[8f]

Internal Combustion Engine: Atkinson Cycle

Efficiency and Power Comparison to Otto Cycle

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Abstract

The most common thermodynamic cycle used in modern internal combustion engines is the Otto cycle. [1] This cycle provides satisfactory work for the majority of driving situations, and has become the most popular cycle for automobiles. The downside to the Otto cycle is in efficiency. The mechanics of typical Otto cycle engines constrain the compression ratio to be the same as the expansion ratio because of crank design and valve timing. Shortly after the Otto cycle's conception in the mid 1800's, James Atkinson proposed a similar cycle that altered the crank design and allowed for greater efficiency. This cycle was aptly named the 'Atkinson Cycle.' [2] Atkinson designed an asymmetric crank design that allowed for a longer expansion stroke compared to the compression stroke. The downside to the Atkinson cycle is that it has low torque output compared to the Otto cycle, so it has been largely disregarded as an internal combustion engine cycle. However, with the onset of hybrid gasoline-electric cars, the low torque output of the Atkinson cycle can be supplemented by the high torque of electric motors, specifically at low RPM. In this report, the advantages and disadvantages of the Atkinson cycle are compared with those of the conventional Otto cycle, and proposals for improvements, namely by incorporating superchargers, are made that make the Atkinson cycle the superior cycle for typical civilian use. We will use thermodynamic efficiency and financial costs as our selection criteria for deciding which configurations are the most desirable.

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P	Power		T	Temperature
rpm	Rotations per minute		p	Pressure
t	Temperature		u	Specific Internal Energy
ε	Exergetic Efficiency		U	Internal Energy
u	Internal energy		h	Specific Enthaply
v_r	Relative specific volume		H	Enthalpy
r_c	Compression ratio		h_f^0	Enthalpy of Formation
r_e	Expansion ratio		v	Specific Volume
m	Mass		V	Volume
AF	Air/fuel ratio		v_r	Reduced Specific Volume
R	Ideal gas constant		w	Specific Work
M	Molar mass		W	Work

n_{cyl}	Number of cylinders in engine		q	Specific Heat
W_{cyc}	Work per cycle		Q	Heat
W_{sc}	Work used by supercharger		η_1	First Law Efficiency
W_{act}	Actual work out, accounting for supercharger		η_2	Second Law Efficiency
q_{in}	Heat in per mass		s	Entropy
Q_{in}	Heat in		e_d	Exergy Destruction
y_i	Molar ratio of the i-th component		$\bar{e}_{ch}$	Chemical Exergy

3 – Introduction

3.1 – Otto Cycle Overview

Modern cars use engines designed for the Otto cycle. An ideal Otto cycle consists of four processes: two isentropic and two isochoric. When the Otto cycle is implemented in an internal combustion engine, two more isobaric processes are added for the intake and exhaust. Figure 1 shows the full cycle on a P-v diagram. In the ideal four-process cycle, 2-3 is isentropic compression, 3-4 is isochoric heat addition (combustion), 4-5 is adiabatic expansion (the power stroke), and 5-6 is isochoric heat rejection. For internal combustion engines, the two added isobaric process are 2-1 (exhaust) and 1-2 (intake). The key engine components that define the cycle are the piston, crankshaft, intake valve, exhaust valve, and spark plug. [2] The moving piston acts to both compress and expand the gas, as well as acting as the delivery system for work. The crankshaft determines the maximum and minimum displacements of the piston, controlling volume and

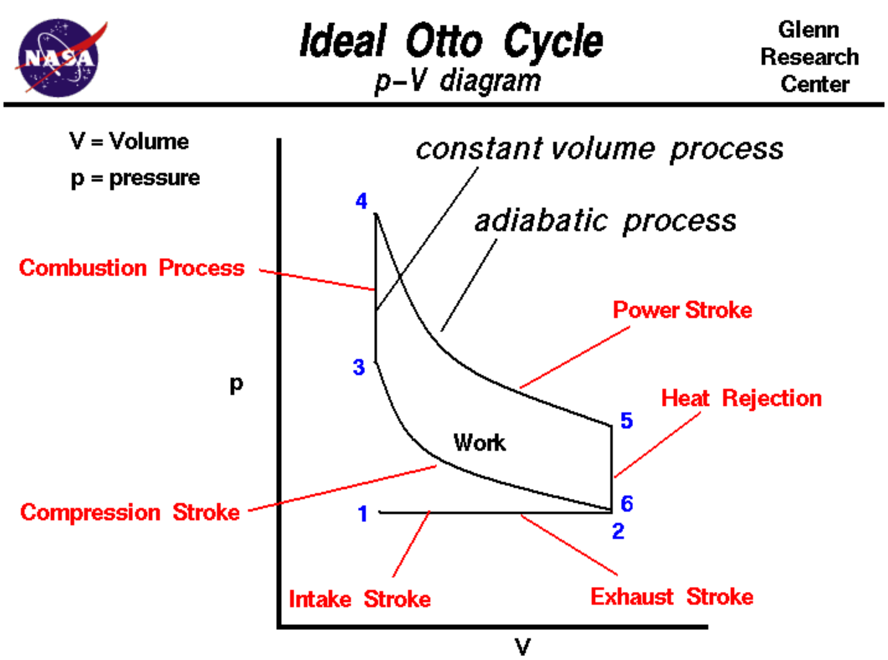


Figure 1 Ideal Otto Cycle [1f]

compression/expansion ratios. The spark plug ignites the fuel, creating the heat input to the cycle, and the intake and exhaust valves determine the timing of the isobaric processes, as well as the supply of fresh fuel and the rejection of spent fuel. [3]

3 – Introduction

3.2 – Atkinson Cycle Overview

In the original 1882 Atkinson model, a complicated cranking mechanism allowed the piston to move further during the expansion stroke than the compression stroke. [5] Figure 2 shows this change relative to the Otto cycle. Point 4O switches to point 4A so the constant volume heat rejection (4O-1) is removed in exchange for a longer power stroke (3-4A) to low pressure. After the exhaust and intake strokes (omitted from Figure 2), process 4A-1 begins the compression stroke at constant pressure, and then follows the rest of the Otto Cycle processes described above. The increase in work can be seen by the larger area enclosed by the cycle, and as the heat input (process 2-3) is the same, the efficiency of the Atkinson Cycle is greater according to

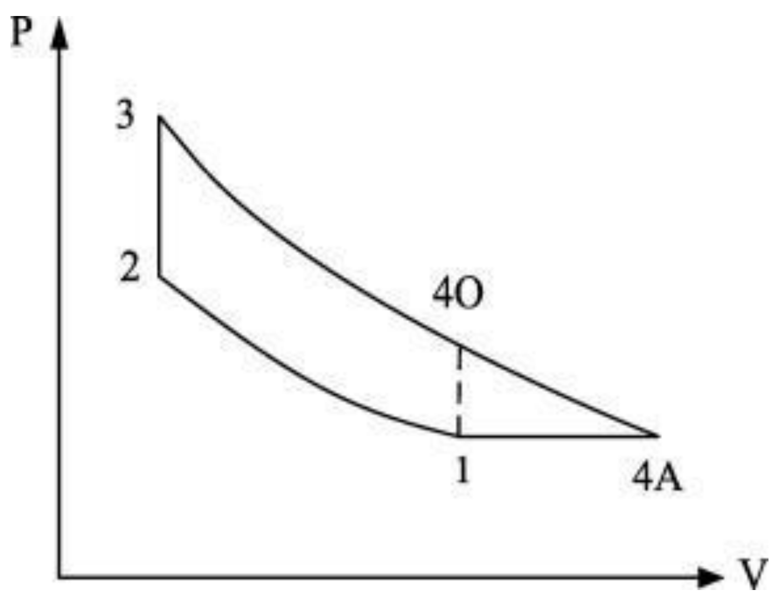


Figure 2 Atkinson vs. Otto Cycle [2f]

$$\eta = \frac{w_{net}}{q_{in}} \qquad \text{eq. 1}$$

This is also what gives it lower torque, as the force of the expansion stroke is lower for the longer expansion distance. The low torque, combined with the large and complicated cranking mechanism, made the Atkinson cycle less appealing than the Otto cycle, which is why the vast majority of modern cars use the Otto cycle. However, in recent years, as oil prices climbed and people became more concerned about efficiency, research into practical applications of the Atkinson cycle began yielding usable results. Today the higher expansion-to-compression ratio can be achieved by implementing only a small modification to an engine designed for an Otto cycle. By leaving the intake valve open for the first part of the compression stroke, a constant pressure process is added, replacing the first part of the compression stroke. The dotted line in figure 3 represents the modification to the cycle obtained by leaving the intake valve open. Process 1-2 becomes processes 1-2A-3A and the compression ratio is effectively reduced from $\frac{V_D}{V_{CC}}$ to $\frac{V_{D2}}{V_{CC}}$. This effectively reduces the compression ratio, while keeping the expansion ratio the same, attaining the increased expansion-to-compression ratio of the Atkinson cycle. The work per cycle decreases, but so does the heat required. The overall effect is to increase efficiency while sacrificing engine output.

