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Engineering Mathematics with TI-Nspire CAS

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ÉTS
Le génie pour l'industrie

Overview

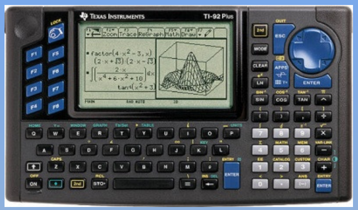
- About ETS (engineering school)
- Why Nspire CAS?
- Why Computer Algebra?
  - Examples in pre-calculus
  - Examples in single variable calculus
  - Examples in multiple variable calculus
  - One example in ODEs
  - One example in linear algebra

About ETS

- Engineering school in Montréal, Québec, Canada.
- “Engineering for Industry”.
- More than 7000 students (1600 at graduate level), 1500 new students each year.
- All math teachers and students have the same calculator (TI-Nspire CX CAS) and textbook.
- Texas Instruments technology is used, campus wide, since 1999.

About ETS

- 1999: TI-92 Plus CAS handheld.
- 2002 : TI Voyage 200.
- 2011 : TI-*nspire CX* CAS.



- Different softwares (Derive, Maple, Matlab, DPGraph, Geogebra) are also used.

- Only CAS calculators are allowed during *exams* (laptops allowed otherwise).

Why Nspire CAS?

- Our engineering school started to use TI products (TI-92 Plus) in september 1999: every student attending ETS had to buy his own handheld. The reason behind this was the fact that some mathematics professors wanted to use computer algebra in the classrooms *without having to go to the computer labs for 15 or 20 minutes!*

- Our experience continued and we moved to Voyage 200 during fall 2002. We were quite satisfied with this product but non mathematics professors were not so enthusiastic because the handheld was mainly a CAS.

- When Nspire CAS was introduced (at the Dresden conference in 2006), we were not convinced that the product was suitable for engineering mathematics: for instance, we were missing differential equations graphing mode and 3D plotting.

Why Nspire CAS?

- So, at ETS, we continued to use V200 during the period from September 2002 until August 2011. But the slow processor, the limited 3D capabilities and the fact that no new version of the operating system of V200 has hit the market since July 2005 were becoming good reasons to make a move. Also, Nspire CAS is much more attractive for physics, chemistry and finance professors for example than was V200.
- During the winter 2011, Gosia Brothers showed me a preview of TI-Nspire CAS: color screen, rechargeable batteries, ODEs and 3D parametric plotting to come!
- We were so convinced by the product that Nspire CAS became the mandatory machine in September 2011. Moreover, many students are using their handheld in the classroom and are switching to their PC at home. And, in the classroom, especially for 3D graphing, they prefer to use their laptop.
- "We have the best of two worlds".

Why Nspire CAS?

- TI-Nspire CAS technology is much more than a single computer algebra system: future engineers can use it for geometry, statistics, physics, chemistry and so on. They can collect and analyze data using the Vernier system. In their finance course, they can use the spreadsheet instead of using Excel. This was not the case with V200.

- These future engineers often need some mathematical refreshments… In fact, high school mathematics subjects need to be revised by almost 50% of new students entering engineering studies at ETS — and this is probably also the case in many parts of Canada and the US.

- A software like Nspire CAS has definitely a great advantage over Maple or Mathematica when ~~comes~~ it's time to talk about and teach high school/college mathematics; many or our examples will focus on subjects that are definitely best covered by Nspire CAS than by the big CAS.

Why Nspire CAS?

- Despite its “calculator look”, Nspire CAS is able to compete with other big CAS as long as you are teaching at undergraduate level; heavy computations required in engineering mathematics can be done. And, in many cases, the results shown on the screen are much more compact and simple. Some examples will be on this.

- Some users would say that special mathematical functions are not implemented into Nspire CAS: this is not so important *at undergraduate level*: instead of seeing this as a lack, one can use it to reinforce important mathematical concepts. We will show this later.

- A big advantage Nspire CAS has over other systems is its ease of use: this remains very important, especially for one who wants to make a daily use of it.

Why Nspire CAS?

- The fact that Nspire CAS technology comes in two parts (handheld and software) is fundamental: *students won't become familiar with computer algebra if they are not allowed to use it from their desk.*

  - During exams, we don't want students to be able to communicate among each other: so they are allowed to use their handheld but not to bring their laptop in the classroom.

  - During a course, many will prefer to use Nspire CAS on the laptop (especially for 3D graphing). There is almost *no reason* to bring the students to a computer lab: they can experiment and use computer algebra in the classroom.

Why Nspire CAS?

- Finally, our experience of using TI technology for the past 14 years has had many consequences on the way to teach mathematics to future engineers.

  - The ease of use, the user friendly interface has allowed us to use different approaches when it was time to solve a particular problem (not for every problem but for many ones, optimization problems for example).
  - And this is a good way for *an old mathematics teacher to stay enthusiastic despite the fact that, year after year, he has to teach the same courses!*
  - It is true that different approaches for solving a problem can be tried with other CAS but, most of the time, *the price to pay will be higher because the user will need to perform many operations and apply commands only specialists know*.

Why Computer Algebra?

- Let us be clear: you can continue to teach mathematics with a great success WITHOUT using computer algebra!

- This is probably the case for the majority of mathematics high school teachers (and also, college teachers).

- But, if you are teaching to future engineers (at university level), you miss a lot if you don't make use of computer algebra.

- Students coming to engineering school can afford spending 200$ for a CAS (handheld and software) if they are told that it will be used in many different courses. The situation is quite different at high school level.

