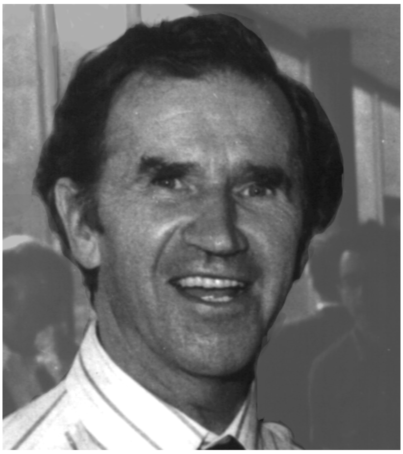
# Intelligence, Learning and Understanding in Mathematics
## A Tribute to Richard Skemp

Edited by
David Tall and Michael Thomas

# RICHARD SKEMP
# March 10, 1919 – June 22, 1995

### A Tribute by David Tall and Michael Thomas



Richard pictured at Karlsruhe, 1976, at the International Congress of Mathematics Education.

Richard Skemp was a unique figure in mathematics education – a significant inspiration to a vast number of teachers and educators who have gained insight through reading his works and a moving spirit in the foundation of the International Group for the Psychology of Mathematics Education.

His background peculiarly fitted him to the role of a father figure in the psychology of mathematics education, for he was qualified as a psychologist, a mathematician and an educator, and one should also add as a practising teacher, an empirical researcher, and a theorist.

He was born in Bristol on March 10th, 1919, the son of Professor A. R. Skemp of the University of Bristol. He was educated as a Foundation scholar at Wellington College, Berkshire (1932–7), taking up an Open Scholarship in Mathematics at Hertford College, Oxford in 1937. The war intervened and he joined the Royal Signals, serving in India and attaining the rank of Captain before returning at the end of the war, sitting nervously on the bomb doors of a Lancaster bomber.

He then completed his degree at Hertford College (1945–1947) and became a mathematics teacher for two years at Oundle School, followed by two years at Rye St Antony, Oxford. His increasing interest in how children learn caused him to return to Hertford College once again in 1952 at the age of 33 to study for a second bachelor's degree in psychology. He completed his PhD in psychology at Manchester University in 1959 where he was first a Lecturer in Psychology (1955–1962) and subsequently Senior Lecturer (1962–1973), directing the Child Study Unit. In 1973, at the age of 54, he became Professor of Educational Theory at the University of Warwick, where he remained until his retirement in 1986.

Richard prided himself on the quality of his output, polishing his work for some time before releasing it for publication. His mould-breaking paper "Instrumental understanding and Relational understanding" was presented in talks for several years before it reached its final form; it is no wonder it continues to be a seminal paper so many years after it was written.

Richard had a special way of dealing with students' work. For instance, he would read a piece of writing with tape-recorder in hand, making comments that he passed on to the student to study at leisure. It was so valuable to have his on-going commentary and the possibility of re-hearing subtle comments several times over.

He took delight in communicating with people of all ages, especially young children, whom he treated with respect as if they were his colleagues. His desire to communicate is evident in all his writings, both practical and theoretical. He aimed for elegant, simple expression of profound ideas, declaring that, "there is nothing as practical as a good theory." He exemplified this duality of purpose by producing both theories of learning, including *Intelligence, Learning and Action* (Wiley, 1979) and corresponding practical curriculum materials such as *Understanding Mathematics* at secondary level and *Mathematics in the Primary School*.

He had a special gift for expressing the essence of ideas in simple language. For instance, he said "it is easy to make simple things difficult but difficult to make hard things easy." He gently criticised curriculum reformers who introduced the "new mathematics" as a logical development by saying that this "teaches the product of mathematical thought, not the process of mathematical thinking."

Many of his ideas have passed into the folk-lore of the subject, especially his use of the distinction between "instrumental" and "relational understanding". But there are also other things that many find especially valuable, for instance, his simple descriptions of the notions of "concept" and "schema", his use of the terms "expansion" and "reconstruction" of schemas instead of "assimilation" and "accommodation", his "three modes of building and testing mathematical concepts", his ideas on "reflective thinking" and his links between the cognitive and affective sides of mathematics in his theory of goals and anti-goals.

What is less well-known is that for 21 years he spent five weeks every summer running camps for up to 45 boys a week, taking delight not only in teaching camping, cooking and sailing, but also leading the camp-fire singing in his mellifluous, well-rounded voice. Sean Neill – who was one of those boys, and later became a lecturer in education with Richard at Warwick – wrote:

> At that time Milford Haven was little developed, and the cottage at Burton looked out over a deserted estuary. The steep hillside was covered with bracken, with some shelter from patches of young trees. Five or six bell-tents accommodated the boys; cooking was by primus stoves of uncertain age and temper, and several tents regularly fried their tent-pole for breakfast. As four of the tents were pitched in a line down the slope, when the weather broke, gravity asserted itself and the bottom tent filled up with peaty slurry and boys in wet sleeping bags. (In later years the tents were set on level standings and cooking was done outside, which considerably reduced the sporting element.)

Richard will be remembered with affection by those who knew him, not only for the rich legacy of ideas he has left in the psychology of mathematics education but also for his unfailing old-fashioned English courtesy and charm, and for his clarity in presentation which made him a great favourite on international speaking tours.

He continued to lecture around the world until he was taken ill at the end of 1994 and diagnosed in January 1995 with non-Hodgkin's lymphoma. He passed peacefully away at noon on Thursday June 22nd with his wife Valerie at his side. He was laid to rest on June 27th. It was a beautiful service, with a touching eulogy by a minister who knew him well, ending with a children's hymn, reflecting Richard's care and joy in communicating with young children. After the service his relatives and a few friends had tea and cakes in the warm sunshine at Pickwick's Cake Shop, where he had delighted in going regularly for cream cakes, a particular favourite since he was a child.

We are conscious of the personal effect he had, both those who were privileged to know him, and the many more who know him through his publications and seminal ideas. As representatives of successive generations of his research students—David Tall (as his last PhD student) and Michael Thomas (as a student of David Tall), we are honoured to present this tribute to his memory.

Richard Skemp was a great pioneer theorist in the Psychology of Learning Mathematics. With his passing a chapter closes, but his legacy lives on.

David Tall M O J Thomas

# Relational Understanding and Instrumental Understanding

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## Faux Amis

*Faux amis* is a term used by the French to describe words which are the same, or very alike, in two languages, but whose meanings are different. For example:

| *French word* | *Meaning in English* |
|---|---|
| histoire | story, not history |
| libraire | bookshop, not library |
| chef | head of any organisation, not only chief cook |
| agrément | pleasure or amusement, not agreement |
| docteur | doctor (higher degree) not medical practitioner |
| médecin | medical practitioner, not medicine |
| parent | relations in general, including parents |

One gets *faux amis* between English as spoken in different parts of the world. An Englishman asking in America for a biscuit would be given what we call a scone. To get what we call a biscuit, he would have to ask for a cookie. And between English as used in mathematics and in everyday life there are such words as field, group, ring, ideal.

A person who is unaware that the word he is using is a *faux ami* can make inconvenient mistakes. We expect history to be true, but not a story. We take books without paying from a library, but not from a bookshop; and so on. But in the foregoing examples there are cues which might put one on guard: difference of language, or of country, or of context.

If, however, the same word is used in the same language, country and context, with two meanings whose difference is non-trivial but as basic as the difference between the meaning of (say) 'histoire' and 'story', which is a difference between fact and fiction, one may expect serious confusion. Two such words can be identified in the context of mathematics; and it is the alternative meanings attached to these words, each by a large following, which in my belief are at the root of many of the difficulties in mathematics education today.

One of these is ‘understanding’. It was brought to my attention some years ago by Stieg Mellin-Olsen, of Bergen University, that there are in current use two meanings of this word. These he distinguishes by calling them ‘relational understanding’ and ‘instrumental understanding’. By the former is meant what I have always meant by understanding, and probably most readers of this article: knowing both what to do and why. Instrumental understanding I would until recently not have regarded as understanding at all. It is what I have in the past described as ‘rules without reasons’, without realising that for many pupils *and their teachers* the possession of such a rule, and ability to use it, was why they meant by ‘understanding’.

Suppose that a teacher reminds a class that the area of a rectangle is given by $A = L \times B$. A pupil who has been away says he does not understand, so the teacher gives him an explanation along these lines. “The formula tells you that to get the area of a rectangle, you multiply the length by the breadth.” “Oh, I see,” says the child, and gets on with the exercise. If we were now to say to him (in effect) “You may think you understand, but you don’t really,” he would not agree. “Of course I do. Look; I’ve got all these answers right.” Nor would he be pleased at our devaluing of his achievement. And with his meaning of the word, he does understand.

We can all think of examples of this kind: ‘borrowing’ in subtraction, ‘turn it upside down and multiply’ for division by a fraction, ‘take it over to the other side and change the sign’, are obvious ones; but once the concept has been formed, other examples of instrumental explanations can be identified in abundance in many widely used texts. Here are two from a text used by a former direct-grant grammar school, now independent, with a high academic standard.

**Multiplication of fractions** To multiply a fraction by a fraction, multiply the two numerators together to make the numerator of the product, and the two denominators to makes its denominator.

E.g. $\frac{2}{3}$ of $\frac{4}{5} = \frac{2\times4}{3\times5} = \frac{8}{15}$

$$\frac{3}{5} \times \frac{10}{13} = \frac{30}{65} = \frac{6}{13}$$

The multiplication sign $\times$ is generally used instead of the word ‘of’.

**Circles** The circumference of a circle (that is its perimeter, or the length of its boundary) is found by measurement to be a little more than three times the length of its diameter. In any circle the circumference is approximately 3.1416 times the diameter, which is roughly $3\frac{1}{7}$ times the diameter. Neither of these figures is exact, as the exact number cannot be expressed either as a fraction or a decimal. The number is represented by the Greek letter $\pi$.

$$\begin{aligned} \text{Circumference} &= \pi d \text{ or } 2\pi r, \\ \text{Area} &= \pi r^2. \end{aligned}$$

The reader is urged to try for himself this exercise of looking for and identifying examples of instrumental explanations, both in texts and in the classroom. This will have three benefits. (i) For persons like the writer, and most readers of this article, it may be hard to realise how widespread is the instrumental approach. (ii) It will help, by repeated examples, to consolidate the two contrasting concepts. (iii) It is a good preparation for trying to formulate the difference in general terms. Result (i) is necessary for what follows in the rest of the present section, while (ii) and (iii) will be useful for the others.

If it is accepted that these two categories are both well-filled, by those pupils and teachers whose goals are respectively relational and instrumental understanding (by the pupil), two questions arise. First, does this matter? And second, is one kind better than the other? For years I have taken for granted the answers to both these questions: briefly, 'Yes; relational.' But the existence of a large body of experienced teachers and of a large number of texts belonging to the opposite camp has forced me to think more about why I hold this view. In the process of changing the judgement from an intuitive to a reflective one, I think I have learnt something useful. The two questions are not entirely separate, but in the present section I shall concentrate as far as possible on the first: does it matter?

The problem here is that of a mis-match, which arises automatically in any *faux ami* situation, and does not depend on whether A or B's meaning is 'the right one'. Let us imagine, if we can, that school A send a team to play school B at a game called 'football', but that neither team knows that there are two kinds (called 'association' and 'rugby'). School A plays soccer and has never heard of rugger, and vice versa for B. Each team will rapidly decide that the others are crazy, or a lot of foul players. Team A in particular will think that B uses a mis-shapen ball, and commit one foul after another. Unless the two sides stop and talk about what game they think they are playing at, long enough to gain some mutual understanding, the game will break up in disorder and the two teams will never want to meet again.

Though it may be hard to imagine such a situation arising on the football field, this is not a far-fetched analogy for what goes on in many mathematics lessons, even now. There is this important difference, that one side at least cannot refuse to play. The encounter is compulsory, on five days a week, for about 36 weeks a year, over 10 years or more of a child's life.

Leaving aside for the moment whether one kind is better than the other, there are two kinds of mathematical mis-matches which can occur.

1. Pupils whose goal is to understand instrumentally, taught by a teacher who wants them to understand relationally.
2. The other way about.

The first of these will cause fewer problems short-term to the pupils, though it will be frustrating to the teacher. The pupils just won't want to know all the careful groundwork he gives in preparation for whatever is to be learnt next, nor his careful explanations. All they want is some kind of rule for getting the answer. As soon as this is reached, they latch on to it and ignore the rest.

If the teacher asks a question that does not quite fit the rule, of course they will get it wrong. For the following example I have to thank Mr. Peter Burney, at that time a student at Coventry College of Education on teaching practice. While teaching area, he became suspicious that the children did not really understand what they were doing. So he asked them: "What is the area of a field 20 cms by 15 yards?" The reply was: "300 square centimetres". He asked: "Why not 300 square yards?" Answer: "Because area is always in square centimetres."

To prevent errors like the above the pupils need another rule (or, of course, relational understanding), that both dimensions must be in the same unit. This anticipates one of the arguments which I shall use against instrumental understanding, that it usually involves a multiplicity of rules rather than fewer principles of more general application.

There is of course always the chance that a few of the pupils will catch on to what the teacher is trying to do. If only for the sake of these, I think he should go on trying. By many, probably a majority, his attempts to convince them that being able to use the rule is not enough will not be well received. ‘Well is the enemy of better,’ and if pupils can get the right answers by the kind of thinking they are used to, they will not take kindly to suggestions that they should try for something beyond this.

The other mismatch, in which pupils are trying to understand relationally but the teaching makes this impossible, can be a more damaging one. An instance which stays in my memory is that of a neighbour’s child, then seven years old. He was a very bright little boy, with an I.Q. of 140. At the age of five he could read *The Times,* but at seven he regularly cried over his mathematics homework. His misfortune was that he was trying to understand relationally teaching which could not be understood in this way. My evidence for this belief is that when I taught him relationally myself, with the help of Unifix, he caught on quickly and with real pleasure.

A less obvious mis-match is that which may occur between teacher and text. Suppose that we have a teacher whose conception of understanding is instrumental, who for one reason or other is using a text which aim is relational understanding by the pupil. It will take more than this to change his teaching style. I was in a school which was using my own text[1], and noticed (they were at Chapter 1 of *Book* 1) that some of the pupils were writing answers like

‘the set of {flowers}’.

When I mentioned this to the teacher (he was head of mathematics) he asked the class to pay attention to him and said: “Some of you are not writing your answers properly. Look at the example in the book, at the beginning of the exercise, and be sure you write you answers exactly like that.”

Much of what is being taught under the description of “modern mathematics” is being taught and learnt just as instrumentally as were the syllabi which have been replaced. This is predictable from the difficulty of accommodating (restructuring) our existing schemas[2]. To the extent that this is so, the innovations have probably done more harm than good, by introducing a mismatch between the teacher and the aims implicit in the new content. For the purpose of introducing ideas such as sets, mappings and variables is the help which, rightly used, they can give to relational understanding. If pupils are still being taught instrumentally, then a 'traditional' syllabus will probably benefit them more. They will at least acquire proficiency in a number of mathematical techniques which will be of use to them in other subjects, and whose lack has recently been the subject of complaints by teachers of science, employers and others.

Near the beginning I said that two *faux amis* could be identified in the context of mathematics. The second one is even more serious; it is the word 'mathematics' itself. For we are not talking about better and worse teaching of the same kind of mathematics. It is easy to think this, just as our imaginary soccer players who did not know that their opponents were playing a different game might think that the other side picked up the ball and ran with it because they could not kick properly, especially with such a mis-shapen ball. In which case they might kindly offer them a better ball and some lessons on dribbling.

It has taken me some time to realise that this is not the case. I used to think that maths teachers were all teaching the same subject, some doing it better than others.

I now believe that *there are two effectively different subjects being taught under the same name, 'mathematics'*. If this is true, then this difference matters beyond any of the differences in syllabi which are so widely debated. So I would like to try to emphasise the point with the help of another analogy.

Imagine that two groups of children are taught music as a pencil-and-paper subject. They are all shown the five-line stave, with the curly 'treble sign at the beginning; and taught that marks on the lines are called E, G, B, D, F. Marks between the lines are called F, A, C, E. They learn that a line with an open oval is called a minim, and is worth two with blacked-in ovals which are called crotches, or four with blacked-in ovals and a tail which are called quavers, and so on – musical multiplication tables if you like. For one group of children, all their learning is of this kind and nothing beyond. If they have a music lesson a day, five days a week in school terms, and are told that it is important, these children could in time probably learn to write out the marks for simple melodies such as *God Save the Queen* and *Auld Lang Syne,* and to solve simple problems such as 'What time is this in?' and 'What key?', and even 'Transpose this melody from C major to A major.' They would find it boring, and the rules to be memorised would be so numerous that problems like 'Write a simple accompaniment for this melody' would be too difficult for most. They would give up the subject as soon as possible, and remember it with dislike.

The other group is taught to associate certain sounds with these marks on paper. For the first few years these are audible sounds, which they make themselves on simple instruments. After a time they can still imagine the sounds whenever they see or write the marks on paper. Associated with every sequence of marks is a melody, and with every vertical set a harmony. The keys C major and A major have an audible relationship, and a similar relationship can be found between certain other pairs of keys. And so on. Much less memory work is involved, and what has to be remembered is largely in the form of related wholes (such as melodies) which their minds easily retain. Exercises such as were mentioned earlier ('Write a simple accompaniment') would be within the ability of most. These children would also find their learning intrinsically pleasurable, and many would continue it voluntarily, even after O-level or C.S.E.

For the present purpose I have invented two non-existent kinds of 'music lesson', both pencil-and-paper exercises (in the second case, after the first year or two). But the difference between these imaginary activities is no greater than that between two activities which actually go on under the name of mathematics. (We can make the analogy closer, if we imagine that the first group of children were initially taught sounds for the notes in a rather half-hearted way, but that the associations were too ill-formed and unorganised to last.)

The above analogy is, clearly, heavily biased in favour of relational mathematics. This reflects my own viewpoint. To call it a viewpoint, however, implies that I no longer regard it as a self-evident truth which requires no justification: which it can hardly be if many experienced teachers continue to teach instrumental mathematics. The next step is to try to argue the merits of both points of view as clearly and fairly as possible; and especially of the point of view opposite to one's own. This is why the next section is called Devil's Advocate. In one way this only describes that part which puts the case for instrumental understanding. But it also justifies the other part, since an imaginary opponent who thinks differently from oneself is a good device for making clearer to oneself why one does think this way.

## Devil's Advocate

Given that so many teachers teach instrumental mathematics, might this be because it does have certain advantages? I have been able to think of three advantages (as distinct from situational reasons for teaching this way, which will be discussed later).

1. Within its own context, instrumental mathematics is usually easier to understand; sometimes much easier. Some topics, such as multiplying two negative numbers together, or dividing by a fractional number, are difficult to understand relationally. "Minus times minus equals plus" and "to divide by a fraction you turn it upside down and multiply' are easily remembered rules. If what is wanted is a page of right answers, instrumental mathematics can provide this more quickly and easily.
2. So the rewards are more immediate, and more apparent. It is nice to get a page of right answers, and we must not underrate the importance of the feeling of success which pupils get from this. Recently I visited a school where some of the children describe themselves as 'thickos'. Their teachers use the term too. These children need success to restore their self-confidence, and it can be argued that they can achieve this more quickly and easily in instrumental mathematics than in relational.
3. Just because less knowledge is involved, one can often get the right answer more quickly and reliably by instrumental thinking than relational. This difference, is so marked that even relational mathematicians often use instrumental thinking. This is a point of much theoretical interest, which I hope to discuss more fully on a future occasion.

The above may well not do full justice to instrumental mathematics. I shall be glad to know of any further advantages which it may have.

There are four advantages (at least) in relational mathematics.

4. *It is more adaptable to new tasks.* Recently I was trying to help a boy who had learnt to multiply two decimal fractions together by dropping the decimal point, multiplying as for whole numbers, and re-inserting the

decimal point to give the same total number of digits after the decimal point as there were before. This is a handy method if you know why it works. Through no fault of his own, this child did not; and not unreasonably, applied it also to division of decimals. By this method $4.8 \div 0.6$ came to 0.08. The same pupil had also learnt that if you know two angles of a triangle, you can find the third by adding the two given angles together and subtracting from $180^{\circ}$. He got ten questions right this way (his teacher believed in plenty of practise), and went on to use the same method for finding the exterior angles. So he got the next five answers wrong.

I do not think he was being stupid in either of these cases. He was simply extrapolating from what he already knew. But relational understanding, by knowing not only what method worked but why, would have enabled him to relate the method to the problem, and possibly to adapt the method to new problems. Instrumental understanding necessitates memorising which problems a method works for and which not, and also learning a different method for each new class of problems. So the first advantage of relational mathematics leads to:

5. *It is easier to remember.* There is a seeming paradox here, in that it is certainly harder to learn. It is certainly easier for pupils to learn that 'area of a triangle = $\frac{1}{2}$ base $\times$ height' than to learn why this is so. But they then have to learn separate rules for triangles, rectangles, parallelograms, trapeziums; whereas relational understanding consists partly in seeing all these in relation to the area of a rectangle. It is still desirable to know the separate rules; one does not want to have to derive them afresh every time. But knowing also how they are inter-related enables one to remember them as parts of a connected whole, which is easier.

There is more to learn – the connections as well as the separate rules – but the result, once learnt, is more lasting. So there is less re-learning to do, and long-term the time taken may well be less altogether.

Teaching for relational understanding may also involve more actual content. Earlier, an instrumental explanation was quoted leading to the statement ‘Circumference = $\pi d$’. For relational understanding of this, the idea of a proportion would have to be taught first (among others), and this would make it a much longer job than simply teaching the rules as given. But proportionality has such a wide range of other applications that it is worth teaching on these grounds also. In relational mathematics this happens rather often. Ideas required for understanding a particular topic turn out to be basic for understanding many other topics too. Sets, mappings and equivalence are such ideas. Unfortunately the benefits which might come from teaching them are often lost by teaching them as separate topics, rather than as fundamental concepts by which whole areas of mathematics can be interrelated.

6. *Relational knowledge can be effective as a goal in itself.* This is an empiric fact, based on evidence from controlled experiments using non-mathematical material. The need for external rewards and punishments is greatly reduced, making what is often called the ‘motivational’ side of teacher’s job much easier. This is related to:
7. *Relational schemas are organic in quality.* This is the best way I have been able to formulate a quality by which they seem to act as an agent of their own growth. The connection with 3 is that if people get satisfaction from relational understanding, they may not only try to understand relationally new material which is put before them, but also actively seek out new material and explore new areas, very much like a tree extending its roots or an animal exploring new territory in search of nourishment. To develop this idea beyond the level of an analogy is beyond the scope of the present paper, but it is too important to leave out.

If the above is anything like a fair presentation of the cases for the two sides, it would appear that while a case might exist for instrumental mathematics short-term and within a limited context, long-term and in the context of a child’s whole education it does not. So why are so many children taught only instrumental mathematics throughout their school careers? Unless we can answer this, there is little hope of improving the situation.

An individual teacher might make a reasoned choice to teach for instrumental understanding on one or more of the following grounds.

1. That relational understanding would take too long to achieve, and to be able to use a particular technique is all that these pupils are likely to need.
2. That relational understanding of a particular topic is too difficult, but the pupils still need it for examination reasons.
3. That a skill is needed for use in another subject (e.g. science) before it can be understood relationally with the schemas presently available to the pupil.
4. That he is a junior teacher in a school where all the other mathematics teaching is instrumental.

All of these imply, as does the phrase 'make a reasoned choice', that he is able to consider the alternative goals of instrumental and relational understanding on their merits and in relation to a particular situation. To make an informed choice of this kind implies awareness of the distinction, and relational understanding of the mathematics itself. So nothing else but relational understanding can ever be adequate for a teacher. One has to face the fact that this is absent in many who teach mathematics; perhaps even a majority.

Situational factors which contribute to the difficulty include:

1. *The backwash effect of examinations.* In view of the importance of examinations for future employment, one can hardly blame pupils if success in these is one of their major aims. The way pupils work cannot but be influenced by the goal for which they are working, which is to answer correctly a sufficient number of questions.
2. *Over-burdened syllabi.* Part of the trouble here is the high concentration of the information content of mathematics. A mathematical statement may condense into a single line as much as in another subject might take over one or two paragraphs. By mathematicians accustomed to handling such concentrated ideas, this is often overlooked (which may be why most mathematics lecturers go too fast).

Non-mathematicians do not realise it at all. Whatever the reason, almost all syllabi would be much better if much reduced in amount so that there would be time to teach them better.

3. *Difficulty of assessment* of whether a person understands relationally or instrumentally. From the marks he makes on paper, it is very hard to make valid inference about the mental processes by which a pupil has been led to make them; hence the difficulty of sound examining in mathematics. In a teaching situation, talking with the pupil is almost certainly the best way to find out; but in a class of over 30, it may be difficult to find the time.
4. *The great psychological difficulty for teachers of accommodating (re-structuring) their existing and long-standing schemas,* even for the minority who know they need to, want to do so, and have time for study.

From a recent article[3] discussing the practical, intellectual and cultural value of a mathematics education (and I have no doubt that he means relational mathematics!) by Sir Hermann Bondi, I take these three paragraphs. (In the original, they are not consecutive.)

> So far my glowing tribute to mathematics has left out a vital point: the rejection of mathematics by so many, a rejection that in not a few cases turns to abject fright.
>
> The negative attitude to mathematics, unhappily so common, even among otherwise highly-educated people, is surely the greatest measure for our failure and a real danger to our society.
>
> This is perhaps the clearest indication that something is wrong, and indeed very wrong, with the situation. It is not hard to blame education for at least a share of the responsibility; it is harder to pinpoint the blame, and even more difficult to suggest new remedies.

If for 'blame' we may substitute 'cause', there can be small doubt that the widespread failure to teach relational mathematics – a failure to be found in primary, secondary and further education, and in 'modern' as well as 'traditional' courses – can be identified as a major cause. To suggest new remedies is indeed difficult, but it may be hoped that diagnosis is one good step towards a cure. Another step will be offered in the next section.

## A Theoretical Formulation

There is nothing so powerful for directing one's actions in a complex situation, and for coordinating one's own efforts with those of others, as a good theory. All good teachers build up their own stores of empirical knowledge, and have abstracted from these some general principles on which they rely for guidance. But while their knowledge remains in this form it is largely still at the intuitive level within individuals, and cannot be communicated, both for this reason and because there is no shared conceptual structure (schema) in terms of which it can be formulated. Were this possible, individual efforts could be integrated into a unified body of knowledge which would be available for use by newcomers to the profession. At present most teachers have to learn from their own mistakes.

For some time my own comprehension of the difference between the two kinds of learning which lead respectively to relational and instrumental mathematics remained at the intuitive level, though I was personally convinced that the difference was one of great importance, and this view was shared by most of those with whom I discussed it. Awareness of the need for an explicit formulation was forced on me in the course of two parallel research projects; and insight came, quite suddenly, during a recent conference. Once seen it appears quite simple, and one wonders why I did not think of it before. But there are two kinds of simplicity: that of naivety; and that which, by penetrating beyond superficial differences, brings simplicity by unifying. It is the second kind which a good theory has to offer, and this is harder to achieve.

A concrete example is necessary to begin with. When I went to stay in a certain town for the first time, I quickly learnt several particular routes. I learnt to get between where I was staying and the office of the colleague with whom I was working; between where I was staying and the office of the colleague with whom I was working; between where I was staying and the university refectory where I ate; between my friend's office and the refectory; and two or three others. In brief, I learnt a limited number of fixed plans by which I could get from particular starting locations to particular goal locations.

As soon as I had some free time, I began to explore the town. Now I was not wanting to get anywhere specific, but to learn my way around, and in the process to see what I might come upon that was of interest. At this stage my goal was a different one: to construct in my mind a cognitive map of the town.

These two activities are quite different. Nevertheless they are, to an outside observer, difficult to distinguish. Anyone seeing me walk from A to B would have great difficulty in knowing (without asking me) which of the two I was engaged in. But the most important thing about an activity is its goal. In one case my goal was to get to B, which is a physical location. In the other it was to enlarge or consolidate my mental map of the town, which is a state of knowledge.

A person with a set of fixed plans can find his way from a certain set of starting points to a certain set of goals. The characteristic of a plan is that it tells him what to do at each choice point: turn right out of the door, go straight on past the church, and so on. But if at any stage he makes a mistake, he will be lost; and he will stay lost if he is not able to retrace his steps and get back on the right path.

In contrast, a person with a mental map of the town has something from which he can produce, when needed, an almost infinite number of plans by which he can guide his steps from any starting point to any finishing point, provided only that both can be imagined on his mental map. And if he does take a wrong turn, he will still know where he is, and thereby be able to correct his mistake without getting lost; even perhaps to learn from it.

The analogy between the foregoing and the learning of mathematics is close. The kind of learning which leads to instrumental mathematics consists of the learning of an increasing number of fixed plans, by which pupils can find their way from particular starting points (the data) to required finishing points (the answers to the questions). The plan tells them what to do at each choice point, as in the concrete example. And as in the concrete example, *what has to be done next is determined purely by the local situation.* (When you see the post office, turn left. When you have cleared brackets, collect like terms.) There is no awareness of the overall relationship between successive stages, and the final goal. And in both cases, the learner is dependent on outside guidance for learning each new 'way to get there'.

In contrast, learning relational mathematics consists of building up a conceptual structure (schema) from which its possessor can (in principle) produce an unlimited number of plans for getting from any starting point within his schema to any finishing point. (I say 'in principle' because of course some of these paths will be much harder to construct than others.)

This kind of learning is different in several ways from instrumental learning.

1. The means become independent of particular ends to be reached thereby.
2. Building up a schema within a given area of knowledge becomes an intrinsically satisfying goal in itself.
3. The more complete a pupil's schema, the greater his feeling of confidence in his own ability to find new ways of 'getting there' without outside help.
4. But a schema is never complete. As our schemas enlarge, so our awareness of possibilities is thereby enlarged. Thus the process often becomes self-continuing, and (by virtue of 3) self-rewarding.

Taking again for a moment the role of devil's advocate, it is fair to ask whether we are indeed talking about two subjects, relational mathematics and instrumental mathematics, or just two ways of thinking about the same subject matter. Using the concrete analogy, the two processes described might be regarded as two different ways of knowing about the same town; in which case the distinction made between relational and instrumental understanding would be valid, but not between instrumental and relational mathematics.

But what constitutes mathematics is not the subject matter, but a particular kind of knowledge about it. The subject matter of relational and instrumental mathematics may be the same: cars travelling at uniform speeds between two towns, towers whose heights are to be found, bodies falling freely under gravity, etc. etc. But the two kinds of knowledge are so different that I think that there is a strong case for regarding them as different kinds of mathematics. If this distinction is accepted, then the word 'mathematics' is for many children indeed a false friend, as they find to their cost.

## The State of Play

This is already a long article, yet it leaves many points awaiting further development. The applications of the theoretical formulation in the last section to the educational problems described in the first two have not been spelt out. One of these is the relationship between the goals of the teacher and those of the pupil. Another is the implications for a mathematical curriculum.

In the course of discussion of these ideas with teachers and lecturers in mathematical education, a number of other interesting points have been raised which also cannot be explored further here. One of these is whether the term ‘mathematics’ ought not to be used for relational mathematics only. I have much sympathy with this view, but the issue is not as simple as it may appear.

There is also research in progress. A pilot study aimed at developing a method (or methods) for evaluating the quality of children’s mathematical thinking has been finished, and has led to a more substantial study in collaboration with the N.F.E.R. as part of the TAMS continuation project. A higher degree thesis at Warwick University is nearly finished; and a research group of the Department of Mathematics at the University of Quebec in Montreal is investigating the problem with first and fourth grade children. All this will I hope be reported in due course.

The aims of the present paper are twofold. First, to make explicit the problem at an empiric level of thinking, and thereby to bring to the forefront of attention what some of us have known for a long time at the back of our minds. Second, to formulate this in such a way that it can be related to existing theoretical knowledge about the mathematical learning process, and further investigated at this level and with the power and generality which theory alone can provide.

## References

1. R. R. Skemp: *Understanding Mathematics* (U.L.P.).
2. For a fuller discussion see R. R. Skemp: *The Psychology of Learning Mathematics* (Penguin 1972) pp. 43–46.
3. H. Bondi: *The Dangers of Rejecting Mathematics (Times Higher Education Supplement,* 26.3.76).

# Richard Skemp and Reflective Intelligence

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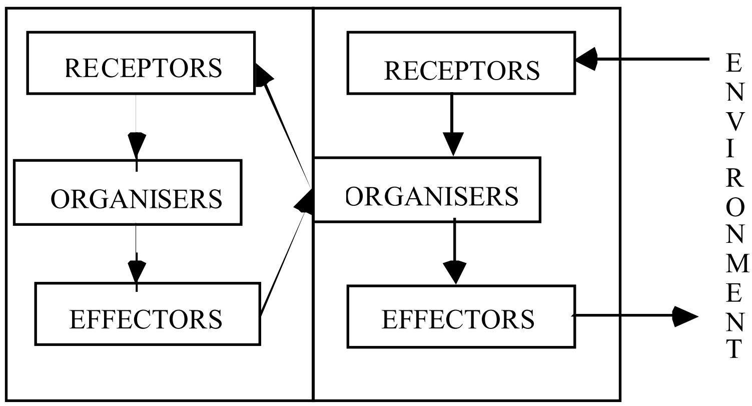
I first met Richard Skemp in Leicestershire, some time in the very late fifties, probably 1959, when I was working with Len Sealey in the County of Leicester. We were trying to introduce more psychologically valid mathematics learning into the elementary schools (known as primary in England). He spent several days looking over my "experimental schools" of which there were over a dozen at the time. The idea of establishing mathematics learning starting from concrete situations was practically unheard of in those days, much less the idea that children should be encouraged to abstract mathematics (how else can you learn it?). Skemp interacted well with the teachers and the children and was keen to have discussions about my "learning theory" on which I had based the work in the experimental schools. We discussed his idea of "reflective learning", which I agreed would follow hard upon the "constructive learning" that we were looking at in the schools. The Piagetian idea of "concrete operations" preceding the more analytical type of thinking, was getting a certain amount of acceptance in psychological circles, but was not known to any great extent in pedagogical circles. I was on the point of accepting a position as Reader (associate professor) of Psychology at Adelaide University, and I was happy to have found someone who could "look after" my baby project. Skemp was keen to do that, although his interests were more linked with the secondary stage of mathematical thinking and learning.

Soon after, in 1961, an important experiment by Dr. Richard Skemp was reported in the British Journal of Educational Psychology, under the title 'Reflective Intelligence and Mathematics'. In this paper Skemp defined different kinds of intelligence: sensory intelligence; motor intelligence; and reflective intelligence. When a child can transcend the differences between sensory stimuli and can think in terms of the relationships between these sensory stimuli, then, according to Skemp, he is using sensory intelligence. In Piaget's preoperational stage, according to this, sensory intelligence would not be fully operative because it is, for example, the look of a set of objects rather than the relationship between their individual members that determines which of them has more in it. Motor intelligence, he suggests, is awareness of relationships between actions, such as between filling up and emptying, putting together and taking apart, and so on. So the child who is learning arithmetic by manipulating objects is using his motor intelligence in realising the relationships between his actions and he is using his sensory intelligence when he is realising the relationship between the sensory output of the result of his action. Skemp went on to say that reflective intelligence is the ability to double back upon ourselves and realise what we were doing when using our sensory and our motor intelligence. He described the differences between each of these kinds of intelligence as follows (Skemp, 1961, pp. 47, 48):

> When a child can transcend [sensory stimuli], and give responses which indicate his perception of the number property independently of configuration, we may say that he shows sensory intelligence – that is to say, awareness of certain relationships between sensory stimuli.
>
> . . . I suggest that the concept of motor intelligence should involve awareness of relationships between *actions*, such as filling up and emptying out, putting together and taking apart, taking away and putting back. For a child learning arithmetic by manipulating objects, awareness of the relationships between (physical) addition and subtraction, successive addition and multiplication, v and division would then all be described as manifestations of motor intelligence.
>
> . . . consider a simple act of reflective thought, such as may be done by a pupil who has been getting wrong a certain kind of algebraic problem, is told his mistake, amends his method at this point, and thereafter solves similar problems correctly. What he has done is to:
>
> (1) reflect on what he did
>
> (2) modify a particular part of the sequence of operations, and
>
> (3) replace the former sequence by the modified sequence.
>
> To do this he must be capable:
>
> (1) of knowing what he did. There must be not only mental representations of his operations, but also a system capable of becoming conscious of these.
>
> (2) (3) of changing some of these mental representations.
>
> This requires a system capable also of acting upon the operations of sensori-motor intelligence.
>
> Reflective thought may thus be regarded as a second order system, aware of and acting on the mental representations of the sensori-motor system in ways which resemble the receptor and effector activity of that system. The following diagram may be helpful in thinking about the two systems.




REFLECTIVE SYSTEM SENSORI-MOTOR SYSTEM

Thus Skemp looked at the situation in two cycles, or two signalling systems, as the Russians would have described it. The environment is received by the receptors, it is organised immediately by the sensory motor intelligence and the effectors put the output back into the environment. This is the response. This sensory motor organisation can again act as a stimulus to the second signalling system, that is to the reflective receptors that can again organise this and the effectors will feed back into the sensory motor organising system which eventually will feed out into the environment through sensory effectors. So the reflective system, according to Skemp, is in some sense built on top of the sensory motor system and operates on it in the way that the sensory motor system operates on the environment.

As regards mathematics learning, Skemp's hypothesis is that in learning arithmetic mostly the sensory motor system is involved whereas in the learning of algebra and the higher reaches of mathematics the reflective system is involved. One cannot help thinking that this hypothesis would not have been made, had arithmetic not been treated as a set of mechanical stimulus response situations, which simply had to be learned. These days, arithmetic is no longer being strictly separated from the rest of mathematics and when we now speak about learning arithmetic, we do not necessarily speak of mechanically learning how to carry out certain processes. We now often mean the learning of not only how arithmetical operations are carried out, but why they are so carried out. In this case, even in the learning of arithmetic, we would need to use our reflective intelligence to quite a little extent. According to Piaget, of course, the reflective system does not really begin to develop or be fully in use until the onset of puberty, so it would seem a sensible assumption to make that arithmetic learned in primary school, that is during the greater part of the concrete operational stage, makes use of the sensory motor system.

It would appear to make a tidy kind of organisation of the theory. Be that as it may with arithmetic it is certainly true that it becomes more and more difficult to use merely sensory motor intelligence, the higher we move into the study of mathematics. Skemp explained the difference between arithmetic and algebra this way (Skemp, 1961, p. 49):

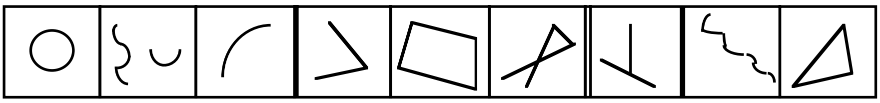
> The ability to do simple arithmetic requires the ability to perceive numbers and their relationships, and make correctly the appropriate responses; but not necessarily the ability to become consciously aware either of the relationships which determined the response, or of the method by which the answer was obtained. … The transition to algebra, however, involves deliberate generalisation of the concepts and operations of arithmetic. . . Such a process of generalisation does require awareness of the concepts and operations themselves. Since these are not physical objects, perceivable by the senses, but are mental, this transition requires the activity of reflective thought. And further, the generalisation requires not only awareness of the concepts and operations but perception of their inter-relations. This involves true reflective intelligence.

Skemp (1961, p. 50) then proposed four hypotheses with, he claimed, far-reaching implications for the teaching of mathematics. These were:

> 1. For mathematical achievement a necessary, though not sufficient, condition is the presence of reflective intelligence as well as sensori-motor intelligence.
>
> 2. (a) The development of number concepts and arithmetical operations arises partly as a cumulative trace representing the invariant properties and relationships of groups of repeated sensori-motor experiences.
>
> (b) The projection of these traces on to subsequent sensori-motor experiences results in perception.
>
> 3. The development of mathematical concepts and operations arises partly as a cumulative trace representing the invariant properties and relationships of groups of repeatedly-employed arithmetical concepts and operations.
>
> 4. The successful use for solving problems, and for further generalisation, of mathematical concepts and operations will be aided by any teaching method which increases pupils' awareness of the concepts and operations which they use.

To investigate these hypotheses he designed a set of tests in which he was able to measure the use of reflective intelligence by presenting subjects with certain exemplars and non-exemplars of concepts in the classical way. From these they would then extract certain concepts, and these concepts would then be used in certain operations which would also be taught non-verbally in the test situation. The examination of the combination and reversals of the operations used were taken as super-ordinates which could not have been learned already. The results of these tests were then correlated to mathematical achievement. In Skemp's test of reflective intelligence he used three stages. Exemplars and non-exemplars were given of a concept. The test figures were given to the subjects who had to decide which of these figures were exemplars and which were non-exemplars. One example given was:

EXAMPLES NOT EXAMPLES TEST FIGURES



Then operations were introduced, again non-verbally, by the use of figures. For example operation C would be turning through a right angle in a clockwise sense. This was not explained verbally, but illustrated by pictorial examples. Similarly operation J was a reflection with respect to a horizontal line. This was also illustrated by three examples. Similarly operation N operated on a figure consisting of two different kinds of things and the number of one kind had to be replaced by the number of the other kind and vice versa. For example two crosses and one circle had to be replaced by one cross and two circles.

In the next part of the experiment, figures were given and subjects were asked to operate on these figures by means of the operations that they had learned. Then finally, combinations of operations, for instance: "Do an operation C followed by an operation J on the following figures", were asked for. To make sure that the super-ordination which arose out of a combination of operations was what was being tested, Skemp made sure that the operations were in fact understood at this stage, namely they were explained to the subjects, in case they had not learned them before this part of the test was given. In the concept formation part, the learning of the concepts was followed by learning of "double" concepts. That is, exemplars were only exemplars if two criteria were satisfied. The subject had to sort out that if only one criterion was satisfied the instance was not an exemplar of the concept, and so had to reflect on whether in the particular instance, only one or neither or both conditions were satisfied. Only in the latter case was it considered an exemplar of the concept. This is, of course, the Bruner type of conjunctive concept formation (Bruner *et al.,* 1956) and it is also what happens with the Vygotsky blocks when conjunctive concepts are being investigated in learning situations.

The correlations Skemp obtained between mathematical ability and the test scores were as given in Table 1.

Table 1
*Correlations Between Mathematical Ability and Test Scores*

*x*=mathematical criterion
*u*=use of operations, not requiring reflective activity
*t*=reflective activity on concepts
*v*= reflective activity on operations

| | Form 5 – age 16 years (N = 50) | Form 4 – age 15 years (N = 88) |
|---|---|---|
| $r_{tx}$ | 0.58 | 0.56 |
| $r_{ux}$ | 0.42 | 0.48 |
| $r_{vx}$ | 0.72 | 0.73 |

Reliability of test scores for V and IV forms together (N=138)

| Test | Reliability |
|---|---|
| $r_{tt}$ | 0.76 |
| $r_{uu}$ | 0.94 |
| $r_{vv}$ | 0.95 |

*X* was called the mathematical criterion, *T* the reflective ability on the concepts, i.e. in this case the ability to handle conjunctions of the concepts, *U* was taken to be the ability to use the operation introduced, that is, not requiring reflective ability and *V* was the reflective ability on operations, that is the ability to combine one operation with another.

It was expected that correlations with the mathematical criterion should be higher between *T* and *X* and between *V* and *X* than would be between *U* and *X* and Skemp hypothesised that it would be highest between *V* and *X*, since that was the highest level where operators were being combined with other operators. The correlations of *VTX* = 0.58, *VUX* = 0.42, *VUX* = 0.72 were obtained for fifth year United Kingdom Grammar School children. The test was also given to fourth year

Grammar School children, and the validation took place a year later, when these children became eligible to take their general certificates of education. The correlations of *VTX*=0.56, *VUX*=0.48, *VUX* = 0.73, were remarkable similar.

The split half reliabilities ranged between 0.76 and 0.95 and so it would seem, therefore, that it was amply validated that reflective activity is certainly connected with the ability to solve mathematical problems, such as are encountered in secondary school mathematics examinations. It is, of course, not established that reflective intelligence is NOT used when handling arithmetical operations. One excellent feature of this experiment was that the testing was not done in any verbal sense, but the "possession of the concept" was regarded as established if this concepts could in fact be used. The reflection on concepts was regarded as having taken place if such reflection did in fact result in an operation being performed which without such reflection could not have been performed. Verbal difficulties of understanding were eliminated as far as possible.

On the basis of this study Skemp concluded that:

> If a child does badly at mathematics, this paper indicates that it may be because he has not formed the necessary concepts and operations, or it may be *because he cannot reflect on them*. . . . If the understanding of mathematics (as here defined) is dependent on reflective ability, then the ages at which its various stages are taught should be related to the stages by which reflective intelligence develops.
>
> (Skemp, 1961, p. 54)

An immediate question that arises out of this study is whether reflective intelligence, if indeed it is a separate entity, can be trained. It would seem likely that one learns to reflect by reflecting and it is also clear that in most of our educational procedures not a great deal of reflection is incorporated, all or most of the reflection is done by the teacher or the textbook. The results of reflections are prepared in the form of lessons which the children learn. It is something of a miracle that in spite of this children learn to reflect. It would seem that the ability to reflect is a necessity for the prosecution of life and presumably the mathematical situations are not the only ones which encourage its formation and development.

It is quite likely that if care were taken to induce children to reflect carefully on what they were doing, a much more rapid development of the reflective ability could result than that which takes place now. Researches in the use of "logical" materials to teach young children logical relationships would seem to indicate that this was so (Dienes, 1963). The reason young children do not use reflective intelligence is probably because they can get along very nicely without it.. There are no situations which they encounter which they cannot handle just using their “childish” logic. Piaget describes this as operating on concrete situations, and using juxtaposition rather than implication and so on. If situations were created in which children were highly motivated to engage, which for their efficient prosecution would require the use of reflective intelligence, it is very possible that such reflective abilities would thereby be helped to develop in children rather sooner than later.

The second time Richard Skemp and I met was at the UNESCO Conference on Mathematics Learning in 1962 held in Budapest. We had three official languages, French, Hungarian and English, and Skemp of course gave his contribution in English, while I made mine in Hungarian with the discussions mostly in French. Fortunately there was simultaneous translation, so that Skemp was able to follow my train of thought, and of course I was able to follow his. We cemented our professional relationship by exchanging views, and surprisingly, we seemed to agree with each other on what was really wrong with mathematics education, which for me, and no doubt for Skemp, was a rare occasion.

This was a meeting held in the “heroic times” of the struggle to put mathematics learning on to the right track. Papy and Servais were there, Krygowska was there from Poland, Pescarini from Italy and many others who were making attempts to improve the situation in their own countries, but most of them only touching the periphery of the problem.

I shall never forget the time when Skemp and I were having a drink together in one of the bars, and Skemp exclaimed:” You know, Zed, you and I are the only serious contenders in trying to pinpoint and to solve the problems!” It seemed somewhat arrogant to say so, but having listened to the “bella lingua” of Pescarini, to the theatrical acrobatics performed by Papy in his explanation of the virtues of arrows and most of the others munching platitudes, I must say I was bound to agree with him.

The International Study Group for Mathematics Learning made a contract with UNESCO, to draw up a report on what was then going on in the reform of mathematics education, over which work I presided. In the final report approved at the Hamburg meeting of UNESCO in

1964 (I think), I gave Skemp's effort a good deal of space, explaining how he was one of the few who tried to get down to the basics of the problem. I was pleased to be able to write in one of my books (Dienes, 1963, p. 175) about his work that:

> Apart from work in these two major fields and, my own contributions, the work of Skemp at Manchester University has shown a promising lead. His predictions for thirteen year-olds are more accurate than any previous predictions on percentage passes in mathematics examinations at sixteen. According to him, mathematical understanding depends almost entirely on the exercise of reflective intelligence (approximately what has been called analysis here). … Skemp's work opens up another avenue of attack in this most intractable field of research.

On another, later, occasion I had occasion to witness reflective intelligence first hand when Skemp was present with me. I had been doing the isometries of the square with some ten year olds, of course using the human body, cardboard squares, logic blocks and so on. An inspector came into the room as Skemp and I were looking round and seeing how the children were getting on. He was rather taken aback at the sight of all the activity and asked me what the children were doing. "Geometry," I replied, "they are studying space." "Oh", replied the inspector. As he seemed so uncertain I asked him to ask the children what they were doing. He went to a little girl who was flipping her square around and asked her. She tried to explain by doing two consecutive flips and showing that one other flip (or turn) would have had the same effect, but she saw that the inspector was still non-plussed.

Having learned some "modern pedagogy", she proceeded to use it with the big burly inspector. (Skemp was all the time watching developments with increasing interest). The little girl got out of her desk and got hold of the inspector and turned him round one quarter of a turn clockwise, and then followed this by turning him round through a half turn. "You see, Sir, if I had turned you one quarter turn anti clockwise, you would have finished up the same as you are now". But then she continued: "Actually, what I have done with you is not the same as what I showed you with the square. To show you the same thing, I would have needed to put you on your head, and, Sir, I am sure you would not have liked that." We all burst out laughing, and the inspector as well as Skemp had had a lesson in "experiential learning". The girl had clearly been engaged in reflective thinking as her postulation showed.

It was through a number of shared experiences of the above kind that Skemp and I cemented our professional and personal relationship, which unfortunately was not to continue in any continuous manner because of my departure for Harvard and then for Adelaide.

If you look around today, Skemp's fairly recent book written for practising and future teachers is possibly the only "textbook" that I could recommend as one for a course on mathematics education. The world has not yet caught up with Skemp, and even less so has it caught up with me! I wish Skemp were around so we could confabulate together again on these problems as we did in Leicestershire and in Budapest! Since I have passed my eightieth birthday, I may soon have the privilege again in the Elysian fields!

## References

Bruner, J. S., Goodnow, J. J., & Austin, G. A. (1956). *A Study of Thinking*, New York: Wiley.

Dienes, Z. P. (1963). *An Experimental Study of Mathematics Learning*, London: Hutchinson.

Skemp, R. R. (1961). Reflective Intelligence and Mathematics, *The British Journal of Educational Psychology*, 31, 45–55.

# Similarities and Differences between the Theory of Learning and Teaching of Skemp and the Van Hiele Levels of Thinking

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## Problems in Learning Caused by Information Overload

In connection with the teaching of mathematics, the Van Hiele levels (van Hiele, 1986; Wirszup, 1974) usually refer to geometry. However my first attempts to improve teaching were in algebra. For many years I was a teacher in a secondary school with Montessori principles: according to these principles teachers had to help children to find solutions by themselves, hence, in Skemp's (1976) terms, to learn not instrumentally but relationally.

When I wanted to learn algebra myself for the first time – I was ten years old and I found a textbook on that subject – I was overwhelmed by the vast number of rules to apply. You were told that $3a+4a=7a$, but $3a\times 4a=12a^2$. To make it more difficult, $a^3\times a^4=a^7$. In secondary school I got another textbook of algebra but it was not much better. However I had no more difficulties with algebra because I had overcome them when I was ten. Already, even at that time, the idea of writing a better textbook of algebra had come into my mind.

I know now that in order to write a good textbook of mathematics you need more courage, more experience and more knowledge of developmental psychology than an inexperienced teacher usually can have. The first issue to address is motivation: 'why should a pupil want to learn the stuff the teacher wants to teach him?' It is very unlikely that a child wants to learn all those rules concerning the beginning of algebra. So my advice is: do not start to teach it! With caution and cleverness a teacher can arouse interest using problems solvable by introducing an unknown quantity $x$ upon which operations can be performed in a natural way. After some time $x$ may also be introduced as a variable in a function. However that is enough for the first year.

Information overload is bad, but common. The inexperienced user of a word-processor has problems understanding the manual, for example a terminology may be used that was never introduced to him. Moreover at the beginning of his study the information concerns subjects he cannot be interested in for he does not know of their existence. The most irritating situation at the start is not knowing how to stop a wrong process in which he has landed. Many people can tell about the horrors they experienced beginning to learn something. It is common for learners starting to learn algebra to suffer likewise.

In 1991 I attended an arithmetic lesson in Stellenbosch (South Africa). Children were asked to find the results of divisions, such as 4536 divided by 72. They were approximately nine years old and of average intelligence. They knew how to find the result with a calculator but this time they were asked to do it without. They had never learned long division but each pupil used his own method to get the result. I was much impressed by the comprehension they showed. They had better notion of arithmetic than candidates for the secondary school entrance examination in my own country, The Netherlands. When I returned to South Africa in 1994, I recalled my visit to the school in Stellenbosch. People told me that the same pupils had had disappointing algebra results in secondary school. When I asked about the textbooks that were using, it became clear that these had overloaded the pupils with information. They did not get enough time to comprehend what they were doing.

## Why Did Skemp and I Never Exchange Ideas?

From the above it may be clear that there are many similarities between the aims of the theory of Skemp and my theory. So we need to find an explanation why cooperation has failed to occur. It is all the more remarkable because a meeting was arranged in Utrecht some thirty years ago as a result of a succession of misunderstandings. The people who invited us together wrongly assumed that Richard Skemp had encountered something like the Van Hiele levels in practice and had found a way to move from one level to the next. However this was not the case. The problems in algebra that cause instrumental thinking have nothing to do with level elevation since the Van Hiele levels do not apply to that part of algebra. People applied terms such as ‘abstraction’ and ‘reflection’ to the stages leading from one level to the next. This resulted in a confusion of tongues: we were talking about completely different things.

## The Discovery of My Levels of Thinking

I was thirteen when I first encountered geometry. Right from beginning there were axioms and definitions but they were harmless because in the proofs you were taught how to use them. Of course you had to learn those proofs partly by heart, it was impossible to find out for yourself what to do. Sometimes you had to take up a shape and rotate it, how could one know that it was allowed? But soon you had to prove congruence of triangles which was easy: you had to find three things the triangles had equal, you wrote them down, a bracket behind them then SAS or something of the sort and after that it was finished.

With time it became clearer however that proving in geometry meant more than we had thought it did at the very beginning. Sometimes you saw something at once and yet it had to be proven. Then there was the incomprehensible topic of theoretical arithmetic. Luckily for me, my teacher only read the theory with us without examination. A friend of mine in a parallel class, whose teacher did test the subject, got an unsatisfactory mark for it. Later as a professor of mathematics he wrote an important book of modern algebra.

After some time new axioms were introduced in geometry of space; this disappointed me because in some proofs reference was made to both sides of a plane. It played an important part in them and it was never mentioned in the axioms or definitions. My next problem was therefore: how can I find better axioms and definitions? When I became a teacher I solved that problem and then tried to put that knowledge into practice. Time was ripe for such ideas: in the Netherlands teachers like Beth and Dijksterhuis tried to teach mathematics in a more rigid form. I discovered soon that results were poor. At the Montessori school pupils really tried to understand but it was as if I spoke a different language. It took them many months to understand what I was talking about and then suddenly they said: “Yes it is easy, but why did you bring it in such a difficult way?” Now they had learned to speak the same language as I did.

At first I was convinced that geometry was a topic that could not be taught, children learned it by their own accord. In the thesis of Jan Koning (1948) on the didactics of physics the stages of Piaget were mentioned. My first article of the levels of geometry (van Hiele, 1955) was based on this part of the thesis. I wrote:

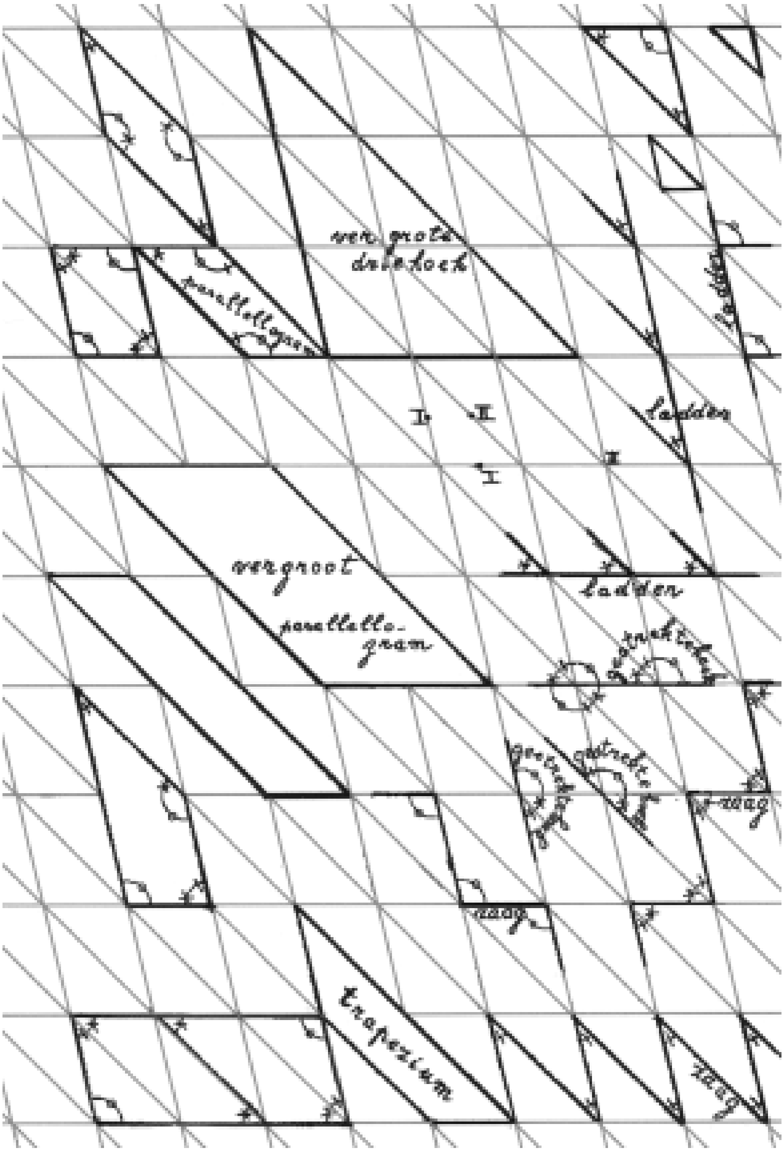
> You can say somebody has reached a higher level of thinking when a new order of thinking enables him, with regard to certain operations, to apply these operations on new objects, The attainment of the new level cannot be effected by teaching, but still, by a suitable choice of exercises the teacher can create a situation for the pupil favorable to the attainment of the higher level of thinking.

The higher level of thinking I wanted the children to bring on had still nothing to do with axioms. I had discovered very soon in my lessons that it was quite impossible to begin geometry with axioms. In the beginning children do not understand that in geometry shapes are determined by their properties. A square is a square because it has four equal sides and four right angles. But if the square was drawn with a vertical diagonal the pupils called it a rhomb. Some years before one of my pupils (twenty five years old) asked seeing an isosceles triangle with one leg horizontal. "Do you really call this an isosceles triangle?" She could only be convinced when I asked her: "Do you call a dog on his back still a dog?" These are no exceptions: the painter Mondriaan has made many pictures showing squares with a vertical diagonal. In the catalogues they are all called rhombs.

## Stages During Transition From One Level to the Next

In the above we have seen the difference between two levels: on the lowest level, the *visual level*, shapes are recognised by seeing: 'This is a square because I see that it is one'. On the higher level a shape is recognised by its properties: 'This is an isosceles triangle because it has three sides and two of them are equal'. This level I have called the *descriptive level*. If a teacher wants something better than instrumental thinking he will have to take account of the difference between the two levels. In the beginning of geometry the names of shapes are important, just by seeing, and children learn that those names do not change if the shapes are put in another position.

In her thesis, my wife Dieke van Hiele-Geldof (1957) describes how she invited pupils to come to the descriptive level. She gave each child a set of congruent triangles and asked them whether it was possible to make a tessellation of them. It was an exciting problem and the children soon found the solution. Afterwards they were asked what they saw in the achieved pattern.