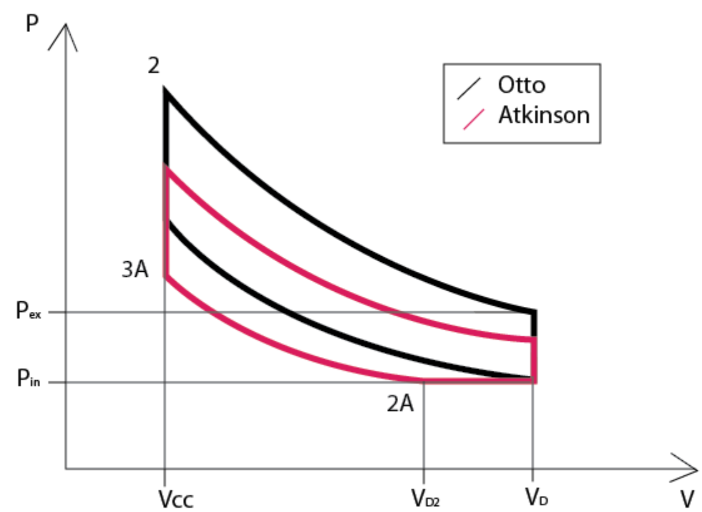


Figure 3 Atkinson Cycle Achieved Through Late Valve Closure

This poor output of the Atkinson cycle is its major drawback, and the reason why it is not widely used as an internal combustion engine. The most popular solution to this problem is coupling the gasoline engine with an electric motor to assist it in the low-torque rpm ranges. These gasoline-electric hybrid cars have much better fuel economy than traditional vehicles, and are still able to provide the torque necessary for quick acceleration. The first successful hybrid car in the United States was the Honda Insight in 1999, followed shortly by the very popular Toyota Prius in 2000. [4] To see how hybrid cars can compete with Otto cycle engines in terms of torque while still maintaining a significantly higher efficiency, we will look at the Toyota Prius in most of our calculations. We will examine ways to supplement the Atkinson cycle's low power and torque output, as well as ways to further improve its efficiency, showing that it is superior to the Otto cycle for normal civilian use. We will supplement both cycles with a supercharger and perform the necessary calculations and observe the performance of each cycle with a supercharger.

4 – First Law Thermo-mechanical Analysis of Internal Combustion Engine

4.1 – Introduction and Assumptions

To analyze the performance of the Otto cycle, we performed a thermodynamic analysis based on the First and Second Laws of Thermodynamics. Our aim was to find the value of the total work output of the engine, given a known compression ratio and a heat input. Before we begin the actual analysis, we list out all of our assumptions.

1. Compression and expansion stages are isentropic
2. Dry-air is the working fluid in the closed Otto cycle. Fuel is ignored
3. Kinetic and potential energy effects are neglected
4. Air is treated as an ideal gas
5. Air comes in at 298 K and 1 atm

Starting from point 1 in Figure 2, knowing the compression ratio and heat input, and assuming isentropic compression and expansion, we can find the temperatures and internal energy values at each point. The total specific work of the cycle can be evaluated as

$$w_{cycle} = |w_T| + |w_c| = (u_3 - u_4) - (u_2 - u_1) \qquad \text{eq. 2}$$

We evaluated the thermal efficiency as a measure of the performance of the cycle. Thermal efficiency is generally defined as the ratio of the desired value to the input cost to obtain the desired quantity. For power cycles, this is simply the total work of the cycle divided by the heat input, or

$$\eta = \frac{w_{cycle}}{q_{in}} \qquad \text{eq. 3}$$

The value of q_{in} for an Otto cycle is

$$q_{in} = u_3 - u_2 \qquad \text{eq. 4}$$

In order to compare the Otto cycle modeled after the Toyota Camry, the Otto cycle with a supercharger, the Atkinson cycle with an electric motor modeled after the Toyota Prius, and the same Atkinson cycle engine with a supercharger, we decided on two different conditions of engine performance, a high power, high speed load and then a more moderate load. Through the following calculations, we are able to determine how much heat is needed from the combustion.

$$P_1 = 120\ hp \qquad rpm_1 = 6000 \frac{rotations}{minute}$$

$$P_2 = 70\ hp \qquad rpm_2 = 3000 \frac{rotations}{minute}$$

4 – First Law Thermo-mechanical Analysis of Internal Combustion Engine

4.2 – Otto Cycle

Starting with dry air at ambient conditions, we can find the following values for internal energy using thermodynamic tables[6].

$$t_1 = 536.37°R \qquad p_1 = 1atm$$

$$u_1(@t_1) = 91.42\frac{btu}{lb} \qquad v_{r1}(@t_1) = 145.22$$

With isentropic compression from state 1, we determine the internal energy at state 2 by determining the relative volume with the compression ratio, which is equal to the expansion ratio in the Otto cycle, to find the internal energy through interpolation with the chart of air properties. The compression and expansion ratio is taken from the product information for the 2013 Camry, published by Toyota[7].

$$r_c = r_e = 10.4$$

$$v_{r2} = \frac{v_{r1}}{r_c} = \frac{145.22}{10.4} = 13.96 \qquad \text{eq. 5}$$

$$u_2 = u_2(@v_{r2}) = 233.34\frac{btu}{lb}$$

The mass of the air taken into the cycle, assuming the fuel makes an insignificance difference, can be determined using the ideal gas law for each cycle.

$$m = v * \frac{p_1 * M}{R * t_1} = \frac{37.99in^3 * 1atm * 28.97\frac{mole}{lb}}{1545\frac{ft-lbf}{R*lb-mole} * 536.37R} * \frac{14.696\frac{lb}{in^2}}{1atm} * \frac{1ft}{12in} = 0.001627lb \qquad \text{eq. 6}$$

The amount of work produced by each cycle can be determined by the power, rotation speed, number of cylinders, and mass of the air.

$$W_{cyc} = \frac{2*P}{rpm*n_{cyl}*m} = \frac{2*120hp}{6000\frac{rotations}{minute}*4*0.001625lb}*\frac{42.42\frac{btu}{min}}{1hp} = 260.725\frac{btu}{lb} \quad \text{eq. 7}$$

The amount of heat produced by combustion per mass of air can be found using the internal energies of state 3 and 4 with eq. 4. The amount of work per cycle is also equal the difference in internal energy between states 3 and 4. Using eq. 2 and combining it with eq. 4, we get the following equation.

$$W_{cyc} = u_3 - u_2 - u_4 + u_1 = q_{in} - u_4 + u_1 = 260.725\frac{btu}{lb} \quad \text{eq. 8}$$

To find the internal energy of state 4, we need to relate it with a known value. We know that there is isentropic expansion between state 3 and 4. Using the three equations below and eq. 8, we find eq. 10

$$v_{r3} = v_r(@u_3) \qquad u_4 = u(@v_{r4})$$

$$v_{r4} = v_{r3} * r_e \quad \text{eq. 9}$$

$$u_4 = u(@[r_e * v_r(@[q_{in} + u_2])]) \quad \text{eq.10}$$

With eq. 10 and eq. 8, we can use Matlab to perform iterations to solve for the qin and u4, as there are two unknowns and two equations. The internal energy of state 4 is then used to interpolate the temperature of the gas at state 4, the exhaust.

$$q_{in} = 470.368\frac{btu}{lb} \qquad t_4 = t(@u_4) = 1675.1°R$$

With the heat entering the system and the work being done calculated, we can determine the efficiency of the cycle using eq. 1.