- So we are not saying ETS experience can be exported everywhere at anytime!

Why Computer Algebra?

- During this talk, we will perform some live examples for convincing you that computer algebra is a natural tool for teaching mathematics to future engineers.
- In fact, we are convinced that *without* computer algebra many parts of the curriculum are not well covered.
- When you are using Nspire CAS, you get automatic simplifications on the screen. A math teacher can use many commands to explain what happened: this is a way to teach mathematics.
- Doing so, a lot of mathematical concepts are shown. At the end, links between subjects are made and mathematics can become more attractive to students.

Why Computer Algebra?

➢ **Example 1 in pre-calculus**

Nspire CAS is able to compute sin(75°): $\sin(75^\circ) = \dfrac{\left(\sqrt{3}+1\right)\sqrt{2}}{4}$

And simplifies $\dfrac{2^{4/3}\,7^{2/3}}{3^{2/3}\,5^{2/3}}$ with no radicals left in the denominator:

$$\sin(75°) \longrightarrow \frac{\left(\sqrt{3}+1\right)\cdot\sqrt{2}}{4}$$

How can we show this?

$$\frac{2^{\frac{4}{3}}\cdot 7^{\frac{2}{3}}}{3^{\frac{2}{3}}\cdot 5^{\frac{2}{3}}} \longrightarrow \frac{2\cdot 30^{\frac{1}{3}}\cdot 7^{\frac{2}{3}}}{15}$$

Why Computer Algebra?

➢ Example 2 in pre-calculus

Why does Nspire CAS automatically expand some expressions?

If g is the inverse function of f, we should have $f(g(x)) = x$. Is this always true?

Is there a simple way to explain inverse functions to students?

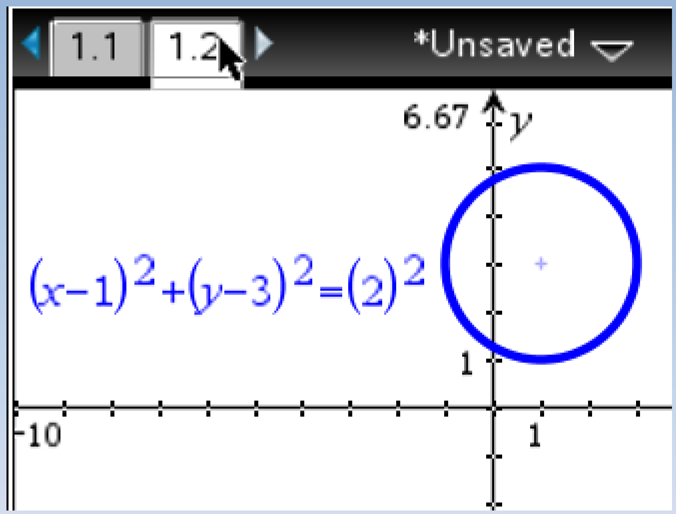
$(x+1)^3+2$	Expansion! →	$x^3+3\cdot x^2+3\cdot x+3$
$(x+1)^{100}$	No expansion! →	$(x+1)^{100}$
$\sin(\sin^{-1}(x))$		x
$\sin^{-1}(\sin(x))$		$\sin^{-1}(\sin(x))$
⚠ $e^{\ln(x)}$		x
$\ln(e^x)$		x

$$f\left(f^{-1}(x)\right) = f^{-1}(f(x)) = x\,??????$$

Why Computer Algebra?

➢ **Example 3 in pre-calculus**

I want to plot the graph of the circle centered at the point (1, 3) and having a radius of 2. The "Equation" Entry/Edit mode is an easy way:



Are there any other ways to plot this circle?

Why Computer Algebra?

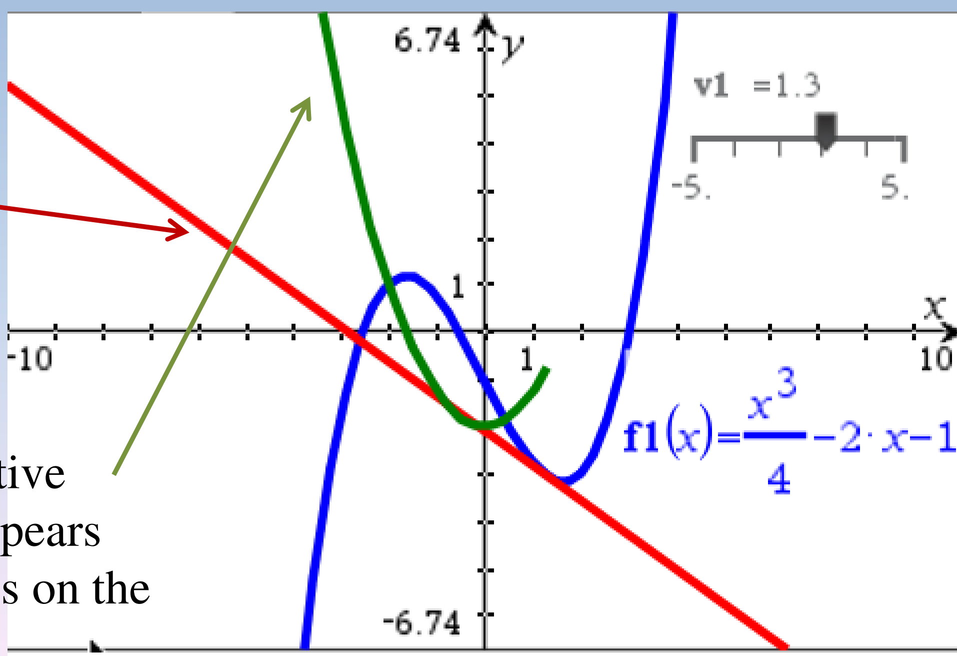
Let's perform the last 3 examples on Nspire CAS!