In the picture above you see what the children observed. I will give some translations:

*driehoek* means *triangle*
*gestrekte* means *straight*
*hoek* means *angle*
*trapezium* means *trapezoid*
*vergroot* means *enlarged*
*zaag* means *saw.*

The designations saw and ladder were invented by the children themselves in the discussion. This was the beginning of a development of the language of the descriptive level. At first this language only had the function to designate new parts of the structure they saw at the visual level, but gradually they used it to solve problems on the descriptive level.

This procedure just shows us the means to bring pupils from one level to the next.

Stage 1 *Information*: The children were told to use congruent triangles for a tessellation.

Stage 2 *Guided orientation*: The pupils were busy to find out whether the problem had a solution.

Stage 3 *Explicitation*: After having found a solution the pupils discussed the results. A technical language was born.

Stage 4 *Free orientation*: New problems were given to make use of results that were obtained in the explicitation.

Stage 5 *Integration*: An overview of all that had been learned of the subject, of the network of relations now at the disposal of the pupils was given.

The five stages mentioned above are very similar to stages that can be found in the work of Dienes. This is not purely accidental: in a discussion people using the ideas of Dienes contributed to my work.

## The Theoretical Level

It is easy to understand now that starting geometry with axioms causes insurmountable problems for the pupils. The theoretical level to which the axioms belong can only be reached by starting from the descriptive level with the five stages and in the beginning the children have not even attained the descriptive level. If they are treated in that way, the best solution for their problems is learning parts of geometry by heart and that means only instrumental understanding. Many teachers were very content with such a course of events. They said: “I have taught in this way for many years and the results are excellent.” This conclusion is understandable for geometry had been taught for many years in this way and there were always pupils who liked mathematics from the very beginning and found their own way to the higher levels. But a great part of the pupils developed a dislike for geometry and after their study was finished forgot practically all of it.

## Statements on Different Levels

It is worth now comparing what I have written above with the ideas of Skemp. There is an obvious similarity between my visual level and the intuitive start of the solution of a problem Skemp mentions. In my model intuition is a great deal 'thinking on the visual level', Skemp also believes in the value of discussion just as I do in my stage of the explicitation. However I have not found in the papers of Skemp that judgments on the descriptive and the theoretical level are not better than those on the visual level. The distinction between judgments on different levels obviously does not exist in the papers of Skemp, he uses his own language.

On the visual level a square is such because we see it, on the descriptive level it is one because it is assumed that the sides are equal and the angles are right. When in practice a square has to be drawn it may be better to see if it is a right one than to start a reasoning about the equal sides and the right angles. In modern examinations you find argumentations on the visual level too. In one of them candidates were told that discs with a diameter of 20 cm had to be made out of a piece of cardboard with a length of 54 cm and a width of 40 cm. How many discs could be made of it? The answer that was graded as right was 'five': four discs in the four corners and one in the middle. If a pupil used Pythagoras' theorem he would find that the middle disc had a diameter of 19.7 cm. The right answer was inspired by visual rather than by descriptive thinking.

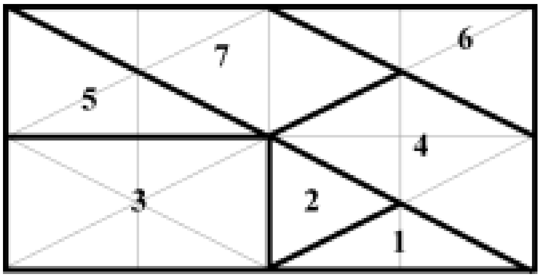
## Reflection and Abstraction

I have difficulties with the use of 'reflection' for the transition from the visual level to the descriptive one. For a pupil it is not a real problem that he is only thinking on the visual level, he has the impression that this thinking just is true thinking. He is not able to reason on the descriptive level, but for him this is not a problem to be solved for he does not understand the language. If he is not helped by suitable exercises it may last perhaps a year before he reaches the descriptive level. Reflection fails because the pupil only disposes of concepts of the visual level and those concepts do not lead to a result on the descriptive level. In the discussion that is part of the explicitation properties discovered in the guided orientation are emphasised, so there the ‘abstraction’ obtained in the orientation is worked out.

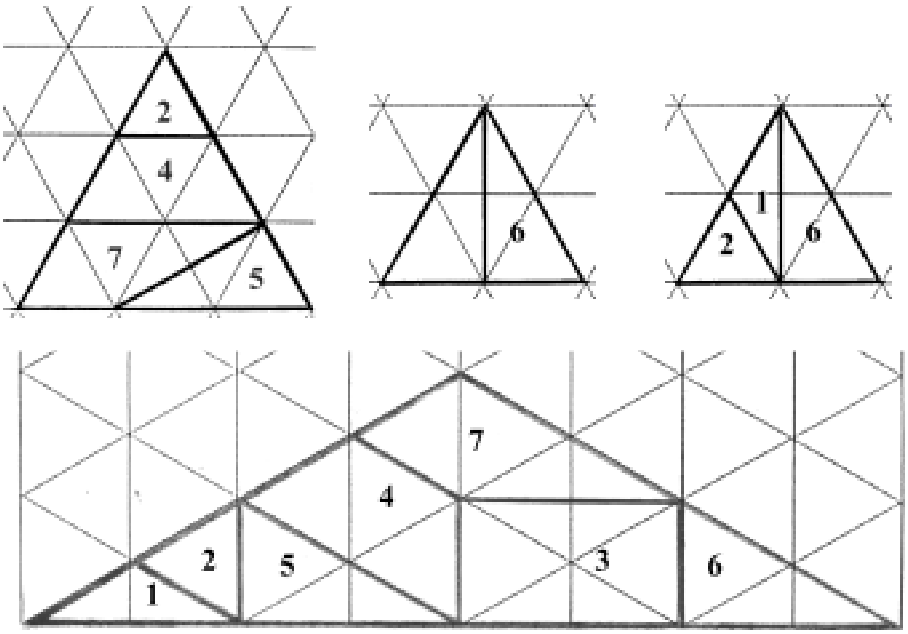
Although Skemp did not see the gap of lack of understanding that exists between two levels of thinking he warns many times for the bad results of learning without building up concepts. Usually this leads to instrumental understanding which really is no understanding at all. In an entrance examination for a higher class of the secondary school I asked a girl if she could compute the length of a diagonal of a rectangle with sides of 12 cm and 5 cm. She said that she had no idea how to do it. I said: “Look at the two triangles you have got by drawing the diagonal.” It did not help. By this I concluded that she did not know the Pythagoras’ theorem. Later that day I was called by the father of the girl. He said: “My daughter did know Pythagoras’ theorem, for when I asked for it she promptly said it.” The father too, a teacher of classical languages, considered the knowledge of an uncomprehended formula as an understanding of geometry.

## Beginning Geometry as a Game

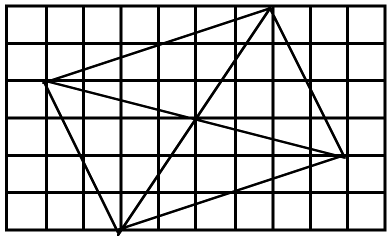
Russian researchers after having heard of my levels concluded that results would be better if the education of the visual level of geometry would be started with children at the age of about nine. It is right to begin the education of the visual level of geometry at an early age on condition that one works under two limitations. The first is that one does not present problems which must be solved on the descriptive level. The second is of a more universal character. No new subject must be introduced if the child does not like it. So geometry only must be brought into the child’s life if we are sure that it will enjoy it. Sometimes this last condition causes problems. From Russia too originates a beautiful game that may help to overcome resistances. It is something like TANGRAM but it has more possibilities.




It consists of seven pieces: a rectangle, two right-angled triangles, an equilateral triangle, an isosceles triangle, a trapezoid and an isosceles one. The shapes are based on equilateral triangles and therefore it is easy to see that the areas of the different pieces are of 1, 2, 3 or 4 equilateral triangles. At first children can be invited to put some pieces together in order to get a new shape. Some examples:




On the visual level children can learn many things: the names of shapes in different positions, composition of shapes, area, enlargement, angles (only multiples of 30 degrees), symmetry (from the very beginning children see that some of the shapes do not change if they are put upside down and others do). The material has specific properties and therefore after some time it has to be replaced by something else. Then we can introduce squared paper and after that coordinates and vectors.



Using coordinates the picture above can be drawn by the following instruction: Start at point (2, 0), join it with (8, 2), this with (6, 6) this with (0, 4) and this with (2, 0). Draw the diagonals of the quadrangle.

If we use vectors we get the following instruction: Start at point (4, 3), start from there and use the four vectors: $\begin{pmatrix} 4 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -4 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \end{pmatrix}$. Join the ends of the vectors. The vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ means go 2 to the right and 3 upwards. In the visual field many instructions of this kind can be given that will be liked by the pupils.

## Levels in Arithmetic and Algebra

Many psychologists of mathematics have looked for a succession of levels in algebra of the secondary school. In normal circumstances such a level-transition is absent. Later in this article I will give an example of an abnormal circumstance in which a succession of levels does exist. In arithmetic however an important level-transition is to be found. It is described in Piaget's book: *The Child's Conception of Number*. Opposite to five dolls are placed five umbrellas. They are all set up on a table between a child and a research worker. The child is asked: "What are more, dolls or umbrellas?" The child says: "There are just as many dolls as umbrellas". "Will each doll have his own umbrella?" "Yes". Then the umbrellas are shoved together and the child is asked again: "What are more, dolls or umbrellas?" The child answers: "Now there are more dolls". "Will each doll have his own umbrella?" "No, there are not enough umbrellas." If children are young, till about five years, this test always has this result, even when the child can count till perhaps ten. Being able to count does not include that the child understands that a set of umbrellas has a constant cardinal number.

Skemp has observed that Piaget in his investigations always prevents children from learning in the situation. His results are based on what children usually know and understand and not on what they are able to understand. This opinion is confirmed by a variation of the above test that I carried out in 1953 with a daughter of mine of three years (3; 8). She too had the opinion that here were less umbrellas (shown together) than dolls. But then I showed a set of three dolls opposite to a set of five umbrellas shown together. Now she said: "Of course there are more umbrellas". "Why?" "That is easy to see, there are five umbrellas and only three dolls. Only you have shown the umbrellas together." After that I repeated the test with the five dolls and the five umbrellas. Now she said that there were just as many umbrellas as dolls. "Are there no less umbrellas?" "No, you have pushed them together." After a short time my daughter of five (5; 1) came home from the Montessori pre-school. I subjected her to the same test and she too was deceived by the five umbrellas shown together. Then the younger daughter said: "Annely, how can you be so stupid, don't you see that there are still five umbrellas?" And then Annely said: "O, I see, you have pushed them together."

This experience of mine was published in 1958 (van Hiele, 1958) and there were many responses. This sign of interest is not astonishing for my test partly undermines Piaget's theory. Piaget always maintained that the transition from the stage in which a child does not have the concept of cardinal number to the stage where it has, is a question of natural development, of as it were a biological growth. In my opinion however it is a learning process. It has the character of a Van Hiele level, which means that it is not possible to bring a child from the visual level to the descriptive one by reasoning. You will have to bring in suitable material to help the child find its own way to the higher level. Following up the ideas of Piaget you will have to wait until the child has reached the higher stage whereas my advice is: give the child the right material. If you do, you can accelerate the process of the level transition, if you don't there is a possibility that the child is still on the visual level when it goes to school and has to learn addition and multiplication. Teachers have reported from their failures in such cases. A child had to learn how to add 8 to 7. First he counted 7 objects of one set and then 8 elements of the other one. Afterwards he was invited to count the number of the two sets together. He found the right number of fifteen. Then the child was invited to do it in a quicker way. After the two separate sets had been counted, the child was asked to put his hand over the first set and to count further. “You know already that under the hand are 7 elements. So now go on.” But the child was not able to count the remaining elements as 8, 9, 10, ... Counting 7 elements did not mean that under the hand was a set with the cardinal number 7.

## The Importance of the Level Transition in Arithmetic

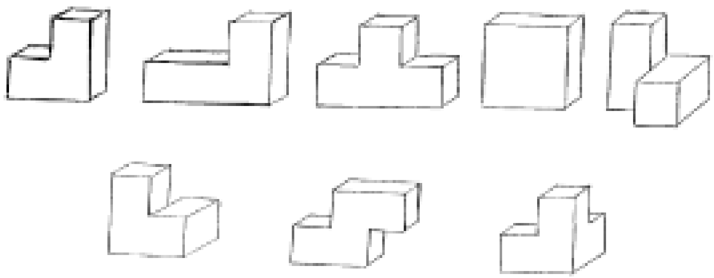
Even though I oppose the meaning of Piaget that there is a gap in the development of the concept of number that only can be conquered by waiting till the child is old enough, his tests are of great importance. The transition of the gap means, in my model, the passage from the visual level of arithmetic to the descriptive one. If teachers do not take this passage into account it is very likely that the pupil will never be able to understand arithmetic. He is condemned to instrumental thinking because a reflection in the sense of Skemp’s theory is quite impossible.

Yet most pupils will, after a pretty long time, pass to the descriptive level. They will no more be deceived by umbrellas shown together. But their knowledge of addition and multiplication remains instrumental for they have learned it before the level transition. Reflection can be used as a means to change this, but it is far more difficult than it would have been if the tables were learned at the right moment.

## Material to Help the Level Transition in Arithmetic

The understanding of the cardinal number as a property of set can be helped by choosing suitable sets. The set of ‘two’ never gives problems: it always is clear that a set of two objects is something else than a set of only one. ‘Three’ is already different: you can have ‘three in a row’ with one element in the middle, but you can also have three elements forming a triangle. The configuration of the numbers on a die demonstrates how higher numbers can be recognised. This does not mean however that if five in the configuration on a die is recognised as a ‘five’, it also has the concept of a cardinal number for a child. It is a step in the direction of such a concept, but it is not enough. Making various buildings with five cubes and comparing the results with buildings of four cubes may help to constitute the concept of the cardinal number 'five'.

The forming of the concept of the cardinal number 'four' can be stimulated by blocks consisting of three and four cubes such as shown in the picture underneath.



If children are invited to play with these blocks they will after some time certainly call the blocks consisting of four cubes 'fours' and those consisting of three cubes 'threes'. In this way passing from the visual level to the descriptive level of arithmetic can be encouraged.

## An Example of a Level Transition in Algebra

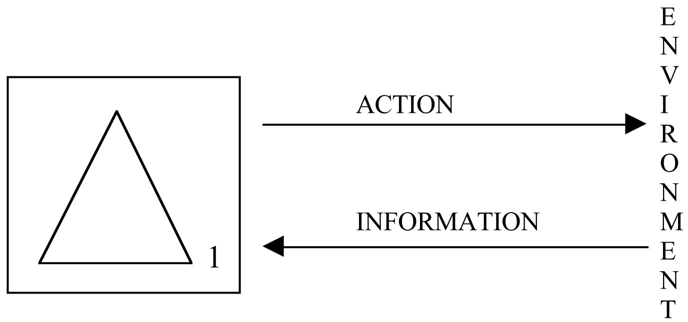
The transition from arithmetic to algebra can not be considered the transition to a new level. Letters can be used to indicate variables, but with variables children are acquainted already. Letters can be used to indicate an unknown quantity, but this too is not new. It is possible to introduce a theoretical level of algebra in secondary school but meeting such a case is exceptionable. In 1987 I encountered a secondary school in New York where algebra was started with groups. The group was defined by axioms and definitions and no examples of numbers and addition or multiplication were given. Elements were just elements, no more and the operator was just given by the necessary axioms and indicated by *. This seemed to me the most direct way to cause instrumental learning.

The normal instruction of algebra is just a continuation of arithmetic; then no problems are expected unless the children are overloaded with information.

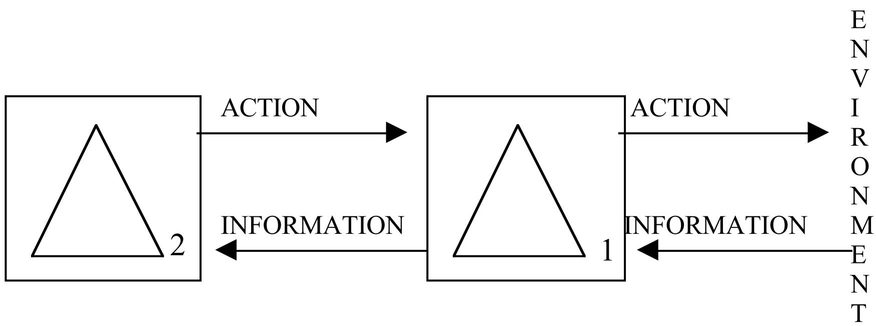
## Six Kinds of Understanding in Skemp's Theory

In the paper: 'Goals of Learning and Qualities of Understanding' Skemp (1979) elucidates his theory so clearly that it is a good starting-point to compare his theory of learning with mine. In the beginning of his article he gives his well-known picture of a 'goal-directed activity operating on the physical environment'.

> For goal directed activity operating on the physical environment, we have a director system, delta-one, which receives information about the present state of the operand (what is being acted on), compares this with a goal state, and with the help of a *plan* which it constructs from its available *schemas*, takes the operand from its present state to its goal state and keeps it there. We may, if we like call delta-one a sensori-motor system.




> Delta-two is another director-system, with a difference. Its operands are not in the outside environment, but in delta-one. They are not physical objects but mental objects. *The functioning of delta-two is to optimise the functioning of delta-one.*




According to the diagrams above Skemp discerns two modes of mental activity: *intuitive* and *reflective*.

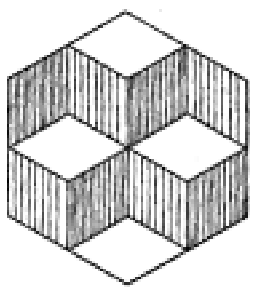
This is just in accordance with my theory: The first level, the visual one, is intuitive. In my view however reflection does not elevate thinking to a higher level. But in his paper Skemp too objects to identifying reflective mental activity with a higher level. Later in his paper he presents the following model:

**KINDS OF UNDERSTANDING**

| | | INSTR. | REL. | LOG. |
|---|---|---|---|---|
| **MODES OF MENTAL ACTIVITY** | INT. | $I_1$ | $R_1$ | $L_1$ |
| | REFL. | $I_2$ | $R_2$ | $L_2$ |

The two kinds of mental activity: *intuitive* and *reflective* are both divided into *instrumental*, *relational* and *logical* and with this we have six different modes of mental activity.

In this model $I_1$ represents a mental activity just directed by delta-one without interference of delta-two. The designation ‘instrumental’ for such understandings has a too negative meaning: in daily life we see an object in a wrong place and we put it in the right one. Such an action belongs to delta-one but is quite adequate.



I asked my daughter of four years old: “What do you see in this picture?” “Oh, those are my blocks.” “How many blocks are there?” “Three, no four, no five, no six!” It was clear that she saw the ambivalence in the picture, one can consider the white rhombs as three tops or three bottoms of cubes. I think this is a good example of $R_1$. It belongs to a direct seeing of delta-one ‘directly assimilated to an appropriate scheme’.

I must interrupt this analysis here for a discussion of my view on ‘intuition’. Just like Skemp I consider intuition to be the beginning of all thinking. Essentially it is non-verbal. Language can help non-verbal thinking to be recalled at appropriate moments and to nominate phenomena that have been observed. Most of human life is filled with non-verbal thinking: if we look around us, we observe many things, we may draw conclusions without using words in our mind. We are called by the telephone, someone speaks to us without mentioning her name and we say: “Oh it is you, Annely”. A voice, a face, all are recognised by their structure without interference of a language. It is not new what I emphasise here, but it is important and in psychology it is too often overlooked.

At the base of all our thinking lays intuition, (non-verbal thinking), so we want to know what the leading powers of intuition are. Here I propose to use ‘structure’. My daughter saw the picture, the structure of it reminded her of blocks and thus she could give adequate answers. The possibilities of a structure are determined by the prop-erties of structure.

In my book *Structure and Insight* (1986) I have written down those properties:

1. It is possible to extend a structure. Whoever knows a part of the structure also knows the extension of it. The extension of a structure is subjected to the same rules as the visible part of it.

2. A structure may be seen as a part of a finer structure. The original structure is not affected by this. The rules of the game are not changed, they are only enlarged. Thus it is possible that more details build up the structure.

3. A structure may be seen as a part of a more inclusive structure. This more-inclusive structure also has more rules. Some of them define the original structure.

4. A given structure may be isomorphic with another structure. In this case the two structures are defined by rules that correspond with each other. So if you have studied the given structure, you also know how the other structure is built up.

Some psychologists were astonished about the fact that I have dug up the structure (Gestalt) psychology again, but it is helpful to understand animal thinking and to extend the human non-verbal thinking.

A striking example of $L_1$ I met some years ago. It is a problem posed by T. Ehrenfest-Afanassjewa. The problem is: ‘If you look in a mirror you see that right and left are exchanged. This however is not the case with below and above. Why not?’ A boy of ten years gave an excellent answer: ‘It is not true that right and left are exchanged, it is the front and the back that are. ‘In his mind he saw a person standing in front of the mirror and seeing this situation he saw the solution. For a boy of ten it is courageous daring to say that an assertion is not true.

In my model, reflective thinking is necessary to attain a higher level, but it is not sufficient. Reaching a higher level is possible with help of adequate material which stimulates reflection in the right direction. Skemp observed in children a dislike to reflect but there are many reasons for such a dislike. First the teachers and the system in which they live are the cause of it. Reflection leads to a new network of relations between concepts but if a child does not know the structure of the topic the teacher is aiming at it takes much time to bring it to the right reflection. The teacher looking at his time-table is doubtful about the time available for such a reflection. After a short time the pupils too understand that reflection will be only moderately appreciated.

The examples Skemp mentions in his article about $I_2$, $R_2$ and $L_2$ do not have any relations with a level transition. They are part of algebra in which topic, as I have emphasised before, normally level transitions do not occur.

For $I_2$ he gives the example of someone who has to differentiate $1/x^3$. He does not know how to differentiate such a quotient, but he remembers that $1/x^3$. After that he uses the rule:

$$y = x^n,$$
$$y' = nx^{n-1},$$

and so he gets:

$$y = x^{-3},$$
$$y' = -3x^{-4}.$$

Skemp objects to such an instrumental action, He would like the student to verify it by some extensive mathematical schemes for the differentiation of fractions. I would be content with the above calculation if I were sure that the student once in his study had comprehended a series of proofs demonstrating that the formula for the differentiation of powers is true for all real exponents. A great part of algebra *is* an instrument and it is recommendable to check it is reliable before using it.

## Strong and Feeble Structures in my Theory

In some of his papers Skemp gives the impression that mathematics is a perfect way of thinking. I am not convinced that mathematical thinking is in most respects superior to intuitive thinking. Mathematics is of great importance in the theory of physics, chemistry and technology but there are many disciplines in which mathematics is of little importance. Sometimes we must even suspect that after introducing mathematics in such a discipline it has become less reliable. An example is test psychology.

The attractiveness of mathematics is that it has a strong structure. With a given set of premises you can be practically sure of the conclusion, so, if you have translated your problem in a mathematical model you can expect a univocal result. The main problem however is how to translate the original structure in a mathematical one. One of the most reliable laws of physics is Archimedes law on upwards force in fluids. It seems that this law has a universal validity and with the proof of Stevin we can even understand why the law is true. We must however not apply the law to a sugar cube in a glass of water for sugar dissolves in water. This exception must not be labelled as childish for it belongs to a group of a more serious character and we must be careful to take them into account. In the proof of Stevin it is supposed that the forces of the water on the object are the same as the forces of the water on the piece of water the object has replaced. With a sugar cube this is not the case.

I need not go on providing examples from physics in which the use of mathematics has to be adjusted. Physics is full of wonders, thus repeatedly unexpected things happen. Sometimes it was already in the mathematical support but very often it generates a new piece of mathematics.

By extending the theory we are able to understand more and more parts of physics in an increasingly exact way. Sometimes a practical application of the theory fails because of human errors. Many of those can be explained by a wrong evaluation of the constituent conditions. Very often these are brought in by intuitive, non-verbal thinking. This mental activity underlies all thinking, but we also rely upon it when the theory lets us down.

Many structures in human life are so strong that it is easy to transform them to pieces of mathematics. But other structures are so feeble that it is useless to apply mathematics to them. For instance the human voice. Many people who call you by telephone do not need to call their name because you recognise the voice directly. But you can not explain how you do it. Yet mistakes occur: once my wife left the house after saying: “See you later.” After a quarter of an hour I heard her voice again. “I didn’t expect you so soon,” I said, but it was not her, it was her sister. The same goes for the recognition of a face: you see that someone is your friend without being able to give an accurate description of him.

Whether we like feeble structures or not, yet we have to deal with them. On a trip in India I said to a lady behind me in the car: “I just saw a hoopoe”. “Yes”, she said, “There were two of them.” “It was the second time in my life I saw one,” I said. “Oh for me it was the first time.” How did we know that it was a hoopoe? Only by having seen pictures in a book. The recognition of birds only by seeing is a feeble structure but the study of bird migration would be impossible if we could not use this structure.

Driving a car is working in a feeble structure. The signposting along the road may be more or less strong, but the behaviour of the other road users causes the whole situation to have an apparent feeble structure. Most car drivers have learned to work with this structure and because of that we have relative few car accidents.

The main part of psychology has a feeble structure. We can infer this from the totally different descriptive levels that exist in the various parts of psychology. The psychology of thought developed by Selz has quite another structure than the developmental psychology of Piaget and these two structures have little resemblance with the structure of test psychology. But in each of those psychologies the structure is feeble. It may be that in test psychology only one answer is considered as right but here an originally feeble structure has been replaced by a strong structure. If the tester really wanted to know whether a given answer is right or wrong, he would ask for the reasons why the answer has been given. It may be that the given answer has a greater value than the one he himself has in mind.

Feeble structures ought to deserve special attention. They descend from the non-verbal intuitive thinking and therefore fill a great part of our daily life. Often they are the origin of an important discipline on a higher level of thinking which may have a strong structure or perhaps still a rather feeble one. If disciplines with a feeble structure would be rejected an important part of human thinking would be discarded.

## Summary

a. I share Skemp's opinion that 'instrumental' thinking can not be considered as real thinking.
b. The main cause of instrumental thinking in algebra is information overload.
c. In most disciplines there are different levels of thinking: the visual level, the descriptive level and the theoretical level. Each level has its own network of relations and its own judgments of truth. The transition from one level to the next can not be realised by reasoning or reflecting.
d. Just like Skemp I see intuitive thinking as the base of all rational thinking.
e. Piaget has found an important level transition in arithmetic. On the visual level children may be able to count but this does not imply that they have a concept of the cardinal number. On the descriptive level they have.
f. With the help of suitable blocks the transition from the visual level to the descriptive level in arithmetic can be stimulated.
g. Geometry can be started in young children. Therefore material must be introduced by which they can see geometry as a game.
h. No new subject ought to be introduced in primary and secondary school if the child does not like it.
i. Many properties of thinking can be understood by using the concept of 'structure'.

j. Many disciplines are based on feeble structures. Very often the usefulness of such disciplines will not be improved by introducing mathematics in them as a strong structure.

## References


Koning, J. (1948). Enige problemen uit de didaktiek der natuurwetenschappen, *in het bijzonder van de scheikunde.* Dordrecht.

Skemp, R. R. (1976). Relational understanding and instrumental understanding, *Mathematics Teaching*, 77, 20–26.

Skemp, R. R. (1979). Goals of learning and qualities of understanding. *Mathematics Teaching*, 88, 44–49.

Skemp, R. R. (1982). Symbolic understanding. *Mathematics Teaching*, 99.

Skemp, R. R. (1986). *The Psychology of Learning Mathematics (2nd edition)*. Harmondsworth: Penguin.

van Hiele, P. M. (1955). *De niveau's in het denken, welke van belang zijn bij het onderwijs in de meetkunde in de eerste klasse van het* V.H.M.O. Paed. Stud. Groningen.

van Hiele, P. M. (1958). Piaget's Beitrag zu unserer Einsicht in die kindliche, *Zahlbegriffsbildung Westermann*, Braunschweig.

van Hiele, P. M. (1986). *Structure and Insight. A Theory of Mathematics Education*. London: Academic Press Inc.

van Hiele-Geldof, D. (1957). *De Didaktiek van de Meetkunde in de Eerste Klas van het V.H.M.O. (The didactics of geometry in the lowest class of the secondary school),* Groningen.

Wirszup, I. (1974). *Some Breakthroughs in the Psychology of Learning and Teaching Geometry.* Address given at the closing of the general session of the 52nd annual meeting of the National Council of Teachers of Mathematics. Atlanta City

# Richard Skemp and his Conception of Relational and Instrumental Understanding: Open Sentences and Open Phrases

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I met Richard Skemp for the first time during the second ICME Congress in Exeter, UK, in 1972. We both attended a workshop devoted to the Psychology of Mathematics Education which was organised through the initiative of Hans Freudenthal. The discussions between the participants touched a host of extremely interesting problems, both theoretically and didactically relevant. We felt that mathematics education could not genuinely improve its achievements without carefully and systematically investigating the psychological dimension of mathematical reasoning, teaching and learning.

Also among the participants was Richard Skemp, who proved to be one of the most competent in the domain. It was obvious that he was a professional mathematician, but with a very fine and deep understanding of the psychological aspects of mathematical activity. He used terminology which was then relatively new in mathematics education – for example, intuitive and reflective intelligence, schemata and their role, mathematical solving strategies, the interaction between concepts and imagery in mathematical reasoning, etc. We continued the discussions after the regular sessions of the workshop and I was delighted to discover a man who was exceptionally original and fresh in his thoughts. There were not – and are not today – many people in the world so competent in both mathematics and psychology. This was Richard Skemp. He mentioned to me his book (first printed in 1971), The *Psychology of Learning Mathematics* and also had the kindness to send me a copy of it. Since then the book has been greatly used by my students and myself.

In 1979, Richard published his second book in the same area, *Intelligence, Learning and Action*, in which he continued to develop and deepen his insights – based on both his didactical experience and reflections. Richard Skemp's ideas inspired, and continue to inspire, my work in the domain of mathematics education. E. F.

## Relational and Instrumental Understanding

In his second book, Richard Skemp refers explicitly to two basic forms of learning and understanding: relational and instrumental. He writes:

> Relational understanding in a mathematical situation consists of recognizing a task as one of a particular class for which one already knows a rule. To find the area of a rectangle, multiply the length by the breadth.
>
> For a triangle, calculate half time the base time the perpendicular height. If the figure is a parallelogram, multiply the length of one pair of parallel sides by the perpendicular distance between them. The kind of learning which makes this possible, which I call instrumental learning, is the memorizing of such rules.
>
> (Skemp, 1979, p. 259)

Instrumental learning (and understanding) represents then, the assimilation of a certain rule somehow isolated from a larger meaningful context. The rule of action is triggered by a certain task, a certain situation.

The advantage of storing such rules mentally is the possibility of translating them more or less automatically into reactions. The disadvantage is that considering a broader context, the reaction may reveal itself to be inadequate. This broad context is represented by *relational learning* and *understanding*.

"Relational understanding, in contrast, writes Skemp, consists primarily in relating a task to an appropriate schema" (Skemp, *ibid*) In his first book, Skemp describes a schema as "a mental structure". The term includes not only the complex conceptual structures of mathematics which coordinate sensori-motor activity. A schema has two main functions. It integrates existing knowledge, and is a mental tool for acquisition of new knowledge" (Skemp, 1971, p. 391). See also Skemp's paper published some years later (Skemp, 1976).

The integrative function of a schema is exemplified by Skemp in the following way:

> When one sees a car, one identifies the respective object by relating it to the class of cars. But the process of integration may go further: We may ask ourselves questions regarding the safety of the respective category of cars, speed, cost, etc. This process of integration may help us to estimate the value of the car, to take a decision with regard to its possible acquisition, etc.
>
> Schemata then, make possible, the process of integration of a certain stimulus in a class of stimuli and thus identifying them. Schemata tend to organize themselves in hierarchical structures. This way our mind does not work as a

> conglomerate of independent acquisitions, but as a systematic, meaningful, associative, adaptable, complex behavioural device. In instrumental learning (and understanding), one tends to adapt the reactions – according to particular, non-integrated rules – to particular elementary stimuli. In relational learning (and understanding) the reactions are controlled by complex schemata by a broad, integrative system which connects, meaningfully, the various aspects relevant to the situation.
>
> Another important role of schemata mentioned by Skemp is that they are also "...indispensable tools for the acquisition or further learning. Almost everything we learn depends on knowing something else already."
>
> (Skemp, 1971, p. 40)

Piaget has also insisted on the adaptive role of schemata: A child cannot learn the operation of arithmetical addition for instance, if his mind has not reached – by maturation and exercise – the general ideas of arithmetical addition (as an interiorization of the act of combining two sets).

Both functions of schemata – the integrative one and the assimilatory ones – are of fundamental importance for the learning of mathematics. But the present paper focuses especially on the integrative role of schemata. Our main theses are: a) Mathematics – as an integrative intellectual system of concepts and operations should be taught by strengthening – in the didactical process – the structural-integrative nature of the mathematical-conceptual system; b) the school reality is that this aim is not fulfilled in the process of mathematical education. Students learn mathematics, mainly as a conglomerate of non-integrated concepts, definitions, theorems and solving rules, and c) teachers are usually not aware of, and not concerned with this aspect of the mathematical knowledge of their students because, usually, the questions used in tests point to how well their students are able to solve a certain class of problems according to a certain standard strategy. The rationale, the theoretical deductive justifications are not invoked and not required.

The teaching of mathematics is generally reduced to *instrumental* learning and understanding. It does not require, basically, the *relational* type of learning and understanding. The effect is disastrous. First, many students do not get an integrative image of mathematics, that is an image in which axioms, theorems, definitions, general concepts and properties control the interpretation and use of more specific concepts and solving procedures. Secondly, and as an effect of the previous phenomenon, even the particular procedures ("the instrumental" aspect) are often mistakenly applied under the impact of superficial similarities. This is, in fact, the fundamental thesis we have learned from Skemp: *relational and instrumental understanding should be structurally integrated in every mathematical activity.* If one reads Skemp's papers and books, one feels strongly that this is the image. Skemp does not only tell us about the general properties, principles and concepts of mathematics. In his work, the general analysis (psychological and mathematical) is combined with analysis of particular problem-solving processes and the respective difficulties and obstacles.

We would like to add to what has been said above, an example from our own work. The example refers to geometry. Students were presented with various types of parallelograms (oblique ones, squares, rectangles) and other figures. They were asked to define the term "parallelogram" and to identify among the figures presented, those which were parallelograms. We found that 90% of 10th grade students were able to define the term correctly, but only 68% defined *and identified* correctly, the parallelograms. For many students a square was *not* a parallelogram. In grade 11, 88% defined correctly, but only 65% defined *and* identified correctly, all the parallelograms. Relational learning would imply, in the above example, that the students possess the schema (the concept) of a parallelogram, *and* are also able to apply it in every particular case, *despite the figural, apparent discrepancies.*

For many students, the *particular* association of a certain category of figures with the corresponding definition (instrumental learning) is stronger than the schema, the concept of parallelogram which should control the consistent application of the schema (Fischbein, Nachlieli, 1997). As Skemp has written:

> The process which we subjectively experience as understanding is conceived, in terms of the present theory, as making connections with and appropriate schema ... The importance of understanding is that it brings new experience into the domains of our director system. Understanding is an essential contributor to their functioning, to the achievement of our goal states and thereby to survival.
>
> (Skemp, 1979, p. 165)

## The Present Research

The research described in the present paper is intended to illustrate the above ideas concerning the respective role and properties of instrumental and relational learning and understanding in mathematics.

A special methodology is required. Students may be able to answer correctly standard questions, and by this, mask the absence of a structural understanding – in Skemp's terminology an adequate schema. We have used the term structural schema for this kind of schema with general relevance.

In the present investigation, the central concept considered is that of "equivalence" and specifically, "the equivalence of equations". The question we have asked was: Do students, who are able to solve a simple linear equation, understand (in the sense of Skemp) what and why they are acting as they do? Do they understand that the successive transformations of equations until the final step, $x = a$ are performed under a number of constraints – the aim of which being to conserve the equivalence of the various steps of the transformation?

Our hypothesis is that many students do *not* possess the concepts of equivalence and equivalence of equations, that their knowledge is basically instrumental. Consequently they will be ready to produce, in certain circumstances, transformations which would *apparently* be safe as an effect of superficial similarities, but which would be, as a matter of fact, incorrect. This would prove that these students, lacking the control of the adequate schemata (which would guarantee a relational type of understanding) proceed, in fact, instrumentally, reducing their operations to elementary instrumental steps. We considered that our findings, if confirming our hypothesis, would, in fact, point to a more general deficiency in mathematics education: mathematics education is usually reduced to an instrumental, procedural type of learning and thus it misses its fundamental aims.

The central technique used by us has been to compare the behaviour of students (grades 9 and 10) in solving linear equations (open sentences) and in dealing with similar expressions, but which do not contain the equality sign (open phrases).

For instance, a student is asked to solve the equation:

$$\frac{x}{2} + \frac{x-3}{4} - \frac{5x}{8} = 1$$

He solves correctly, producing the successive correct transformations. The same student is asked to simplify the expression

$$\frac{x}{2}+\frac{x-3}{4}-\frac{5x}{8}$$

He is looking for the common denominator (as he does when solving the equation) and he writes

$$4x+2(x-3)-5x.$$

The common denominator is eliminated just as for the the equation, where the elimination is justified.

This student has solved automatically. Because he has not learnt algebra in a relational way, but merely in a instrumental spirit, his procedural steps have also been negatively affected under certain circumstances.

Several authors have studied the degree of understanding of the concept of equivalence of equations. Their main finding was that many students, even when able to perform adequate transformations for solving equations, were not aware of the underlying conceptual constraints. Especially, it was the concept of equivalence of equations which was deficient (Davis & Cooney, 1977; Kieran, 1984; Steinberg, Sleeman & Ktorza, 1991). Steinberg, Sleeman and Ktorza concluded that:

> Many of the students in the study knew how to use transformations to solve simple equations, yet many of them did not use this knowledge to judge that the use of a simple transformation gives an equivalent equation.
>
> (Steinberg, Sleeman & Ktorza, 1991, p. 119)

The above authors used pairs of simple linear equations and the student had to decide whether the equations of the pair are equivalent and justify their answer. For instance:

$$3x=5+4 \text{ and } 3x-4=5+4-4.$$

In our research, the students were, generally, asked to compare transformations performed on equations (in order to reach the solution) with transformations performed on similar algebraic expressions, which were *not* equations (open phrases). As already mentioned, the central idea of the technique used, was that students who are able to solve an equation but are not aware of the justifying properties will also tend to use, by analogy, but inadequately, the same transformations in the case of similar open phrases.

## Methodology

*Subjects*

The subjects were students enrolled in grades 9 and 10 – 52 students in grade 9 and 53 students in grade 10. In each grade, two levels of mathematical competence were considered: group A, the high level and group B, the lower level. In all, the research population was consequently divided into four groups: 30 subjects in group 9A and 22 subjects in group 9B; 30 subjects in group 10A and 23 subjects in group 10B.

The levels of competence were established by the school itself and are expressed in the quantity and subject-matter of the mathematics taught to the respective students.

The schools in which the research was conducted were situated in the centre of the country in a region with a population of average and high socio-economic status.

*Instruments*

a) A questionnaire was administered in which equations and open phrases appeared. The subjects had to estimate whether the operations performed (and indicated in the questionnaire) were mathematically correct. The central concept was that of equivalence. The subjects had to estimate whether the successive steps of successive transformations were correct and led to equivalent expressions. After each question, the subjects were asked to justify their answers.
b) A number of subjects, different from those to whom the questionnaire was administered, were interviewed with regard to the same questions. We preferred to interview different subjects because it was assumed that those students who already had to cope with the respective questions, may have been interested in inquiring further (asking colleagues or teachers) with respect to the correct answers.

*Hypotheses*

a) The main hypothesis of the present research was that many students (in line with Skemp's theory) perform and estimate the successive algebraic transformations as mere technical operations and *not* under the control of structural schemata – in the present case the general concept of equation with its laws of equivalence and transformations. More specifically, we assumed that many student will apply particular rules of transformation adequate in solving equations, also to open phrases in which such rules lose their legitimacy.

b) A second hypothesis was that age and level of mathematical competence have an impact on the students' mathematical behaviour. Namely, we have assumed that students in grade 10 and students in group A would perform better than younger and weaker subjects.

The superiority will be expressed in the stronger control exerted by the structural schema of equation (with its properties and rules). As already mentioned, the concept of equivalence is at the centre of this competence.

## Results

Let us first consider a number of equations as they appear in the questionnaire. The subjects are asked whether the transformations are valid ones. In Table 1, there are three such equations given, along with the percentages of correct evaluations. For instance:

"Given $\frac{5x}{3}+1=0$

Tal writes:

$$\frac{5x}{3}+1=0 \Leftrightarrow 5x+3=0$$

Is Tal correct? Yes/No. Justify your answer."

Table 1 contains the results obtained along with the three equations and their transformations. Except for the weaker students in grade 9, almost all the students in groups 9A, 10A and 10B estimated correctly that the transformations were legitimate.

The correct answers to all three questions should be "yes" ("The transformations are correct"). Inspecting Table 1, one may conclude that most of the students, except those in group 9B, seem to have a good command of the equation concept and that they are able to evaluate the correctness of the presented transformations.

Table 1

*Percentages of Correct Answers*

$$\frac{5x}{3}+1=0 \Leftrightarrow 5x+3=0$$

| Grade 9 | | Grade 10 | |
|---|---|---|---|
| A | B | A | B |
| 97 | 50 | 90 | 91 |

$$\frac{x}{2}+\frac{x}{3}=10 \Leftrightarrow 3x+2x=60$$

| Grade 9 | | Grade 10 | |
|---|---|---|---|
| A | B | A | B |
| 97 | 50 | 90 | 87 |

$$\frac{x}{2}+\frac{x-3}{4}-\frac{5x}{8}=1 \Leftrightarrow 4x+2x-6-5x=8$$

| Grade 9 | | Grade 10 | |
|---|---|---|---|
| A | B | A | B |
| 93 | 46 | 90 | 91 |

That is, in terms of schema one may assume, from the above data, that the respective students possess both the structural schema of equation and the specific procedures leading to the solution.

The questionnaire also presented two questions in which the transformations were incorrect (see Table 2a and 2b):

2a) "Given $p(x)=\frac{2-x}{3}+\frac{2+x}{2}+4$

Dan has written:

$$p(x)=\frac{2-x}{3}+\frac{2x+x}{2}+4 \Leftrightarrow 2(2-x)+3(2+x)+24$$

Was Dan correct? Yes/No."

It is evident that the two expressions are not equivalent. The common denominator has simply disappeared. This time many students did not answer correctly (see Table 2, a and b). That is, they accepted the elimination of the common denominator.

2b) “Given $\frac{5}{3-x}+\frac{7}{3+x}=y$

Dan wrote:

$$\frac{5}{3-x}+\frac{7}{3+x}=y \Leftrightarrow 5(3+x)+7(3-x)=y$$

Was Dan correct? Yes/No.”

Many students answered incorrectly that the transformation was adequate (Table 2, question 2b).

Table 2

*Percentages of Correct Answers*

| Question 2a | | | | Question 2b | | | |
|---|---|---|---|---|---|---|---|
| Grade 9 | | Grade 10 | | Grade 9 | | Grade 10 | |
| A | B | A | B | A | B | A | B |
| 27 | 45 | 60 | 22 | 67 | 27 | 63 | 50 |

It is clear that many students, in estimating the correctness of the transformations, mechanically apply *some* rules suitable for equations without the general conceptual control, that is the relational understanding in Skemp’s terminology. In question 2a) one deals with an open phrase which does not allow the elimination of the denominators. In question 2b) one has an equation. This time the elimination of the denominators would be possible with the condition that the right side of the equation (y) is also adequately multiplied.

These results contradict our former assumption that most of the students possess the structural schema of equation. The correct transformations they perform looking for the solution are *not* controlled, tacitly or explicitly, by the formal constraints which justify these transformations. (For instance, that the two sides of an equation may be multiplied by the same number and thus obtaining an equivalent transformed equation.)

Except for the question 2a) (Grade 9) the highly competent students performed better. The age level does not seem to be relevant. The transformations accepted by these students express only particular sets of technical procedures.

Consequently, one has to admit that the mathematical knowledge of these students is not genuine mathematics, but a set (or sets) of procedures. Mathematical knowledge implies a structural organization in which general, formal constraints (axioms, theorems and definitions) control and justify thoroughly the validity of all the operations and transformations performed.

But even when considering the highly competent students in both grades 9 and 10, one finds that only about 60% of the subjects give correct responses when the aspect of the problem does not fit the standard exercises. The common denominator simply disappears without the subjects considering the validity of the transformations.

The questionnaire included more open phrases together with some transformations. The subjects had to answer whether the transformations were performed correctly.

a) $\frac{5x}{3}+1 \rightarrow 5x+3?$

b) $\frac{x}{2}+\frac{x}{3} \rightarrow 3x+2?$

c) $\frac{x}{2}+\frac{x-3}{4}-\frac{5x}{8}+1 \rightarrow 4x+2x-6-5x+8?$

Most of the students gave wrong answers (see Table 3). They relied on the equation model in which the common denominator of *both sides* is eliminated (which is equivalent to multiplying both expressions with the same number). Evidently, the disappearance of the common denominator in an open phrase is not justified and this, most of the students did not know. But even in the case of an equation in which the right side is a simple letter or number, many of the students did not adequately perform the multiplication.

Table 3

*Percentages of Correct Answers*

| Transformation a | | | | Transformation b | | | | Transformation c | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Grade 9 | | Grade 10 | | Grade 9 | | Grade 10 | | Grade 9 | | Grade 10 | |
| A | B | A | B | A | B | A | B | A | B | A | B |
| 13 | 32 | 43 | 17 | 17 | 41 | 47 | 13 | 33 | 27 | 40 | 31 |

The next questions were intended to check whether the subjects could distinguish between open sentences and *similar* open phrases. Therefore, pairs of questions were presented. In each pair, there was one equation (for instance: $\frac{x}{2}+\frac{x}{3}=10$) and a similar open phrase (for instance: $\frac{x}{2}+\frac{x}{3}$ ) Do the students distinguish between them so as to understand that the treatment of the two expressions should be different? We present the results separately for each pair. The subjects were asked if the transformation corresponds to the mathematical rules. The term "correct" implies that the subjects answered correctly to both questions (that is, "yes" for the first question and "no" for the second).

Clearly enough, $3x=9$ yields $x=3$, so the answer "yes" is correct. On the contrary, $3x+9$ cannot be transformed into $x+3y$. The division by 3 is not admissible. So, in this second example, the correct answer should be: "no".

Table 4a

*Percentages of Correct Answers*

| Equation | | Open Phrase | |
|---|---|---|---|
| $3x=9$ (given)<br>$x=3$ | | $3x+9y$ (given)<br>$x=3y$ | |
| Grade 9 | | Grade 10 | |
| A | B | A | B |
| 24 | 23 | 40 | 28 |

Table 4b

*Percentages of Pairs of Correct Answers*

| Open Phrase | | Equation | |
|---|---|---|---|
| $\frac{x}{2}+\frac{x}{3}$ (given)<br>$3x+2x=5$ | | $\frac{x}{2}+\frac{x}{3}=10$ (given)<br>$3x+2x+60$ | |
| Grade 9 | | Grade 10 | |
| A | B | A | B |
| 30 | 13 | 33 | 26 |

Table 4c

*Percentages of Pairs of Correct Answers*

| Equation | | Open Phrase | |
|---|---|---|---|
| $\frac{x}{2}+\frac{x-3}{4}-\frac{5x}{8}=1$ | | $\frac{x}{2}+\frac{x-3}{4}-\frac{5x}{8}=$ | |
| $\Leftrightarrow 4x+2x-6-5x=8$ | | $8(\frac{x}{2}+\frac{x-3}{4}-\frac{5x}{8}+1)=$ | |
| $\Leftrightarrow x=14$ (given) | | $4x+2x-6-5x+8$ (given) | |
| Grade 9 | | Grade 10 | |
| A | B | A | B |
| 30 | 13 | 33 | 26 |

It is obvious from the data in Tables 4a, b, c that most of the students were not able to distinguish the two fundamentally different expressions. Most of the students were ready to accept that transformations suitable for equations (including the elimination of the common denominator) are also suitable for an open phrase (that is, where the equality sign did not appear).

All these findings lead us to the same conclusion. Most of the 9th and 10th graders (even those who belong to the "high competence" level) do not distinguish between an open sentence and open phrase. They do not seem to understand what they are doing. They do not seem to understand that mathematical transformations may be performed only by respecting the formal, basic constraints on which these transformations are based. It seems that for these students, the concept of equation as a structural mathematical schema (with its meaning and its properties) does not exist. *What does exist, is a number of particular procedures lacking any mathematical meaning*, that is, their understanding is mainly instrumental. The great contribution of Skemp in this respect was to analyse and clarify the profound connection between schemata (we would call them "structural schemata") and the process of understanding. "To understand" means basically, to assimilate certain information by integrating it in the person's own mental system. But in order to achieve this aim, certain hierarchically organized schemata are required. The concept of "understanding" cannot be genuinely understood if the nature and dynamics of schemata are not clearly elucidated.

*Interviews*

In order to check further this interpretation, we carried out a number of interviews. In this way we hoped not only to check the above findings, but also to get a more complete image with regard to the students' mathematical knowledge. Let us quote from a number of interviews. The subjects belonged to all four groups investigated: 9A and 9B; 10A and 10B.

*Interview No. 1: Student R – Grade 9B*. T stands for teacher; S stands for student.

T: Given the following equation: $\frac{x}{2} = \frac{x}{3} = 10$. Would you like to solve it?

S: I do not like to solve equations with fractions.

T: Nevertheless, please try.

S: O.K. I will eliminate the denominators. She writes: $3x + 2x = 10$.

T: How do you justify this transformation?

S: We have been taught this. This is the method.

T: Can you explain this method?

S: If I multiply $3 \times 2$ I get 6 and this is the common denominator.

T: Again, on what grounds do you perform this transformation?

S: We have learned.

T: Can you go on?

S: I add and I get $5x + 10$.

T: Can you go on?

S: I wanted to reach equality.

T: What do you mean?

S: I do not know more.

*The same student R:*

T: Consider the expression: $\frac{x}{2}+\frac{x-3}{4}-\frac{5x}{8}+1$. Can you simplify this expression?

S: Yes, to find the common denominator and to eliminate it.

T: Let us see.

The student writes: $4x+2x-6-5x+1=x-5$

T: On what grounds did you perform the transformation from the first expression into the second?

S: According to the formula. I do not act according to my personal view.

T: How do you know that this formula fits the data?

S: According to the examples in the classroom, because of the fractions. Always when there are fractions in a exercise, one should find the common denominator.

T: Can you check whether the expression you have written is correct?

S: I do not know how to check. If the common denominator is correct, then the solution is correct.

*Interview 2: Student S – Grade 9*: a very good student.

T: Given the equation: $\frac{x}{2}+\frac{x}{3}=10$. Would you like to solve it?

S: I have to multiply by 6. I have to find the common denominator and to eliminate the fractions.

She writes: $3x+2x=60$

T: On what do you rely when you operate this transformation?

S: On the fact that we may multiply or divide an expression by the same number. We have been taught this.

T: Can you explain more?

S: It is what we have been taught.

T: Can you go on?

$5x+60=5.$

S: The student writes

$x = 12$

T: On what did you rely when you divided by 5?

S: One may multiply and one may divide by the same number. We have been taught this. The teachers have decided what we have to know. I do not know more.

The student seems to understand what she is doing, that is, the formal rationale of the transformations operated on the given equation. Her competence is certainly higher than that of student R. But how high? Let us see the continuation of the interview.

T: Given the expression $\frac{x}{2}+\frac{x-3}{4}-\frac{5x}{8}+1$. Could you simplify that expression?

S: I can eliminate the denominator. She writes:
$4x+2(x-3)-5+8$.

T: On what did you rely when you transformed the given expression into the second one?

S: This is what I have been taught in order to eliminate fractions. I am acting according to rules.

T: According to which rules are you acting?

S: The rules for the elimination of denominators. The student continues to write:

$4x+2x-6-5x+8=x+2$

T: Can you check whether the expression you have obtained at the end results from a correct transformation?

S: It is an easy exercise. It is not necessary to check.

Let us remember that this student is noted to be a very good one. This is the illusion that many teachers live with. If the student used correct techniques for solving an equation (or any other type of problem) one may assume that he not only knows the rules, but that he also possesses the formal, structural constraints guiding and justifying these rules, that is, the corresponding "relational understanding". As a matter of fact, what we discover from the second part of the interview is that student S does not possess the schema of equation as a math-ematical, integral concept. Mathematics is a strictly deductive body of knowledge. Teaching rules, we do not reach mathematics. We simply teach rules. Knowing mathematics implies the possession of a ensemble of integrating and hierarchically organized schemata. Without understanding the deep meaning of the concept of equivalence – with its properties – one cannot understand, genuinely, the concept of equation and the meaning and limitations of its transformations.

*Interview 3: Student T – Grade 9A*

T: Given the equation $\frac{x}{2}+\frac{x}{3}=10$. Would you like to solve it?

S: First we find the common denominator and after this one multiplies accordingly.

The student writes: $3x+2x=60$.

T: On what did you rely when you operated this transformation?

S: We have been taught. We all act in the same way.

T: Would you like to continue solving the equation?

S: The student writes: $5x=60 \quad x=\frac{60}{5}$.

T: On what did you rely when you divided by 5?

S: One cannot divide by $x$. One has to find $x$.

T: On what did you rely when you did this?

S: According to what we have been taught.

T: (Writes): $\frac{1}{5}\times 5x=\frac{1}{5}\times 60$.

What is your opinion with regard to this multiplication?

S: It does not seem to me to be correct. I do not know this method.

T: Would you like to check this method I have shown you?

S: It does not seem to be correct. We have not been taught this way.

Here the student applies a rule, which leads to the correct solution. Like the former student interviewed, she has no justification for her procedure. Moreover, when the interviewer suggests the corresponding justification, the student does not agree. Not only does she not feel the

need for any formal justification, but when presented with the respective justification, she rejects it as something unconnected, unknown, unlike the usual procedure.
The same student: (T)

T: Given the expression: $\frac{x}{2}+\frac{x-3}{4}-\frac{5x}{8}+1$. Can you simplify it?

S: She writes: $^{4)}\frac{x}{2}+^{2)}\frac{x-3}{4}-^{1)}\frac{5x}{8}+^{8)}1$

$4x+2x-6-5x+8=0$

She explains: I have found the common denominator in order to eliminate it and I add and subtract. I have eliminated the common denominator in order not to be disturbed by it when I have to add or subtract.

T: On what did you rely when you acted this way?

S: All are doing the same. This means that it is correct.

T: Why have you introduced the equality with zero?

S: I think that we have to move the numbers of the left side to the right side.

T: But I have not written "equal with zero" in the given expression.

S: We are used to acting this way according to the quadratic equations we have learned. We write equal with 0 and we solve.

The student introduces a new element in our discussion which has not been revealed so far. She *needs* the zero in order to get the entire justification of this procedure. One may assume that what the students have in mind when eliminating – incorrectly – the common denominator, is *the model of equations* and the usual techniques for solving them. The model is usually applied and generalised tacitly. When some element is absent, the student simply produces it in order to fit the mental model.

In other terms: it seems that all the students act in conformity with the equation model when eliminating the common denominator in an open phrase. One may assume that the students do not act blindly without any justification. What guides their actions in "simplifying" an open phrase (finding and eliminating the common denominator) is tacitly justified by the equation model. It is not the structural schema of equation which inspires their operation, but some external aspects of it, the apparent similarity between the open phrase and a certain open sentence (an equation). Again, just because these students do not possess the structural schema of equation, with all its meanings, properties and rules of transformation, these superficial similarities may mislead them in their algebraic operations.

*Interview 4: Student U – Grade 10B*

T: Given the equation. $\frac{x}{2}+\frac{x}{3}=10$. Would you like to solve it?

S: $3)\frac{x}{2}+2)\frac{x}{3}=6)\ 10$

$3x+2x=60;\ \ 5x=60=;\ \ x=12$

T: On what did you rely when transforming the given equation into the equation $3x+2x=60$?

S: I have simplified using multiplication in order to eliminate the denominator and find *x*.

T: On what did you rely when performing this transformation?

S: I have been taught this way. I know it.

T: What have you done in continuation?S: I had to add the terms with *x*.

T: How did you get from $5x=60, x=12$?

S: Dividing by 5.

T: How do you justify this division?

S: When there is *x* multiplied by a number, one has to divide by this number.

T: On what do you rely when operating this division?

S: I have two answers:

1. Because mathematics is built this way.
2. I have been trained this way, and I do not know another method.

*The same student U:*

T: Given the expression: $\frac{x}{2}+\frac{x-3}{4}-\frac{5x}{8}+1$. Can you simplify it?

S: The student writes:

$$4x+2(x+3)-5x+8=4x+2x+6-5x+8=1x+14$$

T: On what did you rely etc.?

S: I wanted to eliminate the common denominator.

T: On what did you rely when performing the above transformations?

S: I have always been taught this way. In order to render the expression easier, more beautiful and more concentrated.

T: Is it admissible to use every method in order to get a more simple expression?

S: Everything is admissible in order to eliminate the common denominator.

The student's conception is that the purpose justifies the means! She does not know another justification. The deductive structure on which the respective operations rely simply do not exist for her. They probably never existed in the student's mind. What does exist, as we mentioned above, is the model, the equation model from which the conceptual, axiomatic content, the formal properties have been eliminated.

*Interview No. 5: Student V – Grade 10A*

T: $\frac{x}{2}+\frac{x-3}{8}-\frac{5x}{8}=1$

S: The student writes:

$$\frac{4x+(x-3)-5x=8}{8}$$

$$4x+x-3-5x=8$$

$$-3=8$$

$$0=11$$

T: On what did you rely when you performed the first transformation?

S: I have been taught this way. One looks for the common denominator.

T: On what did you rely for the next transformation?

S: It is the way in which one solves an equation, but I do not understand what happened.

T: Would you try to explain?

S: No.

*The same student V:*

T: Given the expression: $\frac{x}{2}+\frac{x}{3}$. Can you simplify it?

S: The student writes: $\frac{3x+2x}{6}=5x$

T: On what did you rely etc.?

S: One cannot simply add the two fractions. One has to find the common denominator.

T: On what did you rely when you eliminated the common denominator?

S: Because I have already used it and I did not need to any more.

This is really fantastic. A grade 10 student belonging to the high competence group eliminates the common denominator because “it has already been used” (as you use a pair of scissors, for instance and puts them away afterwards!) No deductive reasoning, no formal constraints and no justification! The student’s empirical way of reasoning is inspired, wrongly, by the equation model. Is this mathematical reasoning?

*Interview 6: Student X – Grade 9B*

T: Given the equation: $\frac{x}{2}+\frac{x-3}{4}-\frac{5x}{8}=1$. Can you solve it?

S: The student writes:

$$1)\frac{x}{2}+2)\frac{x-3}{4}=4)\frac{5x}{8}=1$$

$$x+2x-6-20x=1$$

T: On what did you rely when you operated the above transformation?

S: One has to find the common denominator.

T: Can you explain what you mean by common denominator?

S: Common denominator is something which is common for a number of things.

T: On what did you rely when you looked for the common denominator?

S: One has to eliminate the common denominators.

T: Can you explain more?

S: No, it is the way to solve it and that is that.

*The same student X:*

T: Given the expression: $\frac{x}{2}+\frac{x}{3}$. Can you simplify it?

S: (Writes): $3x+2x=5x$

T: On what did you rely when you passed from the first to the second expression?

S: I made the common denominator.