$$\eta = \frac{260.725 \frac{btu}{lb}}{470.368 \frac{btu}{lb}} = 0.5543$$

Since the mass entering the engine varies and q_{in} is dependent on mass, we calculate the amount of heat per cycle.

$$Q_{in} = q_{in} * m = 470.368 \frac{btu}{lb} * 0.001627 lb = 0.7651 btu \qquad \text{eq. 11}$$

4 – First Law Thermo-mechanical Analysis of Internal Combustion Engine

4.3 – Atkinson Cycle with Electric Motor

As mentioned before, the main difference between the Atkinson cycle and the Otto cycle is the fact that the expansion ratio is higher than the compression ratio, rather than being the same. This is done by leaving the intake valve open during part of the intake stroke so some of the air fuel mixture is pushed back out and the compression stroke is shorter, decreasing the compression ratio. Thus, the mass that would be calculated with eq. 6 is decreased by the ratio of the compression ratio to expansion ratio. The expansion ratio is taken from the 2013 Prius product information[8], and the compression ratio is a variable we will investigate.

$$r_c = 7.5 \qquad r_e = 13$$

$$m_{atk} = m * \frac{r_c}{r_e} = 0.00118lb * \frac{7.5}{13} = 0.000688lb \qquad \text{eq. 12}$$

Since the Toyota Prius and several other hybrid models use an electric motor, which is powered by a battery that stores the electricity generated by the engine, in addition to an engine, we need to account for the additional power from the motor and loss of power from charging the battery. We assume that the motor operates at maximum power and the battery is removing the maximum power it can from the engine, taken from the Prius product information[8] as well, and used to adapt eq. 8.

$$W_{cyc,motor} = \frac{2*\left(P-\left(P_{motor}-P_{battery}\right)\right)}{rpm*n_{cyl}*m} \qquad \text{eq. 13}$$

$$= \left(\frac{2 * \big(120hp - (80hp - 36hp)\big)}{6000 \frac{rotations}{minute} * 4 * 0.000688lb} \right) * \frac{42.42 \frac{btu}{min}}{1hp} = 395.907 \frac{btu}{lb}$$

The remainder of the calculations is the same as that of the Otto cycle.

4 – First Law Thermo-mechanical Analysis of Internal Combustion Engine

4.4 – Supercharger as Potential Improvement

The supercharger compresses the air fuel mixture entering the cylinder during intake to increase the amount of fuel that can be combusted for more heat and thus more power. We model the supercharger as a compressor that uses a set amount of work produced by the engine at a set efficiency.

$$W_{sc} = 60\frac{btu}{lb} = W_{cyc} - W_{act} \qquad \text{eq. 14}$$

$$\eta_{sc} = 0.7$$

With the supercharger, the pressure now entering the engine is determined using eq. 15, with ambient air being state 0 and k, the specific heat ratio, being 1.4. The new temperature at state is found using the ideal gas law.

$$p_1 = p_0 * \left(W_{sc} * \eta_{sc} * \frac{k-1}{k*R*t_0} + 1\right)^{\frac{k-1}{k}} \qquad \text{eq. 15}$$

$$= 1atm * \left(60\frac{btu}{lb} * 0.7 * \frac{1.4-1}{1.4 * 0.06855\frac{btu}{lb*R} * 536.37°R} + 1\right)^{\frac{1.4-1}{1.4}} = 2.687atm$$

Since the supercharger is not 100% efficient, we calculate the temperature accounting for the inefficiency.

$$t_{1s} = t_0 * \left(\frac{p_1}{p_0}\right)^{\frac{k-1}{k}} = 536.37°R * \left(\frac{2.687atm}{1atm}\right)^{\frac{1.4-1}{1.4}} = 711.42°R \qquad \text{eq. 16}$$

$$t_1 = t_0 + \frac{t_{1s} - t_0}{\eta_{sc}} = 711.42°R + \frac{580.13°R - 536.37°R}{0.7} = 786.45°R \qquad \text{eq. 17}$$

With the new temperature and pressure found for the air entering the engine, we can proceed with the same analysis as for the Otto cycle or the Atkinson cycle.

4 – First Law Thermo-mechanical Analysis of Internal Combustion Engine

4.5 – Optimization and Comparison of Atkinson and Otto Cycle

Since the compression ratio for the Atkinson cycle can be varied, we optimize the ratio to require the least heat in needed to meet the specifications as possible. Looking at figure 4 and figure 5, we see that for each condition, Qin will drop before rising again as compression ratio increases, fluctuating as the mass and qin in eq. 11 vary with compression ratio.

Figure 4 Compression Ratio vs. Qin for 120hp, 6000rpm

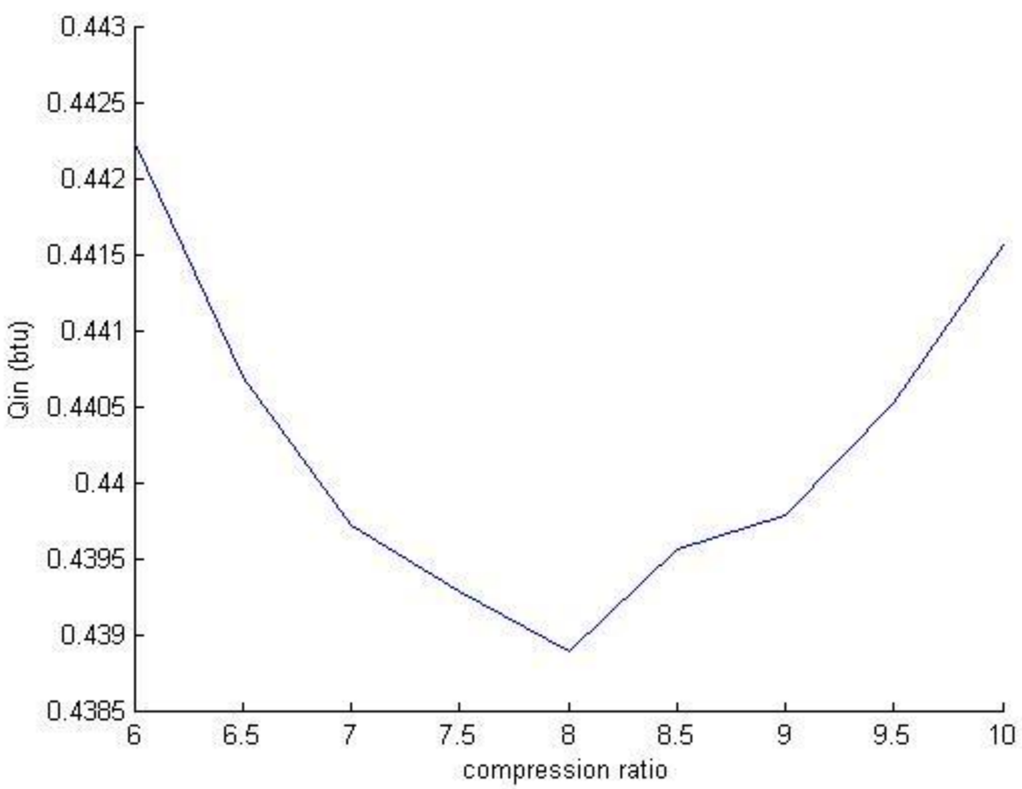
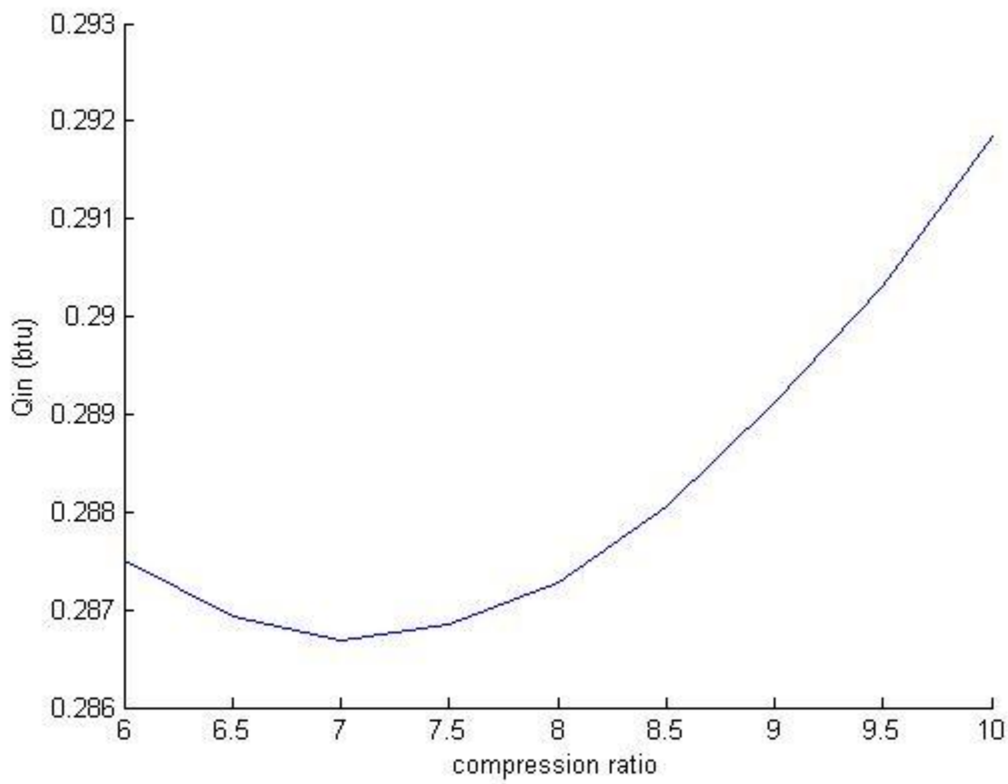