Why Computer Algebra?

➢ **Example 1 in single variable calculus**

One important application of the derivative is finding the equation of the tangent line to a function y = $f(x)$ at the point $x = a$. With an animation, students can get a better understanding of this concept.

Tangent line
to the blue curve



Graph of the derivative
of the blue curve appears
as the tangent moves on the
curve

Why Computer Algebra?

➢ **Example 2 in single variable calculus**

Sometimes Nspire CAS can't compute the exact value of a given definite integral. Is there something we can learn from this?

$$\int_1^8 \sqrt{1+x^3}\,dx \rightarrow 72.6254$$

Only a floating point approximation

$$\int \sqrt{1+x^3}\,dx$$

Nspire is using a certain reduction formula …

$$\frac{3\cdot\int \frac{1}{\sqrt{x^3+1}}\,dx}{5}+\frac{2\cdot x\cdot\sqrt{x^3+1}}{5}$$

Why Computer Algebra?

➢ **Example 2 (cont.) in single variable calculus**

#1: $\int \sqrt{(1 + x^3)}\, dx$

Derive shows us this reduction formula

$$\int (a + b\cdot x^n)^p\, dx \rightarrow \frac{x\cdot(a + b\cdot x^n)^p}{n\cdot p + 1} + \frac{n\cdot p\cdot a\cdot \int (a + b\cdot x^n)^{p-1}\, dx}{n\cdot p + 1}$$

#2: $\frac{2\cdot x\cdot\sqrt{(x^3 + 1)}}{5} + \frac{3\cdot \int \frac{1}{\sqrt{(x^3 + 1)}}\, dx}{5}$

$$\int \sqrt{1+x^3}\, dx$$

$$\frac{3\cdot \int \frac{1}{\sqrt{x^3+1}}\, dx}{5} + \frac{2\cdot x\cdot \sqrt{x^3+1}}{5}$$

But this integral is not implemented into Nspire CAS.

This becomes an opportunity to define a *new* function!

Why Computer Algebra?

- **Example 2 (cont.) in single variable calculus**

  Nspire CAS was *unable* to compute the indefinite integral $\int \sqrt{1+x^3}\,dx$.
  What about the "big" CAS? Here is Mathematica's answer:

$$\frac{2\left(x + x^4 + (-1)^{1/6}\, 3^{3/4} \sqrt{-(-1)^{1/6}\left((-1)^{2/3} + x\right)}\ \sqrt{1 + (-1)^{1/3}x + (-1)^{2/3}x^2}\ \text{EllipticF}\left[\text{ArcSin}\left[\frac{\sqrt{-(-1)^{5/6}(1+x)}}{3^{1/4}}\right], (-1)^{1/3}\right]\right)}{5\sqrt{1+x^3}}$$

Here is Maple's answer:

$$\frac{2}{5}x\sqrt{1+x^3} + \frac{6}{5}\frac{1}{\sqrt{1+x^3}}\left(\left(\frac{3}{2} - \frac{1}{2}\mathrm{I}\sqrt{3}\right)\sqrt{\frac{x+1}{\frac{3}{2}-\frac{1}{2}\mathrm{I}\sqrt{3}}}\sqrt{\frac{x-\frac{1}{2}-\frac{1}{2}\mathrm{I}\sqrt{3}}{-\frac{3}{2}-\frac{1}{2}\mathrm{I}\sqrt{3}}}\sqrt{\frac{x-\frac{1}{2}+\frac{1}{2}\mathrm{I}\sqrt{3}}{-\frac{3}{2}+\frac{1}{2}\mathrm{I}\sqrt{3}}}\ \text{EllipticF}\left(\sqrt{\frac{x+1}{\frac{3}{2}-\frac{1}{2}\mathrm{I}\sqrt{3}}}, \sqrt{\frac{-\frac{3}{2}+\frac{1}{2}\mathrm{I}\sqrt{3}}{-\frac{3}{2}-\frac{1}{2}\mathrm{I}\sqrt{3}}}\right)\right)$$

Why Computer Algebra?

$$david(x) := \int_0^x \frac{1}{\sqrt{1+t^3}}\,dt$$

Done

Students will need to use the following result:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\int \sqrt{1+x^3}\,dx$$

So, we have: $\int_1^8 f(x)\,dx = \int_0^8 f(x)\,dx - \int_0^1 f(x)\,dx$

$$\frac{3 \cdot \int \frac{1}{\sqrt{x^3+1}}\,dx}{5} + \frac{2 \cdot x \cdot \sqrt{x^3+1}}{5}$$

$$g(x) := \frac{2 \cdot x \cdot \sqrt{x^3+1}}{5}$$

Done

$$g(8) - g(1)$$

$$\frac{48 \cdot \sqrt{57}}{5} - \frac{2 \cdot \sqrt{2}}{5}$$

$$\frac{3}{5} \cdot (david(8) - david(1)) + \frac{48 \cdot \sqrt{57}}{5} - \frac{2 \cdot \sqrt{2}}{5}$$

This answer is « better » than the complicated ones of Mathematica and Maple!