T: Can you explain more: On what did you rely?

S: I eliminated the denominator, it is what we do in class.

The same ignorance as so far. In addition, being a low competence student, her errors are expressed even in the techniques used when looking for the common denominator for the initial equation. But basically, with regard to the *justification* of the operations performed, there is no difference between a highly competent student in grade 10 and a weak student in grade 9!

*Interview No. 7 Student Y – Grade 10A*

T: Given the equation: $\frac{x}{2}+\frac{x-3}{4}-\frac{5x}{8}=1$. Can you solve it?

S: One has to multiply by 8. The student writes:

$$^{4)}\frac{x}{2}+^{2)}\frac{x-3}{4}-^{1)}\frac{5x}{8}=^{8)}1$$

$4x + 2x - 6 - 5x = 8$

T: On what did you rely when you passed from the first to the last equation?

S: One has to multiply both sides so that the relations do not change and one multiplies by the denominator in order to solve easier.

*The same student Y:*

T: Given the expression: $\frac{x}{2} + \frac{x}{3}$ Can you simplify it?

S: One cannot simplify.

T: One student called Bill has written $3x + 2x$.What is you opinion?

S: Not correct.

T: Can you explain?

S: Because it is not an equation.

T: Can you explain more?

S: There are no relations, there are no two sides and then there is no motive to eliminate the denominator.

T: Is it a problem of "motive"?

S: Yes. There is no need to do it.

Until the final sequence of the interview, one may have the feeling that this is a student who understands what she is doing. And yet! What does she claim? She does not affirm that one cannot eliminate the common denominator because this would imply that the expression is multiplied by 6. What she claims is that there is "no motive" ("Ein ta-am" in Hebrew) to get rid of the denominator. For her there is no problem of a formal constraint which prevents the elimination of the common denominator. It is a problem of empirical, technical utility. The entire mathematical knowledge of this student seems to be governed not by formal constraints, but rather by empirical, practical justifications.

*Interview No. 8: Student Z – Grade 10B*

The same equation as above. The student produces the common denominator and writes:

$$4x + 2x - 6 - 5x = 8.$$

T: On what did you rely, etc?

S: I have multiplied both sides and eliminated the denominator.

T: Why is it necessary to multiply both notes?

S: We have been taught. If one multiplies one side, one has to multiply also the other.

T: Can you explain why?

S: We always do the same and we do not ask why.

*The same student, Z;*

The expression: $\frac{x}{2} + \frac{x}{3}$.

T: Can you simplify it?

S: She writes: $3x + 2x = 5x$.

T: On what did you rely when you passed from the initial expression to the next one (the interviewer indicates $3x + 2x = 5x$)?

S: I have multiplied everything by the common denominator.

T: You cannot explain more on what did you rely?

S: One has to eliminate the denominator in order to obtain a simplified expression.

Let us try to summarise what we have learned from the above interviews. The responses and justifications do not vary much comparing a weak student in grade 9 with a strong, highly competent student in grade 10. All the students except one (student X, interview 6) were able to obtain the correct common denominator in the given equation. Focusing only on their ability to solve the equation – as teachers usually do in standard examinations – one may conclude that all these students except one possess the structural schema of equation with both its technical solving rules and its formal, axiomatic constraints. The data obtained using the questionnaire and those obtained from interviews point to the same conclusion. But when we pass to the open phrase, one gets a completely different image. Not one of the students interviewed has a complete understanding of what she or he is doing. To most of them, the same justification is: “We act this way because we have learned to act this way.” The student has in mind a model: the solution of an equation including fractions using a number of techniques. The fundamental idea that the procedures used are justified deductively by formal properties – this idea does not exist in the students’ minds.

## Discussion

At first inspection of the data concerning the solution of equations, one may conclude that at least some of the 9th and 10th grade students are able to perform the transformations requested for solving linear equations. They seem to understand that both sides have to be simplified, fractions eliminated, algebraic entities containing $x$ grouped and added at the left side and the numbers grouped and added at the right side until one gets an equality of the form $ax=b$ and thus obtaining the value of $x$ for which the equality is true.

Not all the students were able to go all the way correctly, but at least some of them, as we said, were able to do it. Solving equations – in our examples, linear equations – is an important part of algebraic competence. Things seem to be fine until we deviate from the standard type of questions. But when the students were asked to simplify expressions like:

$$p(x) = \frac{2-x}{3} + \frac{2+x}{2} + y \qquad \text{or} \qquad \frac{x}{2} + \frac{x}{3} \text{ etc.}$$

one obtains a fundamentally different picture. The students applied the same programme they were using when solving equations. They looked for the common denominator, they eliminated it, the equals sign appears, completing the image of an equation and thus the students reached an apparently simplified version of the given expression. As a matter of fact, those students who solved the equations correctly followed a programme of transformations *mechanically*: they simply did not know *what* they were doing, because they did not know *why* they were doing it; what justified their operations. The mathematical understanding of these students is, essentially, procedural, instrumental. In mathematics, if you don't know *why* you are doing something, you, in fact, don't know *what* you are doing. The first consequence is that there is high probability that the respective student will make mistakes in his technical solving endeavours. This is an extremely important consequence of Skemp's theory. If the relat-ional understanding is absent, the student does not remain, as a matter of fact, even with a solid instrumental competence. His instrumental competence is vulnerable to every shift, every modification of the context.

We attempted to describe the above situation in terms of schemata – structural and specific (procedural) schemata. The term schema signifies a programme, a cognitive structure, the behavioural aim of which is to process and integrate (interpret) information and to prepare, release and control adequate reactions. The entire adaptive activity of living organisms is made possible by such structures. Without schemata, without such *programmes* of interpretation and reaction, no understanding, learning and solving would be possible, as Skemp has emphasised.

We distinguish between structural and specific schemata. Structural schemata are basic, very general complex, cognitive behavioural structures. They are the "building blocks" of cognition (Rumelhart, 1980, p. 33) or of 'relational understanding' in Skemp's terminology. *They are pre-conditions for assimilating, understanding and reacting correctly in various domains of human activity.* The idea of causal relationships, the concepts of experimentation, of inductive, deductive, and conditional reasoning, of formal proof versus empirical proof, stochastic versus deterministic reasoning are examples of structural schemata.

Basic mathematical concepts like number, function, equivalence, equation, limit, the group concept, the concept of bijection, etc., are also examples of structural schemata. They are defined by formal properties and by rules of operation, deductively organized.

Specific schemata are those procedures by which a structural schema acts upon a specific content, leading to the solution which one has been looking for. In Skemp's terminology, they are expressed in instrumental learning and understanding. In the case of equations, the identification and elimination of a common denominator, the separation of the terms containing the unknown from the free terms, represent specific schemata.

Skemp did not distinguish explicitly between structural and specific schemata. But, in our opinion, his theory implies such a distinction (which in fact is never absolute). When Skemp speaks about relational understanding and the role played by schemata in the process of understanding, he in fact refers to what we call 'structural schemata': general, powerful, extensively relevant, integrating devices. 'Instrumental understanding' is reducible to specific, particular procedures.

If the specific schemata become autonomous and the general control (conceptual and procedural) of the structural schema is lost, the specific procedures become blind and may be used wrongly.

When the structural schema loses its control, there is another class of factors which intervene. *The impact of the structural schema – more complex, more demanding – is replaced by a lower level influence, the influence of a model, an analogy, a metaphor.* In our case it is the usual model of solving equations which intervenes. There is a superficial similarity between an open phrase and open sentence. The fact that one does not deal any more with a given equality does not disturb the student. The concept of equivalence, the idea of performing transformations which will lead to *equivalent* equations (having the same truth sample) is too subtle for an insufficiently trained mind. *It is the superficial similarity with the inadequate model which dictates the solving behaviour, and not the much more subtle idea of conceptual equivalence.*

This seems to be a general phenomenon: when the particular, specific, procedural schemata are no more under the control of a higher order adequate structural schema, it is, usually, a model, an analogy, a metaphor which takes over the control of the solving strategy. Certainly, models, metaphors are of highest importance in scientific and mathematical reasoning. But this, with the essential condition that the adequate higher order systems of conceptual constraints keep exerting their control.

In mathematics, there is a system of axioms, of general properties, theorems and definitions which must guarantee the validity of the subjacent interpretations and rules of operation.

One may claim, considering many of the students' responses to the items of the questionnaire, that they simply forgot to consider that, in some cases, they dealt with one-sided expressions – open phrases – and not with equations.

But the interviews present a more complete picture of what happened in reality. In these interviews the subjects were asked to *justify explicitly* the operations performed: on what formal basis did they perform the respective operations? Not one of the students interviewed, weak or strong, 9th graders and 10th graders, those who solved correctly and those who did not solve correctly, offered the required formal justification. Everything is carried out at the low level of instrumental understanding. What determines the students' behaviour is mainly composed of technical rules without a conceptual perspective, but influenced by a well rooted, familiar model. Let us remember the beautiful reaction of a student quoted above: Teacher: "On what grounds did you eliminate the common denominator?" Student: "Because it was no more useful." The justification was empirically correct, but the formal justification was absent.

We discussed the above problem with a number of teachers. Some answered that there is simply no time to explain *why* some transformations may be performed. Others argued that, initially, the formal properties and rules have been mentioned in the classroom, but later on one has not insisted any more on these formal justifications.
What we have found is not an isolated symptom. The respective students are, simply, not used to considering the operations they perform as based on a formal deductive system which constitutes the essence of mathematics. It is this deductive hierarchy, starting from axioms, theorems, definitions and leading to practical rules which characterises mathematical reasoning. Practical solving strategies are *not* mathematics. We think that this point should be made very clear.

It may be that one decides that students should not learn mathematics, but only systems of solving procedures. What we claim is that, in this case, one should be morally honest and affirm this loudly and explicitly: our society does not need mathematics – except for a few professional mathematicians.

Our view is different, in agreement with Skemp's view. Mathematics is a fundamental component of modern culture and students have to learn it as such – not only a number of practical solving techniques. Didactically, this does not imply that the student should be asked to mention at every step of an algorithm, the formal justification. He should be able to perform the algorithm automatically. What we claim is that the student should be able to justify formally, *on request*, the steps of his solving operations. This is relational learning and understanding.

This means that the teachers (and also the tests used) should ask the student, from time to time, to explain, to justify why, on what formal ground, does he act as he does. This behaviour should be included in the usual didactical strategy. If this is not done, the student will not learn mathematics and moreover, he will be exposed to the possibility of committing, also, very often, systematic technical errors.

## References


Davis, E., & Cooney, I. J. (1977). Identifying errors in solving certain linear equations *MATYC Journal,* 11*,* 170–176.

Kieran, C. (1984). A comparison between novice and more expert algebra students on tasks dealing with the equivalence of equations. In J. M. Moser (Ed.), *Proceedings of the Sixth Annual Meeting of the Psychology of Mathematics Education in North America,* Madison, WI, 83–91.

Rumelhart, D. (1980). Schemata: The building blocks of cognition. In R. T. Spiro, B. C. Bruce & W. F. Brewer (Eds.), *Theoretical Issues in Reading Comprehension.* Hillsdale, NJ: Erlbaum.

Skemp, R. R. (1971). *The Psychology of Learning Mathematics.* Harmondsworth, UK: Penguin Books.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teacher, 77,* 20–26.

Skemp, R. R. (1979). *Intelligence, Learning and Action.* New York: Wiley & Sons.

Steinberg, R., Sleeman, D. H., & Ktorza, D. (1991). Algebra students' knowledge of equivalence of equations. *Journal for Research in Mathematics Education,* 22(2), 112–121.

# Thinking in Metaphors and Metaphors for Thinking

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Human learning as the natural growth of a plant, understanding as constructing a map, mathematising as participating in a ball game, doing pioneering research as erecting a building in an empty field – these are but a few of many metaphors found in Richard Skemp's writings[1]. The parallels, analogies, images, and comparisons with which Skemp's books and papers are so tightly packed help in conveying the author's message, making understanding easy and transforming the reading into an unforgettable intellectual experience.

Indeed, there is something special about metaphors and the way they help. However, viewing them merely as an ancillary, explanatory device would be a mistake. Try to take the metaphors out of Skemp's writings – try to say whatever he says using only direct, 'literal' expressions – and you will soon find out that the task is daunting, if not outright impossible. Skemp's metaphors are much more than literary tricks: they are as substantial to his message as is colour and shape to a picture and as are sounds to a melody. In a sense, metaphors are the stuff of which Skemp's theories are made.

Following this last remark, a comment should immediately be added: Skemp's metaphors may be particularly felicitous and imaginative, but their appearance in a serious scholarly text is by no means an unusual phenomenon. According to what becomes more and more obvious to philosophers, linguists, and psychologist, using metaphors in any discourse, and in scientific discourse in particular, is only natural, indeed inevitable. In fact, we simply cannot do without them. At a closer look it turns out that analogies, parallels, and comparisons are essential to the progress of our thinking at large, and to the progress of science in particular. Let me say a few more words about it before turning to the analysis of some of Skemp's metaphors.

## Thinking in Metaphors

Quite often, when we choose a concept, say *verbal communication*, and then look carefully at the language in which we use to talk about it, we are able to notice a striking phenomenon: while there may be a great variety of common expressions – both colloquial and scientific – concerned with this concept, a sizeable subset of these expressions takes us in a systematic way to a certain well-defined domain which does not seem to be a "natural setting" for the notion at hand. Thus, for example, whether we talk about 'conveying ideas', 'delivering [getting] a message' or 'putting thoughts into words', we make it clear that our image of communicating is borrowed from the domain of *transport*: the words are conceived as vehicles and meaning is a cargo to be transmitted to the listener.

Similar phenomena can be noticed in any domain of knowledge. Let us take education as our next example. To begin with, it is easy to see the decisive role of metaphorical thinking in both Piaget's and Vygotsky's monumental works on the development of human intellect. It was Piaget himself who pointed out to the metaphorical roots of his theory when he admitted to his being inspired by the Darwinian idea of biological evolution. The now famous 'organic metaphor', according to which intellectual development may be compared to the growth of a plant, followed as a natural consequence. Richard Skemp may not be the only writer who pursued the idea, but he is certainly among those who managed to present it in the most explicit and compelling way:

> ... learning is like natural growth. A plant's development is a cooperation between the roots, which draw its nourishment from the soil, and the leaves which draw energy from the sun. The sun, the rain, and the temperature provide the environment, the soil provides the nutrient. In a similar way, intelligent learning requires a combination of several factors. The school has to provide the environment – not a desert, not an Arctic waste. It has to provide a micro-climate within which intelligent learning can take place. It also has to provide nutrients in the form of specifically devised materials, and this is the soil from which mathematical learning can draw its nourishment. (in Sfard, 1990).

This, indeed, is a perfect example of a systematic metaphorical mapping between two seemingly unrelated conceptual domains: To each element in one domain a certain element from the other domain can be matched. To use a mathematical metaphor, this mapping is clearly a homeomorphism in that it preserves relations between elements of the source domain while matching them to those of the target domain. As a result, any metaphorical correspondence, and the correspondence between the process of natural growth and that of intellectual development in particular, may appear as almost self-evident; however, it is important to note that, in fact, it has been *created* rather than merely documented by those who first spoke of learning in terms of biological maturation. Moreover, it is the metaphor which dictates the way we think about education rather than the other way round.

It is noteworthy that in mathematics metaphors are not less ubiquitous than in any other domain, although their massive presence may be much less obvious. One can easily overlook the figurative nature of mathematical concepts for the simple reason that in mathematics, metaphorical language is often the only one we have. As a consequence, the 'metaphorical' expressions count also as 'literal'. This, by the way, may be the principal reason why many people, mathematicians included, remain unaware of the metaphorical nature of mathematical concepts.

Let's look at an example. The common ways of talking about numbers reveal a geometric connection: at a closer scrutiny, our thinking about numbers turns out to be firmly grounded in the metaphor of numbers as points on a line.

Here are but a few out of the many possible examples:

> Think of a number *between* 0 and 10.
>
> The values of the function *g approach* 5.
>
> The values of *f are very close to* 0.
>
> Even in *the smallest neighbourhood* of a rational number there are infinitely many irrational numbers.

Clearly, the points are mapped onto the members of number set in such a way that spatial relations between the former (points) are isomorphically translated into the order relations between the latter (numbers). Thus, the inequality $0 < x < 10$ may be translated into the sentence: "a number $x$ lies *between* 0 and 10". The word "between" originally refers to spatial relationship and is used now to present the mutual "placement" of the numbers 0 and 10 (a metaphorical express-ion again! Indeed, trying to talk without metaphors about anything – metaphors included! – turns out to be an impossible task).

These examples show clearly what has been stated at the outset: systematic conceptual metaphors are much more than a literary gimmick; neither in everyday life, nor in science can they be regarded as mere tools for the enhancement of understanding and memorising. Figurative expressions clearly have a constitute power, that is, they are instrumental in constructing new concepts rather than just help in conveying meanings of established notion. As such, metaphors must now be recognised as the primary cognitive constructions underlying all our concepts (Lakoff & Johnson, 1980; Johnson, 1987; Lakoff, 1987, 1993; Ortony, 1993; Lakoff & Nunes, 1997).

Once the metaphor is understood this way, its centrality and importance become immediately clear. A scrutinising look around would now suffice to notice that metaphors are ubiquitous both in everyday, intuitive thought, and in scientific theories. In mathematics and in science metaphor is genuinely indispensable, and this is what makes it even more transparent. As Scheffler (1991, p. 45) put it, "The line, even in science, between serious theory and metaphor is a thin one – if it can be drawn at all.... there is no obvious point at which we may say, 'Here the metaphors stop and the theories begin'". Indeed, there are no clear boundaries which would separate the metaphorical from the literal; there is no background of genuinely non-figurative expressions against which the metaphorical nature of such terms as "cognitive strain" or "closed set" would stand in full relief.

Let me finish this very brief exposition of the modern theory of metaphor with stressing what may be the most important implication of this theory for those who try to fathom the workings of human mind: Since metaphors bring with them certain well-defined expectations as to the possible features of target concepts, the choice of a metaphor is a highly consequential decision. Different metaphors may lead to different ways of thinking and to different activities. We can say, therefore, that *we live by the metaphors we use.* Indeed, metaphors are far from neutral with respect to the ways we reason, understand, talk and do things. We must always remain alert to the possibility that even the most useful of our metaphors may turn into obstacles to thinking if they are taken too literally or if we interpret them uncritically. Almost any metaphor can be shown to have some adverse implications along with many useful entailments. Thus, for example, Sheffler (1991) issues a caveat against the organic metaphor – the very same metaphor which Skemp presented so beautifully:

> ... when the organic metaphor is transplanted into practical context in which social policy is at stake, it may become positively misleading, since it makes no room for distinctions that are of highest importance in practical issues... Such distinctions are expressed, for example, in the separation of teaching from force, propaganda, threat, and indoctrination. In addition, biological regenerative processes are not, in general, considered to be subject to choice and control, whereas social processes, to a significant extent, are.

Obviously, it is not Sheffler's intention to say that the organic metaphor should be rejected altogether. Rather, his aim is to encourage only those application of the metaphor which advance our thinking, while warning us against metaphorical abuses and against limitations which a given metaphor may impose if given a full exclusivity.

Metaphorical pluralism may be the answer to the dangers outlined by Sheffler in this last example. Sheffler should therefore be pleased with Skemp, who evidently liked some figurative expressions more than others and yet displayed a great flexibility and diversity in his choice of metaphors. The particular power of Skemp's writing stems from his ability to make agile transitions from one metaphor to another whenever the other metaphor seems more appropriate for the purpose at hand. None of Skemp's metaphors, therefore, has been elevated to the status of an exclusive truth, none of them was taken to be any more than it was from the beginning: a metaphor. This flexibility, openness and the plurality of outlook may be the principal source of Skemp's theory's special appeal. In the reminder of this paper I will be survey two of his most important metaphors, and then scrutinise their different entailments.

## Skemp's Metaphors for Thinking

### *Relational Understanding as Having a Map*

Relational understanding and instrumental understanding – these may well be the first terms which come to mind when psychology of mathematics education is being mentioned. Since their inception in mid-seventies, the two terms became the hall-marks of the professional discourse and penetrated our language so deeply that it seems as if they have always been there. Today, we all speak of children 'with only instrumental understanding' or about 'successful problem-solvers who capitalise on their relational understanding', but not all of us may be aware that the centrally important distinction was first proposed by Skemp in his seminal paper of 1976. It is the characteristic feature of the most successful of scientific ideas that at a certain point they start looking as timeless and self-evident, and that their use is not always accompanied by a reference to their conceivers. Skemp's famous distinction is clearly one of those.

To explain his idea, Skemp urges the reader to imagine a person trying to learn his way around a foreign city. Basically, there are two ways in which the task can be accomplished: one can learn 'an increasing number of fixed plans [paths]', or one can try 'to construct in [one's] mind a cognitive map of the town' (Skemp, 1976, p. 25). These two types of learning, and of the resulting understanding, are given the names 'instrumental' and 'relational', respectively. 'The analogy between the foregoing and learning mathematics is close', explains Skemp. Moreover, the metaphor is so powerful that one hardly needs a guidance to understand the great difference between the two types of learning and understanding. An exploration of the metaphor's entailments is best done with examples.

Let me begin with a story of a 16 year old girl – let us call her Naomi – who was asked to solve a non-standard system of linear equations:

$$\begin{cases} 2(x-3) = 1 - y \\ 2x + y = 7 \end{cases}$$

While solving the problem, Naomi arrived at a stage where the first of the two equations turned into 1 = 1. The following exchange followed:

**Naomi:** One equals one. It's true, but it gives us nothing. Maybe I shouldn't have opened the brackets [in the first equation]... I don't know...

**Interviewer:** What can you say now about the solution of the system of equations?

**Naomi:** That maybe there is one solution.

**Interviewer:** What is it?

**Naomi:** One. The number 1. Or in fact... in fact I don't think so.

**Interviewer:** So?

**Naomi:** There is no solution. An empty set.

**Interviewer**: Where do you infer it from?

**Naomi:** Because we did all this... we isolated $2x$ etc.... to arrive at the value of y. We substituted this [$7 - y$ instead of $2x$ in the first equation] and we were left without $x$, only with $y$. And then there was no y as well. Our goal was to find the value of y and we didn't succeed. So I think that we have here an empty [truth] set.

Poor Naomi. Her brave attempt to apply the well known algebraic technique led her to nowhere. Moreover, after she realised that the standard algorithm would not work as expected, she found herself helpless and confused. A situation not altogether unknown from our classrooms, one may say, but what does it have to do with Skemp's tourist in a foreign city?

Well, a lot. It is helpful to think of Naomi as a person who learned a number of constant paths in the city, but who has no means of dealing with the situation when faced with an unusual goal. She feels quite helpless when the place she wants to go to proves unreachable through any of the roads she knew. Like a tourist without a map and with only limited knowledge of certain concrete paths, Naomi finds herself 'unable to retrace [her] steps and get back on the right path', even though she tries (see her first utterance above: 'Maybe I shouldn't have opened the brackets. . . '). She seems too poorly equipped to even start solving the problem. Until now, she used a very limited range of clues to decide whether her problem has been solved. Like a tourist who has been instructed to stop when a particular sign-board appears on her way, so did Naomi learn to deem her equations as solved only when she reaches the 'atomic' equalities $x=a$, $y=b$, where $a$ and $b$ are concrete numbers. Neither the tourist, nor Naomi, know what to do when the 'halting signal' does not appear.

In situations like this, having a map can certainly help. It does not matter whether the map is a real overall picture of the city or a cognitive schema of the 'problem domain'. The special power of maps and schemes lies in their generative power: they make it possible for us to find paths which we have never travelled before.

An episode from a study recently completed by Carolyn Kieran and myself in Montreal aptly illustrates this last point. In the Montreal experiment, a group of 12–year old children made their first steps in algebra. Our approach was functional and the learning was massively supported with computer graphics. In the final interview, a boy named George was asked to solve the equation $7x+4 = 5x+8$. The children did not learn the algebraic methods of solving equations, but they did learn to see linear functions through formulae such as $7x+4$ or $5x+8$. Here is our exchange with George:

> **George:** Well, you could see, it would be like, … Start at 4 and 8, this one would go up 7, hold on, 8 and 7, hold on... no, 4 and 7; 4 and 7 is 11 … they will be equal at 2 or 3 or something like that.

At first sight, it is rather difficult to understand what George is talking about and where his numbers and calculations are coming from. It looked like George was able to see more than the symbols – more than the formulae on the paper would show. He spoke as if he was looking at something none of us could see with an equal ease. Indeed, by the end of the conversation, the following exchange between the boy and the startled interviewer took place:

> **Interviewer:** How are you getting that 2 or 3?
>
> **George:** I am just graphing in my head.

In our attempt to interpret the episode, the metaphor of relational understanding as having a map proved handy. It helped us greatly to make sense of what was going on. Like a tourist with a mental map of a city, who is able to find his way between two locations even though he never walked this path before, so was George able to imagine a web of relations between two abstract objects represented by the expressions $7x$+4 and $5x$+8. This 'mental map' helped him to cope with the problem which, from his point of view, was rather unusual. Indeed, Skemp's description of a person with a mental map of the town seems to fit George's story perfectly: such person, says Skemp, 'has something from which he can produce, when needed, an almost infinite number of plans by which he can guide his steps from any starting point to any finishing point, provided only that both can be imagined on his mental map.' Moreover, 'if he does take a wrong turn, he will still know where he is, and thereby be able to correct his mistake without getting lost', (Skemp, 1976, p. 25). This is certainly true about George, who used trial and error method, but who was perfectly able to correct his own steps by confronting the intermediate results with what his mental map was showing.

Powerful metaphors are those which can lead to new insights and enhance one's understanding of investigated phenomena beyond what may be read directly form the empirical data. Let us now explore certain entailments of the 'map in a foreign city' metaphor which Skemp did not necessarily mention himself.

While elaborating on the advantages of relational understanding, Skemp never considered the possible *dangers* of 'having a map'.

*Figure 1a.*

The very idea of a fixed plan which shows city's overall structure and, at the same time, overlooks lots of details, implies that as useful as such plan may be, it can also lead us astray. In order to illustrate this point, I conducted the following experiment in one of my university classes. Half of the class was shown the picture presented in Figure 1a. The students were instructed to pay a special attention to the encircled part of the image. After being exposed to the picture for not more than 30 seconds, and after the picture has been taken away by the teacher, the students were asked to reproduce the encircled part from memory. Figure 1b displays a representative sample of the drawings produced by the members of this group.

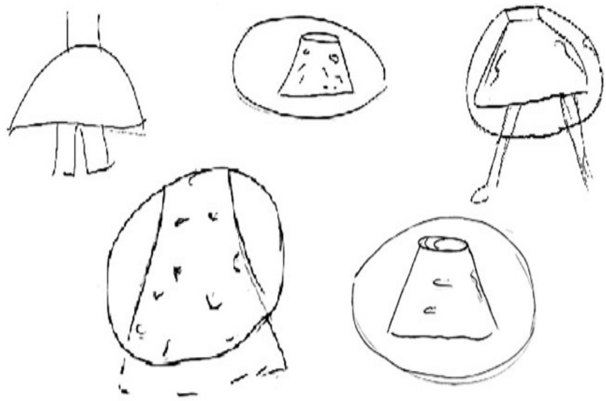

*Figure 1b.*

The other half of the class was presented at the same time with a slightly different picture. Their following task was exactly like that of the first group. The picture given to this second group and the results of students' reproductive efforts are displayed in Figures 2a and 2b.

The interesting – even if not quite surprising – result is that the drawings produced by the two groups seem rather different while, in fact, the shapes the two groups were supposed to reproduce were exactly the same, although embedded in different pictures.

*Figure 2a.*

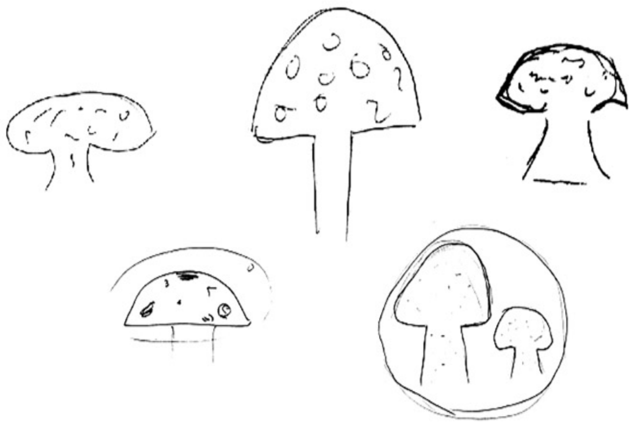

*Figure 2b.*

Clearly, the pictures of the girl and of the mushroom that have been shown to the two groups played a central role in the process of memorising and then recalling the shapes. This means that the way the students *perceived* these shapes in the first place depended very much on the context within which the shapes have been shown. In fact, what the participants saw and remembered was their own interpretation of the shapes. It was the meanings the students gave to the shapes rather than the shapes 'as such'. Clearly, this meaning depended very much on the general scheme within which the encircled shapes were embedded.

Thus, the pictures fulfil the same important function as maps: they confer meaning or, to put it in other words, they help in sense-making. With the help of the two pictures, that of the girl and of the mushroom, like with the help of a map, the students could find out without difficulty where they were and then decide quickly how they should respond to the task at hand. Whether we are talking about a place in a city or a part of a picture, the entity one is dealing with can effectively be handled only if it is recognised as a part of a greater familiar whole.

The point I wish to make is now that this easiness, however important, clearly has its price. The two groups did not reproduce the original picture in the same way: those students who were presented with the picture of a girl have drawn shapes which can easily be recognised as representing a part of a female body, whereas those who dealt with the mushroom produced drawings which displayed all the characteristics of a mushroom. It is therefore clear that the students did not always notice all that could be noticed, while not all the things they did see and memorise were, indeed, a part of the original shape.

This example, therefore, shows how a particular map one has constructed in the past may direct, shape and, eventually, limit one's understanding. While making remembering easier, it may close our eyes to potentially important details. In consequence, it often makes us impervious to novelties – to the things which have not been taken under consideration in the process of constructing the maps. After all, be a map as precise as it may, it will never contain all the possible details.

Maps, therefore, may sometimes be a hurdle to creativity. They make us notice only what we already know and ignore what does not fit into the existing schema. They force us to see a new thing as an instance of something we already know – if at all. Moreover, probably just because of their effectiveness, they are also quite durable and resistant to change. History of mathematics is replete with stories of mathematicians who were reluctant to abandon well-documented paths for the sake of routs which did not fit into the familiar territory. In mathematics, like in geography, an all new area cannot be explored with well established maps. More often than not, something has to be destroyed if something new is to emerge. Thus, Columbus had the good fortune to discover a new land only because he did not follow the available map too carefully; and Bolyai could think of an all new organisation of space only because in contrast to Sacceri he had the power to set his imagination free from the constrains of the all too familiar Euclidean geometry.

Let me finish this section with a double conclusion. First, here is what we have learned from Skemp's metaphor of understanding as having a map: It certainly takes much intellectual power to build a map of a new territory and arrive at relational understanding; it takes not less creativity and intellectual courage to set oneself free from the constrains of maps already established. Second, there is a meta-level conclusion: A good metaphor can promote our understanding of thinking processes in a much more effective way than a most strictly scientific, 'literal' expositions.

*Learning Mathematics as a Participation in a Game*

Let me repeat: metaphors are ubiquitous, and not only in Skemp's writings. Psychologists and educators may be among those writers who use explicit comparisons and parallels more often than others. Not all metaphors, however, are explicit. Some of them, mainly those overarching 'big' metaphors which underlie whole families of 'smaller' metaphor, may be quite difficult to detect. Among the most elusive, and at the same time most essential educational metaphors are those we use in our thinking on learning. Right now, the field seems to be caught between two such metaphors: one that views learning as an *acquisition* of something (knowledge, concepts, schemes), and the other one that equates learning with a steadily improving *participation* in a certain kind of practice or discourse. I once called these metaphors *Acquisition Metaphor* and *Participation Metaphor*, respectively (Sfard, 1998). Both these metaphors are simultaneously present in most recent texts, but while the Acquisition Metaphor is likely to be more prominent in older writings, more recent studies are often dominated by the Participation Metaphor.

Skemp's metaphor of the process of understanding as building a map clearly belongs to the first school of thought; however, some other metaphors in his books and papers may be deemed as participational rather than acquisitional. Here is a representative example.

In his attempt to account for the ineffectiveness of mathematics teaching, as can be witnessed only too often in today's schools, Skemp proposed to view the relations between teachers and students as parallel, in a sense, to the relations between two teams of football players. First implication of this comparison is that one can conceptualise the situation of learning mathematics in terms of game playing. This means, among others, that the two teams are expected to accept a set of well-defined rules and then follow these rules faithfully all along the way. Another entailments of the metaphor is that the whole situation becomes highly ineffective if, for some reason, the two parties have disparate understanding of the rules themselves. The situation becomes positively hopeless if, in addition, the two teams remain unaware of the differences. Skemp's (1976, p. 21) description is quite dramatic:

> Let us imagine, if we can, that school A send a team to play school be at a game called 'football', but neither team knows that there are two kinds (called 'association' and 'rugby'). School A plays soccer and has never heard of rugger, and vice versa for B. Each team will rapidly decide that the other team is crazy, or a lot of foul players. Team A in particular will think that B uses a mis-shapen ball, and commit one foul after another. Unless the two sides stop and talk about what game they think they are playing, long enough to gain some mutual understanding, the game will break in disorder and the two teams will never want to meet again.

The entailments of this metaphor are rich and many. Although Skemp has discussed them only in the special context of instrumental versus relational approach to mathematics, the metaphor of mathematical activity as game playing may be useful and eye-opening in a wide range of greatly varying situations. It applies equally well to the communication between teachers and students, as to the communication among mathematicians themselves. The metaphor sheds as much light on the reasons for ineffectiveness of instruction as on the sources of loud protests some mathematicians voiced when some others presented such outrageously counter-intuitive inventions as negative numbers or a straight line with infinitely many parallels crossing in one point. In both these cases, the situation is characterised by the fact that different participants hold greatly varying meta-mathematical beliefs on such basic meta-mathematical questions as what counts as a legit-imate definition, where mathematical ideas come from, what is acceptable as mathematical proof, what counts as a mathematically sound argument, what is the meaning of the term 'axiom', etc. Clearly, we are talking here about the basic rules according to which the game called mathematics is being played. Since, however, all these rules are mostly tacit, the players do remain ignorant of the need to negotiate them rather than just engage in the game itself. This, of course, makes the situation extremely difficult and diminishes the chances for an effective and satisfying play.

The metaphor of soccer and rugby works, indeed, extremely well if applied to school situation. The teacher may juggle with such entities as –5 or –2 in a matter-of-fact tone without ever realising that for the student the negative and complex numbers are not the kind of objects with which the game called mathematics can be played. For the student, mathematics is about numbers, and numbers are about quantities; however, can any question that begins with the words 'How many' or 'How much' be answered with –5 or –2?

The impossible situation of the football match between soccer and rugby also gives a good sense of those historical junctures where mathematicians were discussing truly innovative ideas. For example, the metaphor helps to understand why Sacceri would be unable to communicate with Bolyai or Lobachevsky, had the two parties ever met. The reason, indeed, would be not so much their disagreement on mathematical ideas as the fact that the very word 'mathematics' would bring to them messages as disparate as the word 'football' brings to soccer and to rugby. If, for Sacceri, mathematics dealt with physical, mind-independent reality and, as such, would not admit of conflicting statements on one and the same object, for Bolyai and Lobachevsky, the idea that some entities (e.g. parallels to the given line) may be regarded as either existent or non-existent, depending only on our choice, was perfectly admissible. While this discrepancy in meta-mathematical assumptions was the principal reason for the historical fight for recognition of non-Euclidean geometries, the situation was exacerbated because the two camps remained unaware of the true sources of their disagreement: they thought they were fighting for access to the ball, while, in fact, they were negotiating the rules of the game.

The metaphor of mathematising as game-playing proves itself times and again as useful and enlightening. It is also noteworthy that it is very much in the spirit of today's research on learning which, inspired by Wittgenstein's ideas of 'language games' and by Foucault's concept of knowledge as a discursive practice, emphasises the aspects of communication and of norms construction, and pays particular attention to the way in which the student becomes a skilful follower of the rules that render mathematical discourse its unique identity. In this sense, Skemp seems to have got ahead of his times.

## Yet Another Metaphor: Pioneering Research as Building a House in an Empty Field

In early winter of 1989 I had the good fortune to meet Richard Skemp personally. On a gloomy rainy day I came to his office at the department of mathematics at the University of Warwick. Retired already for a few years, Skemp was as interested in research as ever. We soon found ourselves engrossed into a lively conversation on the state of research in mathematics education in general, and on his own work in particular. It did not take much effort to convince this retired professor but never-tired thinker to render me an interview (Sfard, 1990).

Asked how he would summarise the last two decades of the developments in research on mathematics education, and his own role in it, Skemp responded with yet another metaphor:

> In the domain of psychology of learning mathematics, we have come a long way in the last twenty years... In general, the discipline can, and probably should, develop through creative dialogues and discussions... This, I think, is the best way of building knowledge. It happens only too often that everybody puts his bricks in a different part of the field, so we never have a wall or a house. In the present case, I put some bricks down, and then others put some bricks on top of mine. Then I rearranged their bricks just a little, to make a surface which could support my next bricks; and so on. (ibid, p. 49)

Skemp, indeed, came to an empty field and left it with an impressive construction. Like first settlers, he put foundations for a whole new community. Being a member of this community, I feel indebted to its founding father. I regard my own research as growing out of the ground work he did for us all. In a sense, our whole activity as researchers may be seen as a continuing attempt to build on, refine, and occasionally rearrange, this fine layer of bricks he put in the then empty field.

When I was trying to make my first steps in this field, Skemp's ideas made me think. Inspired by the observation on a spectrum of different types and levels of understanding I set to myself as a goal to look for the reasons why people's learning efforts may bring so dramatically differing results. I emerged from this search with the idea of *reification* – an ability to conceive of mathematical notions such as number, set or function, as referring to certain *objects*. Of course, I was well aware that these objects of mathematics were nothing like the tangible objects we meet in the real life. Still, the way we speak of number, derivative, set or function stroke me as not much different from the way we speak about tables, trees, and boxes. It soon became clear to me that what we use to refer to as a *mathematical object* is a metaphor created through projection of certain linguistic forms from the real-life discourse to the mathematical discourse. And if mathematical objects are but metaphors, I concluded, then there may well be another way of thinking and talking about functions, numbers, and sets.

Indeed, a closer look at what we do and how we speak when mathematising has shown that in addition to *structural* thinking, that is thinking in terms of objects, there was an *operational* mode which appeared whenever the same notions were treated as referring to processes rather than objects. I conjectured that in the course of learning of a new mathematical concept, one usually proceeds from operational to structural conceptions, that is, *reifies* the processes that underlie this concept. The relation between the structural and operational conceptions on the one hand, and different types of understanding proposed by Skemp on the other hand, could then been described as follows:

> … we can say that in mathematics, transition from processes to objects enhances our sense of understanding mathematics. After all, reification increases problem-solving and learning abilities, so the more structural our approach, the deeper our confidence in what we are doing.... [S]tructural conception is probably what underlies the relational understanding, defined by Skemp (1976) as 'knowing both what and why to do', or having both

> rules and reasons. Purely operational approach would usually give no more than instrumental understanding, once presented by Skemp as having rules without reasons. (Sfard, 1991, pp. 29–30)

Now, let us go back for a moment to the two students, Naomi and George, whom we met above while talking about instrumental and relational understanding. Let us compare the language of Naomi, who struggles hopelessly with the singular system of linear equations, with that of George who copes successfully with an equation for the solution of which he has no algebraic method. While it is quite clear that Gorge is acting as if he was scrutinising the relations of some objects which he is able to 'see through symbols', it is equally obvious that Naomi can only see the symbols themselves. Clearly, it is the ability to interpret algebraic expressions as representing certain mathematical objects which makes George successful, and it is the lack of this ability which accounts for Naomi's failure. It seems, therefore, that the 'maps' which, according to Skemp, may be held responsible for the relational understanding have, indeed, much to do with one's capacity for structural thinking.

To put it differently, mathematical objects and the web of their mutual relations may be just the stuff Skemp's imaginary maps are made of. To make this message clearer, I once proposed a metaphor which, in a sense, elaborates on Skemp's idea of the map: I proposed to look at the learner of mathematics as a person who wants to take part in a virtual reality game. This game, played so skilfully by mathematicians and by mathematics teachers, is inaccessible to those who cannot re-create for themselves the virtual world that the experienced players can see with ease. The game is practically out of reach for a person who does not have an ability to construct a map of the space replete with mathematical objects where the game called mathematics is taking place. Without the map, thus without being able to locate the objects among which she is supposed to move when playing the game, Naomi was unable to make a step. Unlike her, George acted as if he was endowed with this special equipment through which the virtual landscape can be seen and operated upon with ease.

Building on one's ideas does not necessarily mean accepting all the other person says. Thus, some of my personal 'sequels' to Skemp's work are modifications rather than simple additions or refinements. For example, I decided there is room to reconsider the idea of instrumental understanding and to ask ourselves whether our tendency to view it as a rather undesirable phenomenon is fully justified. The argument I presented a few years ago goes approximately as follows: If one agrees that relational understanding has much to do with person's capacity for structural thinking, and if one accepts the claim that operational conceptions are a necessary step toward structural conceptions, one is compelled to conclude that instrumental understanding might have been somehow underrated. This kind of understanding, I claimed, 'is both valuable and inevitable at certain stages of learning'. In the end, I decided 'to push Skemp's original idea a little further, and talk about a third kind of understanding: *reasons without rules'*. This third type was presented as:

> purely intuitive understanding, attained in those rare cases when the vague structural conception is achieved before the operational basis has fully developed... having 'reasons without rules' may be not enough for creating a fully-fledged mathematical theory, but it is certainly most helpful in discovering theorem and in deciding about directions of further developments (p. 30).

This is how I managed to put my own modest layer of bricks on Skemp's solid foundations. Admittedly, the similarity between the original plan and the present construction may be sometimes a bit difficult to see. It is important, however, to recognise the fact that even if our modern buildings look different from those the founding fathers had in mind, Skemp's basic layer of bricks is still there, in the ground. Lest the difference in the facade fool us, then. Lest it conceal the fact that our high-rising buildings would probably never be the same – and perhaps would never be at all – without Skemp's founding effort.

## References

Johnson, M. (1987). *The Body in the Mind: The Bodily Basis of Meaning, Imagination, and Reason*, Chicago: The University of Chicago Press.

Lakoff, G. (1987). *Women, Fire and Dangerous Things: What Categories Reveal about the Mind*, Chicago: The University of Chicago Press.

Lakoff, G. (1993). The contemporary theory of metaphor. In A. Ortony, (Ed.), *Metaphor and Thought, Second edition* (pp. 202–250). Cambridge: Cambridge University Press.

Lakoff, G., & Johnson, M. (1980). *The Metaphors We Live By*. Chicago: The University of Chicago Press.

Lakoff, G., & Nunez, R. E. (1997). The metaphorical structure of mathematics: Sketching out the cognitive foundations for a mind-based mathematics. In L. English (Ed.), *Mathematical Reasoning: Analogies, Metaphors, and Image* (pp. 21–89). Mahwah, NJ: Erlbaum.

Ortony, A. (Ed.) (1993). *Metaphor and Thought, Second edition* . Cambridge: Cambridge University Press.

Reddy, M. (1979). The conduit metaphor: A case of frame conflict in our language about language. In A. Ortony (Ed.), *Metaphor and Thought* (pp. 284–324). Cambridge: Cambridge University Press.

Scheffler, I. (1991). *In Praise of Cognitive Emotions*. New York: Routledge.

Sfard, A. (1990). An interview with Richard Skemp, *Mathematics Teaching*, 133, 49–51.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the coin. *Educational Studies in Mathematics*, ***22***, 1–36.

Sfard, A. (1998). On two metaphors for learning and on the dangers of choosing just one. *Educational Researcher.*

Sfard, A., & Linchevsky, L. (1994). The gains and the pitfalls of reification – the case of algebra. *Educational Studies in Mathematics*, 26, 191–228.

Sfard, A., & Thompson, P. (1994). Problems of reification: Representations and mathematical objects. In D. Kirshner (Ed.), *Proceedings of the Annual Meeting of North American Chapter of the International Group for the Psychology of Mathematics Education*, Vol. 1 (pp. 3–34). Baton Rouge, Louisiana

Skemp, R. R. (1971). *Psychology of Learning Mathematics.* Harmondsworth: Penguin.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching*, *77*, 44–49.

Skemp, R. R. (1979a). Goals of learning and qualities of understanding. *Mathematics Teaching*, *88*, 20–26.

Skemp, R. R. (1979b). *Intelligence, Learning and Action.* Chichester: Wiley.

# Schemes, Schemas and Director Systems

## (An Integration of Piagetian Scheme Theory with Skemp's Model of Intelligent Learning)

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> "All action that is repeatable or generalized through application to new objects engenders by this very fact a 'scheme'"
> (Piaget, 1980, p. 24)

> "All schemes consist of 3 parts: (1) Recognition of a certain situation and/or experience; (2) association of a specific activity with that kind of item; (3) expectation of a certain result that can turn into a prediction." (von Glasersfeld, 1995)

> "The first part of a scheme can be regarded as a 'recognition template' [that] involves interpreting a task or problem for which the goal of the scheme's second part is directed." (Steffe, 1994)

These three quotes define "schemes" as goal-directed activity in ways remarkably similar to Skemp's description of Director Systems in his model of intelligent learning (Skemp, 1979, 1987). We identify schemes through observing children recurrently use a goal-directed activity on several different occasions in what to us are related situations.

Skemp (1979, 1987) used the word "schema" to convey the idea of personal knowledge structures, that he likened to the notion of a cognitive map. Skemp later (1987, 1989) described a special kind of schema that is the result of intensive conceptual analysis of a particular mathematical topic (such as whole number arithmetic). Whereas our personal knowledge structures may be non-hierarchical, these "concept maps" are partially ordered schemas that provide possible learning trajectories that build from lower-order concepts to higher-order concepts. As such, "they can be used in at least two ways [by a teacher]: for planning a teaching sequence, and for diagnosis." (Skemp, 1987, p. 122) In our work on children's construction of fraction knowledge we make distinctions among children's mathematics (that are the children's first-order models of their own mathematical knowledge; i.e. their personal mathematical schemas), mathematics of children (that are the second-order models that we create of children's mathematics; i.e. our schemas of the children's mathematical knowledge), and mathematics for children (that are the Concept Maps that we construct based on our schemas of the mathematics of children).

While schemas are *representational* structures, schemes and director systems are *action* structures. Both kinds of structure are necessary for understanding intelligent learning. This chapter will attempt to explain how schemes, schemas and director systems are related, and how they can be used to build a viable, theoretical model of children's constructive activity in the context of learning about fractions. In the process we construct a modified model of intelligent learning that integrates scheme theory with Skemp's Model of Intelligence.

## Skemp's Model of Intelligence – A General Framework

Underlying Skemp's model of intelligence is the assumption that much of human (and animal) behavior is goal directed and that survival is a goal state on which all other states depend. People's actions move them toward or away from particular goal or anti-goal states. Borrowing from the field of cybernetics, Skemp used the metaphor of a director system for explaining how an individual might organize and direct these actions. A director system keeps things heading in the right direction until the goal state is reached.

Skemp (1979) suggested that an individual has many innate director systems which are the result of evolution and which are essential to the physical survival of the individual. New director systems, constructed within the lifetime of an individual, are the result of learning. Other animals can also develop new director systems as a result of learning, but humans "have available a more advanced kind of learning, which is qualitatively different from any of those available to other species," (Skemp, 1979, p. 7). Skemp described this qualitatively different learning as intelligent learning, brought about by the action of a second order director system acting on the first order system.

In the description of director systems and their components, the following terms are used:

1. *Operand:* the physical or mental object of concern which is brought to the desired goal state and kept there.
2. *Operator*: that which actually does the work of changing the state of the operand from the initial to the chosen (goal) state.
3. *Comparator*: that which compares the mental realizations of present state and goal state of the operand.
4. *Plan*: the mental plan of action for changing the state of the operand.

Skemp provided several examples of mechanical director systems to help the reader make sense of these terms but cautioned that the mental analog was not to be regarded as purely mechanistic in nature. We offer one such example: that of an electric oven. The goal is to raise the interior of the oven to a desired temperature (set by the dial on the front of the oven). The *operand* in this example is the interior of the oven (or more precisely, the air or food inside the oven). The *operator* is the heating coil inside the oven, and the *comparator* is a device that compares the present temperature of the oven to the goal temperature. In order to make that comparison, a *sensor* is required that indicates the current temperature. This sensor is the oven thermostat. The *plan* that has been (literally) hard wired into the system is to provide electricity to the heating element while the oven temperature is less than the goal temperature and to turn off the electricity when the oven temperature is greater than or equal to the goal temperature.

In using this metaphor for the explanation of goal-directed activity in humans several complex issues arise. The first issue is that of perceived goal state and present state of the system. Skemp made a distinction between “actuality” and personal “reality.” Actuality is the physical world in which we live; our personal realities are the ways in which we perceive that physical world. Another issue is the relationship that any particular operand has with the environment within which it exists. In the oven example the interior of the oven is the environment of the operand; the particular relationship is the temperature of the interior (not, for instance, how clean it is!). It is this relationship between operand and environment that determines the particular goal state to be achieved. For human activity we need to have some way of abstracting the relevant relationships and sensing them. The oven example presents a fixed director system, whereas intelligent learning is concerned with “teachable” director systems. Skemp addressed all of these issues (and many more) in his model of intelligent learning. Of particular importance in this chapter is the distinction that Skemp made between a *First Order Director System* (Delta-one) that directs physical activity and a *Second Order Director System* (Delta-two) that directs mental activity.

*First-Order Director Systems (Delta-one)*

Delta-one directs the actions of operators acting on operands which exist within the physical environment. Figure 1 illustrates the components of a director system.

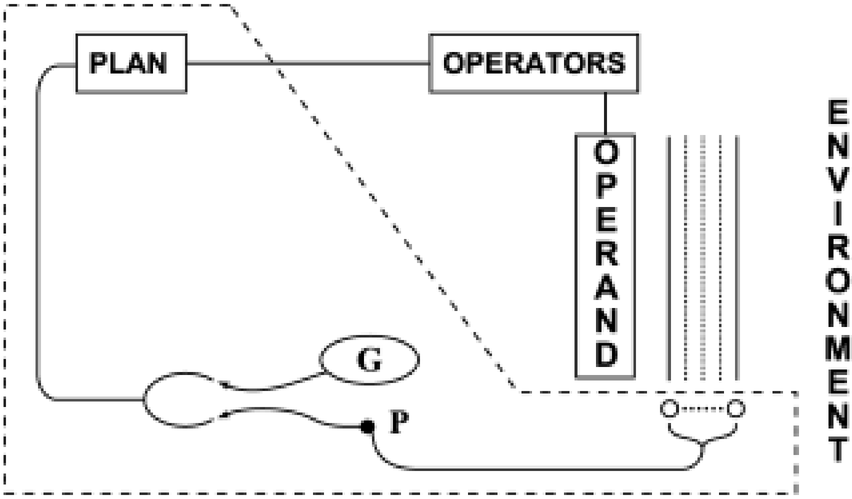


*Figure 1*. Components of a director system,
(from p. 49 of Skemp, 1979).

Actuality is represented within a director system in the form of conceptual structures or schemas. As illustrated in Figure 1, both the present state, indicated at point **P**, and the goal state, **G** of the operand must be realizable within a schema.

The vertical lines between operand and environment represent relationships between these two. The small circles (○······○) represent sensors which abstract these relationships, making the realization of both the present state and goal states of the operand possible within a schema. The comparator  compares present state and goal state. The plan converts this information into appropriate action by directing the energies of the operators, acting on the operand, in such a way as to bring the operand to the goal state. The director system as a whole has been enclosed by a dotted boundary line.

Thus it is from the individual's schemas (in which actuality is realized) that the director system develops plans for progressing from present state to goal state. The accuracy and completeness of one's schemas will directly affect the quality of the plans derived from them.

*Qualities of Schemas*

Schemas can be regarded as networks of connected concepts. They are our mental models of actuality, "or rather, by a process of abstraction, those qualities of actuality which are relevant to the functioning of particular director systems" (Skemp, 1979, p. 113).

Several aspects of a schema affect its quality:

i) The type of concepts which make up the schema (e.g. object concepts, class concepts). The more abstract the concept, the more extensive the domain of the director system of which it forms part.
ii) The quantity of connections between concepts. The more complete the network, the greater variety of plans that may be produced, the more adaptable the director system becomes.
iii) The quality of connections. Concepts may be linked either associatively or conceptually. An example of an associatively linked (A-linked) schema is the schema containing a person's name, address and phone number. The individual concepts have no conceptual relationship with one another; they are linked only by association with a particular person and must be learned by either a process of rote memorizing or by consciously being taken together as one thing. In many cases, a non-conceptual relation may be established (such as a mnemonic) to aid memory. We regard the unitizing of information (taking together) and the establishing of relations as different from "rote memorization."

Skemp (pp. 187–189) gives the following examples of conceptual connections:

a) 2–5–8–11–14–17,
b) 1–2–4–7–11–16,
c) parler – je parle, aimer – j'aime, donner – je donne.

In each example the link between each component can be conceptualized. In (a) the link is "a difference of 3"; in (b) it is a more complex concept – "a difference which increases by one for each successive link;" in (c) the link is a regularity, or rule, consistent for regular verbs of the same declension. In contrast to (c), the link between the verb etre and je suis is purely associative.

According to Skemp, other factors important for the effectiveness of a schema include:

iv) Relevance of content to the task in hand.
v) The accuracy with which it represents actuality[1].

[1] From our perspective as radical constructivists the notion of "accurately representing actuality" appears to assume that an actuality exists totally independent of any "knower." We would argue with this assumption and prefer to replace the notion of "accurately representing actuality" with the notion of "viability." A schema is viable if it fits with our experiences of actuality (i.e. provides us with ways of explaining our experiences).

vi) The quality of organization which makes it possible to use concepts of lower or higher order as required.
vii) The strength of the connections.
viii) The content of ready-to-hand plans, which remain integrated with the parent schema.
ix) Penetration – the degree to which it can function in high noise conditions.
x) Assimilatory power – the degree to which it can assimilate new experience, as well as the possibility of mutual assimilation with other schemas.
xi) The extent of its domain.

**Schema Construction**

Skemp formulated three modes by which individuals both build and test their schemas. The following table (taken from Skemp, 1989, p. 74) outlines these three modes.

Table 1
*Schema Construction*

| BUILDING | | TESTING |
|---|---|---|
| | Mode 1 | |
| from our own encounters with the physical world: *experience* | | against expectations of events in the physical world: *experiment* |
| | Mode 2 | |
| from the schemas of others: *communication* | | comparison with the schemas of others: *discussion* |
| | Mode 3 | |
| from within, by formulation of higher-order concepts: by extrapolation, imagination, intuition: *creativity* | | comparison with one's own existing knowledge and beliefs: *internal consistency* |

As mathematics educators we need to provide learning situations in which all six modes (three modes of building and three modes of testing) are brought into use if children are to be successful in constructing their own schemas. Mode 1 points to the importance of structured practical activities; mode 2 points to the value of co-operative learning situations; and mode 3 emphasizes the need for creativity in the learning of mathematics.

*The Domain of a Director System*

*The Domain of a Director System* refers to a well-defined set of situations or states within a particular universe of discourse. As these states must be realized within a person's available schemas, the quantity and quality of these schemas define the extent of the domain. The director system can function effectively (i.e. is able to develop plans to move the operand from a present state to a goal state) if it is operating within its domain. It cannot function outside its domain. It is important for teachers to know whether a student's inability to reach a goal state is because the goal state is outside the domain of the student's director system or because it is beyond the capacity of the student's operators. Skemp termed that set of states which are both within the domain of the director system and the capacity of the operators the *prohabitat* of the individual. Olive has used the term *comfort zone* in a similar way to that of *prohabitat*. Skemp (1979) provided an example of *prohabitat* in terms of a swimmer and non-swimmer. An experienced swimmer would be in the domain of her swimming director system when swimming across a river, but a non-swimmer would be outside of his domain if the river was too deep for wading across. The swimmer, however, might also get into trouble if the river was running too swiftly for the swimmer to reach her goal on the opposite bank. Here the problem is not the domain of the director system, but the capacity of the operators – the swimmer's muscles are not sufficiently strong to swim against the current. Thus, the swimmer's *prohabitat* consists of those situations in which she can swim effectively (and safely!).

*Frontier Zones* are those regions on the boundary of a domain within which the director system can function but not with complete reliability. One expands one's domains by operating within these frontier zones until they are made reliable regions of the domain; then new frontiers are established. (In the case of the swimmer, she builds up her muscle strength and swimming technique to be able to function reliably in more swiftly flowing rivers.) Thus learning is an activity which may benefit greatly from challenging situations under protected conditions in which the learner can make mistakes without too serious consequences. Steffe's notion of *zones of potential construction* is compatible with Skemp's notion of *frontier zones.*

There are many other details of the model concerned with the nature and use of director systems – the role of emotions as feedback mechanisms, interaction between first order director systems, the concept of motivation, internal and external organization and cooperation between individuals. The goal of an individual's second-order director system is improved functioning of the individual's director systems. This should also be the goal of education. Thus it is only in cooperation with a student's second-order director system that a teacher may improve the functioning of a student's first-order director system. This implication of Skemp's model of intelligent learning becomes critical when we examine children's construction of mathematical *schemes.*

### Second Order Director Systems (Delta-two)

It is the existence (and use) of these second-order systems which, according to Skemp, separates humans from other species and makes possible intelligent learning. The operands of Delta-two are teachable Delta-one director systems. The function of Delta-two is to take a teachable Delta-one from a present state in which Delta-one does not function optimally to a goal state of optimal functioning. Components of the state of a director system include its domain, the viability and completeness of its schemas, and the skill with which it translates plans into action. Successful learning is thus indicative of having more extensive domains, having more viable and complete schemas, and possessing greater skills in putting plans into action.

Prohabitat at the Delta-two level refers to the intellectual regions within which a Delta-two functions; that is, brings Delta-one systems to better states. Emotional signals also come into play at the Delta-two level. If one is operating within one's intellectual prohabitat one experiences confidence in one's ability to learn, whereas, if one is faced with a task outside the intellectual prohabitat one experiences anxiety and frustration. These emotions will now be signaled from two levels, Delta-one and Delta-two, and the signals may not always be complementary.

#### *Intuitive and Reflective Processing*

We now turn to the processing by which plans for action by a director system may be produced. Prerequisite for the production of these plans is understanding, which Skemp (1979) defined as "the realization of present state and goal state within an appropriate existing schema." (p.170). The existence of such a schema implies the existence of one or more paths between these states but does not guarantee that this path can be thought of. However, the better the quality of the schema, the more likelihood of discerning a path and thus producing a plan.

*Intuitive path finding*. As stated earlier, a schema can be regarded as a network of connected concepts. The strength and quality of those connections (A-links or C-links) have been described as important aspects of a schema. The proximity of connected concepts is another important aspect; in the non-spatial sense, proximity can be regarded as “closely related”. Activation of one concept can activate or heighten awareness of other concepts; those closely related will be more activated than distantly related ones.

In Figure 2 the thin lines represent the schema as a whole; the thicker lines radiating from **P** (present state) and **G** (goal state) represent the activation of “neighboring” concepts resulting from heightened consciousness of present state and goal state.

