematics Course 3 to those encountered in previous courses as well as to other disciplines. Students should be encouraged to be creative and innovative in their approach to problems, to be productive and persistent in seeking solutions, and to use multiple means to communicate their insights and understanding.

The Major Concepts below provide the focus for Integrated Mathematics Course 3. They should be taught using a variety of methods and applications so that students attain a deep understanding of these concepts. Maintenance Concepts should have been taught previously and are important foundational concepts that will be applied in this course. Continued facility with and understanding of the Maintenance Concepts is essential for success in the Major Concepts defined for this course.

MAJOR CONCEPTS	MAINTENANCE CONCEPTS
➢ Reasoning and Proof	➢ Fundamental Logic
➢ Geometric Reasoning and Proof	➢ Basic Geometric Proof
➢ Iteration and Its Applications	➢ Power and Polynomial Functions
➢ Piecewise-Linear and Exponential Functions	➢ Expressions and Equations
➢ Characteristics and Transformations of Function and Equation Families	➢ Coordinate Transformations of Linear and Proportional Functions
➢ Mathematical Modeling with Data	➢ Reasoning from Data

A. Reasoning and Proof

Extending the fundamentals of mathematical reasoning introduced in Integrated Mathematics Course 1, students will formalize their understanding of mathematical logic and proof. In this course, reasoning is applied to numeric as well as geometric properties.

Successful students will:

A1 Use geometric examples to illustrate the relationships among undefined terms, axioms/postulates, definitions, theorems, and various methods of reasoning.

a. Analyze and illustrate the effect of changing a definition or an assumption.

b. Analyze the consequences of using alternative definitions; apply this especially to definitions of geometric objects.

c. Demonstrate the effect that changing an assumption has on the validity of a conclusion.

A2 Present and analyze direct and indirect proofs using paragraphs or two-column or flow-chart formats.

A3 Establish simple facts about rational and irrational numbers using logical arguments and examples.

Examples: Explain why, if *r* and *s* are rational, then both *r* + *s* and *rs* are rational (for example, both ¾ and 2.3 are rational in $\frac{3}{4}+2.3=\frac{3}{4}+\frac{23}{10}=\frac{15}{20}+\frac{46}{20}=\frac{61}{20}$, which is the ratio of two integers, hence rational); give examples to show that, if *r* and *s* are irrational, then *r* + *s* and *rs* could be either rational or irrational (for example, $\sqrt{3}+\frac{\sqrt{3}}{2}$ is irrational whereas $\left(5+\sqrt{2}\right)-\sqrt{2}$ is rational); show that a given interval on the real number line, no matter how small, contains both rational and irrational numbers.

A4 Given a degree of precision, determine a rational approximation for an irrational number.

Example: Use arithmetic methods to determine that $2.23 < \sqrt{5} < 2.24$ when two-decimal precision is desired.

B. Geometric Reasoning and Proof

Work with circles provides opportunities to prove and apply important and more complex geometric theorems than those encountered in previous courses in the integrated course sequence. Geometric reasoning is extended to three dimensions, assisting students in developing better spatial sense and analysis skills. An optional section applying possible changes to the parallel postulate of Euclidean geometry may be used to introduce students to a very practical example of non-Euclidean space.

Successful students will:

B1 Know and apply the definitions and properties of a circle and the radius, diameter, chord, tangent, secant, and circumference of a circle.

B2 Recognize, verify and apply statements about the properties of a circle.

a. Recognize and apply the fact that a tangent to a circle is perpendicular to the radius at the point of tangency.

b. Recognize, verify, and apply the relationships between central angles, inscribed angles, and circumscribed angles and the arcs they define.

   Example: Show that a triangle inscribed on the diameter of a circle is a right triangle.

c. Recognize, verify, and apply the relationships between inscribed and circumscribed angles of a circle and the arcs and segments they define.

   Example: Prove that if a radius of a circle is perpendicular to a chord of the circle, then it bisects the chord.

B3 Determine the length of line segments and arcs, the magnitude of angles, and the area of shapes that they define in complex geometric drawings.

Examples: Determine the amount of glass in a semi-circular transom; identify the coverage of an overlapping circular pattern of irrigation; determine the distance for line of sight on the earth's surface.

B4 Interpret and use locus definitions to generate two- and three-dimensional geometric objects.