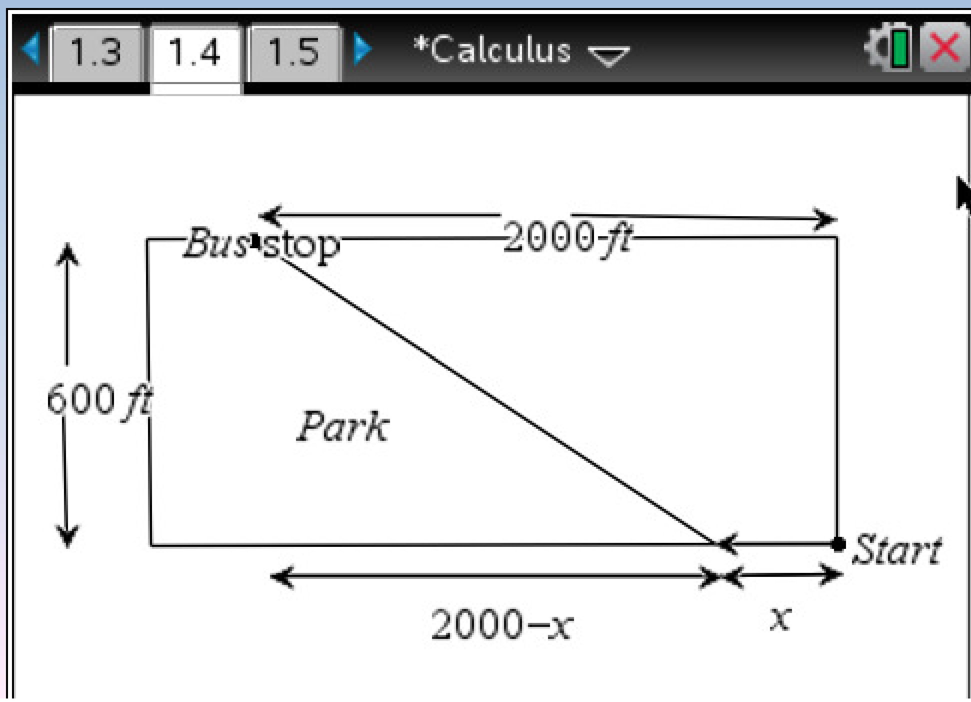
72.6254

$$\int_1^8 \sqrt{1+x^3}\,dx = \frac{3}{5}\big(\text{david}(8) - \text{david}(1)\big) + \frac{48\sqrt{57}}{5} - \frac{2\sqrt{2}}{5} \approx 72.6254$$

Why Computer Algebra?

- **Example 3 in single variable calculus**

Here is a classical optimization problem (from Hughes-Hallet, Gleason, et al.): Alaina wants to get to the bus stop as quickly as possible. The bus stop is across a grassy park, 2000 feet west and 600 feet north of her starting position. Alaina can walk west along the edge of a park on the sidewalk at a speed of 6 ft/s. She can also travel through the grass in the park, but only at a rate of 4 ft/s. What path will get her to the bus stop the fastest?



Why Computer Algebra?

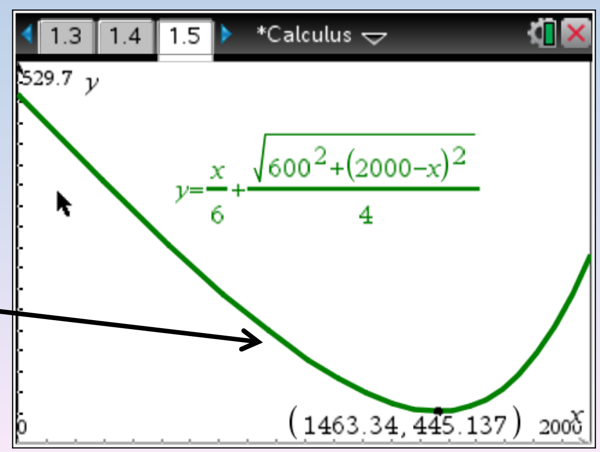
➢ Example 3 (cont.) in single variable calculus

When students have access to a CAS handheld on their desk, this kind of problem can be solved using all the tools of the device.

