Interval Mathematics

Foundations, Algebraic Structures, and Applications

By

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To my family.

Abstract

STUDENT NAME: Hend Dawood Mohamed.
THESIS TITLE: Interval Mathematics: Foundations, Algebraic Structures, and Applications.
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We begin by constructing the algebra of classical intervals and prove that it is a *nondistributive abelian semiring*. Next, we formalize the notion of *interval dependency*, along with discussing the algebras of two alternate theories of intervals: *modal intervals*, and *constraint intervals*. With a view to treating some problems of the present interval theories, we present an alternate theory of intervals, namely the "*theory of optimizational intervals*", and prove that it constitutes a *rich $\mathcal{S}$-field algebra*, which extends the *ordinary field* of the reals, then we construct an *optimizational complex interval algebra*. Furthermore, we define an order on the set of interval numbers, then we present the proofs that it is a *total order*, *compatible* with the interval operations, *dense*, and *weakly Archimedean*. Finally, we prove that this order *extends* the usual order on the reals, *Moore's partial order*, and *Kulisch's partial order* on interval numbers.

Keywords. Classical interval arithmetic; Machine interval arithmetic; Interval dependency; Constraint intervals; Modal intervals; Classical complex intervals; Optimizational intervals; Optimizational complex intervals; $\mathcal{S}$-field algebra; Ordering interval numbers.

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Publications from the Thesis

During the course of the thesis work, we published the following book that provides a bit of perspective on some topics of this research. Some ideas and figures of the thesis have appeared previously in this book.

> Hend Dawood; *Theories of Interval Arithmetic: Mathematical Foundations and Applications*; LAP Lambert Academic Publishing; Saarbrücken; 2011; ISBN 978-3-8465-0154-2.
>
> http://books.google.com/books?id=Q0jPygAACAAJ

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Notation and Conventions

Most of our notation is standard, and notational conventions are characterized, in detail, on their first occurrence. However, we have *many* theories of intervals being discussed throughout the text, with each theory has naturally its *own peculiar* notation; along with some basic logical, set-theoretic, and order-theoretic symbols. So, for the purpose of legibility, we give here a consolidated list of symbols for the entire text.

Logical Symbols

$=$	Identity (*equality*).
$\neg$	Logical Negation (*not*).
$\Rightarrow$	Implication (*if ..., then ...*).
$\Leftrightarrow$	Equivalence (*if, and only if*).
$\wedge$	Conjunction (*and*).
$\vee$	Inclusive disjunction (*or*).
$\forall$	Universal quantifier (*for all*).
$\exists$	Existential quantifier (*there exists*).
$Q \in \{\forall, \exists\}$	A quantifier variable symbol (*with or without subscripts*).
$\mathcal{Q}$	A quantification matrix, $(Q_1x_1) \dots (Q_nx_n)$, where $x_1, \dots, x_n$ are variable symbols and each Q_i is $\forall$ or $\exists$.
$(\forall_{k=1}^{n} x_k)$	The universal quantification matrix, $(\forall x_1) \dots (\forall x_n)$.
$(\exists_{k=1}^{n} x_k)$	The existential quantification matrix, $(\exists x_1) \dots (\exists x_n)$.
φ	A quantifier-free formula.
σ	A prenex sentence, $\mathcal{Q}\varphi$, where $\mathcal{Q}$ is a quantification matrix and φ is a quantifier-free formula.

Set-Theoretic and Order-Theoretic Symbols

$\varnothing$	The empty set.
$\mathcal{S}, \mathcal{T}, \mathcal{U}$	Set variable symbols (*with or without subscripts*).
$\wp(\mathcal{S})$	The powerset of a set $\mathcal{S}$.
$\mathcal{S}^{\langle n \rangle}$	The n-th Cartesian power of a set $\mathcal{S}$.
$\in$	The set membership relation.
$\subset$	The set inclusion relation.
$\cap$	The set intersection operator.
$\cup$	The set union operator.
$\cap_{k=1}^{n} \mathcal{S}_k$	The finitary set intersection $\mathcal{S}_1 \cap \ldots \cap \mathcal{S}_n$.
$\cup_{k=1}^{n} \mathcal{S}_k$	The finitary set union $\mathcal{S}_1 \cup \ldots \cup \mathcal{S}_n$.
$\Re$	A relation variable symbol (*with or without subscripts*).
$\operatorname{dom}(\Re)$	The domain of a relation $\Re$.
$\operatorname{ran}(\Re)$	The range of a relation $\Re$.
$\operatorname{fld}(\Re)$	The field of a relation $\Re$.
$\widehat{\Re}$	The converse of a relation $\Re$.
$\mathrm{Id}_{\mathcal{S}}$	The identity relation on a set $\mathcal{S}$.
$\Re\text{-}\inf(\mathcal{S})$	The infimum of a set $\mathcal{S}$ relative to an ordering relation $\Re$.
$\Re\text{-}\sup(\mathcal{S})$	The supremum of a set $\mathcal{S}$ relative to an ordering relation $\Re$.
$\curlyvee$	The lattice binary join operator.
$\curlywedge$	The lattice binary meet operator.

Interval-Theoretic Symbols

$\mathbb{R}$	The set of real numbers.
x, y, z	Real variable symbols (*with or without subscripts, and with or without lower or upper hyphens*).

a, b, c	Real constant symbols (*with or without subscripts, and with or without lower or upper hyphens*).
$\circ \in \{+, \times\}$	A binary algebraic operator.
$\diamond \in \{-, ^{-1}\}$	A unary algebraic operator.
$[\mathbb{R}]$	The set of classical interval numbers.
$[\mathbb{R}]_p$	The set of point (*singleton*) classical interval numbers $[x, x]$.
$[\mathbb{R}]_s$	The set of symmetric classical interval numbers $[-x, x]$.
$[\mathbb{R}]_{\widetilde{0}}$	The set of zeroless classical interval numbers.
X, Y, Z	Interval variable symbols (*with or without subscripts*).
A, B, C	Interval constant symbols (*with or without subscripts*).
$\inf(X)$	The infimum of an interval number X.
$\sup(X)$	The supremum of an interval number X.
$w(X)$	The width of an interval number X.
$r(X)$	The radius of an interval number X.
$m(X)$	The midpoint of an interval number X.
$\|X\|$	The absolute value of an interval number X.
$d(X, Y)$	The distance (*metric*) between two interval numbers X and Y.
$\mathbb{M} \subset \mathbb{R}$	The set of machine-representable real numbers.
$\mathbb{M}_n$	The set of machine real numbers with n significant digits.
$[\mathbb{M}]$	The set of machine interval numbers.
$\bigtriangledown$	Downward rounding operator.
$\bigtriangleup$	Upward rounding operator.
$\Diamond$	Outward rounding operator.
$\mathrm{I}_f(X_1, ..., X_n)$	The image of the real closed intervals $X_1, ..., X_n$ under a real-valued function f.
$Y \mathfrak{D} X$	The interval variable Y is *dependent* on the interval variable X.
$Y \mathfrak{I} X$	The interval variable Y is *independent* on the interval variable X.

$\Im_{k=1}^{n}(X_k)$	All the interval variables $X_1, ..., X_n$ are pairwise mutually independent.
$\mathcal{M}$	The set of modal intervals.
$\mathcal{M}_{\exists}$	The set of existential (*proper*) modal intervals.
$\mathcal{M}_{\forall}$	The set of universal (*improper*) modal intervals.
${}^{m}X, {}^{m}Y, {}^{m}Z$	Modal interval variable symbols (*with or without subscripts*).
${}^{m}A, {}^{m}B, {}^{m}C$	Modal interval constant symbols (*with or without subscripts*).
$\mathrm{mode}({}^{m}X)$	The mode of a modal interval ${}^{m}X$.
$\mathrm{set}({}^{m}X)$	The set of a modal interval ${}^{m}X$.
$\mathrm{dual}({}^{m}X)$	The dual of a modal interval ${}^{m}X$.
$\mathrm{proper}({}^{m}X)$	The proper of a modal interval ${}^{m}X$.
$\mathrm{improper}({}^{m}X)$	The improper of a modal interval ${}^{m}X$.
$\inf({}^{m}X)$	The infimum of a modal interval ${}^{m}X$.
$\sup({}^{m}X)$	The supremum of a modal interval ${}^{m}X$.
${}^{\mathrm{t}}[\mathbb{R}]$	The set of constraint intervals.
${}^{\mathrm{t}}[\mathbb{R}]_p$	The set of point constraint intervals.
${}^{\mathrm{t}}[\mathbb{R}]_s$	The set of symmetric constraint intervals.
${}^{\mathrm{t}}[\mathbb{R}]_{\widetilde{0}}$	The set of zeroless constraint intervals.
${}^{\mathrm{o}}[\mathbb{R}]$	The set of optimizational interval numbers.
${}^{\mathrm{o}}[\mathbb{R}]_p$	The set of point optimizational interval numbers.
${}^{\mathrm{o}}[\mathbb{R}]_s$	The set of symmetric optimizational interval numbers.
${}^{\mathrm{o}}[\mathbb{R}]_{\widetilde{0}}$	The set of zeroless optimizational interval numbers.
$\mathbb{C}$	The set of ordinary complex numbers.
$i = \sqrt{-1}$	The ordinary imaginary unit.
$\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$	Complex variable symbols (*with or without subscripts*).
$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$	Complex constant symbols (*with or without subscripts*).
$[\mathbb{C}]$	The set of classical complex intervals.
$[\mathbb{C}]_p$	The set of point classical complex intervals.

$[\mathbb{C}]_{\widetilde{0}}$	The set of zeroless classical complex intervals.
$\boldsymbol{i} = [i, i]$	The interval imaginary unit.
$\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$	Complex interval variable symbols (*with or without subscripts*).
$\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$	Complex interval constant symbols (*with or without subscripts*).
$\mathrm{In}\,(\boldsymbol{X})$	The interval part of a complex interval $\boldsymbol{X}$.
$\mathrm{Im}\,(\boldsymbol{X})$	The imaginary part of a complex interval $\boldsymbol{X}$.
$\widetilde{\boldsymbol{X}}$	The conjugate of a complex interval $\boldsymbol{X}$.
${}^{\mathrm{o}}[\mathbb{C}]$	The set of optimizational complex intervals.
${}^{\mathrm{o}}[\mathbb{C}]_p$	The set of point optimizational complex intervals.
${}^{\mathrm{o}}[\mathbb{C}]_s$	The set of symmetric optimizational complex intervals.
${}^{\mathrm{o}}[\mathbb{C}]_{\widetilde{0}}$	The set of zeroless optimizational complex intervals.
$<_{\mathrm{M}}$	Moore's partial ordering for interval numbers.
$\leq_{\mathrm{K}}$	Kulisch and Miranker's partial ordering for interval numbers.
$\leq_{\mathrm{T1}}, \leq_{\mathrm{T2}}$	Ishibuchi and Tanaka's partial orderings for interval numbers.
$\preceq_{\mathcal{I}}$	A total compatible ordering for interval numbers.

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Chapter 1

Introduction and Motivation

You cannot be certain about uncertainty.

–Frank Knight (1885-1972)

Scientists are, all the time, in a struggle with *error* and *uncertainty*. Uncertainty is the quantitative estimation of errors present in measured data; all measurements contain some uncertainty generated through many types of error. Error ("*mistaken result*", or "*mistaken outcome*") is common in all scientific practice, and is always a serious threat to the search for a trustworthy scientific knowledge and to certain epistemic foundations of science.

This introductory chapter is intended to provide a bit of motivation and perspective about the field of interval arithmetic, its history, and how it is a potential weapon against uncertainty in science and technology (For further background, the reader may consult, e.g., [Allchin2000], [Allchin2007], [Dawood2011], [Kline1982], and [Pedrycz2008]). After this motivation for the subject, section 1.4 provides an outline of this work.

1.1 What Interval Arithmetic is

In real-life computations, uncertainty naturally arises because we process values which come from *measurements* and from *expert estimations*. From both the *epistemological* and *physical* viewpoints, neither measurements nor expert estimations can be exact for the following reasons:

- The actual value of the measured quantity is a real number; so, in general, we need *infinitely* many bits to describe the exact value, while after every measurement, we gain only a *finite* number of bits of information (e.g., a finite number of binary digits in the binary expansion of the number).

- There is always some difficult-to-delete noise which is mixed with the measurement results.
- Expert estimates cannot be absolutely exact, because an expert generates only a finite amount of information.
- Experts are usually even less accurate than are measuring instruments.

So, there are usually three sources of error while performing numerical computations with real numbers; *rounding*, *truncation* and *input errors*. A mistaken outcome is always a concern in scientific research because errors inevitably accumulate during calculations. Interval arithmetic keeps track of all error types simultaneously, because an interval arithmetic operation produces a *closed interval* within which the true *real-valued* result is guaranteed to lie.

Interval arithmetic (also known as "*interval mathematics*", "*interval analysis*", and "*interval computation*") is an arithmetic defined on sets of real intervals, rather than sets of real numbers. It specifies a precise method for performing arithmetic operations on closed intervals (*interval numbers*). The concept is simple: in the interval number system, each interval number represents some fixed real number between the *lower* and *upper endpoints* of the closed interval. So, an interval arithmetic operation produces two values for each result. The two values correspond to the lower and upper endpoints of the resulting interval such that the true result certainly lies within this interval, and the *accuracy* of the result is indicated by the *width* of the resulting interval.

As a result of error, we all the time have to face situations in which scientific measurements give *uncertain values*. Let x be a real number whose value is uncertain, and assume a measurement gives adequate information about an acceptable range, $\underline{x} \leq x \leq \overline{x}$, in which the true value of x is estimated to lie. The closed interval (interval number),

$$[\underline{x}, \overline{x}] = \{x \in \mathbb{R} | \underline{x} \leq x \leq \overline{x}\},$$

is called the "*interval of certainty*" (or the "*interval of confidence*") about the value of x. That is, we are certain that the true value of x lies within the interval $[\underline{x}, \overline{x}]$.

If it is the case that $\overrightarrow{x_n} = \langle x_1, x_2, ..., x_n \rangle$ is a *multidimensional quantity* (*real-valued vector*) such that for each x_i there is an interval of certainty $X_i = [\underline{x}_i, \overline{x}_i]$, then the quantity $\overrightarrow{x_n}$ has an "*n-dimensional parallelotope of certainty*", $\mathcal{X}_n$, which is the Cartesian product of the intervals $X_1, X_2, ..., X_n$.

Figure 1.1 illustrates the case $n = 2$; with $\overrightarrow{x_2} = \langle x_1, x_2 \rangle$, $x_1 \in [\underline{x}_1, \overline{x}_1]$, $x_2 \in [\underline{x}_2, \overline{x}_2]$, and $\mathcal{X}_2$ is a *rectangle of certainty.*

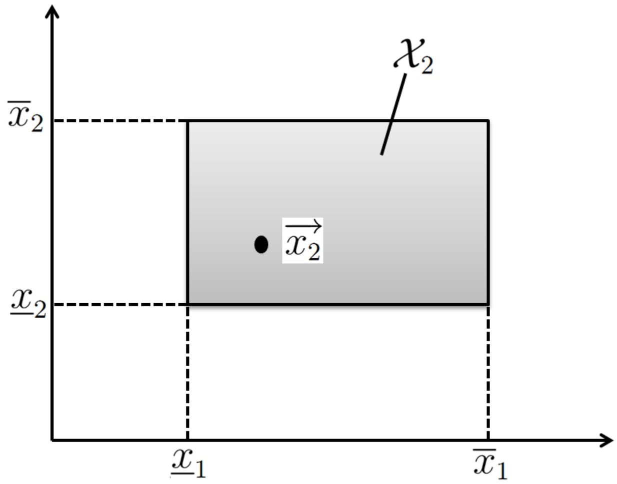

Figure 1.1: A 2-dimensional parallelotope of certainty.

To illustrate this, we give two numerical examples.

Example 1.1 *The Archimedes's constant, π, is an irrational number, which means that its value cannot be expressed exactly as a fraction. Consequently, it has no certain decimal representation because its decimal representation never ends or repeats. Since $314 \times 10^{-2} \le \pi \le 315 \times 10^{-2}$, the number π can be represented as the interval number*

$$[\underline{\pi}, \overline{\pi}] = \left[314 \times 10^{-2}, 315 \times 10^{-2}\right].$$

That is, we are certain that the true value of π lies within the interval $[\underline{\pi}, \overline{\pi}]$ whose width indicates the maximum possible error,

$$\begin{aligned} Error &= width([\underline{\pi}, \overline{\pi}]) \\ &= \overline{\pi} - \underline{\pi} \\ &= 315 \times 10^{-2} - 314 \times 10^{-2} \\ &= 10^{-2}. \end{aligned}$$

Example 1.2 *Suppose two independent scientific measurements give different uncertain results for the same quantity q. One measurement gives $q = 1.4 \pm 0.2$. The other gives $q = 1.5 \pm 0.2$. These uncertain values of q can be represented* *as the interval numbers* $X = [1.2, 1.6]$ *and* $Y = [1.3, 1.7]$*, respectively. Since* q *lies in both, it certainly lies in their intersection* $X \cap Y = [1.3, 1.6]$*. So, if* $X \cap Y \neq \varnothing$*, we can get a better (tighter) "interval of certainty". If not, we can be certain that at least one of the two measurements is wrong.*

In chapters 2, 3, 4, and 5, we shall present formal constructions of the mathematical foundations of interval arithmetic, along with more advanced examples being provided.

1.2 A History Against Uncertainty

The term "*interval arithmetic*" is reasonably recent: it dates from the 1950s, when the works of Paul S. Dwyer, R. E. Moore, R. E. Boche, S. Shayer, and others made the term popular (see, [Dwyer1951], [Moore1959], [Boche1963], and [Shayer1965]). But the notion of calculating with intervals is not completely new in mathematics: in the course of history, it has been invented and reinvented several times, under different names, and never been abandoned or forgotten. The concept has been known since the third century BC, when Archimedes used guaranteed lower and upper bounds to compute his constant, π (see [Archimedes2002]).

Early in the twentieth century, the idea seemed to be rediscovered. A form of interval arithmetic perhaps first appeared in 1924 by J. C. Burkill in his paper "*Functions of Intervals*" ([Burkill1924]), and in 1931 by R. C. Young in her paper "*The Algebra of Many-Valued Quantities*" ([Young1931]) that gives rules for calculating with intervals and other sets of real numbers; then later in 1951 by Paul S. Dwyer in his book "*Linear computations*" ([Dwyer1951]) that discusses, in a heuristic manner, certain methods for performing basic arithmetic operations on real intervals, and in 1958 by T. Sunaga in his book "*Theory of an Interval Algebra and its Application to Numerical Analysis*" ([Sunaga1958]).

However, it was not until 1959 that new formulations of interval arithmetic were presented. Modern developments of the interval theory began in 1959 with R. E. Moore's technical report "*Automatic Error Analysis in Digital Computation*" ([Moore1959]) in which he developed a number system and an arithmetic dealing with closed real intervals. He called the numbers "*range numbers*" and the arithmetic "*range arithmetic*" to be the first synonyms of "*interval numbers*" and "*interval arithmetic*". Then later in 1962, Moore developed a theory for exact or infinite precision interval arithmetic in his very influential dissertation titled "*Interval Arithmetic and Automatic Error Analysis in Digital Computing*" ([Moore1962]) in which he used a modified digital (*rounded*) interval arithmetic as the basis for automatic analysis of total error in a digital computation. Since then, thousands of research papers and numerous books have appeared on the subject.

Interval arithmetic is now a broad field in which rigorous mathematics is associated with scientific computing. The connection between computing and mathematics provided by intervals makes it possible to solve problems that cannot be efficiently solved using floating-point arithmetic and traditional numerical approximation methods. Today, the interval methods are becoming rapidly popular as a prospective weapon against uncertainties in measurements and errors of numerical computations. Nowadays, interval arithmetic is widely used and has numerous applications in scientific and engineering disciplines that deal intensely with *uncertain data* (or *range data*). Practical application areas include electrical engineering, structure engineering, control theory, remote sensing, quality control, experimental and computational physics, dynamical and chaotic systems, celestial mechanics, signal processing, computer graphics, robotics, and computer-assisted proofs.

1.3 Motivation and Problem Statement

Three important aspects of interval mathematics have motivated the research presented in this thesis. These aspects are summarized as follows.

The Interval Algebras

The algebraic systems of classical interval arithmetic and its present alternates are *primitive algebraic structures*, if compared to the *totally ordered field* of real numbers. So, it is not an easy matter to perform arithmetic or to solve equations in such algebras.

The Interval Dependency Problem

Another main drawback of the present interval theories is that when computing the image of real-valued functions using interval arithmetic, we usually get overestimations that inevitably produce meaningless results, if the variables are *functionally dependent*. This persisting problem, known as the "*interval dependency problem*", is the main problem of interval computations. Despite the fact that there are many special methods and algorithms, based on the classical interval theory, that successfully compute useful narrow bounds to the desirable image (see, e.g., [Moore1966], [Moore1979], [Moore2009], and [Hansen2003]), the problem of evaluating the accurate image, using classical interval arithmetic, is proved to be, in general, NP-hard (see, e.g., [Gaganov1985], [Pedrycz2008], and [Rokne1984]).

Ordering Interval Numbers

The notion of *order* plays an important and indispensable role, as important as that of *size*, not only in mathematics and its applications but also in almost all scientific disciplines. Because of its great importance in both fundamental research and practical applications of the interval theory, the problem of ordering interval numbers has been attempted by many researchers, but with the notion of *size* (or *value*) is considered as the *all-important* aspect of order (which contradicts the defining properties of the notion of order in order theory). Despite extensive research on the subject, *no total ordering* for interval numbers is presented, and the important question of *compatibility* of an ordering with the interval algebraic operations is not touched upon, except for the *partial* ordering by the inclusion relation, $\subseteq$, and its well-known theorem of *inclusion monotonicity* (see theorem 2.8, on page 21).

1.4 Outline of the Thesis

This thesis is structured, in *seven* chapters, as follows.

This introductory chapter has provided a bit of perspective on the field of interval mathematics and its history. It has also formulated the motivations for this research.

Chapter 2 opens with the key concepts of the *classical interval theory* and then introduces the algebraic and point operations for interval numbers. In section 2.3, we carefully construct the algebraic system of classical interval arithmetic, deduce its fundamental properties, and finally prove that it is a *nondistributive abelian semiring*. In section 2.4, we describe the key concepts of machine real arithmetic, then we construct the algebra of *machine interval arithmetic* and deduce some of its fundamental properties.

In chapter 3, we *formalize* the notion of *interval dependency*, along with discussing the algebraic systems of two important alternate theories of interval arithmetic: *modal interval arithmetic*, and *constraint interval arithmetic*. In section 3.2, we introduce the basic concepts of modal interval arithmetic, deduce its fundamental algebraic properties, and finally uncover the algebraic system of modal intervals to be a *nondistributive abelian ring*. In section 3.3, we pay great attention, in particular, to studying the foundations of the theory of constraint intervals, and deduce some important results about its underlying algebraic system.

Based on the idea of representing a real closed interval as a convex set, along with our formalization of the notion of interval dependency, we attempt, in chapter 4, to present an alternate theory of intervals, namely the "*theory of optimizational intervals*", with a mathematical construction that tries to avoid some of the defects in the current theories of interval arithmetic, to provide a richer interval algebra, and to better account for the notion of interval dependency. In sections 4.1 and 4.2, we begin by defining the key concepts of the optimizational interval theory, and then we formulate the basic operations and relations for optimizational interval numbers. In section 4.3, we carefully construct the algebraic system of optimizational interval arithmetic, deduce its fundamental properties, and then prove that the optimizational interval theory constitutes a *rich $\mathcal{S}$-field algebra*, which extends the *ordinary field structure* of real numbers. In the final section of this chapter, we discuss some further consequences and future prospects concerning the obtained results.

In the first section of chapter 5, we construct the algebraic system of a *classical* complex interval arithmetic, defined in terms of the classical interval theory, and deduce its fundamental properties. Sections 5.2 and 5.3 are devoted to presenting a new systematic construction of complex interval arithmetic, based on the theory of optimizational intervals. In section 5.2, we define the key concepts of the *optimizational* theory of complex intervals, and then we formulate the basic operations and relations for *optimizational complex intervals*. In section 5.3, we carefully construct the algebraic system of optimizational complex interval arithmetic, deduce its fundamental properties, and then prove that optimizational complex interval arithmetic possesses a rich $\mathcal{S}$-field algebra, which extends the field structure of ordinary complex numbers and the $\mathcal{S}$-field of optimizational interval numbers.

Chapter 6 opens with some order-theoretical preliminaries concerning ordering relations and their properties. Section 6.2 gives a survey of the existing *set-theoretic* approaches for ordering interval numbers, along with discussing their *compatibility* with the interval algebraic operations. In section 6.3, we characterize the class $\mathfrak{O}_{(\wp(\mathbb{R}),\leq_{\mathbb{R}})}$ of all possible ordering relations on the powerset $\wp(\mathbb{R})$ of the reals, in terms of a binary quantification matrix, two real variable symbols, and the standard ordering relation $\leq_{\mathbb{R}}$ on $\mathbb{R}$. Then, we present the proofs that *neither* the set $[\mathbb{R}]$ of interval numbers *nor* the powerset $\wp(\mathbb{R})$ of the reals *can be totally ordered* by any of the relations in the class $\mathfrak{O}_{(\wp(\mathbb{R}),\leq_{\mathbb{R}})}$. In section 6.4, we define a *set-theoretic* ordering relation $\preceq_{\mathcal{I}}$ on the set $[\mathbb{R}]$ of interval numbers, then we present the proofs that the relation $\preceq_{\mathcal{I}}$ is a *non-strict total ordering* on $[\mathbb{R}]$, *compatible* with interval addition and multiplication, *dense* in $[\mathbb{R}]$, and *weakly Archimedean* in $[\mathbb{R}]$. Furthermore, we prove that the relation $\preceq_{\mathcal{I}}$ induces a *distributive lattice structure* for interval numbers, and that $\preceq_{\mathcal{I}}$ is an extension of the usual ordering $\leq_{\mathbb{R}}$ on the reals, *Moore's partial ordering* $<_{\mathrm{M}}$ on $[\mathbb{R}]$, and *Kulisch's partial ordering* $\leq_{\mathrm{K}}$ on $[\mathbb{R}]$.

Finally, the thesis closes with chapter 7, which explicitly delineates the contributions of this research and outlines some directions and perspectives for future research.

Chapter 2

The Classical Theory of Interval Arithmetic

Algebra is the metaphysics of arithmetic.

–John Ray (1627-1705)

A very simple and natural idea is that of enclosing real numbers in real closed intervals. Based on this idea, this chapter is devoted to rigorously defining and constructing the number system of *classical interval arithmetic* and developing the most fundamental tools for classical interval analysis (For other constructions of the classical interval theory, the reader may consult, e.g., [Jaulin2001], [Moore1962], [Moore2009], and [Shayer1965]).

The chapter opens with the key concepts of the classical interval theory and then introduces the algebraic and point operations for interval numbers. In section 2.3, we carefully construct the algebraic system of classical interval arithmetic, deduce its fundamental properties, and finally prove that it is a *nondistributive abelian semiring*. Finally, in section 2.4, we describe the key concepts of machine real arithmetic, then we construct the algebra of *machine interval arithmetic* and deduce some of its fundamental properties.

In chapter 6, we endow the algebraic system of classical interval arithmetic with a *total ordering*, and prove that it is *compatible* with the algebraic operations. In chapter 3, we formalize the notion of interval dependency, along with discussing the algebraic systems of some alternate theories of interval arithmetic. Furthermore, with a view to treating some problems of the present theories of interval arithmetic, chapter 4 is devoted to present an alternate theory of intervals, namely the "*theory of optimizational intervals*".

2.1 Algebraic Operations for Classical Interval Numbers

The basic algebraic operations for real numbers can be extended to classical interval numbers. In this section, we shall formulate the basic relations and algebraic operations for classical interval numbers. Hereafter and throughout this work, the machinery used, and assumed priori, is the *standard* (*classical*) predicate calculus and axiomatic set theory. Moreover, in all the proofs, elementary facts about operations and relations on the real numbers are usually used without explicit reference.

We first define what a classical interval number is.

Definition 2.1 *Let* $\underline{x}, \overline{x} \in \mathbb{R}$ *such that* $\underline{x} \leq \overline{x}$. *A classical interval number* $[\underline{x}, \overline{x}]$ *is a closed and bounded nonempty real interval, that is*

$$[\underline{x}, \overline{x}] = \{x \in \mathbb{R} | \underline{x} \leq x \leq \overline{x}\},$$

where $\underline{x} = \min([\underline{x}, \overline{x}])$ *and* $\overline{x} = \max([\underline{x}, \overline{x}])$ *are called, respectively, the lower and upper bounds (endpoints) of* $[\underline{x}, \overline{x}]$.

The set of classical interval numbers shall be denoted[1] by $[\mathbb{R}]$. It is a proper subset of the powerset of $\mathbb{R}$, that is

$$[\mathbb{R}] = \{X \in \wp(\mathbb{R}) \,|\, (\exists \underline{x} \in \mathbb{R}) (\exists \overline{x} \in \mathbb{R}) (\underline{x} \leq \overline{x} \wedge X = [\underline{x}, \overline{x}])\}.$$

Since, corresponding to each pair of real numbers $\underline{x}, \overline{x}$ ($\underline{x} \leq \overline{x}$) there exists a real closed interval $[\underline{x}, \overline{x}]$, the set of classical interval numbers is *infinite*.

For simplicity of the language, throughout this chapter, we shall usually use the expression "*interval numbers*" instead of the expression "*classical interval numbers*" and the expression "*interval arithmetic*" instead of the expression "*classical interval arithmetic*". Moreover, in order to be able easily to speak of different types of interval numbers, it is convenient to introduce some notational conventions.

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[1] Although we have *many* theories of intervals being discussed throughout the text, we always deal with the *same* set of real closed intervals. However, for legibility and brevity; we shall employ *different notations* for the set of interval numbers, for example, $[\mathbb{R}]$, ${}^{\mathrm{t}}[\mathbb{R}]$, ${}^{\mathrm{o}}[\mathbb{R}]$, and so forth, according to what theory of intervals is being discussed. So, we can write, for instance, the brief expressions "*addition on* $[\mathbb{R}]$" and "*addition on* ${}^{\mathrm{t}}[\mathbb{R}]$" instead respectively of the expressions "*classical interval addition*" and "*constraint interval addition*".

Notation 2.1 *The set of all zeroless interval numbers (interval numbers that do not contain the real number* 0*) is denoted by* $[\mathbb{R}]_{\tilde{0}}$*, that is*

$$[\mathbb{R}]_{\tilde{0}} = \{X \in [\mathbb{R}] \,|0 \notin X\}.$$

Notation 2.2 *The set of all symmetric interval numbers is denoted by* $[\mathbb{R}]_s$*, that is*

$$\begin{aligned} [\mathbb{R}]_s &= \{X \in [\mathbb{R}] \,|(\exists x \in \mathbb{R})(0 \leq x \wedge X = [-x, x])\} \\ &= \{X \in [\mathbb{R}] \,|\min(X) = -\max(X)\}. \end{aligned}$$

Notation 2.3 *The set of all singleton (point) interval numbers is denoted by* $[\mathbb{R}]_p$*, that is*

$$\begin{aligned} [\mathbb{R}]_p &= \{X \in [\mathbb{R}] \,|(\exists x \in \mathbb{R})(X = [x, x])\} \\ &= \{X \in [\mathbb{R}] \,|\min(X) = \max(X)\}. \end{aligned}$$

The set $[\mathbb{R}]_p$ is an infinite proper subset of $[\mathbb{R}]$ and is *isomorphically equivalent* to the set $\mathbb{R}$ of real numbers (see theorem 2.11). That is, every element $[x, x] \in [\mathbb{R}]_p$ is an *isomorphic copy* of an element $x \in \mathbb{R}$. By convention, and being less pedantic, we agree to *identify* a point interval number $[x, x] = \{x\}$ with its *real* isomorphic copy x. So, if confusion is not likely to ensue; throughout the text, we may write x for $[x, x]$.

Hereafter, the upper-case Roman letters X, Y, and Z (with or without subscripts), or equivalently $[\underline{x}, \overline{x}]$, $[\underline{y}, \overline{y}]$, and $[\underline{z}, \overline{z}]$, shall be employed as variable symbols to denote elements of $[\mathbb{R}]$.

The following theorem, concerning the equality relation on $[\mathbb{R}]$, is an immediate consequence of the axiom of extensionality[2] of axiomatic set theory.

Theorem 2.1 (Equality on $[\mathbb{R}]$). *The equality relation for interval numbers is formulated in terms of the intervals' endpoints as*

$$[\underline{x}, \overline{x}] = [\underline{y}, \overline{y}] \Leftrightarrow \underline{x} = \underline{y} \wedge \overline{x} = \overline{y}.$$

[2] The axiom of extensionality asserts that two sets are equal if, and only if they have precisely the same elements (see, e.g., [Causey1994], [Devlin1993], and [Kleene1952]), that is

$$(\forall \mathcal{S})\,(\forall \mathcal{T})\,(\mathcal{S} = \mathcal{T} \Leftrightarrow (\forall z)\,(z \in \mathcal{S} \Leftrightarrow z \in \mathcal{T}))\,.$$

Interval numbers are sets of real numbers. It is therefore not surprising that the first proposed ordering relation[3], for interval numbers, was the ordinary set inclusion, $\subseteq$, which presented by Young in [Young1931]. Another ordering relation, $<_{\mathrm{M}}$, that extends the standard strict ordering $<_{\mathbb{R}}$ on the reals, was presented by Moore in [Moore1966]. Both the inclusion relation $\subseteq$ and Moore's relation[4] $<_{\mathrm{M}}$ are partial orderings[5] on $[\mathbb{R}]$. This is made precise in the following two easily derivable theorems.

Theorem 2.2 (The Ordering $\subseteq$ on $[\mathbb{R}]$). *Let $\subseteq$ be a binary relation on $[\mathbb{R}]$ defined by*

$$[\underline{x}, \overline{x}] \subseteq \left[\underline{y}, \overline{y}\right] \Leftrightarrow \underline{y} \leq_{\mathbb{R}} \underline{x} \wedge \overline{x} \leq_{\mathbb{R}} \overline{y}.$$

Then $\subseteq$ is a non-strict partial ordering on $[\mathbb{R}]$.

Proof. The proof is easy by showing that the relation $\subseteq$ is reflexive, antisymmetric, and transitive in $[\mathbb{R}]$. ■

Theorem 2.3 (The Ordering $<_{\mathrm{M}}$ on $[\mathbb{R}]$). *Let $<_{\mathrm{M}}$ be a binary relation on $[\mathbb{R}]$ defined by*

$$[\underline{x}, \overline{x}] <_{\mathrm{M}} \left[\underline{y}, \overline{y}\right] \Leftrightarrow \overline{x} <_{\mathbb{R}} \underline{y}.$$

Then $<_{\mathrm{M}}$ is a strict partial ordering on $[\mathbb{R}]$.

Proof. The proof is easy by showing that the relation $<_{\mathrm{M}}$ is asymmetric and transitive in $[\mathbb{R}]$. ■

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[3] Because of its great importance in both fundamental research and practical applications of the interval theory, the problem of *ordering interval numbers* shall be dealt with, in detail, in chapter 6.

[4] Since we have *many* ordering relations on $[\mathbb{R}]$ being discussed throughout the text, we shall deviate from the standard notation "$<_{[\mathbb{R}]}$" and give explicitly relation subscripts peculiar to each ordering.

[5] If a relation $\Re$ is a partial ordering on a set $\mathcal{S}$, then $\mathcal{S}$ has at least one pair which is *non-comparable*, in symbols

$$(\exists x \in \mathcal{S})\,(\exists y \in \mathcal{S})\,(\neg\,(x \Re y) \wedge \neg\,(y \Re x))\,.$$

If a relation $\Re$ is a total ordering on a set $\mathcal{S}$, then $\mathcal{S}$ has *no non-comparable* pairs (see, e.g, [Causey1994], [Devlin1993], and [Gleason1992]), in symbols

$$(\forall x \in \mathcal{S})\,(\forall y \in \mathcal{S})\,(x \Re y \vee y \Re x)\,.$$

Unlike the case with $<_{\mathrm{M}}$, the partial ordering by the set inclusion is *compatible*[6] with the interval algebraic operations (see theorem 2.8, on page 21); and this is the reason why it plays an important role in Moore's foremost work in interval analysis (see, e.g. [Moore1966], [Moore1979], and [Moore2009]).

Example 2.1 *For four given interval numbers* $A = [1, 2]$, $B = [1, 2]$, $C = [1, 3]$, *and* $D = [4, 7]$, *we have* $A = B <_{\mathrm{M}} D$ *and* $A \subseteq B \subseteq C$.

We now proceed to define the basic algebraic operations *for*[7] interval numbers: two binary operations, namely *addition* ("+") and *multiplication* ("×"), and two unary operations, namely *negation* ("−") and *reciprocal* ("$^{-1}$").

According to the fact that interval numbers are *sets*, the binary and unary classical algebraic operations for interval numbers can be characterized, respectively, in the following two *set-theoretic* definitions.

Definition 2.2 (Binary Operations in[8] $[\mathbb{R}]$). *For any two interval numbers* X *and* Y, *the binary classical algebraic operations are defined by*

$$X \circ_{\mathrm{c}} Y = \{z \in \mathbb{R} |\, (\exists x \in X)\, (\exists y \in Y)\, (z = x \circ_{\mathbb{R}} y)\},$$

where $\circ \in \{+, \times\}$.

Definition 2.3 (Unary Operations in $[\mathbb{R}]$). *For any interval number* X, *the unary classical algebraic operations are defined by*

$$\diamond_{\mathrm{c}} X = \{z \in \mathbb{R} |\, (\exists x \in X)\, (z = \diamond_{\mathbb{R}} x)\},$$

where $\diamond \in \{-, ^{-1}\}$ *and* $0 \notin X$ *if* $\diamond$ *is* "$^{-1}$".

Hereafter, if confusion is unlikely, the subscript "c", which stands for "*classical* interval operation"[9], and the subscript "$\mathbb{R}$", in the real relation and operation symbols, may be suppressed.

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6 For a detailed characterization of the notion of *order compatibility* with the algebraic operations and other order-theoretic notions, see section 6.1, on page 127.

7 For a set $\mathcal{S}$ and an algebraic operation $\circ_{\mathcal{S}}$, we write the expressions "the operation $\circ_{\mathcal{S}}$ *for* $\mathcal{S}$" and "the operation $\circ_{\mathcal{S}}$ *in* $\mathcal{S}$" to mean that "$\circ_{\mathcal{S}}$" is a *partial or total* operation, in the *general* sense. To indicate that "$\circ_{\mathcal{S}}$" is a *total* operation in $\mathcal{S}$, we write "the operation $\circ_{\mathcal{S}}$ *on* $\mathcal{S}$".

8 See footnotes 1 and 7 of this section.

9 Since we have *many* theories of intervals being discussed throughout the text, with each theory has its *own* operations for the set $[\mathbb{R}]$ of interval numbers; we shall deviate from the standard notation "$\circ_{[\mathbb{R}]}$" and give explicitly operation subscripts peculiar to each theory.

By means of the above set-theoretic definitions and from the fact that interval numbers are *ordered sets* of real numbers, the following four theorems are derivable.

Theorem 2.4 (Addition on $[\mathbb{R}]$). *For any two interval numbers $[\underline{x}, \overline{x}]$ and $\left[\underline{y}, \overline{y}\right]$, classical interval addition is a total*[10] *operation, on $[\mathbb{R}]$, formulated in terms of the intervals' endpoints as*

$$[\underline{x}, \overline{x}] + \left[\underline{y}, \overline{y}\right] = \left[\underline{x} + \underline{y}, \overline{x} + \overline{y}\right] .$$

Proof. Since addition on $\mathbb{R}$ is continuous, it follows, by definition 2.2, that addition in $[\mathbb{R}]$ is continuous, and $[\underline{x}, \overline{x}] + \left[\underline{y}, \overline{y}\right]$ attains its minimum and maximum values.

Then, by definition 2.1, we have

$$\begin{aligned} \min\left([\underline{x}, \overline{x}] + \left[\underline{y}, \overline{y}\right]\right) &= \min_{\substack{x \in [\underline{x}, \overline{x}] \\ y \in [\underline{y}, \overline{y}]}} (x + y) = \underline{x} + \underline{y}, \\ \max\left([\underline{x}, \overline{x}] + \left[\underline{y}, \overline{y}\right]\right) &= \max_{\substack{x \in [\underline{x}, \overline{x}] \\ y \in [\underline{y}, \overline{y}]}} (x + y) = \overline{x} + \overline{y}. \end{aligned}$$

We thus obtain

$$[\underline{x}, \overline{x}] + \left[\underline{y}, \overline{y}\right] = \left[\underline{x} + \underline{y}, \overline{x} + \overline{y}\right] ,$$

and therefore classical interval addition is a *total* binary operation *on* $[\mathbb{R}]$. ∎

Theorem 2.5 (Multiplication on $[\mathbb{R}]$). *For any two interval numbers $[\underline{x}, \overline{x}]$ and $\left[\underline{y}, \overline{y}\right]$, classical interval multiplication is a total operation, on $[\mathbb{R}]$, formulated in terms of the intervals' endpoints as*

$$[\underline{x}, \overline{x}] \times \left[\underline{y}, \overline{y}\right] = \left[\min\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}, \max\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}\right] .$$

Proof. Since multiplication on $\mathbb{R}$ is continuous, it follows, by definition 2.2, that multiplication in $[\mathbb{R}]$ is continuous, and $[\underline{x}, \overline{x}] \times \left[\underline{y}, \overline{y}\right]$ attains its minimum and maximum values.

[10] Let $\mathcal{S}^{\langle n \rangle}$ be the n-th Cartesian power of a set $\mathcal{S}$. An n-ary (total) operation on $\mathcal{S}$ is a total function $t_n : \mathcal{S}^{\langle n \rangle} \longmapsto \mathcal{S}$. An n-ary partial operation in $\mathcal{S}$ is a partial function $p_n : \mathcal{U} \longmapsto \mathcal{S}$, where $\mathcal{U} \subset \mathcal{S}^{\langle n \rangle}$. The ordinal n is called the arity of t_n or p_n.

Then, by definition 2.1, we have the following cases for $\min\left([\underline{x},\overline{x}]\times\left[\underline{y},\overline{y}\right]\right)$ and $\max\left([\underline{x},\overline{x}]\times\left[\underline{y},\overline{y}\right]\right)$.

$$\min\left([\underline{x},\overline{x}]\times\left[\underline{y},\overline{y}\right]\right)=\begin{cases}\underline{x}\underline{y} & \text{if } 0<\underline{x}\wedge 0<\underline{y},\\ \overline{x}\underline{y} & \text{if } 0<\underline{x}\wedge 0>\overline{y},\\ \underline{x}\overline{y} & \text{if } 0>\overline{x}\wedge 0<\underline{y},\\ \overline{x}\overline{y} & \text{if } 0>\overline{x}\wedge 0>\overline{y},\\ \underline{x}\overline{y} & \text{if } \underline{x}<0<\overline{x}\wedge 0<\underline{y},\\ \overline{x}\underline{y} & \text{if } \underline{x}<0<\overline{x}\wedge 0>\overline{y},\\ \overline{x}\underline{y} & \text{if } 0<\underline{x}\wedge \underline{y}<0<\overline{y},\\ \underline{x}\overline{y} & \text{if } 0>\underline{x}\wedge \underline{y}<0<\overline{y},\\ \min\left(\overline{x}\underline{y},\underline{x}\overline{y}\right) & \text{if } \underline{x}<0<\overline{x}\wedge \underline{y}<0<\overline{y};\end{cases}$$

$$\max\left([\underline{x},\overline{x}]\times\left[\underline{y},\overline{y}\right]\right)=\begin{cases}\overline{x}\overline{y} & \text{if } 0<\underline{x}\wedge 0<\underline{y},\\ \underline{x}\overline{y} & \text{if } 0<\underline{x}\wedge 0>\overline{y},\\ \overline{x}\underline{y} & \text{if } 0>\overline{x}\wedge 0<\underline{y},\\ \underline{x}\underline{y} & \text{if } 0>\overline{x}\wedge 0>\overline{y},\\ \overline{x}\overline{y} & \text{if } \underline{x}<0<\overline{x}\wedge 0<\underline{y},\\ \underline{x}\underline{y} & \text{if } \underline{x}<0<\overline{x}\wedge 0>\overline{y},\\ \overline{x}\overline{y} & \text{if } 0<\underline{x}\wedge \underline{y}<0<\overline{y},\\ \underline{x}\underline{y} & \text{if } 0>\underline{x}\wedge \underline{y}<0<\overline{y},\\ \max\left(\underline{x}\underline{y},\overline{x}\overline{y}\right) & \text{if } \underline{x}<0<\overline{x}\wedge \underline{y}<0<\overline{y}.\end{cases}$$

That is,

$$\begin{aligned}\min\left([\underline{x},\overline{x}]\times\left[\underline{y},\overline{y}\right]\right) &= \min\{\underline{x}\underline{y},\underline{x}\overline{y},\overline{x}\underline{y},\overline{x}\overline{y}\},\\ \max\left([\underline{x},\overline{x}]\times\left[\underline{y},\overline{y}\right]\right) &= \max\{\underline{x}\underline{y},\underline{x}\overline{y},\overline{x}\underline{y},\overline{x}\overline{y}\}.\end{aligned}$$

We therefore obtain

$$[\underline{x},\overline{x}]\times\left[\underline{y},\overline{y}\right]=\left[\min\{\underline{x}\underline{y},\underline{x}\overline{y},\overline{x}\underline{y},\overline{x}\overline{y}\},\max\{\underline{x}\underline{y},\underline{x}\overline{y},\overline{x}\underline{y},\overline{x}\overline{y}\}\right],$$

and classical interval multiplication thus is a *total* binary operation *on* $[\mathbb{R}]$. ■

Theorem 2.6 (Negation on $[\mathbb{R}]$). *For any interval number $[\underline{x},\overline{x}]$, classical interval negation is a total operation, on $[\mathbb{R}]$, formulated in terms of the interval endpoints as*

$$-[\underline{x},\overline{x}]=[-\overline{x},-\underline{x}].$$

Proof. Since negation on $\mathbb{R}$ is continuous, it follows, by definition 2.3, that negation in $[\mathbb{R}]$ is continuous, and $-[\underline{x}, \overline{x}]$ attains its minimum and maximum values.

Then, by definition 2.1, we have

$$\begin{aligned} \min\left(-[\underline{x}, \overline{x}]\right) &= -\overline{x}, \\ \max\left(-[\underline{x}, \overline{x}]\right) &= -\underline{x}. \end{aligned}$$

We thus obtain

$$-[\underline{x}, \overline{x}] = [-\overline{x}, -\underline{x}],$$

and therefore classical interval negation is a *total* unary operation *on* $[\mathbb{R}]$. ■

Theorem 2.7 (Reciprocal in $[\mathbb{R}]$). *For any interval number* $[\underline{x}, \overline{x}] \in [\mathbb{R}]_{\tilde{0}}$ *(that is,* $0 \notin [\underline{x}, \overline{x}]$*), classical interval reciprocal is a partial operation, in* $[\mathbb{R}]$*, formulated in terms of the interval endpoints as*

$$[\underline{x}, \overline{x}]^{-1} = \left[\overline{x}^{-1}, \underline{x}^{-1}\right].$$

Proof. Since real reciprocal is continuous for nonzero elements of $\mathbb{R}$, it follows, by definition 2.3, that interval reciprocal is continuous on the subset $[\mathbb{R}]_{\tilde{0}}$ of $[\mathbb{R}]$, and $[\underline{x}, \overline{x}]^{-1}$ attains its minimum and maximum values.

Then, by definition 2.1, we have

$$\begin{aligned} \min\left([\underline{x}, \overline{x}]^{-1}\right) &= \overline{x}^{-1}, \\ \max\left([\underline{x}, \overline{x}]^{-1}\right) &= \underline{x}^{-1}. \end{aligned}$$

We thus obtain

$$[\underline{x}, \overline{x}]^{-1} = \left[\overline{x}^{-1}, \underline{x}^{-1}\right],$$

and therefore classical interval reciprocal is a *partial* unary operation *in* $[\mathbb{R}]$. ■

In accordance with the above theorems, we can now define the *total* operation of "*subtraction*", and the *partial* operations of "*division*" and "*integer exponentiation*", for classical interval numbers.

Definition 2.4 (Subtraction on $[\mathbb{R}]$). *For any two interval numbers* X *and* Y*, classical interval subtraction is defined by*

$$X - Y = X + (-Y).$$

Definition 2.5 (Division in $[\mathbb{R}]$). *For any $X \in [\mathbb{R}]$ and any $Y \in [\mathbb{R}]_{\tilde{0}}$, classical interval division is defined by*

$$X \div Y = X \times \left(Y^{-1}\right).$$

Definition 2.6 (Integer Exponentiation in $[\mathbb{R}]$). *For any interval number X and any integer n, the integer exponents of X are defined, in terms of multiplication and reciprocal in $[\mathbb{R}]$, by the following recursion scheme:*

(i) $X^0 = [1, 1]$,

(ii) $0 < n \Rightarrow X^n = X^{n-1} \times X$,

(iii) $0 \notin X \wedge 0 \leq n \Rightarrow X^{-n} = \left(X^{-1}\right)^n$.

