Integration By U- Substitution

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Definition

- Integrals which are computed by change of variables is called U-substitution.

- In this we have to change the basic variable of an integrand (like ‘x’) to another variable (like ‘u’).

- This make the integral easy to determine.

Why U-Substitution

- It is one of the simplest integration technique.
- It can be used to make integration easier.
- It is used when an integral contains some function and its derivative, when

$$I = \int f(x) f^{1}(x)$$

Let u= f(x)

du=f'(x) dx

Why U-Substitution

Hence the integration becomes

$$I = \int u\,du$$

$$= \frac{u^2}{2} + c$$

$$= \frac{f^2(x)}{2} + c$$

When to use U-Substitution

- We have **function** and its **derivative** together.
- Where by use of simpler methods like **POWER RULE , CONSTANT MULTIPLE RULE** etc its difficult to solve integration.

When not to use U-Substitution

- If you fail to see such a pair of quantities, abandon this method.

Example 1

- Lets Compute the following integral

$$I = \int e^{x^2} (2x) dx$$

Put $u = x^2$

$du = 2x dx$

Example 1(continued)

- the *indefinite* integral becomes

$$I = \int e^u du$$

$$= e^u + c$$

$$= e^{x^2} + c$$

Example 2

- Consider the Following Example

$$I = \int \frac{1}{x \ln x} dx$$

Put lnx=u

1/x dx=du

Hence the given Integral becomes

Example 2(Continued)

- Hence the given Integral becomes

$$I = \int \frac{1}{u} du$$

$$= \ln u$$

$$= \ln(\ln x) + c$$

Example 3

- Consider the following Integral

$$I = \int \cos(2x)\sin(2x)dx$$

Put $\sin(2x)=u$

$2\cos(2x)\,dx=du$

$\cos(2x)\,dx=du/2$

Example 3(Continued)

- Hence the given Integral becomes

$$I = \int \frac{1}{2} u du$$

$$= \frac{1}{2}\left(\frac{u^2}{2}\right) + c$$

$$= \frac{u^2}{4} + c$$

$$= \frac{\sin^2 2x}{4} + c$$

Definite Integral Using U-Substitution

- When evaluating a definite integral using u-substitution, one has to deal with the **limits of integration .**
- So by substitution, **the limits of integration also change**, giving us new Integral in new Variable as well as new limits in the same variable.
- The following example shows this.

Eample4 (Definite Integral)

- Consider the following Integral

$$I = \int_0^1 x^2 (x^3 + 1)^4 dx$$

$$Put \rightarrow x^3 + 1 = u$$

$$3x^2 dx = du$$

$$x^2 dx = \frac{1}{3} du$$

Example 4(Continued)

- Hence the definite integral becomes

  When x=0, u=0

  When x=1,u=1

Hence I=

$$\frac{1}{3}\int_{0}^{1} u^4 du$$

$$= \frac{1}{3}[\frac{u^5}{5}]_0^1$$

Example4(continued)

$$I = \frac{1}{15} u^5]_0^1$$

$$= \frac{1}{15} [1^5 - 0^5]$$

$$= \frac{1}{15}$$

Practice Problems

$$1)\, I = \int \cos^3(x)\sin(x)\,dx$$

$$2)\, I = \int_0^2 -2x\sqrt{4-x^2}\,dx$$

$$3)\, I = \int 9(x^2+3x+5)^8(2x+3)\,dx$$

$$4)\, I = \int x^3 e^{x^4}\,dx$$

$$5)\, I = \int x(x+1)^8\,dx$$

Answer To Practice Problems

$$1) I = -\frac{cox^4 x}{4} + c$$

$$2) I = -\frac{16}{3}$$

$$3) I = (x^2 + 3x + 5)^9 + c$$

$$4) I = \frac{e^{x^4}}{4} + c$$

$$5) I = \frac{1}{10}(x+1)^{10} - \frac{1}{9}(x+1)^9 + c$$

Important Tips for Practice Problem

- If you see a function and its derivative put **function=u** e.g. in question 1 put sinx=u and then solve .
- Same is the case with question 2 and 3.
- For question 2 Put **$4-x^2$=u** and then solve.
- For question 3 Put **x^2+3x+5=u** and then solve.
- For question 4 Put **x^4=u** and then solve.
- For question 5 power rule fails because there is additional x.

Important Tips for Practice Problem

- So we can **reduce** the integral in such a way so that **power rule** works by using substitution.
- So in question 5 put **x+1=u** and then solve the integral in variable u by using simple power rule.