Figure 5 Compression Ratio vs. Qin for 70hp, 3000rpm



From these two figures, we see that the compression ratio with the lowest Qin is about 8 at the higher load and 7 for the moderate load. We take the average of the two, 7.5 and use that for our compression ratio for the Atkinson cycle in our analysis; the results are shown in table 1.

Table 1: Otto and Atkinson cycle analysis results					
		Otto: Camry	**Otto +Super Charger**	**Atkinson +Motor: Prius**	**Atkinson +Super Charger**
	r_c	10.4	10.4	7.5	7.5
	r_e	10.4	10.4	13	13
	m (lb/cycle)	0.001627	0.002981	0.000688	0.001243
	W_{sc} (btu/lb)	0	60	0	60
120 hp 6000 rpm	**W_{act} (btu/lb)**	260.725	142.253	395.907	341.068
	q_{in} (btu/lb)	470.368	370.803	647.549	654.834
	Q (btu)	0.7651	1.1054	0.4393	0.8142
	η	0.5543	0.3836	0.6114	0.5208
70 hp 3000 rpm	**W_{act} (btu/lb)**	304.179	165.962	270.884	397.913
	q_{in} (btu/lb)	553.782	416.212	422.849	760.311
	Q (btu)	0.9007	1.2408	0.2869	0.9453
	η	0.5493	0.3987	0.6406	0.5234

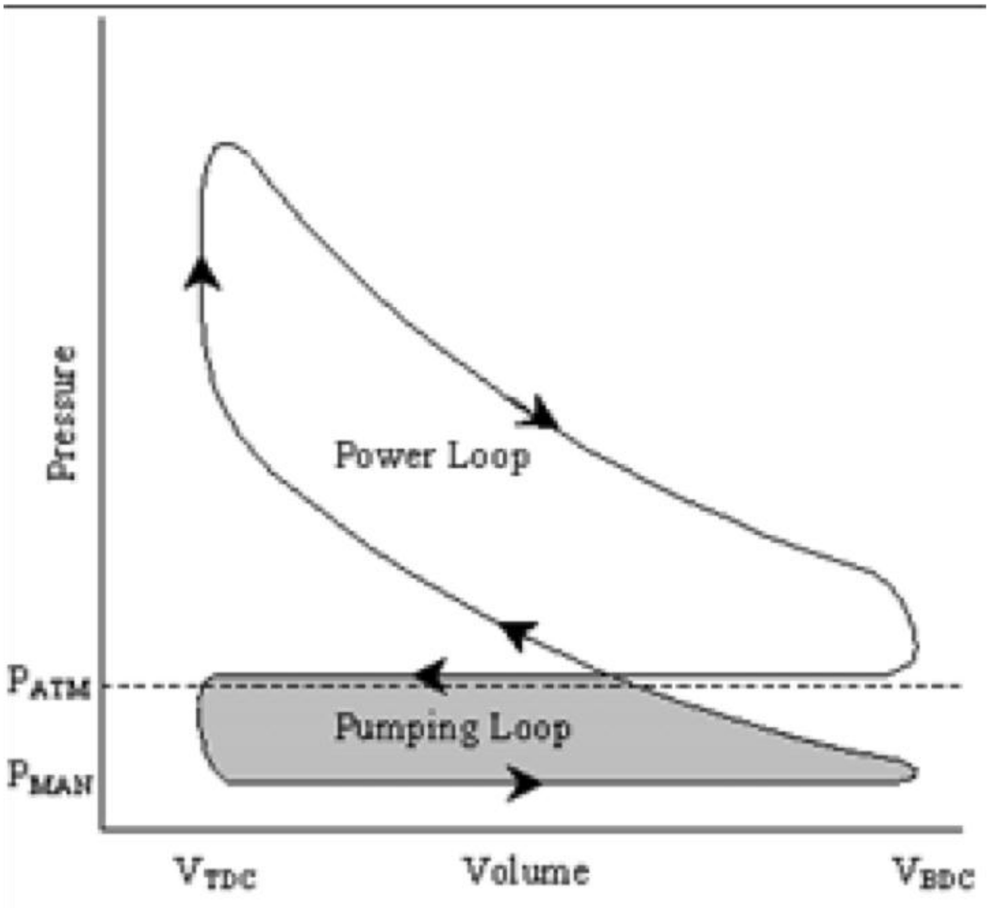
From Table 1, we can see that the Atkinson cycle engine with the motor has the best efficiency, followed by the standard Otto cycle engine. By adding the supercharger to the engines, we are able to decrease the amount of heat needed from combustion per pound of air, but because more compressed air and fuel enters the cycle at a time, bringing more heat. The supercharger also extracts some work from the engine to compress the air, further decreasing the efficiency.

4 – First Law Thermo-mechanical Analysis of Internal Combustion Engine

4.6 – Variable Valve Timing – Pumping Losses

One of the big efficiency drains in Otto cycle engines is pumping losses. Pumping losses refer to work the engine has to do to compress or intake charge. In the Otto Cycle, the pumping losses for intake are unavoidable. In the Atkinson cycle however, they can be essentially eliminated if the valve timing is made adjustable in real time. First, let's see how pumping losses change with engine load. Engines are designed for a maximum output – some torque at some rpm. However, the full extent of this load is only realized during acceleration up slopes. For the vast majority of the time – during acceleration or cruising on flat ground – the engine is at partial load, meaning that less power output is required of it. [9] To reduce the power output, fuel intake must be reduced. In traditional engines, this is achieved through a throttle attached to the intake valve that restricts intake according to the load of the engine. The reduced mass of gas within the piston during intake creates lower pressure, and forces the engine to suck in air, using work to pull the piston against the pressure differential. This can be a significant amount of work. Figure 4 shows the pumping losses associated with partial loading. The exhaust stroke occurs at atmospheric pressure and the intake stroke occurs below atmospheric pressure due to the throttle. The grey area represents negative work being done, contributing to losses in efficiency.