Examples: The locus of points in the plane equidistant from two fixed points is the perpendicular bisector of the line segment joining them; the parabola defined as the locus of points equidistant from the point (5, 1) and the line y = −5 is $y = \frac{1}{12}\left(x-5\right)^2 - 2$; the locus of points in space equidistant from a fixed point is a sphere.

B5 Analyze cross-sections of basic three-dimensional objects and identify the resulting shapes.

a. Describe all possible results of the intersection of a plane with a cube, prism, pyramid, or sphere,

B6 Describe the characteristics of the three-dimensional object traced out when a one- or two-dimensional figure is rotated about an axis.

B7 Analyze all possible relationships among two or three planes in space and identify their intersections.

a. Identify a physical situation that illustrates two distinct parallel planes; identify a physical situation that illustrates two planes that intersect in a line.

b. Demonstrate that three distinct planes may be parallel; two of them may be parallel to each other and intersect with the third, resulting in two parallel lines; or none may be parallel, in which case the three planes intersect in a single point, a single line, or by pairs in three parallel lines.

B8 Recognize that there are geometries other than Euclidean geometry, in which the parallel postulate is not true.

B9 Analyze and interpret geometry on a sphere.

[OPTIONAL ENRICHMENT UNIT]

a. Identify the parallel postulate as key in Euclidean geometry and analyze the effect of changes to that postulate

b. Know and apply the definition of a great circle.

A great circle of a sphere is the circle formed by the intersection of the sphere with the plane defined by any two distinct, non-diametrically opposite points on the sphere and the center of the sphere.

Example: Show that arcs of great circles subtending angles of 180 degrees or less provide shortest routes between points on the surface of a sphere.

Since the earth is nearly spherical, this method is used to determine distance between distant points on the earth.

c. Use latitude, longitude, and great circles to solve problems relating to position, distance, and displacement on the earth's surface.

Displacement is the change in position of an object and takes into account both the distance and direction it has moved.

Example: Given the latitudes and longitudes of two points on the surface of the Earth, find the distance between them along a great circle and the bearing from one point to the other.

Bearing is the direction or angle from one point to the other relative to North = 0°. A bearing of N31°E means that the second point is 31° East of a line pointing due North of the first point.

d. Interpret various two-dimensional representations for the surface of a sphere (e.g., two-dimensional maps of the Earth), called projections, and explain their characteristics.

Common projections are Mercator (and other cylindrical projections), Orthographic and Stereographic (and other Azimuthal projections), pseudo-cylindrical, and sinusoidal. Each projection has advantages for certain purposes and has its own limitations and drawbacks.

e. Describe geometry on a sphere as an example of a non-Euclidean geometry in which any two lines intersect.

In spherical geometry, great circles are the counterpart of lines in Euclidean geometry. All great circles intersect. An angle between two great circles is either of the two angles formed by the intersecting planes defined by the great circles.

Examples: Show that, on a sphere, all lines intersect—that is, the parallel postulate does not hold true in this context; identify and interpret the intersection of lines of latitude with lines of longitude on a globe; recognize that the sum of the degree measures of the interior angles of a triangle on a sphere is greater than 180°.

C. Iteration and Its Applications

Recursive thinking is an important mathematical idea that naturally connects to the study of sequences and series. Sequences and series is included here as an optional topic and may be omitted if time or other constraints make its inclusion difficult.

Successful students will:

C1 Analyze, interpret, and describe relationships represented iteratively and recursively including those produced using a spreadsheet.

Examples: Recognize that the sequence defined by "First term = 5. Each term after the first is six more than the preceding term" is the sequence whose first seven terms are 5, 11, 17, 23, 29, 35, and 41; recognize that the result of repeatedly squaring a number between −1 and 1 appears to approach zero while the result of repeatedly squaring a number less than −1 or greater than 1 appears to continue to increase; determine empirically how many steps are needed to produce four-digit accuracy in square roots by iterating the operations *divide* and *average*.

C2 Generate and describe sequences having specific characteristics; use calculators and spreadsheets effectively to extend sequences beyond a relatively small number of terms.

a. Generate and describe the factorial function or the Fibonacci sequence recursively.

b. Generate and describe *arithmetic* sequences recursively; identify arithmetic sequences expressed recursively.

Arithmetic sequences are those in which each term differs from its preceding term by a constant difference. To describe an arithmetic sequence, both the starting term and the constant difference must be specified.