Since the classical interval operations are defined in terms of the corresponding real algebraic operations, and as long as division by zero is disallowed; it follows that the result of a classical interval operation is always an interval number.

Some numerical examples are shown below.

Example 2.2 *For three given interval numbers $[1, 2]$, $[3, 4]$, and $[-2, 2]$, we have*

(i) $[1, 2] + [3, 4] = [4, 6]$,

(ii) $[1, 2] \times [3, 4] = [3, 8]$,

(iii) $[1, 2]^{-1} = [1/2, 1]$,

(iv) $[-2, 2]^2 = [-2, 2] \times [-2, 2] = [-4, 4]$.

The result $[-4, 4]$ of $[-2, 2]^2$, in the above example, is not natural in the sense that a square is always nonnegative. Generally, for any *non-point* interval number $[\underline{x}, \overline{x}]$, with $0 \in [\underline{x}, \overline{x}]$, the square of $[\underline{x}, \overline{x}]$ is given by

$$\begin{aligned} [\underline{x}, \overline{x}]^2 &= [\underline{x}, \overline{x}] \times [\underline{x}, \overline{x}] \\ &= \left[\min\{\underline{x}^2, \underline{x}\overline{x}, \overline{x}^2\}, \max\{\underline{x}^2, \underline{x}\overline{x}, \overline{x}^2\}\right] \\ &= \left[\underline{x}\overline{x}, \max\{\underline{x}^2, \overline{x}^2\}\right], \end{aligned}$$

which is consistent with classical interval multiplication (theorem 2.5), but it is *not consistent* with the fact that a square is always nonnegative, for the case when $\underline{x}\overline{x} < 0$. However, if we changed it to be

$$\begin{aligned}[\underline{x}, \overline{x}]^2 &= \{z \in \mathbb{R} | (\exists x \in [\underline{x}, \overline{x}]) \left(z = x^2\right)\} \\ &= \left[0, \max\{\underline{x}^2, \overline{x}^2\}\right],\end{aligned}$$

then it would be inconsistent with classical interval multiplication. This is not a problem if interval arithmetic is regarded as a *numerical approximation method*, for real-valued problems, such that the result of an interval operation contains all possible solutions. But if interval arithmetic is regarded as a *definitional extension*[11] of the theory of real numbers (this is the case in almost all interval literature), in the logical sense; then the theory of interval arithmetic is *not consistent*. In this work, we shall regard interval arithmetic as a numerical approximation tool of guaranteed efficiency against computation errors, in the sense we discussed in chapter 1.

---

[11] A theory T_E is called a *definitional extension* of a theory T iff T_E is obtained from T by adding new relation symbols and function symbols defined in terms of symbols of T (see, e.g., [Kleene1952], [Rasiowa1963] and [Tarski1965]).

Defining relation symbols. Let $\varphi(x_1, ..., x_n)$ be a formula of T such that $x_1, ..., x_n$ occur free in φ. A new theory T_E can be obtained from T by adding a new n-ary relation symbol R, the logical axioms featuring R, and the new *definitional axiom* of R

$$(\forall x_1) ... (\forall x_n) (R(x_1, ..., x_n) \Leftrightarrow \varphi(x_1, ..., x_n)).$$

An example of such a definition of relation symbols is the definition of a *closed interval* in terms of the order relation $\leq$ of the theory of real numbers.

Defining function symbols. Let $\varphi(y, x_1, ..., x_n)$ be a formula of T such that $y, x_1, ..., x_n$ occur free in φ. Assume that the sentence

$$(\forall x_1) ... (\forall x_n) (\exists! y) (\varphi(y, x_1, ..., x_n)),$$

is provable in T; that is, for all $x_1, ..., x_n$, there exists a unique y such that $\varphi(y, x_1, ..., x_n)$. A new theory T_E can be obtained from T by adding a new n-ary relation symbol F, the logical axioms featuring F, and the new definitional axiom of F

$$(\forall x_1) ... (\forall x_n) (\varphi(F(x_1, ..., x_n), x_1, ..., x_n)).$$

An example of such a definition of function symbols is the definition of the *interval algebraic operations* in terms of the algebraic operations of the theory of real numbers.

If T_E is a *consistent* definitional extension of T, then for any formula ψ of T_E we can form a formula φ of T, called a translation of ψ into T, such that $\psi \Leftrightarrow \varphi$ is provable in T_E. Such a formula is *not unique*, but any two formulas φ_1 and φ_2 can be proved to be equivalent in T. That is, for T_E to be a consistent definitional extension of T; if $\psi_1 \Leftrightarrow \varphi_1$, $\psi_2 \Leftrightarrow \varphi_2$, and $\varphi_1 \Leftrightarrow \varphi_2$, then $\psi_1 \Leftrightarrow \psi_2$.

2.2 Point Operations for Classical Interval Numbers

A *point* (or *scalar*) interval operation is an operation whose operands are interval numbers, and whose result is a *point* interval (or, equivalently, a real number). This is made precise in the following definition.

Definition 2.7 (Point Interval Operations). *Let* $[\mathbb{R}]^{\langle n\rangle}$ *be the* n*-th Cartesian power of* $[\mathbb{R}]$. *An* n*-ary point interval operation,* ω_n*, is a function that maps elements of* $[\mathbb{R}]^{\langle n\rangle}$ *to the set* $[\mathbb{R}]_p$ *of point interval numbers, that is*

$$\omega_n : [\mathbb{R}]^{\langle n\rangle} \longmapsto [\mathbb{R}]_p .$$

Several point interval operations can be defined. Next we define some *unary* and *binary* point interval operations.

Definition 2.8 (Interval Infimum). *The infimum of an interval number* $[\underline{x}, \overline{x}]$ *is defined to be*

$$\inf([\underline{x}, \overline{x}]) = \min([\underline{x}, \overline{x}]) = \underline{x}.$$

Definition 2.9 (Interval Supremum). *The supremum of an interval number* $[\underline{x}, \overline{x}]$ *is defined to be*

$$\sup([\underline{x}, \overline{x}]) = \max([\underline{x}, \overline{x}]) = \overline{x}.$$

Definition 2.10 (Interval Width). *The width of an interval number* $[\underline{x}, \overline{x}]$ *is defined to be*

$$w([\underline{x}, \overline{x}]) = \overline{x} - \underline{x}.$$

Thus, the width of a point interval number is *zero*, that is

$$(\forall x \in \mathbb{R})\,(w([x, x]) = x - x = 0)\,.$$

Definition 2.11 (Interval Radius). *The radius of an interval number* $[\underline{x}, \overline{x}]$ *is defined to be*

$$r([\underline{x}, \overline{x}]) = \frac{w([\underline{x}, \overline{x}])}{2} = \frac{(\overline{x} - \underline{x})}{2}.$$

Definition 2.12 (Interval Midpoint). *The midpoint (or mean) of an interval number* $[\underline{x}, \overline{x}]$ *is defined to be*

$$m([\underline{x}, \overline{x}]) = \frac{(\underline{x} + \overline{x})}{2}.$$

Hence, the midpoint of a point interval number is its *real isomorphic copy*, that is

$$(\forall x \in \mathbb{R}) \left(m\left([x, x]\right) = \frac{(x + x)}{2} = x \right).$$

We observe that any interval number X can be expressed, in terms of its width and its midpoint, as the sum of its midpoint and a corresponding symmetric interval, that is

$$X = m\left(X\right) + \left[-\frac{w\left(X\right)}{2}, \frac{w\left(X\right)}{2} \right],$$

where, by convention, $m\left(X\right) = \left[m\left(X\right), m\left(X\right)\right]$.

Definition 2.13 (Interval Absolute Value). *The absolute value of an interval number* $[\underline{x}, \overline{x}]$ *is defined, in terms of the absolute values of its real endpoints, to be*

$$\left|[\underline{x}, \overline{x}]\right| = \max\{\left|\underline{x}\right|, \left|\overline{x}\right|\}.$$

Thus, the absolute value of a point interval number is the usual absolute value of its *real isomorphic copy*, that is

$$(\forall x \in \mathbb{R}) \left(\left|[x, x]\right| = \max\{|x|, |x|\} = |x| \right).$$

All the above point interval operations are unary operations. An important definition of a binary point interval operation is that of the *interval distance* (*interval metric* , or *Moore's metric*[12]).

Definition 2.14 (Interval Distance). *The distance (or metric) between two interval numbers* $[\underline{x}, \overline{x}]$ *and* $\left[\underline{y}, \overline{y}\right]$ *is defined to be*

$$d\left([\underline{x}, \overline{x}], \left[\underline{y}, \overline{y}\right] \right) = \max\{\left|\underline{x} - \underline{y}\right|, \left|\overline{x} - \overline{y}\right|\}.$$

The importance of this definition is that starting from the distance function for interval numbers, we can verify that it induces a *metric space*[13] for interval

[12] *Moore's metric* is named after the American mathematician Ramon Edgar Moore, who was the first to characterize the interval metricity and proved that it induces an interval metric space (see [Moore1959]).

[13] A *metric space* is an ordered pair (S, d), where S is a set and d is a metric on S, that is

$$d : S^{\langle 2 \rangle} \longmapsto \mathbb{R},$$

numbers which is a generalization of the usual metric space of real numbers. Thus, the notions of a sequence, convergence, continuity, and a limit can be defined for interval numbers in the standard way. These notions give rise to what we may call a "*measure theory for interval numbers*". An interval measure theory is beyond the scope of this work.

Example 2.3 *For three given interval numbers* $[1,2]$, $[3,4]$, *and* $[-5,3]$, *we have*

(i) $w([1,2]) = w([3,4]) = 1$,

(ii) $m([1,2]) = 3/2$, $m([3,4]) = 7/2$,

(iii) $|[-5,3]| = \max\{|-5|, |3|\} = 5$,

(iv) $d([1,2],[3,4]) = \max\{|1-3|, |2-4|\} = 2$.

2.3 Algebraic Properties of Classical Interval Arithmetic

By means of the notions prescribed in sections 2.1 and 2.2, we shall now inquire into some fundamental theorems concerning interval arithmetic. By virtue of our definition of an interval number, the properties of real numbers are naturally assumed priori.

A first important theorem we shall now prove is the *inclusion monotonicity* theorem for classical interval arithmetic, which asserts that the partial ordering by the set inclusion relation is *compatible*[14] with the algebraic operations on the set $[\mathbb{R}]$ of interval numbers.

Theorem 2.8 (Inclusion Monotonicity in $[\mathbb{R}]$). *Let* X_1, X_2, Y_1, *and* Y_2 *be interval numbers such that* $X_1 \subseteq Y_1$ *and* $X_2 \subseteq Y_2$. *Then for any classical interval operation* $\circ \in \{+, \times\}$, *we have*

$$X_1 \circ X_2 \subseteq Y_1 \circ Y_2.$$

---

such that for any $x, y, z \in S$, the following holds (see [Bryant1985]):

- $x = y \Leftrightarrow d(x,y) = 0$,
- $d(x,y) = d(y,x)$,
- $d(x,z) \leq d(x,y) + d(y,z)$.

[14] For a precise characterization of the notion of *order compatibility* with the algebraic operations, see definition 6.21, on page 133.

Proof. By hypothesis, we have $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$. Then, according to definition 2.2, we have

$$\begin{aligned} X_1 \circ X_2 &= \{r \in \mathbb{R} | (\exists x_1 \in X_1)(\exists x_2 \in X_2)(r = x_1 \circ x_2)\} \\ &\subseteq \{s \in \mathbb{R} | (\exists y_1 \in Y_1)(\exists y_2 \in Y_2)(s = y_1 \circ y_2)\} \\ &= Y_1 \circ Y_2, \end{aligned}$$

and the theorem follows. ■

In consequence of this theorem, from the fact that $[x, x] \subseteq X \Leftrightarrow x \in X$, we have the following important special case.

Corollary 2.1 *Let X and Y be interval numbers with $x \in X$ and $y \in Y$. Then for any classical interval operation $\circ \in \{+, \times\}$, we have*

$$x \circ y \in X \circ Y.$$

In contrast to the case for $\subseteq$, Moore's partial ordering $<_{\mathrm{M}}$ is not compatible with the algebraic operations on $[\mathbb{R}]$.

Theorem 2.9 *The partial ordering $<_{\mathrm{M}}$ is not compatible with the algebraic operations on $[\mathbb{R}]$.*

Proof. To prove the theorem, it suffices to give a counterexample.

Let $X = [-1, 2]$, $Y = [3, 4]$, and $Z = [1, 2]$ be interval numbers. According to theorem 2.3, we have $X <_{\mathrm{M}} Y$ and $[0, 0] <_{\mathrm{M}} Z$. But

$$X \times Z = [-2, 4] \not<_{\mathrm{M}} [3, 8] = Y \times Z,$$

and therefore the ordering $<_{\mathrm{M}}$ is not compatible with the algebraic operations on $[\mathbb{R}]$. ■

Thus, an *ordered* interval algebra, relative to the ordering $<_{\mathrm{M}}$, is *undefinable*[15]. However, we have the following easy-derivable special case for the set $[\mathbb{R}]_p$ of point interval numbers.

Theorem 2.10 (Monotonicity of $<_{\mathrm{M}}$ in $[\mathbb{R}]_p$). *The partial ordering $<_{\mathrm{M}}$ is compatible with the algebraic operations on the set $[\mathbb{R}]_p$ of point interval numbers.*

[15] For a characterization of the notion of *definability* of an *ordered* algebra, see section 6.1, on page 127.

By virtue of the compatibility of the relation $<_{\mathrm{M}}$ with the algebraic operations on $[\mathbb{R}]_p$, it follows that the *ordered* structure $\left\langle [\mathbb{R}]_p ; +_{\mathrm{c}}, \times_{\mathrm{c}}; <_{\mathrm{M}} \right\rangle$ is *definable*, and we then have the *isomorphism theorem* for classical interval arithmetic.

Theorem 2.11 *The structure* $\left\langle [\mathbb{R}]_p ; +_{\mathrm{c}}, \times_{\mathrm{c}}; <_{\mathrm{M}} \right\rangle$ *is isomorphically equivalent to the ordered field* $\langle \mathbb{R}; +_{\mathbb{R}}, \times_{\mathbb{R}}; <_{\mathbb{R}} \rangle$ *of real numbers.*

Proof. Let $\imath : \mathbb{R} \hookrightarrow [\mathbb{R}]_p$ be the mapping from $\mathbb{R}$ to $[\mathbb{R}]_p$ given by

$$\imath\,(x) = [x, x]\,.$$

The following conditions for $\imath$ are satisfied.

- $\imath$ is a bijection from $\mathbb{R}$ onto $[\mathbb{R}]_p$ since the range of $\imath$ is $[\mathbb{R}]_p$, and, by theorem 2.1,
$$(\forall x \in \mathbb{R})\,(\forall y \in \mathbb{R})\,(\imath\,(x) = \imath\,(y) \Rightarrow x = y)\,.$$
- $\imath$ is function-preserving for "+" since, by theorem 2.4, we have
$$\begin{aligned} \imath\,(x +_{\mathbb{R}} y) &= [x +_{\mathbb{R}} y, x +_{\mathbb{R}} y] \\ &= [x, x] +_{\mathrm{c}} [y, y] \\ &= \imath\,(x) +_{\mathrm{c}} \imath\,(y)\,. \end{aligned}$$
- $\imath$ is function-preserving for "×" since, by theorem 2.5, we have
$$\begin{aligned} \imath\,(x \times_{\mathbb{R}} y) &= [x \times_{\mathbb{R}} y, x \times_{\mathbb{R}} y] \\ &= [x, x] \times_{\mathrm{c}} [y, y] \\ &= \imath\,(x) \times_{\mathrm{c}} \imath\,(y)\,. \end{aligned}$$
- $\imath$ is relation-preserving for "<" since, by theorem 2.3, we have
$$\begin{aligned} x <_{\mathbb{R}} y &\Leftrightarrow [x, x] <_{\mathrm{M}} [y, y] \\ &\Leftrightarrow \imath\,(x) <_{\mathrm{M}} \imath\,(y)\,. \end{aligned}$$

The mapping $\imath$ thus is an isomorphism from $\mathbb{R}$ onto $[\mathbb{R}]_p$ and $\langle \mathbb{R}; +_{\mathbb{R}}, \times_{\mathbb{R}}; <_{\mathbb{R}} \rangle$ is isomorphically equivalent to $\left\langle [\mathbb{R}]_p ; +_{\mathrm{c}}, \times_{\mathrm{c}}; <_{\mathrm{M}} \right\rangle$. ■

That is, up to isomorphism, the sets $\mathbb{R}$ and $[\mathbb{R}]_p$ are equivalent, and each element of $[\mathbb{R}]_p$ is an *isomorphic copy* of an element of $\mathbb{R}$. In other words, *everything that is true for real numbers is certainly true for point interval numbers.*

The properties of the classical interval operations are similar to, but not the same as, those of the real operations. The algebraic properties of the classical interval operations are prescribed by the following theorems.

Theorem 2.12 (Absorbing Element in $[\mathbb{R}]$). *The interval number* $[0, 0]$ *is an absorbing element for classical interval multiplication, that is*

$$(\forall X \in [\mathbb{R}])\left([0,0] \times X = X \times [0,0] = [0,0]\right).$$

Proof. For any interval number X, according to definition 2.2 and assuming the properties of real multiplication, we have

$$\begin{aligned}
[0,0] \times X &= \{r \in \mathbb{R} | (\exists x \in X)(\exists y \in [0,0])(r = y \times x)\} \\
&= \{r \in \mathbb{R} | (\exists x \in X)(\exists y \in [0,0])(r = x \times y)\} \\
&= \{r \in \mathbb{R} | (\exists x \in X)(r = x \times 0)\} \\
&= X \times [0,0] = [0,0],
\end{aligned}$$

and therefore, the point interval number $[0, 0]$ *absorbs* any interval number X by classical interval multiplication. ∎

Theorem 2.13 (Identity for Addition in $[\mathbb{R}]$). *The interval number* $[0, 0]$ *is both a left and right identity for classical interval addition, that is*

$$(\forall X \in [\mathbb{R}])\left([0,0] + X = X + [0,0] = X\right).$$

Proof. For any interval number X, according to definition 2.2 and assuming the properties of real addition, we have

$$\begin{aligned}
[0,0] + X &= \{r \in \mathbb{R} | (\exists x \in X)(\exists y \in [0,0])(r = y + x)\} \\
&= \{r \in \mathbb{R} | (\exists x \in X)(\exists y \in [0,0])(r = x + y)\} \\
&= \{r \in \mathbb{R} | (\exists x \in X)(r = x + 0)\} \\
&= X + [0,0] = X,
\end{aligned}$$

and therefore, the point interval number $[0, 0]$ is both a left and right identity for classical interval addition. ∎

Theorem 2.14 (Identity for Multiplication in $[\mathbb{R}]$). *The interval number* $[1,1]$ *is both a left and right identity for classical interval multiplication, that is*

$$(\forall X \in [\mathbb{R}])\,([1,1] \times X = X \times [1,1] = X)\,.$$

Proof. For any interval number X, according to definition 2.2 and assuming the properties of real multiplication, we have

$$\begin{aligned}
[1,1] \times X &= \{r \in \mathbb{R} |\, (\exists x \in X)\,(\exists y \in [1,1])\,(r = y \times x)\} \\
&= \{r \in \mathbb{R} |\, (\exists x \in X)\,(\exists y \in [1,1])\,(r = x \times y)\} \\
&= \{r \in \mathbb{R} |\, (\exists x \in X)\,(r = x \times 1)\} \\
&= X \times [1,1] = X,
\end{aligned}$$

and therefore, it is shown that $[1,1]$ is both a left and right identity for classical interval multiplication. ■

Theorem 2.15 (Commutativity in $[\mathbb{R}]$). *Both classical interval addition and multiplication are commutative, that is*

(i) $(\forall X, Y \in [\mathbb{R}])\,(X + Y = Y + X)$,

(ii) $(\forall X, Y \in [\mathbb{R}])\,(X \times Y = Y \times X)$.

Proof. *(i)* For any two interval numbers X and Y, according to definition 2.2 and assuming the properties of real addition, we have

$$\begin{aligned}
X + Y &= \{r \in \mathbb{R} |\, (\exists x \in X)\,(\exists y \in Y)\,(r = x + y)\} \\
&= \{r \in \mathbb{R} |\, (\exists x \in X)\,(\exists y \in Y)\,(r = y + x)\} \\
&= Y + X.
\end{aligned}$$

(ii) In a manner analogous to *(i)*, assuming the properties of real multiplication, we have

$$\begin{aligned}
X \times Y &= \{r \in \mathbb{R} |\, (\exists x \in X)\,(\exists y \in Y)\,(r = x \times y)\} \\
&= \{r \in \mathbb{R} |\, (\exists x \in X)\,(\exists y \in Y)\,(r = y \times x)\} \\
&= Y \times X.
\end{aligned}$$

Therefore, both addition and multiplication are commutative in $[\mathbb{R}]$. ■

Theorem 2.16 (Associativity in $[\mathbb{R}]$). *Both classical interval addition and multiplication are associative, that is* *(i)* $(\forall X, Y, Z \in [\mathbb{R}])\,(X + (Y + Z) = (X + Y) + Z)$,

(ii) $(\forall X, Y, Z \in [\mathbb{R}])\,(X \times (Y \times Z) = (X \times Y) \times Z)$.

Proof. *(i)* For any three interval numbers X, Y, and Z, according to definition 2.2 and assuming the properties of real addition, we have

$$\begin{aligned} X + (Y + Z) &= \{r \in \mathbb{R} |\, (\exists x \in X)\,(\exists s \in (Y + Z))\,(r = x + s)\} \\ &= \{r \in \mathbb{R} |\, (\exists x \in X)\,(\exists y \in Y)\,(\exists z \in Z)\,(r = x + (y + z))\} \\ &= \{r \in \mathbb{R} |\, (\exists x \in X)\,(\exists y \in Y)\,(\exists z \in Z)\,(r = (x + y) + z)\} \\ &= \{r \in \mathbb{R} |\, (\exists t \in (X + Y))\,(\exists z \in Z)\,(r = t + z)\} \\ &= (X + Y) + Z. \end{aligned}$$

(ii) In a manner analogous to *(i)*, assuming the properties of real multiplication, we have

$$\begin{aligned} X \times (Y \times Z) &= \{r \in \mathbb{R} |\, (\exists x \in X)\,(\exists s \in (Y \times Z))\,(r = x \times s)\} \\ &= \{r \in \mathbb{R} |\, (\exists x \in X)\,(\exists y \in Y)\,(\exists z \in Z)\,(r = x \times (y \times z))\} \\ &= \{r \in \mathbb{R} |\, (\exists x \in X)\,(\exists y \in Y)\,(\exists z \in Z)\,(r = (x \times y) \times z)\} \\ &= \{r \in \mathbb{R} |\, (\exists t \in (X \times Y))\,(\exists z \in Z)\,(r = t \times z)\} \\ &= (X \times Y) \times Z. \end{aligned}$$

Therefore, both addition and multiplication are associative in $[\mathbb{R}]$. ■

Theorem 2.17 (Cancellativity of Addition in $[\mathbb{R}]$). *Classical interval addition is cancellative, that is*

$$(\forall X, Y, Z \in [\mathbb{R}])\,(X + Z = Y + Z \Rightarrow X = Y)\,.$$

Proof. Let $[\underline{x}, \overline{x}]$, $[\underline{y}, \overline{y}]$, and $[\underline{z}, \overline{z}]$ be in $[\mathbb{R}]$. Assume that

$$[\underline{x}, \overline{x}] + [\underline{z}, \overline{z}] = [\underline{y}, \overline{y}] + [\underline{z}, \overline{z}].$$

Then, by theorem 2.4, we immediately have

$$[\underline{x} + \underline{z}, \overline{x} + \overline{z}] = [\underline{y} + \underline{z}, \overline{y} + \overline{z}]\,,$$

which, according to theorem 2.1 and by the cancellation property of real addition, yields $\underline{x} = \underline{y} \wedge \overline{x} = \overline{y}$, that is $[\underline{x}, \overline{x}] = [\underline{y}, \overline{y}]$, and therefore addition is cancellative in $[\mathbb{R}]$. ■

In contrast to the case for addition, the following theorem asserts that multiplication is not always cancellative in $[\mathbb{R}]$.

Theorem 2.18 (Cancellativity of Multiplication in $[\mathbb{R}]$). *A classical interval number is cancellable for multiplication if, and only if, it is a zeroless interval, that is*

$$(\forall X, Y, Z \in [\mathbb{R}])\left((X \times Z = Y \times Z \Rightarrow X = Y) \Leftrightarrow 0 \notin Z\right).$$

Proof. Let $[\underline{x}, \overline{x}]$, $[\underline{y}, \overline{y}]$, and $[\underline{z}, \overline{z}]$ be in $[\mathbb{R}]$. Assume that

$$[\underline{x}, \overline{x}] \times [\underline{z}, \overline{z}] = [\underline{y}, \overline{y}] \times [\underline{z}, \overline{z}] \Rightarrow [\underline{x}, \overline{x}] = [\underline{y}, \overline{y}].$$

Then, by theorems 2.1 and 2.5, we have

$$\begin{aligned}\min\{\underline{xz}, \underline{x}\overline{z}, \overline{x}\underline{z}, \overline{xz}\} &= \min\{\underline{yz}, \underline{y}\overline{z}, \overline{y}\underline{z}, \overline{yz}\} \wedge \\ &\max\{\underline{xz}, \underline{x}\overline{z}, \overline{x}\underline{z}, \overline{xz}\} = \max\{\underline{yz}, \underline{y}\overline{z}, \overline{y}\underline{z}, \overline{yz}\} \Rightarrow \underline{x} = \underline{y} \wedge \overline{x} = \overline{y},\end{aligned}$$

which yields $0 < \underline{z} \leq \overline{z}$ or $\underline{z} \leq \overline{z} < 0$, that is $0 \notin [\underline{z}, \overline{z}]$.

The converse direction is easy to prove, and therefore multiplication is not cancellative in $[\mathbb{R}]$ except for the case when $0 \notin [\underline{z}, \overline{z}]$. ■

The following theorem concerning the additive and multiplicative properties of point interval numbers is derivable.

Theorem 2.19 (Algebraic Operations in $[\mathbb{R}]_p$). *Let X and Y be two classical interval numbers. Then:*

(i) The sum $X + Y$ is a point interval number iff each of X and Y is a point interval number, that is

$$(\forall X, Y \in [\mathbb{R}])\,(X + Y \in [\mathbb{R}]_p \Leftrightarrow X \in [\mathbb{R}]_p \wedge Y \in [\mathbb{R}]_p).$$

(ii) The product $X \times Y$ is a point interval number iff each of X and Y is a point interval number, or at least one of X and Y is $[0,0]$, that is

$$\begin{aligned}(\forall X, Y \in [\mathbb{R}])\,(X \times Y \in [\mathbb{R}]_p \Leftrightarrow (X \in [\mathbb{R}]_p \wedge Y \in [\mathbb{R}]_p) \\ \vee (X = [0,0] \vee Y = [0,0])).\end{aligned}$$

Proof. For *(i)* and *(ii)*, let $X = [\underline{x}, \overline{x}]$ and $Y = [\underline{y}, \overline{y}]$ be any two interval numbers.

(i) According to theorem 2.4, we have

$$X + Y = \left[\underline{x} + \underline{y}, \overline{x} + \overline{y}\right].$$

Assume that $X + Y \in [\mathbb{R}]_p$. Then $\underline{x} + \underline{y} = \overline{x} + \overline{y}$, which yields that each of X and Y is a point interval number.

The converse direction is easy to prove.

(ii) In a manner analogous to *(i)*, according to theorem 2.5, we have

$$X \times Y = \left[\min\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}, \max\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}\right].$$

Assume $X \times Y \in [\mathbb{R}]_p$. Then

$$\min\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\} = \max\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\},$$

which yields that each of X and Y is a point interval number, or at least one of X and Y is $[0, 0]$.

The converse direction is easy to prove. ■

An important property peculiar to the classical interval theory figures in the following theorem.

Theorem 2.20 (Inverses in $[\mathbb{R}]$). *A classical interval number is invertible if, and only if, it is a point interval, that is*

(i) $(\forall X, Y \in [\mathbb{R}]) \Big(X + Y = [0, 0] \Leftrightarrow X \in [\mathbb{R}]_p \wedge Y = -X\Big),$

(ii) $(\forall X, Y \in [\mathbb{R}]) \Big(X \times Y = [1, 1] \Leftrightarrow X \in [\mathbb{R}]_p \wedge Y = X^{-1} \wedge 0 \notin X\Big).$

Proof. The proof is immediate by theorems 2.19 and 2.11. ■

The result formulated in the following theorem is an important property of classical interval arithmetic.

Theorem 2.21 (Subdistributivity in $[\mathbb{R}]$). *The distributive law does not always hold in classical interval arithmetic. Precisely, for any three classical interval numbers X, Y, and Z*

$$Z \times (X + Y) = Z \times X + Z \times Y,$$

if, and only if,

(i) Z *is a point interval number, or*

(ii) $X = Y = [0, 0]$*, or*

(iii) $(\forall x \in X)\,(\forall y \in Y)\,(xy \geq 0)$.

In general, classical interval arithmetic has only the subdistributive law:

$$(\forall X, Y, Z \in [\mathbb{R}])\,(Z \times (X + Y) \subseteq Z \times X + Z \times Y)\,.$$

Proof. Let $[\underline{z}, \overline{z}]$, $[x, x]$, and $[-x, -x]$ be in $[\mathbb{R}]$. First adding and then multiplying, we have

$$[\underline{z}, \overline{z}] \times ([x, x] + [-x, -x]) = [\underline{z}, \overline{z}] \times [0, 0] = [0, 0]\,.$$

But if we first multiply and then add, we get

$$\begin{aligned}
&([\underline{z}, \overline{z}] \times [x, x]) + ([\underline{z}, \overline{z}] \times [-x, -x]) \\
&= [\min\{\underline{z}x, \overline{z}x\}, \max\{\underline{z}x, \overline{z}x\}] + [\min\{-\underline{z}x, -\overline{z}x\}, \max\{-\underline{z}x, -\overline{z}x\}] \\
&\neq [0, 0]\,,
\end{aligned}$$

unless $\underline{z} = \overline{z}$, or $x = -x = 0$, or both.

Thus, there are some interval numbers for which the distributive law does not hold.

Toward proving the equivalence, let us first assume the cases *(i)*, *(ii)*, and *(iii)*, respectively.

(i) Let $Z = [a, a]$ be a point interval number, for some real constant a. According to definition 2.2, we have

$$\begin{aligned}
Z \times (X + Y) &= \{r \in \mathbb{R} |\,(\exists z \in [a, a])\,(\exists s \in (X + Y))\,(r = z \times s)\} \\
&= \{r \in \mathbb{R} |\,(\exists s \in (X + Y))\,(r = a \times s)\} \\
&= \{r \in \mathbb{R} |\,(\exists x \in X)\,(\exists y \in Y)\,(r = a \times (x + y))\} \\
&= \{r \in \mathbb{R} |\,(\exists x \in X)\,(\exists y \in Y)\,(r = a \times x + a \times y)\} \\
&= \{r \in \mathbb{R} |\,(\exists t \in (Z \times X))\,(\exists u \in (Z \times Y))\,(r = t + u)\} \\
&= Z \times X + Z \times Y.
\end{aligned}$$

(ii) By theorems 2.12 and 2.13, we immediately have

$$\begin{aligned} Z \times ([0,0] + [0,0]) &= Z \times [0,0] \\ &= [0,0] \\ &= Z \times [0,0] + Z \times [0,0] . \end{aligned}$$

(iii) Let $X = [\underline{x}, \overline{x}]$ and $Y = [\underline{y}, \overline{y}]$. Without loss of generality, we consider only the case when $\underline{x} \geq 0$ and $\underline{y} \geq 0$. We have three cases for $Z = [\underline{z}, \overline{z}]$.

Case 1. If $\underline{z} \geq 0$, then we have

$$\begin{aligned} Z \times (X + Y) &= \left[\underline{z}\left(\underline{x} + \underline{y}\right), \overline{z}\left(\overline{x} + \overline{y}\right)\right] \\ &= \left[\underline{zx}, \overline{zx}\right] + \left[\underline{zy}, \overline{zy}\right] \\ &= Z \times X + Z \times Y. \end{aligned}$$

Case 2. If $\overline{z} \leq 0$, we obtain the same result by considering $-Z$.

Case 3. If $\underline{z}\overline{z} < 0$, then we have

$$\begin{aligned} Z \times (X + Y) &= \left[\underline{z}\left(\overline{x} + \overline{y}\right), \overline{z}\left(\overline{x} + \overline{y}\right)\right] \\ &= \left[\underline{z}\overline{x}, \overline{zx}\right] + \left[\underline{z}\overline{y}, \overline{zy}\right] \\ &= Z \times X + Z \times Y, \end{aligned}$$

and the last case is thus proved.

The converse direction is analogously derivable by assuming

$$Z \times (X + Y) = Z \times X + Z \times Y,$$

and taking some similar routine steps.

Now, to prove the subdistributivity, let $r \in (Z \times (X + Y))$. From the distributive property of real numbers, we have

$$r = z(x + y) = zx + zy,$$

for some $x \in X$, $y \in Y$, and $z \in Z$.

Thus $r \in (Z \times X + Z \times Y)$, and therefore

$$(\forall X, Y, Z \in [\mathbb{R}]) \left(Z \times (X + Y) \subseteq Z \times X + Z \times Y\right),$$

which completes the proof. ■

It should be borne in mind that classical interval arithmetic does not even have the distributive law

$$Z \times ([x, x] + [y, y]) = Z \times [x, x] + Z \times [y, y] ,$$

for point interval numbers $[x, x]$ and $[y, y]$.

We shall now prove three results about, respectively, the *additive* structure $\langle [\mathbb{R}] ; +_{\mathrm{c}}; [0, 0] \rangle$, the *multiplicative* structure $\langle [\mathbb{R}] ; \times_{\mathrm{c}}; [1, 1] \rangle$, and the *ring-like*[16] structure $\langle [\mathbb{R}] ; +_{\mathrm{c}}, \times_{\mathrm{c}}; [0, 0] , [1, 1] \rangle$ of classical interval arithmetic.

Theorem 2.22 *The additive structure* $\langle [\mathbb{R}] ; +_{\mathrm{c}}; [0, 0] \rangle$ *is a cancellative abelian monoid*[17]*.*

Proof. For $+_{\mathrm{c}}$, the following criteria are satisfied.

- Closure. By theorem 2.4, the set $[\mathbb{R}]$ is closed under classical interval addition.
- Associativity. Classical interval addition is associative, by theorem 2.16.
- Commutativity. Classical interval addition is commutative, by theorem 2.15.
- Identity Element. The interval number $[0, 0]$ is an identity element for classical interval addition, by theorem 2.13.
- Cancellativity. Classical interval addition is cancellative, by theorem 2.17.

Therefore, the set $[\mathbb{R}]$ of interval numbers forms a cancellative abelian monoid under classical interval addition. ■

Theorem 2.23 *The multiplicative structure* $\langle [\mathbb{R}] ; \times_{\mathrm{c}}; [1, 1] \rangle$ *is a noncancellative abelian monoid.*

[16] A *ring-like* algebra is a set equipped with two binary operations, *addition* and *multiplication*, such that multiplication has an *absorbing element* by either an axiom or a theorem.

A *shell* is a ring-like algebra such that multiplication has an identity element and an absorbing element, which is also the identity for addition.

A *ringoid* is a ring-like algebra whose multiplication distributes over addition.

A *semiring* is a ringoid which is also a shell, with addition and multiplication are associative, and addition is commutative.

An *abelian semiring* is a semiring whose multiplication is commutative.

A *nondistributive semiring* is a semiring whose multiplication does not distribute over addition (see, e.g., [Barnes1975], [Levi1961], [Menini2004], and [Steen2008]).

[17] A *monoid* is a semigroup with an *identity* element (see, e.g., [Menini2004]).

Proof. For $\times_c$, the following criteria are satisfied.

- Closure. By theorem 2.5, the set $[\mathbb{R}]$ is closed under classical interval multiplication.
- Associativity. Classical interval multiplication is associative, by theorem 2.16.
- Commutativity. Classical interval multiplication is commutative, by theorem 2.15.
- Identity Element. The interval number $[1, 1]$ is an identity element for classical interval multiplication, by theorem 2.14.
- Non-cancellativity. Classical interval multiplication is not cancellative, by theorem 2.18.

Therefore, the set $[\mathbb{R}]$ of interval numbers forms a noncancellative abelian monoid under classical interval multiplication. ■

Theorem 2.24 *The ring-like structure* $\langle [\mathbb{R}] ; +_c, \times_c; [0, 0] , [1, 1] \rangle$ *is a nondistributive abelian semiring*[18].

Proof. By theorems 2.22 and 2.23, $+_c$ and $\times_c$ are each both commutative and associative, and $\times_c$ has an identity element, $[1, 1]$, and, by theorem 2.12, an absorbing element, $[0, 0]$, which is, according to theorem 2.13, also the identity element for $+_c$. By theorem 2.21, $\times_c$ does not distribute over $+_c$.

Therefore, the structure $\langle [\mathbb{R}] ; +_c, \times_c; [0, 0] , [1, 1] \rangle$ is a nondistributive abelian semiring. ■

If we endow the classical interval algebra with the *compatible* partial ordering $\subseteq$, then we have the *partially-ordered nondistributive abelian semiring*, $\langle [\mathbb{R}] ; +_c, \times_c; [0, 0] , [1, 1] ; \subseteq \rangle$.

Finally, an important immediate result that the preceding theorem implies is the following.

Corollary 2.2 *The theory of classical intervals defines a nondistributive number system*[19] *on the set* $[\mathbb{R}]$.

[18] We may also call it a *subdistributive abelian semiring* (see theorem 2.21, on page 28).

[19] A *number system* is an algebra $\mathfrak{N} = \langle \mathcal{N}; +_{\mathcal{N}}, \times_{\mathcal{N}} \rangle$ with $+_{\mathcal{N}}$ and $\times_{\mathcal{N}}$ are each both commutative and associative, and $\times_{\mathcal{N}}$ distributes over $+_{\mathcal{N}}$. A *nondistributive number system* is a number system whose $\times_{\mathcal{N}}$ is not distributive (see, e.g., [Levi1961] and [Steen2008]).

The name "*numbers*" thus is verified, and therefore we can talk of "*classical interval numbers*".

Two important properties, peculiar to the classical theory of interval arithmetic, figure in the theorems of this section: additive and multiplicative inverses do not always exist for classical interval numbers, and there is no distributivity between classical addition and multiplication except for certain special cases. Then, we have to sacrifice some useful properties of ordinary arithmetic, if we want to use the interval weapon against uncertainty.

2.4 Machine Interval Arithmetic

There are numerous software implementations for classical interval arithmetic, which are usually provided as class libraries. Interval class libraries and language extensions are available for many *numeric* and *symbolic* programming languages such as C++, Java, Fortran, Mathematica, Maple, Lisp, Macsyma, and Coq (For further details, see, e.g., [Fateman2009], [Hansen2003], [Jaulin2001], and [Keene1988]).

However, existing software packages for interval arithmetic, in which interval calculations is simulated with floating-point routines, are often too slow for numerically intensive calculations. Therefore, interval arithmetic is about *five times slower* than floating-point arithmetic, if no special hardware implementations are provided such that interval arithmetic is directly supported on the machine level. Fortunately, computers are getting faster and most existing *parallel processors* provide a tremendous computing power. So, with little extra hardware, it is very possible to make interval computations as fast as floating point computations (For further reading about the hardware support for interval arithmetic, see, e.g., [Kolev1993], [Muller2009], and [Neumaier1991]).

In this section, we begin with some key concepts of machine real arithmetic, then we carefully construct the algebraic system of *machine interval arithmetic* and deduce some of its fundamental properties (For other constructions of machine real arithmetic, the reader may consult, e.g., [Kulisch1981], [Moore1959], and [Moore1962]).

The arithmetic of intervals defined in the preceding sections may be called an *exact*[20] *interval arithmetic*, in the sense that no rounding or approximation is involved. However, when interval arithmetic is realized on a computer, we get some loss of accuracy due to round-off errors. Therefore, due to the fact

[20] See footnote 2, on page 77.

that there is only a *finite* subset $\mathbb{M} \subset \mathbb{R}$ of *machine-representable numbers*, special care has to be taken to guarantee a proper hardware implementation of interval arithmetic. Thus, we need a *machine interval arithmetic* in which interval numbers have to be rounded so that the interval result computed by a machine always contains the exact interval result.

2.4.1 *Rounded-Outward Interval Arithmetic*

The algebraic operations of the classical theory of interval arithmetic are defined in such a way that they satisfy the property of *inclusion monotonicity* (see theorem 2.8, on page 21). An important immediate consequence of the inclusion monotonicity is that given two interval numbers $[\underline{x}, \overline{x}]$ and $[\underline{y}, \overline{y}]$ with $x \in [\underline{x}, \overline{x}]$ and $y \in [\underline{y}, \overline{y}]$, then for any unary operation $\diamond \in \{-, ^{-1}\}$ and any binary operation $\circ \in \{+, \times\}$, the real and interval results shall satisfy

$$\begin{aligned} \diamond x &\in \diamond [\underline{x}, \overline{x}] , \\ x \circ y &\in [\underline{x}, \overline{x}] \circ [\underline{y}, \overline{y}] . \end{aligned}$$

That is, guaranteed enclosures of the real-valued results can be obtained easily by computing on interval numbers. The following membership formulas can be deduced immediately from the property of inclusion monotonicity.

$$\begin{aligned} -x &\in [-\overline{x}, -\underline{x}] , \\ x^{-1} &\in \left[\overline{x}^{-1}, \underline{x}^{-1}\right], \quad \text{if } 0 \notin [\underline{x}, \overline{x}] , \\ x + y &\in [\underline{x} + \underline{y}, \overline{x} + \overline{y}] , \\ x \times y &\in [\min\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}, \max\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}] . \end{aligned}$$

The preceding formulas use the arithmetic of real numbers that are not machine-representable. However, using *outward rounding* for interval numbers, we can obtain alternate formulas that use floating-point arithmetic, and still satisfy the property of inclusion monotonicity.

Two definitions we shall need are those of the *downward* and *upward* rounding operators.

Definition 2.15 (Downward Rounding). *Let x be any real number and let x_m denote a machine-representable real number. Then there exists a machine-representable real number $\bigtriangledown x$ such that*

$$\bigtriangledown x = \sup\{x_m \in \mathbb{M} | x_m \leq x\},$$

where $\bigtriangledown$ *is called the downward rounding operator.*

Definition 2.16 (Upward Rounding). *Let x be any real number and let x_m denote a machine-representable real number. Then there exists a machine-representable real number $\bigtriangleup x$ such that*

$$\bigtriangleup x = \inf\{x_m \in \mathbb{M} | x \leq x_m\},$$

where $\bigtriangleup$ *is called the upward rounding operator.*

On the basis of these definitions, we can obtain a *finite* set $[\mathbb{M}] \subset [\mathbb{R}]$ of *machine interval numbers* by rounding interval numbers *outward*.

Definition 2.17 (Outward Rounding). *Let $[\underline{x}, \overline{x}]$ be any interval number. Then there exists a machine-representable interval number $\Diamond\,[\underline{x}, \overline{x}]$ such that*

$$\Diamond\,[\underline{x}, \overline{x}] = [\bigtriangledown \underline{x}, \bigtriangleup \overline{x}]\,,$$

where $\Diamond$ *is called the outward rounding operator.*

With outward rounding, a *machine interval arithmetic* can be defined such that the result of a machine interval operation is a machine interval number which is guaranteed to contain the exact result of an interval operation. In this manner, the classical interval operations can be redefined, in the language of machine interval arithmetic, as follows.

Definition 2.18 (Machine Interval Operations). *Let $[\underline{x}, \overline{x}]$ and $[\underline{y}, \overline{y}]$ be interval numbers. The unary and binary machine interval operations are defined as*

$$\begin{aligned}
\Diamond\left(-\,[\underline{x}, \overline{x}]\right) &= \left[\bigtriangledown\left(-\overline{x}\right), \bigtriangleup\left(-\underline{x}\right)\right], \\
\Diamond\left([\underline{x}, \overline{x}]^{-1}\right) &= \left[\bigtriangledown\left(\overline{x}^{-1}\right), \bigtriangleup\left(\underline{x}^{-1}\right)\right], \quad \text{if } 0 \notin [\underline{x}, \overline{x}]\,, \\
\Diamond\left([\underline{x}, \overline{x}] + [\underline{y}, \overline{y}]\right) &= \left[\bigtriangledown\left(\underline{x} + \underline{y}\right), \bigtriangleup\left(\overline{x} + \overline{y}\right)\right], \\
\Diamond\left([\underline{x}, \overline{x}] \times [\underline{y}, \overline{y}]\right) &= \left[\bigtriangledown\left(\min\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}\right), \bigtriangleup\left(\max\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}\right)\right].
\end{aligned}$$

With the help of the above definitions, it is not difficult to prove the following two theorems and their corollary.

Theorem 2.25 *For any two real numbers x and y, we have* *(i)* $x \leq y \Rightarrow \triangledown x \leq \triangledown y$,

(ii) $x \leq y \Rightarrow \triangle x \leq \triangle y$.

Theorem 2.26 *For any two interval numbers X and Y, we have*

(i) $X \subseteq Y \Rightarrow \Diamond X \subseteq \Diamond Y$,

(ii) $X \circ Y \subseteq \Diamond (X \circ Y)$,

(iii) $\diamond X \subseteq \Diamond (\diamond X)$.

Corollary 2.3 *For any two interval numbers X and Y with $x \in X$ and $y \in Y$, we have*

(i) $\diamond x \in \Diamond (\diamond X)$,

(ii) $x \circ y \in \Diamond (X \circ Y)$.

Thus, outward rounding provides an efficient implementation of interval arithmetic, with the property of inclusion monotonicity still satisfied.

To illustrate this, we give two numerical examples.

Example 2.4 *Let $\mathbb{M}_3$ be the set of machine-representable real numbers with three significant digits.*

(i) We have

$$\begin{aligned} \Diamond_3 ([1,2] \div [2,3]) &= [\triangledown_3 (1/3), \triangle_3 (1)] \\ &= [0.333, 1], \end{aligned}$$

and

$$([1,2] \div [2,3]) \subset [0.333, 1].$$

(ii) We have

$$\begin{aligned} \Diamond_3 ([0,1] + [2.7182, 3.3841]) &= [\triangledown_3 (2.7182), \triangle_3 (4.3841)] \\ &= [2.718, 4.385], \end{aligned}$$

and

$$([0,1] + [2.7182, 3.3841]) \subset [2.718, 4.385].$$

Unlike the sets $\mathbb{R}$ and $[\mathbb{R}]$, the sets $\mathbb{M}$ and $[\mathbb{M}]$, of machine real numbers and machine interval numbers, are obviously *countable*[21]. Moreover, the sets $\mathbb{M}$ and $[\mathbb{M}]$ are not *dense*[22], as is proved by the following theorem and its corollary.

Theorem 2.27 *Let $\mathbb{M}_n$ be the set of machine real numbers with n significant digits. The set $\mathbb{M}_n$ is not dense with respect to the order relation $<$, that is*

$$(\exists x_m \in \mathbb{M}_n)\,(\exists y_m \in \mathbb{M}_n)\,(x_m < y_m \wedge \neg\,((\exists z_m \in \mathbb{M}_n)\,(x_m < z_m \wedge z_m < y_m)))\,.$$

Proof. Let x_m be an element of $\mathbb{M}_n$. Then x_m can be written as

$$x_m = x_0 + \frac{x_1}{10} + \frac{x_2}{10^2} + ... + \frac{x_n}{10^n} = \sum_{k=0}^{n} \frac{x_k}{10^k},$$

where $x_0, x_1, x_2, ..., x_n$ are nonnegative integers.

Accordingly, if y_m is an element of $\mathbb{M}_n$ such that

$$y_m = x_0 + \frac{x_1}{10} + \frac{x_2}{10^2} + ... + \frac{x_n + 1}{10^n} = \frac{1}{10^n} + \sum_{k=0}^{n} \frac{x_k}{10^k},$$

then y_m is the element of $\mathbb{M}_n$ exactly next to x_m, and therefore, there is no $z_m \in \mathbb{M}_n$ such that $x_m < z_m \wedge z_m < y_m$. ■

Corollary 2.4 *Let $[\mathbb{M}_n]$ be the set of machine interval numbers with n significant digits. The set $[\mathbb{M}_n]$ is not dense with respect to Moore's partial ordering $<_{\mathrm{M}}$.*

Let, for instance, $\mathbb{M}_1$ be the set of machine real numbers with one significant digit. Obviously, there is no $z_m \in \mathbb{M}_1$ such that $1.1 < z_m < 1.2$, and there is no $Z_m \in [\mathbb{M}_1]$ such that $[1.1, 1.1] <_{\mathrm{M}} Z_m <_{\mathrm{M}} [1.2, 1.2]$.