Figure 6. Pumping Losses at Partial Loading *[6f]*



Using the Prius engine specs for data on engine displacement, we will calculate the pumping losses associated with an Otto cycle using about 40% of maximum fuel intake. This corresponds to slightly above cruising speed at 65 mph for a typical sedan. [2] This means that only 40% of the maximum air/fuel mixture is drawn in during the intake process. The throttle maintains a pressure of about 0.4 bar as opposed to 1 bar to accomplish this, which is where the loss of work comes from:

$$W = \int PdV \qquad \text{eq. 18}$$

A couple key assumptions we will make are first, that these are two constant pressure pumping processes (exhaust and intake) and that their endpoints are the maximum and minimum volumes of the piston-cylinder assembly. The second assumption is that these two processes do not affect the four-process Otto Cycle analyzed previously; they are acting as a separate two-process cycle in series with the Otto Cycle. Now, working in gage pressures, exhaust at 1 bar becomes 0 bar, and intake at 0.4 bar becomes negative 0.6 bar. Pressure is constant for each process, so it can come out of the integral:

$$W = \left\{0[bar] * \int_{exhaust} dV\right\} - \left\{0.6[bar] * \int_{intake} dV\right\} \quad \text{eq. 19}$$

Now converting bars to kilopascals and using the maximum and minimum volumes for the Prius engine of 1.798E-3 and 1.383E-4 cubic meters:

$$W = \{0\} - \left\{0.6[bar] * \frac{100}{1}\frac{[kPa]}{[bar]} * (0.001798[m^3] - 0.0001383[m^3])\right\}$$

$$= 0.09958\ [\text{kJ}] = 99.58[\text{ J}]$$

So for freeway driving using an Otto cycle engine of similar size to the Prius', 99.58 Joules are lost per cycle per cylinder. The first law efficiency reported for the four-process Otto cycle must be reduced when we consider the extra exhaust and intake processes. We assume this to be an average engine load for most people, as during cruising in city driving, the load will be less, but during acceleration, the load will be greater. As people tend to be inefficient drivers, the 40% of maximum fuel use is actually about 10% higher than needed during freeway cruising. Because we assume that these processes are independent of the four-process Otto cycle, we subtract this work done by the engine in "pumping losses" from the originally calculated work to recalculate the first law efficiency. As calculated above, the full load engine output is 1146 KJ/kg. To find the energy per cycle, we must multiply this by the mass of air per cycle. Using the temperature of air at intake (298 K), the density of air is 1.184 kg/m^3 . Using the volume per cylinder calculated above of 1.66E-3 m^3 ,

$$m = \rho V \qquad \text{eq. 20}$$

$$m = (1.184)\frac{[kg]}{[m^3]} * (0.00166)[m^3] = 0.001965\ [kg]$$

Multiplying this by the engine output per kg at full load, the output per cycle is

$$1146\frac{[KJ]}{[kg]} * 0.001965[kg] = 2.252\ \frac{KJ}{cycle}$$

Now looking at the 40% of maximum fuel consumption typical of highway driving, we calculate the engine output using our first law Otto Cycle analysis above to be 20% of the maximum, or 0.45 $\frac{KJ}{cycle}$. This means that pumping losses create a loss of work of

$$\frac{0.0996\ [KJ]}{0.45\ [KJ]} = 22.1\%$$

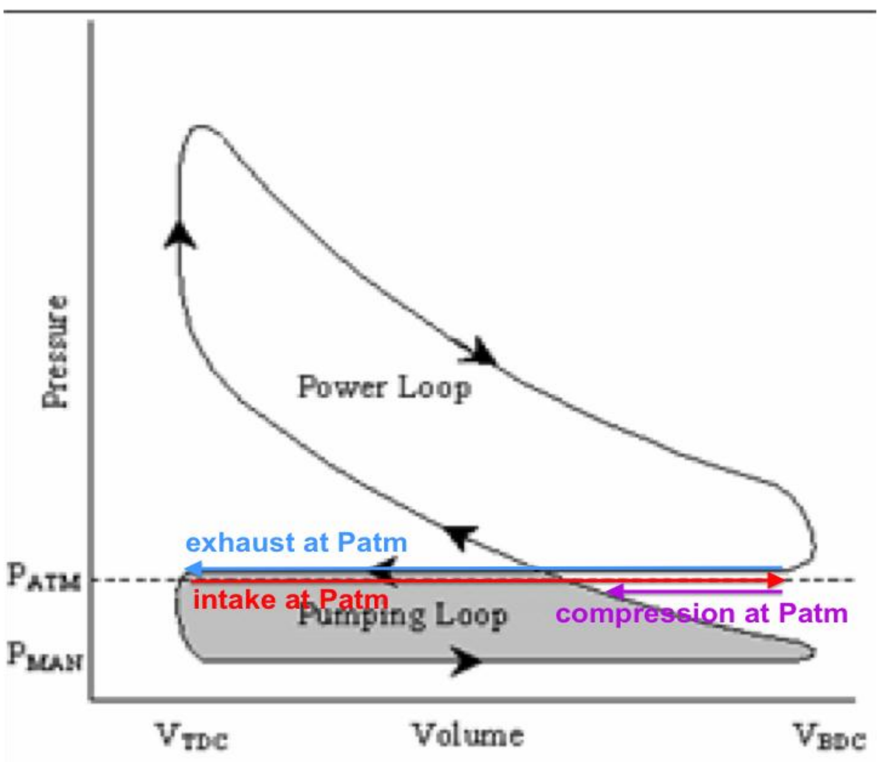
This is a significant number, typical of the Otto Cycle, and one of the main complaints with the Otto Cycle in an internal combustion engine. [2] This brings our first law efficiency down to

$$\eta = \frac{W_{power\ stroke} - W_{compression\ stroke} - W_{intake}}{Q_{in}} \qquad \text{eq. 21}$$

So the addition of the intake work into this equation reduces the efficiency from 54.9% down to 42.8%.

Now let's look at how variable valve timing in an Atkinson cycle eliminates these losses. As stated before, the way an Atkinson cycle is achieved in modern automobiles is by delaying the closure of the intake valve, allowing for a constant pressure process to replace the beginning of the compression stroke (Figure 3). Recalling that the reason for the pumping losses is that lower fuel intake is desired for partial engine loading, the Atkinson cycle's low power output is now used to its advantage. It allows for intake to occur at atmospheric pressure, filling the cylinder with surplus fuel, but expelling this surplus as the intake valve is left open during the beginning of compression. The variable valve timing system allows the valve to close precisely when the required amount of fuel is present in the cylinder, initiating compression. This eliminates the pumping losses that so negatively affected static-valve systems. Figure 7 shows the variable valve timing strokes superimposed over the static-valve system.

Figure 7. Variable Valve Timing Solution *[7f]*



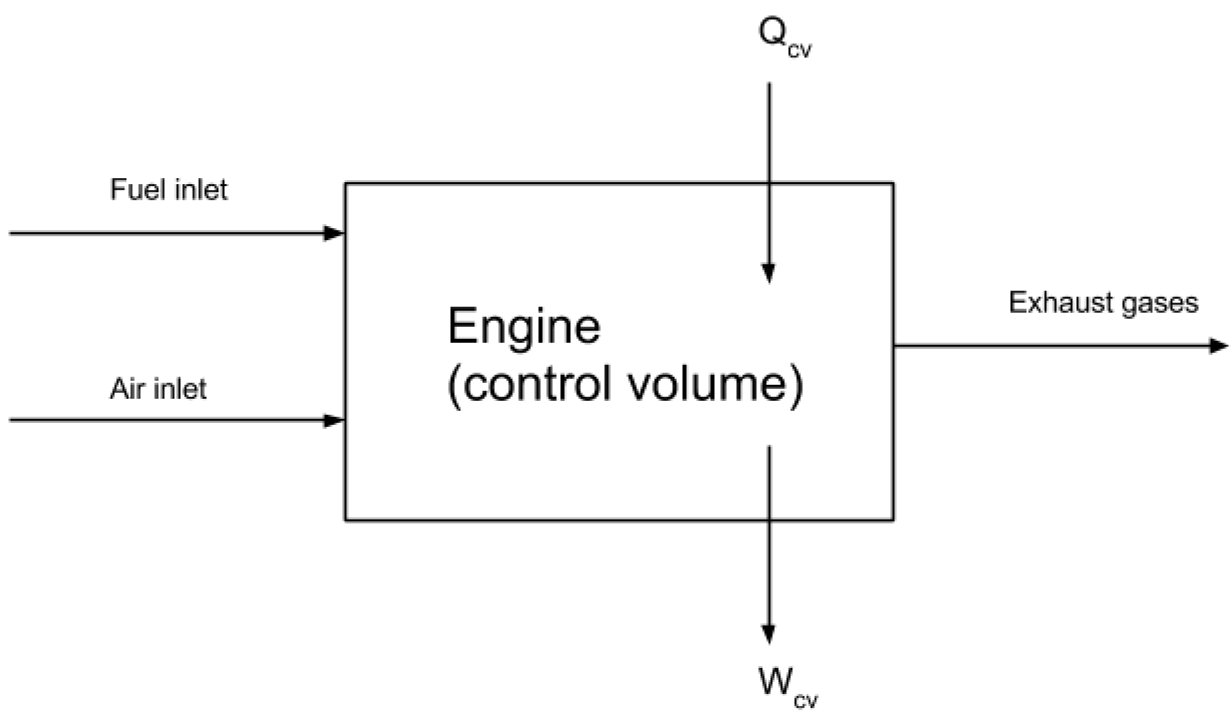
5 – Combustion Analysis

From the first law analysis, we also obtain T_4, which is the temperature of the exhaust gases. Based on this, we can perform a combustion analysis, and Figure 8 shows a schematic of the control volume used for the combustion analysis. We list out all of our assumptions for our combustion analysis.