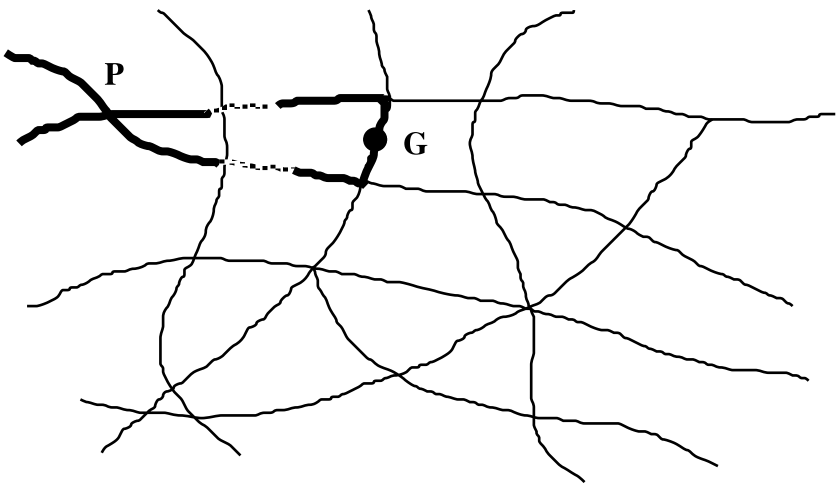


*Figure 2.* An activated schema.
(from Skemp, 1979, p. 171)

“If these neighborhoods connect, then one or more paths from P to G will result. This represents path finding at an intuitive level.” (p. 171). Just by thinking about the problem a solution rises to consciousness. Skemp described the intuitive process in more detail in terms of a resonance model – the mutual excitation of tuned structures resulting in the amplification of a shared frequency due to the phenomena of resonance. During intuitive processing consciousness is centered in Delta-one.

*Reflective processing*. Intuitive path finding has a fundamental weakness. It does not always perform to order. If the goal state and present state are far apart (either literally or metaphorically) then a connected pathway may not surface into consciousness. One may choose a succession of present states which one hopes will eventually bring one within the neighborhood of the goal state. However, if, by a process of reflection, the schema as a whole can be brought into consciousness the planning function becomes much more powerful.

The difference may be illustrated by regarding one's schemas as road maps. If the whole map is available, one can select an optimal route from present state to goal state; and if one encounters unforeseen obstacles on the way, alternative routes can be planned given a map with sufficient detail! If, on the other hand, the whole map is not available but one can recall certain neighboring streets and has a general idea of which direction the goal state lies, one can head off on a trial and error basis; this may, however, result in getting oneself completely lost!

Bringing the whole schema, or as much of it as possible, into consciousness is a function of Delta-two operating on Delta-one. Following Piaget, Skemp has termed this ability to make one's own mental processes the object of conscious observation "reflective intelligence." An immediate consequence of this ability is the possibility of separating the activity of thought from physical action. "This not only allows the preparation of plans in advance of action: it also makes it possible to construct several possible plans for the achievement of a particular purpose, to consider their respective merits, and to settle on that which seems to be the best; then either immediately or at a later time, to put this plan into action." (Skemp, 1979). This process could be regarded in a problem solving situation as the ability to withhold closure.

Another consequence of the shifting of consciousness to the Delta-two level is diminished functioning at the Delta-one level. In effect, Delta-one ceases to direct action. The diminished functioning of Delta-one would explain Romberg's findings (1982), when investigating the acquisition of higher-level algebraic abilities, with respect to already established abilities: "The first step toward progress is regress." The established ability actually diminishes while the student is progressing to the higher level ability (Delta-two activation). When the new ability is acquired the performance of the old ability stabilizes (Delta-one is back in action).

Reflective processing serves many important functions; the following is a brief summary:

Awareness of one's schemas
– improving them by formulating new concepts and connections
– testing their predictive powers and internal consistencies
– revising them if necessary.

Generalizing from our concepts and schemas
– forming higher-order concepts and schemas.

Improving and systematizing the knowledge we already have.

(DIRECTOR SYSTEMS)

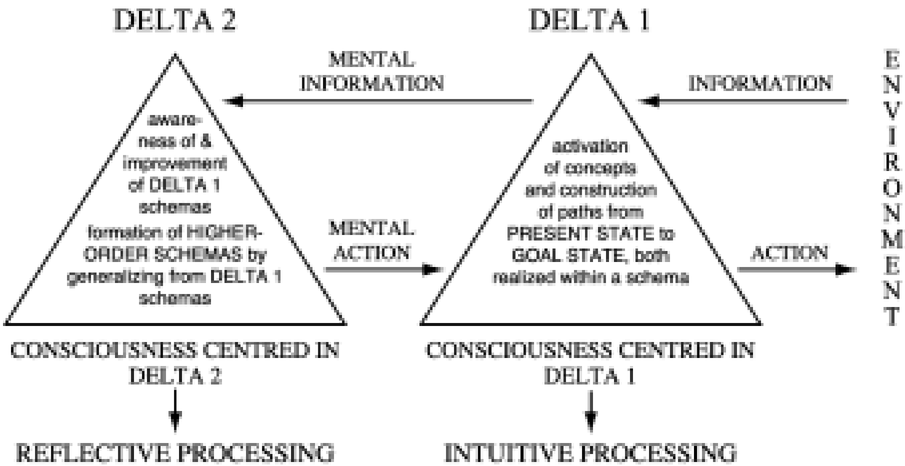


*Figure 3*. Skemp's model of intelligence.

There are many other important aspects of Skemp's model of intelligence not included in this summary. The reader is encouraged to fill in these gaps for him/herself by referring to the original source (1979). Figure 3 is a representation of those parts of the model touched upon here.

## Building Models of Children's Mathematical Thinking

The main goal of our work in the research project on children's construction of the rational numbers of arithmetic is to build models of the children's mathematical thinking as they struggle to make sense of situations and problems involving fractional quantities. These models can be regarded as our schemas of the children's fractional knowledge.

They are what we call second-order models (not to be confused with Skemp's Second-Order Director System!). The schemas that the children construct as they develop understandings of fractions in many different situations are what we call first-order models. We also have our own first-order models of fraction knowledge (our schemas of fractions), but as adults and mathematics educators our first-order models of fraction knowledge are likely to be very different from the children's schemas.

Distinguishing between first- and second-order models is critical in avoiding a conflation between children's mathematical concepts and operations and what is regarded as mathematics *for* children. Traditionally, both have been understood as first-order models – mathematical knowledge from the point of view of contemporary school mathematics. In our developing framework, both of these things are regarded as second-order models. Second-order models are constructed through social processes and we thereby refer to them as social knowledge. In terms of Skemp's three modes of schema building and testing this social knowledge would be the result of *Mode 2* building and testing: building from the schemas of others (through communication and observation), and testing by comparison with the schemas of others (through discussion). Regarding school mathematics as social knowledge is a fundamental shift in belief that is yet to be fully appreciated. Included in this shift in belief is Skemp's implication that a teacher must cooperate with children's second-order director systems in order to foster intelligent learning.

In making the above distinction between what we call first and second-order models of children's mathematical knowledge we add a new quality to Skemp's notion of *Schema* that is important for educators to understand. As educators we cannot know directly the children's own schemas, we must construct our second-order schemas of that knowledge based on our interactions with and observations of the children's mathematical activity. In what follows we elaborate on these different levels of schemas and introduce the Piagetian notion of *schemes of action* in an effort to explain children's mathematical behavior in ways that we believe go beyond Skemp's cybernetic model of director systems, and yet are compatible with that model.

*Mathematics of children*

We attribute mathematical realities to children that are independent of our own mathematical reality (Kieren, 1993; Steffe, Cobb & von

Glasersfeld, 1988). Although the attribution of such realities to children is essential, it does not provide models of what these realities might be like. We adults are obliged to construct models of children's mathematical realities. In this, we use the phrase "children's mathematics" to mean whatever constitutes children's mathematical realities (*their* mathematical schemas) and "mathematics of children" to mean our models of children's mathematics (*our* schemas of *their* mathematical knowledge). We regard the mathematics of children as legitimate mathematics to the extent that we find a rational basis for what children do.

The first-order models that constitute children's mathematics (*their* schemas) are essentially inaccessible to us as observers. By saying this, we don't mean that we don't try to understand children's mathematical realities. Quite to the contrary, we spend a substantial part of our time, during and after teaching children, analyzing their evolving mathematics. What we do mean is that regardless of what the results of those analyses might be, we make no claim that the first-order models that constitutes the children's mathematics corresponds piece-by-piece to what we have established as second-order models (*our* schemas).

### *Mathematics for Children*

We usually find it inappropriate to attribute even our most fundamental mathematical concepts and operations to children. For example, a set of elements arranged in order is a basic element in ordinal number theory. Although children establish units of units and later use these composite units in further operating (e.g. four iterated six times), these composite units are not constituted as ordered sets by the children in the way we understand ordered sets. The observer might regard the composite units that are attributed to the children as early forms of ordered sets, but to call them ordered sets would be a serious conflation of our idea of an ordered set and the idea of a composite unit as a concept that we have found useful in understanding children's mathematics. Ordinal number theory can be orienting, but it is not explanatory in the sense that it can be used to account for children's numerical concepts and operations. In this, it might seem that what adults intend for children to learn remains unspecified.

Mathematics for children consists of those concepts and operations that children might learn (Steffe, 1988). But rather than regard these concepts as being a part of our own mathematical knowledge, we base mathematics for children on the mathematics that we have observed children actually learn. Essentially, mathematics for children cannot be specified a priori and must be experientially abstracted from the observed modifications children make in their schemes (or, in Skemp's terminology, their first-order director systems). That is, mathematics for children can be known only through interpreting changes in children's mathematical activity. Specifically, the mathematics for a group of children with respect to a particular mathematical scheme is initially determined by the modifications that we have observed other children make in the particular scheme. We call this a zone of potential construction. As will be seen, this *zone of potential construction* is analogous to Skemp's *frontier zone* with respect to the domain of a particular director system.

An initial zone of potential construction serves as a guide in the selection of learning situations. As a result of actually interacting with the particular children, the zone of potential construction may be reconstituted as a zone of actual construction (the *prohabitat* of a director system). The two zones usually diverge, however, because in the course of actually interacting with the children, they may make unanticipated contributions and new situations of learning may need to be formulated. Through establishing actual zones of construction, new possibilities may arise for a zone of potential construction with respect to the particular scheme. Some elements may be retrospectively of less importance than originally thought or may be reorganized in new relations with the more novel elements. In short, we recognize the necessity to modify our models of children's mathematics according to the children's work and maintain that teaching mathematics is constitutively adaptive.

Our first-order models of mathematics do play fundamental roles in formulating the second-order models that we call the mathematics of children. Perhaps the most fundamental of these roles is in orienting us as we formulate mathematics *for* children and how to interact with them. Our focus in the remainder of this chapter is on developing a central conceptual construct – schemes – that we use in building models of children's mathematics. Our focus on schemes is appropriate because we regard children's fractional knowledge as consisting of schemes of action and operation that are functioning reliably and effectively. While schemes have many similar features to those of a *Director System,* we shall show how the two constructs differ and how they might complement each other.

Because of the close similarity of the two words “schema” and “scheme” it is necessary to emphasize their functional difference: Schemas are *representational* structures – they represent knowledge in the form of networks of connected concepts, whereas schemes are *action* structures. Piaget (1980) based the idea of scheme on repeatable and generalized action. “All action that is repeatable or generalized through application to new objects engenders … a ‘scheme’.” (p. 24) From this, we identify schemes through observing children recurrently use a goal-directed activity on several different occasions in what to us are related situations. Through such observations, it is possible to describe a scheme. These descriptions are usually interesting and often contain insightful behavior on the part of the child. For the practice of teaching, the descriptions are usually sufficient and there is little need to go beyond them if there is at least an intuitive understanding of the mental operations that may have given rise to the goal-directed activity.

For example, Kieren (1993) described three seven-year-old girls as characterizing one of seven children’s share of four pizzas in Figure 4 as “a half and a bite”.

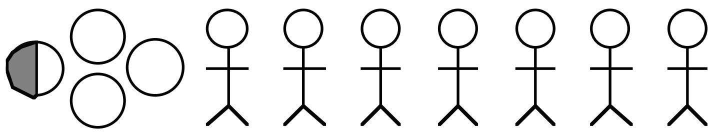

*Figure 4*. A sharing situation.

As researchers, it is our intention to go beyond this description in an attempt to understand and formulate plausible conceptual operations used by the children as they established one child’s share as “a half and a bite.” In this, we infer that the children’s assimilated situation, which involved a question of how much pizza one child would get as well as the picture of the seven children and the four pizzas, constitutes what we interpret as a sharing situation. This inference is based on the result of the children’s activity – “a half and a bite”. We infer that the children would need to establish a goal and engage in a sharing activity in order to reply as they did.

### *The Parts of a Scheme*

This intuitive understanding of the mental operations involved in sharing is enough to qualify the sharing activity as a scheme in the Piagetian sense if we could observe the three children engage in similar sharing activity in other situations. Focusing only on the activity of sharing, however, does not provide a full account of the concept of scheme. von Glasersfeld (1980), in a reformulation of the classical Stimulus → Response schema in associationist learning theories in terms of Piaget's concept of scheme, elaborated the concept of scheme in a way that opens the possibility of focusing on what may go on prior to observable action. It also opens the possibility that the action of a scheme is not sensory-motor action, but interiorized action that is executed with only the most minimal sensory-motor indication. Finally, it opens the possibility to focus on the results of the scheme's action and how those results might close the child's use of the scheme. In these ways, schemes modify the notion of a director system as shall be shown later in this chapter.

According to von Glasersfeld, a scheme consists of three parts. First, there is an experiential situation; an activating situation as perceived or conceived by the child, with which an activity has been associated. Second, there is the child's specific activity or procedure associated with the situation. Third, there is a result of the activity produced by the child.

The first part of a scheme involves the operation of assimilation. For Piaget (1964), assimilation constituted the fundamental relation involved in learning, and he defined it as follows:

> I shall define assimilation as the integration of any sort of reality into a structure, and it is this assimilation which seems to me fundamental in learning, and which seems to me the fundamental relation from the point of view of pedagogical or didactic applications. … Learning is possible only when there is active assimilation. (p. 18)

When we speak of assimilating operations of a scheme, we do not assume that an experiential situation "exists" somewhere in the mind in its totality as an object that a child retrieves. Rather, we assume that records of operations used in past activity are activated in assimilation. In Skemp's model these *records of operations* could be the pathways in an activated schema. In this respect, the first part of a scheme can be regarded as a schema in which the experiential situation has been assimilated. We further assume that the operations comprise a "recognition template" which, when implemented, creates an "experiential situation" that may have been experienced before. When it is clear from context, we refer to the recognition template as an assimilating structure (i.e. a schema).

So, the first part of a scheme consists of a “recognition template” (or schema), which contains records of operations used in past experience. In this respect, the first part of a scheme is very compatible with the way in which a *Director* System must represent the present state (assimilated situation) within a *schema*. Recognition of a situation may activate the scheme’s activity (*Intuitive Processing*). The activity of a cognitive scheme may consist of an implementation of the assimilating operations, and the result of the scheme may consist of an experienced situation. In the case of partitioning operations, mental partitioning activity might constitute the activity of the child’s scheme and an imagined partitioned entity might constitute the scheme’s result. This is in contrast to the sharing scheme alluded to in Figure 4. There, sharing activity was actually implemented, and the situation implied by the comment “a half and a bite” constituted the scheme’s result. In either case we regard both the activity and the result as parts of the scheme, rather than as being external to the scheme, as is the case with Skemp’s model of a director system.

*The Structure of a Scheme*

Figure 5 is a diagram of the idea of a scheme. This diagram is static and as such it can be grossly misleading in interpretation. But it does help to highlight the essential aspects of a scheme. The Generated Goal can be regarded as the apex of a tetrahedron. The vertices of the base of the tetrahedron constitute the three components of a scheme proposed by von Glasersfeld. We include the *generated goal* in our idea of a scheme for reasons similar to Skemp’s inclusion of the goal of a director system (needing to be realized) within a schema that also includes the present state of the situation on which the system is operating.

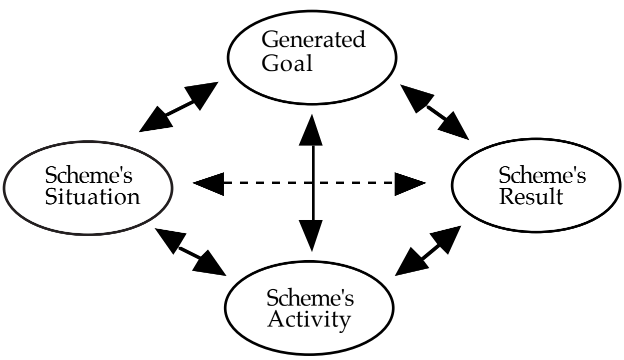


*Figure 5*. The structure of a scheme.

The generated goal has to be associated with the situation of the scheme, the scheme's activity is directed toward that goal, and the results of the scheme are compared to the goal. In these respects, our idea of a scheme is analogous to Skemp's director system but, as will be shown, the ways in which these components may be related are more complex than in Skemp's director system.

The double arrows linking the three base components are to be interpreted as meaning that it is possible for any one of them to be in some way compared or related to either of the two others. The dashed arrow from *situation* to *result* is to be interpreted as an expectation of the scheme's result. The dashed arrow in the opposite direction indicates that the result could modify the situation of the scheme.

The *generated goal* is compared to both the *situation* and *result* (these double arrows are similar to Skemp's *Comparator* in Figure 1). The *generated goal* frames the *activity* of the scheme (arrow from *goal* to *activity* ). It is also possible that the activity of a scheme can generate new goals (or modify an existing goal).

Whereas in Skemp's model *present state* (**P**) and *goal state* (**G**) are locations within a schema, and the action plan of the *director system* is to take an operand from **P** to **G**, the *Generated Goal* in von Glasersfeld's structure of a scheme could be thought of as *framing* the activity of the scheme, and the *result* of the activity is compared to the goal. The result is a new state and if it satisfies the goal then the scheme is closed. For example, in the problem presented in Figure 4, the goal is to find how much of a pizza is the share of one of seven people who are sharing four pizzas. The result is "a half and a bite." For the child, the goal is satisfied and the "sharing scheme" is closed off by the production of the result. (For the teacher, whose goal may have been more mathematically exact, the goal may not be satisfied!)

The goal to find the share for one of the seven stick figures drives the sharing activity during the activity. In this way the goal frames the activity. Partial results (partial from the point of view of the goal) feed back into the goal and we assume that they are compared with the generated goal. This feedback system is indicated by the double arrow between the generated goal and the results. Skemp's notion of a "comparator" is crucial in this situation; however, the comparison is not simply made between "present state" and "goal state" but between successive results of the scheme's activity and the framing goal of the scheme. For us, the "present state" is always changing rather than some fixed position in a cognitive map.

Given a generated goal and a result of a scheme, in some cases it is possible for a child to establish a situation of the scheme if the scheme's activity is reversible. For example, a basic reason why 58 percent of the 11-year-old children in the study of mathematical development conducted by the National Foundation for Educational Research in England & Wales (Foxman, et al., 1980) could not say that 1/4 of 1/2 is 1/8 is understandable when considering the possibility that their fractionizing schemes were not reversible schemes. The students were first given a piece of string and then were asked to cut it in half. The students were then presented with one of the halves and were asked to "cut off 1/4 of this piece". The question "what fraction of the whole string that you started with is that little piece"? was then asked.

The students who were successful in cutting off 1/4 of 1/2 of the whole string had produced a result of their fractionizing scheme and their goal of making 1/4 had been satisfied. When the last question of the series of three was asked, this would serve to establish a new goal and a new situation using the results of the old scheme. Finding 1/8 might involve the students reassembling the four pieces in thought and seeing this as 1/2 of the string partitioned into four equal pieces. If the child then produced another 1/2 of the string in thought also partitioned into four equal pieces to produce the whole string as two equal pieces each partitioned into four equal pieces, the student's fractionizing schemes would be reversible schemes in that the student would be able to start from a result and reestablish the situation using inverse operations. So, in the case a scheme is reversible, its result can be used in establishing a situation of the scheme via the scheme's reversible activity. We stress, however, that these relations are only possible for some schemes. They are not a necessary aspect of all schemes. Some schemes are entirely "one-way" schemes that proceed from situation to activity to result.

The double arrow between the scheme's activity and the scheme's results indicates that the results or partial results may modify the activity, which in turn may modify the results. Likewise, a modification of either of the scheme's activity or results may lead to a modification of the recognition template (the assimilating schema). In further uses of the scheme, the latter modification may in turn lead to a change in the scheme's activity. Of course, the generated goal may also change as the scheme is being used. Thus our schemes are, by their very nature, dynamic and liable to modification whenever they are used in non-routine situations.

*Reflective Processing and Interiorization*

The potential of our schemes for modification when used in non-routine situations points to the very essence of intelligent learning. Skemp's model of intelligence provides us with a possible mechanism for effecting such modifications – a second-order director system that operates on our first-order systems in order to bring these first-order systems into more effective and more powerful conditions. Skemp described the modifications of our first order director systems in terms of improvements in our schemas by formulating new concepts and connections. The improvement could also be in terms of greater *interiority* of the concepts in the schema. "A schema in its general form contains many levels of abstraction, concepts with interiority, and represents possible states (conceivable states) as well as actual states." (Skemp, 1979, p. 190) In terms of scheme-theory, the second-order director system has as its goal the improvement or modification of our schemes of action. As Skemp described in his model, an individual may bring about modifications in his or her own schemes through Reflective Processing. There are several ways in which a scheme may be modified. The recognition template (assimilating schema) may be expanded, thus making the scheme more widely applicable. This would be analogous to extending the domain of a director system. Another, more radical modification concerns the activity of the scheme. We hypothesize that through a process of reflective abstraction the activity of a scheme becomes *interiorized*. Piaget (1964) called such interiorized action an operation: "An operation is … the essence of knowledge: it is an interiorized action which modifies the object of knowledge" (p. 8). This process of interiorization can continue to produce more abstract operations (reinteriorized operations) that have wider applications. In the construction of mathematical knowledge, this process of reinteriorization is obviously necessary for the construction of highly abstract schemes involved in algebraic reasoning; Steffe (1988), however, has shown that for children to reason multiplicatively requires the construction of highly abstract number sequences (schemes) that are the results of several levels of reinteriorization of their initial counting activities. The following example illustrates the process of interiorization for a child who can only count perceptual items (items that the child can actually see and touch).

*From Perceptual to Figurative counting*. A child is given four checkers and asked to find out how many checkers there are in front of him. The request to find "how many" activates the child's perceptual counting scheme, which has as its framing goal "to count the visible objects." The child counts the checkers by pointing to each one in turn uttering the number words "one, two, three, four" and then looks up and says "four." The assimilating structure of the child's counting scheme is a *perceptual plurality* (visible objects to be counted). The activity of the scheme is a coordination of pointing to each item in turn while uttering number words in a set sequence until all items have been touched. The result of the scheme is a *counted collection* and a *final number word.* The four checkers are now covered by a cloth or piece of paper and the child is asked "How many checkers are hidden under the cover?" The *perceptual counter* cannot answer this question even though he has just counted the checkers. The prior result is no longer available. It is only available as a result of the activity of counting, but the counting scheme was closed when the result was obtained. However, the scheme may still be *activated* and in order to *implement* his counting scheme the child needs some objects to count. The assimilating structure of this perceptual counting scheme has nothing to assimilate – there are no visible checkers.

After a few moments, the child points to a position on the cover, staring intently, and utters the word "one." He then continues this activity pointing to another part of the cover and uttering the word "two" – this process continues until the previous result of counting ("four") is obtained. Our explanation of this activity is that the child generated a new temporary goal "create something to count." Because the child had a concept of "checker" the child was able to "re-present" in imagination a *figurative checker.* This was an act of reflective abstraction from the immediate past experience of a checker. The child, however, could not yet re-present more than one checker in imagination (a *figurative plurality*). The child's counting scheme, however, was still activated, thus the *action* of the scheme could be applied to the imagined checker, and another imagined checker can then be produced for continued application of the action of the scheme. The child knew when to stop counting imagined checkers because of the trace of the immediate past counting activity that ended in the number word "four." In essence the child has reenacted his counting activity using a figurative checker in place of the perceptual checkers. What has happened to the child's counting scheme? It has been *internalized* in that it can now be applied to imagined checkers and not just perceptual checkers. Further applications of this internalized scheme will lead to an *interiorization* of the scheme whereby the *results* of the counting scheme can be taken as *given* – the final number word will now represent the whole scheme and the child *does not have to recount even imagined checkers*. The number word "four" can stand for the act of counting "one, two, three, four." The child can now "count-on" and has constructed what Steffe, von Glasersfeld, Richards and Cobb (1983) have termed an "initial number sequence" – *an Interiorized Counting Scheme.*

From the point of view of Skemp's theory, we could say that the blocked implementation of the child's perceptual counting scheme caused a shift of consciousness from functioning at the level of Delta 1 (implementation) to Delta 2 functioning (modification). The first modification was to generate a new but temporary goal: *form items to count*. This, in turn led to a modification of both the recognition template (assimilating schema) and the activity of the perceptual counting scheme so that it could operate on one imagined item at a time as well as perceptual items.

While schemas may contain *interiorized* concepts (that are themselves schemas) (Skemp, 1979, p 141), higher-order schemes contain *interiorized actions* that are themselves schemes. An interiorized action can be taken as given – that is, the child does not have to carry out the action to obtain a result. As seen in the example above, a child with an "Initial Number Sequence" (an interiorized counting scheme) the number four stands for the result of having counted from one to four. Such a child can "count-on," whereas a child with only a figurative or perceptual counting scheme cannot count-on. For the perceptual or figurative counter, when three new checkers are added to a previously counted set of four checkers, the child has to "count-all" to find out how many checkers there are altogether. The child with the interiorized counting scheme does not have to count-all. This child points to the set of previously counted checkers and says "four" then continues counting on "five, six, seven" while pointing to each of the new checkers.

## Interiorized Schemes: A Modified Model of Intelligent Learning

Both interiorized concepts (schemas) and interiorized actions are needed for intelligent learning. Are interiorized actions accounted for in Skemp's model of intelligent learning? Not exactly. Skemp suggested that by bringing a delta 1 schema into conscious awareness (by action of delta 2) several plans of action can be formulated in thought and their results anticipated without actually carrying out the plans, and thus, an optimal plan could be chosen and then put into action by delta 1. This is not the same as having an *interiorized action* – that is, a conceptual symbol for the "plan-in-action" that is its *result.* In the following modification of the structure of a scheme we have attempted to include both *interiorized concepts* (Skemp's schemas) and *Interiorized Action*. Figure 6 illustrates a modification of Figure 5 in order to incorporate some critical aspects of Skemp's Director System into the structure of a scheme.

In Figure 6 the dashed arrows indicate possible modifying relations. The solid arrows indicate a direct relationship. **P** represents the *present state* of the assimilated situation. The generated goal and the result also need to be realized within a schema. For the scheme to function effectively, all three (goal, situation and result) must be part of the same schema. It is conceivable that some schemes may realize these three components in disjoint schemas (for instance, when the goal is set by another person and not fully understood, or when the result is unexpected and connects with a different schema from the one used in the assimilation of the situation). The arrows between Goal and Situation, and between Goal and Result have been replaced by Skemp's *comparators* as the relations between these components are primarily comparison relations.

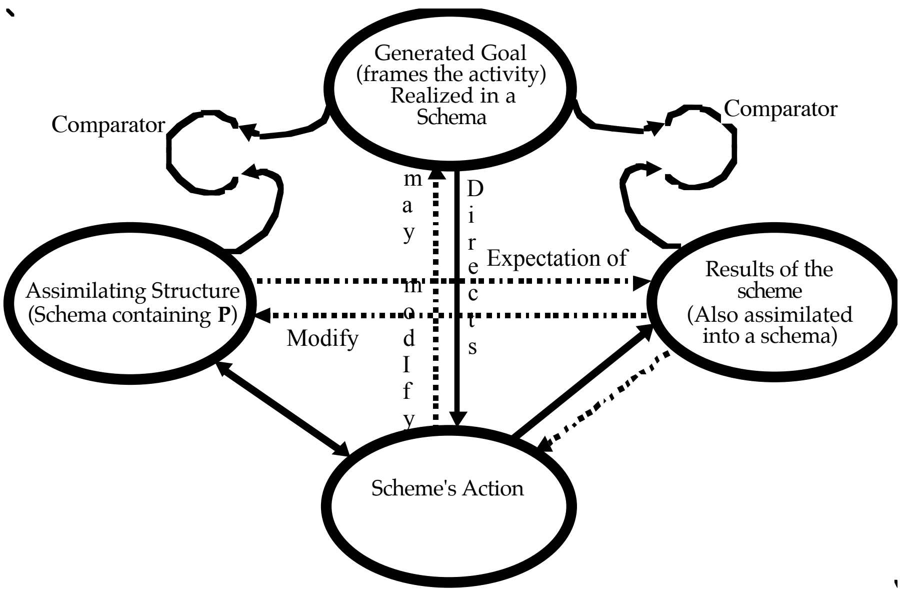


*Figure 6*. Modified structure of a scheme.

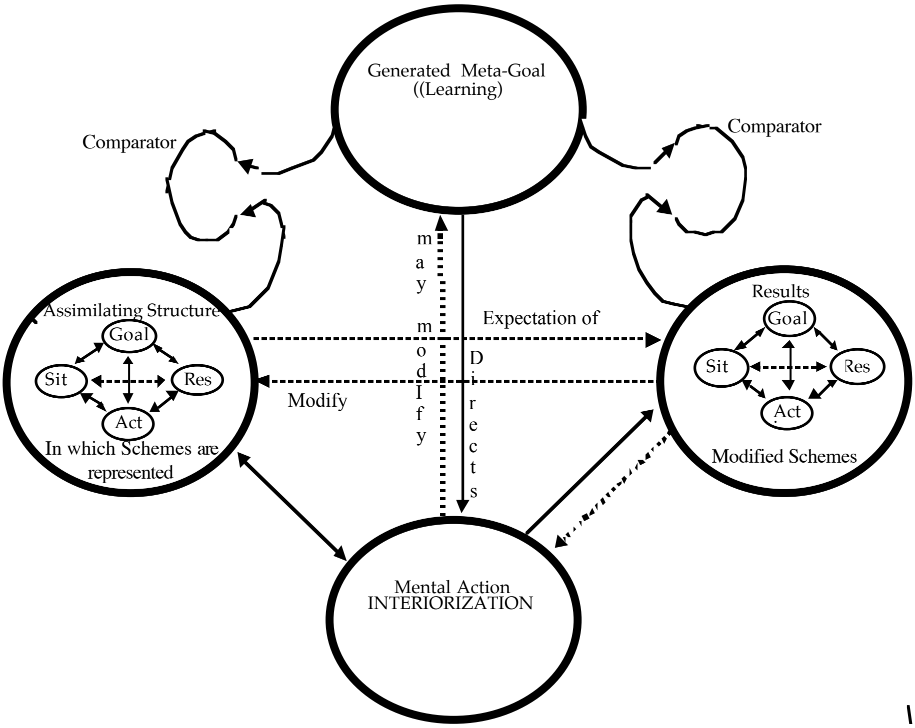


*Figure 7.* An interiorizing scheme.

In Figure 7 we represent what we think of as an *Interiorizing Scheme,* which modifies Skemp's notion of a *Second-Order Director System* through the structure of a scheme that operates on "lower-order" schemes. Thus the *Assimilating Structure* of an interiorizing scheme is a schema in which the person's schemes of action can be represented. The *Mental Action* of the interiorizing scheme is *Interiorization* of the actions of the lower-order scheme that has been assimilated. The *Results* of the interiorizing scheme are modified (interiorized) schemes, and the generated goal is *Learning*. The power of this modified model lies in its *recursive* structure, in that the resulting interiorized schemes can be then assimilated by the interiorizing scheme for reinteriorization. This process of reinteriorization, of course, is not automatic but can only be brought about through perturbations encountered through the use of a scheme (as was the case when the child's counting scheme was blocked). This idea of a recursive model of intelligent learning is compatible with the recursive model of mathematical understanding developed by Kieren and Pirie (1991).

What Skemp's model adds to the Scheme Structure is the importance of two distinct levels of goal directed activity: the first level is involved with the *implementation* of our schemes of action, the second is involved with the *modification* of our schemes through a process of interiorization brought about through reflective abstraction. However, we do not limit Delta 1 to directing only sensory-motor activity, but to the implementation of any action scheme (physical or mental activity). Delta 2 directs the mental activity involved in modifying our schemes – that is, accommodation and expansion. Accommodation is the result of interiorized action, whereas expansion is accomplished through interiorized concepts (expanding the schemas of the scheme).

When Delta 1 is in control, the activity of the scheme is directed outward – to something or some situation that the scheme is acting on. When Delta 2 is in control, the activity of the scheme is directed *inward* – to some aspect of the scheme's structure: the *recognition template,* the *activity,* the *result* or the *generated goal.* In this way, Delta 2's activity is always "reflective." The *interiorization process* is a Delta 2 activity for both *schemas* and *schemes*.

What we have attempted to do in our research on children's fraction knowledge is to find ways of collaborating with the children's Interiorizing Schemes (or second-order director systems) to bring about yet further reinteriorizations of their number sequences that produce schemes for operating on fractional quantities. The following section is an example of this kind of collaboration that resulted in the reinteriorization of the children's partitioning schemes.

## Collaboration with and Modification of Children's Partitioning Schemes

Our collaboration with children's interiorizing schemes took place in the context of a *teaching experiment* with 12 children over a period of three years. We started working with the children when they were in their third grade of elementary school (8 to 9 years of age). We worked with them in pairs for approximately 45 minutes each week for approximately 20 weeks each of the three years (through their fifth grade of elementary school). The contexts for our teaching episodes were established using computer environments that we developed especially for the project. These *Tools for Interactive Mathematical Activity (TIMA)* microworlds provided the children with on-screen manipulatives and actions on those manipulatives that were designed to provide means of enacting their fractionizing operations. There were three such computer environments: *Toys, Sticks* and *Bars.* (Olive & Steffe, 1994) All teaching episodes were videotaped using two cameras – one focused on the computer screen and one on the children and teacher.

The coherence of a teaching experiment resides in what we can say about bringing forth children's mathematical schemes and about sustaining and modifying those schemes. The process we engage in can be regarded as an instantiation of Skemp's three modes of Schema Construction outlined in Table 1 above. We *generate and test hypotheses in our teaching episodes*. The hypotheses may be generated on the fly while interacting with the children (mode 1) or in between two teaching episodes through discussion with the observers of the teaching episode (mode 2). Through generating and testing hypotheses, boundaries of the children's ways and means of operating can be formulated. These boundaries arise from the children's failure in making adaptations and correspond to the boundaries of Skemp's *domain of a director system.* What are inside of the boundaries constitute living models of children's mathematics – the actual mathematical activity of children that can be judged to be independently contributed by the children (the *prohabitat* of the children's particular scheme). Our second-order *schemas* of the children's mathematical activity are the product of retrospective analyses of the videotapes of the teaching episodes (mode 3). As illustration, we present an example of a basic partitioning scheme which we were able to engender in the children through our teaching actions.

*The Equi-Partitioning Scheme*

We did not decide a priori how to base children's construction of partitioning schemes on their number sequences. Rather, our method arose as an insight in the context of teaching two children, Jason and Patricia, when they independently used their numerical concepts in partitioning what to us was a continuous unit. Not only did our method arise in the context of teaching these two children, but what we now refer to as the equi-partitioning scheme was constructed by us as mathematics *for* Jason and Patricia as we taught the two children. Prior to our teaching experience, we had a concept that we called "partitioning", but it was based on mathematical analysis of equivalence relations. We found it necessary when teaching Jason and Patricia to construct a concept of partitioning that consisted of less abstract and of more specific conceptual operations than equivalence relations. Our concept of equivalence relations served us in defining the rational numbers of arithmetic as equivalence classes, but it did not help in explaining Jason and Patricia's partitioning behavior.

Using the computer program TIMA: Sticks, it was our goal for Jason and Patricia to break a stick into two sub-sticks of equal length. After being asked to draw a stick and to share it equally, Jason drew a stick that spanned the screen as shown in Figure 8 and cut it into two pieces. The two pieces were obviously of different lengths and the two children made a visual comparison between them. The teacher then asked, "How do you know that the two pieces are of the same size?" There were several suggestions for how to test to find if the two pieces were of the same size, such as Jason's comment to "copy the biggest one and then copy them again". He then said, "no," shaking his head. After Jason's suggestion, the two children sat in silent concentration.

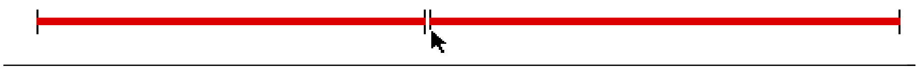

*Figure 8*. A stick cut into two pieces in TIMA: Sticks.

The teacher's question seemed essential in provoking a perturbation within the two children. Their perturbation is indicated by Jason saying "no" and shaking his head as well as by the children sitting in silent concentration. They had formed a goal of justifying why the two pieces were of equal size, but they seemed to have no action they could use to reach their goal. This, along with Protocol I illustrates our role in bringing forth the children's numerical concepts in a situation where there were no countable items in the children's visual field.

*Activation of the children's numerical concepts.* The teacher suggested to Patricia that she draw a shorter stick that would be easier to divide visually. After Patricia drew this stick, the actions the children contributed that are reported in Protocol I were not suggested by the teacher. Rather, they arose independently from the children, which illustrates our understanding of the children as self-regulating and self-organizing.

*Protocol I*

T: (After Patricia had drawn a stick about one decimeter in length) Now, I want you to break that stick up into two pieces of the same size.

> P: (Places her right index finger on the right endpoint of the stick, then places her right middle finger to the immediate left of her index finger. Continues on in this way, walking her two fingers along the stick in synchrony with uttering) one, two, three, four, five. (Stops when she is about one-half of the way across the stick).
>
> J: (Places his right index finger where Patricia left off; uses his right thumb rather than his middle finger to begin walking along the stick. Changes to his left index finger rather than his right thumb after placing his thumb down once. Continues on in this way until he reaches the left endpoint of the stick) six, seven, eight, nine, ten. (Then) there're five and five (Smiles with satisfaction).
>
> P: (Smiles also).

Patricia independently introduced the action of walking her fingers along the stick until she arrived at a place she regarded as one-half of the way. Jason picked up counting where Patricia left off, which solidly indicates that he assembled meaning for Patricia's method of establishing equal pieces of the stick[2] . Patricia's counting activity was meaningful to him and he could be said to engage in cooperative mathematical activity with Patricia (Mode 2 *scheme* construction). Patricia, as well as Jason, now had a way to at least justify where the stick should be cut so that the two pieces would be of the same size. The pleased look on their faces indicated that they had achieved their goal.

Establishing a blank stick as a situation of counting suggests that Patricia projected units into the stick in such a way that she imagined the stick broken into two equal sized pieces, where each piece was in turn broken into an indefinite numerosity of pieces of the same size. The fact that Patricia counted indicates that she was aware of an indefinite numerosity of pieces prior to counting. In this, she satisfied two essential conditions for a counting scheme to be activated. The first is that the child establishes a plurality of unit items to be counted even though there may be no such items in the child's visual field, and the second is that the child is aware of an indefinite numerosity of such items. But rather than use her unitizing operations to establish a collection of discrete items, she used them as partitioning operations. This opened a path for the construction of fractionizing schemes.

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[2] Using "five and five" as a mechanism for establishing two equal pieces seemed to evolve in the joint activity of the two children. It did not seem to be the original goal of either child although one could argue that Patricia originally intended to count five times. In watching the video tape, it seemed that she stopped counting at "five" because she reached a place that she regarded as one-half of the way across the stick.

The children's method of justifying where the stick should be cut may seem to have achieved our goal of basing fractionizing schemes on children's number sequences. But it only established the goal as a possibility. Although the children could use their composite units as templates for partitioning a stick into equal and connected parts, we regarded these templates as only assimilating structures of a possible fractionizing scheme. The children had no reliable activity they could use to partition a stick into any definite numerosity of pieces.

*A new concept of partitioning*. Based on the insight that the children independently used their number concepts in partitioning, we constructed a new concept of partitioning that was based on our social models of children's numerical concepts rather than on our concept of mathematical equivalence. Because each element of the composite units Jason and Patricia used as templates in breaking a stick was an iterable unit[3] , we *hypothesized that the operations of iterating and partitioning are parts of the same psychological structure*. The hypothesis would be confirmed if any single part of a partitioning could be used to reconstitute the unpartitioned whole by iterating the part.

For a child to be judged to have constructed an iterable unit, the child must be able to experientially repeat the unit. But, unless the observer can make two more inferences, the simple repetition of a unit would not imply its iterability. For a unit to be iterable, there must be indication that the child can use the operation of repetition in re-presentation to produce a sequence of unit items that can be taken as a composite unit. In this, the operation of repetition is carried out at the figurative level rather than at the sensory-motor level. Second, as the child repeats the unit, there must be the possibility that the child can unite a current repetition with those preceding, forming a nested sequence of units. If there is solid indication of these two conditions, the child could be judged to have constructed an anticipatory iterating scheme with an expected outcome. Being able to make these two inferences was necessary in a test of our hypothesis.

*Constructing an equi-partitioning scheme.* Based on our concept of partitioning as a psychological structure that included both operations of breaking a continuous unit into equal sized parts and iterating any of

[3] In a teaching experiment, the researcher builds a case for an interpretation by consideration of what the child may or may not do in problem solving episodes. An interpretation is rarely based on only one observation even though singular observations usually are the contexts in which insight occurs. The claim that the two children had constructed their units of one as iterable is warranted by our observations of them solving a myriad of numerical situations.

the parts to reconstitute the whole, we designed a situation of learning in a test of our hypothesis. If our concept of partitioning was viable, then we should have been able to engender an accommodation in the children's number sequence in such a way they would use the parts of the stick they produced in partitioning as countable items rather than their fingers as in Protocol I.

*Protocol II*

T: Let's say that the three of us are together and then there is Dr. Olive over there. Dr. Olive wants a piece of this candy (the stick), but we want to have fair shares. We want him to have a share just like our shares and we want all of our shares to be fair. I wonder if you could cut a piece of candy off from here (the stick) for Dr. Olive.

J: (Using MARKS, makes three marks on the stick, visually estimating the place for the marks.)

P: How do you know they are even? There is a big piece right there.

J: I don't know. (Clears all marks and then makes a mark indicating one share. Before he can continue making marks, the teacher-researcher intervenes.)

T: Can you break that somehow (the teacher-researcher asks this question to open the possibility of iterating)?

J: (Using BREAK, breaks the stick at the mark; then makes three copies of the piece; aligns the copies end-to-end under the remaining piece of the stick starting from the left endpoint of the remaining piece as in Figure 9.)

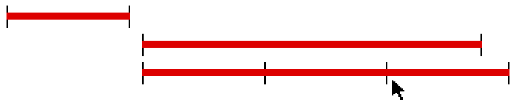

*Figure 9.* Jason testing if one piece is one of four equal pieces.

T: Why don't you make another copy (this suggestion was made to explore if Jason regarded the piece as belonging to the three copies as well as to the original stick)?

J: (Makes another copy and then aligns it with the three others. He now has the four copies aligned directly beneath the original stick which itself is cut once. The four pieces joined together were slightly longer than the original stick as in Figure 10.)

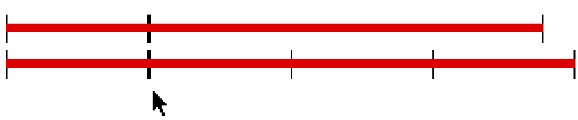

*Figure 10.* Jason's completed test.

Jason copied the part he broke off from the stick three times in a test to find if three copies would reconstitute the remaining part. This was crucial because, along with his completed test, it allowed us to make the two inferences that we stated above. This way of operating was a modification of his countable items displayed in Protocol I[4] , and it solidly indicated that the part he broke off from the stick was for Jason an iterable unit. The iterability of his units was presaged by his comment "copy the biggest one and copy them again" preceding Protocol I. Patricia, in the same teaching episode, demonstrated that she too could operate in the way Jason operated in Protocol II. We regard the scheme that Jason and Patricia constructed as an equi-partitioning scheme. It is crucial to understand that the children's language and actions served in the test of our hypothesis. It is crucial because Jason's test to find if the piece he broke off was a fair share was contributed by him with only a minimal intervention by us.

We now have a *living model* for what it might mean for children to construct the equi-partitioning scheme as an accommodation of their number sequences. It was produced in a retrospective analysis of the two teaching episodes in which it was observed. However, we still have to specify the accommodations that must have occurred in the unit items that constitute the elements of the children's numerical concepts when they used their numerical concepts in partitioning. We also need to establish whether the change that was observed from Protocol I to Protocol II constituted an accommodation or whether it was only a temporary modification that was constrained to the situation of observation.

## Final Comments

Children's mathematics can be stressed in their mathematics education without the kinds of models that we propose here. But without such models, children's mathematics remains situated within the contexts of observation and can be at most described. Without the explanations that we are attempting to build through conceptual analysis, children's mathematics stands little chance of becoming taken-as-shared in the community of mathematics educators, including teachers. If children's

[4] Although it is possible to regard Jason's behavior in Protocol I as unrelated to his behavior in Protocol II, this would miss Jason's use of his number concepts in partitioning.

mathematics is to become the focus of their mathematics education, a language must be developed for its communication.

As did Skemp, we focus on children's mathematics as a constitutive part of a living conceptual system. This way of understanding children's mathematics has great advantages for mathematics education and puts us in education, we think, in an appropriate frame of reference. No longer is the sole focus on the abstracted adult concepts and operations, and no longer is children's mathematical development conflated with those abstracted concepts and operations. Rather, the focus is on the living systems that children comprise and the problem is to understand how to bring the subsystems called mathematics of these living systems forth, and how to bring modifications in these subsystems forth. We believe that the integration of scheme theory (as we are using it) within a modified framework of Skemp's model of intelligent learning provides a powerful schema for generating this understanding.

## References


Foxman, D., Cresswell, M. J., Ward, M., Badger, M. E., Tuson, J. A., & Bloomfield, B. A. (1980). *Mathematical development: Primary survey report no. 1*. London: Her Majesty's Stationery Office.

Kieren, T. (1993). Rational and fractional numbers: From quotient fields to recursive reasoning. In T. Carpenter, E. Fennema, & T. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 49–84). Hillsdale, NJ: Lawrence Erlbaum Associates.

Kieren, T. E., & Pirie, S. E. B. (1991). Recursion and the mathematical experience. In L. P. Steffe (Ed.), *Epistemological foundations of mathematical experience* (pp. 1–13). New York: Springer-Verlag.

Piaget, J. (1964). Development and learning. In R. E. Ripple & V. N. Rockcastle (Eds.), Piaget rediscovered: A report of a conference on cognitive studies and curriculum development (pp. 7–19). Ithaca, NY: Cornell University Press.

Piaget, J. (1980). Opening the debate. In M. Piattelli-Palmarini (Ed.), *Language and learning: The debate between Jean Piaget and Noam Chomsky* (pp. 23–34). Cambridge: Harvard University Press.

Romberg, T. A. (1982). *The development and validation of a set of mathematical Problem-solving superitems*. The University of Wisconsin, Madison: Wisconsin Center for Education Research.

Skemp, R. R. (1979). *Intelligence, Learning, and Action*. Chichester and New York: Wiley.

Skemp, R. R. (1987). *The Psychology of Learning Mathematics*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Skemp, R. R. (1989). *Mathematics in the Primary School.* London: Routledge.

Steffe, L. P. (1994). Children's multiplying schemes. In G. Harel & J. Confrey (Eds.). *The development of multiplicative reasoning in the learning of mathematics* (pp. 3–40). Albany: State University of New York Press.

Steffe, L. P. (1988). Children's construction of number sequences and multiplying schemes. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 119–140). Hillsdale, NJ: Lawrence Erlbaum Associates.

Steffe, L. P., Cobb, P., & von Glasersfeld, E. (1988). *Construction of arithmetical meanings and strategies*. New York, NY: Springer-Verlag.

Steffe, L. P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory, and application*. New York: Praeger Scientific.

von Glasersfeld, E. (1980). The concept of equilibration in a constructivist theory of knowledge. In F. Benseler, P. M. Hejl, & W. K. Kock (Eds.), *Autopoisis, communication, and society* (pp. 75–85). Frankfurt, West Germany: Campus Verlag.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Washington, D. C.: The Falmer Press.

# What is a Scheme?

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This chapter is dedicated to, and fundamentally influenced by, Richard Skemp's pioneering work on schemes. Both of us were present in Warwick when Richard died, and we sang at his funeral. This is no maudlin sentiment but a deeply felt gratitude for the man and his work. David Tall occupies Skemp's Chair of Mathematics Education at Warwick, and both of us were influenced, as have been many mathematics educators, by Skemp's highly polished works. His contributions have an enduring quality because he tackled basic issues of mathematical intelligence.

## Schemes: Psychology to Mathematics Education

The terms "schema" and "schemata" were apparently introduced into psychology by Bartlett (1932), in his study of memory. Bartlett took the term from the neurologist Henry Head who had used it to describe a person's conception of their body or the relation of their body to the world. Bartlett used the term schema in much the same way as Skemp, following him, did: as an organised structure of knowledge, into which new knowledge and experience might fit. The utilisation of Head's notion of schema in psychology was reviewed by Oldfield and Zangwill (1942a, 1942b, 1943). Bartlett's notion of schema was picked up by Skemp (1962, 1971), and then Rumelhart (1975) also resurrected Bartlett's idea and terminology, once again in the study of memory.

Minsky (1975) introduced his idea of "frames" and Schank (1975) the idea of "scripts", both of which are similar to Bartlett's schemata. Davis' (1984) influential book on cognitive science methods in mathematics learning leant heavily on the idea of schemes. It is fair to say that whilst the term scheme has been used in mathematics education (see, for example, Steffe, 1983, 1988; Davis, 1984; Dubinsky, 1992; Cottrill *et al.*, 1996) there have not been many attempts to define more precisely what might constitute a scheme.

Notable exceptions, apart from Skemp's own writings, are the articles by Dubinsky (1992) and Cottrill *et al.* (1996), the first of which at least acknowledged that some sort of definition would help people working in the field, and the second of which attempted a recursive definition.

Skemp (1986) made it clear that schemes play a pivotal role in relational understanding:

> *To understand something means to assimilate it into an appropriate schema.* (p. 43; author's italics).

Chapter 2 of *The Psychology of Learning Mathematics* is entitled "The Idea of a Schema". In that chapter Skemp makes it clear that he views schema as a connected collection of hierarchical relations. It is this point of view that we wish to explore.

## Structuring the World: Categories

A necessary condition for higher order mental functioning is the ability of an individual to categorise things in the world. In order to count, for example, we need things to count. These things are categorised by us as instances of the same thing for the purposes of counting. Do you have enough clean shirts to take on holiday? Are there enough chairs to seat all our guests? In order to answer questions like these we necessarily have to see different things, such as different shirts or chairs, as instances of the same category for the purpose of counting them.

This ability to categorise is possessed by many animals (Edelman, 1989) and is fundamental to the ways in which human beings structure their worlds. It is critical in mathematics learning because counting, the first mathematics that most of us engage in, is so clearly predicated on an ability to categorise.

The patterned records of actions we use in mathematical activity are themselves instances of categorisation. The categorisation involved in the formation of schemes is the brain categorising its own activities. In primary categorisation, perceptual events are categorised, whereas in secondary categorisation – the type of categorisation in which schemes are formed – the brain's own responses to perceptual categorisation are themselves categorised. This reflective activity of the brain is an essential part of Piaget's theories of the development of logico-mathematical structures. Recently, it has been given a more detailed functional and structural form by Edelman (1989, 1992):

> … the theory of neuronal group selection suggests that in forming concepts, the brain constructs maps of its own activities, not just of external stimuli, as in perception. According to the theory, the brain areas

> responsible for concept formation contain structures that categorize, discriminate, and recombine the various brain activities occurring in different kinds of global mappings. Such structures in the brain, instead of categorizing outside inputs from sensory modalities, categorize parts of past global mappings according to modality, the presence or absence of movement, and the presence or absence of relationships between perceptual categorizations. (1992, p. 109, author's italics)

## Action Schemes and Higher Order Schemes

Apart from the ability to subitize numbers in suitably configured arrays, number involves counting, which is an action scheme – a sequence of actions performed to achieve a goal. Using counting, addition is an extended action scheme to obtain the total in two collections. Children use the initial "count-all" scheme until they recognise certain parts of the process as redundant. They then make new connections by omitting the initial counting. This new reconstructed scheme misses out the first part of the count and performs a "count-on". If they sufficiently well manage the "count-on" to allow a neuronal trace of the input to link with the output (or the results are re-corded to *see* the links), they may make a connection between input and output to give the known fact.

Essential to an understanding of schemes is the focus of attention of a learner. What it is a person focuses on in an action scheme determines the consequent structure of that scheme for them. The first attempt to use attentional systems as the basis for mathematical development appears to be von Glasersfeld's (1981) attentional model of unitizing operations. These operations play a major part in his theoretical model for an operation of the human mind that creates a unitary item from perceptual experience. They are still our only model for how it is humans form units and pluralities. So, in this sense, attention is fundamental to how we understand numerosity. von Glasersfeld stated very clearly what he meant by attention for this purpose:

> ... I want to emphasize that 'attention' in this context has a special meaning. Attention is not to be understood as a state that can be extended over longish periods. Instead, I intend a pulselike succession of moments of attention, each one of which may or may not be 'focussed' on some neural event in the organism. By 'focussed' I intend no more than that an attentional pulse is made to coincide with some other signal (from the multitude that more or less continuously pervades the organism's nervous system) and thus allows it to be registered. An 'unfocussed' pulse is one that registers no content. (p. 85)

and he cites neurophysiological experiments that support the view that:

> … attention operates above, and independently of, sensation and can, therefore, function as an organizing principle; second, if attention can, indeed, shift from one place in the experiential field to another, it must have a means of regarding ... these places and disregarding what lies in between. (p. 85).

Von Glasersfeld's model is fundamental to the Steffe-Cobb-von Glasersfeld-Richards theory of children's counting types and arithmetic operations (Steffe, von Glasersfeld, Richards & Cobb 1983; Steffe, Cobb & von Glasersfeld, 1988). von Glasersfeld provides a model of how records of experience can be obtained by a focus of attention on the perceptual stream of data. It is a basic tenet of a constructivist theory of learning, as we understand it, that these records of experience – the memories – are not veridical records of any *thing* in the world external to the recorder. Rather they are chemical traces of neuronal activity occurring as a result of perceptual interaction with the world. Importantly, for our theme, these traces are themselves capable of being taken as the primary material for experience:

> Perceptual categorization, for example, is non-conscious and can be carried out by classification couples, or even by automata. It treats *signals from the outside world* – that is, signals from sensory sheets and organs. By contrast, conceptual categorization works from within the brain, requires perceptual categorization and memory, and treats *the activities of portions of global mappings* as its substrate. (Edelman, 1984, p. 125, author's italics)

This, for us, is in essence how schemes come into being, so we make the following definition:

- An action scheme (or 0-order scheme) is a sequence of actions performed to achieve a goal.
- An $n^{\text{th}}$-order scheme is a categorization of lower order schemes.

Our brains take as basic data for reflection the records of our previous experience, and just as we categorize visual perceptions from cup-like things into the category of cups, so we categorize patterns of action such as directed movement with synchronized standard utterances into the category of counting.

## Examples of Scheme Formation

A well-studied action scheme is children's dealing, or distributive counting, to allocate fair shares (Miller, 1984; Davis and Pitkethly, 1990; Davis and Hunting, 1990; Davis and Pepper, 1992). This is a sequence of actions that children in many, if not all, cultures can carry out to allocate fair shares , but one which, at a young age, 4– 5 years, they usually do not internalize as a 1st order scheme. By the time children are about 8 years of age, however, it is not uncommon for them to refer to dealing as "just like when you play cards". At this point, we infer, they have conceptually categorized various action schemes of dealing and so established a 1st-order scheme for dealing.

An example of higher order scheme formation appears in the thought of a student, Stephanie, described by Maher and Speiser (1997). Carolyn Maher at Rutgers University, and her colleague Robert Speiser, at Brigham Young University, reported a beautiful series of episodes involving a young high school student, Stephanie. She found the formula for calculating the binomial coefficients by relating the problem to one of counting towers of blocks – a problem she, along with many other children, had studied in elementary school.

In 1989 Maher, and her colleague Al Martino, began working with a small group of first graders to encourage the children to explore and explain their differences in thinking as they solved problems. Stephanie was one of the children in this group and when she was in grade 4 worked on the problem: how many different towers of a fixed height can we make from blocks of two colors? (Maher and Martino, 1996; 1997). In the fall of 1995 – coincidentally, just after Richard Skemp passed away – Stephanie had moved to a different school and was in grade 8. Maher and Speiser worked with Stephanie on her reasoning in mathematics problems, which now revolved around algebra. Stephanie had calculated the binomial coefficients for $(a+b)^2$ and $(a+b)^3$. One of the researchers asked her about these numbers for $(a+b)^3$ and Stephanie replied: "So there's *a* cubed. ... And there's three *a* squared *b* and there's three *ab* squared and there's *b* cubed." She then said: "Isn't that the same thing?" The researcher asked what she meant, and Stephanie replied: "As the towers."

It's fair to say that the researchers were surprised at this sudden, unannounced, appearance of towers in an algebra problem. Stephanie had not systematically counted towers of blocks in school since elementary classes. Maher and Speiser's hunch was that Stephanie was visualizing towers of height 3 in order to organize the products of powers $a^i b^j$ of $a$ and $b$. They got support for this hunch when, in questioning Stephanie further about the terms in the expansion of $(a+b)^3$, she said: “I don’t want to think of $a$’s. I want to think of red.” In the same session Stephanie was asked: “What about $b$ squared?” and she replied: “Um. Two yellow.”

Stephanie seemed to interpret the numbers $a$ and $b$ in the expression $(a+b)^3$ as colours of blocks – red or yellow – and the index “3” as the height of a tower of red or yellow blocks. She could then bunch together those towers that had the same number of red (and so same number of yellow) blocks. This was because, in her interpretation, the order of the red and yellow in the towers didn’t matter: the towers were standing in for *products* of $a$’s and $b$’s, just as the individual red and yellow blocks were standing in for individual $a$’s and $b$’s.

Imagery can act as a *generator* of mathematical thought, when the imagery acts to reduce the load on working memory. This is what happened to Stephanie when she formed images of the expansion of $(a+b)^3$ as a row of towers of red and yellow blocks. Far from being a dead weight, filling up working memory with irrelevant detail, the images acted as a powerful engine for her, allowing her to accelerate rapidly and easily through some complicated counting.

Two weeks after the episode reported above, Stephanie was asked to explain to a researcher who was not previously present, what she had said about binomial coefficients and towers. As she began to explain Stephanie described, in vivid detail, episodes from a grade 4 class in which she and two classmates had figured out how to build towers of a given height from those of height one less. Stephanie was able to write a recursive formula for the binomial coefficients by utilising her recalled images of towers of blocks.

That Stephanie had established a 1$^{st}$-order scheme from conceptual categorization of the action schemes of building blocks was evident from her previous conversations about them. At the time of the first interview, above, there does not seem to be evidence that she had similarly schematized the action schemes of expanding a power of a sum of two terms. However Stephanie made the remarks: “Isn’t that the same thing?” and “As the towers” in her explanation of her algebraic calculations. At this point we infer that she has, by analogy, established a 1st-order scheme for binomial expansions. When, two weeks later, she is able to write down the recurrence relation for the binomial coefficients by reference to counting towers we infer that Stephanie is able to express clearly that she sees essentially the same process occurring in these two 1st-order schemes. In other words, she has established a 2nd-order scheme through conceptual categorization of 1st-order schemes.