Example: $a_1 = 5, \quad a_{n+1} = a_n + 2$ describes the arithmetic sequence 5, 7, 9, 11, . . .

c. Generate and describe *geometric* sequences recursively; identify geometric sequences expressed recursively.

Geometric sequences are those in which each term is a constant multiple of the term that precedes it. To describe a geometric sequence both the starting term and the constant multiplier (often called the ***common ratio****) must be specified.*

Example: $a_1 = 3, \quad a_{n+1} = -2a_n$ describes the geometric sequence 3, -6, 12, -24, . . .

C3 Represent, derive, and apply sequences and series. [OPTIONAL ENRICHMENT UNIT]

a. Know and use subscript notation to represent the general term of a sequence and summation notation to represent partial sums of a sequence.

b. Derive and apply the formulas for the general term of arithmetic and geometric sequences.

c. Derive and apply formulas to calculate sums of finite arithmetic and geometric series.

d. Derive and apply formulas to calculate sums of infinite geometric series whose common ratio r is in the interval $(-1, 1)$.

e. Model, analyze, and solve problems using sequences and series.

Examples: Determine the amount of interest paid over five years of a loan; determine the age of a skeleton using carbon dating; determine the cumulative relative frequency in an arithmetic or geometric growth situation.

D. Piecewise-Linear and Exponential Functions

Linear, proportional, reciprocal, quadratic, power, and polynomial functions have been studied in previous courses. This course rounds out the function toolkit with the introduction of piecewise-linear and exponential functions and their applications.

Successful students will:

D1 Identify key characteristics of absolute value, step, and other piecewise-linear functions and graph them.

a. Interpret the algebraic representation of a piecewise-linear function; graph it over the appropriate domain.

b. Write an algebraic representation for a given piecewise-linear function.

c. Determine vertex, slope of each branch, intercepts, and end behavior of an absolute value graph.

d. Recognize and solve problems that can be modeled using absolute value, step, and other piecewise-linear functions.

Examples: Postage rates, cellular telephone charges, tax rates.

D2 Graph and analyze exponential functions and identify their key characteristics.

a. Describe key characteristics of the graphs of exponential functions and relate these to the coefficients in the general form $f(x) = ab^x + c$ for $b > 0$, $b \neq 1$.

Examples: Know that, if *b* > 1, exponential functions are increasing and that they approach a lower limit if *a* > 0 and an upper limit if *a* < 0 as *x* decreases; know that, if 0 < *b* < 1, exponential functions are decreasing and that they approach a lower limit if *a* > 0 and an upper limit if *a* < 0 as x increases.

b. Explain and illustrate the effect that a change in a parameter has on an exponential function (a change in *a*, *b*, or *c* for $f(x) = ab^x + c$).

D3 Demonstrate the effect of compound interest, decay, or growth using iteration.

Examples: Using a spreadsheet, enter the amount of a loan, the monthly interest rate and the monthly payment in a spreadsheet. The formula (loan amount) • (1+interest rate) – (monthly payment) gives the amount remaining monthly on the loan at the end of the first month and the iterative "fill down" command will show the amount remaining on the loan at the end of each successive month; a similar process using past data about the yearly percent increase of college tuition and annual inflation rate will provide an estimate of the cost of college for a newborn in current dollar equivalents.

a. Identify the diminishing effect of increasing the number of times per year that interest is compounded and relate this to the notion of instantaneous compounding.

D4 Determine the composition of simple functions, including any necessary restrictions on the domain. [OPTIONAL ENRICHMENT UNIT]

a. Know the relationship among the identity function, composition of functions, and the inverse of a function, along with implications for the domain.

D5 Determine and identify key characteristics of inverse functions. [OPTIONAL ENRICHMENT UNIT]

a. Analyze characteristics of inverse functions.

b. Identify the conditions under which the inverse of a function is a function.

c. Determine whether two given functions are inverses of each other.

d. Explain why the graph of a function and its inverse are reflections of one another over the line $y = x$.

e. Determine the inverse of linear and simple non-linear functions, including any necessary restrictions on the domain.

f. Determine the inverse of a simple polynomial or simple rational function.

D6 Identify characteristics of logarithmic functions; apply logarithmic functions. [OPTIONAL ENRICHMENT UNIT]

a. Identify a logarithmic function as the inverse of an exponential function.

If $x^y = z$, $x > 0$, $x \neq 1$, y an integer and $z > 0$, then y is the logarithm to the base x of z. The logarithm $y = \log_x z$ is one of three equivalent forms of expressing the relation $x^y = z$ (the other being $x = \sqrt[y]{z}$).