Thus, we can easily determine the number of machine interval numbers between any two elements of $\mathbb{M}$. This is made precise in the following easy-provable theorem.

---

[21] A set S is *countable* if there is an injective mapping from S to the set $\mathbb{N} = \{0, 1, 2, 3, ...\}$ of natural numbers. Otherwise, S is *uncountable*.

[22] For a precise characterization of the notion of *density* and other order-theoretic notions, see section 6.1, on page 127.

Theorem 2.28 *Let $\mathbb{M}_n$ be the set of machine real numbers with n significant digits, and let x_m and y_m be elements of $\mathbb{M}_n$ such that $x_m \leq y_m$. Then the count of machine interval numbers between x_m and y_m is given by*

$$C_{[\mathbb{M}](x_m,y_m)} = \sum_{k=1}^{C_{\mathbb{M}(x_m,y_m)}} k = \frac{C^2_{\mathbb{M}(x_m,y_m)} + C_{\mathbb{M}(x_m,y_m)}}{2},$$

where

$$C_{\mathbb{M}(x_m,y_m)} = 10^n \times (y_m - x_m) + 1,$$

is the count of machine real numbers between x_m and y_m.

The following example makes this clear.

Example 2.5 *Let $\mathbb{M}_2$ be the set of machine real numbers with two significant digits. The count of machine real numbers between 1.23 and 1.32 is*

$$\begin{aligned} C_{\mathbb{M}(1.23,1.32)} &= 10^2 \times (1.32 - 1.23) + 1 \\ &= 10, \end{aligned}$$

and the count of machine interval numbers between 1.23 and 1.32 is

$$C_{[\mathbb{M}](1.23,1.32)} = \frac{10^2 + 10}{2} = 55.$$

2.4.2 *Rounded-Upward Interval Arithmetic*

Outward rounding of interval numbers involves performing computations with *two rounding modes* (upward and downward). This can be much costlier than performing the computations with one single rounding direction.

If, as usual, we have

$$(\forall x_m)\left(x_m \in \mathbb{M} \Rightarrow (-x_m) \in \mathbb{M}\right),$$

then the relation

$$(\forall x \in \mathbb{R})\left(\triangledown(-x) = -\vartriangle(x)\right),$$

which then holds, makes it possible to use upward rounding as one single rounding mode.

In this manner, for instance, machine interval addition can be reformulated as

$$\Diamond\left([\underline{x}, \overline{x}] + [\underline{y}, \overline{y}]\right) = \left[-\vartriangle\left((-\underline{x}) - \underline{y}\right), \vartriangle\left(\overline{x} + \overline{y}\right)\right].$$

Similar optimal roundings can be applied to other interval operations so that we get more efficient implementations of interval arithmetic.

Chapter 3

Interval Dependency and Alternate Interval Theories

Everything should be made as simple as possible, but no simpler.

–Albert Einstein (1879-1955)

It is not easy to perform arithmetic and solve equations in an algebraic system which is only a non-distributive semiring. The algebraic system of classical interval arithmetic, as we proved in theorem 2.24, on page 32, is only a *non-distributive abelian semiring*, which is a primitive algebraic structure, if compared to the totally ordered field of real numbers. Two useful properties of ordinary real arithmetic fail to hold in classical interval arithmetic: additive and multiplicative inverses do not always exist for interval numbers, and there is no distributivity between addition and multiplication except for certain special cases. For instance, the solutions of the algebraic interval equations

$$\begin{aligned} [\underline{x}, \overline{x}] + [\underline{a}, \overline{a}] &= [\underline{b}, \overline{b}] \ , \\ [\underline{y}, \overline{y}] \times [\underline{a}, \overline{a}] &= [\underline{b}, \overline{b}] \ , \end{aligned}$$

are not generally expressible in terms of the interval operations, due to the lack of inverse elements.

Another main drawback of the classical interval theory is that when estimating the image of real functions using classical interval arithmetic, we usually get overestimations that inevitably produces meaningless results, if the variables are *functionally dependent*. This persisting problem is known as the "*interval dependency problem*".

A considerable scientific effort is put into developing special methods and algorithms that try to overcome the difficulties imposed by the algebraic disadvantages of the classical interval arithmetic structure. Various proposals for possible alternate theories of interval arithmetic were introduced to reduce the dependency effect or to enrich the algebraic structure of interval numbers (For further details, the reader may consult, e.g., [Gardenyes1985], [Hansen1975], [Hayes2009], [Kulisch2008a], [Lodwick1999], and [Markov1995]).

In the present chapter, we formalize the notion of interval dependency, along with discussing the algebraic systems of two important alternate theories of interval arithmetic: *modal interval arithmetic*, and *constraint interval arithmetic*. In section 3.2, we introduce the basic concepts of modal interval arithmetic, deduce its fundamental algebraic properties, and finally uncover the algebraic system of modal intervals to be a *nondistributive abelian ring*. In section 3.3, we pay great attention, in particular, to studying the foundations of the theory of constraint intervals, and deduce some important results about its underlying algebraic system. With the help of these results, along with our formalization of the notion of interval dependency, we construct, in chapter 4, our theory of *optimizational intervals*. In chapter 5, after formalizing the classical theory of complex intervals, we present an *optimizational theory of complex interval arithmetic*.

3.1 A Formalization of the Notion of Interval Dependency

The notion of *dependency* comes from the notion of a *function*. There is scarcely a mathematical theory which does not involve the notion of a function. In ancient mathematics the idea of *functional dependence* was not expressed explicitly and was not an independent object of research, although a wide range of specific functional relations were known and were studied systematically. The concept of a function appears in a rudimentary form in the works of scholars in the Middle Ages, but only in the work of mathematicians in the 17th century, and primarily in those of P. Fermat, R. Descartes, I. Newton, and G. Leibniz, did it begin to take shape as an independent concept.

Later, in the 18th century, Euler had a more general approach to the concept of a function as "*dependence of one variable quantity on another*" [Euler2000]. By 1834, Lobachevskii was writing: "The general concept of a function requires that a function of x is a number which is given for each x and gradually changes with x. The value of a function can be given either by an analytic expression or by a condition which gives a means of testing all numbers and choosing one of them; or finally a *dependence* can exist and *remain unknown*" [Lobach1951].

In the theory of real closed intervals, the notion of interval dependency naturally comes from the idea of functional dependence of real variables. Despite the fact that dependency is an essential and useful notion of real variables, interval dependency is the *main unsolved problem* of the classical theory of interval arithmetic and its modern generalizations. Although the notion of interval dependency is widely used in the interval literature, no attempt has been made to put on a systematic basis its meaning, that is, to indicate formally the criteria by which it is to be characterized. Our objective in this section, therefore, is to put on a systematic basis the notion of interval dependency. This systematic basis shall play an essential role in our discussion of the interval theory, in this chapter and later on[1].

In order to be able to formalize the notion of interval dependency, we should deal first with some set-theoretical and logical notions of particular importance to our purpose.

We begin with the notions of an *n-ary*[2] *relation* and an *n-ary function*, along with some related concepts (For further details on these notions within the set-theoretical framework, see, e.g., [Causey1994], [Devlin1993], [Fomin1999], and [Suppes1972]).

Definition 3.1 (n-ary Relation). *Let $\mathcal{U}^{\langle n\rangle}$ be the n-th Cartesian power of a set $\mathcal{U}$. A set $\Re \subseteq \mathcal{U}^{\langle n\rangle}$ is an n-ary relation on $\mathcal{U}$ iff $\Re$ is a binary relation from $\mathcal{U}^{\langle n-1\rangle}$ to $\mathcal{U}$. That is, for $\mathbf{v} = (x_1, ..., x_{n-1}) \in \mathcal{U}^{\langle n-1\rangle}$ and $y \in \mathcal{U}$, an n-ary relation $\Re$ is defined to be*

$$\Re \subseteq \mathcal{U}^{\langle n\rangle} = \{(\mathbf{v}, y) \,|\, \mathbf{v} \in \mathcal{U}^{\langle n-1\rangle} \wedge y \in \mathcal{U}\}.$$

In this sense, an n-ary relation is a binary relation[3] (or simply a relation); then its *domain*, *range*, *field*, and *converse* are defined, as usual, to be, respectively

$$\begin{aligned}
\operatorname{dom}(\Re) &= \{\mathbf{v} \in \mathcal{U}^{\langle n-1\rangle} | \, (\exists y \in \mathcal{U})\, (\mathbf{v}\Re y)\}, \\
\operatorname{ran}(\Re) &= \{y \in \mathcal{U} | \left(\exists \mathbf{v} \in \mathcal{U}^{\langle n-1\rangle}\right) (\mathbf{v}\Re y)\}, \\
\operatorname{fld}(\Re) &= \operatorname{dom}(\Re) \cup \operatorname{ran}(\Re), \\
\widehat{\Re} &= \{(y, \mathbf{v}) \in \mathcal{U}^{\langle n\rangle} | \mathbf{v}\Re y\}.
\end{aligned}$$

[1] The notions, notations, and abbreviations of this section are indispensable for our mathematical discussion throughout the succeeding chapters, and hereafter are assumed priori, without further mention.

[2] All relations and functions considered in this text are *finitary*, that is, n-ary relations and functions, for some *finite* ordinal n (see footnote 13, on page 142).

[3] *Binary relations* and their properties are discussed, in detail, in section 6.1, on page 127.

It is obvious that $y\widehat{\Re}\mathbf{v} \Leftrightarrow \mathbf{v}\Re y$ and $\widehat{\widehat{\Re}} = \Re$.

Two important notions, for the purpose at hand, are the *image* and *preimage* of a set, with respect to an n-ary relation. These are defined as follows.

Definition 3.2 (Image and Preimage of a Relation). *Let $\Re$ be an n-ary relation on a set $\mathcal{U}$, and for $(\mathbf{v}, y) \in \Re$, let $\mathbf{v} = (x_1, ..., x_{n-1})$, with each x_k is restricted to vary on a set $X_k \subset \mathcal{U}$, that is, $\mathbf{v}$ is restricted to vary on a set $\mathbf{V} \subset \mathcal{U}^{\langle n-1\rangle}$. Then, the image of $\mathbf{V}$ (or the image of the sets X_k) with respect to $\Re$, denoted $\mathrm{I}_{\Re}$, is defined to be*

$$\begin{aligned} Y = \mathrm{I}_{\Re}(\mathbf{V}) &= \mathrm{I}_{\Re}(X_1, ..., X_{n-1}) \\ &= \{y \in \mathcal{U} |\, (\exists \mathbf{v} \in \mathbf{V})\, (\mathbf{v}\Re y)\} \\ &= \{y \in \mathcal{U} |\, \left(\exists_{k=1}^{n-1} x_k \in X_k\right) ((x_1, ..., x_{n-1})\, \Re y)\}, \end{aligned}$$

where the set $\mathbf{V}$, called the preimage of Y, is defined to be the image of Y with respect to the converse relation $\widehat{\Re}$, that is

$$\begin{aligned} \mathbf{V} &= \mathrm{I}_{\widehat{\Re}}(Y) \\ &= \{\mathbf{v} \in \mathcal{U}^{\langle n-1\rangle} |\, (\exists y \in Y) \left(y\widehat{\Re}\mathbf{v}\right)\}. \end{aligned}$$

In accordance with this definition and the fact that $y\widehat{\Re}\mathbf{v} \Leftrightarrow \mathbf{v}\Re y$, we have the following obvious result.

Theorem 3.1 *$Y = \mathrm{I}_{\Re}(\mathbf{V}) \Leftrightarrow \mathbf{V} = \mathrm{I}_{\widehat{\Re}}(Y)$.*

A completely general definition of the notion of an *n-ary function* can be formulated, within this set-theoretical framework, as follows.

Definition 3.3 (n-ary Function). *A set f is an n-ary function on a set $\mathcal{U}$ iff f is an $(n+1)$-ary relation on $\mathcal{U}$, and*

$$\left(\forall \mathbf{v} \in \mathcal{U}^{\langle n\rangle}\right) (\forall y, z \in \mathcal{U})\, (\mathbf{v} f y \wedge \mathbf{v} f z \Rightarrow y = z)\,.$$

Thus, an n-ary function is a *many-one* $(n+1)$-ary relation; that is, a relation, with respect to which, any element in its domain is related exactly to one element in its range. Getting down from relations to the particular case of functions, we have at hand the standard notation: $y = f(\mathbf{v})$ in place of $\mathbf{v} f y$.

From the fact that an n-ary function is a special kind of relation, then all the preceding definitions and results, concerning the *domain*, *range*, *field*, *converse*, *image*, and *preimage* of a relation, apply to functions as well.

With some criteria satisfied, a function is called *invertible* (or has an *inverse function*). The notion of invertibility of a function is made precise in the following definition.

Definition 3.4 (Invertibility of a Function). *A function f has an inverse, denoted f^{-1}, iff its converse relation $\widehat{f}$ is a function, in which case $f^{-1} = \widehat{f}$.*

In other words, f is invertible if, and only if, it is an *injection* from its domain to its range, and obviously the inverse f^{-1} is *unique*, from the fact that the converse relation is always *definable* and *unique*.

Now we turn to deal with some semantical and syntactical preliminaries concerning the logical formulation of the notion of functional dependence and some related notions.

When scientists observe the world to formulate the defining properties of some physical phenomenon, these defining properties figure as *attributes* (*variables*) depending on some other attributes. Translating this dependence into a formal mathematical language, gives rise to the notion of functional dependence: "*a variable y is absolutely determined by some given variables $x_1, ..., x_n$*", or "*a variable y is a function of some given variables $x_1, ..., x_n$*", symbolically $y = f(x_1, ..., x_n)$. In some cases, such a translation can deterministically result in a certain rule for the function f, for instance $y = x_1 + ... + x_n$. In other cases, we have an approximate rule for f, or we know that a dependence exist but the rule cannot be determined, in which case we write the general usual notation $y = f(x_1, ..., x_n)$, without specifying explicitly a rule for the function f. So, in mathematics, *a dependence is formally a function* (For further exhaustive details about the notion of dependence, from the logical and epistemological viewpoints, see, e.g., [Armstrong1974], [Hintikka1996], [Vaananen2001], and [Vaananen2007]).

As is well known, the most elementary part of all mathematical sciences is formal logic. So, getting down to the most elementary fundamentals, it can be clarified that in all mathematical theories, any type of dependence can be reduced to the following simple logical definition (see, e.g., [Kleene1952], [Shoenfield1967], and [Vaananen2001]).

Definition 3.5 (Quantification Dependence). *Let $\mathcal{Q}$ be a quantification matrix*[4] *and let $\varphi(x_1, ..., x_m; y_1, ..., y_n)$ be a quantifier-free formula. For any universal quantification $(\forall x_i)$ and any existential quantification $(\exists y_j)$ in $\mathcal{Q}$, the variable y_j is dependent on the variable x_i in the prenex sentence*[5] *$\mathcal{Q}\varphi$ iff $(\exists y_j)$ is in the scope of $(\forall x_i)$ in $\mathcal{Q}$. Otherwise x_i and y_j are independent.*

That is, the order of quantifiers in a quantification matrix determines the mutual dependence between the variables in a sentence.

Let us illustrate this by the following two examples.

Example 3.1 *Consider the prenex sentence*

$$(\exists x)(\forall y)(\exists z)(y = x \circ y \wedge x = z \circ y),$$

which asserts that there exists an identity element x, for the operation $\circ$, with respect to which every element possesses an inverse z.

According to the order in which quantifiers are written, the variable z depends only on y, while there is no dependency between x and y or between x and z.

Example 3.2 *In the prenex sentence*

$$(\forall x)(\exists y)(\forall z)(\exists u)\,\varphi(x, y, z, u),$$

the variable y depends on x, and the variable u depends on both x and z.

By means of a *Skolem equivalent form* or a *Skolemization*[6], a quantification dependence is translated into a functional dependence. The notion of a Skolem equivalent form is characterized in the following definition (see, e.g., [Feferman2006], [Hodges1997], [Vaananen2001], and [Vaananen2007]).

Definition 3.6 (Skolem Equivalent Form). *Let σ be a sentence that takes the prenex form*

$$(\forall_{i=1}^{m} x_i)\left(\exists_{j=1}^{n} y_j\right)\varphi(x_1, ..., x_m; y_1, ..., y_n).$$

[4] A *quantification matrix* $\mathcal{Q}$ is a sequence $(Q_1 x_1) ... (Q_n x_n)$, where $x_1, ..., x_n$ are variable symbols and each Q_i is $\forall$ or $\exists$.

[5] A *prenex sentence* is a sentence of the form $\mathcal{Q}\varphi$, where $\mathcal{Q}$ is a quantification matrix and φ is a quantifier-free formula.

[6] *Skolemization* is named after the Norwegian logician Thoralf Skolem (1887–1963), who first presented the notion in [Skolem1920].

where φ is a quantifier-free formula.

The Skolem equivalent form of σ is defined to be

$$\left(\exists_{j=1}^{n} f_j\right)\left(\forall_{i=1}^{m} x_i\right) \varphi\left(x_1, ..., x_m; f_1, ..., f_n\right),$$

where $f_j\left(x_1, ..., x_m\right) = y_j$ are the dependency functions of y_j upon $x_1, ..., x_m$, for $i \in \{1, ..., m\}$ and $j \in \{1, ..., n\}$.

It comes therefore as no surprise that in all mathematics, any instance of a dependence is, in fact, a functional dependence.

In order to clarify the matters, let us consider the following example.

Example 3.3 *Let a sentence σ take the prenex form*

$$(\forall x)(\exists y)(\forall z)(\exists u)\,\varphi(x, y, z, u).$$

The Skolem equivalent form of σ is

$$(\exists f)(\exists g)(\forall x)(\forall z)\,\varphi(x, f(x), z, g(x, z)).$$

Before we proceed, it is convenient to introduce the following notational convention.

Notation 3.1 *The left-superscripted letters ${}^{\mathbb{R}}f$, ${}^{\mathbb{R}}g$, ${}^{\mathbb{R}}h$ (with or without subscripts) shall be employed to denote real-valued functions, while the letters ${}^{\mathrm{c}}f$, ${}^{\mathrm{c}}g$, ${}^{\mathrm{c}}h$ (with or without subscripts) shall be employed to denote classical interval-valued functions.*

If the type of function is clear from its arguments, and if confusion is not likely to ensue, we shall usually drop the left superscripts "$\mathbb{R}$" and "c"[7]. Thus, we may, for instance, write $f(x_1, ..., x_n)$ and $f(X_1, ..., X_n)$ for, respectively, a real-valued function and an interval-valued function, which are both defined by the same rule.

[7] The left superscript "c" stands for "*classical* interval function". Since we have *many* theories of intervals being discussed throughout the text, with each theory has its *own* interval functions; we shall deviate from the standard notation "${}^{[\mathbb{R}]}f$" and give explicitly function superscripts peculiar to each theory, for example, ${}^{\mathrm{c}}f$, ${}^{\mathrm{t}}f$, ${}^{\mathrm{o}}f$, and so forth, according to what theory of intervals is being discussed.

An important notion we shall need is that of the image set of real closed intervals, under an n-ary real-valued function. This notion is a special case of that of the corresponding $(n+1)$-ary relation on $\mathbb{R}$. More precisely, we have the following definition.

Definition 3.7 (Image of Real Closed Intervals). *Let f be an n-ary function on $\mathbb{R}$, and for $(\mathbf{v}, y) \in f$, let $\mathbf{v} = (x_1, ..., x_n)$, with each x_k is restricted to vary on a real closed interval $X_k \subset \mathbb{R}$, that is, $\mathbf{v}$ is restricted to vary on a set $\mathbf{V} \subset \mathbb{R}^{\langle n \rangle}$. Then, the image of the closed intervals X_k with respect to f, denoted I_f, is defined to be*

$$\begin{aligned} Y = \mathrm{I}_f(\mathbf{V}) &= \mathrm{I}_f(X_1, ..., X_n) \\ &= \{y \in \mathbb{R} | (\exists \mathbf{v} \in \mathbf{V})(\mathbf{v} f y)\} \\ &= \{y \in \mathbb{R} | (\exists_{k=1}^{n} x_k \in X_k)(y = f(x_1, ..., x_n))\} \subseteq \mathbb{R}, \end{aligned}$$

where the set $\mathbf{V}$, called the preimage[8] *of Y, is defined to be the image of Y with respect to the converse relation $\widehat{f}$, that is*

$$\begin{aligned} \mathbf{V} &= \mathrm{I}_{\widehat{f}}(Y) \\ &= \{\mathbf{v} \in \mathbb{R}^{\langle n \rangle} | (\exists y \in Y)\left(y \widehat{f} \mathbf{v}\right)\}. \end{aligned}$$

The fact formulated in the following theorem is well-known.

Theorem 3.2 (Extreme Value Theorem). *Let X_k be real closed intervals and let $f(x_1, ..., x_n)$ be an n-ary real-valued function with $x_k \in X_k$. If f is continuous in X_k, in symbols $\mathrm{Cont}(f, X_k)$, then f must attain its minimum and maximum value, that is*

$$\begin{aligned} (\forall f)(\mathrm{Cont}(f, X_k) \;\Rightarrow\; & (\exists_{k=1}^{n} a_k \in X_k)(\exists_{k=1}^{n} b_k \in X_k)(\forall_{k=1}^{n} x_k \in X_k) \\ & (f(a_1, ..., a_n) \le f(x_1, ..., x_n) \le f(b_1, ..., b_n))), \end{aligned}$$

where $\min_{x_k \in X_k} f = f(a_1, ..., a_n)$ and $\max_{x_k \in X_k} f = f(b_1, ..., b_n)$ are respectively the minimum and maximum of f.

An immediate consequence of definition 3.7 and theorem 3.2, is the following important property.

[8] From the fact that the converse relation $\widehat{f}$ is always definable, the preimage of a function f is always definable, regardless of the definability of the inverse function f^{-1}.

Theorem 3.3 *Let an n-ary real-valued function f be continuous in the real closed intervals X_k. The (accurate) image $\mathrm{I}_f(X_1, ..., X_n)$, of X_k, is in turn a real closed interval such that*

$$\mathrm{I}_f(X_1, ..., X_n) = \left[\min_{x_k \in X_k} f(x_1, ..., x_n), \max_{x_k \in X_k} f(x_1, ..., x_n) \right].$$

A cornerstone result from the above theorem, that should be stressed at once, is that the *best* way to evaluate the *accurate* image of a continuous real-valued function is to apply minimization and maximization directly to determine the exact lower and upper endpoints of the image. For rational[9] real-valued functions, this optimization problem is, in general, computationally solvable, by applying *Tarski's algorithm*, which is also known as *Tarski's real quantifier elimination* (see, e.g., [Chen2000] and [Tarski1951]). For algebraic[10] real-valued functions, the problem is computable, by applying the *cylindrical algebraic decomposition algorithm* (*CAD algorithm*, or *Collins' algorithm*)[11], which is a more effective version of Tarski's algorithm (see, e.g., [Collins1975] and [McCallum2001]).

Before turning to the notion of interval dependency, we first prove the following indispensable result.

Theorem 3.4 *Let σ_1 and σ_2 be the two prenex sentences such that*

$$\begin{aligned} \sigma_1 &\Leftrightarrow (\forall_{i=1}^m x_i \in X_i) \left(\exists_{j=1}^n y_j \in Y_j\right) (\exists z) \left(z = f(x_1, ..., x_m; y_1, ..., y_n)\right), \\ \sigma_2 &\Leftrightarrow (\forall_{i=1}^m x_i \in X_i) \left(\forall_{j=1}^n y_j \in Y_j\right) (\exists z) \left(z = f(x_1, ..., x_m; y_1, ..., y_n)\right), \end{aligned}$$

where X_i and Y_j are real closed intervals, and f is a continuous real-valued function with $x_i \in X_i$ and $y_j \in Y_j$.

If $\mathrm{I}_f^{\sigma_1}$ and $\mathrm{I}_f^{\sigma_2}$ are the images of f, respectively, in σ_1 and σ_2, then $\mathrm{I}_f^{\sigma_1} \subseteq \mathrm{I}_f^{\sigma_2}$.

Proof. According to definition 3.5, in the sentence σ_1, all y_j are dependent upon all x_i, and in the sentence σ_2, all x_i and y_j are pairwise independent.

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[9] A *rational* real-valued function is a function obtained by means of a finite number of the basic real algebraic operations $\circ_{\mathbb{R}} \in \{+, \times\}$ and $\diamond_{\mathbb{R}} \in \{-, ^{-1}\}$.

[10] An *algebraic* function is a function that satisfies a polynomial equation whose coefficients are polynomials with rational coefficients.

[11] The *CAD algorithm* is efficient enough for being one of the most important optimization algorithms of computational real algebraic geometry (see, e.g., [Basu2003]).

By definition 3.6, there are some functions $g_j(x_1, ..., x_m)$ such that σ_1 has the Skolem equivalent form

$$\left(\exists_{j=1}^{n} g_j\right)\left(\forall_{i=1}^{m} x_i \in X_i\right)(\exists z)\left(z = f\left(x_1, ..., x_m; g_1, ..., g_n\right)\right).$$

Finally, employing theorem 3.3, we therefore have $\mathrm{I}_f^{\sigma_1} \subseteq \mathrm{I}_f^{\sigma_2}$. ■

The following example makes this clear.

Example 3.4 *Let σ_1 and σ_2 be the two prenex sentences such that*

$$\begin{aligned}\sigma_1 &\Leftrightarrow (\forall x \in [1,2])(\exists y \in [1,2])(\exists z \in \mathbb{R})(z = f(x,y) = y - x),\\ \sigma_2 &\Leftrightarrow (\forall x \in [1,2])(\forall y \in [1,2])(\exists z \in \mathbb{R})(z = f(x,y) = y - x).\end{aligned}$$

In the sentence σ_1, the variable y depends on x, and therefore there is some function $g(x)$ such that σ_1 has the Skolem equivalent form

$$(\exists g)(\forall x \in [1,2])(\exists z \in \mathbb{R})(z = f(x, g(x)) = g(x) - x).$$

Let g be the identity function. Consequently, the image of f in σ_1 is $\mathrm{I}_f^{\sigma_1} = \{0\}$.

Obviously, the image of f in σ_2 is $\mathrm{I}_f^{\sigma_2} = [-1,1]$, and therefore $\mathrm{I}_f^{\sigma_1} \subseteq \mathrm{I}_f^{\sigma_2}$.

Next we define the notion of an *exact* (or *generalized*) interval operation.

Definition 3.8 (Exact Interval Operation). *Let $\circ_{\mathbb{R}} \in \{+, \times\}$ be a binary real operation, and let $\mathrm{I}_f = \mathrm{I}_f^{\sigma_{\mathrm{Dep}}} \vee \mathrm{I}_f = \mathrm{I}_f^{\sigma_{\mathrm{Ind}}}$, where $\mathrm{I}_f^{\sigma_{\mathrm{Dep}}}$ and $\mathrm{I}_f^{\sigma_{\mathrm{Ind}}}$ are the images of a function f for two real closed intervals X and Y in, respectively, two prenex sentences σ_{Dep} and σ_{Ind} such that*

$$\begin{aligned}\sigma_{\mathrm{Dep}} &\Leftrightarrow (\forall x \in X)(\exists y \in Y)(\exists z \in \mathbb{R})(z = f(x,y) = x \circ_{\mathbb{R}} y),\\ \sigma_{\mathrm{Ind}} &\Leftrightarrow (\forall x \in X)(\forall y \in Y)(\exists z \in \mathbb{R})(z = f(x,y) = x \circ_{\mathbb{R}} y).\end{aligned}$$

Then, an exact interval operation $\circ_{\mathcal{I}} \in \{+, \times\}$ is defined by

$$X \circ_{\mathcal{I}} Y = \mathrm{I}_f(X, Y).$$

We have then the following obvious result for the classical interval operations.

Theorem 3.5 *The value of a classical interval operation $X \circ_{\mathrm{c}} Y$ is exact only when the real variables $x \in X$ and $y \in Y$ are independent, that is*

$$X \circ_{\mathrm{c}} Y = \mathrm{I}_f^{\sigma_{\mathrm{Ind}}}(X, Y).$$

Proof. The theorem is immediate from definition 2.2 of the classical interval operations. ■

With the help of the preceding notions and the deductions from them, we are now ready to pass to our formal characterization of the notion of interval dependency.

Definition 3.9 (Interval Dependency). *Let $S_1, ..., S_m$ be some arbitrary real closed intervals. For two interval variables X and Y, we say that Y is dependent on X, in symbols $Y \mathfrak{D} X$, iff there is some* given *real-valued function f such that Y is the image of $(X; S_1, ..., S_m)$ with respect to f. That is*

$$Y \mathfrak{D} X \Leftrightarrow Y = \mathrm{I}_f\left(X; S_1, ..., S_m\right),$$

where f is called the dependency function of Y on X.

Otherwise Y is not dependent on X, in symbols $Y \Im X$, that is

$$\begin{aligned} Y \Im X &\Leftrightarrow \neg Y \mathfrak{D} X \\ &\Leftrightarrow \neg Y = \mathrm{I}_f\left(X; S_1, ..., S_m\right). \end{aligned}$$

From now on, and throughout the text, the following notational convention shall be adopted.

Notation 3.2 *We write $Y \mathfrak{D}_f X$ (with the subscript f) to mean that Y is dependent on X by some given dependency function f, and we write $\Im(X, Y)$ to mean that X and Y are mutually independent. In general, the notation $\Im(X_1, ..., X_n)$ shall be employed to mean "all X_1, ...,X_n are pairwise mutually independent". Hereafter, for simplicity of the language, we shall always make use of the following abbreviation.*

$$\Im_{k=1}^{n}\left(X_k\right) \Leftrightarrow \Im(X_1, ..., X_n).$$

So, to say that an interval variable Y is dependent on an interval variable X, we must be *given* some real-valued function f such that Y is the image of X under f. This characterization of interval dependency is completely compatible with the concept of functional dependence of real variables: for two real variables x and y, the variable y is *functionally* dependent on x if there is some *given* function f such that

$$y = f\left(x\right),$$

and to keep the dependency information, between x and y, in an algebraic expression $x \circ_{\mathbb{R}} y$, it suffices to write

$$x \circ_{\mathbb{R}} f(x).$$

If x and y are mutually dependent by an *idempotence* $y = f(x)$ and $x = g(y)$, then, to keep the dependency information, it suffices to write only *one* of

$$x \circ_{\mathbb{R}} f(x) \quad \text{and} \quad g(y) \circ_{\mathbb{R}} y.$$

In case there is neither such a given function f nor such a given function g, then it is obvious that the real variables x and y are *not* functionally dependent. Definition 3.9 extends this concept to the set of real closed intervals.

The preceding definition, along with two deductions that we shall presently make (theorem 3.6 and corollary 3.1), touches the notion of interval dependency in a way which copes with all possible cases. This shall be shown, in detail, in section 3.3 of this chapter, and in chapter 4. For now, to illustrate, let us give the following example.

Example 3.5 *Let X and Y be two interval variables that both are assigned the same individual constant* $[0, 1]$*. Then, we may have one of the following cases.*

(i) Y is not dependent on X (there is no given dependency function).

(ii) Y is dependent on X, by the identity function $y = f(x) = x$.

(iii) Y is dependent on X, by the square function $y = f(x) = x^2$.

This example shows that if two interval *variables* X and Y both are assigned the same *individual constant* (both have the same *value*), it does not necessarily follow that X and Y are *identical*, unless they are dependent by the identity function. This shall be made precise in definition 4.2 of section 4.1.

As a consequence of our characterization of interval dependency, we have the next immediate theorem that establishes that the interval dependency relation is a *quasi-ordering* relation.

Theorem 3.6 *The interval dependency relation is a quasi-ordering relation on the set of real closed intervals. That is, for any three interval variables X, Y, and Z, the following statements are true:*

(i) $\mathfrak{D}$ is reflexive, in symbols $(X \mathfrak{D} X)$,

(ii) $\mathfrak{D}$ *is transitive, in symbols* $(X\mathfrak{D}Y \wedge Y\mathfrak{D}Z \Rightarrow X\mathfrak{D}Z)$.

In accordance with this theorem and definition 3.8, we also have the following corollary.

Corollary 3.1 *For any interval operation* $\circ_{\mathcal{I}}$*, and for any three interval variables* X*,* Y*, and* Z*, the following two assertions are true:*

(i) $(X \circ_{\mathcal{I}} Y)\, \mathfrak{D}X$,

(ii) $(X \circ_{\mathcal{I}} Y)\, \mathfrak{D}Y$.

The interval dependency problem can now be formulated in the following theorem.

Theorem 3.7 (Dependency Problem). *Let* X_k *be real closed intervals and let* $f(x_1, ..., x_n)$ *be a continuous real-valued function with* $x_k \in X_k$*. Evaluating the accurate image of* f *for the interval numbers* X_k*, using classical interval arithmetic, is not always possible if there exist* X_i *and* X_j *such that* $X_j\mathfrak{D}X_i$ *for* $i \neq j$*. That is,*

(i) $(\exists f)\,(\mathrm{I}_f(X_1, ..., X_n) \neq f(X_1, ..., X_n))$.

In general,

(ii) $(\forall f)\,(\mathrm{I}_f(X_1, ..., X_n) \subseteq f(X_1, ..., X_n))$.

Proof. For *(i)*, it suffices to give a counterexample.

For two interval variables X_1 and X_2 that both are assigned the same individual constant $[-a, a]$, let f be a function defined by the rule $f(x_1, x_2) = x_1 x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$. If $X_2\mathfrak{D}_g X_1$, with g is the identity function $x_2 = g(x_1) = x_1$, then f has the equivalent rule $f(x) = x^2$, with $x \in [-a, a]$.

According to theorem 3.3, the (accurate) image of $[-a, a]$ under the real-valued function f is

$$\mathrm{I}_f([-a, a]) = \left[\min_{x\in[-a,a]} x^2, \max_{x\in[-a,a]} x^2\right] = \left[0, a^2\right].$$

If we evaluate the image of $[-a, a]$ using classical interval arithmetic, by theorem 2.5, we obtain the interval-valued function,

$$\begin{aligned} f([-a, a]) &= [-a, a] \times [-a, a] \\ &= \left[-a^2, a^2\right], \end{aligned}$$

which is not the actual image of $[-a, a]$ under f, that is, there is some function f, for which

$$\mathrm{I}_f (X_1, ..., X_n) \neq f (X_1, ..., X_n),$$

and therefore evaluating the accurate image of real-valued functions is not always possible, using classical interval arithmetic.

Toward proving *(ii)*, let

$$\mathrm{I}_f (X_1, ..., X_n) = \mathrm{I}_f^{\sigma_1} (X_1, ..., X_n) \vee \mathrm{I}_f (X_1, ..., X_n) = \mathrm{I}_f^{\sigma_2} (X_1, ..., X_n),$$

where $\mathrm{I}_f^{\sigma_1}$ and $\mathrm{I}_f^{\sigma_2}$ are the images of f, respectively, in two prenex sentences σ_1 and σ_2 such that in σ_1, there exist X_i and X_j such that $X_j \mathfrak{D} X_i$ for $i \neq j$, and in σ_2, all X_k are pairwise independent, that is $\Im_{k=1}^{n} (X_k)$. Employing theorem 3.4, We accordingly have

$$\mathrm{I}_f^{\sigma_1} (X_1, ..., X_n) \subseteq \mathrm{I}_f^{\sigma_2} (X_1, ..., X_n).$$

According to definitions 2.2 and 2.3, of the classical interval operations, all interval variables are assumed to be independent. We consequently have

$$f (X_1, ..., X_n) = \mathrm{I}_f^{\sigma_2} (X_1, ..., X_n).$$

Thus

$$(\forall f) (\mathrm{I}_f (X_1, ..., X_n) \subseteq f (X_1, ..., X_n)),$$

and therefore *(ii)* is verified. ■

Obviously, the result $\left[-a^2, a^2\right]$, obtained using classical interval arithmetic, has an overestimation of

$$\left| w \left(\left[-a^2, a^2\right] \right) - w \left(\left[0, a^2\right] \right) \right| = a^2.$$

This overestimated result is due to the fact that the classical interval theory assumes independence of all interval variables, even when dependencies exist.

A numerical example is shown below.

Example 3.6 *Consider the real-valued function*

$$f(x) = x(x-1),$$

with $x \in [0,1]$.

The actual image of $[0,1]$ *under* f *is* $[-1/4, 0]$. *Evaluating the image using classical interval arithmetic, we get*

$$f([0,1]) = [0,1] \times ([0,1]-1) = [-1,0],$$

which has an overestimation of

$$|w([-1,0]) - w([-1/4,0])| = 3/4.$$

The problem of computing the image $\mathrm{I}_f(X_1, ..., X_n)$, using interval arithmetic, is the main problem of interval computations. Despite the fact that there are many special methods and algorithms, based on the classical interval theory, that successfully compute useful narrow bounds to the desirable accurate image (see, e.g., [Moore1966], [Moore1979], [Moore2009], and [Hansen2003]), the problem is, in general, NP-hard[12] (see, e.g., [Gaganov1985], [Pedrycz2008], and [Rokne1984]). That is, for the classical interval theory, there is no efficient algorithm to make the identity

$$(\forall f)\left(\mathrm{I}_f(X_1, ..., X_n) = f(X_1, ..., X_n)\right),$$

always hold unless NP = P, which is widely believed to be false.

3.2 Modal Interval Arithmetic

The theory of modal intervals, constructed by Ernest Gardenyes in 1985 (see [Gardenyes1985]), is a *definitional extension*[13] of the classical interval theory that provides a set of interval arithmetic sentences which is *semantically equivalent* to a larger *subset* of the sentences of real arithmetic. A basic problem of the classical interval theory is that the semantic of quantification over real variables is lost in classical interval arithmetic.

To illustrate this, let $\circ \in \{+, \times\}$, and let σ_1 and σ_2 be two sentences of real

[12] In principle, this result is not necessarily applicable to other theories of interval arithmetic (present or future) because each theory has its peculiar set of algorithms, where each algorithm is a sequence of elementary relations and functions of the foundational level of the theory (see footnote 2, on page 77).

[13] See footnote 11, on page 18.

arithmetic such that

$$\begin{aligned}\sigma_1 &\Leftrightarrow (\forall x \in [\underline{x}, \overline{x}]) \left(\forall y \in \left[\underline{y}, \overline{y}\right]\right) (\exists z \in \mathbb{R}) (z = x \circ y),\\ \sigma_2 &\Leftrightarrow (\forall x \in [\underline{x}, \overline{x}]) \left(\exists y \in \left[\underline{y}, \overline{y}\right]\right) (\exists z \in \mathbb{R}) (z = x \circ y).\end{aligned}$$

In the sentence σ_1, the variables x and y are independent, and in the sentence σ_2, the variable y depends on x. However, the classical interval theory, which assumes *independence of all variables*, has the following *single* translation for the above two real sentences,

$$(\exists Z \in [\mathbb{R}]) \left(Z = [\underline{x}, \overline{x}] \circ \left[\underline{y}, \overline{y}\right]\right),$$

which is semantically equivalent to the first real sentence. The *meaning* of the second sentence, along with the dependency information of the variables x and y, is lost in classical interval arithmetic, because the semantic of the existential quantification over $\left[\underline{y}, \overline{y}\right]$ is not kept by the set-theoretic definition of classical interval operations (definition 2.2). Modal interval arithmetic, which is based entirely on predicate logic and set theory, is conceived as an attempt to solve this problem.

In the sequel, we introduce the basic concepts of modal interval arithmetic, deduce its fundamental algebraic properties, and finally uncover what exactly the algebraic system, of modal intervals, is.

Next we define what a modal interval is.

Definition 3.10 *Let* $[\underline{x}, \overline{x}]$ *be a classical (set-theoretic) interval number, and let* $Q_X \in \{\forall, \exists\}$ *be a logical quantifier. A modal interval* ${}^mX$ *is an ordered pair such that*

$${}^mX = ([\underline{x}, \overline{x}], Q_X) = \begin{cases} {}^m[\underline{x}, \overline{x}] & \textit{iff} \;\; Q_X \textit{ is } \exists, \\ {}^m[\overline{x}, \underline{x}] & \textit{iff} \;\; Q_X \textit{ is } \forall, \end{cases}$$

such that

$$\begin{aligned}{}^m[\underline{x}, \overline{x}] &= \{\varphi | \varphi \Leftrightarrow (\exists x \in [\underline{x}, \overline{x}]) (P(x))\},\\ {}^m[\overline{x}, \underline{x}] &= \{\varphi | \varphi \Leftrightarrow (\forall x \in [\underline{x}, \overline{x}]) (R(x))\},\end{aligned}$$

where $P(x)$ *and* $R(x)$ *are some real predicates for which the existential and universal sentences are true respectively.*

That is, a modal interval (the term "*modal pair*" or "*modal set*" is better) is *not* a set-theoretical interval. In a manner analogous to how a real number x can be represented as a pair $(|x|, \pm)$ whose first element is the absolute value of

x and whose second element is a negative or positive sign, a modal interval can be represented as a pair of a set-theoretical interval and a logical quantifier.

Thus, a modal interval $([\underline{x}, \overline{x}], Q_X)$ is the set of *all* true real sentences with respect to the quantification $Q_X x \in [\underline{x}, \overline{x}]$. For example,

$$\begin{aligned} ([2,3], \exists) &= {}^m[2,3] = \{\varphi | \varphi \Leftrightarrow (\exists x \in [2,3]) (P(x))\}, \\ ([2,3], \forall) &= {}^m[3,2] = \{\varphi | \varphi \Leftrightarrow (\forall x \in [2,3]) (R(x))\}, \end{aligned}$$

where the sentences

$$\begin{aligned} &(\exists x \in [2,3]) (x > 2), \\ &(\forall x \in [2,3]) (x \geq 2), \end{aligned}$$

are, respectively, elements of $([2,3], \exists)$ and $([2,3], \forall)$.

The set $\mathcal{M}$ of modal intervals is characterized as follows.

Definition 3.11 $\mathcal{M} = \{(X, Q_X) | X \in [\mathbb{R}] \wedge Q_X \in \{\forall, \exists\}\}$.

Hereafter, the left-superscripted Roman letters ${}^mX$, ${}^mY$, and ${}^mZ$ (with or without subscripts), or equivalently ${}^m[\underline{x}, \overline{x}]$, ${}^m[\underline{y}, \overline{y}]$, and ${}^m[\underline{z}, \overline{z}]$, shall be employed as variable symbols to denote elements of $\mathcal{M}$. Classical interval numbers shall be denoted as usual.

To further simplify the exposition, it is convenient to introduce the following notational conventions.

Notation 3.3 *A modal interval $(X, \forall)$ is called a universal (or improper) modal interval. The set of all universal modal intervals shall be denoted by $\mathcal{M}_\forall$.*

Notation 3.4 *A modal interval $(X, \exists)$ is called an existential (or proper) modal interval. The set of all existential modal intervals shall be denoted by $\mathcal{M}_\exists$.*

Notation 3.5 *A modal interval (X, Q) with $X \in [\mathbb{R}]_p$ is called a point modal interval. The set of all point modal intervals shall be denoted by $\mathcal{M}_p$.*

Next we define the "mode", "set", "dual", "proper", "improper", "inf", and "sup" operators for modal intervals.

Definition 3.12 *The mode of a modal interval* $^m[\underline{x}, \overline{x}]$ *is defined to be*

$$\text{mode}\left({}^m[\underline{x}, \overline{x}]\right) = \begin{cases} \exists & \textit{iff} \quad \underline{x} \le \overline{x}, \\ \forall & \textit{iff} \quad \underline{x} \ge \overline{x}. \end{cases}$$

Definition 3.13 *The set of a modal interval* $^m[\underline{x}, \overline{x}]$ *is defined to be*

$$\text{set}\left({}^m[\underline{x}, \overline{x}]\right) = \left[\min\{\underline{x}, \overline{x}\}, \max\{\underline{x}, \overline{x}\}\right].$$

Definition 3.14 *The dual of a modal interval* $^m[\underline{x}, \overline{x}]$ *is defined to be*

$$\text{dual}\left({}^m[\underline{x}, \overline{x}]\right) = {}^m[\overline{x}, \underline{x}].$$

Definition 3.15 *The proper of a modal interval* $^m[\underline{x}, \overline{x}]$ *is defined to be*

$$\text{proper}\left({}^m[\underline{x}, \overline{x}]\right) = {}^m\left[\min\{\underline{x}, \overline{x}\}, \max\{\underline{x}, \overline{x}\}\right].$$

Definition 3.16 *The improper of a modal interval* $^m[\underline{x}, \overline{x}]$ *is defined to be*

$$\text{improper}\left({}^m[\underline{x}, \overline{x}]\right) = {}^m\left[\max\{\underline{x}, \overline{x}\}, \min\{\underline{x}, \overline{x}\}\right].$$

Definition 3.17 *The infimum of a modal interval* ^mX *is defined to be*

$$\inf\left({}^mX\right) = \begin{cases} \min\left(\text{set}\left({}^mX\right)\right) & \textit{iff} \quad \text{mode}\left({}^mX\right) = \exists, \\ \max\left(\text{set}\left({}^mX\right)\right) & \textit{iff} \quad \text{mode}\left({}^mX\right) = \forall. \end{cases}$$

Definition 3.18 *The supremum of a modal interval* ^mX *is defined to be*

$$\sup\left({}^mX\right) = \begin{cases} \max\left(\text{set}\left({}^mX\right)\right) & \textit{iff} \quad \text{mode}\left({}^mX\right) = \exists, \\ \min\left(\text{set}\left({}^mX\right)\right) & \textit{iff} \quad \text{mode}\left({}^mX\right) = \forall. \end{cases}$$

It should be noted that the inf and sup operators of a modal interval ^mX are *canonical*, that is, they are *not* necessarily the *infimum* and *supremum* of the corresponding set-theoretical interval X.

Some numerical examples are shown below.

Example 3.7 *For two given modal intervals* $^m[1, 2]$ *and* $^m[4, 3]$*, we have*

(i) $^m[1, 2] = ([1, 2], \exists)$, $^m[4, 3] = ([3, 4], \forall)$;

(ii) $\text{mode}\left({}^m[1,2]\right) = \exists$, $\text{mode}\left({}^m[4,3]\right) = \forall$;

(iii) $\text{set}\left({}^m[1,2]\right) = [1,2]$, $\text{set}\left({}^m[4,3]\right) = [3,4]$;

(iv) $\text{dual}\left({}^m[1,2]\right) = {}^m[2,1] = \left([1,2], \forall\right)$;

(v) $\text{dual}\left({}^m[4,3]\right) = {}^m[3,4] = \left([3,4], \exists\right)$;

(vi) $\text{proper}\left({}^m[1,2]\right) = {}^m[1,2]$, $\text{improper}\left({}^m[1,2]\right) = {}^m[2,1]$;

(vii) $\inf\left({}^m[1,2]\right) = 1$, $\inf\left({}^m[4,3]\right) = 4$;

(viii) $\sup\left({}^m[1,2]\right) = 2$, $\sup\left({}^m[4,3]\right) = 3$.

The result of a modal algebraic operation is a set of true real arithmetic sentences. The algebraic operations for modal intervals are characterized in the following definition.

Definition 3.19 *(Modal Algebraic Operations). Let* ${}^mX = (X, Q_X)$*,* ${}^mY = (Y, Q_Y)$*, and* ${}^mZ = (Z, Q_Z)$ *be modal intervals. For any algebraic operator* $\circ$*, let* $P(x,y,z) \Leftrightarrow z = x \circ_{\mathbb{R}} y$ *be a true real predicate with respect to the quantifications*

$$Q_X x \in X,\ Q_Y y \in Y,\ \text{and}\ Q_Z z \in Z,$$

where $Q_X, Q_Y, Q_Z \in \{\forall, \exists\}$*.*

Then the modal algebraic operations are defined as

$$\begin{aligned} {}^mZ &= {}^mX \circ_{\mathcal{M}} {}^mY \\ &= \{\varphi | \varphi \Leftrightarrow (Q_X x \in X)(Q_Y y \in Y)(Q_Z z \in Z)(z = x \circ_{\mathbb{R}} y)\}. \end{aligned}$$

Hereafter, if confusion is not likely to ensue, the subscript "$\mathcal{M}$", in the modal operation symbols, the left-superscript "m", in modal variable symbols, and the subscript "$\mathbb{R}$", in the real relation and operation symbols, may be suppressed.

The properties of the modal algebraic operations are different than those of the classical interval operations. In the following series of theorems, we formalize some well-known algebraic properties of modal interval arithmetic (see, e.g., [Gardenyes1985], [Gardenyes2001], and [Hayes2009]).

Theorem 3.8 (Absorbing Element in $\mathcal{M}$). *The modal interval* $[0,0]$ *is an absorbing element for modal multiplication, that is*

$$(\forall X \in \mathcal{M})\left([0,0] \times X = X \times [0,0] = [0,0]\right).$$

Theorem 3.9 (Identity for Addition in $\mathcal{M}$). *The modal interval* $[0,0]$ *is both a left and right identity for modal addition, that is*

$$(\forall X \in \mathcal{M})\left([0,0] + X = X + [0,0] = X\right).$$

Theorem 3.10 (Identity for Multiplication in $\mathcal{M}$). *The modal interval* $[1,1]$ *is both a left and right identity for modal multiplication, that is*

$$(\forall X \in \mathcal{M})\left([1,1] \times X = X \times [1,1] = X\right).$$

Theorem 3.11 (Commutativity in $\mathcal{M}$). *Both modal addition and multiplication are commutative, that is*

(i) $(\forall X, Y \in \mathcal{M})\,(X + Y = Y + X)$,

(ii) $(\forall X, Y \in \mathcal{M})\,(X \times Y = Y \times X)$.