1. The control volume encloses the engine, operating at steady state
2. Kinetic and potential energy effects are neglected
3. Dry air as an ideal gas is the combustion air (21% O_2, 79% N_2 on a molar basis)
4. Fuel is treated as pure octane, C_8H_{18}, with no impurities
5. The combustion process is adiabatic (no heat losses)
6. Air and fuel enter separately at atmospheric conditions (298 K, 1 atm)
7. Complete combustion with no carbon monoxide products
8. Water exits the exhaust as water vapor
9. The reference environment is the atmospheric environment (298 K, 1 atm)
10. The engine has 100% volumetric efficiency (piston-cylinder always filled with air)

Note that assuming adiabatic combustion is generally inaccurate, since in reality most engines release significant quantities of heat. However, we made this assumption for two reasons. Firstly, it greatly simplifies calculations. Finding the heat transfer from the engine would require complicated heat transfer analysis, which is outside the focus of this analysis, as well as extensive knowledge of the engine's geometry and material properties. Furthermore, we would need to know the temperature of the boundary where heat transfer occurs in order to calculate entropy, which we will show later. Secondly, assuming adiabatic combustion means that no heat is lost to the environment, which results in obtaining the greatest amount of heat input into the air. This gives us an upper bound on the performance of our system, which we can use to compare to our other configurations.

Figure 8 Schematic of engine as a control volume for combustion analysis



Combustion analysis allows us to calculate the air/fuel ratio based on a given q_{in}. From the ideal gas law

$$pv=RT \quad \text{eq. 22}$$

we can determine the specific volume, so we can determine the mass of air passing through a single cycle given a cylinder volume. With the air/fuel ratio, we can then determine the mass of fuel per cycle by the definition of air fuel ratio,

$$AF = \frac{mass\ of\ air}{mass\ of\ fuel} \qquad \text{eq. 23}$$

The chemical equation for the complete combustion of octane with dry air is given as

$$C_8H_{18} + 12.5x(O_2 + 3.76N_2) \rightarrow 8CO_2 + 9H_2O + 47xN_2 + (x\text{-}1)O_2$$

Here, we define x to be the fraction of theoretical air. Note that for this equation, we must have $x \geq 1$ in order to have complete combustion. In the case of incomplete combustion, we would have the chemical equation

$$C_8H_{18} + 12.5x(O_2 + 3.76N_2) \rightarrow 9H_2O + (12.5x - 4.5)CO_2 + 12.5(1\text{-}x)CO + 47N_2$$

where $0.36 \leq x \leq 1$. This assumes hydrogen prioritizes over carbon during oxidation, so all the hydrogen atoms will be fully oxidized to water molecules, and that we have enough air to oxidize all the hydrogen atoms. We will show only the equations for complete combustion, but the analysis would follow similarly for incomplete combustion, with the main difference being that we would have to include the enthalpy and entropy values for carbon monoxide, a product of incomplete combustion not present in complete combustion. The value x is directly related to the air/fuel ratio, and we can calculate the air fuel ratio from x as

$$AF = x(12.5 \cdot 4.76)\frac{\text{kmol air}}{\text{kmol fuel}} \cdot \frac{28.97\text{ kg air.}}{1\text{ kmol air}} \cdot \frac{1\text{ kmol air}}{114.2\text{ kg air}} = 15.09x\ \frac{\text{kg air}}{\text{kg fuel}} \qquad \text{eq. 24}$$

Based on the first law on a per kilomole basis, an energy balance gives

$$\bar{q}_{cv} - \bar{w}_{cv} + \sum_r n_r \bar{h}_r - \sum_p n_p \bar{h}_p = 0 \qquad \text{eq. 25}$$

For each species, its enthalpy value is given as

$$\bar{h}_i = h_f^{\,0} + \Delta\bar{h} = h_f^{\,0} + (\bar{h}(T_4) - \bar{h}(T_{ref})) \qquad \text{eq. 26}$$

The reference environment is 298 K, and after expanding the summation expressions and disregarding terms that equal 0, we can express the total energy balance by

$$\bar{w}_{cv} + 8\bar{h}_{CO_2} + 9\bar{h}_{H_2O} + 47x\bar{h}_{N_2} + 12.5(x-1)\bar{h}_{O_2} - \bar{h}^0{}_{f_{C_8H_{18}}} = q_{cv} \qquad \text{eq. 27}$$

We are given T_4 and the cycle work per kg of air, so we can solve for x, and thus solve for the air/fuel ratio if we assume adiabatic combustion by setting $q_{cv} = 0$. Rearranging terms, we can get an equation solving for x.

$$x = \frac{-\bar{w}_{cv} - 8\bar{h}_{CO_2} - 9\bar{h}_{H_2O} + 12.5\bar{h}_{O_2} + \bar{h}^0{}_{f_{C_8H_{18}}}}{47\bar{h}_{N_2} + 12.5\bar{h}_{O_2}}$$

Once we have calculated x, we can directly calculate the air/fuel ratio from eq. 24, and then calculate the mass of fuel given the mass of air using eq. 23.

As a sample, from the standard Otto cycle analysis, our exhaust temperature is 2243.3 °R = 1246 K, and the work per kg of air is 707 kJ/kg. This gives us enthalpies for each relevant component of

$$\bar{h}_{CO_2} = -393522 + (29661) = -363860 \text{ kJ/kmol}$$

$$\bar{h}_{H_2O(g)} = -241826 + (23147) = -218679 \text{ kJ/kmol}$$

$$\bar{h}_{N_2} = 0 + (19200) = 19200 \text{ kJ/kmol}$$

$$\bar{h}_{O_2} = 0 + (20294) = 20294 \text{ kJ/kmol}$$

We can then calculate x and obtain a value of x = 2.00, an air/fuel ratio of 30.2. Given a mass of 0.74 grams of air per cycle based on the ideal gas law given in eq. 22 and the assumption of 100% volumetric efficiency, we calculate a mass of fuel of 24.4 micrograms per cycle. Note that calculating x requires iterations; in order to obtain the cycle work per kmol of fuel from the cycle work per kg of air, we need the air/fuel ratio, but we also need the cycle work per kmol of fuel to determine the air/fuel ratio. Here are the results from our combustion analysis, showing the fuel consumption rates for each engine condition.

Table 2. Fuel Consumption From Combustion Analysis

		Otto	**Otto+SC**	**Atkin**	**Atkin+SC**
120 hp 6000 rpm	kg fuel/cycle	$2.204 * 10^{-5}$	$3.19 * 10^{-5}$	$1.25 * 10^{-5}$	$2.33 * 10^{-5}$
	kg fuel/s	$11.0 * 10^{-4}$	$15.9 * 10^{-4}$	$6.25 * 10^{-4}$	$11.7 * 10^{-4}$
	kg fuel/hr	3.96	5.74	2.25	4.20
70 hp 3000 rpm	kg fuel/cycle	$2.44 * 10^{-5}$	$3.42 * 10^{-5}$	$1.00 * 10^{-5}$	$2.56 * 10^{-5}$
	kg fuel/s	$6.10 * 10^{-4}$	$8.56 * 10^{-4}$	$2.50 * 10^{-4}$	$6.39 * 10^{-4}$
	kg fuel/hr	2.20	3.08	0.901	2.30

From our results, we see that the Otto cycle requires more fuel overall, but it also has a higher work output, which we showed in Table 1. However, between the lower and higher loads, the fuel consumption increases more for the Atkinson cycle. The Otto cycle requires about 80% more fuel for the higher load, while the Atkinson cycle without the supercharger requires about 250% more fuel for the higher load. We also observe that the supercharger significantly increases fuel consumption for both cycles at both loads.