Examples: If If $5^a = b$, then $\log_5 b = a$.

b. Know and use the definition of logarithm of a number and its relation to exponents.

Examples: $\log_2 32 = \log_2 2^5 = 5$; if $x = \log_{10} 3$, then $10^x = 3$.

c. Prove basic properties of logarithms using properties of exponents (or the inverse exponential function).

d. Use properties of logarithms to manipulate logarithmic expressions in order to extract information.

e. Use logarithms to express and solve equations and problems.

Example: Explain why the number of digits in the binary representation of a decimal number *N* is approximately the logarithm to base 2 of *N*.

f. Solve logarithmic equations; use logarithms to solve exponential equations.

Examples: $\log(x - 3) + \log(x - 1) = 0.1$; $5^x = 8$.

E. Characteristics and Transformations of Function and Equation Families

Students are expected to refresh their knowledge of all function relationships and deepen their understanding by distinguishing among them and identifying the result when simple coordinate transformations are applied. Building on prior experience with linear, simple polynomial, power, and exponential equations, students will solve rational and radical equations.

Successful students will:

E1 Distinguish among the graphs of linear, exponential, power, polynomial, or rational functions by their key characteristics.

Be aware that it can be very difficult to distinguish graphs of these various types of functions over small regions or particular subsets of their domains. Sometimes the context of an underlying situation can suggest a likely type of function model.

a. Decide whether a given exponential or power function is suggested by the graph, table of values, or underlying context of a problem.

b. Distinguish between the graphs of exponential growth functions and those representing exponential decay.

c. Distinguish among the graphs of power functions having positive integral exponents, negative integral exponents, and exponents that are positive unit fractions ($f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$, $n > 0$, n an integer *).*

*Power functions having exponents that are positive unit fractions are called **root** or **radical** functions.*

d. Identify and explain the symmetry of an even or odd power function.

e. Where possible, determine the domain, range, intercepts, asymptotes, and end behavior of linear, exponential, power, polynomial, or rational functions.

Range is not always possible to determine with precision.

E2 Distinguish among linear, exponential, polynomial, rational, and power expressions; equations; and functions by their symbolic form.

a. Identify linear, exponential, polynomial, rational, or power expressions, equations, or functions by their general form and the position of the variable.

Examples: $f(x) = 3^x$ is an exponential function because the variable is in the exponent while $f(x) = x^3$ has the variable in the position of a base and is a power function; $f(x) = x^3 - 5$ is a polynomial function but not a power function because of the added constant.

b. Distinguish among power expressions, equations, and functions by the type of exponent.

Examples: $f(x) = 3x^5$ is a polynomial function because the exponent is a positive integer, $f(x) = 3x^{-5}$ is a rational or reciprocal function because the exponent is a negative integer, and $f(x) = 3x^{1/5}$ is a radical function because the exponent is a unit fraction.

E3 Solve simple rational and radical equations in one variable.

a. Use algebraic, numerical, graphical, and/or technological means to solve radical and rational equations.

b. Know which operations on an equation produce an equation with the same solutions and which may produce an equation with fewer or more solutions (lost or extraneous roots) and adjust solution methods accordingly.

E4 Recognize and solve problems that can be modeled using exponential or power functions; interpret the solution(s) in terms of the context of the problem.

a. Use exponential functions to represent *growth* functions, such as $f(x) = an^x$ ($a > 0$ and $n > 1$), and *decay* functions, such as $f(x) = an^{-x}$ ($a > 0$ and $n > 1$).

Exponential functions model situations where change is proportional to quantity (e.g., compound interest, population grown, radioactive decay).

b. Use power functions to represent quantities arising from geometric contexts such as length, area, and volume.

Examples: The relationships between the radius and area of a circle, between the radius and volume of a sphere, and between the volumes of simple three-dimensional solids and their linear dimensions.

c. Use the laws of exponents to determine exact solutions for problems involving exponential or power functions where possible; otherwise approximate the solutions graphically or numerically.