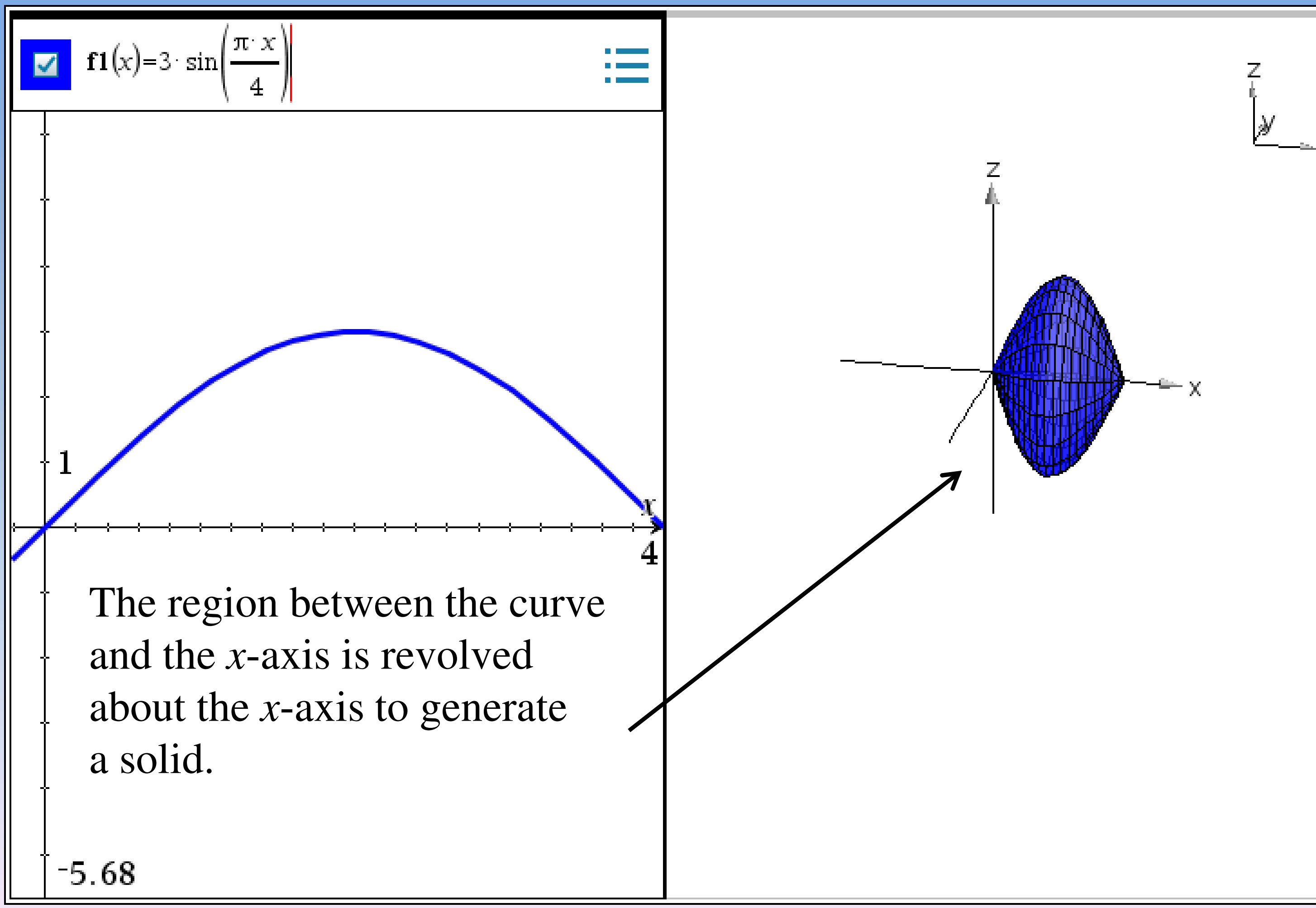
Students have to spend their time ~~on~~ finding the good function to optimize. Then, they can use a graph or the “fmin” function to find the minimum value.

If they use a graph, they need to find the domain of the function: “a continuous function defined over a closed bounded interval achieves a minimum value on the interval”.

Graph of the time with respect to x



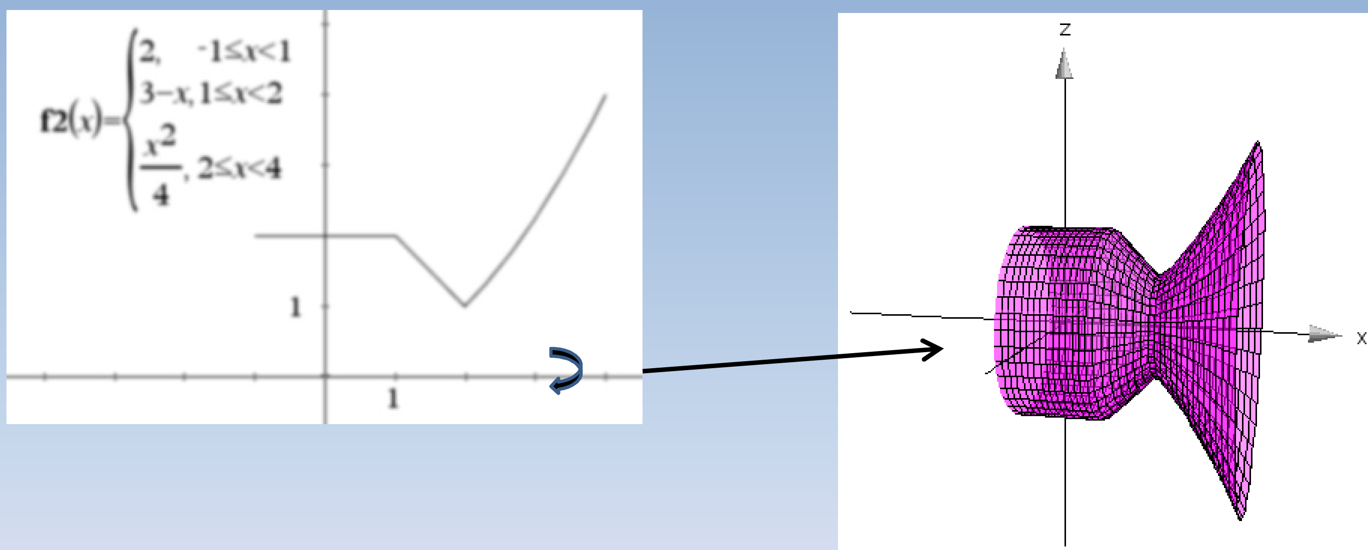
Why Computer Algebra?



Why Computer Algebra?

- **Example 4 (cont.) in single variable calculus**

We would also like to be able to show the solid of revolution generated by revolving a *piecewise* function around the x-axis. Something like this:



Here a surprise will appear … forcing us to do more mathematics!

Why Computer Algebra?