6 – Second Law Analysis – Exergetic Efficiency

Another useful parameter to measure the performance of the engine is to calculate the exergy destruction and the exergetic efficiency of the engine. Exergy is defined as the maximum potential to do work relative to a reference environment. Calculating the exergy destruction allows us to figure out how much potential to do work was lost from the engine. Minimal exergy destruction indicates minimal potential losses, and harnessing the closest amount of work to the theoretical maximum is the most desirable. Here is a list of the assumptions that we will make, along with all of the other assumptions made during the combustion analysis.

1. All irreversibilities occur during combustion
2. Processes for the surrounding are reversible
3. The pressure of the exhaust gas is at atmospheric pressure
4. Exergy of the combustion air at the inlet is negligible

To calculate the exergy destruction, we first must perform an entropy analysis on the system. The total entropy generated during a process is given by

$$S_{gen}=dS_{sys}+dS_{surr} \qquad \text{eq. 28}$$

Since we are assuming the engine to be operating at steady state, the dS_{sys} term is zero, and we only have the dS_{surr} term. Processes for the surrounding are generally assumed to be reversible, so the total specific entropy generation is given by

$$\bar{s}_{gen} = \sum_p n_p \bar{s_p} - \sum_r n_r \bar{s_r} + \frac{q_{surr}}{T_b} \qquad \text{eq. 29}$$

Note that the heat transfer term is relative to the surroundings, meaning heat transfer from the engine to the surroundings means a positive q_{surr}, and vice versa. We are assuming adiabatic combustion for our analysis, so this term goes to 0. The entropy of a given species at a given temperature and partial pressure is given by

$$\bar{s}_i(T, p) = \bar{s}_i{}^0(T) - \bar{R}\ln\left|\frac{y_i P}{P_{ref}}\right| \qquad \text{eqn. 30}$$

We assume the exhaust pressure is the atmospheric pressure, which is also the reference pressure, so this expression simplifies to

$$\bar{s}_i(T, p) = \bar{s}_i{}^0(T) - \bar{R}\ln y_i \qquad \text{eqn. 31}$$

We assume that the fuel and air enter as separate streams, and we know the temperature at each point, so we can calculate the entropy of each species. For example, from our standard Otto cycle analysis, we obtain entropies at T_4 = 1246 K of

$$\bar{s}_{CO_2} = 265.3 + 17.3 = 282.6 \text{ kJ/kmol}$$

$$\bar{s}_{H_2O} = 229.7 + +16.3 = 246.0 \text{ kJ/kmol}$$

$$\bar{s}_{N_2} = 225.7 + 2.6 = 228.3 \text{ kJ/kmol}$$

$$\bar{s}_{O_2} = 241.0 + 71.0 = 311 \text{ kJ/kmol}$$

From this we can calculate the total entropy generated by including the molar coefficients. The specific exergy destruction is then given by

$$e_d = T_0 s_{gen} \qquad \text{eq. 32}$$

The exergetic efficiency is given by the total work performed divided by the exergy given per cycle. For our case this is

$$\epsilon = \frac{\bar{w}_{cv}}{\bar{e}_F} \quad \text{eq. 33}$$

Here we have ignored the exergy accompanying the inlet combustion air, since the combustion air is near the reference state at the inlet. The exergy of the inlet fuel is the sum of the fuel's thermomechanical and chemical exergies, but since the octane enters as a liquid, we can ignore the thermomechanical exergy. The chemical exergy of the inlet octane is approximately equal to its standard chemical exergy, which has a tabulated value of

$$\bar{e}_{C_8H_{18}} = 5{,}413{,}705 \text{ kJ/kmol}$$

From this, we can calculate the exergetic efficiency. Here are our results from our exergy analysis of each of our engine conditions.

Table 3. Exergy Analysis Results

		Otto	**Otto+SC**	**Atkin**	**Atkin+SC**
120 hp 6000 rpm	exergy destruction (kJ/cycle)	1.156	2.36	.401	.748
	total exergy destruction (kW)	57.8	118	20.1	37.4
	exergetic efficiency	0.428	0.297	0.479	0.403
70 hp 3000 rpm	exergy destruction (kJ/cycle)	1.12	2.32	0.437	.708
	total exergy destruction (kW)	27.9	58.1	10.9	17.7
	exergetic efficiency	0.387	0.277	0.598	0.368

From our exergy analysis, we see that the Atkinson cycle has the higher efficiency and lower irreversibilities compared to the Otto cycle. The supercharger decreases the exergetic efficiency of each cycle, and adding the supercharger also increases the exergy destruction. These results parallel our first law analysis; the Atkinson cycle displays the best overall performance in terms of efficiency and exergy destruction. Note that the efficiency for the Atkinson decreases from the lower load to the higher load, while the efficiency of the Otto cycle increases from the lower load to the higher load. While the efficiency of the Atkinson cycle is still higher at the high load, this trend nevertheless demonstrates one of the fundamental differences between the Otto and Atkinson cycles. The Atkinson cycle sacrifices power for efficiency, but its performance decreases at higher loads, while the Otto cycle is capable of performing at high power outputs while maintaining or improving its efficiency.

7 – Cost Analysis and Recommendation

Ultimately, our efficiency analysis is meaningless in industry without a thorough analysis of its economic feasibility. In order for our analysis to be commercially fruitful, it is important to evaluate whether the money invested in our proposed changes pays off thermodynamically. By evaluating the relationship between thermodynamic and economic benefits, we can have a better understanding of costs.

In the case of the thermodynamic analysis of internal combustion engines, consumers are mostly interested in the engine's power and its fuel consumption. Interest in power is not easily quantifiable as it depends largely on individual consumer's preferences and needs. Fuel consumption, however, is largely a financial issue and can be quantified as the fuel cost savings.

Finally, fuel efficiency is an area of high government interest. Because of ecological concerns, the desire to reduce dependency on foreign oil, and numerous other factors, the government takes a role in costs. In addition to funding research in this field, the government also offers incentives for purchasing consumers. These benefits change frequently depending on regulation and are different from state to state. Also, some of the benefits such as California's carpool incentive are not easily quantifiable. Because of this, these factors were left off our calculations. However, the true final cost of any fuel savings could potentially be lower based on these governmental benefits.

In this thermodynamic analysis, we looked at internal combustion engines running on the Otto and Atkinson cycles. We then addressed weaknesses of each design with auxiliary features such as a hybrid-electric system and supercharger. The cost difference between the two different cycles is negligible since the valve timing is easily adjusted through the engine control unit. In the auxiliary systems, however, the costs of the components are significant and may determine the viability of a certain option.

For the hybrid-electric vehicle system, along with the internal combustion engine running an Atkinson cycle the hybrid system requires an electric propulsion motor, a high-voltage battery pack, and a power control module. The costs of these components depends on the amount that the electric system is involved in supplementing the internal combustion engine with power. [10] The classification of these systems range from micro-hybrid to full-hybrid. The Prius we are analyzing falls into the full-hybrid category. [11] [12] The costs for a standard full-hybrid system's components come out to be approximately $3,000.

Table 4: Fuel Consumption Rates								
	120 hp (6000 rpm)				**70 hp (3000 rpm)**			
	Otto	Otto+SC	Atkinson	Atkinson+SC	Otto	Otto+SC	Atkinson	Atkinson+SC
Fuel rate (kg/hr)	3.9670	5.7399	2.2506	4.1997	2.1962	3.0805	0.9010	2.3005

After our thermodynamic analysis of the standard Otto cycle and the hybrid-electric augmented Atkinson cycle, we came up with fuel consumption rates based on given parameters of necessary power and engine rpm. Our analysis featured the engines operating at approximately full power (120 hp) and approximately half power (70 hp). Interestingly, for the two non-supercharged engines, the different necessary power outputs revealed strengths and weaknesses in each of the systems.