## Schemes and Symbols

The operations of arithmetic benefit from being written in signs. These signs enable a subtle form of compression that is not immediately apparent. For many students the signs are simply indicators – for example, the sign "2 + 3" indicates to some students that one should carry out an action, whereas for other students the same sign also functions as a symbol. For these latter students there is a symbolic relation with other signs and a flexibility in interpreting a frame of reference for the signs. This is essentially what Gray and Tall (1994) refer to as a procept. Steffe (1988) has written in relation to symbols that:

> Children's operations seem to be primarily outside their awareness, and, without the use of symbols, they have little chance of becoming aware of them nor can they elaborate those operations beyond their primitive forms.

In answer to the question of whether his use of the word "symbols" referred to conventional mathematical signs, or whether there was a deeper interpretation in which the symbols were records of a process of interaction between a student and teacher, Steffe (personal communication) replied:

> I was thinking of, for example, the way in which a child might use the records of past experience that are recorded in the unit items of a sequence to regenerate something of that past experience in a current context. The figurative material that is regenerated may act as symbols of the operations of uniting or of the results of the operations in that the operations may not need to be carried out to assemble an experiential unit item. The figurative material stands in for the operations or their results. In this I assume that the records are interiorized records – that is, records of operating with re-presented figurative material.
>
> My hypothesis is that these operations will continue to be outside the awareness of the operating child until a stand-in is established in which the operations are embedded. ... awareness to me is a function of the operations of which one is capable. But those operations must become objects of awareness just as the results of operating. To become aware of the operations involves the operations becoming embedded in figurative material on which the operations operate. To the extent that this figurative material can be re-generated the operations become embedded in it.

The figurative material is also operated on again. In this way the operations are enlarged and modified.

The idea that operations are outside conscious awareness until a mental "stand-in", or symbol, is developed upon which the operations can act mentally – is of critical importance in the development of mathematical schemes. For it is these symbols that form 1st-order schemes: not only is the scope of the operations extended, as Steffe says, but what the operations operate on becomes a mental object – a 1st -order scheme.

When Bob Speiser talked about Stephanie's scheme for thinking about the binomial coefficients at the Psychology of Mathematics Education conference in Lahti, Finland, we wondered how much of the connection between the towers and the binomial coefficients Stephanie had taken on board. We asked him if Stephanie had thought about a more complicated situation, such as $(a+b+c)^3$ in terms of towers made from three colours. He affirmed that she had indeed; what's more, Stephanie could write down the recursive definition for the coefficients of the products of powers for *any number* of variables *a, b, c, d , ...* by using images of towers of blocks of the same number of colours as the number of variables.

This is a singular achievement for a grade 8 student, and it illustrates the amazing generative power of vivid images that are tightly coupled to a problem. The complicated looking signs:

$(a+b)^n =$ sum of terms $C(p,q)a^p b^q$ where $p+q=n$, and

$C(p+1,q+1) = C(p,q) + C(p,q+1)$

with $C(n,0)=1$ and $C(n,n)=1$

are not abstruse to a child who has assimilated a model in terms of towers of coloured blocks. These signs simply express a recursive way of building towers. What appears to be complicated mathematics is just a way of writing this recursive relationship. The written mathematics – the marks – are what many students focus their attention on. In so doing they can, and often do, lose sight of the fundamentally simple idea that the marks, or signs, express. Worse, their only terms of reference for the signs is likely to be at an indexical level – a conditioned response. This is how many students see formulas in mathematics: as something upon which something has to be done, such as rearrangement, substitution, cancellation or similar actions. What the signs refer to for students who think this way is the actions that they themselves could carry out. Furthermore, that is all the signs refer to. A student who understands the signs symbolically can also do things like rearrangement, substitution, and cancellation. However, they can do more. They can focus on other aspects of the signs, such as Stephanie's focus on the recursive formula for the binomial coefficients in terms of a recursive procedure for building towers of blocks. This is an essential aspect of Peirce's (see Deacon, 1997) view of the icon-index-symbol relationship as a hierarchy. The ability to operate at one level in this hierarchy implies an ability to operate at a lower level, but not conversely. It is not that children who learn to operate symbolically in mathematics forget the relationship of signs to concrete objects or to remembered processes – it is just that in a particular context some of these memories are not particularly helpful. Indeed, they may well be just so much clutter, filling up the available space in working memory. The sign formulas are useful for programming a machine to calculate the binomial or multinomial coefficients, but these mathematical signs also behave as a very compact symbolic expression of the relationships they embody. So a student capable of interpreting these signs symbolically can choose to think in terms of models, such as towers of blocks, *or can withhold any such interpretation*, knowing that they could interpret it this way if they wished. Their thinking has become proceptual (Gray and Tall, 1994) and they have gained enormous flexibility and economy of thought.

This leads to a curious, and critical, chicken and egg situation. Students may not have a scheme about which they can talk, but their lack of awareness of the operations used in the procedure inhibits them from talking about the procedure in the absence of carrying it out. Talk about the procedure by a teacher would seem to be insufficient for scheme formation, because many students have no awareness of the operations of the procedure on which to hang the talk. They have no symbolic frame of reference for the teacher's words. Repeated carrying out of the procedure by itself is no guarantee that awareness will result, because there may be no necessity to reflect on the procedure in the absence of carrying it out in practice. This dilemma was summed up succinctly by von Glasersfeld (1990), who wrote:

> If it is the case that ... conceptual schemas – and indeed concepts in general – cannot be conveyed or transported from one to the other by words of the language, this raises the question of how language users acquire them. The only viable answer seems to be that they must abstract them from their own experience. (p. 35)

So if talk alone and repeated practice do not suffice for the formation of schemes, and if students must abstract them from their own experience, where does this leave a teacher? Many students, in the process of counting or carrying out elementary operations on numbers, focus their attention on what we, as observers, deem to be peripheral properties of objects, such as their colour or size (Gray, 1991; Gray & Pitta, 1996; Gray & Pitta, 1997a, b; Gray, Pitta & Tall; 1997).This focus of attention necessarily occupies a student's working memory with detail that is known to be irrelevant to the formation of higher order mathematical schemes. We don't usually care whether the chairs we are counting are red or blue, for the purposes of determining whether a given collection of people will be able to be seated. This filling of working memory with irrelevant detail has two effects. First it slows the student considerably: they are able to consider fewer examples in a given time than those students whose focus is not on irrelevant detail. Consequently, such students have considerably fewer examples to categorise. Second, with each example they encounter there is less room in working memory for those aspects that we do consider relevant for the establishment of higher order schemes. As a result these students are seeing fewer examples, and less that is relevant in each example they encounter. Consider, for example, a student at secondary level who is engaged in expansion of the algebraic square of a sum or difference of terms. Examples such as $(a+b)^2$, $(c+d)^2$, $(e-f)^2$, $(s+3t)^2$, $(x-y+z)^2$, might well be seen by a student as unrelated things to do – unrelated actions which they do not categorize as similar instances of a single phenomenon. The most likely purpose of a collection of exercises like this is to give a student practice in algebraic expansion, and to give them enough examples to enable them to categorise the exercises as examples of algebraic squaring of a sum or difference. If a student continues to focus on the actual letters, or whether there is a sum or difference, or a coefficient "3", then they are unlikely to attain the categorization a teacher intended. They remain stuck at the level of individual actions, instead of forming schemes. Consequently, algebra becomes, for such students, a hard subject, with many detailed and unrelated calculations. What a student needs to do is to learn to throw away much of the perceptual information available to them. This involves a focus of attention on a different aspect of the algebraic expressions.

## Schemes as Mental Objects

Perceptual categorization gives rise to our feeling for prototypical named objects in the world. The fact that we can talk about chairs, or a chair, without referring to or pointing at a particular world-thing is a result of a process of perceptual categorization. The concept “chair” is a mental concept, and not a corporeal world-thing. Nevertheless, it is extremely convenient for us to think of “chair” as an object. We cannot even count “chairs” unless we see certain world-things as instances of “chairs”. In other words, we reify our conceptions obtained through perceptual categorization. This should give us cause to suspect that schema formation, which we have defined as categorization based on action schemes, also leads to mental objects. These mental objects are based not on world-things, as in perceptual categorization, but on world-actions.

Dörfler (1993) casts doubt on the nature of mathematical objects. He writes:

> My subjective introspection never permitted me to find or trace something like a mental object for, say, the number 5. What invariably comes to my mind are certain patterns of dots or other units, a pentagon, the symbol 5 or V, relations like 5+5=10, 5*5=25, sentences like five is prime, five is odd, 5/30, etc., etc. But nowhere in my thinking I ever could find something object-like that behaved like the number 5 as a mathematical object does. But nevertheless I deem myself able to talk about the number "five" without having distinctly available for my thinking a mental object which I could designate as the mental object ‘5’. (pp. 146–147)

This, however, is to miss the point of categorization. Where, in our heads do we see the object “chair”? As Dörfler intimates, we may see images of particular chairs, and even be capable of forming images of chairs we have never seen. The point is that “chair” is the name of a category to which we agree that certain world-objects belong. As such, it acquires object status: that of a mental object, a conception, resulting from perceptual categorization. Likewise, the word “dealing” refers to a category of world-actions, and as such it is a mental object resulting from conceptual categorization.

## Perceptual, Social and Conceptual Categorization

Perceptual categorization is, as we have noted, common not only among different peoples, but also among animals of many sorts. Social categorization is also very common among mammals. This form of categorization is exemplified by the "them and us" syndrome: the division of a group into two on the basis of a perceived difference, such as skin color, accent, behaviour, speech, or indeed almost any perceived difference (Harris, 1998). These types of categorization are so common, occurring almost obligatorily in human society, that we often overlook them as mental constructions and take them to represent significant differences given by the phenomenal world.

Why then should the sort of categorization that we postulate as the basis of scheme formation seem to be so difficult for individuals to establish? Why should it require an elaborate apparatus of cultural transmission – classrooms, teachers and textbooks, not to mention psychologists and mathematics education researchers? The answer, one might suspect, lies in the everyday nature of perceptual and social categorization. Perceptual categorization is vital for animals that move around looking for food, shelter, and mates, in a potentially dangerous world. Social categorization is an inevitable consequence for humans who have warring social ancestors (Harris, 1998). Conceptual categorization, one might imagine, arises as a possibility, and only as a possibility, with the development of language. However, the recency, in evolutionary terms, of conceptual development does not provide a sufficient reason why conceptual categorization should be so difficult, at least in the field of mathematics.

What seems, from the empirical evidence, to be a much more compelling reason for the difficulties we see in scheme formation in mathematics is the general lack of awareness that humans have of action schemes. By "lack of awareness" we mean inability to articulate action schemes as distinct from their outcomes. Steffe, among others has remarked on this lack of awareness of action schemes, as we have noted above. Granted this lack of awareness, it is almost clear that we should have difficulty categorizing action schemes: we are not aware of them as schemes. Instead, what we are aware of is the outcome of those schemes: the results of sharing by dealing, or the results of counting, the results of an algebraic calculation. Why this should be the case is, as far as we are cognisant, not known. However, granted that it is the case, it provides a considerable obstacle for conceptual categorization of action schemes. As Steffe has remarked on other occasions, this provides a supremely important role for a teacher of mathematics in helping students to be able to articulate their action schemes regarding number, space, and arrangement – the basic elements of mathematical experience.

The question of why humans should not naturally find it straightforward to articulate an awareness of action schemes therefore assumes a great importance in the study of the acquisition of mathematical conceptual thought. A simple reason suggests itself, namely a separation between the language and motor centres of the brain. The neurologist Ramachandran (1998) highlights the problem of putting actions into words as a translation problem. He writes, a

> … fundamental problem arises when the left hemisphere tries to read and interpret messages from the right hemisphere. … crudely speaking, the right hemisphere tends to use an analogue – rather than digital– medium of representation, emphasizing body image, spatial vision and other functions of the how pathway. The left hemisphere, on the other hand, prefers a more logical style related to language, recognizing and categorizing objects, tagging objects with verbal labels and representing them in logical sequences (done mainly by the what pathway). This represents a profound translation barrier. (p. 283, author's italics)

This translation barrier is particularly evident in adults who have damage to the right brain or a disconnection of the two hemispheres. This happens, for example, when the corpus callosum, the bridge connecting the two hemispheres, is damaged or cut (as used to happen in cases of severe epilepsy). How might this explain why young children have difficulty articulating their action schemes? After all, only in rare cases will children have such sever dislocation between their two brain hemispheres. Yes, but the fact is that the corpus callosum is quite undeveloped in young children: the nerve fibres connecting the two hemispheres have not yet been fully myelinated, so nerve impulses in young children do not conduct between the left and right hemispheres as well as they do for older children and adults (Joseph, 1993, p. 353 *ff*). Whilst this might suggest a reason for young children's relative inability to articulate awareness of their action schemes, it does not explain why older children, and indeed many young adults, are equally incapable of such articulation. It is not uncommon, even in university level mathematics, for students to be able to carry out taught procedures – such as solving simultaneous linear equations by Gaussian elimination – and yet have an almost total inability to articulate how the procedure is carried out. Often the best they can do is to ask for an example, which they then proceed to calculate.

A further clue to the relative difficulty in articulating action schemes comes from work of Ullman *et al* (1997) on language difficulties in sufferers of Alzheimer's disease on the one hand and

Huntington's and Parkinson's on the other. Their work suggests that word memory relies on areas of the brain that handle declarative memory – memory of facts and events. These areas appear to be the temporal or parietal neocortex. However, rules of grammar seem to be processed by areas of the brain that manage procedural memory, the basal ganglia, which are also involved in motor actions. That there seem to be two distinct brain areas for procedural and declarative memory must make us suspicious. In mathematical settings, at least, the region devoted to declarative memory may have difficulty – that is, few mechanisms for – taking as its basic material the activities of the region responsible for procedural memory. If so, the role of teacher becomes even more evident: as an external conduit to allow declarative memories to be formed from the raw material of stored procedural memories.

Let us look again at Stephanie's categorizations in this light, because Stephanie is a child who *was* able to make higher order categorizations beyond the commonplace. First, Stephanie was motivated to seek reasons for things mathematical (Maher & Speiser, 1997). Indeed this was a prime reason for Carolyn Maher and Bob Speiser's focus on Stephanie. However, if we are right about the need for procedural memory to be externalized before it can become declarative then Stephanie must have had some external influence on the formation of her declarative memories of building towers. Did she? Indeed she did: Maher and Speiser report how Stephanie was engaged in elementary school with a group of children who built towers together, and engaged in argument and reasoning about their activities. So Stephanie's external agent in this case was her group of classmates who not only built towers with her, but also argued with her. What about Stephanie's $2^{nd}$-order categorization, in which she linked in considerable detail building towers with the binomial and multinomial theorems? We have to suspect, from the records of interviews, that her external agent for utilising procedural memories to create declarative ones, was the pair of interviewers. Through the questions asked by the interviewers Stephanie was able to take her procedural memories and turn them into objects of reflection, which then created declarative memories for her.

The decisive force in the creation of higher-order schemes, therefore, may be an appropriate agent who can externalize procedural memory and utilize it, consciously or not, so that a child can form declarative memories. The reason for this, we hypothesize, is that the temporal and parietal neocortex has, in young children, or young adults, few mechanisms for taking the memory activities of the basal ganglia as raw data for the formation of new declarative memories. What has to happen, we suspect, is that an external agent needs to *externalize* those memories of motor actions from the basal ganglia and recast them in a form suitable for the temporal or parietal neocortex to process them as procedural memories.

## Connections with APOS Theory

Dubinsky and colleagues (Dubinsky, 1992; Cotterill *et al.*, 1996) have proposed an Action-Process-Object-Schema theory in which schemes feature as the end result of a structural organization. In the APOS setting, an *action* is a physical or mental transformation of objects to obtain other objects. A *process* arises from an action when a person is able to reflect upon and establish conscious control over the action. A process becomes an *object* when "the individual becomes aware of the totality of the process, realizes that transformations can act on it, and is able to construct such transformations." (Cottrill, *et al*, 1996). *Schemas* enter into this theory as structural organizations of actions, process and objects.

Our proposal is that this "structural organization" is obtained through conceptual categorization in the sense of Edelman (1989). The essential point, for us, is that some mechanism must be postulated to facilitate the structural organization central to APOS theory. In line with Skemp's emphasis on brain activity and brain models of mathematical thought we believe that the process of conceptual categorization provides such a mechanism. Skemp discussed the connections between categorization and schema in *Intelligence, Learning and Action*: indeed he regarded them as practically synonymous. The only extra highlight we wish to stress is that, in line with Skemp's emphasis on intelligent, goal-driven, action (an emphasis he shared with many other seminal thinkers in mathematics learning), the focus on scheme formation in mathematics is on categorization beginning with action schemes.

## Dedication

We owe a debt to Richard Skemp. Apart from pioneering work in schema, he began the process of modelling what it is that the brain is doing when it's thinking mathematically. Skemp concentrated on fundamental issues of models for brain operations in mathematical thought, and for intelligent thought more generally. Recent developments in psychology and neurology have been reinforced the gems of principles and models that he elucidated so clearly. He established a solid link between intelligent human actions and the operations of our brains, and regarded the study of mathematics learning as a way to develop models for higher-order intelligence in general. As Anna Sfard has written (this volume) " Skemp ... came to an empty field and left it with an impressive construction." Skemp, himself, wrote:

> ... it seemed that by studying the psychology of learning mathematics, the improved understanding of intelligent learning which can be gained by working in this area should be generalisable to give a better understanding of the nature of intelligence itself: with a potential for applications extending over a very wide range of activities. (1979, p. 288)

This gentle man treated human beings as intelligent creatures who have a capacity to reflect on their actions and learn from them. In so doing he was led to examine models for brain functioning that might allow us to think in this way. His excursion into brain models was an intellectually necessary part of his quest to understand what it is that allows human beings to think as they do, and to behave with the intelligence they are capable of manifesting. We are forever grateful to Richard Skemp for these pioneering efforts. They laid a clear and firm foundation for a subject whose time has now well and truly come – the nature of the mathematical brain, its relation to mathematical intelligence, and to intelligence in general.

## Acknowledgements

Thanks are due to Leslie Steffe for a discussion of his meaning for symbols, to Cath Pearn for discussions on how some children are doing less and seeing less, and to Carolyn Maher for correcting some of our misunderstandings.

## References

Bartlett, F. C. (1932). *Remembering.* Cambridge: Cambridge University Press.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept: beginning with a coordinated process schema. *Journal of Mathematical Behavior*, 15, 167–192.

Davis, G. E., & Hunting, R. P. (1990). Spontaneous partitioning: Pre-schoolers and discrete items. *Educational Studies in Mathematics*, 21, 367–374.

Davis, G., & Pepper, K. (1992). Mathematical problem solving by pre-school children. *Educational Studies in Mathematics*, 23, 397–415.

Davis, G., & Pitkethly, A. (1990). Cognitive aspects of sharing. *Journal for Research in Mathematics Education*, 21 (2), 145–153.

Davis, R. B. (1984). *Learning Mathematics: The Cognitive Science Approach to Mathematics Education*. Norwood, NJ: Ablex.

Deacon, T. (1997). *The Symbolic Species. The co-evolution of language and the human brain*. London: Penguin.

Dörfler, W. (1993). Fluency in a discourse or manipulation of mental objects? *Proceedings of the 17th International Conference for the Psychology of Mathematics Education*, Tsukuba, Japan, II, 145–152.

Dubinsky, E. (1992). Reflective Abstraction in Advanced Mathematical Thinking. In D. Tall (Ed.) *Advanced Mathematical Thinking* (pp. 95–126), Kluwer: Dordrecht.

Edelman, G. M. (1989). *Neural Darwinism: The Theory of Neuronal Group Selection*. New York: Basic Books.

Edelman, G. M. (1992). *Bright Air, Brilliant Fire*. New York: Basic Books.

Gray, E. M. (1991). An analysis of diverging approaches to simple arithmetic: Preference and its consequences. *Educational Studies in Mathematics*, 22 (6), 551 – 574.

Gray, E. M., & Pitta, D. (1996). Number Processing: Qualitative differences in thinking and the role of imagery. In L. Puig and A Guitiérrez (Eds.), *Proceedings of the 20th International Conference for the Psychology of Mathematics Education* (pp. 155–162). Vol.IV.

Gray, E., & Pitta, D. (1997a). In the Mind … What can imagery tell us about success and failure in arithmetic? In G. A. Makrides (Ed.), *Proceedings of the First Mediterranean Conference on Mathematics* (pp. 29–41). Nicosia: Cyprus.

Gray, E., & Pitta, D. (1997b). Evaluating children's mental structures in elementary arithmetic. In M. Hejeny & J. Novotná (Eds.), *Proceedings of the Third International Symposium on Elementary Math Teaching.*

Gray, E., Pitta, D. & Tall, D.O. (1997). The nature of the object as an integral component of numerical processes. In E. Pehkonen (Ed.), *Proceedings of the 21st International Conference for the Psychology of Mathematics Education* (Vol. 1, pp. 115–130),. Lahti: Finland.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity and flexibility: a proceptual view of simple arithmetic*, The Journal for Research in Mathematics Education*, 26 (2), 115–141.

Harris, J. R. (1998). *The Nurture Assumption. Why Children Turn Out the Way They do.* New York: The Free Press, Simon & Schuster.

Joseph, R. (1993). *The Naked Neuron. Evolution and the Languages of the Body and the Brain*. New York: Plenum.

Maher, C. A., & Martino, A. M. (1996). The development of the idea of proof. A five-year case study. *Journal for Research in Mathematics Education*, 27(2), 194–219.

Maher, C. A., & Martino, A. M. (1997). Conditions for conceptual change: From Pattern recognition to theory posing. In H. Mansfield & N. H. Pateman (Eds.), *Young Children and Mathematics: Concepts and their Representations*. Sydney, Australia: Australian Association of Mathematics Teachers.

Maher, C. A., & Speiser, R. (1997). How far can you go with tower blocks?. In E. Pehkonen (Ed.), *Proceedings of the 21st conference of the International Group for the Psychology of Mathematics Education*, (vol. 4, pp. 174–181). Lahti: Finland.

Miller, K. (1984). Child as the measurer of all things: Measurement procedures and the development of quantitative concepts. In C. Sophian (Ed.) *Origins of Cognitive Skills*. Hillsdale NJ: Erlabaum.

Minsky, M. L. (1975). A framework for representing knowledge. In P. H. Winston (Ed.), *The Psychology of Computer Vision* (pp. 211–277). New York: McGraw Hill.

Oldfield, R. C., & Zangwill, O. L. (1942a). Head's concept of the schema and its application in contemporary British psychology. *British Journal of Psychology*, 32, 267–86.

Oldfield, R. C., & Zangwill, O. L. (1942b). Head's concept of the schema and its application in contemporary British psychology. *British Journal of Psychology*, 33, 58–64.

Oldfield, R. C., & Zangwill, O. L. (1943). Head's concept of the schema and its application in contemporary British psychology. *British Journal of Psychology,* 33, 143–149.

Ramachandran, V. S. (1998). *Phantoms in the Brain*. London: Fourth Estate.

Rumelhart, D. E. (1975). Notes on a schema for stories. In D. G. Bobrow & A. Collins (Eds.), *Representation and Understanding* (pp. 211–236). New York: Academic Press.

Schank, R. C. (1975*). Conceptual information processing*. Amsterdam: North Holland.

Skemp, R. R. (1962). The need for a schematic learning theory. *British Journal of Educational Psychology*, 32, 133–142.

Skemp, R. R. (1971). *The Psychology of Learning Mathematics*. Middlesex, England: Penguin.

Skemp, R. R. (1979). *Intelligence, Learning and Action*. Chichester: Wiley.

Skemp, R. R. (1986). *The Psychology of Learning Mathematics*. Second Edition. Middlesex, England: Penguin.

Steffe, L. P. (1983). Children's algorithms as schemes. *Educational Studies in Mathematics*, 14*,* 109–125.

Steffe, L. P. (1988) Children's construction of number sequences and mutliplying schemes. In J. Hiebert & M. Behr (Eds.) *Number Concepts and Operations in the Middle Grades,* (Volume 2, pp. 119–140). Reston, VA: The National Council of Teachers of Mathematics.

Steffe, L. P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's Counting Types: Philosophy, Theory and Application*. New York: Praeger.

Steffe, L. P., Cobb, P., & von Glasersfeld, E. (1988). *Construction of Arithmetical Meanings and Strategies*. New York: Springer Verlag.

Ullman, M. T., Corkin, S., Coppola, M., Hickok, G., Growdon, J. H., Koroschetz, W. J., & Pinker, S. (1997). A neural dissociation within language: Evidence that the mental dictionary is part of declarative memory, and that grammatical rules are processed by the procedural system. *Journal of Cognitive Neuroscience*, 9(2), 266–276.

von Glasersfeld, E. (1981). An attentional model for the construction of units and number. *Journal for Research in Mathematics Education*, 12(2), 83–94.

von Glasersfeld, E. (1990). Environment and communication. In L. P. Steffe & T. Wood (Eds.) *Transforming Children's Mathematics Education. International Perspectives*. New Jersey: Lawrence Erlbaum.

# Continuities and Discontinuities in Long-Term Learning Schemas

## (Reflecting on how relational understanding may be instrumental in creating learning problems)

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> **Master.** … I wil propound here ii examples to you whiche if you often doo practice, you shall be rype and perfect to subtract any other summe lightly …
>
> **Scholar.** Sir, I thanke you, but I thynke I might the better doo it if you did show me the working of it.
>
> **Master.** Yea but you muste prove yourselfe to do som thynges that you were never taught, or els you shall not be able to doo any more than you were taught, and were rather to learne by rote (as they cal it) than by reason.
>
> (Robert Recorde, *The Ground of Artes*, 1543)

The notion of "learning by reason" rather than "learning by rote" has long been a focus of creative teaching. In writing the oft-quoted paper on "instrumental understanding" and "relational understanding", Richard Skemp (1976) is a significant link in the chain of those who developed notions of "meaningful learning". Skemp, however, had wider goals in life. For him, relational learning was part of a broader plan of developing a "long-term learning schema" for life-long learning (Skemp, 1962).

In his publications *Understanding Mathematics* (1964) for secondary school and *Structured Activities in Learning Mathematics* (1993–4) for younger children he developed learning schemas for children's learning over long time periods. He also developed a rich theory of human learning that has proved significant in shedding light on how the cognitive processes of human thinking can lead to the logic and aesthetic beauty of formal mathematics.

It is the purpose of this paper to consider the development of long-term learning. The logic and rigour of mathematics is such that it seems that a curriculum can be built in successive stages where each stage builds on preceding learning. Indeed, the British National Curriculum for children aged 5 to 16 is formulated in ten levels where each level is seen as developing a coherent set of ideas building on earlier levels. This would suggest that a well-designed curriculum could be continuous in the sense that each stage builds smoothly on those already experienced. Such a "self-evident truth" is, in fact, false. This paper will show that there are many points in learning in which cognitive discontinuities occur. Thus the building of a long-term curriculum is likely to need to face various situations in which difficulties arise and need to be conquered.

Skemp was aware that appropriate schemas need to be developed that are appropriate to a given learning task:

> Since new experience which fits into an existing schema is so much better remembered, a schema has a highly selective effect on our experience. What does not fit into it is largely not learnt at all, and what is learnt temporarily is soon forgotten. (Skemp, 1986, p.41)

A major problem occurs when what may be highly appropriate at one stage may be unsuitable later. Skemp formulated this in his overall theoretical position saying:

> … not only are unsuitable schemas a major handicap to our future learning, but *even schemas which have been of real value may cease to become so* if new experience is encountered, new ideas need to be acquired, which cannot be fitted in to an existing schema. A schema can be as powerful a hindrance as help if it happens to be an unsuitable one. (*ibid.* (my italics))

An example of a schema of short-term value is the use of so-called "fruit salad" algebra, in which meaning is given to expressions such as $4a+3b$ by thinking of the letters as standing for actual objects such as "4 apples plus 3 bananas". This will give support in manipulation of expressions such as $4a+3b+2a$ to give $6a+3b$ by simply thinking in terms of manipulating the numbers of each fruit. However, this short-term gain soon leads to difficulties in interpreting expressions such as $7ab$. Does it mean 7 apples and bananas? Is 3 apples plus 4 bananas equal to 7 apples and bananas? Is $3a+4b = 7ab$? Initial simplistic approaches to subjects that subsequently lead to inappropriate links may harm long-term development. My contention takes this observation a step further: even well-designed learning tasks can—at a later stage—harm future understanding.

In the remainder of this paper I begin by considering the general problem of discontinuities in a long-term curriculum, followed by a range of examples of discontinuities occurring at various stages of school mathematics. Since every journey begins with a single step, I start with what I term a *cognitive root*; this is a starting point having meaning for the learner at the beginning of a learning sequence, yet containing the possibility of long-term meaning in the later theoretical development. I then question whether a successful beginning will necessarily lead to long-term success, considering various aspects of the calculus, where visuospatial ideas can act as a long-term foundation in a range of different possible approaches.

But I also reveal that the different representations—symbolic, algebraic and numeric—do not always have obvious links between them. The consequence is that different parts of the subject may benefit from different kinds of representations and pose different kinds of problems for different students. A powerful tool to address these problems is the willingness of the student to reflect carefully on new ideas, to see how they are similar and how they *differ* from earlier meaningful ideas. Learning by rote may allow the student to cope with similar problems, but reflecting on the nature of the mathematics is more likely to support flexible long-term learning.

## Continuities and Discontinuities in Long-Term Learning Schemas

On the assumption that the long-term curriculum designer should be more concerned with ultimate coherence and successful learning than settling for a short-term gain, a good solution would seem to be a long-term curriculum that builds steadily and continuously on previous experience. This seems to be the underlying aim of much curriculum design, after all, mathematics is a coherent and logical subject, so its teaching should be amenable to coherence and logic.

A quarter of a century ago I remember, as an earnest young mathematically oriented educator, suggesting long-term learning schemas based on my interpretation of Skemp's ideas. For me at this stage the quest entailed

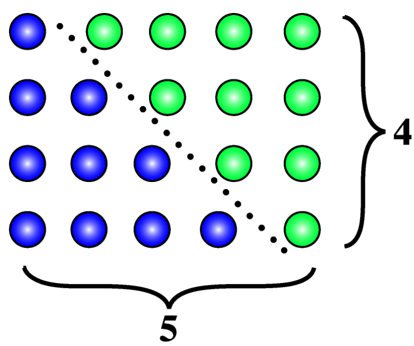


*Figure 1.* 1+2+34 is half of 4 times 5.

looking for ways in which I could present subtle mathematical ideas in simple ways at appropriate times and revisit them later in successively more sophisticated ways—for instance, seeing the sum of an arithmetic sequence 1+2+3+4 as a triangular array of dots which could be fitted to an identical array to give a rectangle of four rows and five columns. Thus the sum 1+2+3+4 is half of 4 times 5. This could be written as an arithmetic sum by adding 1+2+3+4 to 4+3+2+1 in pairs to get 4 lots of 5:

$$\textbf{add columns}\ \Big\downarrow \quad \begin{array}{l} \mathbf{1+2+3+4} \\ \mathbf{4+3+2+1} \\ \hline \mathbf{5+5+5+5 \ = \ 4 \times 5} \end{array}$$

*Figure 2.* Two lots of 1+2+3+4 give 4 lots of 5.

At a later stage this could be generalised algebraically to give

$$1+2+3+\ldots+n = \tfrac{1}{2}n(n+1),$$

and finally proved by induction when, and if, this may be considered appropriate.

The logical development from physical objects, through arithmetic, algebra and formal proof seemed to me an ideal part of a continuously developing long-term learning schema that can be revisited at successive times in more subtle ways. However, such an "ideal" coherent sequence of activities proves to have an Achilles' heel. Although the sequence of development may be apparent to an experienced teacher, at the time of learning the links may not be so readily apparent to the growing child.

The reality is that one must build on the ever-changing cognitive structure of the individual. Achieving a continuous and coherent development proves far more elusive than one might expect. The problem is essentially that the brain works by building connections between neuronal circuits, evoking many internal links, most of which are unconscious:

> Conscious thought is the tip of an enormous iceberg. It is the rule of thumb among cognitive scientists that unconscious thought is 95 percent of all thought—and that may be a serious underestimate. Moreover, the 95 percent below the surface of conscious awareness shapes and structures all conscious thought. If the cognitive unconscious were not there doing this shaping, there could be no conscious thought.
>
> (Lakoff & Johnson, 1999, p. 13)

These links, deeply embedded in our biological brain, produce a complex mental image of concepts (termed the *concept image* by Tall & Vinner, 1981), which operates in many subtle ways. Vinner (1997) speaks of *pseudo-conceptual thinking* in which students learn to do the things that give satisfactory immediate success but may not plumb deeper conceptual issues. These deeper mathematical structures may have no meaning or purpose for the learner at the time so that—even if the curriculum is presented in a manner that is (to the teacher) coherent and logical—it can fail to be understood by the learner.

Skemp was a master at formulating theories to cover such phenomena. He referred to the difference between learning which is an *expansion* of current knowledge—where new ideas fit easily with current schemas—and *reconstruction* of knowledge, where the old schemas must be reflected upon and modified to fit with the new ideas. In a given context, conceptually well-structured learning may be based upon mental images that "fit" the context. Yet these self-same images may later fail in new, as yet unknown, situations.

In designing long-term learning schema, it becomes important to consider if and where reconstruction is likely to be necessary, even in instances where previous learning has been relational. It is my contention that shifts of context causing cognitive conflict occur far more widely in the mathematics curriculum than might at first be apparent.

## Cognitive Discontinuities Throughout the Curriculum

To get a sense of the kind of difficulties that occur, I begin with a number of examples encountered personally at different points in the mathematics curriculum.

John, a "slow learner" aged 12, could not imagine negative numbers. In primary school his early number experience began with counting objects and the number track consisting of distinct unit blocks. He continued to think of number in terms of counting and found it impossible to imagine a "minus number" of objects. How on earth can anyone imagine *minus two cows*? To him, it didn't make sense.

My son Christopher, then aged 8, was easily able to conceive of "minus numbers". The temperature in the winter measured in Centigrade often went below zero, and he enjoyed playing in the snow. But he could not perform arithmetic with minus numbers. When his younger brother tried to explain to him that 5 take away minus 2 is 7, Christopher stamped around in a temper. He *refused* to believe that when something is taken away the result could be *bigger*.

Jane, aged 12, could not make sense of multiplication of fractions. In particular she could not believe that the product of two fractions could give a *smaller* answer. All her previous experience of whole numbers had intimated that multiplication always leads to a (much) larger number.

Rachel, aged 13, did well at arithmetic and could calculate accurately and efficiently. But her first encounter with algebra was a disaster. The teacher explained that a letter such as $x$ could be used to stand for a number, so that if $x$ is a number, then $x+3$ is "the number plus 3". For instance, she explained, if $x$ is 2, then $x+3$ is 5, if $x$ is 3, then $x+3$ is 6, and so on. But Rachel didn't understand why '$x$' had been introduced or what it meant. If she didn't know $x$, she couldn't calculate $x+3$ and if she *did* know $x$, she didn't need algebra, it was far easier just doing arithmetic. Why complicate things by using letters? At a later stage when asked to simplify $3+2x+1$, she wrote $6x$, by adding together all the numbers (after all, they *do* have an addition sign between them) and "leaving the $x$" because she didn't know what to do with it. Algebra, for her, became a meaningless manipulation of symbols using arbitrary rules.

Robert, aged 14, did well with arithmetic and algebra. He understood that $3\times3$ was written as $3^2$, $3\times3\times3$ as $3^3$, $3\times3\times3\times3$ as $3^4$ and so on. He could even see that $3^n$ means $n$ lots of 3 multiplied together and $x^n$ means

$$\underbrace{x \times x \times \ldots \times x}_{n \text{ times}}$$

From this it was a short step to see that

$$\underbrace{x \times x \times \ldots \times x}_{m \text{ times}} \times \underbrace{x \times x \times \ldots \times x}_{n \text{ times}} = \underbrace{x \times x \times \ldots \times x}_{m+n \text{ times}}$$

or that

$$x^m \times x^n = x^{m+n}.$$

He had a clear relational idea about the meaning of the power notation in the context, but when he was shown that $x^{1/2}$ must be $\sqrt{x}$ because

$$x^{1/2} \times x^{1/2} = x^1 = x,$$

he suddenly became confused because he could not make sense of the notation $x^{1/2}$. If $x^n$ means "$n$ lots of $x$ multiplied together", what does "half a lot of $x$ multiplied together" mean? For the expert, the familiar formula is just being "generalised" to apply to fractional exponents. For Robert, such a strategy had no meaning whatsoever.

James was a successful fifteen year old being taught the rudiments of calculus. First the teacher explained ideas in terms of a picture, with a secant approaching the tangential position, then he went through some numerical examples for $y = x^2$, focusing first on the secant through $x = 1$ and $x = 2$, then successively calculating slopes from 1 to 2, from 1 to 1.1 and from 1 to 1.01. He showed that these slopes get closer and closer to 2, and then moved on to consider the ideas algebraically using $(x+\delta x)^2 - x^2$ divided by $\delta x$. After calculating the slope for $y = x^2$ and $y = x^3$, he revealed the general pattern of the derivative for $x^n$ as $nx^{n-1}$ and showed how the rule worked for general polynomials. "That's typical," said James to the teacher after the class, "you always show us the hard ways first before getting down to the simple way to do it." In James's class *all* of the pupils learned to differentiate polynomials but *none* could give any relational explanation of the process. They knew from experience that their teacher would always help them by teaching them a simple rule. The rest of the performance was to please him, not them.

Alec, aged 16, never did understand all that stuff about limits, but he knew that the derivative of $3x^4+5x^2$ is $12x^3+10x$. The derivative of a sum is just worked out by adding the resulting derivatives. He used the "same" rule for a product such as $x^2(x^3+x)$, to give the "derivative of the product" as the "product of the derivatives", namely $2x(3x^2+1)$.

The incidents discussed in this section show children of various ages finding difficulty with a variety of aspects of mathematics in new situations. Even if children have previously been successful through several sequences of learning they may eventually meet a situation that makes no sense. Although these examples arise in a range of different contexts, an underlying story can be uncovered.

## Sources of Cognitive Discontinuities

Several of the examples just considered clearly involve a new context in which previous ideas cause a conflict—for instance, John's inability to conceive of negative numbers, or Christopher's ability to give them a meaning but inability to perform arithmetic with them, or Jane's difficulty with a product of fractions giving an unexpected smaller result. The reality of the growth of neuronal connections reveals a vast complex of growing connections that operate in a range of ways to support our human activities. The coordination required for counting involves seeing, pointing to the things seen, performing it in a sequence that includes each object once and once only, saying the number sequence at the same time. This involves a vast number of neural connections, the majority of which are subconscious. Initially the act of counting involves physical objects in physical situations, linking numbers inextricably to real world referents. When a new situation occurs which causes a dissonance with these connections, the individual may feel confused yet be unable to pinpoint the reason for the confusion. Negative numbers "feel wrong" because "you can't have less than nothing." Even if one can envision positive and negatives as credits and debts, or temperatures above and below freezing point, these particular embodiments carry no sense of a full array of arithmetic operations. One may "start at a positive temperature +2°C and *go down* 3°C to end up at the negative temperature –1°C." This may give a sense of difference between temperatures, perhaps even linking to a conception of subtraction. By talking about "taking away a debt" one may even give a sense that "taking away a minus" is the same as "adding a plus". But these are straining the meaning of the original neuronal links involving combining and removing physical objects. The idea of *multiplying* negative numbers is, for most, a bridge too far.

My son Christopher was able to conceive of negative numbers in terms of temperature at the age of eight but had no sense of how to do arithmetic with them. His younger brother Nic, then only five years old had, without any prompting, actually *asked* how to multiply minus numbers. He too knew about the concept of minus number in terms of temperature. He reasoned that "ordinary numbers" could be added and multiplied so why couldn't you do the same with "minus numbers". On discussing the concept of lending him pocket money and putting pieces of paper with minus numbers in his purse to record the operation (in denominations of –10 pence), I asked him how I could give him 50 pence if I had no money at the time. He said "you could take away five of the 'minus tens'." He then thought for a moment, smiled, and said "… oh, so *two minuses make a plus*." He could then do *any* sum, difference or product involving plus or minus numbers. His one generative idea enabled him to do them all in a consistent manner *without any further teaching*.

How can two children brought up in the same environment be so different? My interpretation of this situation is that Nic happened to think about arithmetic by focusing his attention on the essential detail of the symbols and operations. He probably did not link operatively to physical reality at the time. For him, therefore, it was natural to work in his cognitive context and seek how to do the same operations on minus numbers. I conjecture that such a conception was not possible for his brother Chris because of his neuronal links to physical situations where multiplying negatives clashed with his own meanings. Nic was faced only with a (pleasurable) task of cognitive expansion, building on his existing manipulation of numbers. Chris was faced with a (difficult) task of cognitive reconstruction that challenged his very relationship with the world as he perceived it. For him new ideas did not make sense. They did not fit.

The primitive brain has a way of reacting to perceptions that appear strange or threatening. The lower limbic system unconsciously produces neuro-transmitters that affect the operation of the brain, encouraging some activities and suppressing others. The child who finds certain concepts "do not make sense" is therefore likely to be at a disadvantage in attempting to process the information. I conjecture that, not only are the ideas harder because they are more diffuse and more difficult to co-ordinate, the emotional activity of fear generated by the limbic system suffuses the brain with neurotransmitters that makes the contemplation of these ideas even harder.

The case of Robert reveals an individual with relational understand of the power law for whole number powers, who is then confused when he he is faced with the need for cognitive reconstruction in the new context where the powers are fractional. We may hypothesise that his relational understanding was neuronally connected to the idea that the power represents repeated multiplication. Raising $x$ to the power $\frac{1}{2}$ makes no sense because he has no idea what it means to compute "half an $x$ multiplied together." An idea, which an experienced mathematician sees simply as a"generalisation" of a formula to apply to a larger range of examples, is not meaningful for a learner who has meaningful understanding of the power notation in its whole number manifestation.

It is salutary to realise that children may develop a *relational* understanding in a given context and yet encounter new contexts where the old links no longer work and new links must be forged. This occurs widely when a child meets an extension of the number system, say from counting numbers to negative integers or from whole numbers to fractions, or from real numbers to complex. Old intuitive rules that are part of the essential being of the individual are often deeply embedded in the psyche. The deep belief that "you cannot have *less* than nothing" in an everyday sense may prove too strong even in a world of bank accounts and credit cards where negative quantities become readily available. The practical solution in this case has been to continue to use ordinary numbers but widen the system to allow the same numbers to act separately as debts. In cognitive terms, this practical solution permits a cognitive expansion, rather than requiring a cognitive reconstruction.

In a long-term learning schema, such a focus on expansion rather than reconstruction may work in the short-term, it may even work for the life-time of a bank teller, (or at least until that teller is made redundant by technology). But in the long-term mathematical development towards the full deployment of real numbers, the need for some kind of cognitive reconstruction is inevitable. Faced with a new context which causes internal cognitive conflicts, the child must either make a serious cognitive reconstruction or take the line of least resistance by learning a new rule by rote to "get the right answer." The latter strategy leads to (temporary) survival in the mathematics class. This is seen in the example of the student James, who was perplexed by the teacher's attempt to introduce the limit concept in the calculus but realised that the teacher would later give "the simple way" to differentiate a polynomial using the formula.

Regrettably it does not take many such set-backs before the standard response is to learn "rules without reason" in Skemp's memorable phrase. When Alec learnt to "use the rules" in calculus, he extended them in an inappropriate manner, thinking that the same rule would work for multiplication that worked for addition.

The fall-back to "learning the rules" is widespread.

> I was interviewing a number of students about how they worked through their mathematics. What became very clear was the desire of the students to 'know the rule' or 'the way to do it'. Any attempt on my part to provide some background development or some context was greeted with polite indifference – 'Don't worry about that stuff; just tell me how it goes.' (Pegg, J. 1991, p. 70.)

I conjecture that the *vast majority* of learners reach this point at some stage. I further suggest that the "polite indifference" not to wish to "worry about that stuff" is not just an attitude of mind, but a sign that the student may not be *able* to talk or even think about "it", because there is no "it". A cognitive structure built on procedures to "do" mathematics may not have the mental concepts to "think" about mathematics and therefore may not be amenable to relational understanding.

As an example, consider a straight-line graph through (1,5) and (–3,–3). Its gradient can be found by the formula

$$\frac{(y_2 - y_1)}{(x_2 - x_1)} = \frac{-3-5}{-3-1} = \frac{-8}{-4} = 2$$

which, for many students, involves using meaningless "rules" such as "a minus over a minus is a plus". The equation is now $y = 2x+c$ where $c$ can be found by substituting one of the points. But which point is substituted? One has minus numbers in it, the other has positive numbers, and for students who are struggling, the latter may seem more attractive. This gives

$y = 2x+c$ where $x = 1$, $y = 5$,

so

$5 = 2+c$

$c = 3$

and the equation is

$y = 2x+3$.

Having followed such a sequence of procedures, does the student genuinely identify this equation with "the line through (1,5), (–3,–3)"? Are the two notions different ways of viewing the same conceptual entity, or are they different entities, loosely connected in the brain? Many students appear not to link them at crucial points of an argument. When asked whether a third point, say (2,4), is on the same line, instead of substituting this into the equation, some students, who are struggling, go back to calculating the equation of the line through (2,4), (1,5) to see if it gives the same equation (Crowley & Tall, 1999).

We suggest that these students do not readily see all the different forms of a linear equation as being different versions of one mental entity and so they have no mental entity in their minds to manipulate. They cannot talk about "it" because, for them, there is no "it". As long as they remain fixated on the detail of the procedures, they may work very hard but with little reward. For example, in their case we cannot speak of the various 'representations' of the function concept for there is no function concept to represent. All they can do in this state of mind is to learn an increasing collection of procedures to do specific, but limited, tasks that may grow increasingly difficult to relate in any coherent overall structure.

## Long-term Learning Schemas and Cognitive Roots

In devising initial activities for long-term learning schemas that encourage conceptual ideas rather than just procedural competence, I had the privilege of being Richard Skemp's last PhD student (Tall, 1986). Given the possibility (even probability) of discontinuities in the learning process, a long-term learning schema needs to take into account that cognitive reconstruction is likely to occur at various times. At the outset, I decided that the journey should begin by building an inner cognitive sense of the concept that carried the potential of long-term development. I formulated the notion of a *cognitive root* as a concept that a student meets at the beginning of a period of study that is familiar to the student at this stage, yet contains the seeds of long-term learning of the formal theory (Tall, 1989).

To start from "where the students are" to build to what you wish them to learn is quite different from building a curriculum which focuses on "where the students are desired to get". Many curricula in the calculus build towards a logical meaning of differentiation and integration and therefore decide that they must begin with the limit notion as this is the logical foundation of the formal theory. But it is not a good starting point for the learning of students.

I found that few students would naturally invent the limit concept for themselves. For instance, Tall (1986) presented a graph of the parabola $y=x^2$ with a line drawn through the points (1,1) and $(k,k^2)$ and asked first to write down the gradient of the line and then to explain how they might calculate the tangent at (1,1). Only *one* out of over a hundred students produced a limiting argument as $k \to 1$, and he was part of a minority who had already been *taught* the limiting notion.

On the contrary, every piece of research I have ever seen underlines how difficult the limit concept is to the beginning student (Cornu, 1991). Discussed in a dynamic sense, in terms of "getting close" or "getting small", it builds up intuitive notions of variable quantities that are "arbitrarily small", giving a number line of constants and quantities which act like infinitesimals. This gives a system different from the real number line that is desired by the mathematical community.

For the first step in my own long-term learning schema for the calculus, therefore, I decided to build on human foundations that were widely available to all individuals, the sensori-motor facilities of vision and action. I alighted on the notion of *magnifying* a small portion of the graph to *see* the gradient. Specifically I considered those graphs which, when drawn on a computer screen and magnified highly (maintaining the relative scales on the axes), eventually "look straight".

I emphasise that this is at a more fundamental cognitive level than that adopted by many curriculum builders in the calculus reform movement who start with "local linearity". Local linearity already demands a facility to handle the equation of a straight line that is the "best" approximation to the curve at a point. Local straightness is a direct appeal to perception, in which the limiting notion is implicit in the action of magnification.

When the first group of students who ever used the software were asked to draw any graph they fancied and to see what happens when it is magnified, the first comment was that "it looks less curved as you magnify it". The notion of magnification to "look straight" proved an appealing idea to everyone in the class.

At first I followed the mathematical route of "fixing $x$" to find the gradient at a point, but then realised that the students were able to do something more intuitive—to look along the graph to "see" its changing slope.

Software was designed the student to move a "magnifying glass" along a curve to see the locally straight gradient change (Tall et al, 1990). This aided students to get a gestalt idea of the gradient function using a sensori-motor feeling of "tracing along the graph" of a "locally straight" function.

The visual computer approach drew a numerical chord from $x$ to $x+h$ for fixed $h$ and to plot the gradient of the curve as $x$ moves along the curve. For small $h$ the gradient graph can be seen to stabilise. This can be linked directly to the symbolism. For instance, in the case of $f(x) = x^2$, the gradient of the segment between $x$ and $x+h$ is

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2 - x^2}{h} = 2x + h \text{ (since } h \neq 0\text{).}$$

The graph drawn for the gradient is $y = 2x+h$. For small $h$, this is indistinguishable from the graph $y = 2x$. Thus *at a sensori-motor level*, the student has the opportunity to sense that the gradient of the parabola $f(x) = x^2$ is indistinguishable from $2x$. The subtleties of mathematics are linked to deep primitive modules in the brain that are part of the essential being of the human psyche.

At the time I remember being very pleased with myself, but was surprised that the students simply took up the idea as if it were "obvious". In one school I set up the computer software on a single computer intending to demonstrate it to the students studying calculus. I ended up sitting alone in the corner of the room, completely neglected by a large group of students standing round the computer to "see" how to do their homework. I found to my surprise that many students found it to seem so *simple* as to be "obvious". It was as though my great discovery (which was largely unused by the mathematical community) was evident to the student mind. The explanation, to me, is that it *is* obvious, because it links with deep cognitive structure that is part of the essential being of the individual.

To compare the mental imagery generated with that of students following a more traditional course, both kinds of student were asked to sketch the derivatives of various graphs, including those in Figure 3.

There are at least two possible routes to a solution. One is to proceed symbolically by guessing the formula, differentiating it, and drawing the derivative. This is easier for graph (a) (which looks like the familiar $y=x^2$) than for graph (b) whose formula is not familiar to the student. A second approach is to proceed visually, looking along the graph to see its changing gradient. For instance, graph (b) starts with a positive gradient decreasing to zero and then stays zero.

It is reasonable to hypothesise that students with less visual insight would find problem (b) significantly more difficult and this is supported by the statistics (table 1). Three experimental classes in school (A, B, C) using computer software to visualise the gradient succeed in both problems significantly better than control classes D, E, F, G) following a standard text. The difference in problem (b) was even greater. The experimental classes also perform better than control group H of highly able students studying double mathematics, and on a par with first year university students, group I.

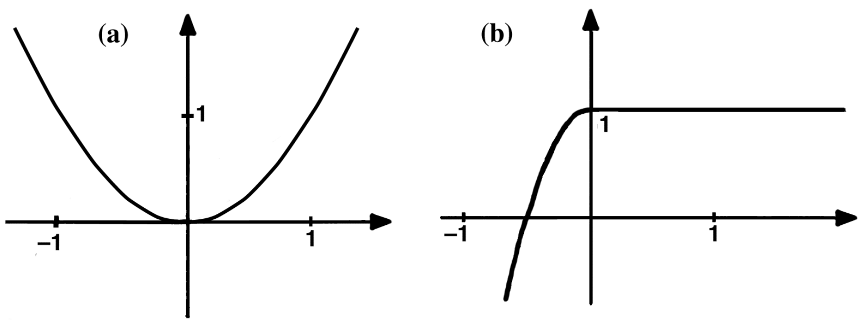



*Figure 3.* Sketching the gradient graphs for given graphs.

Table 1

*Student Responses to Sketching Derivative Graphs*

| Experimental Groups | (a) | (b) |
|---|---|---|
| School 1, group A | 100% | 70% |
| School 2, group B | 99% | 87% |
| School 2, group C | 92% | 89% |

| Control Groups | (a) | (b) |
|---|---|---|
| School 1, group D | 82% | 31% |
| School 2, group E | 73% | 16% |
| School 2, group F | 47% | 3% |
| School 2, group G | 39% | 0% |

| Others (non-experimental) | (a) | (b) |
|---|---|---|
| School 2 (double math), group H | 91% | 56% |
| University year 1, group I | 91% | 89% |

Not only did the idea of "local straightness" help many students to build a richer concept image of the idea of derivative, it can also be used later on in differential equations—"knowing the steepness of the function at each point, and requiring to build a function with that slope." Again, computer software was designed to allow the student to imagine physically building a solution of a (first order) differential equation by placing short line segments of the appropriate gradient end to end (Figure 4).

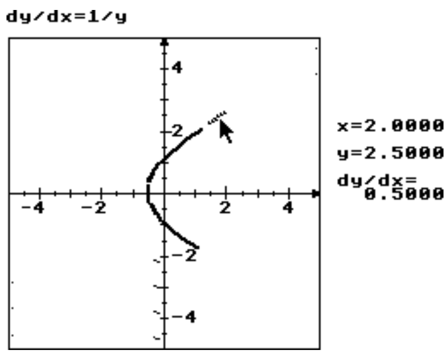



*Figure 4.* Enactively building a solution of a differential equation.

These ideas again provide a visual conception of the solution process, based on sensori-motor activity. They also provide a sense of existence and uniqueness of solutions. Everywhere that the gradient *dy*/*dx* is defined, the direction of the solution is given as a small segment. This supports the sense that, through every point where there is a defined direction there is one, and only one solution. Thus the existence and uniqueness of solutions can be conceptualised before any techniques of solution have been introduced. It allows conceptual links to be formed in an intuitive sensori-motor sense which may later motivate the formal aspects of the theory when and if it is studied.

## Does a Cognitive Root Guarantee Long-Term Success?

Having found a starting point that seems to be able to provide some motivation for later theory, the question arises as to whether this will necessarily lead to later success. The answer is that although a cognitive root has a *potential* for development into a long-term formal theory, it does not *guarantee* that such a foundation will lead to the later formalism for all those who attempt to study it. The cognitive root operates in a sensori-motor manner that can be visualised, verbalised and discussed. But this does not provide a basis for *all* possible formal links.

As an example, consider the symbolic rules for differentiation and integration. The derivative of a product $f(x)g(x)$ is not easily seen by looking at the visual notions of the derivatives of $f(x)$ and $g(x)$. For instance, if one can visualise the derivative of $\sin x$ by looking along its graph to see its changing gradient as $\cos x$, and can similarly "see" the gradient of the graph of $e^x$ how does this allow one to "see" the gradient of their product $e^x \sin x$? It is certainly not obvious by looking at the graphs of the two functions and "seeing" an easy relationship with the graph of the product function.

The usual visual method is to look at the product $uv$ of two quantities as an *area* and to see the increment in the area caused by increas-ing $u$ to $u+\delta u$ and $v$ to $v+\delta v$ as being two strips area $v\delta u+u\delta v$ and a rectangle in the corner of area $\delta u\delta v$ (Figure 5).

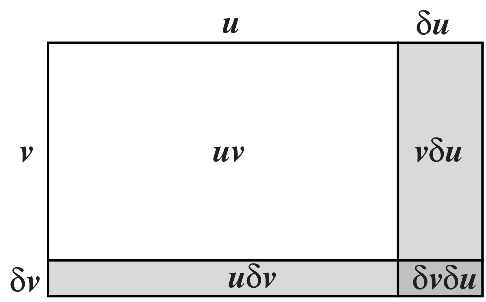



*Figure 5.* The change in $uv$ is $v\delta u+u\delta v+\delta u\delta v$.

This visual representation corresponds precisely to the following symbolic manipulation:

$$\frac{\delta(uv)}{\delta x} = \frac{(u+\delta u)(v+\delta v)-uv}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta v}{\delta x} + \delta u\frac{\delta v}{\delta x}$$
$$\to u\frac{dv}{dx} + v\frac{dv}{dx} + 0\frac{dv}{dx}.$$

There is therefore a direct link between this rectangular representation and the symbolism, but no simple visual link between the locally straight graphs and the gradient of their product function.

The School Mathematics Project asked me to provide such a graphical link for the product function. I could not see one. I *did* provide a graphical link for the composite of two functions, by seeing $y = f(g(x))$ in terms of the functions $u = f(x)$, $y = g(u)$ and plotting the three graphs in three dimensions with axes $x$, $u$, $y$. Figure 6 shows the software I developed to show this idea (Tall, 1991) in the case using

$$u = x^2,\ \ y = sin\, u\ (= sin(x^2)).$$

There are four different windows: the $x$-$u$ plane in the bottom right corner, with $u = x^2$, the $u$-$y$ plane in the bottom left corner, with $y = sin\, u$, and the $x$-$y$ plane in the top right, with $y = sin(x^2)$. Each of these is a projection of the curve in three dimensions in which $u = x^2$ and $y = sin(x^2)$.

Seen as a static picture in a book this may be quite difficult to visualize, but an option to turn the three dimensional graph around gives a sensation of space, making it easier to visualise it as a three dimensional object.

The tangent to the curve has components $dx$, $du$, $dy$ in directions $x$, $u$, $y$ and the chain rule for differentiation becomes:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

as a relationship between the lengths of the components of the tangent (Figure 7).

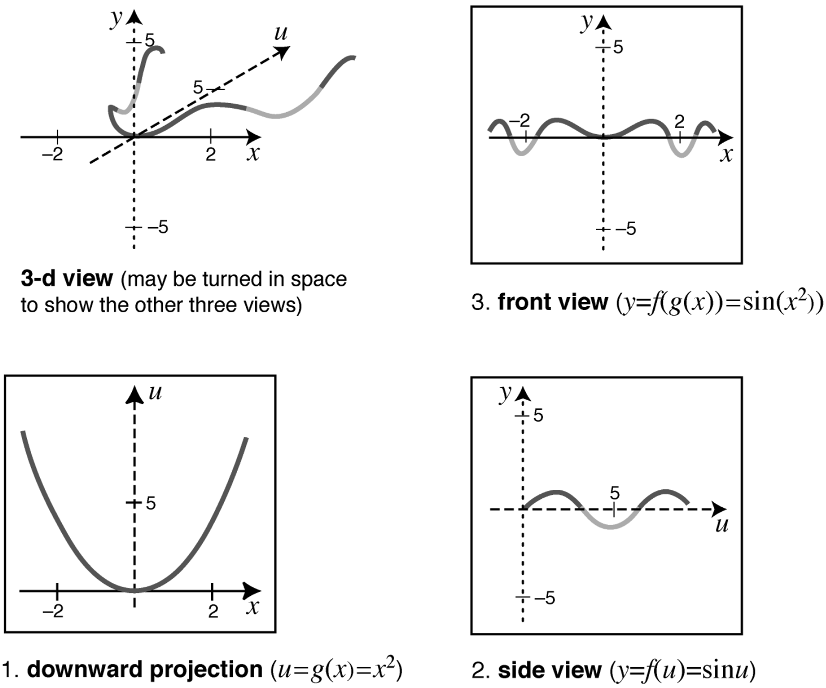



*Figure 6.* A composite function represented in three dimensions.