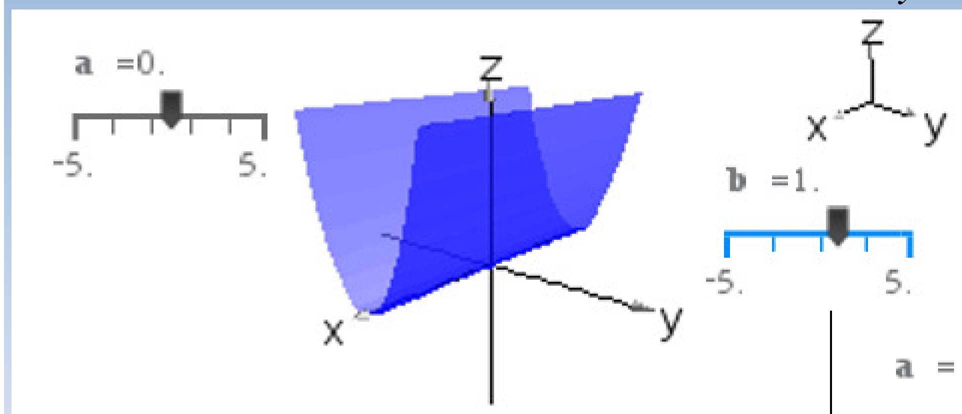
Let's perform the last 4 examples on Nspire CAS!

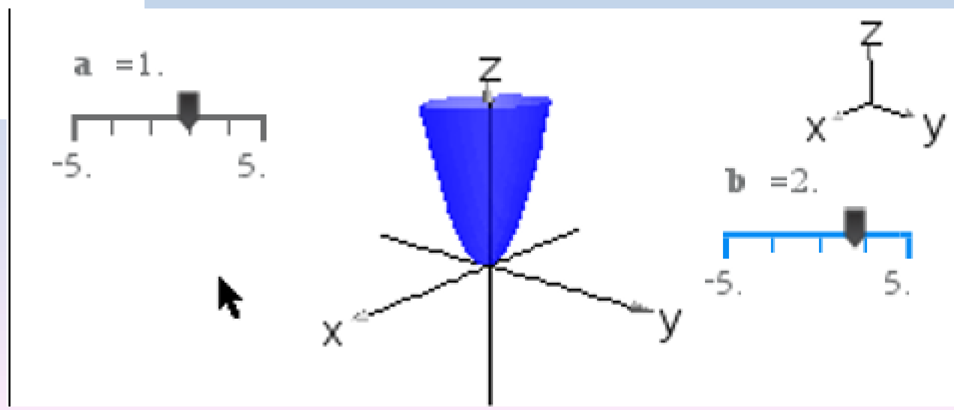
Why Computer Algebra?

➢ **Example 1 in multiple variable calculus**

Students are getting a better understanding of many graphical concepts by being able to *plot themselves* many objects. Again, using sliders becomes a pedagogical tool. A simple example: the graph of

$$z = ax^2 + by^2$$

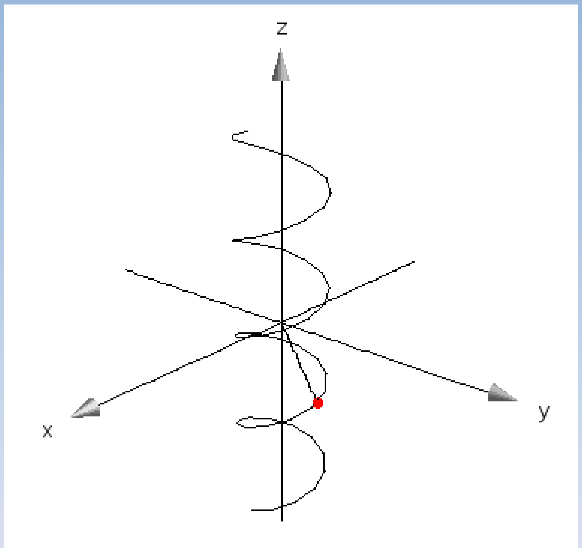




Why Computer Algebra?

- **Example 2 in multiple variable calculus**

How can we produce a graph of a space curve along with the position vector? Something like this:



(thanks to my colleague Robert Michaud)

Why Computer Algebra?

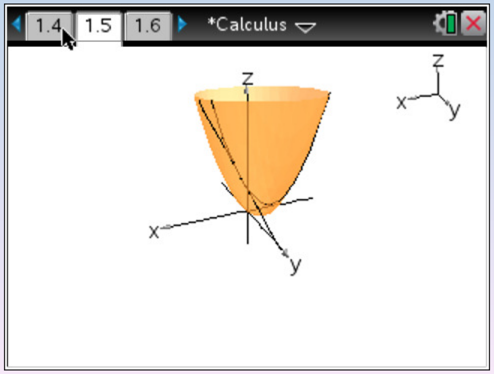
- **Example 3 in multiple variable calculus**

Students learn how to compute a partial derivative. For many years, they were rarely asked to give the geometric interpretation of it. It is now quite easy to show this because Nspire CAS supports 3D parametric plotting (since OS 3.2).

For example, consider $f(x, y) = \frac{x^2}{4} + x + \frac{y^2}{3}$

Then $\frac{\partial f}{\partial x}(2,3) = 2$. What does this number represent?

Certainly a slope of a tangent line: but which tangent line and tangent line to which curve?



Why Computer Algebra?

Let's perform the last 3 examples on Nspire CAS!

Why Computer Algebra?

- **Example 1 in differential equations**

Numerical ODE solvers (Euler's method, RK method) should be used along with analytical method when an application problem is given to students.