Theorem 3.12 (Associativity in $\mathcal{M}$). *Both modal addition and multiplication are associative, that is*

(i) $(\forall X, Y, Z \in \mathcal{M})\,(X + (Y + Z) = (X + Y) + Z)$,

(ii) $(\forall X, Y, Z \in \mathcal{M})\,(X \times (Y \times Z) = (X \times Y) \times Z)$.

Theorem 3.13 (Inverses in $\mathcal{M}$). *Additive and multiplicative inverses exist for modal intervals, that is*

(i) $(\forall X, Y \in \mathcal{M})\,(X - Y = [0,0] \Leftrightarrow Y = \mathrm{dual}\,(X))$,

(ii) $(\forall X, Y \in \mathcal{M})\,(X \div Y = [1,1] \Leftrightarrow Y = \mathrm{dual}\,(X) \wedge 0 \notin \mathrm{set}\,(X))$.

Theorem 3.14 (Subdistributivity in $\mathcal{M}$). *The distributive law does not always hold in modal interval arithmetic, that is*

$$(\exists X, Y, Z \in \mathcal{M})\left(Z \times (X + Y) \neq Z \times X + Z \times Y\right).$$

In general, for all X, Y, *and* Z *in* $\mathcal{M}$, *modal interval arithmetic has only the modal subdistributive law*

$$\begin{aligned} \mathrm{improper}\,(Z) \times X + Z \times Y &\subseteq Z \times (X + Y) \\ &\subseteq \mathrm{proper}\,(Z) \times X + Z \times Y. \end{aligned}$$

We shall now make use of the preceding known properties to prove three results about, respectively, the *additive* structure $\langle \mathcal{M}; +_{\mathcal{M}} \rangle$, the *multiplicative* structure $\langle \mathcal{M}; \times_{\mathcal{M}} \rangle$, and the *ring-like*[14] structure $\langle \mathcal{M}; +_{\mathcal{M}}, \times_{\mathcal{M}} \rangle$ of modal interval arithmetic.

Theorem 3.15 *The additive structure* $\langle \mathcal{M}; +_{\mathcal{M}} \rangle$ *is an abelian group.*

Proof. For $+_{\mathcal{M}}$, the following criteria are satisfied.

- Associativity. Modal addition is associative, by theorem 3.12.
- Commutativity. Modal addition is commutative, by theorem 3.11.
- Identity Element. The modal interval $[0, 0]$ is an identity element for modal addition, by theorem 3.9.
- Inverse Elements. Additive inverses exist for modal intervals, by theorem 3.13.

Therefore, the set $\mathcal{M}$ of modal intervals forms an abelian group under modal addition. ∎

Theorem 3.16 *The multiplicative structure* $\langle \mathcal{M}; \times_{\mathcal{M}} \rangle$ *is an abelian monoid.*

Proof. For $\times_{\mathcal{M}}$, the following criteria are satisfied.

- Associativity. Modal multiplication is associative, by theorem 3.12.
- Commutativity. Modal multiplication is commutative, by theorem 3.11.
- Identity Element. The modal interval $[1, 1]$ is an identity element for modal multiplication, by theorem 3.10.

Therefore, the set $\mathcal{M}$ of modal intervals forms an abelian monoid under modal multiplication. ∎

In consequence of this theorem and theorem 3.13, we have the following corollary.

[14] See footnote 16, on page 31.

Corollary 3.2 *Let $\mathcal{M}_{\tilde{0}}$ be the set of all modal intervals X with $0 \notin \text{Set}(X)$. The multiplicative structure $\langle \mathcal{M}_{\tilde{0}}; \times_{\mathcal{M}} \rangle$ is an abelian group.*

We are now ready to uncover what exactly the algebraic system of modal interval arithmetic.

Theorem 3.17 *The ring-like structure $\langle \mathcal{M}; +_{\mathcal{M}}, \times_{\mathcal{M}}; [0,0], [1,1] \rangle$ is a nondistributive abelian ring*[15]*.*

Proof. By theorem 3.15, the set $\mathcal{M}$ of modal intervals forms an abelian group under modal addition, and, by theorem 3.16, $\mathcal{M}$ forms an abelian monoid under modal multiplication. By theorem 3.8, $[0, 0]$ is an absorbing element for $\times_{\mathcal{M}}$, which is, according to theorem 3.9, also the identity element for $+_{\mathcal{M}}$. According to theorem 3.14, $\times_{\mathcal{M}}$ does not distribute over $+_{\mathcal{M}}$.

Therefore, the structure $\langle \mathcal{M}; +_{\mathcal{M}}, \times_{\mathcal{M}}; [0,0], [1,1] \rangle$, of modal interval arithmetic, is a nondistributive abelian ring. ■

Finally, an immediate consequence of the preceding theorem is the following.

Corollary 3.3 *The theory of modal intervals defines a nondistributive number system*[16] *on the set $\mathcal{M}$.*

Thus, the name "*numbers*" is verified for modal intervals, and therefore we can talk of "*modal interval numbers*".

In conclusion, it should be noted that unlike classical interval arithmetic, modal interval arithmetic has a richer algebraic system with additive and multiplicative inverses. However, distributivity, which is a very useful property of ordinary arithmetic, is not satisfied in the modal interval theory. It should be also mentioned that not every sentence of real arithmetic can be translated into a semantically equivalent sentence of modal arithmetic, without loss of dependency information. Furthermore, in spite of the promising applications of modal intervals, its complicated construction is often misunderstood, which is a drawback for it to have widespread applications.

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[15] A *ring* is an algebra $\langle \mathcal{R}; +_{\mathcal{R}}, \times_{\mathcal{R}} \rangle$ with the additive structure $\langle \mathcal{R}; +_{\mathcal{R}} \rangle$ is an abelian group, the multiplicative structure $\langle \mathcal{R}; \times_{\mathcal{R}} \rangle$ is a monoid, and $\times_{\mathcal{R}}$ distributes over $+_{\mathcal{R}}$.

A *nondistributive ring* is a ring whose multiplication does not distribute over addition.

[16] See footnote 19, on page 32.

3.3 Constraint Interval Arithmetic

An important and promising theory of interval arithmetic is the theory of constraint intervals. Although it is scarcely mentioned in the interval literature, if at all, constraint interval arithmetic has widespread applications in many scientific fields such as artificial intelligence, fuzzy systems, and granular computing, which require more accuracy and compatibility with the *semantic* (*meaning*) of real arithmetic and fuzzy set theory (see, e.g., [Chen2000], [Kahraman2008], [Pedrycz2008], [Pichler2007], and [Yu2004]).

The theory of constraint intervals was presented by Weldon Lodwick in [Lodwick1999], but it perhaps has an earlier root in Cleary's "*logical arithmetic*" which is a logical technique, for real arithmetic in Prolog, that uses constraints over real intervals (see [Cleary1987] and [Cleary1993]). Lodwick presented his theory of constraint intervals as an approach to solving the long-standing dependency problem in the classical interval theory, along with the emphasis that constraint interval arithmetic, unlike Moore's classical interval arithmetic, has additive and multiplicative inverse elements, and satisfies the distributive law.

Our main purpose here is to mathematically examine to what extent the theory of constraint intervals can accomplish these very desirable algebraic properties.

With the elegant idea that a real closed interval is a *convex* subset[17] of the reals, and motivated by the fact that the *best* way to evaluate the *accurate* image of a continuous real-valued function is to apply minimization and maximization directly to determine the exact lower and upper endpoints of the image; Lodwick constructs his constraint interval arithmetic as a simplified type of a min-max optimization problem, with constraints varying in the unit interval.

Lodwick's definition of a constraint interval can be formulated as follows.

Definition 3.20 *Let $\underline{x}, \overline{x} \in \mathbb{R}$ such that $\underline{x} \leq \overline{x}$. A constraint interval is defined to be*

$$[\underline{x}, \overline{x}] = \{x \in \mathbb{R} |\, (\exists \lambda_x \in [0,1])\, (x = (\overline{x} - \underline{x})\, \lambda_x + \underline{x})\},$$

where $\min_{\lambda_x} (x) = \underline{x}$ and $\max_{\lambda_x} (x) = \overline{x}$ are, respectively, the lower and upper

[17] Let $\mathcal{V}$ be a vector space over an ordered field $\langle \mathbb{F}; +_{\mathbb{F}}, \times_{\mathbb{F}}; \leq_{\mathbb{F}} \rangle$. A set $\mathcal{C}$ in $\mathcal{V}$ is said to be convex iff

$$(\forall x, y \in \mathcal{C})\, (\forall \lambda \in [0_{\mathbb{F}}, 1_{\mathbb{F}}])\, (((1 - \lambda)\, x + \lambda y) \in \mathcal{C})\,.$$

bounds (endpoints) of $[\underline{x}, \overline{x}]$.

Obviously, definition 3.20 is equivalent to definition 2.1, and a constraint interval is a closed and bounded nonempty real interval. However, to simplify the exposition, we shall denote the set of constraint intervals by ${}^{\mathrm{t}}[\mathbb{R}]$, and the upper-case Roman letters X, Y, and Z (with or without subscripts), or equivalently $[\underline{x}, \overline{x}]$, $\left[\underline{y}, \overline{y}\right]$, and $[\underline{z}, \overline{z}]$, shall be still employed as variable symbols to denote elements of ${}^{\mathrm{t}}[\mathbb{R}]$. The sets of *point*, *zeroless*, and *symmetric* constraint intervals shall be denoted by the right-subscripted symbols ${}^{\mathrm{t}}[\mathbb{R}]_p$, ${}^{\mathrm{t}}[\mathbb{R}]_{\widetilde{0}}$, and ${}^{\mathrm{t}}[\mathbb{R}]_s$, respectively.

In definition 3.20, a constraint interval is defined as the image of a continuous real-valued function x of one variable $\lambda_x \in [0,1]$ and two constants $\underline{x}$ and $\overline{x}$. The endpoints, $\underline{x}$ and $\overline{x}$, are respectively the minimum and maximum of x (see Figure 3.1) with the constraint

$$\begin{aligned} 0 \le \lambda_x \le 1 \;&\Rightarrow\; 0 \le (\overline{x} - \underline{x})\,\lambda_x \le (\overline{x} - \underline{x}) \\ &\Rightarrow\; \underline{x} \le (\overline{x} - \underline{x})\,\lambda_x + \underline{x} \le \overline{x} \\ &\Rightarrow\; \underline{x} \le x \le \overline{x}. \end{aligned}$$

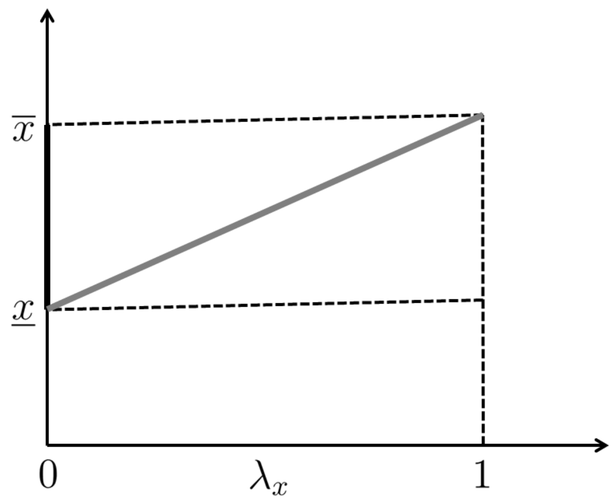

Figure 3.1: A constraint interval as the image of a continuous real function.

Since the endpoints $\underline{x}$ and $\overline{x}$ are known *inputs*, they are parameters whereas λ_x is varying and hence a variable that is constrained between 0 and 1, hence the name "*constraint interval arithmetic*". The binary constraint interval operations can be guaranteed to be *continuous* by introducing two constrained variables $\lambda_x, \lambda_y \in [0,1]$. From the fact that $x \in [\underline{x}, \overline{x}]$ and $y \in \left[\underline{y}, \overline{y}\right]$ are continuous real-valued functions of λ_x and λ_y respectively, the result of a constraint interval operation shall be the image of the continuous function

$$x \circ_{\mathbb{R}} y = \left(\left(\overline{x} - \underline{x}\right) \lambda_x + \underline{x}\right) \circ_{\mathbb{R}} \left(\left(\overline{y} - \underline{y}\right) \lambda_y + \underline{y}\right),$$

with $\lambda_x, \lambda_y \in [0,1]$, and $\circ_{\mathbb{R}} \in \{+,-,\times,\div\}$ such that $y \neq 0$ if $\circ_{\mathbb{R}}$ is "$\div$".

According to the functional dependence of real variables, Lodwick defines two types of constraint interval operations, namely "*dependent operations*" and "*independent operations*". The dependent and independent constraint interval operations are characterized in the following two definitions.

Definition 3.21 (Constraint Dependent Operations). *For any constraint interval $[\underline{x}, \overline{x}]$, there exists a constraint interval $[\underline{z}, \overline{z}]$ such that*

$$\begin{aligned}
[\underline{z}, \overline{z}] &= [\underline{x}, \overline{x}] \circ_{\text{dep}} [\underline{x}, \overline{x}] \\
&= \{z \in \mathbb{R} | \left(\exists x \in [\underline{x}, \overline{x}]\right) \left(z = x \circ_{\mathbb{R}} x\right)\} \\
&= \{z \in \mathbb{R} | \left(\exists \lambda_x \in [0,1]\right) \left(z = \left(\left(\overline{x} - \underline{x}\right) \lambda_x + \underline{x}\right) \circ_{\mathbb{R}} \left(\left(\overline{x} - \underline{x}\right) \lambda_x + \underline{x}\right)\right)\},
\end{aligned}$$

where

$$\begin{aligned}
\underline{z} &= \min_{\lambda_z} (z) = \min_{\lambda_x} \left(x \circ_{\mathbb{R}} x\right), \\
\overline{z} &= \max_{\lambda_z} (z) = \max_{\lambda_x} \left(x \circ_{\mathbb{R}} x\right),
\end{aligned}$$

and $\circ \in \{+,-,\times,\div\}$ such that $0 \notin [\underline{x}, \overline{x}]$ if $\circ$ is "$\div$".

Definition 3.22 (Constraint Independent Operations). *For any two constraint intervals $[\underline{x}, \overline{x}]$ and $\left[\underline{y}, \overline{y}\right]$, there exists a constraint interval $[\underline{z}, \overline{z}]$ such that*

$$\begin{aligned}
[\underline{z}, \overline{z}] &= [\underline{x}, \overline{x}] \circ_{\text{ind}} \left[\underline{y}, \overline{y}\right] \\
&= \{z \in \mathbb{R} | \left(\exists x \in [\underline{x}, \overline{x}]\right) \left(\exists y \in \left[\underline{y}, \overline{y}\right]\right) \left(z = x \circ_{\mathbb{R}} y\right)\} \\
&= \{z \in \mathbb{R} | \left(\exists \lambda_x \in [0,1]\right) \left(\exists \lambda_y \in [0,1]\right) \\
&\qquad \left(z = \left(\left(\overline{x} - \underline{x}\right) \lambda_x + \underline{x}\right) \circ_{\mathbb{R}} \left(\left(\overline{y} - \underline{y}\right) \lambda_y + \underline{y}\right)\right)\},
\end{aligned}$$

where

$$\begin{aligned}
\underline{z} &= \min_{\lambda_z} (z) = \min_{\lambda_x, \lambda_y} \left(x \circ_{\mathbb{R}} y\right), \\
\overline{z} &= \max_{\lambda_z} (z) = \max_{\lambda_x, \lambda_y} \left(x \circ_{\mathbb{R}} y\right),
\end{aligned}$$

and $\circ \in \{+,-,\times,\div\}$ such that $0 \notin \left[\underline{y}, \overline{y}\right]$ if $\circ$ is "$\div$".

It is obvious, from definitions 3.21 and 3.22, that constraint interval arithmetic is a *mathematical programming problem*, and therefore constraint interval operations can be easily performed by any constraint solver such as GeCode[18], HalPPC[19], and MINION[20]. The minimization and maximization are well-defined, attained, and the resultant $[\underline{z}, \overline{z}]$ is in turn a constraint interval.

We now turn to investigate if the theory of constraint intervals accomplishes its objectives. In the first place, we must generally ask: *what exactly is the algebraic structure of constraint interval arithmetic?* Lodwick does not provide an answer for this question. On page 1 in [Lodwick1999], he says:

> "Unlike (classical) interval arithmetic, constrained interval arithmetic has an additive inverse, a multiplicative inverse and satisfies the distributive law. This means that the algebraic structure of constrained interval arithmetic is different than that of (classical) interval arithmetic.",

and then presents proofs for the following three statements:

(i) Additive inverse. $(\forall X \in {}^{\mathrm{t}}[\mathbb{R}])\,(X -_{\mathrm{dep}} X = [0,0])$.

(ii) Multiplicative inverse. $\left(\forall X \in {}^{\mathrm{t}}[\mathbb{R}]_{\widetilde{0}}\right)(X \div_{\mathrm{dep}} X = [1,1])$.

(iii) Distributive law. $(\forall X, Y, Z \in {}^{\mathrm{t}}[\mathbb{R}])\,(Z \times (X + Y) = Z \times X + Z \times Y)$.

The first two statements are derivable by simple substitution in definition 3.21 for constraint dependent operations, hence the subscript "dep". For the third statement, the matter is much more complicated, and therefore we dropped the subscripts for the operation symbols of addition and multiplication.

Getting down to particulars with the above three statements, we must turn to ask, then, the corresponding three questions:

(1) Is the statement "$A -_{\mathrm{dep}} A = [0,0]$" equivalent to "$(-A)$ is the inverse element of A with respect to the operation $+$ on the set ${}^{\mathrm{t}}[\mathbb{R}]$, according to the dependent operation $X +_{\mathrm{dep}} X$"?

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[18] http://www.gecode.org/

[19] http://sourceforge.net/projects/halppc

[20] http://minion.sourceforge.net/

(2) Is the statement "$A \div_{\text{dep}} A = [1,1]$" equivalent to "$\left(A^{-1}\right)$ is the inverse element of A with respect to the operation $\times$ on the set ${}^{\text{t}}[\mathbb{R}]$, according to the dependent operation $X \times_{\text{dep}} X$"?

(3) Does the distributive law hold, according to the dependent and independent operations?

In the sequel, we prove that the answers of the above three questions are all *negative*. On the face of it, the theory of constraint intervals seems to fit squarely into its objectives, but, however the elegance of its underlying idea, we shall argue both that the fit is problematic, and that its mathematical formulation constitutes a serious algebraic defect.

Before setting forth the proofs, we deal first with some algebraic preliminaries we shall need (see, e.g., [Barnes1975], [Levi1961], and [Menini2004]).

Definition 3.23 (Partial and Total Operations). *Let $\mathcal{S}^{\langle n\rangle}$ be the n-th Cartesian power of a set $\mathcal{S}$. An n-ary (total) operation on $\mathcal{S}$ is a total function $t_n : \mathcal{S}^{\langle n\rangle} \longmapsto \mathcal{S}$. An n-ary partial operation in $\mathcal{S}$ is a partial function $p_n : \mathcal{U} \longmapsto \mathcal{S}$, where $\mathcal{U} \subset \mathcal{S}^{\langle n\rangle}$. The ordinal[21] n is called the arity of t_n or p_n.*

A binary operation is an n-ary operation for $n = 2$. Addition and multiplication *on*[22] the set $\mathbb{R}$ of real numbers are best-known examples of binary *total* operations, while division is a *partial* operation *in* $\mathbb{R}$.

Definition 3.24 (Algebraic Structure). *An algebraic structure (or an algebra) is a system $\mathfrak{A} = \left\langle \mathcal{A}; \mathcal{F}^{\mathfrak{A}} \right\rangle$, where*

- *$\mathcal{A}$ is a (possibly empty)[23] set called the individuals universe of $\mathfrak{A}$. The elements of $\mathcal{A}$ are called the individual elements of $\mathfrak{A}$;*
- *$\mathcal{F}^{\mathfrak{A}}$ is a set of n-ary total operations on $\mathcal{A}$. The elements of $\mathcal{F}^{\mathfrak{A}}$ are called the $\mathfrak{A}$-operations.*

[21] See footnote 13, on page 142.

[22] See footnote 7, on page 13.

[23] An algebraic structure with an empty universe of individuals is called an *empty algebraic structure* (see footnote 11, on page 141).

Definition 3.25 (Inverse Elements). *Let $\langle \mathcal{S}; \bullet; \mathbf{e}_\bullet \rangle$ be an algebra with $\bullet$ is a binary operation on $\mathcal{S}$ and $\mathbf{e}_\bullet$ is the identity element for $\bullet$. We say that every element of $\mathcal{S}$ has an inverse element with respect to the operation $\bullet$ iff*

$$(\forall x \in \mathcal{S})\,(\exists y \in \mathcal{S})\,(x \bullet y = \mathbf{e}_\bullet)\,.$$

Definition 3.26 (Number System). *A number system is an algebra $\mathfrak{N} = \langle \mathcal{N}; +_{\mathcal{N}}, \times_{\mathcal{N}} \rangle$ with $+_{\mathcal{N}}$ and $\times_{\mathcal{N}}$ are each both commutative and associative, and $\times_{\mathcal{N}}$ distributes over $+_{\mathcal{N}}$. A nondistributive number system is a number system whose $\times_{\mathcal{N}}$ is not distributive.*

Recalling definition 3.9 of the interval dependency relation, it is also convenient to have at hand the following definitions for two proper subsets of ${}^{\mathrm{t}}[\mathbb{R}]^{\langle 2 \rangle}$, according to interval dependencies.

Definition 3.27 $\mathcal{K}_{\text{dep}} = \{(X, Y) \in {}^{\mathrm{t}}[\mathbb{R}]^{\langle 2 \rangle} \mid Y \mathfrak{D} X\}$.

Definition 3.28 $\mathcal{K}_{\text{ind}} = \{(X, Y) \in {}^{\mathrm{t}}[\mathbb{R}]^{\langle 2 \rangle} \mid \Im\,(X, Y)\}$.

Now we are ready to prove our statements about the theory of constraint intervals. We begin by investigating what type of algebraic operations the constraint operations are.

Theorem 3.18 *Constraint dependent addition and multiplication are partial operations in the set ${}^{\mathrm{t}}[\mathbb{R}]$.*

Proof. For $\circ_{\text{dep}} \in \{+, \times\}$, from definition 3.21, we have

$$\circ_{\text{dep}} : \mathrm{Id}_{{}^{\mathrm{t}}[\mathbb{R}]} \to {}^{\mathrm{t}}[\mathbb{R}],$$

where

$$\mathrm{Id}_{{}^{\mathrm{t}}[\mathbb{R}]} = \{(X, X) \mid X \in {}^{\mathrm{t}}[\mathbb{R}]\}.$$

Obviously the set $\mathrm{Id}_{{}^{\mathrm{t}}[\mathbb{R}]}$ is the *identity* relation on ${}^{\mathrm{t}}[\mathbb{R}]$, which, by definition 3.27, is a proper subset of $\mathcal{K}_{\text{dep}}$, and hence a proper subset of ${}^{\mathrm{t}}[\mathbb{R}]^{\langle 2 \rangle}$.

Therefore, according to definition 3.23, the operations $\circ_{\text{dep}} \in \{+, \times\}$ are partial operations in the set ${}^{\mathrm{t}}[\mathbb{R}]$ of constraint intervals. ■

One immediate result that this theorem implies is that the constraint dependent operations consider only a single special case of interval dependency, namely the *dependency by identity*, $X\mathfrak{D}X$. Other cases of interval dependency, characterized in definition 3.9, are not considered by Lodwick's dependent operations.

Theorem 3.19 *Constraint independent addition and multiplication are partial operations in the set* ${}^{\mathrm{t}}[\mathbb{R}]$.

Proof. For $\circ_{\mathrm{ind}} \in \{+, \times\}$, by definitions 3.22 and 3.28, we have

$$\circ_{\mathrm{ind}} : \mathcal{K}_{\mathrm{ind}} \to {}^{\mathrm{t}}[\mathbb{R}].$$

It is clear, by 3.28, that the set $\mathcal{K}_{\mathrm{ind}}$ is a proper subset of ${}^{\mathrm{t}}[\mathbb{R}]^{\langle 2\rangle}$, and therefore, by definition 3.23, the operations $\circ_{\mathrm{ind}} \in \{+, \times\}$ are partial operations in the set ${}^{\mathrm{t}}[\mathbb{R}]$ of constraint intervals. ■

In accordance with the preceding two results, we are now led to the following three theorems, which answer our questions concerning inverse elements and distributivity.

Theorem 3.20 *Inverse elements for addition cannot be proved to exist in constraint interval arithmetic. In other words, the statement*

$$\left(\forall X \in {}^{\mathrm{t}}[\mathbb{R}]\right)(X + (-X) = [0, 0]),$$

is undecidable in the constraint interval theory.

Proof. For any constraint interval X, define the negation of X, in the standard way, to be

$$-X = \{z \in \mathbb{R} |\, (\exists x \in X)\, (z = -x)\}.$$

Obviously, the relation $(-X)\,\mathfrak{D}X$ is true. But, by theorem 3.18, the pair $((-X), X) \notin \mathrm{Id}_{{}^{\mathrm{t}}[\mathbb{R}]}$ unless $X = [0, 0]$, and the expression $X + (-X)$ thus is not expressible as a constraint dependent operation. On the other hand, by theorem 3.19, $((-X), X) \notin \mathcal{K}_{\mathrm{ind}}$ because the predicate $\Im\,((-X), X)$ is not true, and the expression $X + (-X)$ thus is not expressible as a constraint independent operation.

It follows, therefore, that the existence of additive inverses is undecidable in the constraint interval theory. ■

Theorem 3.21 *Inverse elements for multiplication cannot be proved to exist in constraint interval arithmetic. In other words, the statement*

$$\left(\forall X \in {}^{\mathrm{t}}[\mathbb{R}]\right)\left(0 \notin X \Rightarrow X \times \left(X^{-1}\right) = [1,1]\right),$$

is undecidable in the constraint interval theory.

Proof. For any constraint interval X with $0 \notin X$, define the reciprocal of X, in the standard way, to be

$$X^{-1} = \{z \in \mathbb{R} | \left(\exists x \in X\right)\left(z = x^{-1}\right)\}.$$

Obviously, the relation $\left(X^{-1}\right)\mathfrak{D}X$ is true. But, by theorem 3.18, the pair $\left(\left(X^{-1}\right), X\right) \notin \mathrm{Id}_{{}^{\mathrm{t}}[\mathbb{R}]}$ unless $X = [1,1]$, and the expression $X \times \left(X^{-1}\right)$ thus is not expressible as a constraint dependent operation. On the other hand, by theorem 3.19, $\left(\left(X^{-1}\right), X\right) \notin \mathcal{K}_{\mathrm{ind}}$ because the predicate $\Im\left(\left(X^{-1}\right), X\right)$ is not true, and the expression $X \times \left(X^{-1}\right)$ thus is not expressible as a constraint independent operation.

It follows, therefore, that the existence of multiplicative inverses is undecidable in the constraint interval theory. ∎

Theorem 3.22 *The distributive law does not hold in constraint interval arithmetic. In other words, the statement*

$$\left(\forall X, Y, Z \in {}^{\mathrm{t}}[\mathbb{R}]\right)\left(Z \times (X+Y) = Z \times X + Z \times Y\right),$$

is not provable in the constraint interval theory.

Proof. Obviously, in the left-hand side

$$Z \times (X + Y),$$

all the variables are mutually independent. Then, applying definition 3.22 of the constraint independent operations, we obtain the same result as in classical interval arithmetic, that is

$$Z \times_{\mathrm{ind}} (X +_{\mathrm{ind}} Y) = Z \times_{\mathrm{c}} (X +_{\mathrm{c}} Y).$$

Let us now consider the right-hand side

$$(Z \times X) + (Z \times Y).$$

It is clear that the relations $(Z \times X)\,\mathfrak{D}Z$ and $(Z \times Y)\,\mathfrak{D}Z$ are true. However, by theorem 3.18, the pair $((Z \times X), (Z \times Y)) \notin \mathrm{Id}_{{}^{\mathrm{t}}[\mathbb{R}]}$, and the expression $(Z \times X) + (Z \times Y)$ thus is not expressible as a constraint dependent operation. Then, applying definition 3.22 of the constraint independent operations, we again have the same result as in classical interval arithmetic, that is

$$(Z \times_{\mathrm{ind}} X) +_{\mathrm{ind}} (Z \times_{\mathrm{ind}} Y) = (Z \times_{\mathrm{c}} X) +_{\mathrm{c}} (Z \times_{\mathrm{c}} Y)\,.$$

According to the subdistributivity theorem of the classical interval theory (theorem 2.21, on page 28), we also have only the subdistributive law

$$Z \times_{\mathrm{ind}} (X +_{\mathrm{ind}} Y) \subseteq (Z \times_{\mathrm{ind}} X) +_{\mathrm{ind}} (Z \times_{\mathrm{ind}} Y)\,,$$

for constraint interval arithmetic.

It follows, therefore, that distributivity is not provable in the constraint interval theory. ■

Thus, the preceding three theorems prove that the answers of our questions are all *negative*. The constraint dependent and independent operations do not qualify as *total* operations on ${}^{\mathrm{t}}[\mathbb{R}]$, and in their full extent, do not suffice to cope with interval dependencies except for the special case when the operands are trivially *dependent by identity*, that is, $X\mathfrak{D}X$.

In order to make this clear, we next give an example.

Example 3.8 *Let σ be the prenex sentence such that*

$$\sigma \Leftrightarrow (\forall x \in [-1,1])\,(\exists y \in [0,1])\,(\exists z \in \mathbb{R})\,(z = y - x)\,.$$

In the sentence σ, the variable y depends on x, and therefore there is some function $g(x)$ such that σ has the Skolem equivalent form

$$(\exists g)\,(\forall x \in [-1,1])\,(\exists z \in \mathbb{R})\,(z = g(x) - x)\,.$$

The dependency function g can be, for instance, the quadratic function, that is $y = g(x) = x^2$. It is clear that the relation $[0,1]\,\mathfrak{D}\,[-1,1]$ is true, that is, the interval number $[0,1]$ is dependent on $[-1,1]$. However, the pair $([0,1], [-1,1]) \notin \mathrm{Id}_{{}^{\mathrm{t}}[\mathbb{R}]}$, and the expression $[0,1] - [-1,1]$ thus is not expressible as a constraint dependent operation.

We now pass to our general question concerning the algebraic system of constraint interval arithmetic. The following theorem clarifies an answer.

Theorem 3.23 *The theory of constraint intervals does not define an algebra for addition or multiplication on the set* ${}^{\mathrm{t}}[\mathbb{R}]$.

Proof. By theorems 3.18 and 3.19, the operations $\circ_{\mathrm{dep}}$ and $\circ_{\mathrm{ind}}$, in $\{+, \times\}$, are partial operations in ${}^{\mathrm{t}}[\mathbb{R}]$, and therefore, according to definition 3.24, the algebras $\langle {}^{\mathrm{t}}[\mathbb{R}]; \circ_{\mathrm{dep}} \rangle$ and $\langle {}^{\mathrm{t}}[\mathbb{R}]; \circ_{\mathrm{ind}} \rangle$ are not definable. ■

That is, the structures $\langle {}^{\mathrm{t}}[\mathbb{R}]; \circ_{\mathrm{dep}} \rangle$ and $\langle {}^{\mathrm{t}}[\mathbb{R}]; \circ_{\mathrm{ind}} \rangle$ are *undefinable*, for the requirement that an algebraic operation must be total on the universe set ${}^{\mathrm{t}}[\mathbb{R}]$.

In consequence of the last theorem, we also have the following important result.

Theorem 3.24 *The theory of constraint intervals does not define a number system on the set* ${}^{\mathrm{t}}[\mathbb{R}]$.

Proof. The proof immediately follows from definition 3.26 and theorem 3.23, by the fact that every number system is an algebra. ■

The name "*numbers*" thus is not correct for constraint intervals, and therefore we cannot talk of "*constraint interval numbers*".

From the above discussion, we can conclude that the underlying idea of constraint interval arithmetic seems elegant and simple, but it is *too simple* to fully account for the notion of interval dependency or to achieve a richer algebraic structure for interval arithmetic. It is therefore imperative both to supply the defect in Lodwick's approach and to present an alternative theory with a mathematical construction that avoids the defect. The former was attempted in the present section, and the latter is attempted in the next chapter ("*a theory of optimizational intervals*").

Chapter 4

A Theory of Optimizational Intervals

Nothing at all takes place in the universe in which some rule of maximum or minimum does not appear.

–Leonhard Euler (1707-1783)

In the preceding chapter, we proved that the theory of constraint intervals cannot fully account for the notion of interval dependency, does not define an algebra for interval addition or multiplication, and consequently does not define a number system on the set of constraint intervals. With a view to treating these problems, the present chapter is devoted to constructing a new arithmetic of interval numbers.

Based on Lodwick's idea of representing an interval number as a convex set, along with our formalization of the notion of interval dependency (see section 3.1, on page 42), we attempt, in this chapter, to present an alternate theory of intervals, namely the "*theory of optimizational intervals*", with a mathematical construction that tries to avoid some of the defects in the current theories of interval arithmetic, to provide a richer interval algebra, and to better account for the notion of interval dependency.

We begin, in sections 4.1 and 4.2, by defining the key concepts of the optimizational interval theory, and then we formulate the basic operations and relations for optimizational interval numbers. In section 4.3, we carefully construct the algebraic system of optimizational interval arithmetic, deduce its fundamental properties, and then prove that the optimizational interval theory constitutes a rich algebra, which extends the ordinary field structure of real numbers. Finally, in Section 4.4, we discuss some further consequences and future prospects concerning the results presented in this chapter. In chapter

5, after formalizing the classical theory of complex intervals, we present an "*optimizational theory of complex interval arithmetic*".

4.1 Algebraic Operations for Optimizational Interval Numbers

In this section, we define the key concepts of the optimizational interval theory, and then we formulate the basic operations and relations for optimizational interval numbers. As usual, in all the proofs, elementary facts about operations and relations on the real numbers are usually used without explicit reference. Moreover, the notions, notations, and abbreviations of section 3.1 are indispensable for our mathematical discussion throughout this chapter and the succeeding chapters, and hereafter are assumed priori, without further mention.

Following Lodwick, we begin by defining an optimizational interval number as a type of convex set.

Definition 4.1 *Let $\underline{x}, \overline{x} \in \mathbb{R}$ such that $\underline{x} \leq \overline{x}$. An optimizational interval number is a closed and bounded non-empty convex subset of $\mathbb{R}$, that is*

$$[\underline{x}, \overline{x}] = \{x \in \mathbb{R} |\, (\exists \lambda_x \in [0,1])\, (x = (\overline{x} - \underline{x})\, \lambda_x + \underline{x})\},$$

where $\min\limits_{\lambda_x}(x) = \underline{x}$ and $\max\limits_{\lambda_x}(x) = \overline{x}$ are, respectively, the lower and upper bounds (endpoints) of $[\underline{x}, \overline{x}]$.

We shall denote the set of optimizational interval numbers by ${}^{o}[\mathbb{R}]$. The upper-case Roman letters X, Y, and Z (with or without subscripts), or equivalently $[\underline{x}, \overline{x}]$, $[\underline{y}, \overline{y}]$, and $[\underline{z}, \overline{z}]$, shall be still employed as variable symbols to denote elements of ${}^{o}[\mathbb{R}]$. The sets of *point*, *zeroless*, and *symmetric* optimizational interval numbers shall be denoted, as usual, by the right-subscripted symbols ${}^{o}[\mathbb{R}]_p$, ${}^{o}[\mathbb{R}]_{\tilde{0}}$, and ${}^{o}[\mathbb{R}]_s$, respectively.

By virtue of definition 3.9, we characterize the equality relation on ${}^{o}[\mathbb{R}]$, in terms of the dependency relation $\mathfrak{D}$ and the identity function Id, as follows.

Definition 4.2 (Equality on ${}^{o}[\mathbb{R}]$). *Two optimizational interval variables X and Y are equal (identical) iff they are dependent by identity, that is*

$$X =_{o} Y \Leftrightarrow X \mathfrak{D}_{\mathrm{Id}} Y.$$

In consequence of this definition and definition 4.1, we have the following immediate theorem.

Theorem 4.1 *Let* $[\underline{x}, \overline{x}]$ *and* $\left[\underline{y}, \overline{y}\right]$ *be any two optimizational interval variables. Then*

$$[\underline{x}, \overline{x}] =_{\text{o}} \left[\underline{y}, \overline{y}\right] \Leftrightarrow \underline{x} = \underline{y} \wedge \overline{x} = \overline{y} \wedge (\forall x \in [\underline{x}, \overline{x}]) \left(\exists y \in \left[\underline{y}, \overline{y}\right]\right) (y = \mathrm{Id}\,(x))\,.$$

Thus, if two optimizational interval *variables* X and Y both are assigned the same *individual constant* (*value*), it does not necessarily follow that X and Y are *equal* (*identical*), unless they are dependent by the identity function (recall example 3.5, on page 52).

We then characterize the binary and unary algebraic operations for optimizational interval numbers, respectively, in the following two *set-theoretic* definitions.

Definition 4.3 (Binary Optimizational Operations). *For any two optimizational interval numbers* X *and* Y*, the binary algebraic operations are defined by*

$$X \circ_{\text{o}} Y = \begin{cases} \{z \in \mathbb{R} | \,(\exists x \in X)\,(\exists y \in Y)\,(z = x \circ_{\mathbb{R}} y)\} & \text{if} \quad \Im\,(X, Y)\,, \\ \{z \in \mathbb{R} | \,(\exists x \in X)\,(\exists_{l=1}^{m} s_l \in S_l)\,(z = \\ \qquad\qquad x \circ_{\mathbb{R}} f\,(x; s_1, ..., s_m))\} & \text{if} \quad Y \mathfrak{D}_f X, \end{cases}$$

where $\circ \in \{+, \times\}$.

Definition 4.4 (Unary Optimizational Operations). *For any optimizational interval number* X*, the unary algebraic operations are defined by*

$$\diamond_{\text{o}} X = \{z \in \mathbb{R} | \,(\exists x \in X)\,(z = \diamond_{\mathbb{R}} x)\},$$

where $\diamond \in \{-, {}^{-1}\}$ *and* $0 \notin X$ *if* $\diamond$ *is* "-1".

Hereafter, if confusion is unlikely, the subscript "o", which stands for "*optimizational* interval operation", and the subscript "$\mathbb{R}$", in the real relation and operation symbols, may be suppressed[1].

In comparing definition 4.3 with Lodwick's two definitions of dependent and independent interval operations (definitions 3.21 and 3.22 of section 3.3), it might at first seem that the advantage of simplicity lies with Lodwick's definitions. However, the advantage of definition 4.3 is that it characterizes, as

[1] As regards notation, see footnotes 1 and 9, on pages 10 and 13, respectively.

we shall prove presently, a (single) "*total binary operation*" on $^{\circ}[\mathbb{R}]$, for each $\circ \in \{+, \times\}$, and exhibits a uniform approach applicable to *all* cases of interval dependency (specified in definition 3.9 of section 3.1).

With a view to reaching more profound results, we supplement the two preceding definitions by the following characterization of an optimizational rational function.

Definition 4.5 (Optimizational Rational Functions). *Let X_k be optimizational intervals and let $\mathcal{F}$ be a function variable symbol. An optimizational rational function $^{\circ}\mathcal{F}(X_1, ..., X_i, X_j, ..., X_n)$ is a (multivariate) function obtained by means of a finite number of the optimizational interval operations such that*

$$^{\circ}\mathcal{F} = \begin{cases} \{z \in \mathbb{R} | \left(\exists_{k=1}^{n} x_k \in X_k\right) (z = \\ \qquad {}^{\mathbb{R}}\mathcal{F}(x_1, ..., x_i, x_j, ..., x_n))\} & \text{if} \quad \Im_{k=1}^{n}(X_k), \\ \{z \in \mathbb{R} | \left(\exists_{k=1}^{n} x_k \in X_k\right) \left(\exists_{l=1}^{m} s_l \in S_l\right) (z = \\ \qquad {}^{\mathbb{R}}\mathcal{F}\left(x_1, ..., x_i, {}^{\mathbb{R}}f(x_i; s_1, ..., s_m), ..., x_n\right))\} & \text{if} \quad X_j \mathfrak{D}_f X_i, \end{cases}$$

where $^{\mathbb{R}}\mathcal{F}(x_1, ..., x_i, x_j, ..., x_n)$ is the corresponding real-valued rational function with $x_k \in X_k$.

As we mentioned before, If the type of function is clear from its arguments, and if confusion is unlikely, we shall usually drop the left superscripts "$\mathbb{R}$" and "o" and simply write $\mathcal{F}(X_1, ..., X_n)$ and $\mathcal{F}(x_1, ..., x_n)$ for, respectively, an optimizational rational function and its corresponding real-valued rational function, which are both defined by the same rule.

By virtue of our definition of an optimizational interval number as a type of convex set, the evaluation of an optimizational rational function is a simplified type of *mathematical optimization*, with the constraints are always in the unit interval $[0, 1]$; and hence the name "*optimizational interval arithmetic*". If there is no dependency between interval numbers, the value of an optimizational rational function is the same as in classical interval arithmetic. When dependencies exist, we have a different value. Thus, optimizational interval arithmetic has an algebra different than that of the classical interval theory.

With the help of the notions characterized above, we are now ready to prove the main theorem of this section, and of optimizational interval arithmetic.

Theorem 4.2 *The value of an optimizational rational function $^{\circ}\mathcal{F}(X_1, ..., X_i, X_j, ..., X_n)$ is the (accurate) image $\mathrm{I}_{\mathcal{F}}$ of the corresponding real-valued rational* *function* $\mathcal{F}(x_1, ..., x_i, x_j, ..., x_n)$, *with* $x_k \in X_k$. *That is*

$$^{\mathrm{o}}\mathcal{F} = \begin{cases} \mathrm{I}_{\mathcal{F}}(X_1, ..., X_i, X_j, ..., X_n) & \text{if} \quad \Im_{k=1}^{n}(X_k), \\ \mathrm{I}_{\mathcal{F}}(X_1, ..., X_i, \mathrm{I}_f(X_i; S_1, ..., S_m), ..., X_n) & \text{if} \quad X_j \mathfrak{D}_f X_i. \end{cases}$$

Proof. Since a real-valued rational function is continuous, it follows, by definition 4.5, that an optimizational rational function is continuous and attains its minimum and maximum values.

Then, by definitions 4.1 and 4.5, optimizing with respect to all $\lambda \in [0,1]$, we obtain

$$^{\mathrm{o}}\mathcal{F} = \begin{cases} [\min\limits_{x_k \in X_k} \mathcal{F}(x_1, ..., x_i, x_j, ..., x_n), \\ \quad \max\limits_{x_k \in X_k} \mathcal{F}(x_1, ..., x_i, x_j, ..., x_n)] & \text{if} \quad \Im_{k=1}^{n}(X_k), \\ \\ [\min\limits_{\substack{x_k \in X_k \\ s_l \in S_l}} \mathcal{F}(x_1, ..., x_i, f(x_i; s_1, ..., s_m), ..., x_n), \\ \quad \max\limits_{\substack{x_k \in X_k \\ s_l \in S_l}} \mathcal{F}(x_1, ..., x_i, f(x_i; s_1, ..., s_m), ..., x_n)] & \text{if} \quad X_j \mathfrak{D}_f X_i, \end{cases}$$

from which we conclude, by theorem 3.3, that

$$^{\mathrm{o}}\mathcal{F} = \begin{cases} \mathrm{I}_{\mathcal{F}}(X_1, ..., X_i, X_j, ..., X_n) & \text{if} \quad \Im_{k=1}^{n}(X_k), \\ \mathrm{I}_{\mathcal{F}}(X_1, ..., X_i, \mathrm{I}_f(X_i; S_1, ..., S_m), ..., X_n) & \text{if} \quad X_j \mathfrak{D}_f X_i, \end{cases}$$

and therefore the value of $^{\mathrm{o}}\mathcal{F}$ is the image $\mathrm{I}_{\mathcal{F}}$ of the corresponding real-valued rational function. ■

Thus, optimizational interval operations are *exact* (or *generalized*) interval operations (see definition 3.8, on page 50), and therefore we have an exact algebra[2] of optimizational intervals. That is, it follows, from this theorem, that arithmetical expressions which are *identical* in real arithmetic are identical

[2] It is understood that every formal theory of arithmetic is a *hierarchy* composed of three successive levels. Let T_i be a theory. The three levels of T_i are as follows (For further details about formal theories and their structures, see, e.g., [Malcev1971], [Rasiowa1963], [Shankar1997], and [Smullyan1961]).

(i) The symbolic (*algebraic*, or *axiomatic*) foundation $S(T_i)$. This is the set of *symbolic* sentences (axioms, definitions, propositions, and so forth), with which the theory is constructed. In a formal sense, an arithmetical theory is *identified* with its algebraic foundation, that is, the expression "*the first order theory of real numbers*" is equivalent to the expression "*the set of first order symbolic sentences that define the theory of real numbers*".

in optimizational interval arithmetic, since both sides of the identity relation yield the image of the real arithmetical expression. With this result at our disposal, many identities of optimizational interval arithmetic can be entailed by the corresponding identities of real arithmetic[3]. As examples, we can mention that, for $x \in X$, $y \in Y$, and $z \in Z$, the *commutative* and *distributive* laws for optimizational interval arithmetic can be immediately established from the

(ii) The numeric (or *arithmetical*) level $N(T_i)$. This is the set of numeric statements obtained by endowing the symbolic sentences of the theory with an arithmetical *interpretation* (*model*, or *structure*) that makes them satisfiable, in a way such that every variable symbol is assigned an *individual constant* (*value*) from the universe set of the interpretation.

(iii) The algorithmic (or *computational*) level $A(T_i)$. This is the set of algorithms for performing computations in the numeric level $N(T_i)$, according to the rules prescribed by the sentences of the symbolic foundation $S(T_i)$.

Each of the three levels is *completely peculiar* to its theory, that is, the set $A(T_1)$ of a theory T_1 is completely different than the set $A(T_2)$ of a theory T_2, because each set is based upon different rules.

Exactness in the symbolic foundation of a theory is a *prior necessary* condition for the theory to be *consistent*, while in the numeric and algorithmic levels, *inaccuracy* naturally arises, due to *finiteness* of numerical representations. For instance, in the axiomatic foundation of the theory of real numbers, we know, by a syntactic proof or by means of a model, that the axiom of real addition

$$(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(\exists z \in \mathbb{R})(z = x + y),$$

is *always true* (that is, z is always *identical* to $x+y$), while in the numeric and algorithmic levels, due to finite representations of real numbers, we may tend to accept some looser approximated formulas such as

$$4.1 \approx 1.333... + 2.777... ,$$

and this what makes *numerical real analysis* different from *real algebra*. That is in any theory of arithmetic, inaccuracy arises *only* when getting up to the the numeric level. On the contrary, this is not the case for the present theories of intervals. For instance, the sentence of interval addition

$$(\forall X \in [\mathbb{R}])(\forall Y \in [\mathbb{R}])(\exists Z \in [\mathbb{R}])(Z = X + Y),$$

is *not always true* (see theorem 3.7, on page 53), for the case when the intervals X, Y, and Z are prescribed by the prenex sentence

$$(\forall x \in X)(\exists y \in Y)(\exists z \in \mathbb{R})(z = x + y).$$

Theorem 4.2 asserts that the symbolic foundation of the optimizational interval theory is *exact* and, like other theories of arithmetic, including the theory of real numbers, inaccuracies can arise *only* when getting up to the numeric level.

[3] A similar result for classical interval arithmetic was proved by Moore, for the restricted case when every variable *occurs only once* on each side of the identity relation, or, in other words, when all variables are *functionally independent* (see, e.g., [Moore1966], [Moore1979], and [Moore2009]). Moore's result is entailed by theorem 3.5, on page 50.

corresponding laws of real arithmetic, as follows.

$$
\begin{aligned}
x + y &= y + x \Rightarrow X + Y = Y + X, \\
x \times y &= y \times x \Rightarrow X \times Y = Y \times X, \\
z \times (x + y) &= z \times x + z \times y \Rightarrow Z \times (X + Y) = Z \times X + Z \times Y.
\end{aligned}
$$

In view of this theorem, we have then the following corollary that gives a new reformulation of the dependency relation.

Corollary 4.1 $Y \mathfrak{D}_f X \Leftrightarrow Y = {}^{\mathrm{o}}f\,(X; S_1, ..., S_m)$.