At half load, the hybrid-Ackerman engine consumed fuel at about 41% the rate of that of the Otto cycle engine, and at full load, the Ackerman engine consumes fuel at about 57% of the Otto. This consumption confirms our analysis about the decreasing effectiveness of the Ackerman cycle at high load. However, it is worth noting that normal operating conditions for an engine usually do not reach full load powers. Therefore, under normal (half-load) conditions, the engine uses less fuel. To convert this fuel consumption rate to a more consumer-friendly miles-per-gallon, we look at the following equations.

$$\frac{speed\ of\ vehicle}{rate\ of\ fuel\ consumption(\dot{m}_{fuel})} = distance\ per\ unit\ of\ fuel\ (mpg)$$

$$mpg_{Otto} \times \dot{m}_{fuel_{Otto}} = mpg_{Atkinson} \times \dot{m}_{fuel_{Atkinson}}$$

$$\frac{\dot{m}_{fuel_{Atkinson}}}{\dot{m}_{fuel_{Otto}}} = 0.410$$

$$\frac{mpg_{Otto}}{0.410} = mpg_{Atkinson}$$

Manufacturer's specification for fuel economy of our given Otto cycle engine is 28 mpg.

$$mpg_{Otto} = 28\ mpg$$

$$mpg_{Atkinson} = 68.3\ mpg$$

Taking this data to analyze economic costs, we find that (assuming 15,000 mi/yr and $4 per gal of gasoline):

$$15000\,\frac{mi}{yr} \times \frac{gal}{28\ mi} \times \frac{\$4}{gal} = \$2143\ per\ year$$

Table 5: Mileage and Yearly Costs				
	Otto	Otto+SC	Atkinson	Atkinson+SC
Miles per gallon *(half-load)*	28	20.0	68.3	26.7
Yearly fuel costs *(assume half-load; 15,000 mi; $4/gal)*	$2143	$3006	$879	$2245

With these fuel savings for the hybrid vehicle, it would take approximately two and a half years to recoup the $3000 costs of the hybrid-electric system. Modern automobiles' life cycles are generally much greater than that time period, so the hybrid-electric vehicle running on an Atkinson cycle engine is a very viable option for consumers looking to save on overall costs.

Both of these engine cycles with superchargers, however, got lower fuel economy. Therefore, adding a supercharger is not a relevant in the discussion of economics regardless of its costs. Other factors such as power, responsiveness, etc. that a supercharger may add to the vehicle are not factored into the cost analysis.

8 - Conclusion

After analyzing and comparing the Atkinson Cycle to the Otto Cycle, their differences can be summarized by saying that the Atkinson Cycle has better efficiency and reduced output. By employing variable valve timing, the efficiency of the Atkinson Cycle can be increased over the Otto Cycle even further. Both the first law analysis and the exergy analysis indicate that the Atkinson cycle performs better per unit of energy or exergy input at the price of total power output. The Atkinson cycle also reduces exergy destructions compared to the Otto cycle. However, the Atkinson cycle engine alone does not have the power to meet our minimum power constraints.

By supplementing the Atkinson Cycle with an electric motor, the engine can compete with a similarly sized Otto Cycle engine. This hybrid Atkinson-electric system can easily meet the requirements of normal civilian use. However, these extra system components come at a price. Overall, the equipment needed have significant costs at around $3000, but when offset by the fuel savings, it becomes clear that the Atkinson-electric system is the superior system.

In contrast, our analysis of the supercharger indicates that the supercharger worsens the performance of both cycles. Superchargers decrease first law efficiency and exergetic efficiency, as well as increasing exergy destruction and fuel consumption, so we conclude that incorporating superchargers onto either the Otto or the Atkinson cycle is an undesirable choice based on our selection criteria of thermodynamic efficiency and costs.

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Figure References

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Matlab Code

This code was used to compare the compression ratio to the heat in:

```
% Generates plot to compare compression ratio and Qin
clear; clc; close all;

% Loading air property table
chart = load('chart.txt');
air1.t = chart(:,1);
air1.h = chart(:,2);
air1.pr = chart(:,3);
air1.u = chart(:,4);
air1.vr = chart(:,5);
air1.s0 = chart(:,6);

% Parameters
Pow = [120 70];
RPM = [6000 3000];
er = 13;
rat = (6:0.5:10)./13;
cr = er.*rat;
v = 27.4655;
RG = 0.06855;

% Thermodynamic analysis
a.p1 = 1;
a.t1 = 536.37;
a.u1 = interp1(air1.t,air1.u,a.t1);
a.vr1 = interp1(air1.t,air1.vr,a.t1);
a.vr2 = a.vr1./cr;
a.u2 = interp1(air1.vr,air1.u,a.vr2);

a.m = v.*a.p1./(1545./28.97.*a.t1).*14.696./12.*rat;

% Loops for the two cases
for j=1:2;
a.Pow = Pow(j)-44;
a.W = a.Pow./RPM(j)./4*2./0.0235808867./a.m;

% inital guess for iteration
a.u4 = 255.5;
qin = 426.1;
x = [a.u4; qin];

% loops to solve for qin through iteration
   for i=1:length(cr);
   options = optimset('Display','iter','TolFun',10^(-10),'MaxIter',100);
```

```
    [x_sol0, fval] = fsolve(@(x) lceq(x, a.u2(i), a.u1, a.W(i), er), x, options);
    a.u4(i) = x_sol0(1);
    qin(i) = x_sol0(2);
    end

a.t4 = interp1(air1.u,air1.t,a.u4);
a.n = a.W./qin;
Q = a.m.*qin;

% generates plot showing compression ratio vs. Qin
figure(j)
hold on;
plot(cr, Q)
xlabel('compression ratio'); ylabel('Qin (btu)');
end
```

This code was used to analyze the cycles using the first law:

```
% Performs thermodynamic analysis on designs
clear; clc; close all;

% Loads air property tables
chart = load('chart.txt');
air1.t = chart(:,1);
air1.h = chart(:,2);
air1.pr = chart(:,3);
air1.u = chart(:,4);
air1.vr = chart(:,5);
air1.s0 = chart(:,6);

% Engine parameters
Pow = [100 70];
RPM = [6000 3000];
er = [10.4 10.4 13 13];
rat = [1 1 7.5/13 7.5/13];
cr = er.*rat;
v = [37.99126 37.99126 27.4655 27.4655];
RG = 0.06855;

% Ambient air conditions
a.t0 = 536.37;
a.p0 = 1;

% Supercharger parameters
a.Wsc = [0 60 0 60];
a.Nsc = 0.7;
```

```
a.p1 = (a.Wsc.*a.Nsc.*0.4./1.4./RG./a.t0+1).^(1.4./0.4)*a.p0;
a.t1s = a.t0.*(a.p1./a.p0).^(0.4./1.4);
a.t1 = a.t0+(a.t1s-a.t0)./a.Nsc;
a.u1 = interp1(air1.t,air1.u,a.t1);
a.vr1 = interp1(air1.t,air1.vr,a.t1);
a.vr2 = a.vr1./cr;
a.u2 = interp1(air1.vr,air1.u,a.vr2);

a.m = v.*a.p1./(1545./28.97.*a.t1).*14.696./12.*rat;

% Loops for each case
for j=1:2;
a.Pow = Pow(j)-[0 0 44 0];
a.Wact = a.Pow./RPM(j)./4*2./0.0235808867./a.m;
a.W = a.Wact+a.Wsc;

% Initial guesses for iteration
a.u4 = 255.5;
qin = 426.1;
x = [a.u4; qin];

% Loops to iterate for each design
   for i=1:4;
   options = optimset('Display','iter','TolFun',10^(-10),'MaxIter',100);
   [x_sol0, fval] = fsolve(@(x) Iceq(x, a.u2(i), a.u1(i), a.W(i), er(i)), x, options);
   a.u4(i) = x_sol0(1);
   qin(i) = x_sol0(2);
   end

a.t4 = interp1(air1.u,air1.t,a.u4);
a.n = a.Wact./qin;
end
```

This code is the solver function called by the previous two codes:

```
% Function with equations to solve for qin and u4
function [F] = Iceq(x, u2, u1, W, er)

chart = load('chart.txt');
u = chart(:,4);
vr = chart(:,5);

% sort uknowns
u4 = x(1);
qin = x(2);

% equations for each case to be set to zero
```

```
f1 = qin - u4 + u1 - W;
f2 = u4 - interp1(vr,u,er*interp1(u,vr,qin+u2));

F = [f1, f2];
F = [real(F); imag(F)];

return
```