A good example is the fall of a skydiver: a skydiver weighing 180 lb (including equipment) falls vertically downward from an altitude of 5000 ft and opens the parachute after 10s of free fall. We assume that the air resistance force is $0.75|v|$ when the parachute is closed and $12|v|$ when the parachute is open, where the velocity v is measured in ft/s. Determine in how many seconds the skydiver will reach the ground.

Let $y(t)$ be the height of the skydiver at time t, the ground being $y = 0$. Then Newton's second law yields $\frac{180}{32.2}\frac{dv}{dt} = -180 - k \cdot v,\ v(0) = 0$

where

$$k = \begin{cases} 0.75, & 0 \le t \le 10 \\ 12, & t > 10 \end{cases}$$

Why Computer Algebra?

➢ **Example 1 (cont.) in differential equations**

Using the 2D differential equation graphing mode and a first order system, we can easily plot the graph of the height of the skydiver WITHOUT solving any ODE!

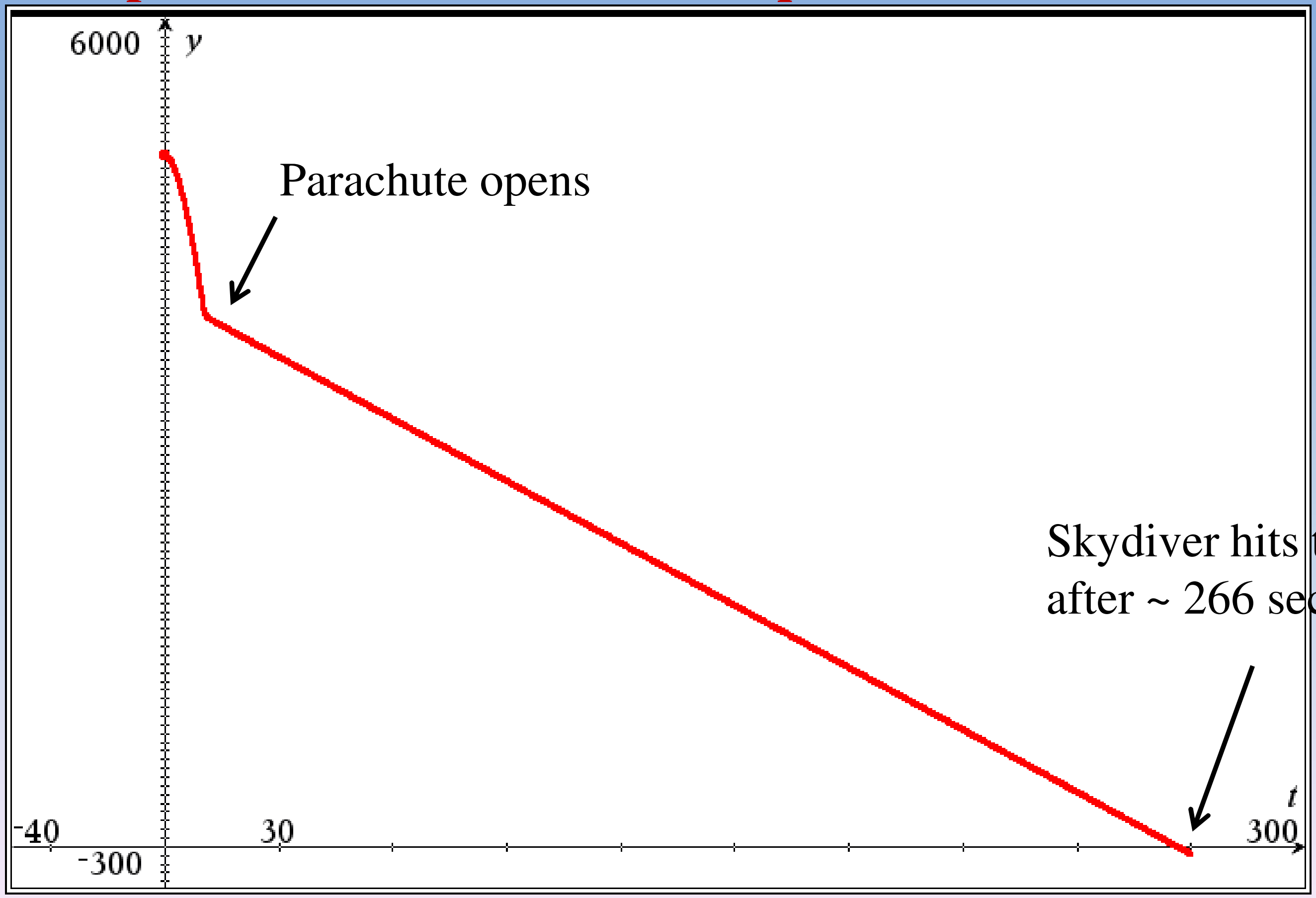
With g = 32.2 and m = 180/32.2, the system is the following:

$$\begin{cases} \dfrac{dy}{dt} = v,\ y(0) = 5000 \\ \dfrac{dv}{dt} = -g - \dfrac{k \cdot v}{m},\ v(0) = 0 \end{cases}$$

Nobody would try solve this system because k is a piecewise function! So, by hand, we need to solve 2 different ODEs: one before the parachute opens and one after.

Why Computer Algebra?