This corollary assures an important and *peculiar* property of optimizational interval arithmetic: when a real-valued function is translated into the corresponding optimizational function, the *semantic* of functional dependence is *completely conserved*, and we have the equivalence

$$y = f\,(x; s_1, ..., s_m) \Leftrightarrow Y = {}^{\mathrm{o}}f\,(X; S_1, ..., S_m)\,.$$

Accordingly, an optimizational n-ary predicate is characterized as follows.

Definition 4.6 (Optimizational Predicate). *Let X_k be optimizational intervals and let $\mathcal{P}$ be an n-ary predicate variable symbol. Then*

$$\mathcal{P}\,(X_1, ..., X_i, X_j, ..., X_n) \wedge X_j \mathfrak{D}_f X_i \Leftrightarrow \mathcal{P}\,(X_1, ..., X_i, f\,(X_i; S_1, ..., S_m)\,, ..., X_n)\,.$$

For instance, let $\mathcal{P}\,(X, T, Y, U)$ be $X + T = Y + U$. Then

$$X + T = Y + U \wedge Y \mathfrak{D}_f X \Leftrightarrow X + T = f\,(X; S_1, ..., S_m) + U.$$

Combining theorem 4.2 with theorem 3.7, we obtain the following result that establishes the relation between an optimizational rational function and its corresponding classical interval function.

Theorem 4.3 *Let $\mathcal{F}_{\Im}$ and $\mathcal{F}_{\mathfrak{D}}$ be the values of an optimizational rational function ${}^{\mathrm{o}}\mathcal{F}(X_1, ..., X_n)$, for, respectively, $\Im_{k=1}^{n}\,(X_k)$ and $X_j \mathfrak{D}_f X_i$, and let ${}^{\mathrm{c}}\mathcal{F}(X_1, ..., X_n)$ be the corresponding classical rational function. Then*

$$\mathcal{F}_{\mathfrak{D}} \subseteq \mathcal{F}_{\Im} = {}^{\mathrm{c}}\mathcal{F},$$

and, in general

$$\mathrm{I}_{\mathcal{F}} = {}^{\mathrm{o}}\mathcal{F} \subseteq {}^{\mathrm{c}}\mathcal{F}.$$

Proof. According to theorem 3.4, we have $\mathcal{F}_{\mathfrak{D}} \subseteq \mathcal{F}_{\Im}$. By theorem 4.2, the value of ${}^{\mathrm{o}}\mathcal{F}$ is the image $\mathrm{I}_{\mathcal{F}}$ of the corresponding real-valued rational function, which is, by theorem 3.7, a subset of the classical rational function ${}^{\mathrm{c}}\mathcal{F}$. The two statements, therefore, are verified. ■

Thus, by virtue of our formalization of the notion of interval dependency (definition 3.9 of section 3.1), optimizational interval arithmetic, unlike the classical interval theory and its alternates, copes with all possible cases of functional dependence between interval variables. This what makes our construction differ fundamentally from the interval theories discussed in the previous chapters.

In particular, theorem 4.2, plus definitions 4.3 and 4.4, implies the following four easily derivable results.

Theorem 4.4 (Addition on ${}^{\mathrm{o}}[\mathbb{R}]$). *For any two optimizational interval numbers* $X = [\underline{x}, \overline{x}]$ *and* $Y = \left[\underline{y}, \overline{y}\right]$*, optimizational interval addition is a total operation, on* ${}^{\mathrm{o}}[\mathbb{R}]$*, formulated as*

$$X + Y = \begin{cases} \left[\min\limits_{\lambda_x,\lambda_y}(x+y), \max\limits_{\lambda_x,\lambda_y}(x+y)\right] \\ \qquad\qquad = \left[\underline{x}+\underline{y}, \overline{x}+\overline{y}\right] & \text{if } \Im(X,Y), \\ \\ \left[\min\limits_{\lambda_x,\lambda_s}(x + f(x; s_1, ..., s_m)),\right. \\ \qquad \left.\max\limits_{\lambda_x,\lambda_s}(x + f(x; s_1, ..., s_m))\right] \\ \qquad\qquad = X + f(X; S_1, ..., S_m) & \text{if } Y\mathfrak{D}_f X. \end{cases}$$

Theorem 4.5 (Multiplication on ${}^{\mathrm{o}}[\mathbb{R}]$). *For any two optimizational interval numbers* $X = [\underline{x}, \overline{x}]$ *and* $Y = \left[\underline{y}, \overline{y}\right]$*, optimizational interval multiplication is a total operation, on* ${}^{\mathrm{o}}[\mathbb{R}]$*, formulated as*

$$X \times Y = \begin{cases} \left[\min\limits_{\lambda_x,\lambda_y}(x\times y), \max\limits_{\lambda_x,\lambda_y}(x\times y)\right] \\ \qquad = \left[\min\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}, \max\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}\right] & \text{if } \Im(X,Y), \\ \\ \left[\min\limits_{\lambda_x,\lambda_s}(x \times f(x; s_1, ..., s_m)),\right. \\ \qquad \left.\max\limits_{\lambda_x,\lambda_s}(x \times f(x; s_1, ..., s_m))\right] \\ \qquad = X \times f(X; S_1, ..., S_m) & \text{if } Y\mathfrak{D}_f X. \end{cases}$$

Theorem 4.6 (Negation on ${}^{\mathrm{o}}[\mathbb{R}]$). *For any optimizational interval number* $[\underline{x}, \overline{x}]$*, optimizational interval negation is a total operation, on* ${}^{\mathrm{o}}[\mathbb{R}]$*, formulated* *as*

$$-[\underline{x}, \overline{x}] = \left[\min_{\lambda_x}(-x), \max_{\lambda_x}(-x)\right] = [-\overline{x}, -\underline{x}].$$

Theorem 4.7 (Reciprocal in $^{\mathrm{o}}[\mathbb{R}]$). *For any optimizational interval number $[\underline{x}, \overline{x}] \in {}^{\mathrm{o}}[\mathbb{R}]_{\tilde{0}}$ (that is, $0 \notin [\underline{x}, \overline{x}]$), optimizational interval reciprocal is a partial operation, in $^{\mathrm{o}}[\mathbb{R}]$, formulated as*

$$[\underline{x}, \overline{x}]^{-1} = \left[\min_{\lambda_x}\left(x^{-1}\right), \max_{\lambda_x}\left(x^{-1}\right)\right] = \left[\overline{x}^{-1}, \underline{x}^{-1}\right].$$

These results, along with theorem 4.3, express an important fact of our development: for the case when $\Im(X, Y)$, the value of an optimizational interval operation is the same as that of the corresponding classical interval operation, that is $X \circ_{\mathrm{o}} Y = X \circ_{\mathrm{c}} Y$; and for the case when $Y \mathfrak{D}_f X$, an optimizational interval operation gives a different value, according to the dependency function f. This is why optimizational interval arithmetic copes with all possible cases of interval dependency. In order to clarify the matters, Let, for example, σ be a real sentence that takes the prenex form

$$(\forall x)(\exists y)(\exists z)(\exists u)(u = xy + z).$$

The Skolem equivalent form of σ is

$$(\exists f)(\exists g)(\forall x)(\exists u)(u = x \times f(x) + g(x)).$$

Such a sentence, as we proved in section 3.3, is not expressible by the *partial* constraint operations, and its dependency relations are not considered by the classical interval operations. In contrast, the sentence σ can be evaluated in optimizational interval arithmetic, with its dependency relations are completely coped with. So, it comes as a matter of fact that the theory of optimizational intervals is *completely compatible* with the *semantic* of real arithmetic. That is, any sentence of real arithmetic can be translated into a *semantically equivalent* sentence of optimizational interval arithmetic.

To complete our construction of optimizational interval arithmetic, we next define the *total* operation of "*subtraction*", and the *partial* operations of "*division*" and "*integer exponentiation*", for optimizational interval numbers.

Definition 4.7 (Subtraction on $^{\mathrm{o}}[\mathbb{R}]$). *For any two optimizational interval numbers X and Y, optimizational interval subtraction is defined by*

$$X - Y = X + (-Y).$$

Definition 4.8 (Division in $^{o}[\mathbb{R}]$). *For any* $X \in {}^{o}[\mathbb{R}]$ *and any* $Y \in {}^{o}[\mathbb{R}]_{\tilde{0}}$, *optimizational interval division is defined by*

$$X \div Y = X \times \left(Y^{-1}\right).$$

Definition 4.9 (Integer Exponentiation in $^{o}[\mathbb{R}]$). *For any optimizational interval number* X *and any integer* n, *the integer exponents of* X *are defined, in terms of multiplication and reciprocal in* $^{o}[\mathbb{R}]$, *by the following recursion scheme:*

(i) $X^0 = [1, 1]$,

(ii) $0 < n \Rightarrow X^n = X^{n-1} \times X$,

(iii) $0 \notin X \wedge 0 \leq n \Rightarrow X^{-n} = \left(X^{-1}\right)^n$.

In view of this definition, we have, as an immediate consequence, the following theorem that prescribes the properties of integer exponents of optimizational interval numbers.

Theorem 4.8 *For any two optimizational interval numbers* X *and* Y, *and any two positive integers* m *and* n, *the following identities hold:*

(i) $X^m \times X^n = X^{m+n}$,

(ii) $(X^m)^n = X^{m \times n}$,

(iii) $(X \times Y)^n = X^n \times Y^n$.

Accordingly, we also have the following easy-deducible corollary.

Corollary 4.2 *The identities of integer exponents (i), (ii), and (iii), in theorem 4.8, are valid for all* $X, Y \in {}^{o}[\mathbb{R}]_{\tilde{0}}$ *and any two integers* m *and* n.

To clarify how interval dependencies are *fully* addressed by the optimizational interval operations, in a way which is *completely compatible* with the *semantic* of real arithmetic, consider the real-valued square function

$$f(x) = x^2,$$

with $x \in [\underline{x}, \overline{x}]$, and $\underline{x}\overline{x} < 0$. The value of the corresponding classical interval function, according to theorem 2.5 and definition 2.6, is given by

$$\begin{aligned} {}^{\mathrm{c}}f([\underline{x}, \overline{x}]) &= [\underline{x}, \overline{x}]^2 \\ &= [\underline{x}, \overline{x}] \times_{\mathrm{c}} [\underline{x}, \overline{x}] \\ &= \left[\min\{\underline{x}^2, \underline{x}\overline{x}, \overline{x}^2\}, \max\{\underline{x}^2, \underline{x}\overline{x}, \overline{x}^2\}\right] \\ &= \left[\underline{x}\overline{x}, \max\{\underline{x}^2, \overline{x}^2\}\right], \end{aligned}$$

which is *not consistent* with the fact that a square is always nonnegative. Strictly speaking, the accurate image of the real-valued function f is a proper subset of the value of the corresponding classical interval function ${}^{\mathrm{c}}f$, that is

$$\mathrm{I}_f([\underline{x}, \overline{x}]) \subset {}^{\mathrm{c}}f([\underline{x}, \overline{x}]).$$

Now the value of the corresponding optimizational interval function, according to theorem 4.5 and definition 4.9, is given by

$$\begin{aligned} {}^{\mathrm{o}}f([\underline{x}, \overline{x}]) &= [\underline{x}, \overline{x}]^2 \\ &= [\underline{x}, \overline{x}] \times_{\mathrm{o}} [\underline{x}, \overline{x}] \\ &= \left[\min_{x \in [\underline{x}, \overline{x}]}\left(x^2\right), \max_{x \in [\underline{x}, \overline{x}]}\left(x^2\right)\right] \\ &= \left[\min_{\lambda_x \in [0,1]}\left(((\overline{x} - \underline{x})\lambda_x + \underline{x})^2\right), \max_{\lambda_x \in [0,1]}\left(((\overline{x} - \underline{x})\lambda_x + \underline{x})^2\right)\right] \\ &= \left[0, \max\{\underline{x}^2, \overline{x}^2\}\right], \end{aligned}$$

which is always nonnegative, and we have the identity

$$\mathrm{I}_f([\underline{x}, \overline{x}]) = {}^{\mathrm{o}}f([\underline{x}, \overline{x}]).$$

Therefore, unlike classical interval exponentiation, optimizational exponentiation is *completely compatible* with the *semantic* of the real-valued function $f(x) = x^2$, with $x \in [\underline{x}, \overline{x}]$.

So, with this construction of the optimizational interval theory at our disposal, it is not surprising that we can formulate and evaluate interval arithmetic expressions in a way analogous to that of real arithmetic. As a further illustration, let us consider some examples.

Example 4.1 *Let σ_1 and σ_2 be the two prenex sentences such that*

$$\begin{aligned}\sigma_1 &\Leftrightarrow (\forall x \in [-2,2])(\forall y \in [-2,2])(\exists z \in \mathbb{R})(z = f(x,y) = x \times y),\\ \sigma_2 &\Leftrightarrow (\forall x \in [-2,2])(\exists y \in [-2,2])(\exists z \in \mathbb{R})(z = f(x,y) = x \times y).\end{aligned}$$

It is apparent that the variables x and y are independent in the sentence σ_1, in which case the image of f is $\mathrm{I}_f^{\sigma_1} = [-4,4]$. The value of the corresponding optimizational interval function, for X and Y both are assigned the same individual constant $[-2,2]$, with $\Im(X,Y)$, is given, according to theorem 4.5, by

$$\begin{aligned}{}^{\mathrm{o}}f(X,Y) &= X \times_{\mathrm{o}} Y\\ &= \left[\min_{\substack{x\in[-2,2]\\ y\in[-2,2]}} (x \times y), \max_{\substack{x\in[-2,2]\\ y\in[-2,2]}} (x \times y)\right]\\ &= \left[\min_{\substack{\lambda_x\in[0,1]\\ \lambda_y\in[0,1]}} ((4\lambda_x - 2) \times (4\lambda_y - 2)), \max_{\substack{\lambda_x\in[0,1]\\ \lambda_y\in[0,1]}} ((4\lambda_x - 2) \times (4\lambda_y - 2))\right]\\ &= [-4,4],\end{aligned}$$

which is the accurate image of f.

In the sentence σ_2, the variable y depends on x, and therefore there is some function $g(x)$ such that σ_2 has the Skolem equivalent form

$$(\exists g)(\forall x \in [-2,2])(\exists z \in \mathbb{R})(z = f(x,g(x)) = x \times g(x)).$$

Let us consider the following two cases for the dependency function g.

(i) g is given to be the identity function $y = g(x) = x$. Then $f(x) = x^2$, with an image of $\mathrm{I}_f^{\sigma_2} = [0,4]$. The value of the corresponding optimizational interval function, for X is assigned the value $[-2,2]$, is given, according to

theorem 4.5 and definition 4.9, by

$$\begin{aligned} {}^{\mathrm{o}}f(X) &= X^2 \\ &= X \times_{\mathrm{o}} X \\ &= \left[\min_{x\in[-2,2]} \left(x^2\right), \max_{x\in[-2,2]} \left(x^2\right) \right] \\ &= \left[\min_{\lambda_x\in[0,1]} \left((4\lambda_x - 2)^2\right), \max_{\lambda_x\in[0,1]} \left((4\lambda_x - 2)^2\right) \right] \\ &= [0,4], \end{aligned}$$

which is the accurate image of f.

(ii) g is given to be the negation function $y = g(x) = -x$. Then $f(x) = -x^2$, with an image of $\mathrm{I}_f^{\sigma_2} = [-4,0]$. The value of the corresponding optimizational interval function, for X is assigned the value $[-2,2]$, is given, according to theorem 4.5 and definition 4.9, by

$$\begin{aligned} {}^{\mathrm{o}}f(X) &= -X^2 \\ &= X \times_{\mathrm{o}} (-X) \\ &= \left[\min_{x\in[-2,2]} \left(-x^2\right), \max_{x\in[-2,2]} \left(-x^2\right) \right] \\ &= \left[\min_{\lambda_x\in[0,1]} \left(-(4\lambda_x - 2)^2\right), \max_{\lambda_x\in[0,1]} \left(-(4\lambda_x - 2)^2\right) \right] \\ &= [-4,0], \end{aligned}$$

which is the accurate image of f.

Example 4.2 *Let σ be the prenex sentence such that*

$$\sigma \Leftrightarrow (\forall x \in [-1,1])\,(\exists y \in [-1,1])\,(\exists z \in \mathbb{R})\,(z = f(x,y) = x + y)\,.$$

It is apparent that the variable y depends on x, and therefore there is some function $g(x)$ such that σ has the Skolem equivalent form

$$(\exists g)\,(\forall x \in [-1,1])\,(\exists z \in \mathbb{R})\,(z = f(x, g(x)) = x + g(x))\,.$$

Let us consider the following two cases for the dependency function g.

(i) g is given to be the identity function $y = g(x) = x$. Then $f(x) = 2x$, with an image of $\mathrm{I}_f^{\sigma} = [-2,2]$. The value of the corresponding optimizational interval function, for X is assigned the value $[-1,1]$, is given, according to

theorem 4.4, by

$$\begin{aligned} {}^{\mathrm{o}}f(X) &= X +_{\mathrm{o}} X \\ &= \left[\min_{x\in[-1,1]} (2x), \max_{x\in[-1,1]} (2x) \right] \\ &= \left[\min_{\lambda_x\in[0,1]} (2(2\lambda_x - 1)), \max_{\lambda_x\in[0,1]} (2(2\lambda_x - 1)) \right] \\ &= [-2, 2], \end{aligned}$$

which is the accurate image of f.

(ii) *g is given to be the negation function $y = g(x) = -x$. Then $f(x) = 0$, with an image of $\mathrm{I}_f^\sigma = [0, 0]$. The value of the corresponding optimizational interval function, for X is assigned the value $[-1, 1]$, is given, according to theorem 4.4, by*

$$\begin{aligned} {}^{\mathrm{o}}f(X) &= X +_{\mathrm{o}} (-X) \\ &= \left[\min_{x\in[-1,1]} (x - x), \max_{x\in[-1,1]} (x - x) \right] \\ &= [0, 0], \end{aligned}$$

which is the accurate image of f.

The identity $X +_{\mathrm{o}} (-X) = [0, 0]$, in the above example, expresses the fact that *additive inverses* exist in optimizational interval arithmetic. This fact, along with the fundamental algebraic properties of optimizational interval arithmetic, shall be established in section 4.3.

Aside from the important fact that optimizational interval arithmetic copes with all possible cases of interval dependency, the cornerstone result from the above construction is that each of the optimizational operations of addition and multiplication, unlike the case with Lodwick's constraint intervals, is a (single) "*total operation*" on ${}^{\mathrm{o}}[\mathbb{R}]$, not a "*partial operation*"; and therefore we can fix the structures $\langle {}^{\mathrm{o}}[\mathbb{R}]; +_{\mathrm{o}} \rangle$ and $\langle {}^{\mathrm{o}}[\mathbb{R}]; \times_{\mathrm{o}} \rangle$ and study their properties in the standard way. In section 4.3, we shall carefully construct the algebraic system of optimizational interval arithmetic and deduce its fundamental properties.

4.2 Point Operations for Optimizational Interval Numbers

In a manner analogous to the classical interval theory, point optimizational operations can be characterized as follows.

Definition 4.10 (Point Optimizational Operations). *Let* ${}^{o}[\mathbb{R}]^{\langle n \rangle}$ *be the* n*-th Cartesian power of* ${}^{o}[\mathbb{R}]$*. An* n*-ary point optimizational operation,* ω_n*, is a function that maps elements of* ${}^{o}[\mathbb{R}]^{\langle n \rangle}$ *to the set* ${}^{o}[\mathbb{R}]_p$ *of point optimizational intervals, that is*

$$\omega_n : {}^{o}[\mathbb{R}]^{\langle n \rangle} \longmapsto {}^{o}[\mathbb{R}]_p.$$

Point operations, for an optimizational interval number $[\underline{x}, \overline{x}]$, are also optimization functions with respect to the argument $\lambda_x \in [0, 1]$, and have the same results as in classical interval arithmetic. Next we define some of the point operations for optimizational interval numbers.

Definition 4.11 (Optimizational Infimum). *The infimum of an optimizational interval number* $[\underline{x}, \overline{x}]$ *is the minimum value of* $x \in [\underline{x}, \overline{x}]$*, that is*

$$\begin{aligned} \inf\left([\underline{x}, \overline{x}]\right) &= \min_{\lambda_x}(x) \\ &= \left((\overline{x} - \underline{x})\,\lambda_x + \underline{x}\right)_{\lambda_x = 0} \\ &= \underline{x}. \end{aligned}$$

Definition 4.12 (Optimizational Supremum). *The supremum of an optimizational interval number* $[\underline{x}, \overline{x}]$ *is the maximum value of* $x \in [\underline{x}, \overline{x}]$*, that is*

$$\begin{aligned} \sup\left([\underline{x}, \overline{x}]\right) &= \max_{\lambda_x}(x) \\ &= \left((\overline{x} - \underline{x})\,\lambda_x + \underline{x}\right)_{\lambda_x = 1} \\ &= \overline{x}. \end{aligned}$$

It is apparent that for any optimizational interval number $[\underline{x}, \overline{x}]$, we have the *dual* property

$$\inf\left([\underline{x}, \overline{x}]\right) = -\sup\left(-[\underline{x}, \overline{x}]\right).$$

Definition 4.13 (Optimizational Width). *The width of an optimizational interval number* $[\underline{x}, \overline{x}]$ *is defined to be*

$$\begin{aligned} w\left([\underline{x}, \overline{x}]\right) &= \max_{\lambda_x}(x) - \min_{\lambda_x}(x) \\ &= \overline{x} - \underline{x}. \end{aligned}$$

Thus, for any optimizational interval number $[\underline{x}, \overline{x}]$, we obviously have

$$w\left([\underline{x}, \overline{x}]\right) = w\left(-[\underline{x}, \overline{x}]\right).$$

Definition 4.14 (Optimizational Radius). *The radius of an optimizational interval number* $[\underline{x}, \overline{x}]$ *is defined to be*

$$\begin{aligned} r\left([\underline{x}, \overline{x}]\right) &= \frac{w\left([\underline{x}, \overline{x}]\right)}{2} \\ &= \frac{(\overline{x} - \underline{x})}{2}. \end{aligned}$$

Definition 4.15 (Optimizational Midpoint). *The midpoint (or mean) of an optimizational interval number* $[\underline{x}, \overline{x}]$ *is defined to be*

$$\begin{aligned} m\left([\underline{x}, \overline{x}]\right) &= \frac{\max\limits_{\lambda_x}(x) + \min\limits_{\lambda_x}(x)}{2} \\ &= \left((\overline{x} - \underline{x})\,\lambda_x + \underline{x}\right)_{\lambda_x = 1/2} \\ &= \frac{(\overline{x} + \underline{x})}{2}. \end{aligned}$$

Definition 4.16 (Optimizational Absolute Value). *The absolute value of an optimizational interval number* $[\underline{x}, \overline{x}]$ *is defined to be*

$$\begin{aligned} \left|[\underline{x}, \overline{x}]\right| &= \max\{\left|\min_{\lambda_x}(x)\right|, \left|\max_{\lambda_x}(x)\right|\} \\ &= \max\{|\underline{x}|, |\overline{x}|\}. \end{aligned}$$

Definition 4.17 (Optimizational Metric). *The distance (or metric) between two optimizational interval numbers* $[\underline{x}, \overline{x}]$ *and* $[\underline{y}, \overline{y}]$ *is defined to be*

$$d\left([\underline{x}, \overline{x}], [\underline{y}, \overline{y}]\right) = \max\{\left|\underline{x} - \underline{y}\right|, |\overline{x} - \overline{y}|\}.$$

4.3 Algebraic Properties of Optimizational Interval Arithmetic

We shall now make use of the part of the theory developed in sections 4.1 and 4.2 to further investigate the algebraic properties of optimizational interval arithmetic. In our definition of an optimizational interval number, the properties of real numbers are naturally assumed in advance.

Let us first mention that as variants of the proofs presented in this section, most of the identities of optimizational interval arithmetic can be established from the corresponding identities of real arithmetic. This fact is immediately entailed by theorem 4.2, on page 76.

In addition, by virtue of theorems 4.4-4.7, along with theorem 4.3, if $\Im(X, Y)$, then the result of an optimizational interval operation is the same as that of the corresponding classical interval operation, that is $X \circ_{\mathrm{o}} Y = X \circ_{\mathrm{c}} Y$. This implies that the algebra of optimizational interval arithmetic will differ than that of classical interval arithmetic for only when $Y \mathfrak{D}_f X$. Accordingly, in the proofs of this section, we shall consider only the case when $Y \mathfrak{D}_f X$. Moreover, for brevity, and without loss of generality, we shall assume that

$$Y \mathfrak{D}_f X \Leftrightarrow \mathrm{I}_f(X) = Y \Leftrightarrow {}^{\mathrm{o}}f(X) = Y.$$

Our first results of this section, concerning the isomorphism properties, follow immediately form this fact, plus theorem 2.11 of section 2.3.

Theorem 4.9 *The structure* $\left\langle {}^{\mathrm{o}}[\mathbb{R}]_p; +_{\mathrm{o}}, \times_{\mathrm{o}}; <_{\mathrm{M}} \right\rangle$ *is isomorphically equivalent to the ordered field* $\langle \mathbb{R}; +_{\mathbb{R}}, \times_{\mathbb{R}}; <_{\mathbb{R}} \rangle$ *of real numbers.*

That is, up to isomorphism, the algebra of point optimizational interval numbers, endowed with Moore's ordering $<_{\mathrm{M}}$, is equivalent to the ordered field of real numbers.

Theorem 4.10 *Let* $+_{\Im}$ *and* $\times_{\Im}$ *be, respectively, the optimizational addition and multiplication restricted to the case* $\Im(X, Y)$. *Then the structure* $\langle {}^{\mathrm{o}}[\mathbb{R}]; +_{\Im}, \times_{\Im} \rangle$ *is isomorphically equivalent to the nondistributive abelian semiring* $\langle [\mathbb{R}]; +_{\mathrm{c}}, \times_{\mathrm{c}} \rangle$ *of classical interval numbers.*

That is, optimizational interval arithmetic extends classical interval arithmetic in the sense that if it is the case when $\Im(X, Y)$, then the algebra of the optimizational interval theory is equivalent to that of the classical interval theory.

Now we turn to investigate the algebra of optimizational interval arithmetic. With the help of the results obtained in sections 4.1 and 4.2, we next deduce the fundamental algebraic properties of the optimizational interval operations.

Theorem 4.11 (Absorbing Element in ${}^{\mathrm{o}}[\mathbb{R}]$). *The optimizational interval number* $[0, 0]$ *is an absorbing element for optimizational multiplication, that is*

$$(\forall X \in {}^{\mathrm{o}}[\mathbb{R}])\,([0, 0] \times X = X \times [0, 0] = [0, 0]).$$

Proof. For any optimizational interval number X, according to theorem 4.5 and assuming the properties of real multiplication, we have

$$\begin{aligned}[0,0]\times X &= \left[\min_{\lambda_x}(0\times x), \max_{\lambda_x}(0\times x)\right]\\ &= \left[\min_{\lambda_x}(x\times 0), \max_{\lambda_x}(x\times 0)\right]\\ &= X\times[0,0]=[0,0],\end{aligned}$$

and therefore, the point optimizational interval $[0,0]$ *absorbs* any optimizational interval number X by optimizational multiplication. ■

Theorem 4.12 (Identity for Addition in $^{o}[\mathbb{R}]$). *The optimizational interval number* $[0,0]$ *is both a left and right identity for optimizational addition, that is*

$$(\forall X\in {}^{o}[\mathbb{R}])\left([0,0]+X=X+[0,0]=X\right).$$

Proof. For any optimizational interval number X, according to theorem 4.4 and assuming the properties of real addition, we have

$$\begin{aligned}[0,0]+X &= \left[\min_{\lambda_x}(0+x), \max_{\lambda_x}(0+x)\right]\\ &= \left[\min_{\lambda_x}(x+0), \max_{\lambda_x}(x+0)\right]\\ &= X+[0,0]=X,\end{aligned}$$

and therefore, the point optimizational interval $[0,0]$ is both a left and right identity for optimizational addition. ■

Theorem 4.13 (Identity for Multiplication in $^{o}[\mathbb{R}]$). *The optimizational interval number* $[1,1]$ *is both a left and right identity for optimizational multiplication, that is*

$$(\forall X\in {}^{o}[\mathbb{R}])\left([1,1]\times X=X\times[1,1]=X\right).$$

Proof. For any optimizational interval number X, according to theorem 4.5 and assuming the properties of real multiplication, we have

$$\begin{aligned}[1,1]\times X &= \left[\min_{\lambda_x}(1\times x), \max_{\lambda_x}(1\times x)\right]\\ &= \left[\min_{\lambda_x}(x\times 1), \max_{\lambda_x}(x\times 1)\right]\\ &= X\times[1,1]=X,\end{aligned}$$

and therefore, it is shown that $[1, 1]$ is both a left and right identity for optimizational multiplication. ■

Theorem 4.14 (Commutativity in $^{o}[\mathbb{R}]$). *Both optimizational interval addition and multiplication are commutative, that is*

(i) $(\forall X, Y \in {}^{o}[\mathbb{R}])\,(X + Y = Y + X)$,

(ii) $(\forall X, Y \in {}^{o}[\mathbb{R}])\,(X \times Y = Y \times X)$.

Proof. For $\Im(X, Y)$, the result holds analogously to theorem 2.15 of the classical interval theory.

The proof for $Y \mathfrak{D}_f X \Leftrightarrow f(X) = Y$ is constructed as follows.

(i) For any two optimizational interval numbers X and Y, according to theorem 4.4 and assuming the properties of real addition, we have

$$\begin{aligned} X + Y &= X + f(X) \\ &= \left[\min_{\lambda_x}(x + f(x)), \max_{\lambda_x}(x + f(x))\right] \\ &= \left[\min_{\lambda_x}(f(x) + x), \max_{\lambda_x}(f(x) + x)\right] \\ &= f(X) + X = Y + X. \end{aligned}$$

(ii) In a manner analogous to *(i)*, according to theorem 4.5 and assuming the properties of real multiplication, we have

$$\begin{aligned} X \times Y &= X \times f(X) \\ &= \left[\min_{\lambda_x}(x \times f(x)), \max_{\lambda_x}(x \times f(x))\right] \\ &= \left[\min_{\lambda_x}(f(x) \times x), \max_{\lambda_x}(f(x) \times x)\right] \\ &= f(X) \times X = Y \times X. \end{aligned}$$

Therefore, both addition and multiplication are commutative in $^{o}[\mathbb{R}]$. ■

Theorem 4.15 (Associativity in $^{o}[\mathbb{R}]$). *Both optimizational addition and multiplication are associative, that is* *(i)* $(\forall X, Y, Z \in {}^{\mathrm{o}}[\mathbb{R}])\,(X + (Y + Z) = (X + Y) + Z)$,

(ii) $(\forall X, Y, Z \in {}^{\mathrm{o}}[\mathbb{R}])\,(X \times (Y \times Z) = (X \times Y) \times Z)$.

Proof. For the case when all variables are pairwise independent, the result holds analogously to theorem 2.16 of the classical interval theory.

For the case when some variables are functionally dependent, without loss of generality, we consider the dependency instance $Y \mathfrak{D}_f X \Leftrightarrow f(X) = Y$, and the proof is constructed as follows.

(i) For any three optimizational interval numbers X, Y, and Z, let $[\underline{s}, \overline{s}] = X + (Y + Z)$ and $[\underline{t}, \overline{t}] = (X + Y) + Z$. According to theorems 4.2 and 4.4, we have

$$\begin{aligned}
[\underline{s}, \overline{s}] &= X + (Y + Z) \\
&= X + (f(X) + Z) \\
&= \left[\min_{\lambda_x, \lambda_z} (x + (f(x) + z)), \max_{\lambda_x, \lambda_z} (x + (f(x) + z))\right],
\end{aligned}$$

and

$$\begin{aligned}
[\underline{t}, \overline{t}] &= (X + Y) + Z \\
&= (X + f(X)) + Z \\
&= \left[\min_{\lambda_x, \lambda_z} ((x + f(x)) + z), \max_{\lambda_x, \lambda_z} ((x + f(x)) + z)\right].
\end{aligned}$$

Optimizing with respect to $\lambda_x, \lambda_z \in [0, 1]$, and assuming associativity of real addition, we thus get

$$\begin{aligned}
\underline{s} &= \min_{\lambda_x, \lambda_z} (x + (f(x) + z)) \\
&= \min_{\lambda_x, \lambda_z} ((x + f(x)) + z) \\
&= \underline{t}.
\end{aligned}$$

and

$$\begin{aligned}
\overline{s} &= \max_{\lambda_x, \lambda_z} (x + (f(x) + z)) \\
&= \max_{\lambda_x, \lambda_z} ((x + f(x)) + z) \\
&= \overline{t},
\end{aligned}$$

Hence, according to theorem 4.1, we have $X + (Y + Z) = (X + Y) + Z$.

(ii) In a manner analogous to *(i)*, let $[\underline{s}, \overline{s}] = X \times (Y \times Z)$ and $[\underline{t}, \overline{t}] = (X \times Y) \times Z$. According to theorems 4.2 and 4.5, we have

$$\begin{aligned} [\underline{s}, \overline{s}] &= X \times (Y \times Z) \\ &= X \times (f(X) \times Z) \\ &= \left[\min_{\lambda_x, \lambda_z} (x \times (f(x) \times z)), \max_{\lambda_x, \lambda_z} (x \times (f(x) \times z)) \right], \end{aligned}$$

and

$$\begin{aligned} [\underline{t}, \overline{t}] &= (X \times Y) \times Z \\ &= (X \times f(X)) \times Z \\ &= \left[\min_{\lambda_x, \lambda_z} ((x \times f(x)) \times z), \max_{\lambda_x, \lambda_z} ((x \times f(x)) \times z) \right]. \end{aligned}$$

Optimizing with respect to $\lambda_x, \lambda_z \in [0, 1]$, and assuming associativity of real multiplication, we thus get

$$\begin{aligned} \underline{s} &= \min_{\lambda_x, \lambda_z} (x \times (f(x) \times z)) \\ &= \min_{\lambda_x, \lambda_z} ((x \times f(x)) \times z) \\ &= \underline{t}. \end{aligned}$$

and

$$\begin{aligned} \overline{s} &= \max_{\lambda_x, \lambda_z} (x \times (f(x) \times z)) \\ &= \max_{\lambda_x, \lambda_z} ((x \times f(x)) \times z) \\ &= \overline{t}, \end{aligned}$$

Hence, according to theorem 4.1, we have $X \times (Y \times Z) = (X \times Y) \times Z$.

Therefore, both addition and multiplication are associative in ${}^{o}[\mathbb{R}]$. ■

Hereafter, in all the succeeding theorems, and if not otherwise stated; it should be understood that any two interval variables X and Y can be dependent or independent. The proofs for the case when X and Y are dependent can be simply obtained, in a manner analogous to the preceding theorems, by employing the equivalence $Y \mathfrak{D}_f X \Leftrightarrow f(X) = Y$. The trivial case of dependence by identity $X \mathfrak{D} X$ is obvious.

Theorem 4.16 (Cancellativity of Addition in $^{\text{o}}[\mathbb{R}]$). *Optimizational interval addition is cancellative, that is*

$$(\forall X, Y, Z \in {}^{\text{o}}[\mathbb{R}])\,(X + Z = Y + Z \Rightarrow X = Y)\,.$$

Proof. Let X, Y, and Z be in $^{\text{o}}[\mathbb{R}]$. Assume that

$$X + Z = Y + Z.$$

Then, by theorem 4.4, we immediately have

$$\left[\min_{\lambda_x,\lambda_z}(x+z)\,,\max_{\lambda_x,\lambda_z}(x+z)\right] = \left[\min_{\lambda_y,\lambda_z}(y+z)\,,\max_{\lambda_y,\lambda_z}(y+z)\right],$$

which, according to theorem 4.1, yields

$$\min_{\lambda_x,\lambda_z}(x+z) = \min_{\lambda_y,\lambda_z}(y+z) \;\wedge\; \max_{\lambda_x,\lambda_z}(x+z) = \max_{\lambda_y,\lambda_z}(y+z)\,.$$

Optimizing with respect to $\lambda_x, \lambda_y, \lambda_z \in [0,1]$, we thus get

$$\min_{\lambda_x}(x) = \min_{\lambda_y}(y) \;\wedge\; \max_{\lambda_x}(x) = \max_{\lambda_y}(y)\,,$$

that is $X = Y$, and therefore addition is cancellative in $^{\text{o}}[\mathbb{R}]$. ■

In contrast to the case for addition, and analogously to the classical interval theory, the following theorem asserts that multiplication is not always cancellative in $^{\text{o}}[\mathbb{R}]$.

Theorem 4.17 (Cancellativity[4] of Multiplication in $^{\text{o}}[\mathbb{R}]$). *An optimizational interval number is cancellable for multiplication if, and only if, it is a zeroless interval, that is*

$$(\forall X, Y, Z \in {}^{\text{o}}[\mathbb{R}])\,((X \times Z = Y \times Z \Rightarrow X = Y) \Leftrightarrow 0 \notin Z)\,.$$

Proof. Let X, Y, and Z be in $^{\text{o}}[\mathbb{R}]$. Assume that

$$X \times Z = Y \times Z \Rightarrow X = Y.$$

[4] The cancellative laws for addition and multiplication are also derivable from theorems 4.18 and 4.19, by the fact that *every invertible element is cancellable.* The cancellative law for multiplication is also entailed by theorem 4.21, from the fact that *an element is not cancellable for multiplication iff it is a zero divisor.*

Then, by theorems 4.1 and 4.5, we have

$$\min_{\lambda_x,\lambda_z}(x \times z) = \min_{\lambda_y,\lambda_z}(y \times z) \wedge \max_{\lambda_x,\lambda_z}(x \times z) = \max_{\lambda_y,\lambda_z}(y \times z)$$
$$\Rightarrow \min_{\lambda_x}(x) = \min_{\lambda_y}(y) \wedge \max_{\lambda_x}(x) = \max_{\lambda_y}(y),$$

which yields $z \neq 0$, that is $0 \notin [\underline{z}, \overline{z}]$.

The converse direction is easy to prove, and therefore multiplication is not cancellative in ${}^{\mathrm{o}}[\mathbb{R}]$ except for the case when $0 \notin [\underline{z}, \overline{z}]$. ■

An important property peculiar to the theory of optimizational intervals is that unlike classical interval arithmetic, optimizational interval arithmetic has inverse elements for addition and multiplication. This property figures in the following two theorems.

Theorem 4.18 (Additive Inverses in ${}^{\mathrm{o}}[\mathbb{R}]$). *Additive inverses exist in optimizational interval arithmetic, that is*

$$(\forall X \in {}^{\mathrm{o}}[\mathbb{R}])\,(X + (-X) = [0,0]).$$

Proof. Let X be any optimizational interval number. According to theorem 4.4, we immediately have

$$X + (-X) = \left[\min_{\lambda_x}(x + (-x)), \max_{\lambda_x}(x + (-x))\right] = [0,0],$$

and therefore for each $X \in {}^{\mathrm{o}}[\mathbb{R}]$, there is an inverse element $(-X) \in {}^{\mathrm{o}}[\mathbb{R}]$ under optimizational addition. ■

Theorem 4.19 (Multiplicative Inverses in ${}^{\mathrm{o}}[\mathbb{R}]$). *Every zeroless optimizational interval number is invertible for multiplication on ${}^{\mathrm{o}}[\mathbb{R}]$, that is*

$$\left(\forall X \in {}^{\mathrm{o}}[\mathbb{R}]_{\widetilde{0}}\right)\left(X \times \left(X^{-1}\right) = [1,1]\right).$$

Proof. Let X be any zeroless optimizational interval number, that is $0 \notin X$. According to theorem 4.5, we immediately have

$$X \times \left(X^{-1}\right) = \left[\min_{\lambda_x}\left(x \times \left(x^{-1}\right)\right), \max_{\lambda_x}\left(x \times \left(x^{-1}\right)\right)\right] = [1,1],$$

and therefore for each $X \in {}^{\mathrm{o}}[\mathbb{R}]_{\widetilde{0}}$, there is an inverse element $\left(X^{-1}\right) \in {}^{\mathrm{o}}[\mathbb{R}]$ under optimizational multiplication. ■

The result formulated in the following theorem establishes the additive and multiplicative properties of point optimizational intervals.

Theorem 4.20 (Algebraic Operations in ${}^{o}[\mathbb{R}]_p$). *Let X and Y be two optimizational interval numbers, and let A be any arbitrary constant in ${}^{o}[\mathbb{R}]_p$. Then:*

(i) The sum $X+Y$ is a point optimizational interval iff each of X and Y is a point optimizational interval, or $Y = A + (-X)$, that is

$$(\forall X, Y \in [\mathbb{R}])\,(X+Y \in {}^{o}[\mathbb{R}]_p \Leftrightarrow (X \in {}^{o}[\mathbb{R}]_p \wedge Y \in {}^{o}[\mathbb{R}]_p) \\ \vee (Y = A + (-X))).$$

(ii) The product $X \times Y$ is a point optimizational interval iff each of X and Y is a point optimizational interval, or at least one of X and Y is $[0,0]$, or $Y = A \times (X^{-1})$ with $0 \notin X$, that is

$$(\forall X, Y \in [\mathbb{R}])\,(X \times Y \in {}^{o}[\mathbb{R}]_p \Leftrightarrow (X \in {}^{o}[\mathbb{R}]_p \wedge Y \in {}^{o}[\mathbb{R}]_p) \\ \vee (X = [0,0] \vee Y = [0,0]) \vee (Y = A \times (X^{-1}) \wedge 0 \notin X)).$$

Proof. For *(i)* and *(ii)*, let X and Y be any two optimizational interval numbers.

(i) According to theorem 4.4, we have

$$X+Y = \left[\min_{\lambda_x,\lambda_y}(x+y), \max_{\lambda_x,\lambda_y}(x+y)\right].$$

Assume that $X+Y \in {}^{o}[\mathbb{R}]_p$. Then $\min_{\lambda_x,\lambda_y}(x+y) = \max_{\lambda_x,\lambda_y}(x+y)$, which yields that each of X and Y is a point optimizational interval, or, by theorem 4.18, $Y = A + (-X)$.

The converse direction is easy to prove.

(ii) In a manner analogous to *(i)*, according to theorem 4.5, we have

$$X \times Y = \left[\min_{\lambda_x,\lambda_y}(x \times y), \max_{\lambda_x,\lambda_y}(x \times y)\right].$$

Assume $X \times Y \in {}^{o}[\mathbb{R}]_p$. Then $\min_{\lambda_x,\lambda_y}(x \times y) = \max_{\lambda_x,\lambda_y}(x \times y)$, which yields that each of X and Y is a point optimizational interval, or at least one of X and Y is $[0,0]$, or, by theorem 4.19, $Y = A \times (X^{-1})$ with $0 \notin X$.

The converse direction is easy to prove. ■

In consequence of this theorem and theorem 4.11, the following important property of the algebra of optimizational intervals is easily derivable.

Theorem 4.21 (Zero Divisors in $^{o}[\mathbb{R}]$). *Zero divisors do not exist in optimizational interval arithmetic, that is*

$$(\forall X, Y \in {}^{o}[\mathbb{R}])\,(X \times Y = [0,0] \Rightarrow X = [0,0] \vee Y = [0,0])\,.$$

Thus, like the algebra of real numbers, the algebra of optimizational intervals has no zero divisors, that is for each $X \neq [0,0]$, there is no $Y \neq [0,0]$ such that the identity $X \times Y = [0,0]$ holds.

Now, we turn to the very desirable algebraic property of distributivity. Distributivity of optimizational interval arithmetic is established in the next theorem.

Theorem 4.22 (Distributivity in $^{o}[\mathbb{R}]$). *Multiplication distributes over addition in optimizational interval arithmetic, that is*

$$(\forall X, Y, Z \in {}^{o}[\mathbb{R}])\,(Z \times (X + Y) = Z \times X + Z \times Y)\,.$$

Proof. For any three optimizational interval numbers X, Y, and Z, let $[\underline{s}, \overline{s}] = Z \times (X + Y)$ and $[\underline{t}, \overline{t}] = Z \times X + Z \times Y$. According to theorems 4.2, 4.4, and 4.5, we have

$$\begin{aligned}[\underline{s}, \overline{s}] &= Z \times (X + Y) \\ &= \left[\min_{\lambda_x, \lambda_y, \lambda_z} (z \times (x + y))\,,\ \max_{\lambda_x, \lambda_y, \lambda_z} (z \times (x + y))\right],\end{aligned}$$

and

$$\begin{aligned}[\underline{t}, \overline{t}] &= Z \times X + Z \times Y \\ &= \left[\min_{\lambda_x, \lambda_y, \lambda_z} (z \times x + z \times y)\,,\ \max_{\lambda_x, \lambda_y, \lambda_z} (z \times x + z \times y)\right].\end{aligned}$$

Optimizing with respect to $\lambda_x, \lambda_y, \lambda_z \in [0,1]$, we thus get

$$\begin{aligned}\underline{s} &= \min_{\lambda_x, \lambda_y, \lambda_z} (z \times (x + y)) \\ &= \min_{\lambda_x, \lambda_y, \lambda_z} (z \times x + z \times y) \\ &= \underline{t}.\end{aligned}$$

and

$$\begin{aligned}\overline{s} &= \max_{\lambda_x,\lambda_y,\lambda_z} (z \times (x+y)) \\ &= \max_{\lambda_x,\lambda_y,\lambda_z} (z \times x + z \times y) \\ &= \bar{t},\end{aligned}$$

Hence, according to theorem 4.1, we have

$$Z \times (X+Y) = Z \times X + Z \times Y,$$

and therefore multiplication distributes over addition in ${}^{\mathrm{o}}[\mathbb{R}]$. ■

Thus, in contrast to the classical interval theory and its present alternates, optimizational interval arithmetic does satisfy the distributive law.

We shall now make use of the preceding results to fix the algebraic system of optimizational intervals. First, we prove two theorems about, respectively, the *additive* structure $\langle {}^{\mathrm{o}}[\mathbb{R}]; +_{\mathrm{o}} \rangle$, and the *multiplicative* structure $\langle {}^{\mathrm{o}}[\mathbb{R}]; \times_{\mathrm{o}} \rangle$ of optimizational interval arithmetic.

Theorem 4.23 *The additive structure* $\langle {}^{\mathrm{o}}[\mathbb{R}]; +_{\mathrm{o}} \rangle$ *is an abelian group.*

Proof. For $+_{\mathrm{o}}$, the following criteria are satisfied.

- Associativity. Optimizational addition is associative, by theorem 4.15.
- Commutativity. Optimizational addition is commutative, by theorem 4.14.
- Identity Element. The optimizational interval $[0, 0]$ is an identity element for optimizational addition, by theorem 4.12.
- Inverse Elements. Additive inverses exist for optimizational intervals, by theorem 4.18.

Therefore, the set ${}^{\mathrm{o}}[\mathbb{R}]$ of optimizational intervals forms an abelian group under optimizational addition. ■

Theorem 4.24 *The multiplicative structure* $\langle {}^{\mathrm{o}}[\mathbb{R}]; \times_{\mathrm{o}} \rangle$ *is an abelian monoid.*

Proof. For $\times_{\mathrm{o}}$, the following criteria are satisfied.

- Associativity. Optimizational multiplication is associative, by theorem 4.15.
- Commutativity. Optimizational multiplication is commutative, by theorem 4.14.
- Identity Element. The optimizational interval $[1, 1]$ is an identity element for optimizational multiplication, by theorem 4.13.

Therefore, the set ${}^{\mathrm{o}}[\mathbb{R}]$ of optimizational intervals forms an abelian monoid under optimizational multiplication. ■

In consequence of this theorem and theorem 4.19, we have the following corollary.

Corollary 4.3 *The multiplicative structure* $\left\langle {}^{\mathrm{o}}[\mathbb{R}]_{\tilde{0}}; \times_{\mathrm{o}} \right\rangle$ *of zeroless optimizational intervals is an abelian group.*

With the preceding two theorems and their corollary at our disposal, we are now ready to prove the following result about the algebraic system of optimizational interval arithmetic.

Theorem 4.25 *The structure* $\langle {}^{\mathrm{o}}[\mathbb{R}]; +_{\mathrm{o}}, \times_{\mathrm{o}}; [0,0], [1,1] \rangle$ *is an integral domain*[5] *with every zeroless element has a multiplicative inverse.*

Proof. By theorem 4.23, the set ${}^{\mathrm{o}}[\mathbb{R}]$ of optimizational intervals forms an abelian group under optimizational addition. By theorem 4.24, ${}^{\mathrm{o}}[\mathbb{R}]$ forms an abelian monoid under optimizational multiplication. According to theorem 4.22, $\times_{\mathrm{o}}$ distributes over $+_{\mathrm{o}}$. Hence, ${}^{\mathrm{o}}[\mathbb{R}]$ forms an abelian unital ring, which has, by theorem 4.21, no zero divisors. By theorem 4.19, every element of ${}^{\mathrm{o}}[\mathbb{R}]_{\tilde{0}}$ has a multiplicative inverse.