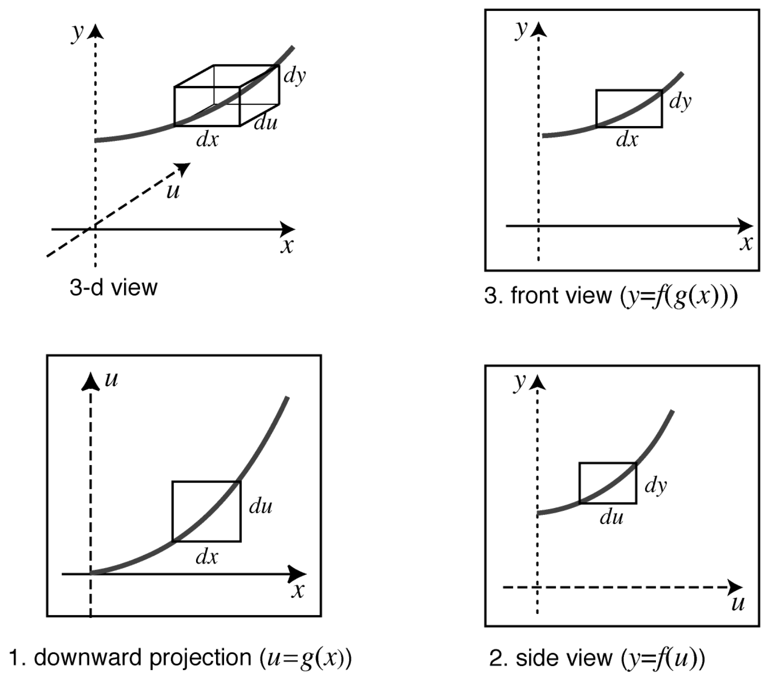



Figure 7. *The gradient box for the composite of two functions.*

This idea is best seen as a perspex model of a cuboid sides *dx*, *du*, *dy*. Such a model could be turned by hand to see the projection of the diagonal onto each of the sides where the latter have gradients *dy*/*dx*, *dy*/*du* and *du*/*dx* respectively.

These examples reveal that, although the various "rules of the calculus" can all be handled in a similar *symbolic* manner, each one has various visual images providing support which do not have simple, direct links between them.

The symbolic differentiation procedures themselves operate on the symbolism seen as a sequence of steps of evaluation (such as sums, products, quotients, composites etc). This calculation is not represented in the gestalt visual picture of the graph. *Thus the processes of "seeing" the relationships between the differentials and the procedures of symbolic use of rules focus on totally different aspects of the activity.*

This difference between the use of symbols and visualisation and the failure of one representation to link easily with the other is widespread and goes back to the earliest operations on functions. For instance, to express the line through (5,3) with gradient 5 in the form $y=mx+c$ begins with

$$y - 3 = 5(x - 4)$$

then the brackets is multiplied out and 3 added to both sides to get:

$$y - 3 = 5x - 20$$
$$y = 5x - 17.$$

We found that students remarked that, when manipulating symbols, they could not "see" what is going on (Crowley & Tall, 1999). Reflection on what is happening reveals that, as the symbols change, the graph *remains the same*. In a very real sense, a student would not be able to "see what is going on" because *nothing is happening visually to the graph*. Of course, to an expert, the fact that nothing is happening to the graph has real meaning because the line remains invariant under the symbolic manipulations. But for the student who lacks the conceptual links between disparate cognitive structures, the situation may be meaningless.

In each of these examples we see that the desired coherence of different ways of looking at a particular mathematical activity do not necessarily fit together in a precise correspondence. Some representations represent some aspects better than others, and some processes in one representation may not be mirrored in a one-to-one way to processes in the others.

The corollary of this discussion is that the cognitive development of the calculus cannot ever hope to lead to a totally coherent theory without encountering the need for various cognitive reconstructions on the way. To cope with the meaning of the calculus requires the student to reflect on the various situations to understand them in their own terms and to fit them together in a way which "makes sense". If this does not happen, then the default position is liable to involve instrumental learning of procedures without supporting conceptual links.

However, the cognitive root of local straightness can be used for a wide range of students to provide a platform with the potential to lead to various different approaches to the calculus. It can provide sensori-motor support for applications in physics, biology, economics and so on, including a mental image of notions such as the solution of a differential equation. It can be used as a based for the theory of formal analysis, either in terms of the standard "epsilon-delta" definition or in terms of an alternative "non-standard" theory using infinitesimals (Tall, 1981).

It may not be possible for all students to come to terms with all of the mathematics, but it may certainly prove possible for students who have great difficulty with some parts of the theory (eg symbol manipulation) to have conceptual insight into the nature of the theory. For instance I have had experiences with students who are not strong in mathematics being able to imagine functions that are continuous everywhere but differentiable nowhere, or to visualise functions that were differentiable everywhere once but nowhere twice, based on the notion of "local straightness" (Tall, 1995). But these self-same students were not necessarily able to cope with complicated symbol manipulation or to make the transition to fluent use of formal proof.

In this sense a cognitive root cannot guarantee that all students will understand all the theory which is to come. However, an idea such as local straightness is a "cognitive root" that appears to make sense to most learners, and also has the *potential* to be used as an introduction to various highly subtle aspects of formal theory encountered later on.

One feature of the use of local straightness as a cognitive root showed a profound difference between the conceptual structures of those that used this approach and those that followed a traditional course.

The idea of a tangent to a curve has long been known to contain cognitive conflicts requiring reconstruction. For instance, in the geometry of a circle, it is a line which touches a circle just once and it is often envisaged as a line which “touches” a curve but does not cross it. When a curve with an inflection point is encountered, students often misconceive the notion of tangent. For instance, the tangent to $y=x^2$ at the origin is seen by many, not as a horizontal line through the origin, but as a line that “touches but does not cross”. This gives what is sometimes termed a “generic tangent” as reported by Vinner (1991), (Figure 8).

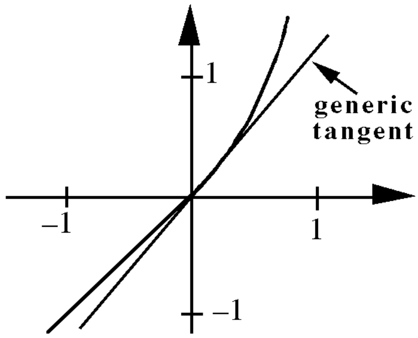



*Figure 9.* A generic tangent.

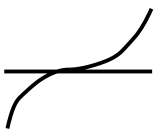
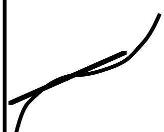
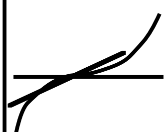
| A | B | C | D | E |
|---|---|---|---|---|
|  |  |  | | |
| The right answer | A generic tangent | two tangents | Another drawing | No drawing |
| 18% | 38% | 6% | 10% | 28% |

*Figure 8.* Drawing the tangent to $y = x^2$ at the origin (Vinner 1991).

The “locally straight” visual approach did not completely eliminate the belief in a generic tangent, for instance, the graph:

$$y = \begin{cases} x & x \le 0 \\ x + x^2 & x \ge 0 \end{cases}$$

presents a cognitive problem at the origin. A tangent exists but it coincides with the graph to the left and therefore does not intuitively “touch” the graph. A generic tangent is drawn a little to the right to show it as “touching” the curve. (Figure 9.)

The students responded as in Table 2. This reveals that the experimental students are significantly more likely to link the nature of the tangent to its standard form with less influence from the earlier concept image of "touching" the graph.

Table 2
*Responses to the Tangent in Figure 9*

| | Yes | | | No | |
|---|---|---|---|---|---|
| | Standard | Generic | Other | 2 | 0 |
| Experimental (N=41) | 31 | 8 | 0 | 2 | 0 |
| Control (N=65) | 22 | 30 | 2 | 15 | 1 |
| University (N=47) | 29 | 14 | 0 | 4 | 0 |

In terms of developing a long-term learning schema, it is helpful for the imagery to be developed in a way that encourages manageable reconstruction at a later stage. An example of this is to look forward to the existence of left and right derivatives, which is usually considered far too technical to discuss in a first course. However, zooming in to see a "corner" with different left and right gradients is available using the graphical software. This allowed discussion of different left and right gradients to occur simply and naturally. To test the effect of this introduction, experimental and control students were asked to consider the graph of the function

$$y = \left|x^2 - x\right| + x + 1$$

given also in the alternative form:

$$y = \begin{cases} (x^2 - x) + x - 1 = x^2 - 1 & \text{(for } x \leq -1 \text{ or } x \geq 1) \\ -(x^2 - x) + x - 1 = x^2 - 2x - 1 & \text{(for } -1 \leq x \leq 1) \end{cases}$$

Formally it has no tangent and the slope and derivative are undefined at $x$=1. The students were asked to say (with reasons) how many tangents, derivatives, slopes the graph has at $x = 1$ from the following possibilities: 0, 1, 2, more than two, other (e.g. "infinity"). The response deemed formally correct was zero in each case. In addition there were many other reasons that were possible to give, say "two" (to refer to the different cases on left and right) or even "infinity" tangents which are capable of touching and not crossing through the point (1,1).

In Table 3 we see that, although only 6 out of 14 students in experimental group A gave the accepted answer (none), in the other two groups there were 9 out of 12 and 13 out of 16. However, in the control groups, only 6 out of a total of 52 gave the accepted answer.

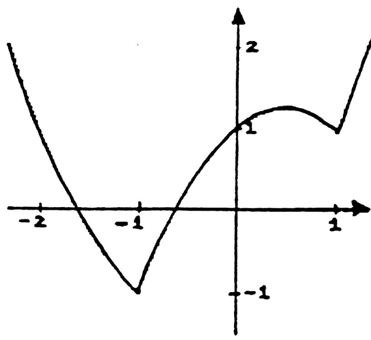



*Figure 10.* The graph of $y = \text{abs}(x(x-1))+x+1$.

The amazing statistic is that, in three out of four control groups, there were almost as many *different* responses as there were students. Pandora's box opens and reveals all the variety of possibilities in the students' minds. In experimental group A the teacher (myself) had encouraged discussion and participated actively, but had not given explicit instruction.

Table 3

*Student Responses to Tangents, Derivatives, Slopes at a "Corner" Point*

| Experimental Groups | none | two | Other solutions | Total number of *different* solutions |
|---|---|---|---|---|
| School 1, group A (*N*=14) | 6 | 1 | 7 | 9 |
| School 2, group B (*N*=12) | 9 | 0 | 3 | 3 |
| School 2, group C (*N*=16) | 13 | 0 | 3 | 4 |
| Control Groups | | | | |
| School 1, group D (*N*=9) | 0 | 0 | 9 | 8 |
| School 2, group E (*N*=13) | 2 | 2 | 9 | 9 |
| School 2, group F (*N*=17) | 3 | 4 | 10 | 10 |
| School 2, group G (*N*=13) | 1 | 1 | 11 | 13 |
| Others (non-experimental) | | | | |
| School 2, group H (*N*=12) | 5 | 0 | 7 | 8 |
| University year 1, (*N*=57) | 14 | 8 | 35 | 14 |

There was a combination of discussion and more explicit instruction in Group C leading to a greater uniformity in (correct) response. We thus see the power of the software to encourage discussion and the value of the directed participation of the teacher. By teaching in the traditional way the students had been left with a wide variety of possible connections as yet unassigned. The response of the university students shows the consequences of this. Some students had discussed left and right derivatives in class and 8 gave a response consistent with this, in addition to the 14 giving the response “none” to all three parts. But there were still 14 different possibilities given in total, leaving a wide range of beliefs to be brought together in teaching the notion of differentiability and non-differentiability at university.

## Conclusions

In this paper we have looked at the long-term building of learning schemas. Many examples have been given of discontinuities in the curriculum in which a change in context requires considerable cognitive reconstruction from many children which may prove too difficult for many of these. These include many contexts which occur in the English National Curriculum which provoke difficulties for children and cause many of them to take the line of least resistance and learn ideas which are meaningless to them by rote.

Even when long-term curricula are carefully designed using a cognitive approach, some of the conflicts persist. If they are directly addressed by the teacher as mentor in a meaningful context then it is possible to give a wider spectrum of students more coherent insights into the nature of the mathematics. A standard curriculum which takes things in a steady order, avoiding difficulties till later can create a Pandora’s box of different ideas that need to be rationalised (but are more probably, ignored) later on.

In all this development of long-term learning schemas, it becomes eminently clear that individual children make cognitive links in a wide variety of ways. To address this requires mutual activity on the part of the teacher and learner. The learner must become increasingly aware of his or her own part in reflective thinking about the concepts and helped to grow in confidence in dealing with the mismatches that occur in new contexts. The teacher, as mentor, must be aware of several different facets that may not be so clear to the pupil. One is the pupil’s need in cognitive growth, the second is an awareness of those things that are mutually agreed in the mathematical community and the third is the balance between pupil need and the focus on those aspects of mathematics that give long-term power.

I do not believe it is sensible or practical to attempt to design a smooth long-term curriculum in which each idea builds easily on previous ones. New contexts will always demand new ways of looking at things and often require significant cognitive reconstruction.

Awareness of the essential nature of cognitive reconstruction is vital in long-term curriculum design. The National Curriculum for England and Wales, for instance, consists of a sequence of ten levels which are to be taught successively over the years of schooling from the age of five to the age of sixteen. It is expected that some children will move quicker and more effectively through the levels whilst others may take it at a slower pace. The aim of the British government is to "raise standards" by getting a greater percentage of children to a given level. There is even a concept of "value added" namely the average improvement in grade level of the children in a given school. This metaphor of development presupposes a sequence of successively more subtle ideas and a naïve assumption that less successful children can move through the same sequence of ideas but at a slower pace.

However, the notion of discontinuity suggests an impediment to such progress. If a child meets a point requiring cognitive reconstruction and this reconstruction does not occur, then any later developments are hampered by subsequent misconceptions. These misconceptions arise not only from mistakes or misunderstandings in earlier mathematics, but also in the failure to adapt to new contexts where the old ideas are no longer completely appropriate. Far from being an occasional problem in learning, I claim that discontinuities in development are widespread in the curriculum and must be taken into account so that more children are able to succeed in personal reconstruction of the powerful mathematical ideas that may bear fruit in later life.

## References


Cornu, B. (1991). Limits. In D. O. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 153–166). Kluwer: Dordrecht.

Crowley, L., & Tall, D. O. (1999). The roles of cognitive units, connections and procedures in achieving goals in college algebra. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of PME* (pp. 225–232). Haifa, Israel, 2.

Lakoff, G., & Johnson, M. (1999). *Philosophy in the flesh: The embodied mind and its challenge to western thought*, New York: Basic Books.

Pegg, J. (1991). Editorial, *Australian Senior Mathematics Journal*, 5 (2), 70.

Skemp, R. R. (1962). The need for a schematic learning theory, *Brit. J. Educ. Psych.,* xxxii, 133–142.

Skemp, R. R. (1964) *Understanding Mathematics*. London: University of London Press.

Skemp, R. R. (1976). Relational understanding and instrumental understanding, *Mathematics Teaching* 77, 20–26.

Skemp, R. R. (1986). *The Psychology of Learning Mathematics*, (2nd edition). Penguin Books Ltd.

Skemp, R. R. (1993–4). *SAIL through Mathematics*, Volumes 1 & 2. Calgary: EEC Ltd.

Tall, D. O. (1981). Comments on the difficulty and validity of various approaches to the calculus, *For the Learning of Mathematics*, 2, 2, 16–21.

Tall, D. O. (1986). *Building and testing a cognitive approach to the calculus using interactive computer graphics,* Ph.D. thesis, University of Warwick.

Tall, D. O. (1989). Concept images, generic organizers, computers & curriculum change, *For the Learning of Mathematics*, 9 3, 37–42.

Tall, D. O. (1991). *Real Functions & Graphs: SMP 16-19* (computer programs and 64 page booklet), Cambridge: Cambridge University Press.

Tall, D. O. (1995). Visual organizers for formal mathematics. In R. Sutherland & J. Mason (Eds.), *Exploiting mental imagery with* *computers in mathematics education* (pp. 52–70). Springer-Verlag: Berlin.

Tall, D. O. (1997). Functions and calculus. In A. J. Bishop et al. (Eds.), *International Handbook of Mathematics Education*, (pp. 289–325), Dordrecht: Kluwer.

Tall D. O., & Vinner S. (1981). Concept image and concept definition in mathematics, with particular reference to limits and continuity, *Educational Studies in Mathematics,* 12, 151–169.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. O. Tall (Ed.) *Advanced Mathematical Thinking* (pp. 65–81). Dordrecht: Kluwer.

Vinner, S. (1997). The pseudo-conceptual and the pseudo-analytical thought processes in mathematics learning, *Educational Studies in Mathematics*, 34, 97–125.

# Versatile Learning of Mathematics

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It was many years ago now, in 1971, that as a young undergraduate in mathematics at Warwick University, I was first introduced to Richard Skemp's ideas by a young lecturer named David Tall. I read the paperback *The Psychology of Learning Mathematics* with great interest. It was not until I was a graduate student of mathematics education, in 1983, (once again working with David) that I re-read some of Richard's work, and yet, as a teacher of mathematics in schools it had remained with me. I read *Intelligence, Learning and Action* and was struck by the power of the ideas expressed in it and the simplicity of the language in which they were presented, especially in comparison with other texts I read. I learned the valuable lesson that powerful ideas can be communicated in simple terms and do not require a facade of convoluted definitions and expressions in their presentation.

Another educationist, Ausubel (1968, p. iv), well-known for his concept of meaningful learning, said, "If I had to reduce all of educational psychology to just one principle I would say this: The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly." Similarly, today, many constructivist mathematics educators would maintain as a central tenet that the mathematics children know should be the basis on which to teach mathematics (e.g. Steffe, 1991). Skemp, agreed, remarking that "our conceptual structures are a major factor of our progress" (1979c, p. 113). Since our existing schemas serve either to promote or restrict the association of new concepts, then the quality of what an individual already knows is a primary factor affecting their ability to understand. The existence of a wide level of agreement on this point indicates very strongly that it is something which mathematics educators should take to heart. However, one does not have to have been a teacher for very long before being faced with a dilemma. Many of us will have experienced the reaction of students when, after we have spent some considerable time trying to develop the ideas and concepts of differentiation, we introduce the rule for antidifferentiation of $x^n$. They may say "well why didn't you just tell us that was how to do it?" Too often it seems, the students' focus is on how 'to do' mathematics rather than on what mathematics is about, what its objects and concepts are. This procedural view of mathematics is, sadly, often reinforced by teachers who succumb to the pressure and only tell students how to do the mathematics.

I am particularly interested in the way in which our conceptual structures enable us to relate the procedural/process aspects of mathematics with the conceptual ideas, such as why the formula is correct. The essentially sequential nature of algorithms often contrast with the more global or holistic nature of conceptual thinking, and, the ways in which we, as individuals, construct schemas which enable us to relate the two in a *versatile* way is of great importance. The full meaning of the term '*versatile',* as used here will emerge during this discussion but, suffice it to say at this stage, that it will be used in a way which will have the essence of the usual English meaning, but, will take on a more precise technical sense, which will be explained during a discussion of the nature of our conceptual structures.

Having acknowledged the importance of our mental schemas in building mathematical understanding, some questions worth considering in any attempt to encourage *versatile* learning of mathematics include:

- How are schemas constructed?
- How can we identify and describe the quality of our constructions?
- How do our schemas influence our perception of the objects and procedures in mathematics, and their relationship?
- What experiences will help us improve the quality of our constructions so that we can build a *versatile* view of mathematics?

Skemp's theory has much to offer towards building answers to these sorts of questions, and in the rest of this paper I will seek to show how valuable it is. First we will consider his model of intelligence and its implications for improving the quality of learning and understanding.

# Skemp's Model of Intelligence

The basis of Skemp's theory of learning (Skemp 1979c, p. 89) is a model which describes intelligence as an activity in which learning is "a goal-directed change of state of a director system towards states which, for the assumed environment, make possible optimal functioning." According to this model of intelligence, we all engage in mental construction of reality by building and testing a schematic knowledge structure (Skemp, 1985). For Skemp (1979c, p. 219), a schema is "a conceptual structure existing in its own right, independently of action.", and he describes (*ibid,* p. 163) three modes which each of us may use to build and test such structures:

> Reality building: from our own encounters with actuality; from the realities of others; from within.
>
> Reality testing: against expectation of events in actuality; comparison with the realities of others; comparison with one's own existing knowledge and beliefs.

This process of mental construction involves two director systems, which Skemp describes as delta-one ($\Delta_1$) and delta-two ($\Delta_2$). The former is a kind of sensori-motor system which "receives information … compares this with a goal state, and with the help of a plan which it constructs from available schemas, takes the operand from its present state to its goal state." (Skemp, 1979b, p. 44). Delta-two on the other hand is a goal directed mental activity, whose operands are in delta-one, and its job is to optimise the functioning of delta-one (Skemp, 1979a). Hence the construction of concepts in a schema, or knowledge structure, may be by abstraction via direct sensory experience from actuality (primary concepts) using delta-one, or by derivation from other concepts (secondary concepts), using delta-two. In turn, the acquisition of new concepts may require expansion or re-construction of the relevant schema, altering it to take account of a concept for which it is relevant but not adequate (Skemp, 1979c, p. 126). Factors such as the frequency of contributory experiences, the existence of *noise* (irrelevant input) and the availability of lower order concepts, may affect one's ability to form concepts using the director systems.

Skemp (1979b, p. 48) outlines two modes of mental or 'intelligent' activity which take place in the context of delta-one and delta-two, namely intuitive and reflective:

> In the intuitive mode of mental activity, consciousness is centred in delta-one. In the reflective mode, consciousness is centred in delta-two. 'Intuitive' thus refers to spontaneous processes, those within delta-one, in which delta-two takes part either not at all, or not consciously. 'Reflective' refers to conscious activity by delta-two on delta-one.

The concept of *reflective intelligence* was one of the earliest which Skemp illuminated (see Skemp, 1961, 1978), and while acknowledging that the term is Piaget's, Skemp views his model of the concept differently, saying "When the concept of reflective intelligence was first introduced...I acknowledged that this term was borrowed from Piaget, and noted also that my model was not the same as his." (Skemp, 1979c, p. 218). For Skemp *reflective intelligence* involves the awareness of our own concepts and schemas, examining and improving them, thereby increasing our ability to understand. The methods available for doing so include physical and mental experimentation, generalisation and systematising knowledge, by looking for conceptual connections. It is this *conscious*, reflective mental activity which Skemp (1961) considered vital for successful building of mathematical knowledge structures, and it is this which he believes increases mathematical performance. He (Skemp, 1961, p. 49) illustrates this with reference to algebra, commenting:

> The transition to algebra, however, involves deliberate generalisation of the concepts and operations of arithmetic...Such a process of generalisation does require awareness of the concepts and operations themselves. Since these are not physical objects, perceivable by the external senses, but are mental, this transition requires the activity of reflective thought. And further, the generalisation requires not only awareness of the concepts and operations but perceptions of their inter-relations. This involves true reflective intelligence.

Thus, for Skemp, the change in the quality of thinking involved in the transition from arithmetic to algebra is brought about by 'awareness of' and 'deliberate generalisation of' the concepts and operations of arithmetic; that is by reflective intelligence.

## The Qualitative Nature of Mathematical Understanding

Having briefly discussed Skemp's view of the construction of mental schemas, we now ask, 'what about the qualitative nature of these schemas?' One of the first of Skemp's ideas I encountered, and which has stayed with me, was a strikingly simple, but extremely useful, definition of understanding, a term which is often used, but less commonly defined or explained. According to Skemp (1979c, p. 148), "to understand a concept, group of concepts, or symbols is to connect it with an appropriate schema." Skemp further clarified what he saw as understanding by describing two types, now well known in mathematics education, as instrumental and relational. This appreciation of the qualitative nature of understanding is no doubt one of the most valuable insights provided by Skemp's model of intelligence. In his landmark paper (Skemp, 1976) he described instrumental understanding as learning 'how to', involving learning by rote, memorising facts and rules. In contrast, relational learning, or learning 'why to', consists primarily of relating a task to an *appropriate* schema. Whilst this has received wide acceptance as a valuable insight, there were attempts to extend and re-shape some of the ideas he presented, and following this Skemp (1979b) created another category of understanding, akin to formal understanding, thus increasing them to three; instrumental, relational and logical. This last type he describes (Skemp, 1979b, p. 47) in these terms:

> Logical understanding is evidenced by the ability to demonstrate that what has been stated follows of logical necessity, by a chain of inferences, from (i) the given premises, together with (ii) suitably chosen items from what is accepted as established mathematical knowledge (axioms and theorems).

Hence the acquiring of logical understanding implies that the individual not only has relational understanding but is able to demonstrate evidence of such understanding to others by means of "a valid sequence of logical inferences" (Skemp, 1979a, p. 200).

The qualitative differences in understanding proposed in Skemp's model of intelligent learning are clarified through his use of a metaphor for mental schemas. Figure 1 is an example of the kind of network Skemp introduced as a metaphor for the understanding of a mental schema.

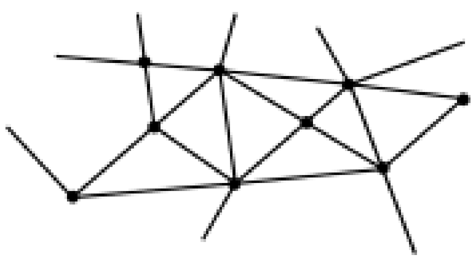


*Figure 1.* Skemp's schema metaphor.

It should be remembered that such diagrams are not intended to be in any sense a physical representation of the structure of the brain or the way in which it stores data, but are simply a metaphor to assist our perception of the cognitive structures of the mind and our discussion of the storage and manipulation of concepts affecting our thinking, learning and understanding. The qualitative differences in understanding of Skemp's model of intelligent learning are represented in this diagrammatic metaphor through references to associative, or A-links, between concepts where one has instrumental understanding, and conceptual, or C-links, for relational understanding. To justify the idea of concept links Skemp argues (1979c, p. 131) that since "activation of one concept can activate, or lower the threshold for, others." then there must be connections between them, and he adds:

> This idea has been developed to include a dimension of strong or weak connections, whereby activation of a particular concept results in the activation of others within quite a large neighbourhood, or within only a small one. Recently I have become interested in another difference which is not quantitative as above, but qualitative. This is between two kinds of connection which I call *associative* and *conceptual:* for short A-links and C-links.

He illustrates the difference between the two types of connections with number sequences. The numbers 2, 5, 7, 0, 6 are concepts which are connected by A-links; there is "no regularity which can give a foothold for the activity of intelligence." (*ibid,* p. 187). In contrast the numbers 2, 5, 8, 11, 14, 17 have a conceptual connection – a common difference of 3 – and so have a common C-link. Clearly, a link may be either associative or conceptual for any given person depending on whether they have formed the connecting concept. I have sometimes illustrated this by asking students whether they could remember, for a week, the following sequence:

7, 8, 5, 5, 3, 4, 4, 6, 9, 7, 8, 8

Most immediately decide, on the basis of A-links, that they could not, but once they are given the concept that these are the number of letters in the months of the year, the C-link is formed and they agree that they could. This demonstrates the principle that A-links can change to C-links, through the use of reflective intelligence, and such changes are accompanied by reconstruction of the appropriate schemas. One of Skemp's endearing teaching points is his use of personal experience and

I have identified in my own experience, something exemplifying this transition. Like many of us, no doubt, I have committed to memory, using A-links, certain results in mathematics. One in particular which I had trouble remembering was from the trigonometric results, namely, which of $\tan^{-1}(\surd 3)$ and $\tan^{-1}(1/\surd 3)$ was $30^\circ$ and which $60^\circ$. My A-link was in a state of constant degeneration. I can remember now my feelings of stupidity when one day I formed the C-link that since $(1/\surd 3) < 1 < \surd 3$, then $\tan^{-1}(1/\surd 3) < \tan^{-1}(1) < \tan^{-1}(\surd 3)$, and since $\tan^{-1}(45°) = 1$ was already a C-link for me, from a $45^\circ$ isosceles triangle, the rest followed instantly. It seems to me that consciously capturing such moments of insight brought about by reflective intelligence is quite rare, but especially valuable.

As the above example illustrates, a disadvantage of A-links is that they have to be memorised by rote, whereas for a C-link "we can put a name to it, communicate it, and make it an object for reflective intelligence: with all the possibilities which this opens up." (*ibid,* p. 188). Unsurprisingly, the quality of our schemas is a key determinant of our success in mathematics and Skemp (*ibid,* p. 189) observes the importance of C-links in this:

> But the larger the proportion of C-links to A-links in a schema, the better it is in several closely related ways. It is easier to remember, since there is only one connection to learn instead of many. Extrapolation is often possible and even inviting, as some will find in the cases of sequences of C-linked numbers. And the schema has an extra set of points at which assimilation, understanding, and thus growth can take place.

Thus, according to Skemp's theory of intelligent learning, a key objective of mathematics teaching should be to provide experiences which encourage and promote the formation of knowledge structures which, wherever possible, comprise conceptual links (C-links) corresponding to relational learning. It is exactly these types of knowledge structures which are necessary to promote *versatile* thinking since it is only through conceptual links that the global and sequential aspects of mathematics, and the different representations of concepts, can be properly related. What may not be so clear is what activities in the mathematics classroom we can use to encourage the formation of such schemas.

The relationship between processes (which I shall try to use as a generic term, reserving procedure for a specific algorithm for a given process) and objects has been the subject of much scrutiny by a number of researchers in recent years. Skemp's (1961 p. 47) insight that:

> For the algebraist will continue, in due course, to develop concepts of new classes of numbers (e.g., complex numbers) and new functions (e.g., gamma functions) by generalising the field of application of certain operations (taking square roots, taking the factorial of a number); and will study the application of the existing set of operations to the new concepts.

describes the construction of new mathematical objects (concepts) by generalising operations. Davis (1984, pp. 29–30) formulated a similar idea:

> When a procedure is first being learned, one experiences it almost one step at time; the overall patterns and continuity and flow of the entire activity are not perceived. But as the procedure is practised, the procedure itself becomes an entity – it becomes a *thing*. It, itself, is an input or object of scrutiny. . . . The procedure, formerly only a thing to be done – a verb – has now become an object of scrutiny and analysis; it is now, in this sense, a noun.

Describing here how a procedure becomes an object he strikes at a key distinction between the two when he mentions the 'one step at a time' nature of procedures when they are first encountered, in contrast with 'the overall. . . flow of the entire activity', or the holistic, object-like nature which they can attain for an individual. More recently others have spoken of how an individual *encapsulates* or *reifies* the process so that it becomes for them an object which can be symbolised as a *procept* (Dubinsky & Lewin, 1986; Dubinsky, 1991, Cottrill *et al.,* 1997; Sfard, 1991, 1994; Gray & Tall, 1991, 1994). However, it seems that there are at least two qualitatively different types of processes in mathematics; those *object-oriented* processes from which mathematical objects are encapsulated (see Tall *et al.* in press) and those *solution-oriented* processes which are essentially algorithms directed at solving 'standard' mathematical problems. An example of the first would be the addition of terms to find the partial sums of a series, which leads to the conceptual object of limit and the second could be exemplified by procedures to solve linear algebraic equations. Much mathematics teaching in schools has concentrated on *solution-oriented* processes to the exclusion of *object-oriented* processes. However *solution-oriented* processes usually operate on the very objects which arise from the *object-oriented* processes. Hence ignoring these is short sighted and will prove counter-productive in the long term. Failure to give students the opportunity to encapsulate *object-oriented* processes as objects may lead them to engage in something similar to the *pseudo-conceptual* thinking described by Vinner (1997, p. 100) as exhibiting "behaviour which might look like conceptual behaviour, but which in fact is produced by mental processes which do not characterize conceptual behaviour." In this case students may appear to have encapsulated processes as objects but this turns out on closer inspection to be what we will call a *pseudo-encapsulation*. For example, in arithmetic students may appear to be able to work with a fraction such as $\frac{4}{5}$ as if they have encapsulated the process of division of integers as fractions. However, many have not done so. I am still intrigued by the first time I encountered such a perception. It was in 1986 (see Thomas, 1988; Tall & Thomas, 1991) when I discovered that 47% of a sample of 13 year-old students thought that $6 \div 2x + 17$ and $\frac{6}{7}$ were not the same, because, according to them, the first was a 'sum' but the second was a 'fraction'. Such students are at a stage where they think primarily in terms of *solution-oriented* processes, and in this case they had not encapsulated division of integers as fractions. Instead they had constructed a *pseudo-encapsulation* of fraction as an object, but not one arising from the encapsulation of division of integers.

Similarly in early algebra many students may work with symbolic literals in simplifying expressions like $3a + 2b - 2a + b$ as if they view them as procepts. However, rather than having encapsulated the process of variation as a procept, symbolised by $a$ or $b$, they are actually thinking of them as *pseudo-encapsulations,* concrete objects similar to 'apples' and 'bananas'. When dealing with expressions such as $2x + 1$, a common perspective enables students to work with it but see it as a single arithmetic result not as a structural object representing the generalisation of adding one to two times any number $x$. When they move on to equations these students can use their *pseudo-encapsulations* to solve equations such as $2x + 1 = 7$. They may appear to be using algebraic methods to get their answer but they are actually thinking arithmetically, and given the answer 7, either use trial and error, or work backwards numerically, to find $x$. This becomes clear when they are unable to solve equations similar to $2x+1=7x-3$ (Herscovics & Linchevski, 1994).

It is possible that students might be assisted to avoid engaging in the construction of *pseudo-encapsulations* by approaching the learning of mathematics (in different representations) in the following way.

Experience the object-oriented process
↓
Encapsulate the conceptual object
↓
Learn solution-oriented processes using the object

This may be illustrated by the learning of the derivative of a function, $f'(x)$ or $\frac{dy}{dx}$. Students should first experience the limiting process of $\frac{f(x+h)-f(x)}{h}$ (or equivalent) tending to a limit as $h$ tends to 0 in the symbolic representation, and the gradient of a chord tending to the limit of the gradient of a tangent in the graphical representation. Then they will be able to encapsulate the derived function as an object encapsulated from this process. This will mean that they are able to operate on it, enabling them to understand the meaning of $\frac{d(\frac{dy}{dx})}{dx}$ in terms of rates of change, which otherwise will prove very difficult. Many students, who learn the derivative as a solution-oriented process, or algorithmic rule, are in precisely the position of having no meaning for the second derivative and are left having to apply it as a repeated process, probably devoid of any other meaning in any representation.

The *versatile* approach to the learning of mathematics which is being espoused here recognises the importance of each of the three stages of the recurring trilogy above and stresses the importance of experiencing each step in as many different representations as possible in order to promote the formation of conceptual or C-links across those representations. Students need personal experience of object-oriented processes so that they can encapsulate the objects. The *versatile* learner gains the ability to think in a number of representations both holistically about a concept or object, and sequentially about the process from which it has been encapsulated. A major advantage of a *versatile* perspective is that, through encapsulation, one may attain a global view of a concept, be able to break it down into components, or constituent processes, and conceptually relate these to the whole, across representations, as required. Without this one often sees only the part in the context of limited, often procedural, understanding and in a single representation.

How though does one encourage versatile thinking? What experiences should students have so that they may build such thinking and conceptual links? We have seen that reflective intelligence is a key to the transition from process to object, and hence to progress in mathematics (although this is not the only way objects may be formed in mathematics – see Davis, Tall & Thomas, 1997; Tall *et al*, 1999) and my contention is that representation-rich environments which allow investigation of the object-oriented processes which give rise to mathematical objects will encourage reflective thinking.

Skemp (1971, p. 32) appreciated that "Concepts of a higher order than those which a person already has cannot be communicated to him by a definition, but only by arranging for him to encounter a suitable collection of examples." Whilst this is a valuable principle, it somewhat oversimplifies the situation. Many higher order concepts are not abstracted from examples but, as we have discussed above, have to be encapsulated by the individual from object-oriented processes (see Tall *et al.*, 1999). Hence the examples which the student needs are not simply those embodying the finished concept itself, but also the processes which give rise to it.

*A Newton-Raphson Example*

Many of the points made in the above discussion can be well exemplified by the Newton-Raphson method for calculating the zeroes of a differentiable function. Given the formula:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)},$$

students may have no difficulty working in a solution-oriented manner to calculate the zeroes of a function, such as a polynomial, to any required accuracy. However, asking them questions about what it is they are doing, and why it works, is likely to elicit blank stares. They might reason that *doing* the mathematics and getting the right answer is the main thing, so why would one want to know anything further?

Furthermore, it is of interest that the formula above, in the usual form it is given, is that best suited to a procedural calculation of $x_2$, given $x_1$. To emphasise understanding rather than convenience of calculation a better form, which follows directly from a graphical representation, such as Figure 1 where $x_2$ can be *seen* to be a better approximation to the zero than $x_1$, would be:

$$f'(x_1) = \frac{f(x_1)}{x_1 - x_2}.$$

This can then easily be rendered in the usual procedural form for calculations. This version equates two symbolic representations of the gradient of a tangent, which are easily linked via the graphical format. The *versatile* approach in this case is to look at the object-oriented or concept-oriented processes first, especially in a visually rich graphical representation, *before* a consideration of the *solution-oriented* process. This can be accomplished by allowing students to experience graphs of functions, such as sine and cubic curves (for example using a graphic calculator or suitable computer graphing package), and to construct meaning for the process of 'zooming in' on a zero by drawing successive tangents at points on the curve. A few minutes experience with diagrams such as that in Figures 2 and 3 can provide tremendous insight into when and why this method works, and where the formula comes from.

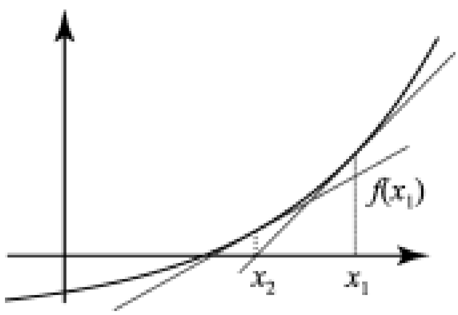



*Figure 2.* A diagram illustrating a successful search for a function's zeroes.

In this way students can reflect on what the process is doing, where the formula for finding a zero comes from, and most importantly under what conditions it may succeed or fail, emphasising, for example, the importance of the first approximation.

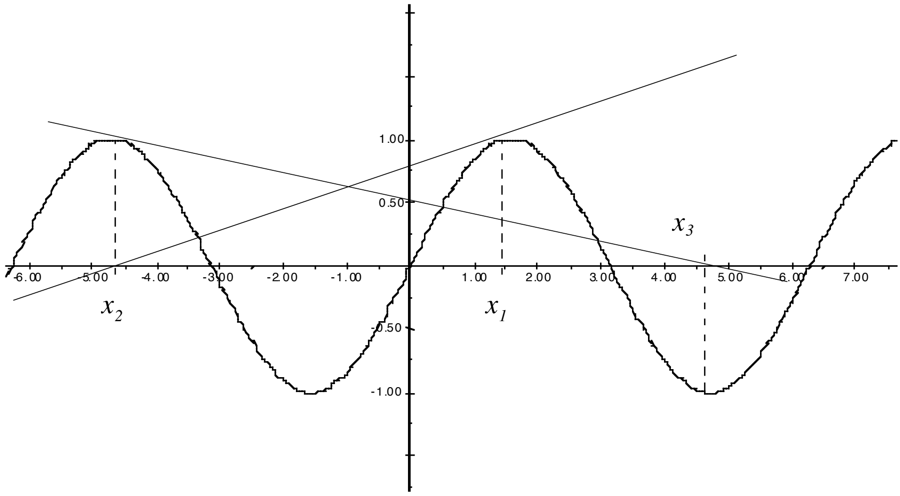



*Figure 3*. An example of what can go wrong with the Newton-Raphson method.

Thus they may visually conceptualise the process of finding the zeroes. Once this is established the students can begin to learn how to calculate values of zeroes of functions using a symbolically based solution-oriented process.

*Other Examples*

In arithmetic children become accustomed to working in an environment where they solve problems by producing a specific numerical "answer" and this leads to the expectation that the same will be true for algebra. It isn't. Such pupils face a number of conceptual obstacles to their progress in algebra (see e.g. Tall & Thomas, 1991; Linchevski & Herscovics, 1996). The concept of variable forms one major barrier between arithmetic and algebra, and yet it is a concept not often explicitly addressed in teaching which focuses on solution-oriented processes. Skemp (1971, p. 227) expressed the same point, commenting that, "The idea of a variable is in fact a key concept in algebra – although many elementary texts do not explain or even mention it." A versatile approach to the learning of variables seeks to build experience of the mathematical process of variation in a manner which blends personal investigation with the incorporation of visual representations. In a study to investigate this (Graham and Thomas, 1997, 2000) a graphic calculator was used to help students to a versatile view of variable, as both a varying process and an object symbolised by a letter. The 13 year-old children were encouraged to ‘see’ the calculator as storing numbers in its lettered stores, with each store comprising a location for the value and a name, a letter. A major advantage of the graphic calculator in this work is that it preserves several of these operations and their results in the view of the student, prolonging their perception time, and the opportunity to reflect on what they have done and hence form C-links. A feature of the research which encouraged such reflective thinking was the use of *screensnaps,* such as that in Figure 4. Here one is required to reproduce the given screen on the graphic calculator. This is a valuable exercise which students attempt, not by using algebraic procedures, but by assigning values to the variables and predicting and testing outcomes.

```
A+B
          11
A–B
          –3
```

*Figure 4*: Example of a *screensnap* from graphic calculator algebra research.

Exercises such as this have the advantage of encouraging beginning algebra students to engage in reflective thinking using variables, requiring them to: reflect on their mental model of variable; ‘see’ the variable’s value as a changeable number; physically change the value of a variable; and relate the values of two variables.

These experiences significantly improved understanding of the way students understood the use of letters as specific unknown and generalised number in algebra. For example, when asked a question similar to that used by Küchemann (1981) in his research, namely:

*Does* $x - y = z - y$ — always, never or sometimes ... when?,

30% of a graphic calculator group of 130 answered ‘sometimes when $x = z$’, compared with 16% of the 129 controls ($\chi^2 = 7.73$, $p < 0.01$), who had learned algebra in a traditional, procedural, solution-oriented way. Correspondingly, one of the students who had used the graphic calculators was asked on interview:

Int: Did you have any previous knowledge of algebra?

N: Yeah, we did some in form 2, a little bit, but I didn’t really understand much of it.

... Int: $L+M+N=L+P+N$. Now what would be the answer there do you think? [ie Are they equal always, never or sometimes ... when?]

N: Sometimes, if *M* and *P* are the same value.

Int: OK good. What about the next one, $A+B=B$?

N: Um, sometimes if $A=0$.

Such questions cannot easily be answered by students who, having constructed a *pseudo-encapsulation,* see letters as concrete objects, since they will not appreciate that two different objects can be the same. Rather they require one to understand that the letters represent values, which may coincide. N had seen this, was happy to let different letters have the same value and could pick out the correct value 0 for A in the second example. It was apparent that the model had assisted in the attainment of this view, since one student remarked:

> I think the STORE button really helped, when we stored the numbers in the calculator. I think it helped and made me understand how to do it and the way the screen showed all the numbers coming up I found it much easier than all the other calculators which don't even show the numbers.

The power of a representation which promotes visualisation is seen here. The way the calculator screen preserves several computations on the variables has assisted this student to build ideas about the use of letters in algebra to represent numbers. These results and comments demonstrate that these students had not only gained the flexibility of insight where they could see two letters as representing a range of numbers which could sometimes coincide, but, as their simplification results showed, they could operate on these encapsulated objects at least as well as students who had spent their time on manipulation techniques. They had a more *versatile* conception of variable.

Many students carry forward a procedural, arithmetic perspective into their acquaintance with expressions and equations in algebra. Substitution of a value into an expression or function is not a problem for such students because they have an arithmetic procedure they can follow. However, many students have not encapsulated expression as a variable-based object and so when faced with simplification of expressions or solving equations they build meaning based on concrete objects and then engage in solution-oriented processes. This view of equation is a real problem when they are required to solve linear algebraic equations of the form $ax + b = cx + d$. For the student who is thinking sequentially and looking for the result of a calculation, such a problem is telling them that two procedures give the same result, but they are not told what that result is. Thus, unable to work either backwards from the answer, or forwards to a known answer, many fall at this obstacle, which has been called the *didactic cut* (Filloy and Rojano, 1984) or *cognitive gap* (Herscovics and Linchevski, 1994).

A *versatile* view of equation can be promoted by a computer environment such as that of the *Dynamic Algebra* program (see Figure 5), which encourages students to construct equations in terms of variable and expression objects which can be simultaneously evaluated (Thomas & Hall, 1998).

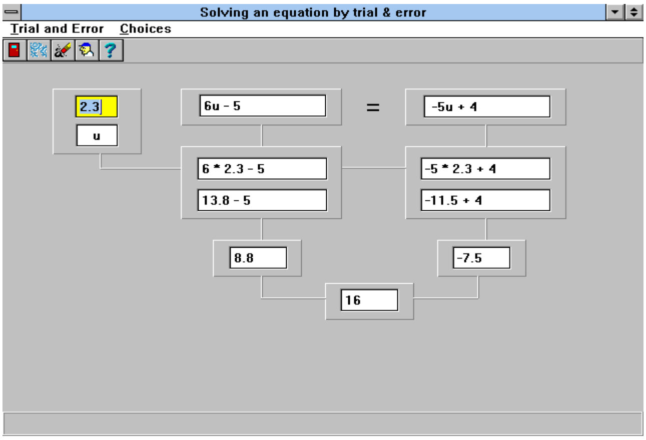



*Figure 5.* A screen from the *Dynamic Algebra* program.

The representation employed in this approach emphasises the visual aspects of variable mentioned above (as a location and label) and equation (as two equal expression boxes) in an environment where a number of processes for equation solving, including trial and error substitution and balancing can be investigated. The trial and error experiences combine the visual model of variable with the idea of substitution in an expression (or function) and equivalence of expression (or function). Students investigate solutions by entering values for the variable, here $u$, until both sides are seen to be equal in value, that is the difference between the two sides is zero. This reinforces in the student's experience, the results of evaluating the expressions constituting each side of the equation. In this way students can conceptualise equation before embarking on solution-oriented processes. One student, who used guess and substitution to solve linear equations before using this program, when interviewed about questions with more than one $n$, such as $2n = n + 6$, showed little conceptual appreciation of equation asking, "Does $n$ have a variety of numbers or are they the same?" After investigating equations using the computer environment she appeared to have no problem with this and her solution to the equation $5n + 12 = 3n + 24$ was written as:

$$\begin{gathered} 5n - 3n + 12 = \cancel{3n} - \cancel{3n} + 24 \\ 2n + 12 = 24 \\ 2n + \cancel{12} - \cancel{12} = 24 - 12 \\ 2n = 12 \\ n = 6 \end{gathered}$$

The computer program was structured by this procedure of performing the same operation on each side of the equation, although the students were not taught it explicitly but had to build up their own understanding. This student had progressed through an investigation of object-oriented processes to an encapsulation of equation as a statement of equality of expressions and hence was able to operate on it using a solution-based procedure. To improve in ability to the point where she could solve such equations after a few hours reflective activity is no small achievement, and indicates the possible value of a versatile approach to the learning of expression and equation.

While introductory algebra finds its place at one end of the secondary school curriculum, the definite (or Riemann) integral is firmly at the other. However, here too, it is not uncommon to find students who are instrumentally engaged in following set solution-oriented procedures for calculating integrals with little idea of what the integrals are or represent. They are *process-oriented* (Thomas, 1994), locked into a mode of calculating answers using antiderivatives without the directing influence of a holistic, structural understanding. The inability of such students to deal with questions which do not appear to have an explicit solution procedure was highlighted in another study (Hong & Thomas, 1997, 1998). For example, a generalised function notation proved particularly difficult for many of these students to cope with, since it did not fit into their process-oriented framework. In the question:

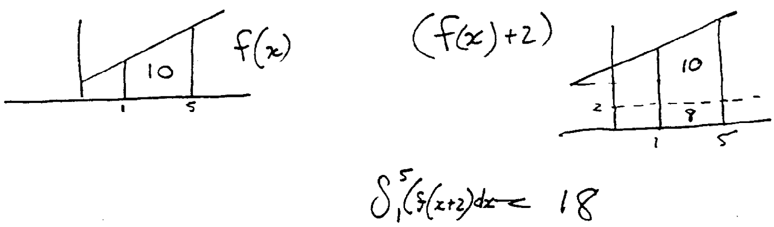
If $\int_1^3 f(t)dt = 8.6$, write down the value of $\int_2^4 f(t-1)dt$, a number of students who were unable to see how to apply a known procedure to deal with $f(t-1)$, used a number of different procedural ways to surmount the obstacle including taking $f$ as a constant, using $f(t-1)$ or attempting to integrate the $t-1$ and obtaining $t^2-t$ or $\frac{(t-1)^2}{2}$.

The students were introduced to visually rich computer environments, using Maple and Excel which combined symbolic, tabular and graphical representations. Students could calculate upper and lower area sums (in more than one representation), find the limits of these sums, relate the sign of the definite integral to the area under the graph of the function, consider the effect of transformations on areas, and begin to think of area as a function. In this way these computer experiences allowed a personal cross-representation investigation of the processes lying behind these concepts of integration, assisting the students to construct conceptual objects associated with Riemann sums and integrals.

Having investiagted the area under the functions $x^2$, $x^2 + 2$ and $(x-2)^2$ they were able to extend their understanding to cope with the question,

If $\int_1^5 f(x)dx = 10$, then write down the value of $\int_1^5 (f(x)+2)dx$,

which considered a transformation parallel to the $y$-axis for a general function $f(x)$. 44% of the school students were able to deal with integrating $f(x) + 2$ after their computer experiences, compared with only 11% before. A student unable to make any attempt at the pre-test, drew a post-test diagram showing clearly that he understood this question visually in terms of the creation of an extra area by the transformation parallel to the $y$-axis.

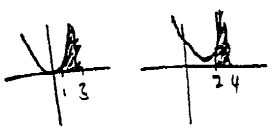


When the transformation (see the question above) was parallel to the $x$-axis, and involved a general function $f(t–1)$, the computer group's performance improved from 11%, to a success rate of 50%. One student demonstrated that she was now thinking in terms of conservation of area and was able to relate this to the procepts by writing:

$\int_1^3 f(t)dt = \int_2^4 f(t-1)dt$. ∵ Area the same.

∴ $\int_2^4 f(t-1)dt = 8.6$.

her reason for the equality being '[because] Area the same'. Another student who gave no response at the pre-test showed her improved conception that the translation of the graph leaves the area unchanged when in the post-test she was able to visualise that after the transformation parallel to the $x$-axis the area would not be changed. This understanding was manifest through her working, when she drew the following diagrams as part of her solution:




It would be the same, the area does not change ⇒ 8.6

These examples show the value of the linking of graphical, visual experiences with other representations, and their effect on the form of some of the students' reasoning and answers. In contrast other students were able to symbolise the relationship very precisely in terms of procepts by writing $\int_2^4 f(t-1)dt = \int_1^3 f(y)dy$, and hence give the solution as 8.6. There was evidence of a versatility in approach from the students who had worked on the computer.

## The Role of Visual Representation: Extending Skemp's Model

The research projects and ideas outlined above relied heavily on the use of representations of concepts using visual imagery; for the graphs of functions, for a variable, as a store with a value and a label, for equations, and for the way in which areas relate to integration. The importance of visual representations to encapsulation of conceptual objects is one which I believe is at the core of learning and one which intrinsically involves the formation of conceptual structures, with their C-links and A-links. Secondly, I believe that imagery plays a far more important part in all our thinking than we are, literally, aware of. Much of our thinking (for example the processing of the majority of data from our visual field) appears to be unconscious, as Johnson-Laird (1993, p. 354) comments "… there are also many benign unconscious processes that underlie perception and cognition."

The majority of such unconscious processing appears to be holistic in nature and extremely fast, compared with the generally sequential and slower processing of the conscious mind (Gazzaniga, 1985). Imaginal input, is unlikely to be randomly stored in the brain but, along with other holistic data, such as aural input, for example voices and music, is also formed into schemas, similar to those of our logical and verbal thinking.

I have designated the consciously accessible schemas as higher level thinking, and the unconscious as lower level thinking, more to emphasise their qualitatively different natures than to elevate one above the other. Rather than being formed into two totally separate and distinct schematic structures there is much circumstantial evidence that the two are linked.

I believe that there is a *cognitive integration* (Thomas, 1988) of the schemas at these two levels of thinking with both C-links and A-links between concepts etc. at the two levels which can be encouraged and enhanced. For example one may see a face in a crowd or hear a piece of music which one appears to recognise but is unable to relate to in any other way, such as by naming. This may signify that an image of the face or music pattern in the lower level schemas is linked associatively (A-link) to those at the higher level which mediate recognition. In contrast one may see the face and exclaim "Hi, uncle John" or hear the music and comment that, "I prefer von Karajan's version of this Beethoven symphony." Such remarks are evidence of conceptual links (C-links) between the schemas at the lower and higher levels. Figure 6, building on Skemp's diagram, is a metaphor for the relationship between the two qualitatively different modes of thinking.

The 'vertical' links between the two levels allow the mental imagery schemas to influence the higher level cognitive functions of the mind. This approach leads to some valuable insights about why we find some intellectual goals hard to achieve and yet others, which we may not wish, seem to thrust themselves upon our consciousness.

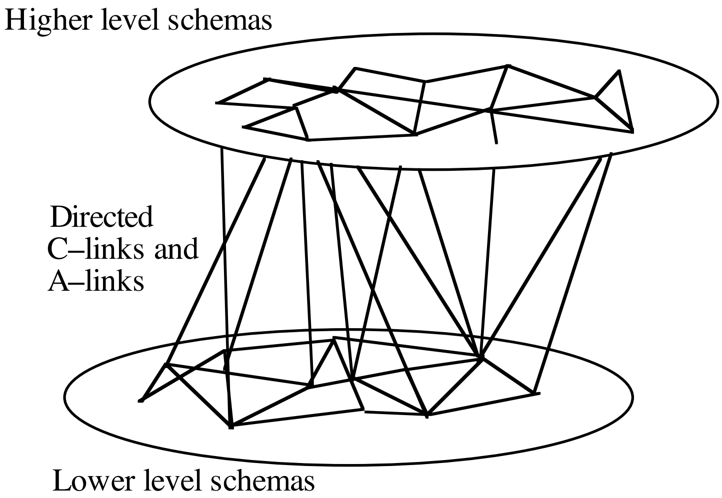



*Figure 6.* A simplified model for Cognitive Integration.

How may this be explained? In the mental process of finding a path from a present state P, to a goal state G, there will be at virtually every stage of the path, different possibilities for the route, many (if not all) of which will involve unconscious processing. It is likely that much of the unconscious processing will occur simultaneously and across representations, where the C-link connections exist.

At any nodal point, the path most likely to be taken is that which has the highest strength factor. Thus these tend to dominate our thinking process at each stage. This is not to say that we must take this particular route, but simply that it is more prominent and thus more likely to be taken.

To illustrate, if we travel from one place to another in a car and wish to get to our destination in the most efficient way we may likely travel along the motorway, if there is one. Periodically we will approach junctions, with roads of varying size leaving the motorway, ranging from small country roads to other motorways. Given a choice, and the desire for the most efficient journey, we will probably choose to use the major route at each junction. We do however have a choice, and other factors may influence our decision, such as our level of tiredness, boredom, interest in a passing feature of the landscape, likely traffic jams etc.

With intelligent learning, just as with this illustration, it is often the case that, in order to follow the most efficient route, indeed sometimes in order to gain access to any major route at all, we must first avail ourselves of minor routes. In the context of our thought processes, to make a positive decision to choose a path of lower strength factor rather than one of a higher factor requires reflective intelligence, since it is this which makes us aware of the choices. A mathematical example which illustrates the problem students may have with finding what for them is often a minor cognitive pathway is given by this question:

Solve the equation $(2x+3)^2 = 9$.

Many students immediately link this question to quadratic equations (which it is of course), but these in turn are often very strongly linked to the 'standard' form

$$ax^2 + bx + c = 0$$

which is in turn strongly linked to the 'formula', a solution-oriented procedure for solving such an equation. Hence they may be pushed along the procedural route of multiplying out the brackets, collecting like terms, simplifying and then using this formula. In doing so they miss the 'minor' cognitive path which, via the step

$$2x + 3 = \pm 3,$$

leads much more directly to the goal state of a solution to the equation. Finding this minor path can be facilitated by versatile thinking. A global view, promoted by suitable imagery, can enable one to see the structure $A^2 = B^2$ in the question, leading to $A = \pm B$.

How may students be encouraged to think in this versatile way? One necessary initiative is for teachers to copy the example of the esoteric, boutique tour guide who sees it as his job not only to make sure that the coach party is taken down the small country roads but that they get to stop off at all the places of interest on the way, take photographs (images) as reminders, observe the structural form of buildings etc. and receive a commentary on why they are of interest. Sadly many teachers who would love to teach in this versatile way on their mathematical 'tour', visiting a number of available representations of each concept they introduce, do not have the luxury of the time to do so. They are constrained by the 'directors' of the tour company who insist that schedules are strictly kept to and standards and deadlines are met.

How then does the concept of *cognitive integration* help us to address the vital question of what may be done to help the solution process-oriented student gain a versatile conception of algebra and other branches of mathematics? In view of the above model, two possible measures, which are not mutually exclusive, have been stressed here. First we may argue that students are best helped to a versatile view of mathematics by an approach which emphasises experience and use of processes leading to mathematical constructs before being presented with finished mathematical objects and corresponding solution processes. The second conclusion is that more emphasis should be given to the global/holistic view of the mathematics by promoting different representations and the links between them at all stages. As educators, the more we incorporate these into our mathematics teaching the more we make available links, which are often visual, to assist understanding and learning, with the possibility that versatile thinking may result. However, one should be conscious to introduce the representations in a way which encourages conceptual or C-links between the symbolic, the procedural and the visual, as in the Newton-Raphson example we started with. Such conceptually linked imagery will increase the possibility of a student being able to vary their cognitive focus in a given mathematical situation from a sequential, left-to-right process-led perspective to a global/holistic concept-driven mode, or vice-versa.

These ideas on versatile learning have been firmly based on the pioneering insights of Richard Skemp. The quality of his thinking and the presentation of his ideas continue to move mathematics education forward even today. The day that Richard died, I later found out, I had spent discussing mathematics education with David Tall, and somehow that seemed very appropriate.

## References

Ausubel D. P. (1968). *Educational Psychology – A cognitive view*, Holt, Reinhart And Winston Inc..

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept:

Beginning with a coordinated process scheme, Journal of Mathematical Behavior, 15, 167–192.

Crowley, L., Thomas, M. O. J., & Tall, D. O. (1994). Algebra, symbols and translation of meaning, *Proceedings of the 18th Conference of the International Group for the Psychology of Mathematics Education*, Portugal, 4, 240–247.

Davis, R. B. (1984). *Learning Mathematics: The Cognitive Science Approach to Mathematics Education*. Norwood, NJ: Ablex.

Davis, G., Tall, D. O., & Thomas, M. O. J. (1997). What is the Object of the Encapsulation of a Process? *Proceedings of the 20th Mathematics Education Research Group of Australasia Conference*, Rotorua, New Zealand, 132–139.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking, In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 95–123). Dordrecht: Kluwer Academic Publishers.

Dubinsky, E., & Lewin, P. (1986). Reflective abstraction and mathematics education: The genetic decomposition of induction and compactness, *Journal of Mathematical Behavior*, 5 (1), 55–92.

Filloy, E. & Rojano, T. (1984). From an arithmetical to an algebraic thought, In J. M. Moser (Ed.) *Proceedings of the Sixth Annual Meeting of PME-NA,* Madison: University of Wisconsin, 51–56.

Gazzaniga, M. S. (1985). *The Social Brain: Discovering the Networks of the Mind*, New York: Basic Books.

Graham, A. T., & Thomas, M. O. J. (1997). Tapping into Algebraic Variables Through the Graphic Calculator, *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education*, Lahti, Finland, 3, 9–16.

Graham, A. T., & Thomas, M. O. J. (2000). Building a versatile understanding of algebraic variables with a graphic calculator, *Educational Studies in Mathematics*, 41 (3), 265–282.

Gray, E., & Tall, D. O. (1991). Duality, ambiguity and flexibility in successful mathematical thinking, *Proceedings of the 15th Conference of the International Group for the Psychology of Mathematics Education*, Assisi, 2, 72–79.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic, *The Journal for Research in Mathematics Education*, 26 (2), 115–141.

Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics*, 27, 59–78.

Hong, Y. Y., & Thomas, M. O. J. (1997). Using the Computer to Improve Conceptual Thinking in Integration, *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education*, Lahti, Finland, 3, 81–88.

Hong, Y. Y., & Thomas, M. O. J. (1998). Versatile Understanding in Integration, *Proceedings of the International Congress of Mathematics Instruction – South East Asian Conference on Mathematics Education*, Seoul, Korea, 255–265.

Johnson-Laird, (1993). *The Computer and the Mind: An introduction to cognitive science*, London: Fontana.

Küchemann, D. (1981). Algebra, in Hart, K. (Ed.) *Children's Understanding of Mathematics: 11–16*. London: John Murray.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin, *Educational Studies in Mathematics*, 22 (1), 1–36.

Sfard, A. (1994). Reification as the birth of metaphor, *For The Learning of Mathematics*, *14*(1), 44–54.

Skemp, R. R. (1961). Reflective intelligence and mathematics, *The British Journal of Educational Psychology*, 31, 45–55.

Skemp, R. R. (1971). *The Psychology of Learning Mathematics*, Middlesex, UK: Penguin.

Skemp, R. R. (1976). Relational understanding and instrumental understanding, *Mathematics Teaching,* 77, 20–26.

Skemp, R. R. (1978). Revised model for reflective intelligence, *Proceedings of the 2nd Conference of the International Group for the Psychology of Mathematics Education*, Osnabrück, Germany, 286–295.

Skemp, R. R. (1979a). Goals of learning and qualities of understanding, *Mathematics Teaching*, 88, 44–49.

Skemp, R. R. (1979b). Goals of learning and qualities of understanding, *Proceedings of the 3rd Conference of the International Group for the Psychology of Mathematics Education*, Warwick, UK, 197–202.

Skemp, R. R. (1979c). *Intelligence, Learning and Action*: – *A Foundation for Theory and Practice in Education,* Chichester, UK: Wiley.

Skemp, R. R. (1985). PMP: A progress report, *Proceedings of the 9th Conference of the International Group for the Psychology of Mathematics Education*, Utrecht Netherlands, 447–452.

Steffe, L. P. (1991). The constructivist teaching experiment, in E. von Glasersfeld (Ed.) *Radical Constructivism in Mathematics Education,* Dordrecht, Holland: Kluwer

Tall, D. O., & Thomas, M. O. J. (1991). Encouraging versatile thinking in algebra using the computer, *Educational Studies in Mathematics*, 22, 125–147.

Tall, D. O., Thomas, M. O. J., Davis, G., Gray, E., & Simpson, A. (1999). What is the object of the encapsulation of a process? *Journal of Mathematical Behavior,* 18 (2), 223–241.

Thomas, M. O. J. (1988). *A conceptual approach to the early learning of algebra using a computer*, Unpublished Ph.D thesis, University of Warwick.

Thomas, M. O. J., & Hall, D. (1998). A computer environment to encourage versatile understanding of algebraic equations, *Proceedings of the 21st Mathematics Education Research Group of Australasia Conference*, Brisbane, 605–612.

Vinner, S. (1997). The pseudo-conceptual and the pseudo-analytical thought processes in mathematics learning, *Educational Studies in Mathematic*s, 34, 97–129.

# As Though the Thinking is Out There on the Table

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> "It is as though the children's thinking is out there on the table."
> A teacher quoted by Richard Skemp.

This quote from Richard Skemp is taken from the Horizon television program "Twice Five Plus the Wings of a Bird" (Campbell-Jones, 1986). The program was made after the influential Cockcroft Report of 1982, and brought together the work of many of Britain's most prominent mathematics educators, painting a picture of how their insights into children's ways of learning could improve the teaching of mathematics. Richard Skemp was filmed in a class of upper primary children who were working on various mathematics games. One of the games, devised by Skemp, was to make rectangular arrays out of various numbers of counters. One child chose a number of counters (say 12) and challenged a partner to make the array (say 2 by 6 or 4 by 3). When they played the game, the children began to see the patterns in the numbers and began to be able to predict when counters could be made into an array and when not. Children began to discover for themselves the role of factors and the special properties of prime numbers. As he watched the children talking to each other and with him, Skemp commented that for children the benefits of games are in the way in which they require the use of intelligence and in the positive features of learning together. However, for teachers, he identified the major benefit was that games give a much enhanced insight into children's thinking. He recalled a teacher who had said, "it is as though the children's thinking is out there on the table."

[1] The research referred to in this chapter was funded by a grant to Kaye Stacey from the Australian Research Council. We thank the teachers and students who contributed their time generously to the project.

One of Skemp's legacies to mathematics education is his contribution to the recognition of the importance of having children's thinking "out there on the table". At a time when most research in mathematics education was concerned mainly with performance, he appreciated the need for a psychology of understanding, learning and thinking about mathematical ideas. This helped bring about the "cognitive revolution" that in turn supported the ideas for a new vision of mathematics teaching put forward in the Horizon program and in so many other places. As Skemp described it in the program, it is a style of mathematics teaching where children are both able and encouraged to bring their intelligence into action for the learning of mathematics.

Skemp's interest in the psychology of learning mathematics was driven by a desire to make mathematics teaching in schools more interesting and beneficial for all the participants, by promoting better understanding. He wanted to have the thinking "out there on the table" to improve teaching. In the first sections of this chapter, we will summarise his reasons for this and how subsequently this quest has been taken up by others. Then we consider Skemp's illustration in the domain of elementary algebra of how insights into thinking and a focus on mathematics as "intelligent" rather than habit learning changes teaching. We then move on to consider how the views that he expressed have evolved in more recent research on algebra learning. Skemp showed the need to analyse the content of mathematics from a psychological as well as from a mathematical point of view.

## Why is Knowledge of Students' Thinking Important?

In his book "The Psychology of Learning Mathematics", Skemp (1986) outlined his theories of learning and understanding. This second edition was a substantial revision of the 1971 version, reflecting the development of Skemp's work. He identified the most important style of learning for mathematics as intelligent learning: the formation of conceptual structures communicated and manipulated by symbols (p. 16). For Skemp, the main task of the teacher is therefore to help students acquire concepts.

> Mathematical concepts are highly abstract and highly general. This gives mathematics its power – because it is about nothing it can be about everything – but it also makes it hard to learn. Because they are several

> layers removed from reality, mathematical concepts cannot be learned directly from the environment but instead have to be built up from other concepts. Two principles of learning mathematics follow from this fact:
>
> (i) Concepts of a higher order than those that people already have cannot be communicated to them by a definition, but only by arranging for them to encounter a suitable collection of examples.
>
> (ii) Since in mathematics these examples are almost invariably other concepts, it must first be ensured that these are already formed in the mind of the learner. (p. 30)

Here then we have a central problem of designing and delivering teaching of mathematics for understanding: firstly to select a suitable collection of examples for pupils, so that they can abstract from them new concepts, and secondly to do it with concepts that are already adequately formed in the mind of the learner. This then is the reason for Skemp's interest in having the thinking out on the table.

As a psychologist studying learning, Skemp (1986, p.68) would have been frustrated and challenged that ideas are "invisible, inaudible and perishable". He noted that the difficulty of observing thinking is compounded in mathematics teaching because it is hard to distinguish the early stages of conceptual learning from merely learning to manipulate symbols to obtain an approved answer. "All the teachers can see (or hear) are the symbols. Not being thought-readers, they have no direct knowledge of whether or not the right concepts, or any at all are being attached" (*ibid,* p. 48). He therefore valued the mechanisms that a teacher or researcher can use to tap into thinking. In the Horizon program, he notes that children's working together on games is one of these mechanisms.

*Mechanisms for Getting Thinking Out on the Table*

Following the work of Skemp and others, mathematics educators have been extremely interested in exploring classroom techniques where the chances of children displaying their thinking are increased. For example, co-operative activity of many types has been extensively studied for many reasons, some of which are relevant here (Good, Mulryan and McCaslin, 1992). A more recent trend is to explore new computer environments for their capacity to provide a catalyst for the sharing of ideas in a collaborative environment. Geiger & Goos (1996) give a current example of these new perspectives. Education development can design teaching methods to assist a teacher to observe each individual student's thinking so that the student can be taught most effectively. Accomplishing this task in real time is clearly a major challenge for teachers, and one that they can hope at best to only partially achieve. The possibility of students helping each other alleviates this, but only to some extent.

However, there is another way in which thinking can be out on the table. This is by researchers documenting typical thinking of children learning a topic, so that teachers are forewarned about what sorts of ideas they may encounter in the classroom and can better work with individuals in limited class time. Putting thinking out on the table in this second way has been a major contribution of the research that has followed from pioneers such as Richard Skemp. Researchers have the opportunity to interact with individual students; watching, asking questions and listening, to a depth which is impossible in day-to-day teaching. Commonalities in the written work of large numbers of students can be discovered by careful analysis. This accumulated knowledge about "typical" students is not the same as the knowledge that a teacher may find out about one of his or her own students and therefore cannot inform action in the same way. It can however, be useful by priming teachers to know what to look for in the few short moments that they do have with each child.

There is a growing body of empirical evidence that teachers who understand about students' thinking teach differently and more effectively than teachers who do not. Moreover, it is now clear that the experience of teaching of itself does not generally produce sufficient understanding to make a difference. Several empirical studies of teaching have shown these results. For example, the Cognitively Guided Instruction Project (Carpenter, Fennema, Peterson, Chiang & Loef, 1989) gave teachers information about research results on the way in which children approach elementary addition and subtraction problems. Teachers learned, for example, about the different types of situations that are modelled by subtraction and how children find some of these types much easier than others. Before the project, the teachers had a good understanding of addition and subtraction, but their knowledge was not organised in a way that helped them to understand children's thinking. To teach, an understanding of content and of children's thinking need to go hand in hand.

## Students' Thinking About Algebra

### *Variables and Unknowns*

Skemp worked from what is now called a constructivist point of view. For him it was axiomatic that the actual construction of a conceptual system is something that individuals have to do for themselves (p. 27), although, as discussed above, he sees the teacher as having a critical role in this. Skemp also appreciated the need for teaching to be based on a psychological analysis of the content as well as a mathematical analysis and began pioneering work in this field. In the remainder of this chapter we will consider some of the ways in which subsequent research has developed from this beginning and how deeper analysis of students' thinking has changed some of the perspectives.

We will illustrate this by examining Skemp's proposals for elementary algebra as outlined in *The Psychology of Learning Mathematics*. The first part of the book outlines a theory of learning mathematics as primarily the learning of conceptual structures. The second section illustrates his principles with reference to several topics, one of which is introductory algebra, up to solving equations. The Editorial Foreword to the book describes this second part as a textbook based on the psychological ideas developed in the first half. "It is a bold step and one which is calculated to give those who already know a little mathematics new insights into their own thinking and into the subject itself".

Skemp identifies algebra as beginning with the generalisation of the number schema in a new direction (1979, p. 211) and identifies the concept of variable as its root. He aims "to show how the number schema, when combined with the idea of variable, leads straight into algebra" (*ibid.*, p. 222). As was common in the new mathematics tradition, he takes the idea of a set as the basic concept in mathematics and aims to build the concept of variable from there. To bring the idea of variable from the intuitive to the reflective level, he provides a number of examples. In doing so, he is following his own two principles of concept learning, quoted above.

Skemp draws on everyday knowledge to give examples of the idea of variable. He notes how when we talk about "a car" we mean an unspecified element of the set of all motor cars, (*ibid.*, p. 213). He notes that talking about "a car" is useful:

(i) if it doesn't matter which car we are referring to (as in "I want to hire a car")

(ii) if we want to make statements that are true for all cars ("a car must be registered") or

(iii) when we do not know which car we are referring to, although we may wish to know (e.g. "a car was involved in the hit-and-run accident").

In this way, with several everyday examples (a car, a handkerchief, a pickpocket) and some geometric ones (a number, a circle etc.) Skemp builds up the mathematical concept of variable from its pre-mathematical precursors, illustrating the theory proposed earlier in the book. The variety of examples also illustrates his principle that it is important that the examples have nothing in common except the concept to be abstracted from them. When he does define a variable in words, it is as "an unspecified element of a given set". We find this definition very unsatisfactory – the meaning of unspecified is itself unspecified yet without this the definition cannot distinguish a variable from a value – but this only serves to illustrate the impossibility of defining root concepts such as variable in words. This is one of the general points that Skemp himself emphasised.

How does identifying algebra as springing from the number concept with the basic concept of variable added appear from a nineties point of view? Seeing the key concept of being a variable as its unspecified nature is clearly insightful. The analysis of how being unspecified is dealt with in everyday speech, and how it is applied for three different purposes, is very apt. It is sad that insights such as this are not very widely used in textbooks today. None of the explanations of variable we have seen in the textbooks widely used in our schools capture this essential characteristic.

In the new mathematics movement, there was a desire to reduce to a minimum the number of undefined terms like variable (which are basic to the subject despite being high up the conceptual hierarchy). Possibly for this reason, Skemp's analysis coalesces the ideas of variable and unknown. However, modern research is tending to separate these ideas once more. *Variable* is the label applied to the first two aspects in Skemp's list of being unspecified whereas *unknown* is applied to the third (when there is a definite car but the speaker does not know which one). In effect research into student's thinking has reinstated the idea of an unknown, which the new mathematics movement saw as unnecessary. Although clear distinctions cannot always be made, most researchers would now agree that there is a significant psychological distinction, apparent in the way in which students approach tasks, between situations which involve a fixed but unknown quantity and situations which involve a quantity which is inherently varying. Skemp puts these ideas together in his notion of variable – it has several different aspects – but learners may be better to meet these ideas separately.

Another difference between Skemp's introduction to the idea of variable and the more recent approach is related to the purpose of the activity. Again in accordance with the spirit of the new mathematics movement, Skemp's first illustration of the use of a variable in algebra is to record the commutative law: If $a$ and $b$ are any numbers, then $a + b = b + a$. The current terminology in Australia is "expressing generality" (AEC, 1991) but the generality which common teaching approaches to algebra in English speaking countries seek to express in the nineties are more concrete. For example, a common approach is to introduce the idea of variables by considering the relationship between the length of a row of squares and the number of unit line segments used to make it. Provided the generality obtained is used to answer some questions about the pattern (for example, how many line segments are needed to make a row of 100 squares) then this has a more obvious use for most young teenagers than Skemp's suggestion that algebra is used to record properties of numbers (the field laws for the real numbers). We have ourselves stressed the importance of deep understanding of the properties of numbers for understanding algebra and for it to be better recognised in today's curriculum (Stacey & MacGregor, 1997). However, showing students the purpose of algebra can be accomplished – in part – by showing them how it can be used to express general statements. It is important that the generalisations that are expressed seem worthwhile and interesting to the students. Situations that appeal to young adolescent learners are more out on the table today.

*Learners Bring More Knowledge to New Learning than was Thought*

One of the main insights arising from the study of students' thinking since Skemp's work is that learners bring more – ideas, concepts, habits, assumptions, misconceptions, knowledge – to the task of learning than was appreciated earlier. This is illustrated by looking at the way in which the next part of Skemp's chapter on elementary algebra tackles some issues of notation. He notes the use of the rules of order of operation in making algebraic expressions easier to read and write (for example $a+b\times c$ means $a+(b\times c)$, not $(a+b)\times c$ and that conjoining (placing symbols adjacent to each other) is used to indicate multiplication (i.e. that $b\times c$ is written as *bc*). In this discussion he notes the conflict of this notation with Hindu-Arabic numeration: if $a$=2 and $b$ = 7, then $a\times b$ can be written as 14, *ab*, *ba*, 2*b*, *a*7 or 7*a,* but not as 27 or 72.

In drawing attention to this conflict, Skemp is illustrating how prior concepts that students bring to a topic may not be helpful. However, this is not a strong theme in his work. Skemp began looking at mathematical content from a psychological perspective as well as a mathematical perspective, but thought mainly about mathematical prior concepts. Subsequent research has shown that as well as the conflict that some students will find with Hindu-Arabic notation, there are many other ways in which the prior experiences of students make standard algebraic notation seem a strange thing. Research that exposes students' thinking about how they perceive notation gives some practical guidance for teachers but in a broader sense it also helps them see the task of learning about algebraic notation from the point of view of a learner rather than that of an expert.

In our research (MacGregor & Stacey, 1997) we have found that students' thinking about algebraic notation is influenced by a great deal of prior knowledge, much of which is not recognised by teachers as being relevant. In particular, we have found students' difficulties in learning to use algebraic letters are influenced by:

(a) intuitive assumptions and sensible pragmatic reasoning about an unfamiliar symbol system;

(b) analogies with other symbol systems used in everyday life, in other parts of mathematics and in other school subjects;

(c) poorly-designed and misleading teaching materials, causing sometimes long-lasting errors (which will not be discussed here).

Large numbers of students from a representative sample of secondary schools across Melbourne, seemed to interpret and write algebra by making analogies with other symbol systems. The analogy with Hindu-Arabic notation that Skemp highlighted was present, although rare in our sample. A range of other analogies were found, some of which have also been identified in research in other countries. For example, some students associated the values of letters with their alphabetical position. There are various ways in which this analogy is used. In a problem asking for a height 10 cm greater than $h$ cm, students might give 18 cm or $r$ cm. This is because $h$ is the eighth letter of the alphabet and $r$ is eighteenth. Another example is giving the answer $x$ for one less than $y$. We found this analogy in beginners and also clustered in several schools at later year levels, which enabled us to trace it back to students' experience with codes and puzzles, often as part of mathematics. When we brought this misunderstanding to the attention of the teachers (who were unaware that it was common in their schools), it was easily corrected. This is an instance where, once teachers are alerted to the thinking of their students, they can quickly act to address misunderstanding, with good chances of success.

The association of letters with values deriving from the position in the alphabet is probably not just from codes and puzzles. In fact, it goes much deeper than this. Evidence is the historical use of letters for numbers by the Greeks (who used alpha as the symbol for one, beta, the second letter, as the symbol for two etc). The mapping of the alphabet onto counting numbers is used implicitly in the way in which we "number" questions in text books as 1(a), 1(b), 1(c) etc or as we "numbered" the points in the list two paragraphs ago. The point that we wish to make is that since Skemp's book was published, there has been a recognition that there are many sign systems, formal and informal, from which students may draw analogies with algebra. What children bring to the learning of mathematics from their previous learning and experiences matters a great deal, and has positive and negative effects. Recent research has shown how many aspects of their previous experience, not just in mathematics, affects new learning.

*Schemas and Misconceptions*

In seeing the major task of the teacher as assisting students to build conceptual structures, Skemp was concerned with the communication of mathematics, a theme that has recently been given increased attention internationally. In *The Psychology of Learning Mathematics* (1986, p. 76), he points out three reasons why explanations might be unsuccessful:

1) The wrong schema may be in use. … Failure of understanding could result from attaching a different meaning from that intended. In this case the explanation needed is simply to activate the appropriate schema.
2) The gap between the new idea and the existing schema may be too great. The explanation here would be to supply more intervening steps, thereby bridging the gap.
3) The existing schema may not be capable of assimilating the new idea without itself undergoing expansion or restructuring. In this case, an explanation is to help the subjects to reflect on their schemas, to detach them from their original sets of examples, which are now having a restrictive effect, and to modify them appropriately.

In this section, we will contrast two misconceptions about the use of letters in algebra, which are superficially similar, but on closer analysis seem to be very different. In one case, it seems that for most students the wrong schema is in use and another schema can be activated, as in the first alternative above. In the second case, the schema seems incapable of expansion, restructuring and sometimes of replacement.

As discussed in the previous section, Skemp pointed out how the algebraic notation of conjoining symbols (placing them alongside each other) for multiplication conflicts with place value notation in arithmetic. In doing this, he was illustrating one of the reasons why explanations might be unsuccessful. It is likely that in pointing out the conflict with Hindu-Arabic place value notation, Skemp is writing to guard against his explanation being unsuccessful for the first of the reasons above. He is concerned that some readers may use or interpret algebraic notation inappropriately because they confuse it with place value numeration, possibly without knowing. Placing letters with numerical values alongside each other and asking for the value of the combination is likely to activate the wrong schema: in this case a number schema. As commented above, our empirical research has shown that there are indeed a small number of children who are confused on this point, but this misinterpretation is quite rare and apparently not long lasting. This provides some support for the idea that simply telling (in Skemp's words, simply activating the appropriate schema) may be an effective way of eliminating this error.

Our study of students' thinking indicates other errors may behave quite differently. One instance which may look as simple as the previous two but is apparently deeper (MacGregor and Stacey, 1997) is the interpretation of a letter as standing for an object, rather than a number. This error has been well documented in studies around the world. In our study, we used a modified version of an item from the CSMS study (Küchemann, 1981) in Figure 1.

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A blue card costs 5 cents and a red card costs 6 cents. I buy some blue cards and some red cards. The total cost is 90 cents.
If $b$ stands for the number of blue cards and $r$ stands for the number of red cards, which if the following is correct?
[Tick your choice or choices]

(a) $b = 5, r = 6$
(b) $5b + 6r = 90$
(c) $b + r = 90$
(d) $b + r = 11$
(e) $6b + 10r = 90$
(f) $12b + 5r = 90$
(g) $b + r = \$0.90$
(h) none of these

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*Figure 1.* Item examining "letter as object" misconception.

The correct answer (b) $5b + 6r = 90$, apparently so simple, was given by only 7% of 188 Year 10 students from two secondary schools in our research project. The wrong answers were based on results of previous studies which indicated that students thought of $b$ standing for the blue cards, rather than the number of blue cards. For example,

(c) could be read as “the blue cards and the red cards together cost 90 cents” and (e) and (f) incorporate possible numbers of cards into the equations (i.e. solutions to the equation appearing in its formulation – a not uncommon phenomenon). Six blue cards and 10 red cards would cost 90 cents: $6b + 5r = 90$. The striking thing about the answers given by the students was not that so few were correct – this was expected – but that most students ticked quite a few of the wrong answers (e.g. (e), (f), (c) and (g)). They could make an imprecise sense from several of them and this was enough to tick them as correct.

As with the error patterns discussed above, there are many instances in everyday life which relate to this use of letters. People use their initials instead of their names. We use abbreviations when we write quick notes or in small spaces (MTWTF, J F M A M J J A S O N D) and for common or technical items (TV, CD, RAM, DNA). Many teachers stress that algebra is a shorthand for writing mathematical statements concisely, as indeed it is. Unfortunately, this may accidentally further strengthening the link between abbreviations and algebraic letters. Algebra is indeed a “shorthand” for writing mathematical statements. Before our notation was invented, words were used in rhetorical algebra. Great progress became possible as soon as the “shorthand” was developed. One of the obstacles for students learning to use algebra is to realise that it may be a shorthand, but it is far from a personal set of abbreviations. Algebraic notation has highly constrained meanings and has to be used in a very precise way – unlike other languages. Precise distinction of meaning, for example between the blue cards and the number of blue cards, is required for learning algebra.

As with the alphabetical analogies, our data has showed that misinterpretation of letters as standing for objects or labelling in a general way also clusters in particular schools. In those schools which we have been able to investigate further, we have found use of teaching materials which deliberately introduce this misconception – the so-called “fruit-salad algebra”. They introduce this misconception for short term gain to make several aspects of learning notation easy, because it avoids any need to think about the properties of number operations. Thinking of c as cat and d as dog, for example, makes it easy to add 5c and 3c to get 8c; easy to see that $5c + 2d$ might not have a simple answer (although students might suggest $7cd$); and easy to picture a bracketed expression such as $8\times(5c+2d)$ and calculate it as $40c+16d$. As an introduction to algebra this is certainly easy and even attractive. One year 7 girl told us that she liked algebra better than other parts of mathematics at school because "it is about animals and flowers and things and not numbers".

The evidence that we have been able to piece together about the aetiology of the "letter as object" error indicates that it behaves differently from the earlier two. Even though they similarly cluster in schools and seem to be sometimes actively taught as well as receiving indirect support from other experiences, this error seems much harder to cure. Straightforward attempts by the teachers to eliminate this seemed much less effective. (MacGregor & Stacey, 1977). Simply alerting students to the fact that they are using the wrong schema as was done with the alphabetical errors was not enough.

In this case, it is probable that the third of Skemp's reasons for failure of an explanation may be operating. These students have been allowed or encouraged to develop a schema of letters in algebra that does not link them tightly to numbers. It is then quite likely that this existing schema may not be capable of assimilating the new idea without itself undergoing expansion or restructuring, as Skemp suggests, or possibly dismantling, as we suggest. In this case, detaching the idea of letters from the original sets of examples, which are having a restrictive or misleading effect would be a major task.

The low incidence of the "letter-as-object" error at other schools indicates to us that it is not an inevitable naive misinterpretation. However, once it is established as a part of a student's schema, there is possibly sufficient reinforcement of the idea from analogy with other symbol and sign systems and reinforcement from the fact that it sometimes works satisfactorily, for the incorrect schema to be very deeply established. Through these studies of student thinking, it has become clear to us that teachers and students very often use algebraic symbols in ways which looks the same on the surface, but which have very different meanings. Uncovering some of these meanings in order that teachers can be better equipped to identify instances of mismatched meaning is our contribution to getting the thinking out on the table.

## Conclusion

In this chapter, it has only been possible to pick up several small ways in which ideas expounded by Skemp have developed from the original. Going back to look at how Skemp had analysed elementary algebra has been an interesting case study, offering instances where perspectives have changed significantly and instances where there has been a steady accumulation of data filling out the original ideas. Sometimes, in education, new insights into learning are not actually new, but come in and out of the collective consciousness.

We began by recalling how Skemp had valued insights into students' thinking and had therefore advocated teaching methods that exposed them to teachers. The emphasis in the chapter has not been on exposing the thinking of individual children as they work, which Skemp identified as – and we agree is – one of the major tasks for a teacher. Instead we see the contribution of research as providing an understanding of children's ideas, knowledge which teachers can use to help them quickly identify how a task is perceived from a child's point of view.

Skemp expounded the need to analyse the content of mathematics from a psychological as well as from a mathematical point of view and he made great strides in doing this. From today's point of view, however, his analysis seems to be predominantly mathematical. This is evidence of how far work has been carried out in his quest to put the thinking out on the table.

## References


Australian Education Council (1991). *A National Statement on Mathematics for Australian Schools.* Melbourne: publisher as author.

Campbell-Jones, S. (Producer) (1986). Twice five plus the wings of a bird. (Video-recording). London: BBC Enterprises.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C., & Loef, M. (1989). Using children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal*, *26*, 499–531.

Geiger, V., & Goos, M. (1996) Number Plugging or Problem Solving? Using technology to support collaborative learning. In P. Clarkson (Ed.) *Technology in Mathematics Education*. (*Proceedings of the Nineteenth Annual Conference of MERGA*,) pp. 229–236. Melbourne: MERGA.

Good, T. L., Mulryan, C., & McCaslin, M. (1992). Grouping for Instruction in Mathematics: A call for programmatic research on small group processes. In D. Grouws (Ed.) *Handbook of Research on Mathematics Teaching and Learning.* (pp. 165 – 196). New York: MacMillan.

Küchemann, D. E. (1981). Algebra. In K. Hart (Ed.) *Children's Understanding of Mathematics 11 – 16* (pp. 102–119). London: Murray

MacGregor, M., & Stacey, K. (1997). Students' Understanding of Algebraic Notation: 11–15. *Educational Studies in Mathematics*, 33, 1–19.

Skemp, R. R. (1986). *The Psychology of Learning Mathematics* (Second Edition). Harmondsworth: Penguin.

Stacey, K., & MacGregor, M. (1997). Building Foundations for Algebra. Mathematics *Teaching in the Middle School*. 2 (4), 252–260.

# Teaching and Learning Mathematics by Abstraction

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We learn a lot about teaching and learning mathematics from student comments. For example, one of us was attempting to explore the principles underlying differential calculus with a first year university calculus class when a number of students complained, "You're not explaining it clearly enough," and "I used to understand it properly, but now I don't," (White, 1991). In his famous 1976 paper, Skemp ascribes this reaction to a mis-match between the teacher's desire for relational understanding and the student's preference for instrumental understanding:

> The pupils just "won't want to know" all the careful ground-work given for whatever is to be learnt next. ... All they want is some kind of rule for getting the answer. As soon as this is reached, they latch on to it and ignore the rest. ... By many, probably a majority, attempts to convince them that being able to use the rule is not enough will not be well received. "Well is the enemy of better", and if pupils can get the right answers by the kind of thinking they are used to, they will not take kindly to suggestions that they should try for something beyond this. (pp. 21–22 in the original paper).

In the calculus class above, we discussed Skemp's ideas. A more peaceful journey ensued, with students writing comments on their assigned tasks such as "please be kind, I'm an instrumental." We were fortunate enough to be able to discuss this incident with Richard Skemp in 1992, when he was guest speaker at the annual conference of the Mathematics Education Research Group of Australasia. His reply was that he was delighted that his influence had been "instrumental" in forming better "relationships" between teacher and students.

This story can be taken to highlight the difficulties both students and teachers face when confronting abstract concepts. Given that abstract concepts occur throughout mathematical development, from the earliest encounters with numbers to advanced topics such as calculus, it is vital to our understanding of the learning and teaching of mathematics that we study the process of abstraction. In this area, Richard Skemp has made a major contribution, which has been built upon with gratitude by many researchers—including ourselves.

*Abstraction and Procepts*

Skemp (1986) provided a definitive treatment of how concepts are formed through an abstraction process. He defined abstracting as "an activity by which we become aware of similarities ... among our experiences" and a concept as "some kind of lasting change, the result of abstracting, which enables us to recognise new experiences as having the similarities of an already formed class" (p. 21). This definition emphasises three important characteristics of concept formation by abstraction:

- its function is to allow more efficient classification of experience;
- it depends on recognising similarities between different contexts; but
- it is qualitatively different from simply identifying patterns in a set of examples.

We shall use the term *base contexts* to denote the contexts from which a concept is abstracted. Abstraction may be seen as a many-to-one function where generalisations about the base contexts are synthesised to form a new abstraction. Dreyfus (1991) summarises the process as a sequence of generalising → synthesising → abstracting.

Several authors have applied abstraction theory to the development of mathematical concepts from mathematical operations. Sfard (1991) describes a three-phase model similar to the sequence of Dreyfus: interiorisation → condensation → reification. The interiorisation phase occurs when some process is performed on familiar mathematical objects. The condensation phase occurs when the process is condensed into more manageable units. Both of these phases are said to be operational because they are process-orientated. Reification is the leap from an operational mode to a structural mode where a process becomes an object in its own right. Dubinsky (1991) also describes this leap from a dynamic process to a static object as a form of reflective abstraction; he calls the leap encapsulation.

Gray and Tall (1994) take the process/object transition further by observing that in mathematics the same symbols are commonly used to represent both process and object. For example, $2+3x$ can be considered as the process of adding 2 to the product of 3 and $x$ and also as the object which is the result of the process. Gray and Tall define the amalgam of the process, the resultant object, and the symbol used to represent both, as an *elementary procept*. They hypothesise that successful mathematical thinkers can think proceptually, that is, they can comfortably deal with symbols as either process or object. An operational orientation would thus interpret $2(a+b)$ and $2a+2b$ quite differently, whereas proceptually the two expressions would be seen as identical.

The proceptual approach to mathematical abstraction flows naturally from Skemp's work and has strong support from subsequent empirical research. However, it must be noted that not all mathematical abstractions are procepts—geometry provides some obvious examples. The concept "triangle", for example, embodies the similarities of closed figure, plane figure, and three straight sides. The abstraction "triangle" depends on recognising these similarities among a range of otherwise different objects, not on proceptualising a triangle process into a triangle object.

### *Abstraction and Connectedness*

Skemp distinguished between primary and higher-order concepts, explaining that higher-order concepts are abstractions of earlier abstractions and so progressively "removed from experience of the outside world", (1986, p. 24). He gives the example of "red", which is a primary concept because it is formed from sensory experience, and "colour", which is a secondary concept abstracted from red, blue, and so on. We note that the higher-order concept encompasses wider experience than does each primary concept.

Once an abstraction has occurred, the generalising $\rightarrow$ synthesising $\rightarrow$ abstracting sequence can be repeated. The sequence certainly occurs in concept formation at all levels in mathematics (as well as in most other areas of life), but it is a feature of advanced concepts that they are based on several repetitions. Each repetition has led to a higher level of abstraction and a further removal from primary concepts. Successive abstraction leads to "abstract" being increasingly seen to mean "removed from reality" or "context free". Nevertheless, the higher-order concepts are connected to all the lower-level concepts in a complex hierarchy of abstractions.

By contrast, many students believe that mathematics consists of a large set of rules with little or no connection to each other and hardly any relevance to their everyday lives (Graeber, 1993; Schoenfeld, 1989). The following is typical of how our student teachers describe their school mathematics:

> When given a situation . . . that relates to a real-life situation, we tended to get confused and have difficulty in responding. We were only happy when given an abstract problem like those given in the school textbooks.

The student seems to be using the term “abstract problem” to mean an exercise in symbolic form which does not refer to any concrete context. Many educators also seem to be referring to disconnectedness when they use the term “abstract”. For example, Nickson (1992, p. 103) calls the heavy emphasis on computational skills in US elementary schools a concentration on “the abstract manipulation of number”, and Brown, Collins, and Duguid (1989, p. 32) characterise most school teaching as “the transfer of ... abstract, decontextualized formal concepts”.

Practitioners clearly intend that abstract ideas should be applied to contextualised problems. Word problems appear consistently at the end of the chapter in both primary and secondary textbooks. Syllabuses are often explicit about the issue. The New South Wales *Mathematics in Society* course for school years 11 and 12 (introduced in 1982 and still current at the time of writing) states that “the mathematics of the core would receive general treatment and then be applied to a variety of situations” (Board of Senior School Studies, 1980, p. 9). The worldwide New Mathematics movement of the 1960s was also predicated on the assumption that the learning of abstract structures would allow their correct usage later (Moon, 1986).

Mathematics teachers also seem to accept the abstract → concrete order of learning. It has been reported that many teachers believe that procedures should be learnt before conceptual understanding (Steinbring, 1989) and that they teach in a corresponding manner (Porter, 1989). Another student teacher’s comment encapsulates this belief. She had been exploring 9-year-old Andrew’s understanding of area, and found that he could calculate the area of a rectangle using the formula but could not explain what the result meant. She wrote, "Andrew understands area in the abstract but cannot yet apply it in practice."

The problem is that applications are often not reached or are dismissed as too difficult. Our student teachers frequently ask, "How can we teach mathematics through applications when we have never learnt it that way?" The separation of mathematics from reality would seem to be a historical fact.

*The Two Faces of Abstraction*

There thus appear to be two usages of the term "abstract ". Much curriculum and instruction treats mathematical concepts in isolation from their base contexts, whereas Skemp had stressed the essential dependence of abstract concepts on these contexts. To understand the distinction, we sought help from the Webster Dictionary (1977):

> *Abstract* (adj): Apart from the concrete; general as opposed to particular; expressed without reference to particular examples (e. g. numbers).
> *Abstract idea*: Mental representation or concept that isolates and generalises an aspect of an object or group of objects from which relationships may be perceived.
> *Abstract* (verb): To consider apart from particular instances; to form a general notion of.

Two key words appear in these definitions: apart and general. We have therefore found it useful to view abstract mathematical concepts in two ways:

(1) Concepts formed in the Skempian manner we call *abstract-general*. There are degrees of generality, depending on the range of contexts which are recognised as similar. Many concepts start with a restricted range of base contexts and gradually become more general as further contexts are recognised as similar.

(2) Concepts formed in isolation from the contexts in which they occur we call *abstract-apart*. Such concepts are formed by definition rather than abstraction, and, because "concepts of higher order … cannot be communicated by a definition" (Skemp, 1986, p. 30), are no more or less abstract than other concepts.

Figure 1, inspired by Skemp (1986, p. 20), is intended to illustrate the difference between abstract-general and abstract-apart ideas. The word "idea" here is to be interpreted in the broadest sense; it could refer to any mathematical object such as a concept, an operation or a relation. The base contexts $C_i$ could involve concrete objects or mathematical ideas. The arrows depicting an abstract-general idea indicate that two-way links exist between the idea and the $C_i$. These links enable the learner both to recognise the idea in each $C_i$ and to call up a variety of contexts in which the abstract idea is found. In other words, the idea is general to all the $C_i$. The absence of the arrows in the case of an abstract-apart idea indicates that the mathematical idea is separated and apart from any of the possible base contexts.

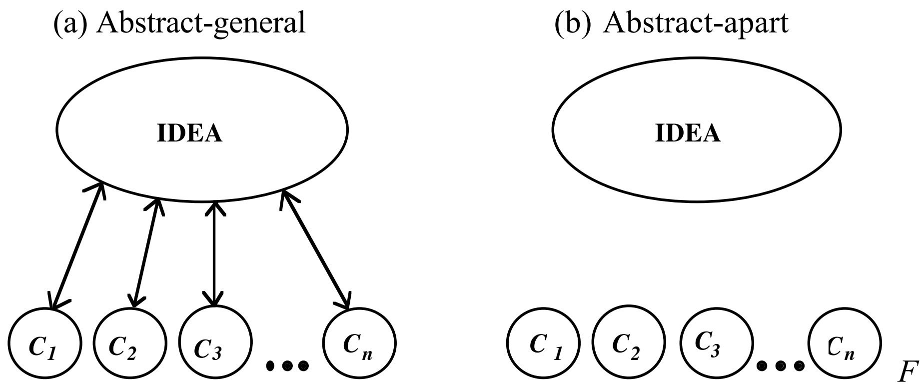


*Figure 1.* The difference between an abstract-general and an abstract-apart idea.

The arrows in Figure 1(a) also have another interpretation. The upward arrows can be taken to represent the process of abstracting a common property from the base contexts as a result of the recognition of a similarity and the formation of a new object at a higher level of abstraction. These arrows therefore define the meaning of the idea. The downward arrows in Figure 1(a) can be taken to represent the process of relating any reference to the idea (such as relationships to other ideas) back to the contexts $C_i$. These links serve two purposes. Firstly, by allowing the learner to interpret new and unfamiliar relations at the higher level of abstraction in terms of the more familiar base contexts $C_i$, they allow the learner to make sense of the more abstract relations. Secondly, they allow procedures developed for manipulating ideas at the higher level to be applied to the solution of problems in the base contexts.

In the case of an abstract-apart idea, there are no links to any base contexts and therefore no meaning in terms of those contexts. An abstract-apart concept cannot be applied to any of the possible base contexts, but only to situations (usually entirely symbolic) which look superficially similar to the form in which it was originally learnt. There is a sense in which an abstract-apart idea is meaningless and useless. The only possible meaning or use for the idea is in terms of its links to other ideas at the same level of abstraction.

The two meanings of "abstract" rest comfortably with Skemp's descriptions of instrumental and relational understanding and the resulting mismatches identified in the introduction to this chapter. Students who have formed abstract-apart concepts would clearly favour an instrumental approach to mathematics and be uncomfortable when ideas at different levels of abstraction are introduced. However, our approach extends Skemp's original description by suggesting that there might be degrees of relational thinking. For example, consider a student who can add fractions and can explain the procedure using equivalent fractions. This is certainly relational—one procedure (adding fractions) is related to another (finding equivalent fractions). However, it is a relation within one level of abstraction, and therefore only draws on abstract-apart ideas. By contrast, a student with an abstract-general concept of fractions can draw on the meaning of fractions, their addition and their equivalence to provide a concrete explanation of the addition procedure. By being able to make connections across levels of abstraction, the second student not only has a deeper understanding but is in a better position to cope with non-routine problems.

*Teaching for Abstraction*

We argued above that the common approach to mathematics curriculum and instruction follows the abstract → concrete sequence. It starts with definitions, may or may not reach applications, and tends to lead to disconnected, abstract-apart concepts. What approach to teaching would be more likely to lead to abstract-general concept formation? We suggest that it is necessary to follow more closely the generalising → synthesising → abstracting process, observing the aphorism "concrete before abstract". In this approach teaching would be deliberately directed at the recognition and abstraction of similarities, and definitions would only appear as a means of making these similarities explicit. We call such an approach *teaching for abstraction*. It may be seen as an extension of the teaching methods advocated by Dienes (1963).

For example, a common approach to teaching the identity $a^m \div a^n = a^{m-n}$ is to analyse the abstract form $\frac{a \times a \times \ldots \times a \ (m \text{ times})}{a \times a \times \ldots \times a \ (n \text{ times})}$. This often leads students to happily say that $m^5 \div \mathrm{m}^3 = \mathrm{m}^2$ but also to believe that $6^5 \div 6^3 = 1^2$. One interpretation of this misconception is that students do not link the abstract relation to a pattern in the base contexts (numbers) which it is supposed to summarise. An alternative teaching method is to begin with the numerical examples, to allow students to "encounter a suitable collection of examples" (Skemp, 1986, p. 32), and to help them recognise the pattern relating the indices and the bases. The identity $a^m \div a^n = a^{m-n}$ would be used to formulate and clarify the pattern. Abstract manipulations such as $m^5 \div m^3 = m^2$ should be constantly linked back to the base contexts—for example, by requiring students to give a numerical example for each simplification. Students would gradually move from a process orientation to $a^m$ as the multiplication of a number by itself $m$ times to a structural orientation in which $a^m$ becomes a mathematical object which can be operated on.

We shall describe further examples of teaching for abstraction in the next section.

## Abstraction as a Research Framework

Skempian abstraction theory can provide a valuable framework for interpreting and designing research on mathematics learning and teaching. We have applied it to the interpretation of the literature on multiplication and fractions (Mitchelmore, 1994) and the use of letters in algebra (Mitchelmore & White, 1995). More importantly, it has been the foundation of our own research program: Our research on angle focuses on young children's recognition of similarities across superficially different angle contexts, and suggests a three-stage abstraction hierarchy. The focus of our work on rate of change is tertiary students' ability to link rate of change contexts to symbolic calculus, and involves the procepts of variables and derivatives. In both areas, we have confirmed the wide-spread existence of abstract-apart concepts and designed methods of teaching for abstraction. The following sections describe some of our results in these two areas in more detail.

*Angles*

The angle concept is fundamental to the whole of high school geometry, but several large-scale surveys in the USA (Usiskin, 1987; Kouba et al., 1988) and the UK (Foxman & Ruddock, 1984) have demonstrated poor understanding of angles topics among 11–15 year olds. Close's claim, as a secondary teacher, that "the large number of children possessing misconceptions about angle is immediately apparent when the topic is discussed in class" (1982, p. 2) has been confirmed in interview studies by herself and others (eg., Krainer, 1989; Davey & Pegg, 1991). For example, it has been found that 30-60% of secondary students judge angle size on the basis of such features as the length and orientation of the arms and the radius of the arc marking the angle. It would appear that children are not leaving primary school with an angle concept which is adequate for starting high school geometry.

We have identified the definition of angle commonly used in primary school as a possible source of this problem. From a Skempian abstraction point of view, a definition of any given concept should be a succinct description of the predominant similarities in all the different situations from which the concept is abstracted. In New South Wales, the curriculum defines angle as "the amount of turning between two lines about a common point" (New South Wales Department of Education, 1989, p.79). This definition derives from the argument that the development of the angle concept consists of establishing a particular relationship between the two arms of the angle; and a turn is a familiar, physical action which relates the two arms (Wilson & Adams, 1992). However, consider a sloping hillside. To interpret this situation in terms of angles, a student must visualise a turning movement between the horizontal and a line giving the slope of the hill. Given that a hill does not involve physical turning and the horizontal line is not visible, this may not be an easy task. Much research (Clements & Battista, 1989; Close, 1982; Scally, 1987) confirms that children have great difficulty coordinating different angle contexts. As a result, we claim, students tend to form an abstract-apart concept of angle which is restricted to angle diagrams—a concept which is inflexible and inapplicable.

Our research has taken a radical Skempian approach. We regard an abstract-general understanding of the angle diagram to be the end-product of a long-term abstraction process by which children interpret and make sense of their angle-related physical experiences. We have therefore investigated the long-term path of development—all the way from early childhood experiences through to a mature, mathematically viable angle concept—looking specifically for factors which help or hinder the recognition of similarities between different angle contexts and therefore the recognition of abstract angles in those contexts. We believe we have found a viable model, consisting of three overlapping stages at increasing levels of abstraction, for describing the development of an abstract-general angle concept (Mitchelmore, 1997; Mitchelmore & White, 1997a). Briefly, the three stages are as follows.

(1) Well before coming to school, children classify physical angle experiences which are superficially similar into *physical angle situations* and abstract from each situation a primary concept which we call a *situated angle concept*. Three examples are hills, roofs and cranes. Initially, children see no connection between different angle situations.

(2) At some stage, children recognise geometrical similarities between physical angle situations and classify them into *physical angle contexts*. For example, they might see that hills, roofs and crane arms all slope and then gradually extend the idea to include other examples of sloping objects. From each angle context, children abstract a secondary concept which we call a *contextual angle concept*. Some examples of contextual angle concepts are slopes, turns and corners.

(3) Later, children become aware of deeper similarities between physical angle contexts. For example, by constructing a fixed (probably horizontal) reference line for the slopes and focussing on the edges of the corners, they may see that slopes and corners are similar. We call such a class of similar angle contexts an *abstract angle domain* and the corresponding concept an *abstract angle concept*. Students appear to form several abstract angle domains (a point, a single sloping line, and the standard angle concept consisting of two lines through a point), some of which only apply to a restricted set of contexts. The mature, standard angle concept includes all angle contexts in a single domain.

Our research on angle conceptual understanding among schoolchildren (Mitchelmore, 1997; Mitchelmore & White, 1997a, 1998; White & Mitchelmore, 1997) has confirmed the general outline of the above theory. It has also shown conclusively that—except in obvious turning situations—children rarely recognise the dynamic similarity (turning) implied by the generally accepted definition of an angle as "an amount of turning between two lines meeting at a point ". Students more easily recognise the static similarity (two lines meeting at a point), the facility depending on the salience of the two lines which form the angle in each context. Over 90% of Grade 2 children in our samples readily recognised the angular similarity between situations such as corners, junctions and scissors, but it was not until Grade 6 that a similar percentage included hand fans and leaning signposts in the same category. Students found it even more difficult to incorporate situations where one or both lines have to be imagined (hills, doors and wheels); even at Grade 8, about one-third of students could not identify the two arms of the angle implied in these situations. The gradual expansion of children's angle concepts to include more and more contexts similar to an initial "core " may be summarised in the following diagram, derived from facet theory (Levy, 1985):

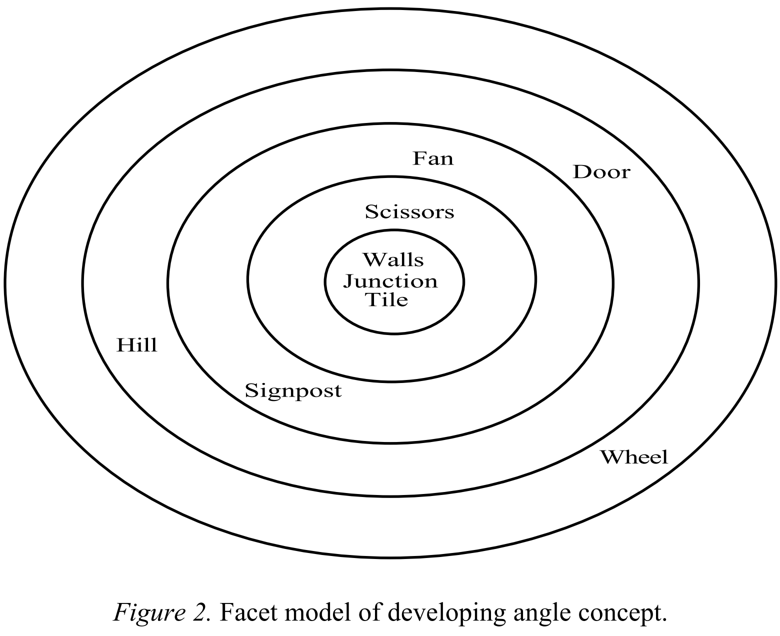


*Figure 2.* Facet model of developing angle concept.

“Teaching for abstraction “ in the case of angles means helping children to recognise the similarities between different angle contexts and then reifying the similarities. Our findings suggest that a focus on identifying the two arms of an angle would allow similarity recognition across a wider range of angle contexts than a focus on visualising turning, and is therefore likely to be more effective. The result is a teaching model which might be summarised as follows:

$$\text{identification of two lines} \rightarrow \text{similarity recognition}$$
$$\rightarrow \text{angle concept by abstraction.}$$

Working outwards from the core in Figure 2 suggests the following approach.

(1) Develop an agreed language (both graphic and verbal) for describing and comparing various *corners*—situations, like walls, junctions and tiles, where both arms of the angle are fixed and clearly visible.

(2) Do the same for what we call *V-hinges*—situations, like scissors and fans, consisting of two arms pivoting about their point of intersection.

(3) Explore the similarity between corners & V-hinges, and develop an abstract angle language which applies to both contexts.

(4) Explore what we call *I-hinges*—situations, like doors and lids, where a single arm rotates about an end-point—and link them to V-hinges and hence to corners.

(5) Explore various *slope* situations and link them to I-hinges and hence to V-hinges and corners.

(6) Explore various *turn* situations and link them to I-hinges, V-hinges and corners.

(7) Extend in a similar way to other angle contexts such as bends, directions and rebounds. Introduce angle measurement.

Because we wanted to assess current understanding, our research to date has avoided any attempt to teach children about angles (even when, for example, it seemed that a child might well have recognised a similarity if it had been pointed out). A large-scale experiment to test the suggested method of teaching for abstraction is currently at the planning stage.

*Rates of Change and Derivative*

Earlier [page 3] we quoted a student comment supporting what every teacher knows: "applied" problems are more difficult than routine symbolic exercises. Further support comes from a study of calculus understanding among first year university students (White & Mitchelmore, 1996). Test items were administered in four versions which varied systematically from a typical contextual problem to a symbolic form which involved exactly the same procedural manipulation. The average success rate on the items rose from 15% for the most contextual to nearly 60% for the most symbolic form. Our explanation for this difference relied on several ideas from Skempian abstraction theory, in particular the ideas of abstract-apart and abstract-general concepts.

Poor performance on contextual items was often attributable to abstract-apart thinking. For example, one item was as follows:

> If the edge of a contracting cube is decreasing at a rate of 2 centimetres per minute, at what rate is the volume contracting when the volume of the cube is 64 cubic centimetres?

The solution basically consists of the following six steps.

1. $\dfrac{dx}{dt} = -2$, 2. $V = x^3$, 3. $\dfrac{dV}{dt} = \dfrac{dV}{dx} \cdot \dfrac{dx}{dt}$,

4. $\dfrac{dV}{dx} = 3x^2$ 5, $V = 64$, so $x = 4$, 6. $\dfrac{dV}{dt} = 3 \cdot (4)^2 \cdot (-2) = -96$.

Many students did not see any connection between derivative and instantaneous rate of change and fell back on to some notion of average rate. Common answers were −8 and $\frac{1}{8}$, where students had apparently tried to find the average decrease in one second. Such responses suggest that students' concepts of rate of change had been abstracted from a severely restricted set of contexts and was not general enough to cope with variable rates of change.

Other responses show how an abstract-apart concept of variable can thwart attempts to solve rate of change items even when some degree of abstract-general notion of rate of change concept is evident.

For example, in the above item, some students who realised $\frac{dV}{dt}$ was required wrote "$V = x^3$, therefore $V' = 3x^2$". No thought of the appropriate independent variable was evident. Others who correctly interpreted the item and even correctly wrote $\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt} = -6x^2$ could not finish because they were confused by $V = 64$. The confusion arose because the students saw no connection between $V = x^3$ and $V = 64$. Some students even gave two answers: $-6x^2$ and 0. Each equation was seen apart from the other, so that there was no general view of 64 being a specific value of a varying $x^3$. This is a clear example of an abstract-apart concept of variable.

The confusion between $V = 64$ and $V = x^3$ by students who correctly stated the chain rule suggests that the chain rule can also be an abstract-apart concept. In fact, several responses showed clearly that students based their decisions on which derivatives to use in the chain rule on which variables were visible and could be "cancelled out" to get the desired $\frac{dV}{dt}$. Apparently, no meaning was attached to the symbols.

There were many other responses which indicated a focus on visible symbols. Several occurred in response to the following item.

> In the special theory of relativity, the mass of a particle moving at velocity $v$ is given by $m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$ where $c$ is the speed of light and $m_0$ is the rest mass.
>
> Find the rate of change of the mass when $v = \frac{1}{2}c$ and the acceleration is $0.01c$ per sec.

The solution involves first finding $\frac{dm}{dv}$, then substituting $v = \frac{1}{2}c$ for $v$, and finally obtaining $\frac{dm}{dt}$ using the chain rule. However, a large number of students first substituted $\frac{1}{2}c$ for $v$ and were surprised when $\frac{dm}{dt}$ turned out to be zero. Others interpreted the item as requiring $\frac{dm}{dv}$ or $\frac{dm}{dc}$ because they "could not see any $t$'s". Their decisions about how to proceed were based on the visible letters which were available as input to manipulation rules—regardless of whether such rules were appropriate. Such a reliance on visible cues seems to be typical of abstract-apart thinking. The concepts used are superficial, with no connection to any base contexts. In such circumstances, the only connections that can be made are those suggested by the appearance of the symbols.

An abstract-apart concept—often the result of learning superficial rules with no connection to the appropriate underlying contexts—can be virtually useless. Responses to the following item illustrate this point:

> Find the area of the largest rectangle with its lower base on the $x$-axis and upper vertices on the curve $y = 12 - x^2$.

Instead of using an expression for the area of the rectangle based on the $x$ and $y$ coordinates of a general point, many students inappropriately solved $\frac{dy}{dx} = 0$. When asked why, several claimed to be simply "following the rule". Their view of the calculus procedure for finding maxima was apparently restricted to contexts in which the variables were called $x$ and $y$. These students appeared to have no general concept of a function having a zero rate of change at its maximum value.

We also found that abstract-apart concepts could be adequate for solving simple contextual problems but not more complex ones. For example, all the items required as their first step the symbolisation of appropriate derivatives. In the relativity item, only 8 out of 40 responses correctly symbolised $\frac{dm}{dt}$ and $\frac{dv}{dt}$ and 7 of these gave a fully correct response. However, in the shrinking cube item, 36 correctly symbolised $\frac{dV}{dt}$ and $\frac{dx}{dt}$ but only 22 gave fully correct responses. The reason would seem to be the differing structural complexity of the two items. In the shrinking cube item, it is easy to identify the appropriate independent and dependent variables; there is only one obvious derivative to choose. Correct answers can therefore be obtained by manipulating the symbols without regard to their meaning—that is, using abstract-apart thinking. In the relativity item, on the other hand, several letters must be given meaning in order to identify the independent and dependent variables and to distinguish between constants, variables, and specific values of variables. Correctly symbolising derivatives in such complex situations requires an abstract-general concept of a variable.

A further example of the power of an abstract-general concept is given by some data on responses to the parabola item: Only 8 out of 40 students correctly symbolised the appropriate derivatives but all 8 went on to give a fully correct response. The main difficulty in this item is translating the given information into a form in which differentiation can be applied, and the crucial first step is to identify the area as the critical variable and assign it a letter. This skill is clearly only available with an abstract-general concept of variable. Other evidence suggests that students who can create their own variables in this way have in fact reified the concept of variable.

It is noticeable from the results for the relativity and parabola items that almost all those students with a sufficiently abstract-general concept of variable to translate the given problem correctly into derivatives also completed the manipulations without error. Although the numbers are small, the implication is clear. We have already argued that abstract-apart concepts are adequate for decontextualised problems but useless for contextual problems; now it would appear that abstract-general concepts are adequate for both contextual and decontextualised problems. In other words: If you understand the concepts, the procedures will look after themselves.

Can introductory calculus be taught by the principles of "teaching for abstraction"? If genuine abstraction is to occur, rates of change need to be investigated in different contexts at the beginning of a course—not at the end as applications, as so often happens. The derivative then becomes an embodiment of the similarities between all rate of change situations (White, 1993). A high school introduction to calculus which focuses on symbolic definitions and manipulations results in an abstract-apart concept of derivative, and students have no sense as to what calculus is about. Moreover, it does not seem possible to remedy this situation by enlightened teaching later. For example, the 40 students involved in the study reported above subsequently studied a 24-hour course intended to make the concept of rate of change more meaningful—but almost the only detectable result was an increase in the number of errors in symbolising a derivative. The most serious block appeared to be students' abstract-apart concepts of variable. One could conclude that an abstract-general concept of variable at or near the point of reification is a prerequisite for studying calculus meaningfully. "Teaching for abstraction" may need to be directed at variables before it can be directed at rate of change (White & Mitchelmore, 1993).

## Conclusion

The discussion presented has shown that the notions of abstract-apart and abstract-general ideas, derived from Skemp's abstraction theory, have considerable potential for clarifying the interpretation and planning of research and practice in mathematics teaching and learning.

We have shown that there is a great deal of misunderstanding about the nature of abstraction in mathematics. Mathematical ideas are not abstract because they are not concrete; they are abstract because they are general. Also, mathematical ideas are not difficult because they are abstract; they are difficult because they are often taught as abstract-apart when their understanding requires abstract-general thinking. We have identified aspects of traditional mathematics teaching which contribute to the development of abstract-apart concepts, and proposed teaching strategies ( "teaching for abstraction") which would be more likely to lead to abstract-general concepts.

We believe that the abstract-apart/abstract-general distinction is more crucial than the distinction variously referred to as procedural/ conceptual or skills/understanding. Procedures and skills are not separate from concepts and understanding, so it is not productive to differentiate them. The fundamental difference is between abstract-general procedures and skills, which are based on and almost indistinguishable from conceptual understanding, and abstract-apart procedures, which are purely superficial manipulations. The former are the stuff of which mathematics is made; the latter are mindless and useless.

Teaching for abstraction in mathematics would look very different from traditional mathematics teaching, completely reversing the standard abstract $\rightarrow$ concrete order. Unfortunately, although some articles on teaching for abstraction are appearing in professional journals (e.g., Mitchelmore & White, 1997b; White, 1993; White & Mitchelmore, 1993) and at least one textbook has been written to incorporate teaching for abstraction (Sattler & White, 1997, 1998), there is as yet no firm research evidence to support its efficacy. That is the next great challenge.

## References


Board of Senior School Studies (1980). *Mathematics Syllabus and Notes to the Syllabus: 2 unit Mathematics in Society course.* Sydney: New South Wales Department of Education.

Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher, 18*(1), 32-42.

Close, G. S. (1982). *Children's Understanding of Angle at the Primary/Secondary Stage*. London: Polytechnic of the South Bank.

Davey, G., & Pegg, J. (1989). Relating descriptions of common 2-D shapes to underlying geometric concepts. Paper read at the 12th Annual Conference of the Mathematics Education Research Group of Australasia, Bathurst.

Dienes, Z. P. (1963). *An Experimental Study of Mathematics-Learning.* London: Hutchinson.

Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 25–41). Dordrecht: Kluwer.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 95–123). Dordrecht: Kluwer.

Foxman, D., & Ruddock, G. (1984). Concepts and skills: Line symmetry and angle. *Mathematics in Schools,* 13 (2), 9–13.

Gray, E., & Tall, D. (1994). Duality, ambiguity, and flexibility: a "proceptual " view of simple arithmetic. *Journal for Research in Mathematics Education*, 25, 116–140.

Graeber, A. (1993). Mathematics and the reality of the student. In R. Davis & C. Maher (Eds.), *School Mathematics and the World of Reality.* New York: Allyn & Bacon.

Kouba, V., Brown, C. A., Carpenter, T. P., Lindquist, M. M., Silver, E. A., & Swafford, J. O. (1988). Results of the fourth NAEP assessment of mathematics: Measurement, geometry, data interpretation, attitudes and other topics. *Arithmetic Teacher,* 35 (9), 10–16.