➢ **Example 1 (cont.) in differential equations**



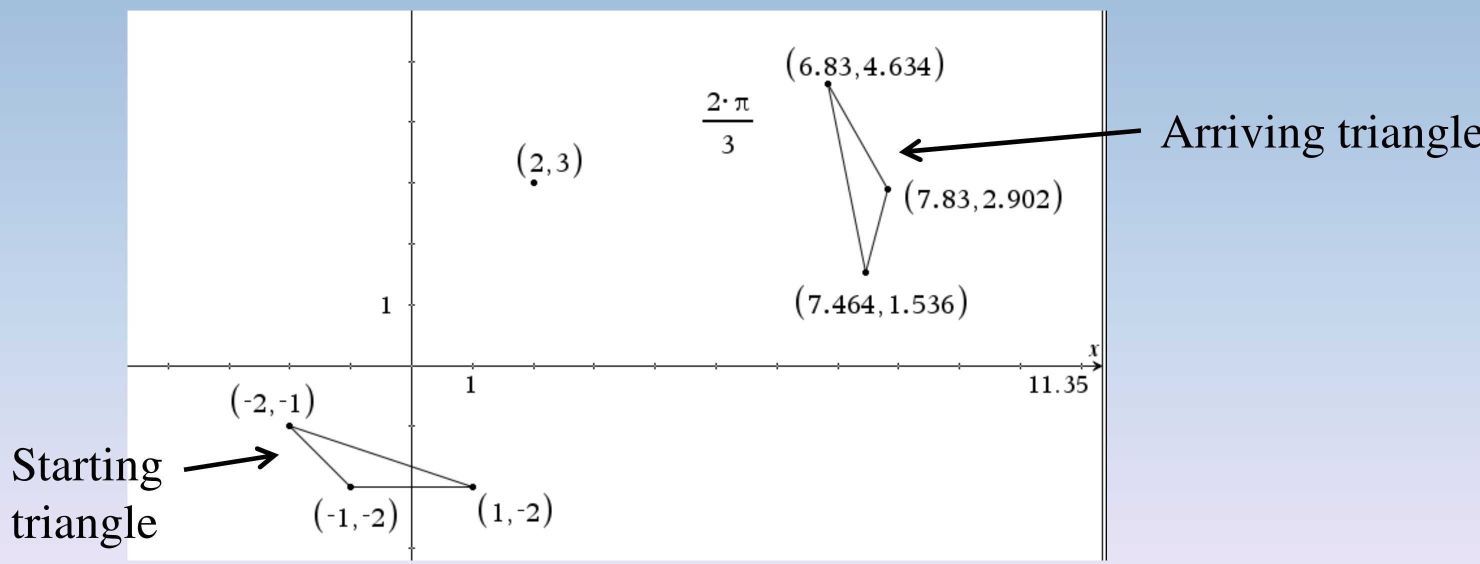
Why Computer Algebra?

Let's perform this last example on Nspire CAS!

Why Computer Algebra?

➢ Example 1 in linear algebra

Matrices can be used in connection with geometry. This is because Nspire CAS is more than just being a CAS. For example, consider the rotation of a triangle around a point. Let suppose a triangle has vertices located at the points (-2, -1), (1, -2) and (-1, -2). We rotate counterclockwise this triangle around the point (2, 3) by an angle of 120°. What will be the new vertices of the image triangle?



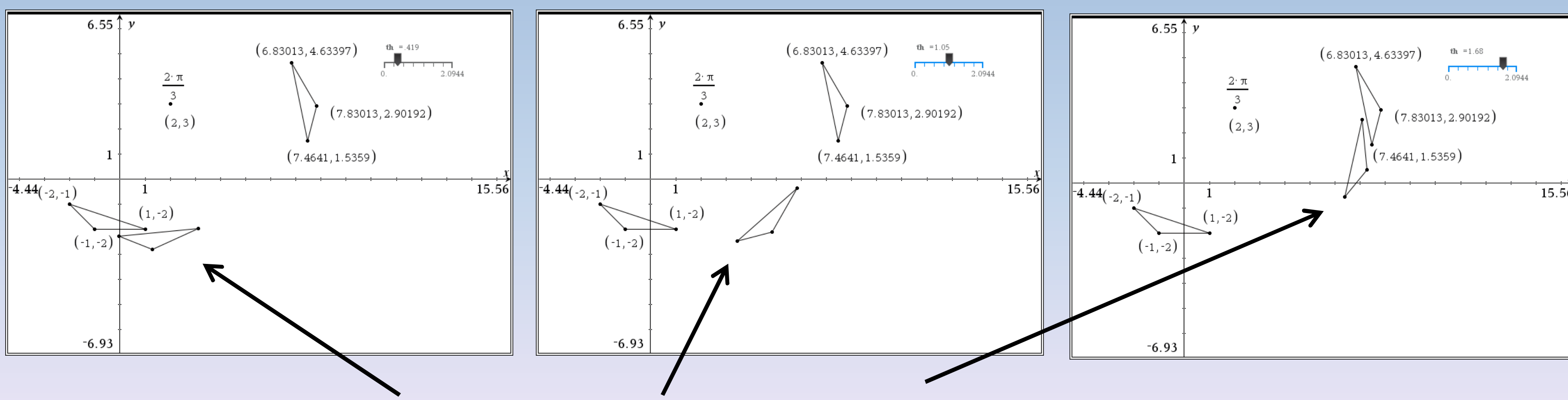
Why Computer Algebra?

➢ Example 1 (cont.) in linear algebra

The mathematics needed to perform this transformation rely upon linear algebra stuff.

We need to translate the triangle by the vector [–2,–3], then rotate around the origin the starting triangle and, finally, translate by the vector [2, 3].

The DG software integrated into Nspire CAS showed us almost instantly where the triangle should be moved: using some matrices product will confirm this result. Here is the funny thing: we can *observe* this using a nice animation.



We will also animate the rotation of the triangle.

Why Computer Algebra?

Let's perform this final example on Nspire CAS!