Therefore, the structure $\langle {}^{\mathrm{o}}[\mathbb{R}]; +_{\mathrm{o}}, \times_{\mathrm{o}}; [0,0], [1,1] \rangle$, of optimizational interval arithmetic, is an integral domain with every zeroless element has a multiplicative inverse. ■

A field structure $\langle \mathcal{F}; +_{\mathcal{F}}, \times_{\mathcal{F}}; 0_{\mathcal{F}}, 1_{\mathcal{F}} \rangle$ is an integral domain in which every element $\alpha \neq 0_{\mathcal{F}}$ has a multiplicative inverse. The difference between the field structure and the structure of optimizational intervals is that for the optimizational algebra, we have the condition $0 \notin \alpha$ instead of $\alpha \neq 0_{\mathcal{F}}$. Such an algebraic

[5] An *integral domain* is an abelian unital ring with no zero divisors.

structure is not usual in mathematics, and it emerges from the fact that the elements of the universe set of the optimizational algebra are themselves *sets* (*interval numbers*). So, to make an explicit stipulation about this new type of algebraic property, it is very convenient to define a new type of algebraic structure, the *set field* (or the $\mathcal{S}$*-field*), that extends the ordinary field structure to the case when elements of the universe set are themselves sets.

Definition 4.18 ($\mathcal{S}$-field). *A set field (or an $\mathcal{S}$-field) is a field structure*

$$\langle \mathcal{F}; +_{\mathcal{F}}, \times_{\mathcal{F}}; \mathbf{0}_{\mathcal{F}}, \mathbf{1}_{\mathcal{F}} \rangle \, ,$$

subject to the following conditions:

(i) $\mathcal{F}$ *is a collection of nonempty sets,*

(ii) $\mathbf{0}_{\mathcal{F}} = \{0\}$ *such that for each* $\alpha \in \mathcal{F}$ *the element* 0 *is the zero element for all* $x \in \alpha$*,*

(iii) $\mathbf{1}_{\mathcal{F}} = \{1\}$ *such that for each* $\alpha \in \mathcal{F}$ *the element* 1 *is the unital element for all* $x \in \alpha$*,*

(iv) the field axiom

$$(\forall \alpha \in \mathcal{F}) \, (\alpha \neq \mathbf{0}_{\mathcal{F}} \Rightarrow (\exists \beta \in \mathcal{F}) \, (\alpha \times_{\mathcal{F}} \beta = \mathbf{1}_{\mathcal{F}})) \, ,$$

is extended to be

$$(\forall \alpha \in \mathcal{F}) \, (0 \notin \alpha \Rightarrow (\exists \beta \in \mathcal{F}) \, (\alpha \times_{\mathcal{F}} \beta = \mathbf{1}_{\mathcal{F}})) \, .$$

In view of this definition and theorem 4.25, the following result can then be concluded.

Theorem 4.26 *The structure* $\langle {}^{\mathrm{o}}[\mathbb{R}]; +_{\mathrm{o}}, \times_{\mathrm{o}}; [0,0] \, , [1,1] \rangle$ *is an* $\mathcal{S}$*-field.*

Finally, an important immediate result that the preceding theorem implies is the following.

Corollary 4.4 *The theory of optimizational intervals defines a number system*[6] *on the set* ${}^{\mathrm{o}}[\mathbb{R}]$*.*

---

[6] See footnote 19, on page 32.

Thus, the name "*numbers*" is verified for optimizational intervals, and therefore we can talk of "*optimizational interval numbers*".

In conclusion, unlike classical interval arithmetic and its present alternates, optimizational intervals have *additive inverses*, *multiplicative inverses* and satisfy the property of *distributivity.* By virtue of the algebraic properties proved in this section, optimizational interval arithmetic possesses a rich $\mathcal{S}$-field algebra, which extends the ordinary field structure of real numbers, and therefore we do not have to sacrifice the useful properties of ordinary arithmetic. In addition, with our formalization of the notion of interval dependency at disposal (see section 3.1, on page 42), optimizational interval operations are defined such that they exhibit a uniform approach applicable to *all cases* of interval dependency. So, in comparing the optimizational interval theory with other theories of intervals, the main advantage that lies with optimizational interval arithmetic, over all other theories of intervals, is that: the theory of optimizational intervals is *completely compatible* with the *semantic* of real arithmetic. That is, any sentence of real arithmetic can be translated into a *semantically equivalent* sentence of optimizational interval arithmetic, without loss of dependency information.

4.4 Limitations of Optimizational Intervals and Future Prospects

To evaluate an optimizational interval expression, the minimization and maximization should usually be applied to the whole corresponding *real* expression, in order not to lose the dependency information. If the problem is too large and the optimization is computationally too costly, we may have to *deviate* from the theoretical construction of the theory by dividing the optimizational problem and hence losing some dependency information. For example, in the interval expression

$$X \times Y + X \times Z = \left[\min_{\lambda_x,\lambda_y,\lambda_z} (xy + xz) \, , \max_{\lambda_x,\lambda_y,\lambda_z} (xy + xz) \right] ,$$

we have to apply optimization to the real expression $xy + xz$ as a whole to get the accurate result. Optimizing each of the products xy and xz *separately*, and then summing the results for the individual optimizations, we shall, with this departure from the theoretical construction, lose the dependency information for x.

An approach we can use for overcoming this issue is to slightly alter our construction of the theory of optimizational intervals by representing an interval by a *3-tuple* $X = [\underline{x}, \overline{x}; \alpha_x]$ called a "*triple interval*", where the third "slot" α_x , that we may call the "*dependency characterizer*", is a symbolic variable used to keep the dependency information for X, as long as there is a *live accessible* value that uses it. The idea is not to lose the dependency information, during a series of calculations, after evaluating *separately* an intermediate interval expression, such that after evaluating a function

$$f\left([\underline{x},\overline{x};\alpha_x],\left[\underline{y},\overline{y};\alpha_y\right]\right)=[\underline{z},\overline{z};f(\alpha_x,\alpha_y)],$$

the third slot $f(\alpha_x,\alpha_y)$ of the resulting interval keeps the dependency information, and a problem which is computationally too costly can be divided into smaller problems without losing the dependency information. Now we can rewrite the dependency and identity between two optimizational *triple* intervals X and Y, respectively, as

$$\begin{aligned} Y\mathfrak{D}_f X &\Leftrightarrow X=[\underline{x},\overline{x};\alpha_x]\wedge Y=\left[\underline{y},\overline{y};f(\alpha_x)\right], \\ X=Y &\Leftrightarrow X\mathfrak{D}_{\mathrm{Id}}Y \\ &\Leftrightarrow \underline{x}=\underline{y}\wedge\overline{x}=\overline{y}\wedge\alpha_x=\alpha_y. \end{aligned}$$

Triple interval representations are used, with the *classical* interval operations, to evaluate the *particular* interval expressions $X-X$ and $X\div X$ as, respectively, $[0,0]$ and $[1,1]$. Although it is limited to such a *dependence by identity*, examples of this usage can be found in algorithmic algebra and artificial intelligence, and are successfully and *neatly implementable* in symbolic languages (computer algebra systems), automated theorem provers, and numeric languages (see, e.g., [Fateman2009], [Keene1988], [Pichler2007], and [Yu2004]). The implementation idea is based on constructing an "*interval data structure*" with *three slots*, rather than the usual representation of an interval as a "pair". In symbolic languages, the "*dependency slots*" are added to a *sequence* or a *list* type. In numerical languages, a *variable name* is constructed for each newly added dependency slot and a *counter incremented* (there is a convenient mechanism that is built-in to "Lisp" for this purpose, a "*weak hash table*"). The dependency information can be also *inherited* in any *object-oriented* language. Since we (humans) ordinarily do not need to see the extra slot, programs ordinarily will not display it in the computer output (see [Fateman2009]).

The following code fragment shows how to perform the operations

$$\begin{aligned} [-1,1;\alpha_x]-[-1,1;\alpha_x] &= [0,0;\alpha_x-\alpha_x], \\ [-1,1;\alpha_x]-[-1,1;\alpha_y] &= [-2,2;\alpha_x-\alpha_y], \end{aligned}$$

in the computer algebra system "Macsyma"[7] (or "Maxima"[8]) (see, e.g., [Chu2010] and [Fateman2009]).

```
block([x:interval(-1,1)], x-x)
block([x:interval(-1,1),y:interval(-1,1)], x-y)
```

The idea can be implemented similarly in other symbolic languages such as "SymbolicC++"[9] and "Maple"[10] (see, e.g., [Robert1996] and [Tan2008]).

In automated theorem provers, the 3-tuple representation of an interval is usually called a "*block interval*" and the dependency slot is called a "*pending identifier*" (see, e.g., [Pichler2007]). in the ATP "Coq"[11], a triple interval *constructor* and a *list* (*sequence*) for the dependency slots are coded by

```
inductive interval: Set:= tuple: nat->nat->nat->interval
inductive seq: Set:= empty: seq | make: interval->seq->seq
```

and in the ATP "Pvs"[12] by

```
interval: TYPE = [nat,nat,nat]
seq: TYPE = list[interval]
```

Using the triple interval representation with classical interval arithmetic provides some nice results, such as $X - X = [0, 0]$, but the outcome is very limited due to the fact that the classical interval operations are not *dependence-aware*, and interval dependencies cannot therefore be fully addressed via the classical interval theory (and in fact, cannot be fully addressed via the present alternate interval theories as well, and this is why the dependency problem is still persisting). In contrast, using the triple interval representation with optimizational interval arithmetic shall yield more profound results, namely:

(1) Interval dependencies are *fully* addressed by the optimizational interval operations, and the accurate result can be computationally obtained, if the problem under concern is not computationally too costly.

[7] http://www.symbolics-dks.com/Macsyma-1.htm

[8] http://maxima.sourceforge.net/

[9] http://issc.uj.ac.za/symbolic/symbolic.html

[10] http://www.maplesoft.com/

[11] http://coq.inria.fr/

[12] http://pvs.csl.sri.com/

(2) If the problem under concern is computationally too costly (such as an engineering problem with a large number of constrained variables); the computational cost can be reduced to a minimum by using the triple interval representation to keep the dependency information, and dividing the problem into smaller ones.

(3) From (1) and (2), the *accurate* result can be obtained, along with the computational cost reduced to a *minimum*, by using the triple interval representation with optimizational interval arithmetic.

(4) Optimizational interval arithmetic has a nice $\mathcal{S}$-*field* algebra, which extends the ordinary field structure of real numbers.

As a future prospect, we shall attempt to *axiomatically* formalize the theory of optimizational intervals with the triple interval representation ("*an axiomatic theory of optimizational triple intervals*").

Chapter 5

An Optimizational Complex Interval Arithmetic

The shortest route between two truths in the real domain passes through the complex domain.

–Jacques Hadamard (1865-1963)

A great extension of number systems is that to the complex numbers. The need for complex numbers in mathematics far transcends the existence of imaginary roots of polynomial equations: there is scarcely a scientific theory which does not involve the notion of a complex number.

As it is the case with computing with real numbers, computing with complex numbers involves *uncertain data*. So, given the fact that an interval number is a real closed interval and a complex number is an ordered pair of real numbers, there is no reason to limit the application of interval arithmetic to the measure of uncertainties in computations with real numbers, and therefore interval arithmetic can be extended, via *complex interval numbers*, to determine regions of uncertainty in computing with ordinary complex numbers.

In the first section of this chapter, we construct the algebraic system of a *classical* complex interval arithmetic, defined in terms of the classical interval theory, and deduce its fundamental properties. Sections 5.2 and 5.3 are devoted to presenting a new systematic construction of complex interval arithmetic, based on the theory of optimizational intervals developed in the preceding chapter. In section 5.2, we define the key concepts of the *optimizational* theory of complex intervals, and then we formulate the basic operations and relations for optimizational complex intervals. In section 5.3, we carefully construct the algebraic system of optimizational complex interval arithmetic, deduce its fundamental properties, and then prove that optimizational complex interval arithmetic possesses a rich $\mathcal{S}$-field[1] algebra, which extends the field structure of ordinary complex numbers and the $\mathcal{S}$-field of optimizational interval numbers.

5.1 Complex Interval Arithmetic: *A Classical Construction*

In this section, we shall construct the algebraic system of a *classical* complex interval arithmetic, defined in terms of the classical interval theory, and deduce its fundamental properties (For other classical constructions of complex interval arithmetic, the reader may consult, e.g., [Alefeld1983], [Boche1966], and [Petkovic1998]).

Hereafter, the boldface small letters $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ (with or without subscripts) shall be employed as variable symbols to denote elements of the set $\mathbb{C}$ of *ordinary* complex numbers, and the boldface capital letters $\boldsymbol{X}$, $\boldsymbol{Y}$, and $\boldsymbol{Z}$ (with or without subscripts) shall be employed as variable symbols to denote elements of the set of classical complex intervals.

We first define what a *classical* complex interval is.

Definition 5.1 *Let X_α and X_β be classical interval numbers. A classical complex interval $\boldsymbol{X}$ is the set of all ordinary complex numbers $x_\alpha + ix_\beta$ for all $x_\alpha \in X_\alpha$ and $x_\beta \in X_\beta$, that is*

$$\begin{aligned} \boldsymbol{X} &= \{\boldsymbol{x} \in \mathbb{C} | \, (\exists x_\alpha \in X_\alpha) \, (\exists x_\beta \in X_\beta) \, (\boldsymbol{x} = x_\alpha + i x_\beta)\} \\ &= X_\alpha + \boldsymbol{i} X_\beta, \end{aligned}$$

where the classical intervals X_α and X_β are called, respectively, the interval and imaginary parts of $\boldsymbol{X}$, and $\boldsymbol{i} = [i, i]$ is the interval imaginary unit.

The set of classical complex intervals shall be denoted by $[\mathbb{C}]$. The set of all $\boldsymbol{X} \in [\mathbb{C}]$ with $0 \notin X_\alpha^2 + X_\beta^2$ is called the set of zeroless classical complex intervals, and shall be denoted by $[\mathbb{C}]_{\tilde{0}}$. The set of all $\boldsymbol{X} \in [\mathbb{C}]$ with X_α and X_β are classical point intervals is called the set of point classical complex intervals, and shall be denoted by $[\mathbb{C}]_p$.

Geometrically, a complex interval may be conceived as a rectangle in the complex plane with sides parallel to the coordinate axes, that is, a complex interval is a *rectangle of certainty* (see Figure 5.1). If it is the case of a point complex interval, the geometric representation shall be a *point* in the complex plane, which is the same as that of the corresponding ordinary complex number.

[1] See definition 4.18, on page 100.

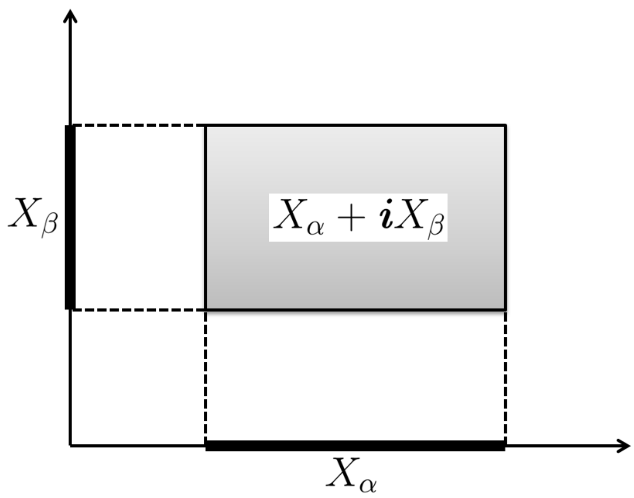

Figure 5.1: Geometric representation of a complex interval.

As it is the case with ordinary complex numbers, classical complex intervals cannot be ordered in a way *compatible*[2] with the algebraic operations. The following theorem, concerning the equality relation on $[\mathbb{C}]$, is an immediate consequence of definition 5.1 and the axiom of extensionality of axiomatic set theory.

Theorem 5.1 (Equality on $[\mathbb{C}]$). *The equality relation for classical complex intervals is formulated in terms of the interval and imaginary parts as*

$$(X_\alpha + \boldsymbol{i}X_\beta) = (Y_\alpha + \boldsymbol{i}Y_\beta) \Leftrightarrow X_\alpha = Y_\alpha \wedge X_\beta = Y_\beta.$$

In a way similar to how ordinary complex arithmetic is defined in terms of real arithmetic, classical complex interval arithmetic is defined in terms of classical interval arithmetic. The binary and unary algebraic operations for classical complex intervals can be characterized, respectively, in the following two *set-theoretic* definitions.

Definition 5.2 (Binary Operations in $[\mathbb{C}]$). *For any two classical complex intervals* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$*, the binary algebraic operations are defined by*

$$\boldsymbol{X} \circ_{\text{cc}} \boldsymbol{Y} = \{\boldsymbol{z} \in \mathbb{C} | \, (\exists \boldsymbol{x} \in \boldsymbol{X}) \, (\exists \boldsymbol{y} \in \boldsymbol{Y}) \, (\boldsymbol{z} = \boldsymbol{x} \circ \boldsymbol{y})\},$$

where $\circ \in \{+, \times\}$.

[2] For a precise characterization of the notion of *order compatibility* with the algebraic operations, see definition 6.21, on page 133.

Definition 5.3 (Unary Operations in $[\mathbb{C}]$). *For any classical complex interval $\boldsymbol{X}$, the unary algebraic operations are defined by*

$$\diamond_{\mathrm{cc}}\boldsymbol{X} = \{\boldsymbol{z} \in \mathbb{C} | \, (\exists \boldsymbol{x} \in \boldsymbol{X}) \, (\boldsymbol{z} = \diamond \boldsymbol{x})\},$$

where $\diamond \in \{-, ^{-1}, \tilde{}\}$ and $\boldsymbol{X} \notin [\mathbb{C}]_{\tilde{0}}$ if $\diamond$ is "$^{-1}$".

The symbol "$\tilde{}$" denotes the *conjugate* operation. Throughout this section, if confusion is unlikely, the subscript "cc", which stands for "*classical complex* interval operation", shall be suppressed, and it shall be assumed, without further mention, that all interval operations are classical operations[3]. For brevity, we shall write $1/X$, XY and X/Y in place of, respectively, X^{-1}, $X \times Y$ and $X \div Y$, by analogy with the ordinary language of arithmetic.

By means of the above set-theoretic definitions and from definitions 2.2 and 2.3 for the classical interval operations, the following five theorems are immediate.

Theorem 5.2 (Addition on[4] $[\mathbb{C}]$). *For any two classical complex intervals $\boldsymbol{X} = X_\alpha + \boldsymbol{i}X_\beta$ and $\boldsymbol{Y} = Y_\alpha + \boldsymbol{i}Y_\beta$, classical complex addition is a total operation, on $[\mathbb{C}]$, formulated as*

$$\boldsymbol{X} + \boldsymbol{Y} = (X_\alpha + Y_\alpha) + \boldsymbol{i}\,(X_\beta + Y_\beta)\,.$$

Theorem 5.3 (Multiplication on $[\mathbb{C}]$). *For any two classical complex intervals $\boldsymbol{X} = X_\alpha + \boldsymbol{i}X_\beta$ and $\boldsymbol{Y} = Y_\alpha + \boldsymbol{i}Y_\beta$, classical complex multiplication is a total operation, on $[\mathbb{C}]$, formulated as*

$$\boldsymbol{X} \times \boldsymbol{Y} = (X_\alpha Y_\alpha - X_\beta Y_\beta) + \boldsymbol{i}\,(X_\alpha Y_\beta + X_\beta Y_\alpha)\,.$$

Theorem 5.4 (Negation on $[\mathbb{C}]$). *For any classical complex interval $\boldsymbol{X} = X_\alpha + \boldsymbol{i}X_\beta$, classical complex negation is a total operation, on $[\mathbb{C}]$, formulated as*

$$-\boldsymbol{X} = (-X_\alpha) + \boldsymbol{i}\,(-X_\beta) = -X_\alpha - \boldsymbol{i}X_\beta.$$

Geometrically, the negation of a complex interval, $\boldsymbol{X}$, is that complex interval which determines a region symmetric to the region determined by $\boldsymbol{X}$ with respect to the origin $(0, 0)$ of the complex plane (see Figure 5.2).

[3] As regards notation, see footnotes 1 and 9, on pages 10 and 13, respectively.

[4] See footnote 7, on page 13.

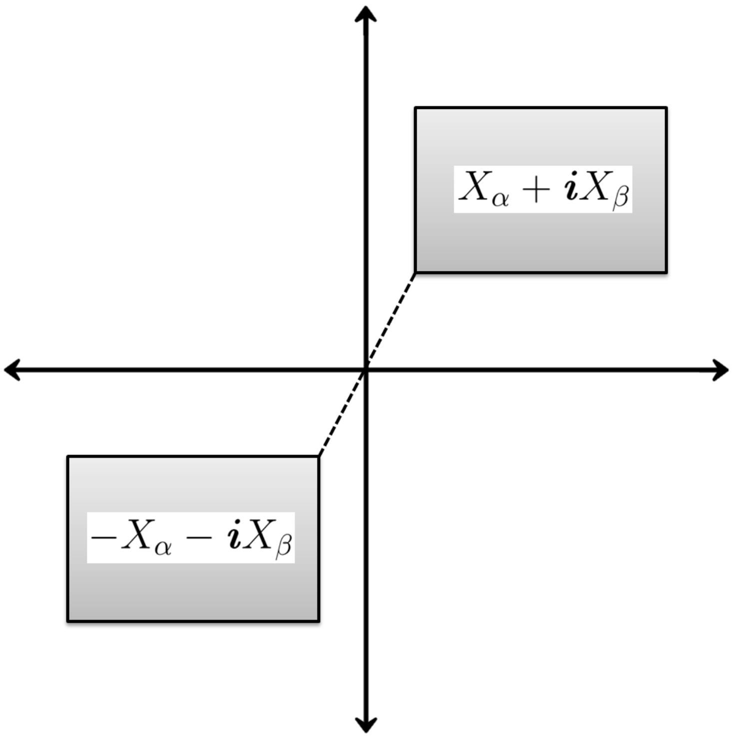

Figure 5.2: Geometric representation of complex interval negation.

An important unary operation peculiar to complex arithmetic is the conjugate operation. This operation is prescribed in the following theorem.

Theorem 5.5 (Conjugate on $[\mathbb{C}]$). *For any classical complex interval* $\boldsymbol{X} = X_\alpha + \boldsymbol{i}X_\beta$, *classical complex conjugate is a total operation, on* $[\mathbb{C}]$, *formulated as*

$$\widetilde{\boldsymbol{X}} = X_\alpha + \boldsymbol{i}\,(-X_\beta) = X_\alpha - \boldsymbol{i}X_\beta.$$

Geometrically, the conjugate of a complex interval, $\boldsymbol{X}$, is that complex interval which determines a region symmetric to the region determined by $\boldsymbol{X}$ with respect to the axis of the reals (see Figure 5.3).

Theorem 5.6 (Reciprocal in $[\mathbb{C}]$). *For any zeroless classical complex interval* $\boldsymbol{X} = X_\alpha + \boldsymbol{i}X_\beta$ *(that is,* $0 \notin X_\alpha^2 + X_\beta^2$*), classical complex reciprocal is a partial operation, in* $[\mathbb{C}]$, *formulated as*

$$\frac{1}{\boldsymbol{X}} = \frac{X_\alpha - \boldsymbol{i}X_\beta}{X_\alpha^2 + X_\beta^2} = \frac{\widetilde{\boldsymbol{X}}}{X_\alpha^2 + X_\beta^2}.$$

In accordance with the above theorems, we can now define *subtraction* and *division* for classical complex intervals, as usual.

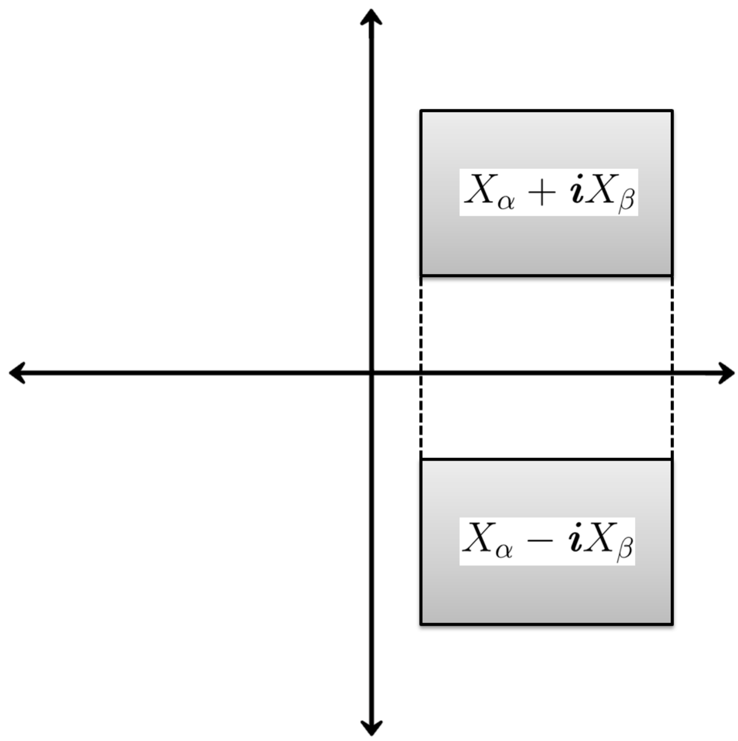

Figure 5.3: Geometric representation of complex interval conjugate.

Definition 5.4 (Subtraction on $[\mathbb{C}]$). *For any two classical complex intervals* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$, *classical complex subtraction is defined by*

$$\boldsymbol{X} - \boldsymbol{Y} = \boldsymbol{X} + (-\boldsymbol{Y}) \, .$$

Definition 5.5 (Division in $[\mathbb{C}]$). *For any classical complex interval* $\boldsymbol{X}$ *and any zeroless classical complex interval* $\boldsymbol{Y}$, *classical complex division is defined by*

$$\frac{\boldsymbol{X}}{\boldsymbol{Y}} = \boldsymbol{X} \times \left(\frac{1}{\boldsymbol{Y}} \right) .$$

By means of the above notions, plus the results of section 2.3, it can be shown that there is no distributivity between addition and multiplication of classical complex intervals except for certain special cases, classical complex multiplication is not always associative, and inverse elements do not always exist for classical complex intervals.

Two other useful properties of ordinary complex arithmetic fail to hold in classical complex interval arithmetic. The following two theorems show that the additive and multiplicative properties, of ordinary complex conjugates, do not hold for classical interval conjugates.

Theorem 5.7 *The additive property of complex conjugates does not always hold for classical complex intervals, that is, for some classical complex interval* $\mathbf{X} = X_\alpha + \boldsymbol{i}X_\beta$

$$(X_\alpha + \boldsymbol{i}X_\beta) + (X_\alpha - \boldsymbol{i}X_\beta) \neq 2X_\alpha.$$

Proof. By theorem 5.2, we have

$$(X_\alpha + \boldsymbol{i}X_\beta) + (X_\alpha - \boldsymbol{i}X_\beta) = 2X_\alpha + \boldsymbol{i}\,(X_\beta - X_\beta)\,,$$

which, due to lack of additive inverse elements (by theorem 2.20), is not equal to $2X_\alpha$ unless X_β is a classical point interval. ■

Theorem 5.8 *The multiplicative property of complex conjugates does not always hold for classical complex intervals, that is, for some classical complex interval* $\mathbf{X} = X_\alpha + \boldsymbol{i}X_\beta$

$$(X_\alpha + \boldsymbol{i}X_\beta) \times (X_\alpha - \boldsymbol{i}X_\beta) \neq \left(X_\alpha^2 - X_\beta^2\right).$$

Proof. By theorem 5.3, we have

$$(X_\alpha + \boldsymbol{i}X_\beta) \times (X_\alpha - \boldsymbol{i}X_\beta) = \left(X_\alpha^2 - X_\beta^2\right) + \boldsymbol{i}\,(X_\alpha X_\beta - X_\alpha X_\beta)\,,$$

which, by theorem 2.20, is not equal to $\left(X_\alpha^2 - X_\beta^2\right)$ unless $X_\alpha X_\beta$ is a classical point interval. ■

With classical complex multiplication is *not associative*, the multiplicative structure $\langle [\mathbb{C}]\,; \times_{cc} \rangle$ does not qualify to be a *monoid*, and therefore the algebraic system of classical complex interval arithmetic is more primitive than that of classical interval arithmetic. This accordingly entails the following obvious theorem.

Theorem 5.9 *The theory of classical complex intervals does not define a number system*[5] *on the set* $[\mathbb{C}]$*.*

The name "*numbers*" thus is not correct for classical complex intervals, and therefore we cannot talk of "*classical complex interval numbers*".

Moreover, as an immediate consequence of theorem 3.7 and the fact that classical complex operations are defined in terms of classical interval operations, we have the following theorem that asserts that the dependency problem is inherited to classical complex interval arithmetic.

[5] See definition 3.26, on page 68.

Theorem 5.10 (Dependency Problem in $[\mathbb{C}]$). *Let $\boldsymbol{X}_k$ be classical complex intervals and let $f(\boldsymbol{x}_1, ..., \boldsymbol{x}_n)$ be an analytic complex-valued function with $\boldsymbol{x}_k \in \boldsymbol{X}_k$. Evaluating the accurate image of f for the classical complex intervals $\boldsymbol{X}_k$, using classical complex interval arithmetic, is not always possible. That is,*

$$(\exists f)\left(\mathrm{I}_f(\boldsymbol{X}_1, ..., \boldsymbol{X}_n) \neq f(\boldsymbol{X}_1, ..., \boldsymbol{X}_n)\right).$$

In general,

$$(\forall f)\left(\mathrm{I}_f(\boldsymbol{X}_1, ..., \boldsymbol{X}_n) \subseteq f(\boldsymbol{X}_1, ..., \boldsymbol{X}_n)\right).$$

Finally, we can conclude that constructing complex interval arithmetic as based on the classical interval theory, we have to sacrifice many useful properties of the *field* of ordinary complex numbers, and moreover we lose the *nondistributive abelian semiring* of the classical interval theory. All of these, and the fact that the dependency problem is inherited from the classical interval theory, have as a consequence that it is not an easy matter to perform arithmetic, solve equations, or evaluate functions in the algebraic system of classical complex intervals.

5.2 Complex Interval Arithmetic: *An Optimizational Construction*

The basic algebraic operations for *optimizational* interval numbers (described in section 4.1) can be extended to complex numbers. In this section and its successor, we set out to present a new systematic construction of complex interval arithmetic, based on the theory of optimizational intervals. In this section, we shall formulate the basic relations and algebraic operations for optimizational complex intervals. In section 5.3, we shall carefully construct the algebraic system of optimizational complex interval arithmetic and deduce its fundamental properties.

With notation mostly as in the preceding section, hereafter, the boldface small letters $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ (with or without subscripts) shall be employed as variable symbols to denote elements of the set $\mathbb{C}$ of ordinary complex numbers, and the boldface capital letters $\boldsymbol{X}$, $\boldsymbol{Y}$, and $\boldsymbol{Z}$ (with or without subscripts) shall be still employed as variable symbols to denote elements of the set of optimizational complex intervals.

We first define what an *optimizational* complex interval is.

Definition 5.6 *Let X_α and X_β be optimizational interval numbers. An opti-*

mizational complex interval $\boldsymbol{X}$ *is defined as*

$$\boldsymbol{X} = X_\alpha +_{\mathrm{o}} \boldsymbol{i} X_\beta,$$

where $\mathrm{In}(\boldsymbol{X}) = X_\alpha$ *and* $\mathrm{Im}(\boldsymbol{X}) = X_\beta$ *are called, respectively, the interval and imaginary parts of* $\boldsymbol{X}$*, and* $\boldsymbol{i} = [i, i]$ *is the interval imaginary unit.*

The set of optimizational complex intervals shall be denoted by ${}^{\mathrm{o}}[\mathbb{C}]$. For the sake of brevity in formulating the statements about optimizational complex interval arithmetic, we shall employ the following abbreviations for some distinguished elements of $\mathbb{C}$ and ${}^{\mathrm{o}}[\mathbb{C}]$.

$$\begin{aligned} 0_{\mathbb{C}} &= 0 + i\,0, \\ 1_{\mathbb{C}} &= 1 + i\,0, \\ \boldsymbol{0}_{[\mathbb{C}]} &= [0,0] + \boldsymbol{i}\,[0,0], \\ \boldsymbol{1}_{[\mathbb{C}]} &= [1,1] + \boldsymbol{i}\,[0,0]. \end{aligned}$$

Moreover, in order to be able easily to speak of different types of optimizational complex intervals, it is also convenient to introduce some notational conventions.

Notation 5.1 *An optimizational complex interval* $\boldsymbol{X}$ *with* $\mathrm{Im}(\boldsymbol{X}) = [0,0]$ *is called a pure interval. The set of all pure intervals shall be denoted by* ${}^{\mathrm{o}}[\mathbb{C}]_{\mathrm{In}}$.

Notation 5.2 *An optimizational complex interval* $\boldsymbol{X}$ *with* $\mathrm{In}(\boldsymbol{X}) = [0,0]$ *is called a pure imaginary. The set of all pure imaginaries shall be denoted by* ${}^{\mathrm{o}}[\mathbb{C}]_{\mathrm{Im}}$.

Notation 5.3 *An optimizational complex interval* $\boldsymbol{X}$ *with* $0_{\mathbb{C}} \notin \boldsymbol{X}$ *(or, equivalently with* $0 \notin \mathrm{In}(\boldsymbol{X}) \wedge 0 \notin \mathrm{Im}(\boldsymbol{X})$*) is called a zeroless optimizational complex interval. The set of all zeroless optimizational complex intervals shall be denoted by* ${}^{\mathrm{o}}[\mathbb{C}]_{\widetilde{0}}$.

Notation 5.4 *An optimizational complex interval* $\boldsymbol{X}$ *with* $\mathrm{In}(\boldsymbol{X})$ *and* $\mathrm{Im}(\boldsymbol{X})$ *are point optimizational intervals is called a point optimizational complex interval. The set of all point optimizational complex intervals shall be denoted by* ${}^{\mathrm{o}}[\mathbb{C}]_p$.

The set ${}^{\mathrm{o}}[\mathbb{C}]_p$ is *isomorphically equivalent* to the set $\mathbb{C}$ of ordinary complex numbers (see theorem 5.19). That is, every element $([x_\alpha, x_\alpha] + \boldsymbol{i}\,[x_\beta, x_\beta]) \in {}^{\mathrm{o}}[\mathbb{C}]_p$

is an *isomorphic copy* of an element $(x_\alpha + i x_\beta) \in \mathbb{C}$. By convention, and being again less pedantic, we agree to *identify* a point optimizational complex interval $[x_\alpha, x_\alpha] + \boldsymbol{i}\,[x_\beta, x_\beta]$ with its isomorphic copy $x_\alpha + i x_\beta$. So, if confusion is not likely to ensue; henceforth, we may write $x_\alpha + i x_\beta$ for $[x_\alpha, x_\alpha] + \boldsymbol{i}\,[x_\beta, x_\beta]$.

Next we characterize the equality relation on $^{\mathrm{o}}[\mathbb{C}]$, in terms of the equality relation on $^{\mathrm{o}}[\mathbb{R}]$ (definition 4.2, on page 74).

Definition 5.7 (Equality on $^{\mathrm{o}}[\mathbb{C}]$). *Two optimizational complex interval variables $\boldsymbol{X}$ and $\boldsymbol{Y}$ are equal (identical) iff they have identical interval and imaginary parts, respectively. That is*

$$\begin{aligned}\boldsymbol{X} =_{\mathrm{oc}} \boldsymbol{Y} \;&\Leftrightarrow\; \mathrm{In}\,(\boldsymbol{X})\,\mathfrak{D}_{\mathrm{Id}}\mathrm{In}\,(\boldsymbol{Y}) \wedge \mathrm{Im}\,(\boldsymbol{X})\,\mathfrak{D}_{\mathrm{Id}}\mathrm{Im}\,(\boldsymbol{Y}) \\ &\Leftrightarrow\; \mathrm{In}\,(\boldsymbol{X}) =_{\mathrm{o}} \mathrm{In}\,(\boldsymbol{Y}) \wedge \mathrm{Im}\,(\boldsymbol{X}) =_{\mathrm{o}} \mathrm{Im}\,(\boldsymbol{Y})\,.\end{aligned}$$

Analogously, in terms of the algebraic operations in $^{\mathrm{o}}[\mathbb{R}]$, we then characterize the binary and unary algebraic operations for optimizational complex intervals, respectively, in the following two definitions.

Definition 5.8 (Binary Operations in $^{\mathrm{o}}[\mathbb{C}]$). *For any two optimizational complex intervals $\boldsymbol{X} = X_\alpha +_{\mathrm{o}} \boldsymbol{i} X_\beta$ and $\boldsymbol{Y} = Y_\alpha +_{\mathrm{o}} \boldsymbol{i} Y_\beta$, the binary algebraic operations are defined by*

$$\boldsymbol{X} \circ_{\mathrm{oc}} \boldsymbol{Y} = (X_\alpha +_{\mathrm{o}} \boldsymbol{i} X_\beta) \circ_{\mathrm{o}} (Y_\alpha +_{\mathrm{o}} \boldsymbol{i} Y_\beta)\,,$$

where $\circ \in \{+, \times\}$.

Definition 5.9 (Unary Operations in $^{\mathrm{o}}[\mathbb{C}]$). *For any optimizational complex interval $\boldsymbol{X} = X_\alpha +_{\mathrm{o}} \boldsymbol{i} X_\beta$, the unary algebraic operations are defined by*

$$\diamond_{\mathrm{oc}} \boldsymbol{X} = \diamond_{\mathrm{o}} (X_\alpha +_{\mathrm{o}} \boldsymbol{i} X_\beta)\,,$$

where $\diamond \in \{-, ^{-1}, \,\tilde{}\,\}$ and $\boldsymbol{X} \notin {}^{\mathrm{o}}[\mathbb{C}]_{\tilde{0}}$ if $\diamond$ is "-1".

Hereafter, if confusion is unlikely, the subscript "oc", which stands for "*optimizational complex* interval operation", and the subscript "o", in the optimizational interval relation and operation symbols, shall be suppressed; and it shall be assumed, without further mention, that all interval operations and relations are optimizational operations and relations[6]. As in the preceding section, for brevity, we shall write $1/X$, XY and X/Y in place of, respectively, X^{-1}, $X \times Y$ and $X \div Y$, by analogy with the ordinary language of arithmetic.

[6] As regards notation, see footnotes 1 and 9, on pages 10 and 13, respectively.

In a manner completely analogous to how the *ordinary* complex operations are deduced from the real operations, the following five results are easily derivable, by means of the above two definitions, plus definitions 4.3 and 4.4 for the optimizational interval operations.

Theorem 5.11 (Addition on $^{\mathrm{o}}[\mathbb{C}]$). *For any two optimizational complex intervals $\boldsymbol{X} = X_\alpha + \boldsymbol{i}X_\beta$ and $\boldsymbol{Y} = Y_\alpha + \boldsymbol{i}Y_\beta$, optimizational complex addition is a total operation, on $^{\mathrm{o}}[\mathbb{C}]$, formulated as*

$$\boldsymbol{X} + \boldsymbol{Y} = (X_\alpha + Y_\alpha) + \boldsymbol{i}\,(X_\beta + Y_\beta)\,.$$

Theorem 5.12 (Multiplication on $^{\mathrm{o}}[\mathbb{C}]$). *For any two optimizational complex intervals $\boldsymbol{X} = X_\alpha + \boldsymbol{i}X_\beta$ and $\boldsymbol{Y} = Y_\alpha + \boldsymbol{i}Y_\beta$, optimizational complex multiplication is a total operation, on $^{\mathrm{o}}[\mathbb{C}]$, formulated as*

$$\boldsymbol{X} \times \boldsymbol{Y} = (X_\alpha Y_\alpha - X_\beta Y_\beta) + \boldsymbol{i}\,(X_\alpha Y_\beta + X_\beta Y_\alpha)\,.$$

Theorem 5.13 (Negation on $^{\mathrm{o}}[\mathbb{C}]$). *For any classical optimizational interval $\boldsymbol{X} = X_\alpha + \boldsymbol{i}X_\beta$, optimizational complex negation is a total operation, on $^{\mathrm{o}}[\mathbb{C}]$, formulated as*

$$-\boldsymbol{X} = (-X_\alpha) + \boldsymbol{i}\,(-X_\beta) = -X_\alpha - \boldsymbol{i}X_\beta.$$

Theorem 5.14 (Conjugate on $^{\mathrm{o}}[\mathbb{C}]$). *For any optimizational complex interval $\boldsymbol{X} = X_\alpha + \boldsymbol{i}X_\beta$, optimizational complex conjugate is a total operation, on $^{\mathrm{o}}[\mathbb{C}]$, formulated as*

$$\widetilde{\boldsymbol{X}} = X_\alpha + \boldsymbol{i}\,(-X_\beta) = X_\alpha - \boldsymbol{i}X_\beta.$$

Since the conjugate of an optimizational complex interval is the negation of its imaginary part, it is apparent that we have the *dual* property $\widetilde{\widetilde{\boldsymbol{X}}} = \boldsymbol{X}$.

Theorem 5.15 (Reciprocal in $^{\mathrm{o}}[\mathbb{C}]$). *For any zeroless optimizational complex interval $\boldsymbol{X} = X_\alpha + \boldsymbol{i}X_\beta$ (that is, $0_{\mathbb{C}} \notin \boldsymbol{X}$), optimizational complex reciprocal is a partial operation, in $^{\mathrm{o}}[\mathbb{C}]$, formulated as*

$$\frac{1}{\boldsymbol{X}} = \frac{X_\alpha - \boldsymbol{i}X_\beta}{X_\alpha^2 + X_\beta^2} = \frac{\widetilde{\boldsymbol{X}}}{X_\alpha^2 + X_\beta^2}.$$

To complete our construction of optimizational complex interval arithmetic, we next define the *total* operation of "*subtraction*", and the *partial* operations of "*division*" and "*integer exponentiation*", for optimizational complex intervals.

Definition 5.10 (Subtraction on $^{o}[\mathbb{C}]$). *For any two optimizational complex intervals $\boldsymbol{X}$ and $\boldsymbol{Y}$, optimizational complex subtraction is defined by*

$$\boldsymbol{X} - \boldsymbol{Y} = \boldsymbol{X} + (-\boldsymbol{Y}) .$$

Definition 5.11 (Division in $^{o}[\mathbb{C}]$). *For any optimizational complex interval $\boldsymbol{X}$ and any zeroless optimizational complex interval $\boldsymbol{Y}$, optimizational complex division is defined by*

$$\frac{\boldsymbol{X}}{\boldsymbol{Y}} = \boldsymbol{X} \times \left(\frac{1}{\boldsymbol{Y}}\right).$$

Definition 5.12 (Integer Exponentiation in $^{o}[\mathbb{C}]$). *For any optimizational complex interval $\boldsymbol{X}$ and any integer n, the integer exponents of $\boldsymbol{X}$ are defined, in terms of multiplication and reciprocal in $^{o}[\mathbb{C}]$, by the following recursion scheme:*

(i) $\boldsymbol{X}^0 = [1, 1]$,

(ii) $0 < n \Rightarrow \boldsymbol{X}^n = \boldsymbol{X}^{n-1} \times \boldsymbol{X}$,

(iii) $0 \notin \boldsymbol{X} \wedge 0 \leq n \Rightarrow \boldsymbol{X}^{-n} = \left(\boldsymbol{X}^{-1}\right)^n$.

In view of this definition, we have, as an immediate consequence, the following theorem that prescribes the properties of integer exponents of optimizational complex intervals.

Theorem 5.16 *For any two optimizational complex intervals $\boldsymbol{X}$ and $\boldsymbol{Y}$, and any two positive integers m and n, the following identities hold:*

(i) $\boldsymbol{X}^m \times \boldsymbol{X}^n = \boldsymbol{X}^{m+n}$,

(ii) $(\boldsymbol{X}^m)^n = \boldsymbol{X}^{m \times n}$,

(iii) $(\boldsymbol{X} \times \boldsymbol{Y})^n = \boldsymbol{X}^n \times \boldsymbol{Y}^n$.

Accordingly, we also have the following easy-deducible corollary.

Corollary 5.1 *The identities of integer exponents (i), (ii), and (iii), in theorem 5.16, are valid for all $\boldsymbol{X}, \boldsymbol{Y} \in {}^{o}[\mathbb{C}]_{\tilde{0}}$ and any two integers m and n.*

Moreover, as an immediate consequence of theorem 4.2 and the fact that optimizational complex operations are defined in terms of optimizational interval operations, we have the following important theorem.

Theorem 5.17 *Let $\boldsymbol{X}_k$ be optimizational complex intervals and let $f(\boldsymbol{x}_1, ..., \boldsymbol{x}_n)$ be a rational complex-valued function with $\boldsymbol{x}_k \in \boldsymbol{X}_k$. Then*

$$(\forall f)\left(\mathrm{I}_f(\boldsymbol{X}_1, ..., \boldsymbol{X}_n) = f(\boldsymbol{X}_1, ..., \boldsymbol{X}_n)\right).$$

That is, as it is the case with optimizational interval operations, optimizational complex operations copes with all possible cases of functional dependence.

By analogy with point operations for optimizational interval numbers, a point operation for optimizational complex intervals is an operation whose operands are optimizational complex intervals, and whose result is a *point* optimizational complex interval (or, equivalently, an *ordinary* complex number). This is made precise in the following definition.

Definition 5.13 (Point Optimizational Complex Operations). *Let ${}^{\mathrm{o}}[\mathbb{C}]^{\langle n\rangle}$ be the n-th Cartesian power of ${}^{\mathrm{o}}[\mathbb{C}]$. An n-ary point optimizational complex operation, ω_n, is a function that maps elements of ${}^{\mathrm{o}}[\mathbb{C}]^{\langle n\rangle}$ to the set ${}^{\mathrm{o}}[\mathbb{C}]_p$ of point optimizational complex intervals, that is*

$$\omega_n : {}^{\mathrm{o}}[\mathbb{C}]^{\langle n\rangle} \longmapsto {}^{\mathrm{o}}[\mathbb{C}]_p.$$

Next we define some point operations for optimizational complex intervals.

Definition 5.14 (Optimizational Complex Infimum). *The infimum of an optimizational complex interval $[\underline{x}_\alpha, \overline{x}_\alpha] + \boldsymbol{i}\left[\underline{x}_\beta, \overline{x}_\beta\right]$ is defined to be*

$$\begin{aligned}\inf\left([\underline{x}_\alpha, \overline{x}_\alpha] + \boldsymbol{i}\left[\underline{x}_\beta, \overline{x}_\beta\right]\right) &= \inf([\underline{x}_\alpha, \overline{x}_\alpha]) + \boldsymbol{i}\inf\left(\left[\underline{x}_\beta, \overline{x}_\beta\right]\right)\\ &= \underline{x}_\alpha + i\underline{x}_\beta.\end{aligned}$$

Definition 5.15 (Optimizational Complex Supremum). *The supremum of an optimizational complex interval $[\underline{x}_\alpha, \overline{x}_\alpha] + \boldsymbol{i}\left[\underline{x}_\beta, \overline{x}_\beta\right]$ is defined to be*

$$\begin{aligned}\sup\left([\underline{x}_\alpha, \overline{x}_\alpha] + \boldsymbol{i}\left[\underline{x}_\beta, \overline{x}_\beta\right]\right) &= \sup([\underline{x}_\alpha, \overline{x}_\alpha]) + \boldsymbol{i}\sup\left(\left[\underline{x}_\beta, \overline{x}_\beta\right]\right)\\ &= \overline{x}_\alpha + i\overline{x}_\beta.\end{aligned}$$

Definition 5.16 (Optimizational Complex Width). *The width of an optimizational complex interval $[\underline{x}_\alpha, \overline{x}_\alpha] + \boldsymbol{i}\left[\underline{x}_\beta, \overline{x}_\beta\right]$ is defined to be*

$$\begin{aligned}w\left([\underline{x}_\alpha, \overline{x}_\alpha] + \boldsymbol{i}\left[\underline{x}_\beta, \overline{x}_\beta\right]\right) &= w([\underline{x}_\alpha, \overline{x}_\alpha]) + \boldsymbol{i}\,w\left(\left[\underline{x}_\beta, \overline{x}_\beta\right]\right)\\ &= (\overline{x}_\alpha - \underline{x}_\alpha) + i\left(\overline{x}_\beta - \underline{x}_\beta\right).\end{aligned}$$

Thus, the width of a point optimizational complex interval is the ordinary complex number $0_{\mathbb{C}}$, that is

$$(\forall x_\alpha, x_\beta \in \mathbb{R})\left(w\left([x_\alpha, x_\alpha] + \boldsymbol{i}\,[x_\beta, x_\beta]\right) = 0 + i\,0 = 0_{\mathbb{C}}\right).$$

Definition 5.17 (Optimizational Complex Radius). *The radius of an optimizational complex interval* $[\underline{x}_\alpha, \overline{x}_\alpha] + \boldsymbol{i}\left[\underline{x}_\beta, \overline{x}_\beta\right]$ *is defined to be*

$$\begin{aligned} r\left([\underline{x}_\alpha, \overline{x}_\alpha] + \boldsymbol{i}\left[\underline{x}_\beta, \overline{x}_\beta\right]\right) &= \frac{w\left([\underline{x}_\alpha, \overline{x}_\alpha] + \boldsymbol{i}\left[\underline{x}_\beta, \overline{x}_\beta\right]\right)}{2} \\ &= \frac{(\overline{x}_\alpha - \underline{x}_\alpha)}{2} + i\frac{(\overline{x}_\beta - \underline{x}_\beta)}{2}. \end{aligned}$$

Definition 5.18 (Optimizational Complex Midpoint). *The midpoint (or mean) of an optimizational complex interval* $[\underline{x}_\alpha, \overline{x}_\alpha] + \boldsymbol{i}\left[\underline{x}_\beta, \overline{x}_\beta\right]$ *is defined to be*

$$\begin{aligned} m\left([\underline{x}_\alpha, \overline{x}_\alpha] + \boldsymbol{i}\left[\underline{x}_\beta, \overline{x}_\beta\right]\right) &= m\left([\underline{x}_\alpha, \overline{x}_\alpha]\right) + \boldsymbol{i}\, m\left(\left[\underline{x}_\beta, \overline{x}_\beta\right]\right) \\ &= \left(\frac{\overline{x}_\alpha + \underline{x}_\alpha}{2}\right) + i\left(\frac{\overline{x}_\beta + \underline{x}_\beta}{2}\right). \end{aligned}$$

Hence, the midpoint of a point optimizational complex interval is its *ordinary complex isomorphic copy*, that is

$$(\forall x_\alpha, x_\beta \in \mathbb{R})\left(m\left([x_\alpha, x_\alpha] + \boldsymbol{i}\,[x_\beta, x_\beta]\right) = x_\alpha + i x_\beta\right).$$

We can also extend the optimizational interval metric to optimizational complex intervals. This is done in the following definition.