Levy, S. (1985). Lawful roles of facets in social theories. In D. Canter (Ed.), *Facet Theory* (pp. 60–96). New York: Springer.

Mitchelmore, M. C. (1994). Abstraction, generalisation and conceptual change in mathematics. *Hiroshima Journal of Education, 2*, 45–56.

Mitchelmore, M. C. (1997). Children's informal knowledge of physical angle situations. *Learning and Instruction*, *7,* 1–19.

Mitchelmore, M. C. (1997). Young students' concepts of turning and angle. *Cognition and Instruction.*

Mitchelmore, M. C., & White P. (1995). Abstraction in mathematics: Conflict, resolution and application. *Mathematics Education Research Journal*, 7, 50–68.

Mitchelmore, M. C., & White, P. (1997a). Development of angle concepts: A framework for research. Manuscript submitted for publication.

Mitchelmore, M. C., & White, P. (1997b). Sharpening up on angles. *Reflections*, 22 (1), 28–30.

Mitchelmore, M. C., & White, P. (1998). Recognition of angular similarities between familiar physical situations. Paper submitted for presentation at the *22nd Annual International Conference for the Psychology of Mathematics Education*, Stellenbosch, South Africa, July 1998.

Moon, B. (1986). *The 'New Maths' Curriculum Controversy.* Barcombe: Falmer.

NSW Department of School Education (1989). *Mathematics K-6*. Sydney: Author.

Nickson, M. (1992). The culture of the mathematics classroom: an unknown quantity? In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 101–114). New York: Macmillan.

Porter, A. (1989). Curriculum out of balance: The case of elementary school mathematics. *Educational Researcher*, 18 (5), 9–15.

Sattler, J., & White, P. (1997). *Inside Mathematics: Year 9 Advanced.* Melbourne: Longman .

Sattler, J., & White, P. (1998). *Inside Mathematics: Year 10 Advanced.* Melbourne: Longman .

Scally, S. P. (1987). The effects of learning LOGO on ninth grade students' understanding of geometric relations. In J. Bergeron, N. Herscovics, & C. Kieren (Eds.), *Proceedings of the 11th International Conference for the Psychology of Mathematics Education.* Montreal, Canada: Program Committee.

Schoenfeld, A. (1989). Explorations of students' mathematical beliefs and behavior. *Journal for Research in Mathematics Education,* 20, 338–355.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22, 1–36.

Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching*, 77, 20–26.

Skemp, R. (1986). *The Psychology of Learning Mathematics* (2nd ed.). Harmondsworth: Penguin.

Steinbring, H. (1989). Routine and meaning in the mathematics classroom. *For the Learning of Mathematics*, 9 (1), 24–33.

Usiskin, Z. (1987). Resolving the continuing dilemmas in school geometry. In M. M. Lindquist & A. P. Shulte (Eds.), *Learning and teaching geometry, K-12: 1987 Yearbook* (pp. 17–31). Reston, VA: National Council of Teachers of Mathematics.

White, P. (1991). Rules versus understanding. *Australian Senior Mathematics Journal*, 5 (1), 9–10.

White, P. (1993). Differential calculus. *Reflections*, 18 (4), 31–34.

White, P. & Mitchelmore, M. C. (1993). Aiming for variable understanding. *Australian Mathematics Teacher*, 49 (4), 31–33.

White, P., & Mitchelmore, M. C. (1996). Conceptual knowledge in introductory calculus. *Journal for Research in Mathematics Education*, 27, 79–95.

White, P., & Mitchelmore, M. C. (1997). Recognition of abstract angles in familiar physical situations. In F. Biddulph & K. Carr (Eds.), *People in Mathematics Education: Proceedings of the 20th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 577–584). Hamilton, New Zealand: MERGA.

White, P., & Mitchelmore, M. C. (1998). Sharpening up on angles. *Australian Primary Mathematics Teacher*, 3(1), 19–21.

Wilson, P., & Adams, V. (1992). A dynamic way to teach angle and angle measure. *Mathematics Teacher*, 39 (5), 6–13.

# The Symbol is and is Not the Object

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## Anecdotal Beginnings

I first met the symbol 'Skemp' before I met the referent. While in secondary school in London in the late 1960s, with mine being the last year throughout the school doing what came to be known as the 'old maths' (but which of course was then simply called 'maths'), I would come across 'Skemp is mad', 'Skemp is hard', 'Skemp is impossible' scrawled on wooden desks. I, of course, had no idea what a 'Skemp' was, nor once I had met him how involved our life trajectories would become for a while.

I was also unaware of the rhetorical term 'metonymy', a general linguistic process based on associative substitution including that whereby the name of the author of a book becomes a substitute for the name of the book itself (in this case, 'Skemp' for the *Understanding Mathematics* text series written by Richard Skemp a few years earlier). Yet it is metonymy in a core sense which is the undoubted theme of this chapter.

It was only later, during 1973-4 – my final year as a mathematics undergraduate at Warwick University in England – that a new professor of education arrived, one Richard Skemp, whom I got to know through his teaching of a general year-long course entitled *Human Learning*. As well as some examples from the learning of mathematics, including his then still-recent work on symbolic learning, one specific element that stayed with me to become part of me was his inclusion of a set of six lectures on Freud, whom Skemp non-standardly framed as the quintessential learning theorist.

This course was actually not my first introduction to issues in mathematics education, for the previous year I had taken a mathematics department option course on this topic, taught and originated by David Tall, whose readings alongside works of Piaget and Dienes included Skemp's recently published book *The Psychology of Learning Mathematics* (1971), one of the early, accessible accounts of mathematics education. From time to time, I look back and recognise how many of my own emerging interests have been shaped by the concerns and attentions of Richard Skemp and David Tall.

In 1979, I returned from the U.S.A. where I had been doing graduate work both in mathematics education and the history of science/mathematics, to take up a two-and-a-bit year position with Richard as his Research Associate on a Leverhulme Trust-funded research project on primary mathematics. (He and I had kept in periodic touch during this intervening period, and I was delighted to be offered a job out of the blue during a return visit home in 1978.)

In fact, my first job on my re-arrival back in England in June 1979 was to help him to organise the 3rd International Group for the Psychology of Mathematics Education conference to be held at Warwick the following month (PME is a group with which I have retained strong links over the past twenty years). I had left England in mathematics and had returned in mathematics education. Sitting on the registration desk, meeting and talking to delegates about organisational matters indirectly provided me with an extraordinary *entrée* as a neophyte into the area in which I have subsequently made my career.

During those two years as his research associate, I worked with Richard on teaching an M.A. by research course whose participants comprised some ten primary teachers working through a manuscript versions of his second book *Intelligence, Learning and Action* (which was to be published in 1981), who then undertook substantial projects in their own classrooms. He and I also carried out our research work in the classrooms of these teachers. Among the teachers on that course was Janet Ainley, who succeeded me as Richard's new research associate (with project funding now provided by the Nuffield Foundation) when I left for a position at the University of Nottingham Shell Centre for Mathematics Education. During this time, and later as Richard's colleague at Warwick University in the newly-formed Dept of Science Education there, Janet also contributed an immense amount of her own experience, time, knowledge, creativity, ingenuity and thought into the development of classroom tasks and materials, which appeared in Skemp (1989a) and particularly in *Structured Activities for Primary Mathematics* (Skemp, 1989b).

## Opening Observations: Fluency and Understanding

Over time, I have increasingly felt that Richard's attempted discrimination between 'instrumental' and 'relational' understanding (Skemp, 1976), arising from an observation of Stieg Mellin-Olsen's, has been mis-taken, literally, by certain others who have misread him. Firstly, he had a lot of respect for efficient, fluent mathematical practice and the role of symbols within it. Next, his specification of these as two different forms of *understanding* (and that alone is something that I feel has been lost, or at least mislaid) was not oppositional (and certainly was not the same discrimination as that which later came unhelpfully to be labelled 'rote' and 'meaningful' learning). One is included in the other: to have relational understanding is to have instrumental understanding *plus* something more!

For a range of reasons, both personal and cultural in terms of the external forces operating on mathematics curricula and practice (including the UK Cockcroft report – DES, 1982 – the various UK national curricula for mathematics and their revisions from 1988 onwards, and developments in technological instruments for carrying out mathematical processes 'instrumentally'), Skemp's concern for fluency *and* understanding (which, I believe, was the intended message of his term 'relational understanding') as the necessary twin goals of mathematics education (and not simply 'understanding' or 'conceptual understanding' as sometimes seems the case today) have stayed with me.

My interest in the importance of language in mathematics date back to these early encounters with Skemp. In 1987, I published my first book *Speaking Mathematically: Communication in Mathematics Classrooms*, which took a language focus on a variety of aspects of mathematics classrooms. It was not until 1995, however, with *Symbols and Meanings in School Mathematics*, that I attempted to explore the mathematics curriculum systematically in terms of symbolic demand and the diversity of symbolic practices. The main theme is quite specifically there about the centrality of the metaphor of 'manipulation' for doing mathematics and how mathematics has problems with its objects. And taking the symbol for the object (a central metonymy lying at the heart of mathematical practice) is an essential mathematical phenomenon and not an epistemological aberration as some would have it. This intentional mis-taking of the 'shadow of the object' (Bollas, 1987) for the object itself is what makes much mathematics possible.

I became growingly aware of the fact that in many different parts of mathematics education there are surface forms (which are public and hence observable, such as spoken or written language) and other phenomena of interest (which are not, such as thinking or learning). At the time, it was these unobservables which were mostly sought after and written about, and the surface forms – what was actually said or written or aspects of notational systems in mathematics – were not considered so important (possibly linked to a belief in deep over surface structures, emanating from the Chomskyan linguistic paradigm). Perhaps the most mathematically telling instance of this occurs with the use of symbols themselves.

## Symbols and Mathematics

> Civilisation advances by extending the number of important operations we can perform *without* thinking about them.
>
> (Whitehead, 1925, p. 59; my emphasis)

Of all that I learnt from working alongside Richard, his focus on and interest in the role of symbols in the doing, teaching and learning of mathematics is that aspect of his work which has primarily served to shape my own. He wrote a lot about different functions of symbols in the service of mathematics, and in the chapter on symbols in *The Psychology of Learning Mathematics*, he lists (on p. 68) ten distinct functions, perhaps an attempt at writing down everything he could conceive of symbols being used for. Sitting in Richard's list is the mathematically-significant number 8, 'making routine manipulations automatic', and this is the one I wish to attend to here.

This chapter developed from a talk (entitled 'The symbol *is* the object') which I gave at Richard's 60th birthday celebration in 1986, immediately following the 10th PME conference in London. It had struck me that one of the insistences present in the then 'new' mathematics writing was that of maintaining and reiterating the distinction between symbol and object, symbol and referent (most particularly between number and numeral, but also, for instance, between subtraction and a negative number). And, in addition, there was a desire to mark the distinction in the surface form (rather than rely on what mathematician Jean Dieudonné has termed an '*abus de notation*'). So, for instance, differentially distinguishing – both by size and vertical height – the symbol for subtraction and that of a negative number.

Many symbol systems are sub-minimal for learners, in the sense of not marking all the distinctions that could helpfully be made. But what users eventually need to be able to do is to contend with the conventional system. During the 1960s and 1970s, the Initial Teaching Alphabet was offered as a symbolic 'manipulative', that is a device to assist with the learning of conventional English orthography, based on the fact that there are 44 different sounds (phonemes) in English but only some 26 letters. (The use of written dots to indicate vowels in semitic languages as an aid to novice readers, where they are not conventionally marked in the surface form, is a continuing practice of this type.)

The intent of the ITA was to be able to enhance the acquisition of spelling through an extensive period of work with a non-standard form, one with a one-to-one match between oral and written discriminations, before condensing down to the conventional written form of English. The ITA is no longer used in schools (different arguments rage about learning to read and explicit attention to genre features currently, as well as 'phonics' versus 'whole language', another unhelpful opposition), but the general issue underlying it remains. Some of the discussions about the role of physical manipulatives in mathematics teaching bear a striking similarity. Elsewhere (Pimm, 1995), I have written about Dave Hewitt's school practice in beginning algebra teaching of making sure there is a complete correspondence between what is said – including operations – and what is written in moving from 'spoken equations' (verbally-posed problems of a think-of-a-number kind) to written counterparts.

The distinction between symbol and object (and use of additional differential markings) is, in some sense, an 'of course' observation and some persistent errors when working and thinking about different mathematical objects and forms can be avoided by maintaining an awareness of the difference. Confusion between symbol and object can even be found in the conventional nomenclature for different types of numbers, whereby attributes of the symbols are talked about as if they were attributes of the objects themselves. Are there different types of numbers called *binary*, *decimal* and *fractional*, in the same way that there are prime, whole and triangular numbers? Or are the former categorisations based on surface phenomena, namely certain aspects of the written *forms* of the conventional symbols used rather than involving a property of the number *per se*?

Certainly some current mathematical usage suggests that there is a particular type or class of numbers called *binary*, rather than this being a particular means of representing whole numbers. Precisely the same form of expression is employed for *decimal* or *fractional* numbers, whereby the adjective seems to be modifying the numbers, rather than referring to a characteristic of the representation. 7/4 is a fraction, 1.75 is a decimal: what is being described and differentiated is the form of symbolic representation rather than a property of the number itself. Numbers being *prime* or *composite*, on the other hand, refer to inherent properties of the numbers themselves, invariant across any representation.

However, over time, I began to think about what the pulls were towards their confusion, including those embodied and held in language. What fusion was there between symbol and object? I began to think anew about fluency with mathematical symbols, what made it possible and how it might be worked on in schools. I believe to work fluently (most particularly when working algebraically) the symbol *is* or at least for a while must *become* the object (or at least be seen as indistinguishable from it). And that part of the efficiency and economy that comes from such symbolic practice is available precisely because of this confusion, the *not* paying of attention to this distinction, rendering it simply a *possible* rather than insistently and always an *actual* distinction.

Richard wrote about 'same and different' and about equivalence, most tellingly about functional equivalence, something I drew on in my later work on mathematical metaphor (see e.g. Pimm, 1981, 1988). The very term 'functional equivalence' reminds us that it is always a choice (I see or treat these as the same for my current purposes), and about perceived sameness: and that the choice can be profitably rescinded and another one taken. I might take the symbol to be the object for a while, stressing equivalence and gaining from the fluency that can arise from this identification. I might later choose to maintain the difference between symbol and object and think hard and long about the nature of mathematical objects and their link to the conventional symbolic form (and even conceive and imagine how it might be different).

Most tellingly, many mathematical theorems and definitions arise from a relation between mathematical form and mathematical meaning, asserting the professional subordination of the latter to the former, despite the immense difficulty that this can lead to for learners. At a very basic level:

> A whole number is *even*, if it ends in a 2, 4, 6, 8, or 0.
>
> A whole number is *even*, if it can be divided exactly into two equal whole numbers.

It is a certain theorem (almost never discussed) that relates the latter to the former (creating another equivalence), thereby allowing the surface functionality of the former to be used. Young children can accurately distinguish between odd and even numbers, using this surface criterion, but when asked why may only make reference to the final digit. And some will confuse adding odd or even numbers with properties of the component digits of a number: for instance, saying that 73 is even because both 7 and 3 are odd and an odd plus an odd is even (where adding numbers is seen as juxtaposition). Or that 57 is odd, because both 5 and 7 are odd.

Algorithms too are frequently taught in terms of operations to be carried out on the surface form, with injunctions such as: 'To multiply be ten add a nought', 'Take it over to the other side and change the sign' or 'Invert and multiply'. 'To divide by a fraction, invert and multiply' provides an example of a precept which purports to indicate how to carry out a numerical operation on numbers, namely division, in terms of an operation on the symbolic form. But that is the desired practice which leads to fluency of functioning. (I recently observed a girl offering the 'rule', to divide two fractions, divide the tops and divide the bottoms: in other words, $a/b \div c/d = a{\div}c/c{\div}d$. It took me a while to think this through, even though the corresponding 'rule' for multiplication of fractions, is something I use 'unthinkingly'.)

'Inversion' is an operation which can be performed on a symbol where there is an inherent orientation of up and down (as well as a notation which makes use of it by breaking with the customary left-right order of writing symbols). However, the numbers represented symbolically by $a/b$ and $b/a$ are intimately connected, in that they are mutual inverses with respect to the operation of multiplication.

Thus 'inverse' has two meanings:

(i) to do with 'turning upside down';
(ii) to do with obtaining 1 from a product.

It is possible however, merely by following the operations on the symbols, to obtain a correct representation of the numerical answer. Fluent users, who actually want to be able to confuse symbol and object when calculating (because it is so much quicker and more efficient) frequently forget that this mode of operation is, in some sense, a pathology, one potentially lethal to many learners. Yet they became fluent users precisely by ignoring this distinction.

## Counterparts and Signs

I have come to realise that mathematics has a problem with its objects, namely what the symbols of mathematics 'point to'. Personally, I find I have less and less grip on what a 'number' is. Robert Schmidt (1986), a historian of mathematics, makes a helpful and for me novel distinction between symbols serving as *signs* and as *counterparts*. A *sign* names or points to something else, but bears no necessary relation to the thing named. A *counterpart* stands for something else, but does not name or point to it (an indicative function): however, there is an actual relation, a resemblance or connection, between the object and its counterpart. These two functions can coalesce on the same symbol, but there may be confusion when this occurs.

Schmidt uses the example of lines drawn on a nautical chart to illustrate the notion of a counterpart: a nautical chart in no way names what it stands for, but it allows computations and actions to be made upon it which can be directly transferred to actions on the actual object represented. Technical drawings, as opposed to geometric diagrams, are counterparts, though when teachers invite pupils to use rulers and protractors on geometric drawings they are shifting the drawing's status to that of a counterpart. Counterparts are to be acted on, and then the results interpreted via the connection. Counterpart forms can also provide substitute images. Schmidt claims:

> It is also the nature of counterparts to draw attention to themselves, while it is in the nature of signs to lead our attention away from themselves and towards the thing signified. [...] Furthermore, it is in the nature of counterparts to turn their object into themselves, while it is the nature of signs to disclose their objects. (p. 1)

I see the signification and counterpart functions of symbols as complementary; neither one suffices by itself for mathematics, yet they seem to conflict with one another, pulling in opposite directions. As Richard Skemp himself drew attention to, mathematical symbols have more than one function: signification and counterpart are two central ones I believe. Counterparts offer visible or tangible substitutes which are then available for 'manipulation', for acting on *as if* they were the object itself. Counterparts seem secular, of this sensible world; signs are often taken as spiritual, that is, other-worldly.

Manipulatives such as Cuisenaire rods or Dienes apparatus are undoubtedly offered as physical counterparts for numbers. Or is it that Cuisenaire rods offer a counterpart for number, while Dienes blocks are actually confusingly used both as a counterpart for numbers and a sign for the place-value numeration system? Written numerical algorithms are seldom identical either in form or structure to the 'corresponding' manipulations with the blocks, in part because the former frequently draw on specific properties of the decimal, place-value numeration system. (In passing, Dienes Blocks provide at least as good a model for the ancient Egyptian numerals, which are non-place-value in nature, as Hindu-Arabic numerals – in consequence, one cannot simply 'read' place value out of the blocks – it has to be induced arbitrarily.) The apparatus allows or encourages certain ways of operation, and this needs to be transformed before the 'traditional' algorithm can come to be seen as a 'mere' record of operations with the apparatus. Thus, records can be seen as relics of actions. (It can be an interesting mathematical question as to whether the residue is sufficient to reconstruct the process.)

Ironically, in order to see connections, pupils must already understand to some degree that which the blocks are supposedly helping them to learn. The operations with the written symbols are what is to be learned and that is what is guiding the way the equipment is being used. Who is in control of the apparatus, and, in particular, the interpretation of the meaning? This is one reason why there seems such a difference between asking about the 'meaning' of a geoboard and the 'meaning' of Dienes blocks. Once again, the danger is that the wrong thing is being seen as primary.[1] One practice that is the hallmark of mathematics occurs when symbols start being used *as if* they were the objects themselves, namely as counterparts.

With algebra, 'manipulation' comes into its own, with symbols as counterparts very much to the fore; the 'true' nature of the algebraic object becomes ever more confused. Fluent users report two awarenesses when working with algebraic expressions: being able to see them as structured strings of symbols (and hence symbolic objects in their own right) *and* seeing them as descriptions connected with some 'reality' or situation they are concerned with. Maintaining this dual perspective, of substituting counterpart and indicating sign, is of central concern when working on mathematical symbols at whatever level, and places a heavy burden on novices. As Schmidt points out, algebra offers both a calculus and a language.

Transformation is a key power of algebra, the most important means for gaining knowledge. Operation symbols in algebra are virtual and not actual as they seem to be in arithmetic. What is transformed is the expressions, viewed as counterparts and not just as descriptions, and different forms reveal different aspects. Algebra invokes forces that transform: algebraic expressions are shape-shifters. But what else is being transformed as I manipulate the symbols? Counterpart algebra involves an echo, a shadow.

Currently, when working on algebraic forms, I am encouraged to suppress 'meaning' in order to automate and become an efficient symbol manipulator. In other words, I am encouraged to ignore the signification function and to see the symbols as counterparts until the very end of a computation. I learn to associate aspects of the forms with aspects of the situation (for instance, the numerical value of the discriminant in order to discriminate among various types of conic section), and work with them instead. This displacement of attention is a mathematical commonplace, and can also lead to very powerful theorems.

How do actions on symbols connect to actions on the actual objects? How do arguments involving symbols relate to arguments about their counterparts? Seeing the symbol as part of the whole produces a reason for the link. Do we ever argue about the whole from the part in mathematics? Despite much turning up of intellectual noses at such an apparent mathematical solecism, there are a number of occasions when such generic arguments are used. Here are three:

- in Euclid's proof of the unlimited nature of primes, where he shows that if there were three primes, then he knows how to construct another one – and doesn't even make the remark that the proof goes likewise for any other number of primes;
- when working on particular elements of equivalence classes, in order to show what happens to the classes as a whole (usually with a general theorem in the background that the operations are well-behaved with regard to class boundaries);
- when deciding where a derivative is positive or negative over an interval by finding the zeros and then testing an individual point (similarly for points in the plane satisfying inequalities).

The key feature here in all of these instances is that the argument presented is a particular argument that apparently does for all. (See Balacheff, 1988, for a discussion of similar modes of thinking in adolescent provers.)

With algebra, the symbol used is *not* an actual instance of what is being talked about, a letter is not a number in the main (*pace* $e$ and $i$, and perhaps $\pi$). So there is more of a question about the relationship – and relating principle – between the symbol and the referent, as well as how an argument conducted on the symbols bears any relation (be it one of mimicry or something else) to what is the case for the actual members of the set. The Greek mathematician Diophantos' symbol for an unknown was a Greek letter s-like mark that is plausibly a relic of the last letter of the word *arithmos* ("number"), and a common choice of variable name is to use the first letter of the corresponding word name ($r$ for radius, $t$ for time). So here it is a part of the word symbol rather than part of the actual object that is used.

With arithmetic, early number symbols were either pictographic (using three strokes or objects for three) or used a residue principle based on the natural language word. (Attic Greek numerals, for instance, had π for five, the first letter of the Greek word *pente*.) It is possible to see numbers both as bearing the same relation to referent experiences as with algebra (e.g. Martin Hughes' (1986) interviews with young pupils who baulk at intransitive, non-adjectival use of number words as things to be added), but also as individual numbers in their own right. (Try to make sensible use of '6' as an algebraic symbol; for instance, 'Let 6 be a group'.)

Generic examples of computations with simple surds such as $\sqrt{2}$ can sometimes be more illustrative than with an algebraic variable, in that they partake of particular number status without being easily reducible in computations, so it is possible also to gain from the structural, placeholder function that algebraic variables serve in highlighting every occurrence of the particular number in focus.

What about geometry? The same duality of symbol relation occurs. A geometric figure is symbolic of the general often, but is perceivable also a particular element. I have wondered whether the ease of transformation of *Cabri-géomètre* or *Geometer's Sketchpad* drawings encouraged viewing a diagram only as an actual particular rather than as a general symbolic? And what of the singular perspective that it is not a single figure being transformed, but a continuous highlighting into attended awareness of distinct figures all equally present in potentiality. The continuity of deformation by the mouse again can act *against* this perception.

With any general argument in mathematics, there is an additional need for symbolisation. The relationship between the symbols and the things symbolised needs to be established (whether by synecdoche or some other means). Does one of the differences between geometry and algebra lie in the difference in relation between symbol and referent, and hence, in the different symbolic nature of arguments that are used to justify general claims?

In conclusion, I want to say a few words about the other strand to my work that has certain roots in what I learnt from Richard. His insistence on seeing Freud as a learning theorist, one who's work has much to say about the practice of schools and learning and teaching, attuned me to this sense of the psychological at a time when I was finding other theories of learning (each apparently claiming itself to be universal, covering all forms of learning as an apparently monolithic activity) too remote or unhelpful.

My year spent at Cambridge University in 1974–5, just after first meeting Richard, had produced in me (admittedly, not unaided) some significant destruction of meaning and a sense of being left with the symbolic forms which no longer spoke to me. And that experience has remained with me, and while it does not keep me awake at nights, a residual curiosity exists as a personal desire to explore – one reason why I do the particular work I do. (Mathematics educators are notoriously bad at proclaiming and insisting theirs to be an utterly selfless endeavour focused solely on the other, those mathematics teachers and learners on whose (often unasked) behalf they ceaselessly toil. Not the whole story, I reckon.)

So a second Richard-inspired thread to the work I do lies in providing occasional psychological accounts for certain, usually invisible, classroom events. In a chapter 'Another psychology of mathematics education' (Pimm, 1994), I wrote of a classroom 'slip', a child saying 'fidelity' instead of 'infinity' in response to a teacher's question. Such symbolic slippage, 'along the chain of signifiers', from one sound or mark to another is the stuff of mathematics, and adds richness to why mathematics is seldom viewed as a neutral subject, one to which it is possible to be indifferent.

A hundred years ago mathematician Paul Souriau queried the feasibility (and I, now, the desirability) of remaining fully in touch with the 'meaning' of algebraic expressions and manipulations.

> Does the algebraist know what becomes of his ideas when he introduces them, in the form of signs, into his formulae? Does he follow them throughout every stage of the operations he performs? Undoubtedly not; he immediately loses sight of them. (cited in Hadamard, 1945/1954, p. 64)

And these two strands of work are actually related, though a psychoanalytically-informed account of the processes of metaphor and metonymy and the corresponding psychodynamic ones of condensation and displacement. And the best account I know of these links in relation to mathematics education can be found in Tahta (1991).

## On Symbols, Metaphor and Metonymy: Holding the Tension of Opposites[2]

> In a Symbol there is concealment and yet revelation …
>
> (Carlyle, 1836; 1987, p. 166)

What is the origin of the word 'symbol' itself? Literary critic Northrop Frye (1987) writes of the Greek verb *symballein* meaning "to put or throw together", but also draws attention to the noun *symbolon*, which he glosses as:

> a token or counter, something that could be broken in two and recognized by the identity of the break. (p. 3)

Symbols and the gap, the separation, between symbol and object are fundamental to the very possibility of mathematics. Frye goes on with his etymological discussion:

> A *symbolon* is something that is not complete in itself, but needs something else, or another half of itself, to make it complete. A *symbolos*, in contrast, links us to something too complex or mysterious to grasp all at once. (p. 4)

We symbolise when we want something that is absent or missing in some way – and then we work on or with the symbol as a substitute, and on occasion as a consolation. Through working with the symbol, we also gain experience of the thing substituted for. In the process, we can often lose sight of the fact that what we have *is* a symbol, and not the thing we originally desired. But Aldous Huxley (1956), like the 'modern' mathematicians of the same period, is at pains to remind us: "However expressive, symbols can never be the things they stand for" (p. 29).

So at the end of this piece I am left with a tension of opposites, the one that constitutes my title. A path to fluency and understanding, to relational understanding, lies in being able to hold in productive tension the two, oppositional claims implicit within it, both accepting and rejecting the identification at different times. I think it would not have surprised Richard that such an ability requires a considerable sophistication in the use of symbols.

# References


Balacheff, N. (1988) Aspects of proof in pupils' practice of school mathematics, In D. Pimm, (Ed.) *Mathematics, Teachers and Children* (pp. 216–235). London, Hodder and Stoughton.

Bollas, C. (1987). *The Shadow of the Object: Psychology of the Unthought Known*, London: Free Association Books.

Breen, C. (1993). Holding the tension of opposites, For the Learning of Mathematics 13 (1), 6–10.

Carlyle, T. (1836; 1987) *Sartor Resartus*, Oxford: Oxford University Press.

Dawson, S. (1991). Learning mathematics *does not (necessarily) mean* constructing the right knowledge, In D. Pimm, & E. Love, (Eds.), *Teaching and Learning School Mathematics* (pp. 195–204.). London: Hodder and Stoughton

DES (1982) *Mathematics Counts (The Cockcroft Report)*, London: HMSO.

Frye, N. (1987). The symbol as a medium of exchange, In J. Leith (Ed.) *Symbols in Life and Art* (pp. 3–16). Montréal, QC: McGill-Queen's University Press.

Hadamard, J, (1945/1954). *The Psychology of Invention in the Mathematical Field*, New York, NY: Dover.

Hughes, M. (1986). *Children and Number*, Oxford: Basil Blackwell.

Huxley, A. (1956). *The Doors of Perception* and *Heaven and Hell*, New York, NY: Harper and Row.

Pimm, D. (1981). Metaphor and analogy in mathematics, *For the Learning of Mathematics* 1 (3), 47–50.

Pimm, D. (1987). *Speaking Mathematically: Communication in Mathematics Classrooms*, London: Routledge and Kegan Paul.

Pimm, D. (1988). Mathematical metaphor', *For the Learning of Mathematics 8*(1), 30–34.

Pimm, D. (1994). Another psychology of mathematics education, In P. Ernest, (Ed.), *Constructing Mathematical Knowledge: Epistemology and Mathematical Education* (pp. 111–124). London: Falmer Press.

Pimm, D. (1995). *Symbols and Meanings in School Mathematics*, London: Routledge.

Schmidt, R. (1986). On the signification of mathematical symbols, Preface to Bonasoni, P. (trans. Schmidt, R.), *Algebra Geometrica*, Annapolis, MD: Golden Hind Press, 1–12.

Skemp, R. (1964–70) *Understanding Mathematics*, London: London University Press.

Skemp, R. (1971) *The Psychology of Learning Mathematics*, Harmondsworth, Middlesex: Penguin.

Skemp, R. (1976). Relational understanding and instrumental understanding, *Mathematics Teaching* 77, 20–26.

Skemp, R. (1979). *Intelligence, Learning, and Action: A foundation for theory and practice in education*, Chichester, W. Sussex: John Wiley.

Skemp, R. (1989a). *Mathematics in the Primary School*, London,: Routledge.

Skemp, R. (1989b). *Structured Activities for Primary Mathematics*, London: Routledge.

Tahta, D. (1991). Understanding and desire, In D. Pimm, & E. Love (Eds.), *Teaching and Learning School Mathematics* (pp. 221–246). London, Hodder and Stoughton.

Whitehead, A. (1925). *Science in the Modern World*, New York, NY: Macmillan.

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[1] This need not be the case. Sandy Dawson (1991) has produced an account of developing a *written* subtraction algorithm which operates from left to right, arising from observations made when working with Cuisenaire rods.

[2] Chris Breen (1993) has written an article with this subtitle as its title.

# Richard Skemp's Fractions In-Service Course for Teachers

**Bruce Harrison**
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From 1987 through 1994, Richard Skemp designed and instructed twelve in-service courses for teachers of elementary school mathematics. No ordinary university summer courses, these unique embodiments of his theory of intelligent learning were enthusiastically received by more than 700 teachers, primarily at The University of Calgary but also at The University of Alberta, The University of Regina, and several rural centres in Canada.

While exploring and preparing ready-to-use, practical learning activities, the teachers were introduced to a theory whereby children can bring their intelligence to bear on the learning of mathematics, the very theory embodied in the activities themselves. Skemp's mathematics learning activities were developed using a mental model beyond common sense to enable teachers to cumulatively nurture the mental processes of growing children. Without such a mental model one can only have *instrumental understanding*, whereas *relational understanding* is necessary for the complex task of teaching and learning mathematics intelligently.

One of Richard's twelve teacher in-service courses covered everything an elementary school child needs to know about **fractions**. Designed to develop relational understanding of key concepts and processes, the fractions in-service course explored the entire **Fractions Network** found in *Structured Activities for Primary Mathematics* (Skemp, 1989b) and *SAIL through Mathematics* (Skemp, 1993, 1994). It was conducted over five days in four-hour morning or afternoon sessions, including a 20-minute break. Each day began with a 35-minute *Plenary session* in which: relevant *concept maps* were explored as a means of integrating children's long term learning of the topic at hand; the learning activities to be worked with that day were introduced and discussed in terms of the underlying theory and classroom application; occasional videotaped activity demonstrations with children were viewed and discussed; and responses were given to comments and questions from the in-service course participants. Then, having read designated chapters from *Mathematics in the Primary School* (Skemp, 1989a), the participants would engage in 40-minute

*Small group discussions* of aspects of Skemp's theory of intelligent learning and its embodiments in the activities. These were followed by *Making* the day's activity materials from printed, coloured card and other supplied materials (50 minutes) and *Doing* the activities by following the activity instructions and discussing how the activities would likely be received by children in a classroom (50 minutes). Finally, a half-hour *Plenary discussion* in which Richard answered questions raised by the teachers would round out the events for the day.

What follows is a brief day-by-day summary of the concepts, processes, and insights shared with teachers who participated in the **Fractions** in-service courses led by Richard Skemp (assisted by educators who counted themselves particularly privileged). The learning activities in the **Fractions Network** are described in general terms here but can be found in detail elsewhere (Skemp, 1989b, 1993, 1994).

## Fractions In-Service, Day 1

Many children have difficulty with fractions. They *are* difficult. Many programs start the teaching of fractions too early and go too far, too quickly. Furthermore, authors of many textbooks confuse the three meanings of fractions. Skemp's **Fractions Network** was designed to remedy these and other problems. It is the result of a thorough conceptual analysis of what is needed for understanding and using fractions by children up to the age of about twelve years. The **Fractions** concept map (see Figure 1) schematically summarises a structured progression of learning activities from beginning physical actions through creative extrapolations to operations with decimal fractions as numbers, considering fractions as quotients, and rounding decimal fractions.

Topic 1 in the **Fractions Network** is *making equal parts* (shown on the extreme left in the concept map above). There are five learning activities in Topic 1, of which the first three facilitate *Mode 1* schema building via the physical experience of making Plasticine objects and cutting them into equal parts. Fractions are not talked about as such. Rather, children *physically* cut *wholes* into *parts* (just the first stage in building the concept of a fraction). *Mode 1* schema building (physical experience) enables the learner to abstract the mathematical operation of division from the physical action of cutting up a whole thing into equal parts (Skemp, 1994, pp. 219–223).

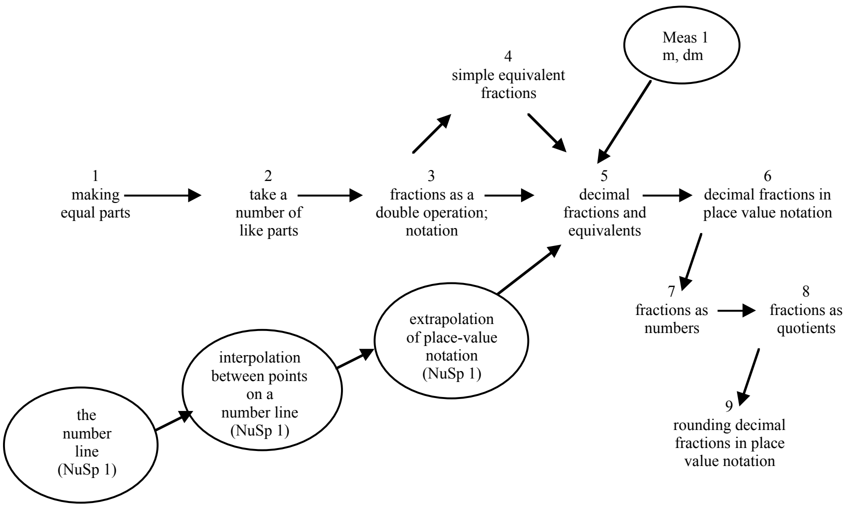


*Figure 1.* Num 7 Fractions (as a double operation, as numbers, as quotients) (Skemp, 1994, p. 42)

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Emphasis on *Mode 2* learning, building via communication and testing predictions through discussion, occurs in the last two activities in Topic 1, with progress to pictorial representations of the physical actions experienced in the earlier activities. Traditionally, textbooks have *started* with the diagrams, but such diagrams, which typically contain a condensation of no less than 6 ideas, are likely to be helpful *if and only if* the children already have the appropriate foundational schema to which these can be assimilated. C*ommunication* and *discussion* come to the fore in the last activity in Topic 1, a fraction-diagram classification game: **Match and mix: parts**. This activity consolidates the notions of equal parts of an object and their names, using a variety of pictorial representations, as illustrated following. A

10-minute videoclip of Richard Skemp introducing the activity to four children is available (Skemp, Edison & Morris, 1991, Video Volume II).

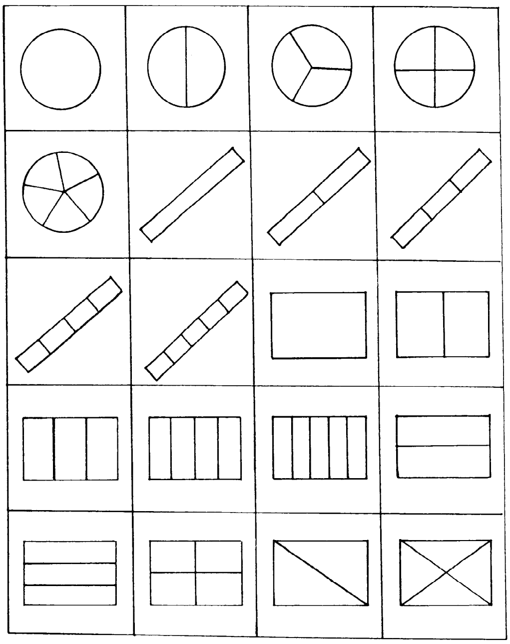


*Figure 2.* Sample cards from Sorting parts & Match and Mix parts (Skemp, 1994, p. 224)

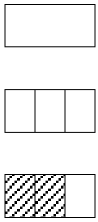
Three related concepts emerge from the *making equal parts* activities:

- the whole of an object (a special part of an object – not the object itself),
- part of an object,
- equal parts of an object and their names.

At this point the activities do not involve fractions, as such, just objects, and parts of objects, and the idea that the parts must be of equal size.

## Fractions In-Service, Day 2

Accurate concepts are easier to grasp than woolly ones. To be accurate and simple is not always easy. A concept cannot be learned with understanding unless the learner has the appropriate schema to which it can be attached. Basic to the concept of a fraction is the notion of carrying out an operation on an object (the operand) and then carrying out another operation on the result. For example, ‘taking two-thirds of something’ can be illustrated in the following way:



Operand

Three equal parts

Take 2 of the 3 equal parts

The Day 1 fraction activities deal with the *denominator*. The Day 2 activities are from Topics 2 and 3 and focus on the *numerator* and the notion of a fraction as a mathematical operation.

The three activities in Topic 2, *take a number of like parts,* have the pupils engaged in a zoo microworld in which they are responsible for feeding various animals specified numbers of *parts* of appropriate “foods” like “eels,” “slabs of meat,” and “biscuits.” They begin by physically forming and cutting up Plasticine “foods.” Then they assemble the *parts* needed to feed certain collections of animals. In the process they experience fractions below and above “one” in a very natural way. These activities continue to build towards the concept of a fraction by combining physical actions and social interaction in the contexts of interesting activities related to feeding the animals in an imaginary zoo. In the last activity in Topic 2 the children are brought a step closer to *fractions as mental operations* by requiring them to *think about* and *describe* the physical actions of making equal parts, and then taking a given number of these (to feed a particular group of animals), instead of actually doing these actions (Skemp, 1994, pp. 223–231).

Topic 3, *fractions as a double operation,* develops the concept of a fraction as a mathematical operation which corresponds to two actions: making a given number of equal parts, and putting together a given number of these. The first activity, **Expanding the diagram**, relates conventional fraction diagrams to the six concepts which they combine, as illustrated in the activity board in Figure 3.

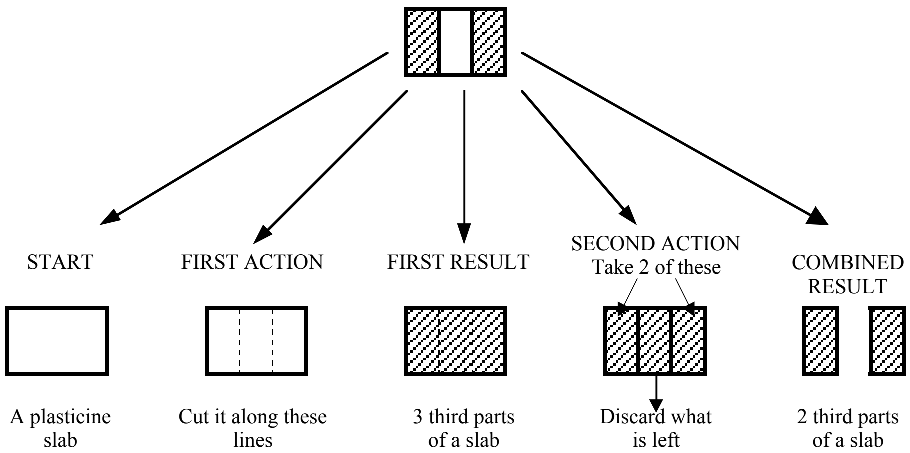



*Figure 3.* Activity board for two-thirds from **Expanding the diagram** (Skemp, 1994, p. 234).

The second activity in Topic 3 links the fraction diagrams to conventional bar notation, and fractions begin to be treated independently of any particular embodiment (the same operator can act on a variety of operands). In this topic, the children are introduced to a notation by which to communicate, manipulate and record the mental concepts that they have learned about fractions. Assimilating the notation and the transition to the concept of a fraction as a combined operation essentially involves (for the child) inventing a new kind of number and testing these for *internal consistency* with the numbers they already know … *Mode 3* schema construction, building using creativity and testing in accordance with existing knowledge (Skemp, 1989a, p. 74).

Schematic learning is essential for intelligent learning. To be able to use what one has learned on future occasions, abstraction and conceptualization are also necessary. Concepts and schemas cannot be communicated directly but must be constructed individually in the learner’s mind (Skemp, 1989a, p. 72).

Because so many children (and adults) have difficulty with fractions, it is important to return to using physical embodiments of both the actions and the operands in order to establish a solid *Mode 1* foundation for the concept of a fraction. The fraction diagrams and notation introduced in Topic 3 exemplify the extremely concentrated nature of mathematical knowledge and representation (symbolism). To facilitate communication and manipulation of the purely mental concepts and operations underlying fractions, traditional 'bar notation' is introduced to record the underlying fraction concepts (which are invisible and intuitive). The 'denominator' of the fraction indicates what kind of parts are being represented and the 'numerator' indicates how many of those parts.

## Fractions In-Service, Day 3

The concept of *equivalent fractions* is a prerequisite for understanding *decimal fractions*. If two different fractions of the same operand (object) give the same amount, they are *equivalent* (e.g., two fourth-parts of a cookie and one-half of the cookie give the same amount, although they do not 'look' alike). It is important to be able to recognize when this is the case, i.e., to match equivalent fractions.

*Equivalence*, an important idea throughout mathematics, facilitates making the bridge between *fractions as a double operation* (Topic 3) and *fractions as numbers* (Topic 7). Since it is the *idea* that is important, the concept of equivalent fractions is constructed with simple examples at three levels:

- concrete objects [e.g., fruit bars],
- diagrams (half way to the abstract) [e.g., pictures or drawings evocative of a fruit bar],
- notation (writing, written symbols … not particularly evocative) [e.g., is notation, a numeral, when written (not spoken)].

The activities in Topic 4 introduce the concept of *equivalent fractions* by the partitioning of Plasticine "fruit bars" in a candy store context and then extend the concept to the more abstract representation of fractions by diagrams and notation. The concepts are consolidated in a match-and-mix game. Topics 4 and 5, *simple equivalent fractions* and *decimal fractions and equivalents*, refresh earlier *Mode 1* work with physical materials and extend the *Mode 2* discussion and representations of fractions in pictorial and notational forms. *Mode 3* thinking comes into play in the process by which the fraction concepts are detached from any particular physical embodiment and the groundwork is laid for *decimal fractions in place-value notation* (Topic 6). Richard maintained that decimal fractions can't really be understood until equivalent fractions are understood in bar notation.

The first two activities in Topic 5, *decimal fractions and equivalents,* introduce *decimal fractions*, meaning ‘related to ten,’ first in verbal form and then in the context of “equivalent fraction diagrams,” as illustrated in Figure 4.

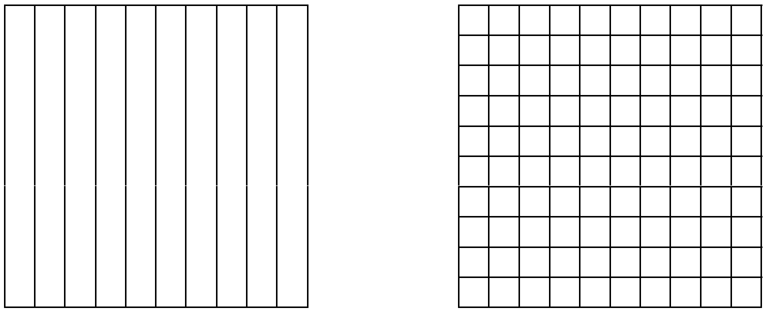


*Figure 4.* Equivalent fraction diagrams (decimal) (Skemp, 1994, p. 242).

By using the diagrams singly or together and shading appropriate combinations of equal parts, equivalences can be seen, for example, between “2 fifth-parts” and “4 tenth-parts,” between “1 fifth-part” and “20 hundredth-parts,” …

When participants posed questions like: “How can learners be helped to construct their own knowledge structures?” and “How is the theory embodied in the activities?” Richard responded in terms of *Modes 1, 2 and 3* schema construction using analogies like “building a brick wall” and activities like **Missing stairs** (Skemp, 1993, pp. 76, 77), **Shrinking and growing** (Skemp, 1994, pp. 376–378), and **Make a set. Make others which match.** (Skemp, 1993, pp. 243, 244). A set of VHS videocassettes have been produced at the University of Calgary containing responses from Richard to questions like these (Skemp, Fountain & Edison, 1992, Video Volume V) and containing model introductions to schoolchildren of many of the activities in *Structured Activities for Primary Mathematics* (Skemp, 1989b) and *SAIL through Mathematics* (Skemp, 1993; 1994).

## Fractions In-Service, Day 4

The final two activities in Topic 5, *decimal fractions and equivalents,* deal entirely with symbols, consolidating the new concepts of equivalent decimal fractions, detached from any particular physical embodiment.

Topic 6, *decimal fractions in place-value notation*, develops the idea that different notational forms can be used to express the same fraction concept. For example, 0.5 and are two different ways of writing the same fraction. The two notations, *place-value notation* and *bar notation*, are useful for different purposes. While many computations are facilitated by the use of place-value notation, the understanding of multiplication of decimal fractions (e.g., $0.1 \times 0.1$) is especially facilitated with bar notation (to cite only one example). Since the concern here is with notation rather than concepts, it is important that the underlying concepts are well formed, which has been the aim of the preceding fraction activities.

Three *number line* topics need to be covered before Topic 5, *decimal fractions and equivalents* (as shown at the bottom of the **Fractions** concept map shown earlier in this chapter). Similarly, experience with *metres* and *decimetres* as encountered in the **Length** network (Skemp, 1994, pp. 385–393) is also prerequisite. The first *number line* topic needed is *unit intervals: the number line* (Skemp, 1994, pp. 359–362) which treats the number line as an infinite mental object that can be represented by points on a line using the notion of a unit interval. The second is *interpolation between points; fractional numbers (decimal)* (Skemp, 1994, pp. 368–372) in which points are located marking tenth-parts of a unit interval, and decimal fraction names (tenth-parts) are applied to these points. Here children begin to form the concept of a new number, a decimal fraction, in the context of the number line. They already have spoken names for these new numbers and now they learn how to write them, first in headed column notation. The third number line topic, *extrapolation of place-value notation* (Skemp, 1994, pp. 373–379), extends place-value notation to include tenth-parts and hundredth-parts for writing mixed numbers, as represented by points on the number line. A 21.5-minute video in which four eleven-year-olds creatively explore the **Shrinking and growing** activities that form the culmination of this topic is available (Skemp & Edison, 1992, Video Volume IIIa).

When asked by the teachers where to start when introducing fraction activities part way through a year or in later grades, Richard recommended that they begin with activities in topics that should already be known, according to the curriculum, and that they use mathematical content that the pupils are already familiar with to help them learn the new approach. A concept map can be used to identify topic areas that need more work. “Projects” can help one to see concepts in different contexts and are valuable for extending, adapting, and applying the mathematics curriculum. True “problem solving,” while generally not useful for *developing* concepts, is certainly valuable for applying, adapting, and extending well-formed concepts and processes. In fact, intelligent mathematical problem solving requires a background of well-structured mathematical concepts.

## Fractions In-Service, Day 5

“Fractions are often treated as numbers right from the start. This almost guarantees confusion for the children, since they are seldom told that we are *expanding* our concept of number to include a *new kind of number*. Place-value notation makes this transition smoother [in fact it greatly simplifies addition of fractions], but does not remove the need to explain what is going on, so that children have a chance to adjust their thinking.” (Skemp, 1994, p. 250)

The first activity in Topic 7, *fractions as numbers,* shows that fractions, in place-value notation, may be added in the same way as whole numbers; and that the results make sense … behaving like the numbers with which the children are already familiar (Skemp, 1994, pp. 251). The second activity explores the question “How do we know that our method [for addition] is still correct?” by means of a teacher-led discussion, with *Mode 3* testing for the more able. In this topic, place-value notation is used to extrapolate the idea of number, from familiar whole numbers to a new kind called fractional numbers. The word “fraction” has now acquired a second meaning as a number that behaves like the familiar whole numbers when added to another number (whole or fractional).

Regarding Topic 8, *fractions as quotients*, Richard has pointed out that one of the ways in which many textbooks have created unnecessary difficulties for children is to confuse fractions as quotients with the other aspects of fractions [as a double operation and as a number]. He argued as follows:

> If we read as ‘two-fifths,’ short for ‘two fifth-parts,’ the corresponding physical actions are: start with an object, make five equal parts, take two of these parts, result two fifth-parts.
>
> Now, however, we are going to think of in a different way, as the result of division. The corresponding physical actions are: take two objects, share equally among five, each share is how much?

> At a physical level, these are quite different, just as physically 3 sets of 5 objects and 5 sets of 3 objects are quite different, and mathematically 3(5) and 5(3) are different; although the results are the same. So we should be more surprised than we are that the division $2 \div 5$ gives as a result. And to teach that is just another way of writing $2 \div 5$ is to beg the whole question. $2 \div 5$ is a mathematical operation, a division. is the result of this operation, a quotient. The distinction becomes even sharper if we write this quotient as 0.4. To put this another way: would we say that 45 was just another way of writing $5 \times 9$? (Skemp, 1994, p. 254).

The first activity in Topic 8, *fractions as quotients*, returns to *Mode 1* to relate a new meaning of fractions to a physical situation in which Plasticine "fruit bars" are shared equally among certain numbers of recipients, the result of which (what each person receives) can be expressed as a fraction of a "fruit bar." Another three activities use calculators to consolidate the relation between fractions in bar notation and in place-value notation, regarding the latter as quotients and developing a critical attitude with respect to whether or not quotient results obtained by calculator give sensible results in the real world.

Finally, Topic 9, *rounding decimal fractions in place-value notation*, extends "rounding whole numbers" to "rounding decimal fractions in place-value notation," to the nearest tenth or hundredth. Richard was certainly in favour of "project work" and "real life" problem solving but often pointed out that these cannot be relied on alone for building mathematical structures, concepts, and operations. Mathematics is knowledge which is in our minds, not just out there in the environment around us, waiting to be discovered. It is not enough to say that you can get a lot of mathematics from observing the gyroscopic effects of a spinning bicycle wheel, for example, but, rather, you need a lot of mathematics in your head to understand what is happening. Often in "projects" the mathematics is very dilute … a lot of time can be invested with only a little math coming into play. (Also see pp. 174, 175 in *Mathematics in Primary School* (Skemp, 1989a) on the importance of 'low noise' while building mathematical structures.) On this topic he also liked to quote Pasteur: "Discoveries come to the prepared mind." Problem solving *is* valuable and important, of course, but it will only be successful to the extent that the child has the appropriate structured knowledge to bring to bear on the problem.

## Participant Perspective

What was it like to take an in-service course on the teaching and learning of fractions from Richard Skemp? The following teacher-participant comments are typical of the overwhelmingly positive feedback given in the course evaluations.

> It has been a delight to be in Dr. Skemp's classes. He is a scholar and a gentleman. The introduction to his writings and activities has been one of enlightenment for me personally. I am eagerly looking forward to using the material in my classroom.
>
> The format of the course seems to fit right in with Dr. Skemp's schema construction. There is experimenting (playing the games/activities), discussion (within discussion and do groups), and internalizing (as we read and then work with the concepts).

Those of us fortunate enough to assist Richard in the delivery of the in-service courses held him in great esteem not only for his impeccable scholarship, for the power of his theory, and for the excellence of the activities that he had developed as embodiments of his theory, but also for his friendship, generosity, and the wonderfully kind and encouraging way that he had with all of us, colleagues and students, alike.

## References


Skemp, R. R. (1989a). *Mathematics in the Primary School.* London: Routledge.

Skemp, R. R. (1989b). *Structured Activities for Primary Mathematics, Volumes 1, 1a, 2, 2a.* London: Routledge.

Skemp, R. R. (1993). *SAIL through Mathematics, Volume 1: Structured Activities for Intelligent Learning.* Activities for the early years (Age 5–7). Calgary, Canada: EEC Ltd.

Skemp, R. R. (1993pm). *SAIL through Mathematics, Volume 1a: Structured Activities forIintelligent Learning.* Photomasters for the early years (Age 5–7). Calgary, Canada: EEC Ltd.

Skemp, R. R. (1994). *SAIL through Mathematics, Volume 2: Structured Activities for Intelligent Learning.* Activities for the later years (Age 8–12). Calgary, Canada: EEC Ltd.

Skemp, R. R. (1994pm). *SAIL through Mathematics, Volume 2a: Structured Activities for Intelligent Learning.* Photomasters for the later years (Age 8–12). Calgary, Canada: EEC Ltd.

Skemp, R. R. (Author, Teacher) & Edison, L. (Producer/Director). (1992). *Mathematics Learning Activities: Skemp videocassette series, Volume IIIa.* [VHS videocassette]. Calgary, Canada: Department of Communications Media, The University of Calgary.

Skemp, R. R. (Author, Teacher), Edison, L., & Morris, P. (Producer/Director). (1991). *Mathematics Learning Activities: Skemp videocassette series, Volume II.* [VHS videocassette]. Calgary, Canada: Department of Communications Media, The University of Calgary.

Skemp, R. R. (Author, Teacher), Fountain, F. & Edison, L. (Producers). (1992). *Mathematics Learning Activities: Skemp videocassette series, Volume V: Question Time with Richard Skemp.* [VHS videocassette]. Calgary, Canada: Department of Communications Media, The University of Calgary.

# The Silent Music of Mathematics

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The thoughts which follow result from a combination of two events which took place last Christmas. One of these was a visit from a niece of mine who has two bright children aged seven and eight. She was worried because all the mathematics they did at school was pages of sums. Shortly before this, we had heard and seen a performance of Benjamin Britten's beautiful Ceremony of Carols. This was introduced by Britten's lifelong friend Peter Pears, who related how it had been composed by Britten at sea, in a cramped cabin with no piano or other musical instrument on which Britten could hear what he was composing. Afterwards, I began to wonder how such music could be composed under these conditions. How could he know how wonderful it would sound in performance? Maybe he sang it to himself, some of it. But he could only sing one part at a time, and what about the harp? My answer cannot be more than a conjecture; but I think we may assume that like many other composers he was able to write music directly from his head onto paper because he could hear the music in his mind. The musical notation represented for him patterns of sound, sequential and simultaneous.

There are others besides composers who can hear music in their minds, and who (we are told) can get pleasure from reading a score in the same way as others enjoy reading a book. But most of us are not like this. We need to hear music performed, better still to sing or play it ourselves, alone or with others, before we can appreciate it.

We would not think it sensible to teach music as a pencil and paper exercise, in which children are taught to put marks on paper according to certain rules of musical notation, without ever performing music, or interacting with others in making music together. To start with, children are not taught to read or write music at all - they sing, listen, and move their bodies to the sounds of music. And when they do learn musical notation, it is closely linked with the performance of music, using their voices, or instruments on which they can play without too much difficulty.

If we were to teach children music the way we teach mathematics, we would only succeed in putting most of them off for life. It is by hearing musical notes, melodies, harmonies and rhythms that even the most musical are able to reach the stage of reading and writing music silently in their minds.

So why are children still taught mathematics as a pencil and paper exercise which is usually somewhat solitary? For most of us mathematics, like music, needs to be expressed in physical actions and human interactions before its symbols can evoke the silent patterns of mathematical ideas (like musical notes), simultaneous relationships (like harmonies) and expositions or proofs (like melodies).

Regretfully I hold Mathematicians (with a capital M) largely to blame for this. They are so good at making silent mathematics on paper for themselves and each other that they have put this about as what mathematics is supposed to be like for everyone.

We are all the losers. Music is something which nearly everyone enjoys hearing at a pop, middle-brow or classical level. Those who feel they would like to learn to perform it are not frightened to have a go, and those who perform it well in any of these varieties are sure of appreciative audiences. But Mathematicians have only minority audiences, consisting mostly or perhaps entirely of other Mathematicians. The majority have been turned off it in childhood. For these, the music of mathematics will always be altogether silent.

May I finish by asking a question? I am hopeful that many, probably most, readers of *Mathematics Teaching* will be in agreement with this general line of thinking. But can anyone tell me how we may get this message to those who need it most? By this, I mean the many who teach mathematics to children in their early years almost entirely as pages of ‘sums’ on paper; and who - because they think that it is for specialist teachers of mathematics, or for some other reason - do not read this journal. I ask this urgently: how?

## References

[1] Gilbert Ryle: *Concept of Mind* (Hutchinson 1949, p. 20)

[2] Jerome Bruner: *Toward a Theory of Instruction* (Norton 1968, p. 113)

[3] See *MT* No. 77

[4] Howell, Walker, Fletcher: *Mathematics for Schools* Teachers' Resource book Level II 5&6 (Second edition. Addison-Wesley 1981, p. 57). I thought perhaps this would be dropped in the new edition. Instead the introductory *Understanding the Series* has been dropped. It contained the following: "...we have stressed the thorough understanding of concepts ..." (1972 edition p.2)

[5] John Holt: *How Children Fail* (Penguin 1965, pp. 73-74)

[6] Margaret Donaldson: *Children's Minds* (Fontana 1978, p. 55)

[7} *Ibid*, p.18.

[8] Frank Smith: *Understanding Reading* (first edition, Holt, Rinehart, Winston 1971, p. 168)

[9] A.P.U.: *Mathematical Development* Primary survey report No 1. (H.M.S.O. 1980, p. 99)

[10] Jan Stanfield & Jerzy Cwirko-Godycki: *Impact Maths* (Evans 1981)

[11] Benchara Branford: *Mathematical Education* (1908, p. 9) - with thanks to Richard Harris

*This article was written before the Cockcroft Report was published! - Ed.