E5 Explain, illustrate, and identify the effect of simple coordinate transformations on the graph of a function.

a. Interpret the graph of $y = f(x - a)$ as the graph of $y = f(x)$ shifted $|a|$ units to the right ($a > 0$) or the left ($a < 0$).

b. Interpret the graph of $y = f(x) + a$ as the graph of $y = f(x)$ shifted $|a|$ units up ($a > 0$) or down ($a < 0$).

c. Interpret the graph of $y = f(ax)$ as the graph of $y = f(x)$ expanded horizontally by a factor of $\frac{1}{|a|}$ if $0 < |a| < 1$ or compressed horizontally by a factor $|a|$ if $|a| > 1$ and reflected over the y-axis if $a < 0$.

d. Interpret the graph of $y = af(x)$ as the graph of $y = f(x)$ compressed vertically by a factor of $\frac{1}{|a|}$ if $0 < |a| < 1$ or expanded vertically by a factor of $|a|$ if $|a| > 1$ and reflected over the x-axis if $a < 0$.

e. Relate the algebraic properties of a function to the geometric properties of its graph.

Examples: The graph of $f(x) = \frac{x-2}{x^2-1}$ has vertical asymptotes at $x = 1$ and $x = -1$ while the graph of $f(x) = \frac{x-2}{x^2-4}$ has a vertical asymptote at $x = -2$ but a hole at (2, ¼); the graph of $f(x) = \sqrt{x+5} - 2$ is the same as the graph of $f(x) = \sqrt{x}$ translated five units to the left and 2 units down.

F. Mathematical Modeling with Data

Now that students have amassed experience with various function prototypes and with the effect of transformations on them, they would benefit from engaging in a project collecting and analyzing data. They will need to understand the differences among the major types of statistical studies. For the purposes of applying what they have learned about functions, a project that generates bivariate data would be most effective. As time permits, an optional section on transformation of data may be included to provide students with an introduction to how statisticians generally develop models for real data.

Successful students will:

F1 Describe the nature and purpose of sample surveys, experiments, and observational studies, relating each to the types of research questions they are best suited to address.

a. Identify specific research questions that can be addressed by different techniques for collecting data.

b. Critique various methods of data collection used in real-world problems, such as a clinical trial in medicine, an opinion poll, or a report on the effect of smoking on health.

c. Explain why observational studies generally do not lead to good estimates of population characteristics or cause-and-effect conclusions regarding treatments.

F2 Plan and conduct sample surveys, observational studies, or experiments.

a. Recognize and explain the rationale for using randomness in research designs; distinguish between random sampling from a population in sample surveys and random assignment of treatments to experimental units in an experiment.

Random sampling is how items are selected from a population so that the sample data can be used to estimate characteristics of the population; random assignment is how treatments are assigned to experimental units so that comparisons among the treatment groups can allow cause-and-effect conclusions to be made.

b. Use simulations to analyze and interpret key concepts of statistical inference.

Key concepts of statistical inference include margin of error and how it relates to the design of a study and to sample size; confidence interval and how it relates to the margin of error; and p-value and how it relates to the interpretation of results from a randomized experiment.

F3 Determine, interpret, and compare linear models for data that exhibit a linear trend.

a. Identify and evaluate methods of determining the goodness of fit of a linear model.

Examples: A linear model might pass through the most points, minimize the sum of the absolute deviations, or minimize the sum of the square of the deviations.

b. Use a computer or a graphing calculator to determine a linear regression equation (least-squares line) as a model for data that suggest a linear trend.

c. Use and interpret a residual plot or correlation coefficient to evaluate the goodness of fit of a regression line.

d. Note the effect of outliers on the position and slope of the regression line; interpret the slope and *y*-intercept of the regression line in the context of the relationship being modeled.

F4 Apply transformations to data that exhibit curvature to analyze the underlying pattern of growth and its characteristics. [OPTIONAL ENRICHMENT UNIT]

a Apply transformations of data for the purpose of “linearizing” a scatter plot that exhibits curvature.

Examples: Apply squaring, square root, reciprocal, and logarithmic functions to input data, output data, or both; evaluate which transformation produces the strongest linear trend.

b. Interpret the results of specific transformations in terms of what they indicate about the trend of the original data.

c. Estimate the rate of exponential growth or decay by fitting a regression model to appropriate data transformed by logarithms.

e. Estimate the exponent in a power model by fitting a regression model to appropriate data transformed by logarithms.

f. Analyze how linear transformations of data affect measures of center and spread, the slope of a regression line, and the correlation coefficient.

g. Use transformation techniques to select, interpret, and apply mathematical functions to summarize and model data; include models involving the functions and relationships found in all three model integrated courses.