Definition 5.19 (Optimizational Complex Metric). *The distance (or metric) between two optimizational complex intervals* $\boldsymbol{X} = [\underline{x}_\alpha, \overline{x}_\alpha] + \boldsymbol{i}\left[\underline{x}_\beta, \overline{x}_\beta\right]$ *and* $\boldsymbol{Y} = \left[\underline{y}_\alpha, \overline{y}_\alpha\right] + \boldsymbol{i}\left[\underline{y}_\beta, \overline{y}_\beta\right]$ *is defined to be*

$$d\left(\boldsymbol{X}, \boldsymbol{Y}\right) = \max\{\left|\underline{y}_\alpha - \underline{x}_\alpha\right|, \left|\overline{y}_\alpha - \overline{x}_\alpha\right|\} + \max\{\left|\underline{y}_\beta - \underline{x}_\beta\right|, \left|\overline{y}_\beta - \overline{x}_\beta\right|\}.$$

We observe that for an optimizational complex interval $\boldsymbol{X}$ with $\mathrm{Im}\left(\boldsymbol{X}\right) = [0, 0]$, the point operations for optimizational complex intervals are reduced to the corresponding point operations for optimizational interval numbers.

5.3 Algebraic Properties of Optimizational Complex Intervals

We shall now make use of the part of the theory of optimizational complex intervals developed in section 5.2 to further inquire into the algebraic properties of optimizational complex interval arithmetic. In our definition of an optimizational complex interval, the properties of optimizational interval numbers are naturally assumed in advance.

In a manner analogous to the proof of theorem 2.11, the following two theorems and their corollary are derivable.

Theorem 5.18 *The structure* $\langle {}^{\mathrm{o}}[\mathbb{C}]_{\mathrm{In}}; +_{\mathrm{oc}}, \times_{\mathrm{oc}} \rangle$ *is isomorphically equivalent to the* $\mathcal{S}$*-field*[7] $\langle {}^{\mathrm{o}}[\mathbb{R}]; +_{\mathrm{o}}, \times_{\mathrm{o}} \rangle$ *of optimizational interval numbers.*

Theorem 5.19 *The structure* $\left\langle {}^{\mathrm{o}}[\mathbb{C}]_{p}; +_{\mathrm{oc}}, \times_{\mathrm{oc}} \right\rangle$ *is isomorphically equivalent to the field* $\langle \mathbb{C}; +_{\mathbb{C}}, \times_{\mathbb{C}} \rangle$ *of ordinary complex numbers.*

Corollary 5.2 *Let* ${}^{\mathrm{o}}[\mathbb{C}]_{\mathrm{Re}} = {}^{\mathrm{o}}[\mathbb{C}]_{\mathrm{In}} \cap {}^{\mathrm{o}}[\mathbb{C}]_{p}$. *The structure* $\langle {}^{\mathrm{o}}[\mathbb{C}]_{\mathrm{Re}}; +_{\mathrm{oc}}, \times_{\mathrm{oc}} \rangle$ *is isomorphically equivalent to the field* $\langle \mathbb{R}; +_{\mathbb{R}}, \times_{\mathbb{R}} \rangle$ *of real numbers.*

That is, up to isomorphism, the algebra of optimizational complex intervals extends the field of real numbers, the field of ordinary complex numbers, and the $\mathcal{S}$-field of optimizational interval numbers.

Now we turn to investigate the algebra of optimizational complex interval arithmetic. The algebraic properties of optimizational complex intervals are directly established on the properties of optimizational interval numbers, in a manner *completely analogous* to how the properties of ordinary complex numbers are derived from the properties of real numbers. So, in proving the theorems of this section, many redundant details are omitted, and only the main aspects of the proofs are briefly, but satisfactorily, outlined.

Unlike the case for classical complex conjugates, the following two theorems show that the additive and multiplicative properties, of ordinary complex conjugates, are valid for optimizational complex conjugates.

Theorem 5.20 *The additive property of complex conjugates holds for optimizational complex intervals, that is*

$$(\forall \boldsymbol{X} \in {}^{\mathrm{o}}[\mathbb{C}]) \left(\boldsymbol{X} + \widetilde{\boldsymbol{X}} = 2 \times \mathrm{In}\left(\boldsymbol{X}\right) \right).$$

[7] See definition 4.18, on page 100.

Proof. The proof is immediate by theorem 5.11, and the fact that optimizational interval arithmetic has additive inverse elements (by theorem 4.18). ■

Theorem 5.21 *The multiplicative property of complex conjugates holds for optimizational complex intervals, that is*

$$(\forall \boldsymbol{X} \in {}^{o}[\mathbb{C}]) \left(\boldsymbol{X} \times \widetilde{\boldsymbol{X}} = \mathrm{In}\,(\boldsymbol{X})^2 - \mathrm{Im}\,(\boldsymbol{X})^2 \right).$$

Proof. The proof is immediate by theorem 5.12, and the fact that optimizational interval arithmetic has additive inverse elements (by theorem 4.18). ■

We also have the following theorem concerning quadratic equations in ${}^{o}[\mathbb{C}]$.

Theorem 5.22 *The optimizational complex intervals* $+\boldsymbol{i}$ *and* $-\boldsymbol{i}$ *are solutions of the equation* $\boldsymbol{X}^2 = -\mathbf{1}_{[\mathbb{C}]}$.

Proof. We have, by definitions 5.12 and 4.9,

$$(\pm\boldsymbol{i})^2 = \boldsymbol{i}^2 = [i, i]^2 = [-1, -1] + \boldsymbol{i}\,[0, 0] = -\mathbf{1}_{[\mathbb{C}]},$$

and the theorem follows. ■

With the help of the results obtained in the preceding section and section 4.3, the following theorems are derivable.

Theorem 5.23 (Absorbing Element in ${}^{o}[\mathbb{C}]$). *The optimizational complex interval* $\mathbf{0}_{[\mathbb{C}]}$ *is an absorbing element for optimizational complex multiplication, that is*

$$(\forall \boldsymbol{X} \in {}^{o}[\mathbb{C}]) \left(\mathbf{0}_{[\mathbb{C}]} \times \boldsymbol{X} = \boldsymbol{X} \times \mathbf{0}_{[\mathbb{C}]} = \mathbf{0}_{[\mathbb{C}]} \right).$$

Proof. The theorem follows from theorems 5.12 and 4.11. ■

Theorem 5.24 (Identity for Addition in ${}^{o}[\mathbb{C}]$). *The optimizational complex interval* $\mathbf{0}_{[\mathbb{C}]}$ *is both a left and right identity for optimizational complex addition, that is*

$$(\forall \boldsymbol{X} \in {}^{o}[\mathbb{C}]) \left(\mathbf{0}_{[\mathbb{C}]} + \boldsymbol{X} = \boldsymbol{X} + \mathbf{0}_{[\mathbb{C}]} = \boldsymbol{X} \right).$$

Proof. The theorem follows from theorems 5.11 and 4.12. ■

Theorem 5.25 (Identity for Multiplication in $^{\mathrm{o}}[\mathbb{C}]$). *The optimizational complex interval $\mathbf{1}_{[\mathbb{C}]}$ is both a left and right identity for optimizational complex multiplication, that is*

$$(\forall \boldsymbol{X} \in {}^{\mathrm{o}}[\mathbb{C}])\left(\mathbf{1}_{[\mathbb{C}]} \times \boldsymbol{X} = \boldsymbol{X} \times \mathbf{1}_{[\mathbb{C}]} = \boldsymbol{X}\right).$$

Proof. The theorem follows from theorems 5.12 and 4.13. ∎

Theorem 5.26 (Commutativity in $^{\mathrm{o}}[\mathbb{C}]$). *Both optimizational complex addition and multiplication are commutative, that is*

(i) $(\forall \boldsymbol{X}, \boldsymbol{Y} \in {}^{\mathrm{o}}[\mathbb{C}])\,(\boldsymbol{X} + \boldsymbol{Y} = \boldsymbol{Y} + \boldsymbol{X})$,

(ii) $(\forall \boldsymbol{X}, \boldsymbol{Y} \in {}^{\mathrm{o}}[\mathbb{C}])\,(\boldsymbol{X} \times \boldsymbol{Y} = \boldsymbol{Y} \times \boldsymbol{X})$.

Proof. The theorem is entailed by theorems 5.11 and 5.12, plus commutativity of the optimizational interval operations (by theorem 4.14). ∎

Theorem 5.27 (Associativity in $^{\mathrm{o}}[\mathbb{C}]$). *Both optimizational complex addition and multiplication are associative, that is*

(i) $(\forall \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in {}^{\mathrm{o}}[\mathbb{C}])\,(\boldsymbol{X} + (\boldsymbol{Y} + \boldsymbol{Z}) = (\boldsymbol{X} + \boldsymbol{Y}) + \boldsymbol{Z})$,

(ii) $(\forall \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in {}^{\mathrm{o}}[\mathbb{C}])\,(\boldsymbol{X} \times (\boldsymbol{Y} \times \boldsymbol{Z}) = (\boldsymbol{X} \times \boldsymbol{Y}) \times \boldsymbol{Z})$.

Proof. The theorem is established from theorems 5.11 and 5.12, plus associativity and distributivity of the optimizational interval operations (theorems 4.15 and 4.22, respectively). ∎

An important property peculiar to the theory of optimizational complex intervals is that unlike classical complex interval arithmetic, optimizational complex interval arithmetic has inverse elements for addition and multiplication. This property figures in the following two theorems.

Theorem 5.28 (Additive Inverses in $^{\mathrm{o}}[\mathbb{C}]$). *Additive inverses exist in optimizational complex interval arithmetic, that is*

$$(\forall \boldsymbol{X} \in {}^{\mathrm{o}}[\mathbb{C}])\left(\boldsymbol{X} + (-\boldsymbol{X}) = \mathbf{0}_{[\mathbb{C}]}\right).$$

Proof. The theorem is immediate from the fact that additive inverses exist in optimizational interval arithmetic, by theorem 4.18. ∎

Theorem 5.29 (Multiplicative Inverses in ${}^{o}[\mathbb{C}]$). *Every zeroless optimizational complex interval is invertible for multiplication on ${}^{o}[\mathbb{C}]$, that is*

$$\left(\forall \boldsymbol{X} \in {}^{o}[\mathbb{C}]_{\tilde{0}}\right)\left(\boldsymbol{X} \times \left(\boldsymbol{X}^{-1}\right) = \mathbf{1}_{[\mathbb{C}]}\right).$$

Proof. The theorem is immediate from the fact that multiplicative inverses exist for zeroless optimizational interval numbers, by theorem 4.19. ■

From the fact that every invertible element is cancellable, the preceding two theorems immediately entail the following two theorems concerning the cancellative laws for addition and multiplication in ${}^{o}[\mathbb{C}]$.

Theorem 5.30 (Cancellativity of Addition in ${}^{o}[\mathbb{C}]$). *Optimizational complex addition is cancellative, that is*

$$(\forall \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in {}^{o}[\mathbb{C}])\,(\boldsymbol{X} + \boldsymbol{Z} = \boldsymbol{Y} + \boldsymbol{Z} \Rightarrow \boldsymbol{X} = \boldsymbol{Y}).$$

Theorem 5.31 (Cancellativity of Multiplication in ${}^{o}[\mathbb{C}]$). *An optimizational complex interval is cancellable for multiplication if, and only if, it is a zeroless optimizational complex interval, that is*

$$(\forall \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in {}^{o}[\mathbb{C}])\,((\boldsymbol{X} \times \boldsymbol{Z} = \boldsymbol{Y} \times \boldsymbol{Z} \Rightarrow \boldsymbol{X} = \boldsymbol{Y}) \Leftrightarrow 0_{\mathbb{C}} \notin \boldsymbol{Z}).$$

Moreover, the following important property of the algebra of optimizational complex intervals is derivable.

Theorem 5.32 (Zero Divisors in ${}^{o}[\mathbb{C}]$). *Zero divisors do not exist in optimizational complex interval arithmetic, that is*

$$(\forall \boldsymbol{X}, \boldsymbol{Y} \in {}^{o}[\mathbb{C}])\left(\boldsymbol{X} \times \boldsymbol{Y} = \mathbf{0}_{[\mathbb{C}]} \Rightarrow \boldsymbol{X} = \mathbf{0}_{[\mathbb{C}]} \vee \boldsymbol{Y} = \mathbf{0}_{[\mathbb{C}]}\right).$$

Proof. Assume that $\boldsymbol{X} \neq \mathbf{0}_{[\mathbb{C}]} \wedge \boldsymbol{Y} \neq \mathbf{0}_{[\mathbb{C}]}$ and $\boldsymbol{X} \times \boldsymbol{Y} = \mathbf{0}_{[\mathbb{C}]}$. Then, by theorem 5.29, we have

$$\mathbf{0}_{[\mathbb{C}]} = \boldsymbol{X} \times \boldsymbol{Y} = \boldsymbol{X} \times \boldsymbol{Y} \times \left(\boldsymbol{Y}^{-1}\right) = \boldsymbol{X},$$

which contradicts the assumption that both $\boldsymbol{X}$ and $\boldsymbol{Y}$ are nonzero. ■

Thus, like the algebra of ordinary complex numbers, the algebra of optimizational complex intervals has no zero divisors, that is for each $\boldsymbol{X} \neq \mathbf{0}_{[\mathbb{C}]}$, there is no $\boldsymbol{Y} \neq \mathbf{0}_{[\mathbb{C}]}$ such that the identity $\boldsymbol{X} \times \boldsymbol{Y} = \mathbf{0}_{[\mathbb{C}]}$ holds.

Distributivity of optimizational complex interval arithmetic is established in the next theorem.

Theorem 5.33 (Distributivity in ${}^{\mathrm{o}}[\mathbb{C}]$). *Multiplication distributes over addition in optimizational complex interval arithmetic, that is*

$$(\forall \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in {}^{\mathrm{o}}[\mathbb{C}])\,(\boldsymbol{Z} \times (\boldsymbol{X} + \boldsymbol{Y}) = \boldsymbol{Z} \times \boldsymbol{X} + \boldsymbol{Z} \times \boldsymbol{Y})\,.$$

Proof. The theorem is established from theorems 5.11 and 5.12, plus associativity and distributivity of the optimizational interval operations (theorems 4.15 and 4.22, respectively). ■

Thus, in contrast to classical complex interval arithmetic, optimizational complex interval arithmetic does satisfy the distributive law.

We shall now make use of the preceding results to fix the algebraic system of optimizational complex intervals. In a manner analogous to the proof of theorems 4.23 and 4.24, the following two theorems are derivable.

Theorem 5.34 *The additive structure* $\langle {}^{\mathrm{o}}[\mathbb{C}]; +_{\mathrm{oc}} \rangle$ *is an abelian group.*

Theorem 5.35 *The multiplicative structure* $\langle {}^{\mathrm{o}}[\mathbb{C}]; \times_{\mathrm{oc}} \rangle$ *is an abelian monoid.*

In consequence of this theorem and theorem 5.29, we have the following corollary.

Corollary 5.3 *The multiplicative structure* $\left\langle {}^{\mathrm{o}}[\mathbb{C}]_{\widetilde{0}}; \times_{\mathrm{oc}} \right\rangle$ *of zeroless optimizational complex intervals is an abelian group.*

With the preceding two theorems and their corollary at our disposal, we can conclude the following result, which proof is analogous to that of theorem 4.26.

Theorem 5.36 *The structure* $\left\langle {}^{\mathrm{o}}[\mathbb{C}]; +_{\mathrm{oc}}, \times_{\mathrm{oc}}; \mathbf{0}_{[\mathbb{C}]}, \mathbf{1}_{[\mathbb{C}]} \right\rangle$ *is an* $\mathcal{S}$*-field*[8]*.*

Finally, an important immediate result that the preceding theorem implies is the following.

Corollary 5.4 *The theory of optimizational complex intervals defines a number system*[9] *on the set* ${}^{\mathrm{o}}[\mathbb{C}]$*.*

[8] See definition 4.18, on page 100.

[9] See footnote 19, on page 32.

Thus, the name "*numbers*" is verified for optimizational complex intervals, and therefore we can talk of "*optimizational complex interval numbers*".

From the above development, we conclude that unlike classical complex interval arithmetic, optimizational complex intervals have *additive inverses*, *multiplicative inverses* and satisfy the *distributive law*. By virtue of the algebraic properties proved in this section, optimizational complex interval arithmetic possesses a rich $\mathcal{S}$-field algebra, which extends the field structure of ordinary complex numbers, and therefore we do not have to sacrifice the useful properties of ordinary complex arithmetic. Aside from this, an important difference between the classical approach and the optimizational approach of constructing complex interval arithmetic is that since the optimizational complex operations are defined in terms of the optimizational interval operations, the theory of optimizational complex intervals inherits the capability of coping with *all cases* of interval dependency. All of these have as a consequence that the main advantage that lies with optimizational complex interval arithmetic, over classical complex interval arithmetic, is that: the theory of optimizational complex intervals is *completely compatible* with the *semantic* of ordinary complex arithmetic.

Finally, let us remark that optimizational interval arithmetic can be extended analogously to other multidimensional algebras beyond the ordinary complex algebra, in a manner that conserves their algebraic properties. In this way, one can construct, for example, the algebras of *optimizational interval quaternions* and *optimizational interval octonions*, without having to sacrifice the useful algebraic properties of ordinary quaternions and octonions.

Chapter 6

Ordering Interval Numbers and Other Subsets of the Reals

The chief forms of beauty are order and symmetry and definiteness, which the mathematical sciences demonstrate in a special degree.

–Aristotle (384 BC–322 BC)

The notion of *order* plays an important and indispensable role, as important as that of *size*, not only in mathematics and its applications but also in almost all scientific disciplines. Order and size are usually confused in general knowledge, but the two notions are *fundamentally different*. In other words, as Huntington says in [Huntington1905a]: "the notion of *size* (or *quantity*) is *not involved* in the notion of *order*"[1]. In mathematics and its philosophy, the notion of order emerges as an abstraction of the notion of *ascendance* (*precedence*, or *predominance*) as formulated in such expressions as "*x comes before y*", "*the point a is closer than the point b*", "$S \subset T$", and "$2 > 1$" (For more detailed and exhaustive discussions of the defining properties of the concept of order and related concepts, the interested reader may consult, e.g., [Blyth1994], [Dixon1902], [Huntington1905a], [Huntington1905b], [Russell1901], and [Russell1910]).

In contrast to many texts that deal with the question of ordering interval numbers, this characterization of the notion of order, which is most fundamental in order theory, lies a priori in our discussion in this chapter, and it is therefore assumed, in advance, that the concepts of *size*, *quantity*, *value*, *magnitude*, *width*, and so forth, are not involved in what we mean by "*order*".

[1] In contrary to intuition one may have, every ordering relation in mathematics, including the usual ordering on real numbers, obeys this rule.

Because of its great importance in both fundamental research and practical applications of the interval theory, the problem of ordering interval numbers has been attempted by many researchers, but with the notion of *size* (or *value*) is considered as the *all-important* aspect of order (which contradicts the defining properties of the notion of order in order theory). Despite extensive research on the subject, no *standard*[2] (*set-theoretic* or *order-theoretic*) total ordering for interval numbers is presented, and it seems that the important question of *compatibility* of an ordering with the interval algebraic operations is not touched upon, except for the *partial* ordering by the inclusion relation, $\subseteq$, and its well-known theorem of *inclusion monotonicity* (see theorem 2.8, on page 21).

This chapter opens with some order-theoretical preliminaries concerning ordering relations and their properties, which are important for our purpose. Section 6.2 gives a survey of the existing *set-theoretic* approaches for ordering interval numbers, along with discussing their *compatibility* with the interval algebraic operations. In section 6.3, we characterize the class $\mathfrak{O}_{(\wp(\mathbb{R}),\leq_{\mathbb{R}})}$ of all possible ordering relations on the powerset $\wp(\mathbb{R})$ of the reals, in terms of a binary quantification matrix, two real variable symbols, and the standard ordering relation $\leq_{\mathbb{R}}$ on $\mathbb{R}$. Then, we present the proofs that *neither* the set $[\mathbb{R}]$ of interval numbers *nor* the powerset $\wp(\mathbb{R})$ of the reals *can be totally ordered* by any of the relations in the class $\mathfrak{O}_{(\wp(\mathbb{R}),\leq_{\mathbb{R}})}$. In section 6.4, we define a *set-theoretic* ordering relation $\preceq_{\mathcal{I}}$ on the set $[\mathbb{R}]$ of interval numbers, then we present the proofs that the relation $\preceq_{\mathcal{I}}$ is a *non-strict total ordering* on $[\mathbb{R}]$, *compatible* with interval addition and multiplication, *dense* in $[\mathbb{R}]$, and *weakly Archimedean* in $[\mathbb{R}]$. Furthermore, we prove that the relation $\preceq_{\mathcal{I}}$ induces a *distributive lattice structure* for interval numbers, and that $\preceq_{\mathcal{I}}$ is an extension of the usual ordering $\leq_{\mathbb{R}}$ on the reals, Moore's partial ordering $<_{\mathrm{M}}$ on $[\mathbb{R}]$, and Kulisch's partial ordering $\leq_{\mathrm{K}}$ on $[\mathbb{R}]$.

Let us mention that all the results of this chapter, except the compatibility theorem (theorem 6.18), apply to *any* set of *nonempty* closed intervals on a partially ordered set[3], in the general sense, and hence to real closed intervals,

[2] We adopt the term "*standard ordering*" (or "*set-theoretic ordering*") to mean an ordering relation whose defining properties can be formulated in the language of classical predicate logic and can be studied order-theoretically in the standard way. This is to distinguish it from the *non-standard* relations defined using the *probabilistic* or *fuzzy* approaches.

[3] Let $\mathcal{S}$ be a set equipped with a partial ordering $\leq_{\mathcal{S}}$ and let $\underline{x}, \overline{x} \in \mathcal{S}$. A closed interval between $\underline{x}$ and $\overline{x}$ is denoted and defined as

$$[\underline{x}, \overline{x}]_{\mathcal{S}} = \begin{cases} \{x \in \mathcal{S} | \underline{x} \leq_{\mathcal{S}} x \wedge x \leq_{\mathcal{S}} \overline{x}\} & \textit{if} \quad \underline{x} \leq_{\mathcal{S}} \overline{x}, \\ \varnothing & \textit{if} \quad \overline{x} <_{\mathcal{S}} \underline{x}. \end{cases}$$

regardless of what theory of intervals is considered. The compatibility theorem applies to the classical interval theory and its standard extensions (such as *Hansen's generalized intervals*, *Kulisch intervals*, *constraint intervals*, and *optimizational intervals*), in which the standard notion of a *set-theoretic* closed interval ($[\underline{x}, \overline{x}]$ with $\underline{x} \leq \overline{x}$) is adopted. For other theories of intervals (such as *Kaucher intervals*, *modal intervals*, and *directed intervals*), which allow the existence of *inverted intervals* ($[\underline{x}, \overline{x}]$ with $\underline{x} > \overline{x}$), the compatibility theorem is not valid in general.

6.1 Some Order-Theoretical Preliminaries

Before discussing the question of ordering interval numbers in the succeeding sections of this chapter, we deal here with some order-theoretical preliminaries concerning comparison (ordering) relations and their properties, which are important for our purpose (For further details, the reader may consult, e.g., [Blyth2010], [Carnap1958], [Devlin1993], [Kleene1952], and [Suppes1972]).

We first define what a binary relation is.

Definition 6.1 (Binary Relation). *Let $\mathcal{S}^{\langle 2 \rangle}$ be the binary Cartesian power of a set $\mathcal{S}$. A binary relation on $\mathcal{S}$ is a subset of $\mathcal{S}^{\langle 2 \rangle}$. That is, a set $\Re$ is a binary relation on a set $\mathcal{S}$ iff*

$$(\forall \mathbf{r} \in \Re) \left((\exists x, y \in \mathcal{S}) \left(\mathbf{r} = (x, y) \right) \right).$$

Hereafter, the notations $(x, y) \in \Re$, $\Re(x, y)$, and $x \Re y$ shall be equivalently employed to mean "x is $\Re$-related to y in the set $\mathcal{S}$". The order of the elements in (x, y) is extremely important: if $x \neq y$, then $(x, y) \in \Re$ and $(y, x) \in \Re$ can be, independently of each other, true or false. We shall also adopt the notation $\mathrm{Id}_{\mathcal{S}}$ to denote the *identity* relation $\{(x, x) \,|x \in \mathcal{S}\}$ on a set $\mathcal{S}$.

The *domain*, *range*, and *field* of a binary relation are defined as follows.

Definition 6.2 (Domain, Range, and Field of a Relation). *The domain, range, and field of a binary relation $\Re$ are, respectively, defined by*

$$\begin{aligned} \mathrm{dom}(\Re) &= \{x |\, (\exists y)\, (x \Re y)\}, \\ \mathrm{ran}(\Re) &= \{y |\, (\exists x)\, (x \Re y)\}, \\ \mathrm{fld}(\Re) &= \mathrm{dom}(\Re) \cup \mathrm{ran}(\Re). \end{aligned}$$

In this general sense, a closed interval can be *empty*.

The notion of the *converse* of a binary relation is characterized in the following definition.

Definition 6.3 (Converse of a Relation). *The converse of a binary relation* $\Re$, *in symbols* $\widehat{\Re}$, *is the relation such that for all* x *and* y, $y\widehat{\Re}x \Leftrightarrow x\Re y$. *That is*

$$\widehat{\Re} = \{(y, x)\, |x\Re y\}.$$

The eight basic properties, a binary relation can have, are prescribed by the following definition.

Definition 6.4 (Properties of a Binary Relation). *Let* $\Re$ *be a binary relation on a set* $\mathcal{S}$.

(i) $\Re$ *is* reflexive *in* $\mathcal{S} \Leftrightarrow (\forall x \in \mathcal{S})\,(x\Re x)$.

(ii) $\Re$ *is* irreflexive *in* $\mathcal{S} \Leftrightarrow (\forall x \in \mathcal{S})\,(\neg\,(x\Re x))$.

(iii) $\Re$ *is* symmetric *in* $\mathcal{S} \Leftrightarrow (\forall x, y \in \mathcal{S})\,(x\Re y \Rightarrow y\Re x)$.

(iv) $\Re$ *is* asymmetric *in* $\mathcal{S} \Leftrightarrow (\forall x, y \in \mathcal{S})\,(x\Re y \Rightarrow \neg\,(y\Re x))$.

(v) $\Re$ *is* antisymmetric *in* $\mathcal{S} \Leftrightarrow (\forall x, y \in \mathcal{S})\,(x\Re y \wedge y\Re x \Rightarrow x = y)$.

(vi) $\Re$ *is* transitive *in* $\mathcal{S} \Leftrightarrow (\forall x, y, z \in \mathcal{S})\,(x\Re y \wedge y\Re z \Rightarrow x\Re z)$.

(vii) $\Re$ *is* connected[4] (total) *in* $\mathcal{S} \Leftrightarrow (\forall x, y \in \mathcal{S})\,(x \neq y \Rightarrow x\Re y \vee y\Re x)$.

(viii) $\Re$ *is* strongly connected *in* $\mathcal{S} \Leftrightarrow (\forall x, y \in \mathcal{S})\,(x\Re y \vee y\Re x)$.

Various comparison relations occur in almost all branches of mathematics and its applications. By virtue of the above definitions, we can next consider some of the more useful ones.

Definition 6.5 (Preordering). *A relation* $\Re$ *is a preordering (quasi-ordering) on a set* $\mathcal{S}$ *iff* $\Re$ *is reflexive and transitive in* $\mathcal{S}$.

[4] The connectedness property is a logical variant of the well-known principle of trichotomy, which asserts that

$$(\forall x, y \in \mathcal{S})\,(x = y \vee x\Re y \vee y\Re x)\,.$$

An example of a preordering relation is the logical implication "$\Rightarrow$" on the set of sentences.

Definition 6.6 (Weak Ordering). *A relation $\Re$ is a weak ordering (total preordering) on a set $\mathcal{S}$ iff $\Re$ is reflexive, transitive, and connected in $\mathcal{S}$. That is a weak ordering is a preordering which is connected.*

Definition 6.7 (Equivalence). *A relation $\Re$ is an equivalence on a set $\mathcal{S}$ iff $\Re$ is reflexive, transitive, and symmetric in $\mathcal{S}$. That is an equivalence is a preordering which is symmetric.*

An example of an equivalence relation is the logical equivalence "$\Leftrightarrow$" on the set of sentences.

Definition 6.8 (Non-Strict Partial Ordering). *A relation $\Re$ is a non-strict partial ordering (often referred to as an order or ordering) on a set $\mathcal{S}$ iff $\Re$ is reflexive, antisymmetric, and transitive in $\mathcal{S}$.*

Definition 6.9 (Strict Partial Ordering). *A relation $\Re$ is a strict partial ordering on a set $\mathcal{S}$ iff $\Re$ is asymmetric and transitive in $\mathcal{S}$.*

A set endowed with a partial ordering is called a *partially ordered set* (or a *poset*).

Why a comparison relation $\Re$, on a set $\mathcal{S}$, is called a *partial* ordering is that there are *some* pairs $(x, y) \in \mathcal{S}^{\langle 2 \rangle}$ such that we have none of $x = y$, $x \Re y$, or $y \Re x$. As familiar examples of partial orders we mention the following: the inclusion relation on the powerset $\wp(\mathcal{S})$, of a set $\mathcal{S}$; and the relation of divisibility on the set of natural numbers.

Definition 6.10 (Non-Strict Total Ordering). *A relation $\Re$ is a non-strict total (simple, or linear) ordering on a set $\mathcal{S}$ iff $\Re$ is antisymmetric, transitive, and strongly connected in $\mathcal{S}$.*

Definition 6.11 (Strict Total Ordering). *A relation $\Re$ is a strict total (simple) ordering on a set $\mathcal{S}$ iff $\Re$ is asymmetric, transitive, and connected in $\mathcal{S}$.*

A set endowed with a total ordering is called a *totally ordered set*, a *simply ordered set*, a *linearly ordered set*, or a *chain*.

Why a comparison relation $\Re$, on a set $\mathcal{S}$, is called a *total* ordering is that for *all* pairs $(x, y) \in \mathcal{S}^{\langle 2 \rangle}$, we have $x = y$, $x\Re y$, or $y\Re x$ (or *both* $x\Re y$ and $y\Re x$, in the *non-strict* case). A familiar example of a total order is the relation $\leq$, defined in the standard way, on the set of real numbers.

For a comparison relation $\Re$ on a set $\mathcal{S}$; the ensuing structure, denoted by the pair $\langle \mathcal{S}; \Re \rangle$, is a *relational* structure called an *ordered relational structure* with respect to the relation $\Re$.

Henceforth, we shall use the term "*comparison relation*" to refer to any of the binary relations characterized in definitions 6.5-6.11 (including the *preordering* relation), and we shall use the term "*ordering relation*" ("*order*", or "*ordering*") to mean a relation which is at least a "*partial ordering*".

The results formulated in the following two theorems are derivable and well-known (see, e.g., [Blyth2010], [Dunn2001], [Suppes1972], and [Tarski1941]).

Theorem 6.1 *Let $\Re$, $\Re_1$, and $\Re_2$ be binary relations. The following statements are true.*

(i) $\widehat{\widehat{\Re}} = \Re$.

(ii) $\widehat{\Re_1 \cap \Re_2} = \widehat{\Re}_1 \cap \widehat{\Re}_2$.

(iii) $\widehat{\Re_1 \cup \Re_2} = \widehat{\Re}_1 \cup \widehat{\Re}_2$.

(iv) $\Re$ *is symmetric* $\Leftrightarrow \widehat{\Re} = \Re$.

(v) $\Re$ *is asymmetric* $\Leftrightarrow \Re \cap \widehat{\Re} = \varnothing$.

(vi) $\Re$ *is antisymmetric* $\Leftrightarrow \Re \cap \widehat{\Re} \subseteq \mathrm{Id}_{\mathrm{dom}(\Re)}$.

Theorem 6.2 *Let $\Re$ be a binary relation on a set $\mathcal{S}$. The following statements are true.*

(i) $\Re$ *is a preordering on* $\mathcal{S} \Leftrightarrow \widehat{\Re}$ *is a preordering on* $\mathcal{S}$.

(ii) $\Re$ *is a partial ordering on* $\mathcal{S} \Leftrightarrow \widehat{\Re}$ *is a partial ordering on* $\mathcal{S}$.

(iii) $\Re$ *is a total ordering on* $\mathcal{S} \Leftrightarrow \widehat{\Re}$ *is a total ordering on* $\mathcal{S}$.

A well-known example of converse orderings is the usual ordering relation $\leq_{\mathbb{R}}$ on real numbers and its converse $\geq_{\mathbb{R}}$.

Before characterizing some properties of particular importance for our purpose, let us introduce some definitions we shall need.

Definition 6.12 (Minimal). *Let $\Re$ be an ordering on a set $\mathcal{S}$. x is an $\Re$-minimal element of $\mathcal{S}$ iff*

$$x \in \mathcal{S} \wedge (\forall y \in \mathcal{S}) (\neg (y \Re x)) .$$

Definition 6.13 (Lower Bound). *Let $\Re$ be an ordering on a set $\mathcal{S}$. x is an $\Re$-lower bound of $\mathcal{S}$ iff*

$$(\forall y \in \mathcal{S}) (x \Re y) .$$

Definition 6.14 (Infimum). *Let $\Re$ be an ordering on a set $\mathcal{S}$. x is an $\Re$-infimum of $\mathcal{S}$, in symbols $\Re$-inf, iff x is an $\Re$-lower bound of $\mathcal{S}$ and for all y, if y is an $\Re$-lower bound of $\mathcal{S}$ then $y \Re x$.*

Definition 6.15 (Upper Bound). *Let $\Re$ be an ordering on a set $\mathcal{S}$. x is an $\Re$-upper bound of $\mathcal{S}$ iff*

$$(\forall y \in \mathcal{S}) (y \Re x) .$$

Definition 6.16 (Supremum). *Let $\Re$ be an ordering on a set $\mathcal{S}$. x is an $\Re$-supremum of $\mathcal{S}$, in symbols $\Re$-sup, iff x is an $\Re$-upper bound of $\mathcal{S}$ and for all y, if y is an $\Re$-upper bound of $\mathcal{S}$ then $x \Re y$.*

A notion of a rather special character is that of a *well-ordering* relation. This notion is introduced by the following definition.

Definition 6.17 (Well-Ordering). *A binary relation $\Re$ is a well-ordering on a set $\mathcal{S}$ iff $\Re$ is a strict total ordering on $\mathcal{S}$ and every nonempty subset of $\mathcal{S}$ has an $\Re$-minimal.*

Unlike other ordering relations, If a relation $\Re$ is a well-ordering, it does not follow that $\widehat{\Re}$ is a well-ordering. An example is the set $\mathbb{Z}^+$ of positive integers which is well-ordered by the usual *less-than* relation $<_{\mathbb{Z}}$. On the other hand, $\mathbb{Z}^+$ is not well-ordered by the converse *greater-than* relation $>_{\mathbb{Z}}$. A familiar example of a set which cannot be well-ordered is the set $\mathbb{R}$ of real numbers[5] relative to the ordering relations $<_{\mathbb{R}}$ and $>_{\mathbb{R}}$, since many subsets of $\mathbb{R}$ do not have minimal elements with respect to the relations $<_{\mathbb{R}}$ and $>_{\mathbb{R}}$.

A notion of particular importance in both order theory and universal algebra is that of a lattice. The notions of a *lattice*, a *distributive lattice*, and a *modular lattice* (or *Dedekind lattice*[6]) relative to a relation $\Re$, are characterized in the following three definitions.

Definition 6.18 (Lattice). *A set $\mathcal{S}$ is a lattice with respect to a binary relation $\Re$ (or an $\Re$-lattice) iff $\Re$ is a partial ordering on $\mathcal{S}$ and for all x and y in $\mathcal{S}$, the set $\{x, y\}$ has an $\Re$-supremum and an $\Re$-infimum in $\mathcal{S}$.*

Definition 6.19 (Distributive Lattice). *A set $\mathcal{S}$ is a distributive lattice with respect to a binary relation $\Re$ (or a distributive $\Re$-lattice) iff $\mathcal{S}$ is an $\Re$-lattice and for all x , y, and z in $\mathcal{S}$,*

$$\Re\text{-inf}(x, \Re\text{-sup}\,(y, z)) = \Re\text{-sup}(\Re\text{-inf}\,(x, y)\,, \Re\text{-inf}\,(x, z)).$$

Definition 6.20 (Modular Lattice). *A set $\mathcal{S}$ is a modular lattice with respect to a binary relation $\Re$ (or a modular $\Re$-lattice) iff $\mathcal{S}$ is an $\Re$-lattice and for all x , y, and z in $\mathcal{S}$,*

$$x\Re y \Rightarrow \Re\text{-sup}(x, \Re\text{-inf}\,(z, y)) = \Re\text{-inf}(\Re\text{-sup}\,(x, z)\,, y).$$

Three well-known properties of distributive and modular lattices follow (see, e.g., [Birkhoff1948], [Blyth2010], [Dunn2001], and [Suppes1972]).

Theorem 6.3 *If a set $\mathcal{S}$ is a distributive $\Re$-lattice, then for all x , y, and z in $\mathcal{S}$, the following two identities are equivalent:*

[5] We know that every *countable* set can be well-ordered, by the fact that it can be put into one-to-one correspondence with the set of natural numbers. The set $\mathbb{Q}$ of rational numbers can therefore be well-ordered, but not with the standard ordering relations $<_{\mathbb{Q}}$ and $>_{\mathbb{Q}}$. This means that there exist well-ordering relations on a countable set, but none of them may be *compatible* with the algebraic operations on the set. The question whether *uncountable* sets can be well-ordered depends on the open question whether the *axiom of choice* holds, which implies that *every* set can be well-ordered by *some* relation. So, the question whether the set $\mathbb{R}$ of real numbers can be well-ordered by *some* relation is an open question (For further details, see, e.g., [Carnap1958], [Devlin1993], [Kleene1952], and [Suppes1972]).

[6] A *modular lattice* is also called a "*Dedekind lattice*", in honor of the German mathematician Richard Dedekind (1831–1916), who was the first to characterize modularity and derived some of its properties (see [Dedekind1900]).

(i) $\Re\text{-inf}(x, \Re\text{-sup}(y,z)) = \Re\text{-sup}(\Re\text{-inf}(x,y), \Re\text{-inf}(x,z))$,

(ii) $\Re\text{-sup}(x, \Re\text{-inf}(y,z)) = \Re\text{-inf}(\Re\text{-sup}(x,y), \Re\text{-sup}(x,z))$.

Theorem 6.4 *If a set $\mathcal{S}$ is totally ordered with respect to a relation $\Re$, then $\mathcal{S}$ is a distributive $\Re$-lattice.*

Theorem 6.5 *If a set $\mathcal{S}$ is a distributive $\Re$-lattice, then $\mathcal{S}$ is a modular $\Re$-lattice.*

Now we turn to the important notion of *compatibility of a comparison relation* with the algebraic operations. This notion, which plays an essential role in our discussion, is characterized in the following definition.

Definition 6.21 (Order Compatibility). *Let $\mathfrak{S} = \langle \mathcal{S}; +_{\mathcal{S}}, \times_{\mathcal{S}}; 0_{\mathcal{S}} \rangle$ be a ring-like[7] algebra with $+_{\mathcal{S}}$ and $\times_{\mathcal{S}}$ are respectively the addition and multiplication operations on the universe set $\mathcal{S}$, and $0_{\mathcal{S}}$ is the absorbing element for $\times_{\mathcal{S}}$. We say that a comparison relation $\Re$ on $\mathcal{S}$ is compatible with the addition operation of $\mathfrak{S}$ (or $+_{\mathcal{S}}$-compatible) iff*

$$(\forall x, y, z \in \mathcal{S})\,(x \Re y \Rightarrow (x +_{\mathcal{S}} z)\,\Re\,(y +_{\mathcal{S}} z)),$$

and we say that $\Re$ is compatible with the multiplication operation of $\mathfrak{S}$ (or $\times_{\mathcal{S}}$-compatible) iff

$$\begin{aligned}(\forall x, y, z \in \mathcal{S})\,((x \Re y \wedge 0 \Re z &\Rightarrow (x \times_{\mathcal{S}} z)\,\Re\,(y \times_{\mathcal{S}} z)) \\ \vee (x \Re y \wedge z \Re 0 &\Rightarrow (x \times_{\mathcal{S}} z)\,\Re\,(y \times_{\mathcal{S}} z))).\end{aligned}$$

If the two criteria are satisfied, then $\mathfrak{S}$ is orderable with respect to the relation $\Re$, and the ensuing structure $\langle \mathcal{S}; +_{\mathcal{S}}, \times_{\mathcal{S}}; 0_{\mathcal{S}}; \Re \rangle$ is called an $\Re$-ordered algebra.

The indispensability of the above definition for studying ordered algebraic structures is that a compatible comparison relation is preserved, in all contexts, by the structure operations. In other words, an *ordered algebraic structure* is not *definable* (or an algebraic structure is not *orderable*), unless we have an ordering $\Re$ on the universe set of the structure such that $\Re$ is compatible with the structure operations. A well-known example of an algebra that cannot be

[7] A *ring-like* algebra is a set equipped with two binary operations, *addition* and *multiplication*, such that multiplication has an *absorbing element* by either an axiom or a theorem.

ordered in a way compatible with the algebraic operations is the field of complex numbers (the set of complex numbers can be ordered, in many plausible ways, and we can talk of an *ordered set* of complex numbers $\langle \mathbb{C}; \prec_{\mathbb{C}} \rangle$, but we cannot talk of an *ordered field* of complex numbers $\langle \mathbb{C}; +_{\mathbb{C}}, \times_{\mathbb{C}}; \prec_{\mathbb{C}} \rangle$).

The following theorem is well-known and easily derivable by using the equivalence $y\widehat{\Re}x \Leftrightarrow x\Re y$ in definition 6.21.

Theorem 6.6 *Let* $\circ \in \{+, \times\}$ *be an algebraic operation of a ring-like structure. Then, an ordering relation* $\Re$ *is compatible with* $\circ$ *iff* $\widehat{\Re}$ *is compatible with* $\circ$.

Finally, the following three definitions characterize the notions of *density*, *Archimedeanity*[8], and *Dedekind completeness* with respect to an ordering relation $\Re$.

Definition 6.22 (Density). *A set* $\mathcal{S}$ *is dense with respect to a strict ordering relation* $\Re$ *(or an* $\Re$*-dense) iff*

$$(\forall x, y \in \mathcal{S})\,(x\Re y \Rightarrow (\exists z \in \mathcal{S})(x\Re z \wedge z\Re y)).$$

Definition 6.23 (Archimedeanity). *Let* $\mathcal{S}$ *be a set equipped with an addition operation* $+_{\mathcal{S}}$ *and an identity element* $0_{\mathcal{S}}$ *such that the structure* $\langle \mathcal{S}; +_{\mathcal{S}}; 0_{\mathcal{S}} \rangle$ *is an additive monoid. The set* $\mathcal{S}$ *is Archimedean with respect to an ordering relation* $\Re$ *(or an* $\Re$*-Archimedean) iff*

(i) $\Re$ *is compatible with* $+_{\mathcal{S}}$,

(ii) $(\forall x, y \in \mathcal{S})\,(0\Re x \wedge 0 \neq x \Rightarrow (\exists n \in \mathbb{Z}^+)\,(y\Re nx))$,
where nx *is defined by the additive recursion*

$$nx = \begin{cases} x & \text{iff} \quad n = 1, \\ x +_{\mathcal{S}} (n-1)\,x & \text{iff} \quad n > 1. \end{cases}$$

If the two criteria are satisfied, then the structure $\langle \mathcal{S}; +_{\mathcal{S}}; 0_{\mathcal{S}}; \Re \rangle$ *is called an* $\Re$*-Archimedean monoid.*

[8] The *Archimedean property* is named after the ancient Greek mathematician Archimedes of Syracuse because it appears as the fifth axiom in his "*On the Sphere and Cylinder*". Since Archimedes credited it to Eudoxus of Cnidus, Archimedeanity is also called the "*Eudoxus axiom*" (see [Archimedes2002]).

Definition 6.24 (Dedekind Completeness)[9]. *A set $\mathcal{S}$ is Dedekind complete with respect to a strict ordering relation $\Re$ (or a Dedekind $\Re$-complete) iff every nonempty subset of $\mathcal{S}$ which has an $\Re$-upper bound in $\mathcal{S}$ has an $\Re$-supremum in $\mathcal{S}$, or equivalently, iff every nonempty subset of $\mathcal{S}$ which has an $\Re$-lower bound in $\mathcal{S}$ has an $\Re$-infimum in $\mathcal{S}$.*

6.2 On Existing Orderings for Interval Numbers

Of all the existing ordering relations for interval numbers, none is a total ordering. This section is devoted to providing a quick survey of such partial ordering relations, along with discussing their *compatibility* with the interval algebraic operations (For other surveys, the reader may consult, e.g., [Alolyan2011], [Sengupta2000], and [Sengupta2009]).

Interval numbers are sets of real numbers. It is therefore not surprising that the first proposed ordering relation, for interval numbers, was the ordinary set inclusion, $\subseteq$, which presented by Young in [Young1931]. The partial ordering by the inclusion relation, which was later advocated by Sunaga in [Sunaga1958] and then by Moore in [Moore1959], is characterized as follows.

Definition 6.25 *For any two interval numbers $[\underline{x}, \overline{x}]$ and $[\underline{y}, \overline{y}]$, a non-strict partial ordering on $[\mathbb{R}]$, with respect to the relation $\subseteq$, is defined by*

$$[\underline{x}, \overline{x}] \subseteq [\underline{y}, \overline{y}] \Leftrightarrow \underline{y} \leq_{\mathbb{R}} \underline{x} \wedge \overline{x} \leq_{\mathbb{R}} \overline{y}.$$

The partial ordering by the set inclusion is proved, by Young in [Young1931], to be *compatible* with the interval algebraic operations (see theorem 2.8, on page 21); and this is the reason why it plays an important and indispensable role in Moore's foremost work in interval analysis (see, e.g. [Moore1966], [Moore1979], and [Moore2009]). However, the inclusion ordering does *not* extend the usual ordering $\leq_{\mathbb{R}}$ on the reals, that is, for some x and y in $\mathbb{R}$, it is *not* the case that

$$x \leq_{\mathbb{R}} y \Rightarrow [x, x] \subseteq [y, y].$$

A partial ordering relation, $<_{\mathrm{M}}$, that extends the standard strict ordering $<_{\mathbb{R}}$ on the reals, was presented by Moore in [Moore1966].

[9] *Dedekind Completeness* is named after the German mathematician Richard Dedekind (see [Dedekind1963]). It is also known as *Dedekind continuity*, *conditional completeness*, or *relative completeness*.

Definition 6.26 *For any two interval numbers* $[\underline{x}, \overline{x}]$ *and* $[\underline{y}, \overline{y}]$, *a strict partial ordering on* $[\mathbb{R}]$, *with respect to the relation* $<_{\mathrm{M}}$, *is defined by*

$$[\underline{x}, \overline{x}] <_{\mathrm{M}} [\underline{y}, \overline{y}] \Leftrightarrow \overline{x} <_{\mathbb{R}} \underline{y}.$$

In contrast to the case for $\subseteq$, Moore's partial ordering $<_{\mathrm{M}}$ is *not compatible* with the interval algebraic operations (see theorem 2.9, on page 22).

Another partial ordering relation, $\leq_{\mathrm{K}}$, that extends the usual non-strict ordering $\leq_{\mathbb{R}}$ on the reals, was presented by Kulisch and Miranker in [Kulisch1981].

Definition 6.27 *For any two interval numbers* $[\underline{x}, \overline{x}]$ *and* $[\underline{y}, \overline{y}]$, *a non-strict partial ordering on* $[\mathbb{R}]$, *with respect to the relation* $\leq_{\mathrm{K}}$, *is defined by*

$$[\underline{x}, \overline{x}] \leq_{\mathrm{K}} [\underline{y}, \overline{y}] \Leftrightarrow \underline{x} \leq_{\mathbb{R}} \underline{y} \wedge \overline{x} \leq_{\mathbb{R}} \overline{y}.$$

Kulisch's partial ordering $\leq_{\mathrm{K}}$, which is a special case of the product ordering $\leq^n_{prod}$, for $n = 2$ (see definition 6.33 of section 6.4 below), can be shown to be *compatible* with the interval algebraic operations. The following theorem establishes the compatibility of $\leq_{\mathrm{K}}$.

Theorem 6.7 *The partial ordering* $\leq_{\mathrm{K}}$ *is compatible with the algebraic operations on* $[\mathbb{R}]$.

Proof. We need to prove that the relation $\leq_{\mathrm{K}}$ satisfies the two criteria for a compatible ordering. Below, we suppress the subscripts for the operation symbols for brevity.

Let $X = [\underline{x}, \overline{x}]$, $Y = [\underline{y}, \overline{y}]$, and $Z = [\underline{z}, \overline{z}]$ be any three elements in $[\mathbb{R}]$.

- Compatibility with addition in $[\mathbb{R}]$. Let the formula

  $$X \leq_{\mathrm{K}} Y,$$

  be true. According to definition 6.27, we have

  $$X \leq_{\mathrm{K}} Y \Leftrightarrow \underline{x} \leq_{\mathbb{R}} \underline{y} \wedge \overline{x} \leq_{\mathbb{R}} \overline{y}.$$

  Then, by compatibility of $\leq_{\mathbb{R}}$ with real addition, we get $\underline{x} + \underline{z} \leq_{\mathbb{R}} \underline{y} + \underline{z}$ and $\overline{x} + \overline{z} \leq_{\mathbb{R}} \overline{y} + \overline{z}$. Hence, $X + Z \leq_{\mathrm{K}} Y + Z$. By definition 6.21, the relation $\leq_{\mathrm{K}}$ thus is compatible with addition in $[\mathbb{R}]$.

- Compatibility with multiplication in $[\mathbb{R}]$. Let the formula

$$X \leq_{\mathrm{K}} Y \wedge [0,0] \leq_{\mathrm{K}} Z,$$

be true. According to definition 6.27, we have

$$\begin{aligned} X &\leq_{\mathrm{K}} Y \Leftrightarrow \underline{x} \leq_{\mathbb{R}} \underline{y} \wedge \overline{x} \leq_{\mathbb{R}} \overline{y}, \\ [0,0] &\leq_{\mathrm{K}} Z \Leftrightarrow 0 \leq_{\mathbb{R}} \underline{z} \wedge 0 \leq_{\mathbb{R}} \overline{z}. \end{aligned}$$

Since $0 \leq_{\mathbb{R}} \underline{z}$ and $0 \leq_{\mathbb{R}} \overline{z}$, we obtain

$$\begin{aligned} X \times Z &= \left[\min\{\underline{xz}, \underline{x}\overline{z}\}, \max\{\overline{x}\underline{z}, \overline{xz}\}\right], \\ Y \times Z &= \left[\min\{\underline{yz}, \underline{y}\overline{z}\}, \max\{\overline{y}\underline{z}, \overline{yz}\}\right]. \end{aligned}$$

We have three possible cases for X.

Case 1. $0 \leq_{\mathbb{R}} \underline{x} \leq_{\mathbb{R}} \overline{x}$. Then, with $\underline{x} \leq_{\mathbb{R}} \underline{y}$ and $\overline{x} \leq_{\mathbb{R}} \overline{y}$, we should have $0 \leq_{\mathbb{R}} \underline{y} \leq_{\mathbb{R}} \overline{y}$. So we get

$$\begin{aligned} X \times Z &= \left[\underline{xz}, \overline{xz}\right], \\ Y \times Z &= \left[\underline{yz}, \overline{yz}\right]. \end{aligned}$$

Hence, by compatibility of $\leq_{\mathbb{R}}$ with real multiplication, it is an immediate consequence that $X \times Z \leq_{\mathrm{K}} Y \times Z$.

Case 2. $\underline{x} \leq_{\mathbb{R}} \overline{x} <_{\mathbb{R}} 0$. We then have three subcases for Y.

Subcase 2.1. $0 \leq_{\mathbb{R}} \underline{y} \leq_{\mathbb{R}} \overline{y}$. We get

$$\begin{aligned} X \times Z &= \left[\underline{x}\overline{z}, \overline{x}\underline{z}\right], \\ Y \times Z &= \left[\underline{yz}, \overline{yz}\right]. \end{aligned}$$

Then $\underline{x}\overline{z} \leq_{\mathbb{R}} \underline{yz}$ and $\overline{x}\underline{z} \leq_{\mathbb{R}} \overline{yz}$, and hence $X \times Z \leq_{\mathrm{K}} Y \times Z$.

Subcase 2.2. $\underline{y} \leq_{\mathbb{R}} \overline{y} <_{\mathbb{R}} 0$. We get

$$\begin{aligned} X \times Z &= \left[\underline{x}\overline{z}, \overline{x}\underline{z}\right], \\ Y \times Z &= \left[\underline{y}\overline{z}, \overline{y}\underline{z}\right]. \end{aligned}$$

Hence, by compatibility of $\leq_{\mathbb{R}}$ with real multiplication, we get $X \times Z \leq_{\mathrm{K}} Y \times Z$.

Subcase 2.3. $\underline{y} <_{\mathbb{R}} 0 <_{\mathbb{R}} \overline{y}$. We obtain

$$\begin{aligned} X \times Z &= \left[\underline{x}\overline{z}, \overline{x}\underline{z}\right], \\ Y \times Z &= \left[\underline{y}\overline{z}, \overline{yz}\right]. \end{aligned}$$

Then $\underline{x}\overline{z} \leq_{\mathbb{R}} \underline{y}\overline{z}$ and $\overline{x}\underline{z} \leq_{\mathbb{R}} \overline{yz}$, and hence $X \times Z \leq_{\mathrm{K}} Y \times Z$.

Case 3. $\underline{x} <_{\mathbb{R}} 0 <_{\mathbb{R}} \overline{x}$. We then have two subcases for Y.

Subcase 3.1. $0 \leq_{\mathbb{R}} \underline{y} \leq_{\mathbb{R}} \overline{y}$. We get

$$\begin{aligned} X \times Z &= \left[\underline{x}\overline{z}, \overline{xz}\right], \\ Y \times Z &= \left[\underline{yz}, \overline{yz}\right]. \end{aligned}$$

Then $\underline{x}\overline{z} \leq_{\mathbb{R}} \underline{yz}$ and $\overline{xz} \leq_{\mathbb{R}} \overline{yz}$, and so $X \times Z \leq_{\mathrm{K}} Y \times Z$.

Subcase 3.2. $\underline{y} <_{\mathbb{R}} 0 <_{\mathbb{R}} \overline{y}$. We obtain

$$\begin{aligned} X \times Z &= \left[\underline{x}\overline{z}, \overline{xz}\right], \\ Y \times Z &= \left[\underline{y}\overline{z}, \overline{yz}\right]. \end{aligned}$$

Hence, by compatibility of $\leq_{\mathbb{R}}$ with real multiplication, it is immediate that $X \times Z \leq_{\mathrm{K}} Y \times Z$.

According to definition 6.21, the compatibility of the relation $\leq_{\mathrm{K}}$, with multiplication in $[\mathbb{R}]$, is then proved.

Thus, it is shown that the two criteria are met for the relation $\leq_{\mathrm{K}}$, and therefore $\leq_{\mathrm{K}}$ is compatible with the algebraic operations on $[\mathbb{R}]$. ■

In [Ishibuchi1990], Ishibuchi and Tanaka presented two more non-strict partial orderings; $\leq_{\mathrm{T1}}$ and $\leq_{\mathrm{T2}}$. These orderings are characterized in the following two definitions.

Definition 6.28 *For any two interval numbers* $[\underline{x}, \overline{x}]$ *and* $[\underline{y}, \overline{y}]$*, a non-strict partial ordering on* $[\mathbb{R}]$*, with respect to the relation* $\leq_{\mathrm{T1}}$*, is defined by*

$$[\underline{x}, \overline{x}] \leq_{\mathrm{T1}} [\underline{y}, \overline{y}] \Leftrightarrow m\left([\underline{x}, \overline{x}]\right) \leq_{\mathbb{R}} m\left([\underline{y}, \overline{y}]\right) \wedge r\left([\underline{x}, \overline{x}]\right) \leq_{\mathbb{R}} r\left([\underline{y}, \overline{y}]\right),$$

where m *and* r *are, respectively, the midpoint and radius of an interval number.*

Definition 6.29 *For any two interval numbers* $[\underline{x}, \overline{x}]$ *and* $[\underline{y}, \overline{y}]$*, a non-strict partial ordering on* $[\mathbb{R}]$*, with respect to the relation* $\leq_{\mathrm{T2}}$*, is defined by*

$$[\underline{x}, \overline{x}] \leq_{\mathrm{T2}} [\underline{y}, \overline{y}] \Leftrightarrow \overline{x} \leq_{\mathbb{R}} \overline{y} \wedge m\left([\underline{x}, \overline{x}]\right) \leq_{\mathbb{R}} m\left([\underline{y}, \overline{y}]\right),$$

where m *is the midpoint of an interval number.*

We shall now prove two theorems concerning the compatibility of the partial orderings $\leq_{\mathrm{T1}}$ and $\leq_{\mathrm{T2}}$.

Theorem 6.8 *The partial ordering* $\leq_{\mathrm{T1}}$ *is not compatible with the algebraic operations on* $[\mathbb{R}]$*.*

Proof. To prove the theorem, it suffices to give a counterexample.

Let $X = [-8, -5]$, $Y = [-1, 2]$, and $Z = [1, 2]$ be interval numbers. According to definition 6.28, we have $X \leq_{\mathrm{T1}} Y$ and $[0, 0] \leq_{\mathrm{T1}} Z$. But

$$X \times Z = [-16, -5] \nleq_{\mathrm{T1}} [-2, 4] = Y \times Z,$$

and therefore the ordering $\leq_{\mathrm{T1}}$ is not compatible with the algebraic operations on $[\mathbb{R}]$. ■

Theorem 6.9 *The partial ordering* $\leq_{\mathrm{T2}}$ *is not compatible with the algebraic operations on* $[\mathbb{R}]$*.*

Proof. In a manner analogous to the proof of the preceding theorem, the proof can be easily obtained by taking the interval numbers $X = [-2, -1]$, $Y = [-1, 0]$, and $Z = [-1, 2]$. ■

Because none of the existing ordering relations for interval numbers is a total ordering, Alolyan, on page 2 in [Alolyan2011], goes to claim, without a proof, that:

> "Theoretically, intervals can *only be partially ordered* and hence cannot be compared".

In section 6.4, we shall attempt to *disprove* this claim, by presenting a *total compatible* order for interval numbers.

6.3 Ordering Relations on the Powerset of $\mathbb{R}$

In this section, we characterize the class $\mathfrak{O}_{(\wp(\mathbb{R}), \leq_{\mathbb{R}})}$ of all possible ordering relations on the powerset $\wp(\mathbb{R})$ of the reals, in terms of a binary quantification matrix[10], two real variable symbols, and the standard ordering relation $\leq_{\mathbb{R}}$ on $\mathbb{R}$. Then, we present the proofs that *neither* the set $[\mathbb{R}]$ of interval numbers *nor*

[10] A *quantification matrix* $\mathcal{Q}$ is a sequence $(Q_1 x_1) \ldots (Q_n x_n)$, where $x_1, \ldots, x_n$ are variable symbols and each Q_i is $\forall$ or $\exists$. A binary quantification matrix is a quantification matrix, for $n = 2$.

the powerset $\wp(\mathbb{R})$ of the reals *can be totally ordered* by any of the relations in the class $\mathfrak{O}_{(\wp(\mathbb{R}),\leq_{\mathbb{R}})}$.

The standard ordering relations on the set $\mathbb{R}$ of real numbers can be characterized, in terms of the real operations (see, e.g., [Montague1974], [Montague1980], and [Tarski1994]), in the following two definitions.

Definition 6.30 $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})\left(x \leq_{\mathbb{R}} y \Leftrightarrow (\exists r \in \mathbb{R})\left(r^2 = y - x\right)\right)$.

Definition 6.31 $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(x <_{\mathbb{R}} y \Leftrightarrow x \leq_{\mathbb{R}} y \wedge x \neq y)$.

On the basis of these definitions, the class of *all possible* ordering relations on the powerset $\wp(\mathbb{R})$ of $\mathbb{R}$, in terms of $\leq_{\mathbb{R}}$, can be characterized in the following definition.

Definition 6.32 *Let $Q_{\mathcal{S}}, Q_{\mathcal{T}} \in \{\forall, \exists\}$. For any two subsets $\mathcal{S}$ and $\mathcal{T}$ of the reals, the class of all possible ordering relations on $\wp(\mathbb{R})$, in terms of the ordering $\leq_{\mathbb{R}}$ on $\mathbb{R}$, can be characterized by*

$$\mathfrak{O}_{(\wp(\mathbb{R}),\leq_{\mathbb{R}})} = \{\preceq_{(Q_{\mathcal{S}},Q_{\mathcal{T}})} \mid Q_{\mathcal{S}}, Q_{\mathcal{T}} \in \{\forall, \exists\} \wedge \ \preceq_{(Q_{\mathcal{S}},Q_{\mathcal{T}})} \subseteq \wp^{\langle 2 \rangle}(\mathbb{R})\},$$

where

$$\mathcal{S} \preceq_{(Q_{\mathcal{S}},Q_{\mathcal{T}})} \mathcal{T} \Leftrightarrow (Q_{\mathcal{S}}\ s \in \mathcal{S})(Q_{\mathcal{T}}\ t \in \mathcal{T})(s \leq_{\mathbb{R}} t).$$

By means of definitions 6.4 and 6.10, we can derive the following result for the set $[\mathbb{R}]$ of interval numbers.

Theorem 6.10 *Let $Q_{\mathcal{S}}, Q_{\mathcal{T}} \in \{\forall, \exists\}$. None of the relations $\preceq_{(Q_{\mathcal{S}},Q_{\mathcal{T}})}$ in the class $\mathfrak{O}_{(\wp(\mathbb{R}),\leq_{\mathbb{R}})}$ is a non-strict total ordering on the set $[\mathbb{R}]$ of interval numbers.*

Proof. We demonstrate the proof that none of the relations $\preceq_{(Q_{\mathcal{S}},Q_{\mathcal{T}})}$ is a total ordering on $[\mathbb{R}]$ by giving a counterexample for each relation.

We have four cases.

- The relation $\preceq_{(\forall,\forall)}$. Let $\mathcal{S} = [1,2]$. We have $2 \in \mathcal{S}$ and $1 \in \mathcal{S}$, but $2 \not\leq_{\mathbb{R}} 1$. We thus obtain $\mathcal{S} \not\preceq_{(\forall,\forall)} \mathcal{S}$, and therefore the relation $\preceq_{(\forall,\forall)}$ is *not reflexive*.
- The relation $\preceq_{(\forall,\exists)}$. Let $\mathcal{S} = [1,2]$ and $\mathcal{T} = [2,2]$. We have $\mathcal{S} \preceq_{(\forall,\exists)} \mathcal{T}$ and $\mathcal{T} \preceq_{(\forall,\exists)} \mathcal{S}$, but $\mathcal{S} \neq \mathcal{T}$. The relation $\preceq_{(\forall,\exists)}$, therefore, is *not antisymmetric*.

- The relation $\preceq_{(\exists,\exists)}$. Let $\mathcal{S} = [1,2]$ and $\mathcal{T} = [2,2]$. We have $\mathcal{S} \preceq_{(\exists,\exists)} \mathcal{T}$ and $\mathcal{T} \preceq_{(\exists,\exists)} \mathcal{S}$, but $\mathcal{S} \neq \mathcal{T}$. The relation $\preceq_{(\exists,\exists)}$, therefore, is *not antisymmetric.*
- The relation $\preceq_{(\exists,\forall)}$. Let $\mathcal{S} = [1,1]$ and $\mathcal{T} = [1,2]$. We have $\mathcal{S} \preceq_{(\exists,\forall)} \mathcal{T}$ and $\mathcal{T} \preceq_{(\exists,\forall)} \mathcal{S}$, but $\mathcal{S} \neq \mathcal{T}$. The relation $\preceq_{(\exists,\forall)}$, therefore, is *not antisymmetric.*

From the above cases, it follows that none of the relations $\preceq_{(Q_\mathcal{S},Q_\mathcal{T})}$ satisfies the criteria for a non-strict total ordering on the set $[\mathbb{R}]$ of interval numbers. ■

Thus, the set $[\mathbb{R}]$ of interval numbers *cannot be totally ordered* with respect to any of the relations $\preceq_{(Q_\mathcal{S},Q_\mathcal{T})}$, for $Q_\mathcal{S}, Q_\mathcal{T} \in \{\forall, \exists\}$.

Further, an immediate consequence, of theorem 6.10, is the following important result for the powerset $\wp(\mathbb{R})$ of $\mathbb{R}$.

Theorem 6.11 *Let $Q_\mathcal{S}, Q_\mathcal{T} \in \{\forall, \exists\}$. None of the relations $\preceq_{(Q_\mathcal{S},Q_\mathcal{T})}$ in the class $\mathfrak{O}_{(\wp(\mathbb{R}),\leq_\mathbb{R})}$ is a non-strict total ordering on $\wp(\mathbb{R})$.*

Proof. The proof is immediate from theorem 6.10, by the fact that $[\mathbb{R}]$ is a proper subset of $\wp(\mathbb{R})$.

An alternate proof can be directly obtained, from the properties of the quantification over the empty set[11], as follows.

- The relation $\preceq_{(\forall,\forall)}$. Let $\mathcal{A}$ be any nonempty subset of $\mathbb{R}$. By the fact that all universal quantifications over the empty set $\varnothing$ are true, we have $\mathcal{A} \preceq_{(\forall,\forall)} \varnothing$ and $\varnothing \preceq_{(\forall,\forall)} \mathcal{A}$, but $\mathcal{A} \neq \varnothing$. The relation $\preceq_{(\forall,\forall)}$, therefore, is *not antisymmetric.*
- The relations $\preceq_{(Q_\mathcal{S},Q_\mathcal{T})}$ with $Q_\mathcal{S}$ or $Q_\mathcal{T}$ is $\exists$. By the fact that all existential quantifications over the empty set $\varnothing$ are false, we have $\varnothing \npreceq_{(Q_\mathcal{S},Q_\mathcal{T})} \varnothing$, and therefore the relations $\preceq_{(Q_\mathcal{S},Q_\mathcal{T})}$ are *not reflexive.*

Therefore, none of the relations $\preceq_{(Q_\mathcal{S},Q_\mathcal{T})}$ satisfies the criteria for a non-strict total ordering on the powerset $\wp(\mathbb{R})$ of $\mathbb{R}$. ■

[11] Any *existential* quantification $(\exists x \in \varnothing)(\varphi(x))$ over the empty set is *trivially false*, regardless of the formula $\varphi(x)$, because it implies the *existence* of some object in the empty universe of individuals. Any *universal* quantification over the empty set is *vacuously true*, because $\neg(\exists x \in \varnothing)(\varphi(x)) \Leftrightarrow (\forall x \in \varnothing)(\neg\varphi(x))$. First-order logics with empty structures were first considered by Mostowski in [Mostowski1951], and then studied by many logicians (see, e.g., [Quine1954], [Hintikka1959], and [Amer1989]). Such logics are now referred to as *free logics* (see, e.g. [Lambert2003]).

That is, the powerset $\wp(\mathbb{R})$ of $\mathbb{R}$ *cannot be totally ordered* with respect to any of the relations in the class $\mathfrak{O}_{(\wp(\mathbb{R}),\leq_{\mathbb{R}})}$.

6.4 A Total Compatible Order for Interval Numbers

In the preceding section, we presented the proofs that *neither* the set $[\mathbb{R}]$ of interval numbers *nor* the powerset $\wp(\mathbb{R})$ of the reals *can be totally ordered* with respect to any of the relations in the class $\mathfrak{O}_{(\wp(\mathbb{R}),\leq_{\mathbb{R}})}$ of all possible ordering relations on $\wp(\mathbb{R})$. By virtue of the fact that the set $[\mathbb{R}]$ of interval numbers can be *represented*[12] by the subset $\{(\underline{x},\overline{x}) \in \mathbb{R}^{\langle 2\rangle} | \underline{x} \leq \overline{x}\}$ of $\mathbb{R}^{\langle 2\rangle}$, one can attempt to define an ordering relation on interval numbers, in terms of the possible ordering relations on the n-th Cartesian power of $\mathbb{R}$.

With this fact in mind, in the sequel, we define an ordering relation $\preceq_{\mathcal{I}}$ on the set $[\mathbb{R}]$ of interval numbers, then we present the proofs that the relation $\preceq_{\mathcal{I}}$ is a *non-strict total ordering* on $[\mathbb{R}]$, *compatible* with interval addition and multiplication, *dense* in $[\mathbb{R}]$, and *weakly Archimedean* in $[\mathbb{R}]$. Furthermore, we prove that the relation $\preceq_{\mathcal{I}}$ induces a *distributive lattice structure* for interval numbers, and that $\preceq_{\mathcal{I}}$ is an extension of the standard ordering $\leq_{\mathbb{R}}$ on the reals, Moore's partial ordering $<_{\mathrm{M}}$ on $[\mathbb{R}]$, and Kulisch's partial ordering $\leq_{\mathrm{K}}$ on $[\mathbb{R}]$.

Before we proceed, let us investigate the properties of the possible ordering relations on the n-th Cartesian power of a set $\mathcal{S}$.

For a set $\mathcal{S}$ and an ordinal[13] n, two possible ordering relations on the n-th

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[12] It must be noted that a *representation* does not mean a *definition*. For example, a pair representation, (a, b), can mean a 2-dimensional vector, a real interval, a complex number, and so forth.

[13] For each ordinal n, there exists an ordinal $S(n)$ called the successor of n such that

$$(\forall n)(\forall k)(k = S(n) \Leftrightarrow (\forall m)(m \in k \Leftrightarrow m \in n \vee m = n)),$$

that is,

$$S(n) = n \cup \{n\}.$$

Setting $\mathit{0} = \varnothing$, each ordinal is then equal to the set of all ordinals preceding it, so that

$$\mathit{0} = \varnothing;\ \ \mathit{1} = S(\mathit{0}) = \{\varnothing\};\ \ \mathit{2} = S(\mathit{1}) = \{\varnothing, \{\varnothing\}\};\ \text{etc.}$$

If we keep applying the successor operator infinitely, we reach the first *infinite* (*transfinite*) *ordinal*

$$\omega = \{\mathit{0}, \mathit{1}, \mathit{2}, ...\}.$$

Cartesian power[14] of $\mathcal{S}$ are the *product ordering* (or the *componentwise ordering*) and the *lexicographical ordering*[15] (the *ordinal power*, or *ordering by first difference*). The two types of ordering can be characterized in the following two definitions.

Definition 6.33 (Product Ordering). *Let $\mathcal{S}$ be a set equipped with an ordering relation $\leq_{\mathcal{S}}$ and let $\mathcal{S}^{\langle n\rangle}$ be the n-th Cartesian power of $\mathcal{S}$, for an ordinal $n \geq 1$. For any two elements $(x_1, ..., x_n)$ and $(y_1, ..., y_n)$ in $\mathcal{S}^{\langle n\rangle}$, the product ordering is defined by*

$$(x_1, ..., x_n) \leq^n_{prod} (y_1, ..., y_n) \Leftrightarrow (\forall i \in \{1, ..., n\}) (x_i \leq_{\mathcal{S}} y_i)$$

Definition 6.34 (Lexicographical Ordering). *Let $\mathcal{S}$ be a set equipped with an ordering relation $\leq_{\mathcal{S}}$ and let $\mathcal{S}^{\langle n\rangle}$ be the n-th Cartesian power of $\mathcal{S}$, for an ordinal $n \geq 1$. For any two elements $(x_1, ..., x_n)$ and $(y_1, ..., y_n)$ in $\mathcal{S}^{\langle n\rangle}$, the*

The next two ordinals beyond ω are

$$\begin{aligned} S(\omega) &= \{0, 1, 2, ..., \omega\}, \\ S(S(\omega)) &= \{0, 1, 2, ..., \omega, S(\omega)\}. \end{aligned}$$

All ordinals preceding ω (all elements of ω) are *finite ordinals*. The idea of *transfinite counting* (*counting beyond the finite*) is due to Cantor (See [Cantor1955]).

For an ordinal $n = S(k)$, an n-tuple is any mapping τ whose domain is n. A finite n-tuple is an n-tuple for some finite ordinal n. That is

$$\begin{aligned} \tau_{S(k)} &= \langle \tau(0), \tau(1), ..., \tau(k)\rangle \\ &= \langle (0, \tau(0)), (1, \tau(1)), ..., (k, \tau(k))\rangle. \end{aligned}$$

If $n = 0 = \varnothing$, then, for any set $\mathcal{A}$, there is exactly one mapping (*the empty mapping*) $\tau_{\varnothing} = \varnothing$ from $\varnothing$ into $\mathcal{A}$. So, $\mathcal{A}^{\langle 0\rangle} = \{\varnothing\}$.

14 Let $\varnothing$ denote the empty set. For a set $\mathcal{S}$ and an ordinal n, the n-th Cartesian power of $\mathcal{S}$ is the set $\mathcal{S}^{\langle n\rangle}$ of all mappings from n into $\mathcal{S}$, that is

$$\mathcal{S}^{\langle n\rangle} = \begin{cases} \{\varnothing\} & n = 0, \\ \text{the set of all } n\text{-tuples of elements of } \mathcal{S} & n = 1 \vee 1 \in n. \end{cases}$$

If $\mathcal{S}$ is the empty set $\varnothing$, then

$$\varnothing^{\langle n\rangle} = \begin{cases} \{\varnothing\} & n = 0, \\ \varnothing & n = 1 \vee 1 \in n; \end{cases} \quad \text{and} \quad \varnothing^{\langle \varnothing^{\langle n\rangle}\rangle} = \begin{cases} \varnothing & n = 0, \\ \{\varnothing\} & n = 1 \vee 1 \in n. \end{cases}$$

Amer in [Amer1989] used the n-th Cartesian power of $\varnothing$ to define empty structures, and axiomatized their first-order theory.

15 The lexicographical ordering was first considered by Cantor (see [Cantor1955]), and later by Hausdorff, who presented the definitions for the lexicographical ordering and its converse (see [Hausdorff1978]).

lexicographical ordering (or the ordinal power) is defined by

$$(x_1, ..., x_n) \ \leq_{lex}^{n} \ (y_1, ..., y_n) \Leftrightarrow x_1 <_{\mathcal{S}} y_1 \\ \vee (\exists k \in \{2, ..., n\}) \, (\forall i < k) \, (x_i = y_i \wedge x_k \leq_{\mathcal{S}} y_k) \, .$$

The results formulated in the following two theorems are well-known properties of the product and lexicographical orderings (see, e.g., [Hausdorff1978] and [Kuratowski1976]).

Theorem 6.12 *Let $\mathcal{S}$ be a totally ordered set. Then the product ordering on $\mathcal{S}^{\langle n \rangle}$ is a partial ordering.*

Theorem 6.13 *Let $\mathcal{S}$ be a totally ordered set. Then the lexicographical ordering on $\mathcal{S}^{\langle n \rangle}$ is a total ordering.*

A further well-known property of the lexicographical ordering is the following (see, e.g., [Knuth1997] and [Kuratowski1976]).

Theorem 6.14 *Let $\mathcal{S}$ be a set equipped with an ordering relation $\leq_{\mathcal{S}}$ and let $\mathcal{S}^{\langle n \rangle}$ be the n-th Cartesian power of $\mathcal{S}$, for an ordinal $n \geq 1$. Then we have the following:*

(i) The lexicographical ordering on the set $\mathcal{S}^{\langle n \rangle}$ of all n-tuples from $\mathcal{S}$ is a well-ordering iff $\leq_{\mathcal{S}}$ is a well-ordering.

(ii) The lexicographical ordering on the set $\cup_{i=1}^{n} \mathcal{S}^{\langle n \rangle}$ of all ordered tuples from $\mathcal{S}$ is not a well-ordering.

Now we proceed to define an ordering relation on the set $[\mathbb{R}]$ of interval numbers and investigate its properties. Obviously, every set is the unary Cartesian power of itself. Thus, for $n = 1$, it can be seen at once that the usual ordering $\leq_{\mathbb{R}}$ on the reals is equivalent to the unary lexicographical ordering $\leq_{lex}^{1}$ on $\mathbb{R}^{\langle 1 \rangle}$. By virtue of the fact that interval numbers can be represented by ordered pairs of the reals, it is therefore natural to extend the *unary* lexicographical ordering $\leq_{\mathbb{R}}$ on $\mathbb{R}$, to the set $[\mathbb{R}]$ of interval numbers. This is made precise in the following definition.

Definition 6.35 *Let $\preceq_{\mathcal{I}}$ be a binary relation on $[\mathbb{R}]$ such that*

$$\preceq_{\mathcal{I}} = \{ \big([\underline{x}, \overline{x}] \, , [\underline{y}, \overline{y}] \big) \in [\mathbb{R}]^{\langle 2 \rangle} | \ \underline{x} <_{\mathbb{R}} \underline{y} \vee \big(\underline{x} = \underline{y} \wedge \overline{x} \leq_{\mathbb{R}} \overline{y} \big) \}.$$

That is, for all $[\underline{x}, \overline{x}]$ *and* $[\underline{y}, \overline{y}]$ *in* $[\mathbb{R}]$,

$$[\underline{x}, \overline{x}] \preceq_{\mathcal{I}} [\underline{y}, \overline{y}] \Leftrightarrow \underline{x} <_{\mathbb{R}} \underline{y} \vee \left(\underline{x} = \underline{y} \wedge \overline{x} \leq_{\mathbb{R}} \overline{y}\right).$$

The strict relation $\prec_{\mathcal{I}}$ and the converse relation $\succeq_{\mathcal{I}}$, of $\preceq_{\mathcal{I}}$, are characterized as follows.

Definition 6.36 *For all* X *and* Y *in* $[\mathbb{R}]$, *the strict relation* $\prec_{\mathcal{I}}$ *of* $\preceq_{\mathcal{I}}$ *is defined by*

$$X \prec_{\mathcal{I}} Y \Leftrightarrow (X \preceq_{\mathcal{I}} Y) \wedge (X \neq Y).$$

Definition 6.37 *For all* X *and* Y *in* $[\mathbb{R}]$, *the converse relation* $\succeq_{\mathcal{I}}$ *of* $\preceq_{\mathcal{I}}$ *is defined to be*

$$\succeq_{\mathcal{I}} = \widehat{\preceq_{\mathcal{I}}} = \{(X, Y) \in [\mathbb{R}]^{\langle 2 \rangle} \,|\, (Y, X) \in \preceq_{\mathcal{I}}\}.$$

By means of the above definitions, and according to definition 6.3, the following two theorems are immediate.

Theorem 6.15 *For all* $[\underline{x}, \overline{x}]$ *and* $[\underline{y}, \overline{y}]$ *in* $[\mathbb{R}]$, *the strict relation* $\prec_{\mathcal{I}}$ *of* $\preceq_{\mathcal{I}}$ *is formulated in terms of the intervals' endpoints as*

$$[\underline{x}, \overline{x}] \prec_{\mathcal{I}} [\underline{y}, \overline{y}] \Leftrightarrow \underline{x} <_{\mathbb{R}} \underline{y} \vee \left(\underline{x} = \underline{y} \wedge \overline{x} <_{\mathbb{R}} \overline{y}\right).$$

Theorem 6.16 *For all* $[\underline{x}, \overline{x}]$ *and* $[\underline{y}, \overline{y}]$ *in* $[\mathbb{R}]$, *the converse relation* $\succeq_{\mathcal{I}}$ *of* $\preceq_{\mathcal{I}}$ *is formulated in terms of the intervals' endpoints as*

$$\begin{aligned} [\underline{x}, \overline{x}] \succeq_{\mathcal{I}} [\underline{y}, \overline{y}] &\Leftrightarrow \underline{y} <_{\mathbb{R}} \underline{x} \vee \left(\underline{y} = \underline{x} \wedge \overline{y} \leq_{\mathbb{R}} \overline{x}\right) \\ &\Leftrightarrow \underline{x} >_{\mathbb{R}} \underline{y} \vee \left(\underline{x} = \underline{y} \wedge \overline{x} \geq_{\mathbb{R}} \overline{y}\right). \end{aligned}$$

Hereafter, if confusion is unlikely, the subscript "$\mathbb{R}$", in the real relation and operation symbols, may be suppressed.

Our characterization of the relation $\preceq_{\mathcal{I}}$ on the set $[\mathbb{R}]$ of interval numbers implies a number of results. The first important result of this section is formulated in the following theorem.

Theorem 6.17 (Total Orderness of $\preceq_{\mathcal{I}}$). *The relation* $\preceq_{\mathcal{I}}$ *is a non-strict total ordering on the set* $[\mathbb{R}]$ *of interval numbers.*

Proof. The theorem immediately follows from theorem 6.13, by the fact that the set $\mathbb{R}$ is totally ordered by the relation $\leq_{\mathbb{R}}$.

An alternate proof, based on the properties of interval numbers, can be constructed as follows.

According to definition 6.10, we are required to prove that $\preceq_{\mathcal{I}}$ is antisymmetric, transitive, and strongly connected in $[\mathbb{R}]$. In what follows Let $X = [\underline{x}, \overline{x}]$, $Y = [\underline{y}, \overline{y}]$, and $Z = [\underline{z}, \overline{z}]$ be any three elements in $[\mathbb{R}]$.

- $\preceq_{\mathcal{I}}$ is antisymmetric in $[\mathbb{R}]$. Let the formula

  $$X \preceq_{\mathcal{I}} Y \wedge Y \preceq_{\mathcal{I}} X,$$

  be true. By definition 6.35, we have

  $$\begin{aligned} X &\preceq_{\mathcal{I}} Y \Leftrightarrow \underline{x} < \underline{y} \vee \left(\underline{x} = \underline{y} \wedge \overline{x} \leq \overline{y}\right), \\ Y &\preceq_{\mathcal{I}} X \Leftrightarrow \underline{y} < \underline{x} \vee \left(\underline{y} = \underline{x} \wedge \overline{y} \leq \overline{x}\right). \end{aligned}$$

  Since these cannot hold together unless $\underline{x} = \underline{y}$ and $\overline{x} = \overline{y}$, it follows, according to definition 6.4, that $\preceq_{\mathcal{I}}$ is antisymmetric in $[\mathbb{R}]$.

- $\preceq_{\mathcal{I}}$ is transitive in $[\mathbb{R}]$. Let the formula

  $$X \preceq_{\mathcal{I}} Y \wedge Y \preceq_{\mathcal{I}} Z,$$

  be true. By definition 6.35, we have

  $$\begin{aligned} X &\preceq_{\mathcal{I}} Y \Leftrightarrow \underline{x} < \underline{y} \vee \left(\underline{x} = \underline{y} \wedge \overline{x} \leq \overline{y}\right), \\ Y &\preceq_{\mathcal{I}} Z \Leftrightarrow \underline{y} < \underline{z} \vee \left(\underline{y} = \underline{z} \wedge \overline{y} \leq \overline{z}\right). \end{aligned}$$

  We then have four possible cases.

Case 1. $\underline{x} < \underline{y}$ and $\underline{y} < \underline{z}$. This implies, by transitivity of $<$, that $\underline{x} < \underline{z}$, and hence $X \preceq_{\mathcal{I}} Z$.

Case 2. $\underline{x} < \underline{y}$ and $\left(\underline{y} = \underline{z} \wedge \overline{y} \leq \overline{z}\right)$. This implies that $\underline{x} < \underline{z}$ and $\overline{y} \leq \overline{z}$, which yields $X \preceq_{\mathcal{I}} Z$.

Case 3. $\left(\underline{x} = \underline{y} \wedge \overline{x} \leq \overline{y}\right)$ and $\underline{y} < \underline{z}$. This implies $\underline{x} < \underline{z}$ and $\overline{x} \leq \overline{y}$, which yields $X \preceq_{\mathcal{I}} Z$.

Case 4. $\left(\underline{x} = \underline{y} \wedge \overline{x} \leq \overline{y}\right)$ and $\left(\underline{y} = \underline{z} \wedge \overline{y} \leq \overline{z}\right)$. This implies, by transitivity of $\leq$, that $\underline{x} = \underline{z}$ and $\overline{x} \leq \overline{z}$, which yields $X \preceq_{\mathcal{I}} Z$.

Since, in all the preceding four cases, we have $X \preceq_{\mathcal{I}} Z$, it follows, according to definition 6.4, that $\preceq_{\mathcal{I}}$ is transitive in $[\mathbb{R}]$.

- $\preceq_{\mathcal{I}}$ is strongly connected in $[\mathbb{R}]$. According to definition 6.4, we want to prove that

$$X \preceq_{\mathcal{I}} Y \vee Y \preceq_{\mathcal{I}} X,$$

or, equivalently, we want to prove the following

$$\neg (X \preceq_{\mathcal{I}} Y) \Rightarrow Y \preceq_{\mathcal{I}} X.$$

So let $\neg (X \preceq_{\mathcal{I}} Y)$ be true, and we shall prove that $Y \preceq_{\mathcal{I}} X$. By definition 6.35, we have

$$\begin{aligned}
\neg (X \preceq_{\mathcal{I}} Y) &\Leftrightarrow \neg \left(\underline{x} < \underline{y} \vee \left(\underline{x} = \underline{y} \wedge \overline{x} \leq \overline{y}\right)\right) \\
&\Rightarrow \left(\neg \left(\underline{x} < \underline{y}\right)\right) \wedge \left(\neg \left(\underline{x} = \underline{y} \wedge \overline{x} \leq \overline{y}\right)\right) \\
&\Rightarrow \left(\underline{x} \geq \underline{y}\right) \wedge \left(\underline{x} \neq \underline{y} \vee \overline{x} > \overline{y}\right) \\
&\Rightarrow \left(\underline{x} \geq \underline{y} \wedge \underline{x} \neq \underline{y}\right) \vee \left(\underline{x} \geq \underline{y} \wedge \overline{x} > \overline{y}\right) \\
&\Rightarrow \left(\underline{x} > \underline{y}\right) \vee \left(\underline{x} \geq \underline{y} \wedge \overline{x} > \overline{y}\right) \\
&\Rightarrow \left(\underline{x} > \underline{y}\right) \vee \left(\left(\underline{x} > \underline{y} \vee \underline{x} = \underline{y}\right) \wedge \overline{x} > \overline{y}\right) \\
&\Rightarrow \left(\underline{x} > \underline{y}\right) \vee \left(\left(\underline{x} > \underline{y} \wedge \overline{x} > \overline{y}\right) \vee \left(\underline{x} = \underline{y} \wedge \overline{x} > \overline{y}\right)\right) \\
&\Rightarrow \left(\underline{y} < \underline{x}\right) \vee \left(\left(\underline{y} < \underline{x} \wedge \overline{y} < \overline{x}\right) \vee \left(\underline{y} = \underline{x} \wedge \overline{y} < \overline{x}\right)\right) \\
&\Rightarrow Y \preceq_{\mathcal{I}} X.
\end{aligned}$$

Hence, $\preceq_{\mathcal{I}}$ is strongly connected in $[\mathbb{R}]$.

Thus all the three criteria are met for the relation $\preceq_{\mathcal{I}}$, and therefore $\preceq_{\mathcal{I}}$ is a non-strict total ordering on the set $[\mathbb{R}]$ of interval numbers. ■

To illustrate, we next give some examples.

Example 6.1 *The following instances show the total orderness of the relation* $\preceq_{\mathcal{I}}$.

(i) $[2,3] \preceq_{\mathcal{I}} [3,4]$.

(ii) $[3,5] \preceq_{\mathcal{I}} [3,6]$.

(iii) The subset $\{[3,5], [2,4], [-2,1], [8,9], [-4,-2], [-1,0]\}$ *of* $[\mathbb{R}]$ *can be ordered by* $\preceq_{\mathcal{I}}$ *as follows:*

$$[-4,-2] \preceq_{\mathcal{I}} [-2,1] \preceq_{\mathcal{I}} [-1,0] \preceq_{\mathcal{I}} [2,4] \preceq_{\mathcal{I}} [3,5] \preceq_{\mathcal{I}} [8,9].$$

The $\preceq_{\mathcal{I}}$*-minimum in the set is* $[-4,-2]$ *and the* $\preceq_{\mathcal{I}}$*-maximum is* $[8,9]$.

Each interval number can be represented as a point on, or above, the line $\overline{x} = \underline{x}$ in the 2-dimensional Cartesian space (see Figure 6.1). The ordering $\preceq_{\mathcal{I}}$ on $[\mathbb{R}]$ can therefore be geometrically conceived as follows: if $X_i = [\underline{x}_i, \overline{x}_i]$ then $X_1 \preceq_{\mathcal{I}} X_2$ if X_1 is on the left of X_2, or X_1 and X_2 are on the same vertical line such that X_1 is below X_2, or X_1 and X_2 coincide. Interval numbers which are on the line $\overline{x} = \underline{x}$ are *point intervals* (or *real numbers*) which can be compared by $\preceq_{\mathcal{I}}$ according to their position on the line; lower points are $\preceq_{\mathcal{I}}$ upper points of the line. In Figure 6.1, an arrow form X_1 to X_2 indicates that $X_1 \preceq_{\mathcal{I}} X_2$.

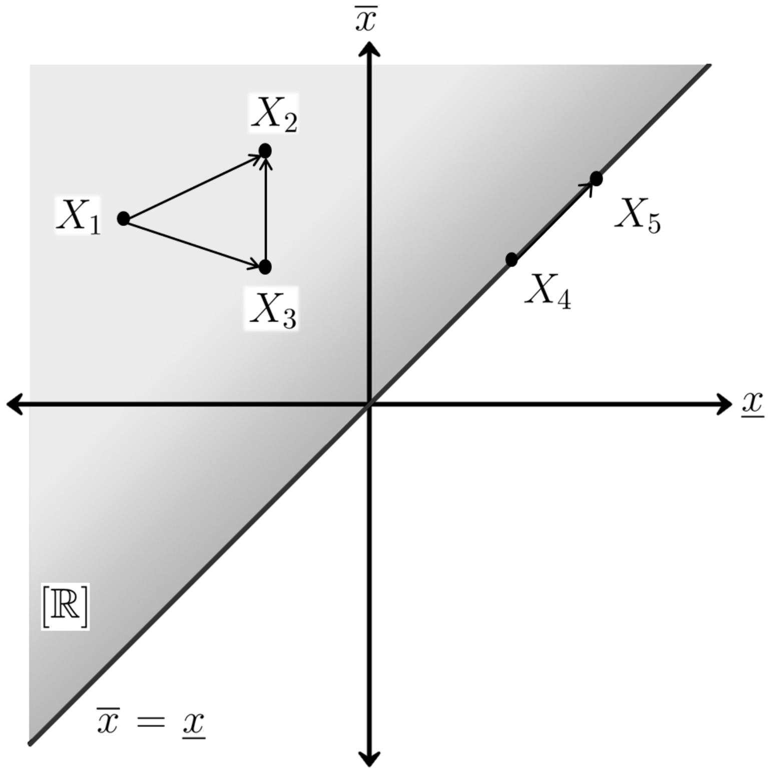

Figure 6.1: $X_1 \preceq_{\mathcal{I}} X_3 \preceq_{\mathcal{I}} X_2 \preceq_{\mathcal{I}} X_4 \preceq_{\mathcal{I}} X_5$.

In consequence of the preceding theorem and by means of theorem 6.2, the following result for the converse relation $\succeq_{\mathcal{I}}$ is provable.

Corollary 6.1 *The converse relation* $\succeq_{\mathcal{I}}$ *is a non-strict total ordering on the set* $[\mathbb{R}]$ *of interval numbers.*

A necessary criterion for an ordered interval algebra to be *definable* is that an ordering relation on the set $[\mathbb{R}]$ of interval numbers must be proved to be *compatible* with the interval algebraic operations. The following theorem says that the criteria for a compatible ordering are met by the relation $\preceq_{\mathcal{I}}$.

Theorem 6.18 (Order Compatibility of $\preceq_{\mathcal{I}}$). *The ordering $\preceq_{\mathcal{I}}$ is compatible with the nondistributive abelian semiring*[16]*, $\langle [\mathbb{R}]\,; +_{\mathrm{c}}, \times_{\mathrm{c}}; [0,0]\,, [1,1]\rangle$, of classical interval arithmetic. That is*

(i) $(\forall X, Y, Z \in [\mathbb{R}])\,(X \preceq_{\mathcal{I}} Y \Rightarrow (X +_{\mathrm{c}} Z) \preceq_{\mathcal{I}} (Y +_{\mathrm{c}} Z))$,

(ii) $(\forall X, Y, Z \in [\mathbb{R}])\,(X \preceq_{\mathcal{I}} Y \wedge [0,0] \preceq_{\mathcal{I}} Z \Rightarrow (X \times_{\mathrm{c}} Z) \preceq_{\mathcal{I}} (Y \times_{\mathrm{c}} Z))$.

Proof. We need to prove that the relation $\preceq_{\mathcal{I}}$ satisfies the two criteria for a compatible ordering. Below, we suppress the subscripts for the operation symbols for brevity.

Let $X = [\underline{x}, \overline{x}]$, $Y = \left[\underline{y}, \overline{y}\right]$, and $Z = [\underline{z}, \overline{z}]$ be any three elements in $[\mathbb{R}]$.

(i) Compatibility with addition in $[\mathbb{R}]$. Let the formula

$$X \preceq_{\mathcal{I}} Y,$$

be true. According to definition 6.35, we have

$$X \preceq_{\mathcal{I}} Y \Leftrightarrow \underline{x} < \underline{y} \vee \left(\underline{x} = \underline{y} \wedge \overline{x} \leq \overline{y}\right).$$

We then have two cases.

Case 1. $\underline{x} < \underline{y}$. Then, by compatibility of $<$ with real addition, we have $\underline{x} + \underline{z} < \underline{y} + \underline{z}$, and therefore $X + Z \preceq_{\mathcal{I}} Y + Z$.

Case 2. $\left(\underline{x} = \underline{y} \wedge \overline{x} \leq \overline{y}\right)$. Then, by compatibility of $\leq$ with real addition, it follows $\overline{x} + \overline{z} \leq \overline{y} + \overline{z}$ and also $\underline{x} + \underline{z} = \underline{y} + \underline{z}$. Hence, $X + Z \preceq_{\mathcal{I}} Y + Z$, and, by definition 6.21, $\preceq_{\mathcal{I}}$ thus is compatible with addition in $[\mathbb{R}]$.

(ii) Compatibility with multiplication in $[\mathbb{R}]$. Let the formula

$$X \preceq_{\mathcal{I}} Y \wedge [0,0] \preceq_{\mathcal{I}} Z,$$

be true. According to definition 6.35, we have

$$\begin{aligned} X &\preceq_{\mathcal{I}} Y \Leftrightarrow \underline{x} < \underline{y} \vee \left(\underline{x} = \underline{y} \wedge \overline{x} \leq \overline{y}\right), \\ [0,0] &\preceq_{\mathcal{I}} Z \Leftrightarrow 0 < \underline{z} \vee (0 = \underline{z} \wedge 0 \leq \overline{z}). \end{aligned}$$

We have four possible cases.

[16] We proved in theorem 2.24, on page 32, that the algebraic system of classical interval arithmetic is a *nondistributive abelian semiring*.

Case 1. $\underline{x} < \underline{y}$ and $0 < \underline{z}$. Then

$$\begin{aligned} X \times Z &= [\min\{\underline{x}\underline{z}, \underline{x}\overline{z}, \overline{x}\underline{z}, \overline{xz}\}, \max\{\underline{x}\underline{z}, \underline{x}\overline{z}, \overline{x}\underline{z}, \overline{xz}\}] \\ &= [\min\{\underline{x}\underline{z}, \underline{x}\overline{z}\}, \max\{\overline{x}\underline{z}, \overline{xz}\}]\,, \end{aligned}$$

since $0 < \underline{z} \leq \overline{z}$. Similarly

$$\begin{aligned} Y \times Z &= \left[\min\{\underline{y}\underline{z}, \underline{y}\overline{z}, \overline{y}\underline{z}, \overline{yz}\}, \max\{\underline{y}\underline{z}, \underline{y}\overline{z}, \overline{y}\underline{z}, \overline{yz}\}\right] \\ &= \left[\min\{\underline{y}\underline{z}, \underline{y}\overline{z}\}, \max\{\overline{y}\underline{z}, \overline{yz}\}\right]. \end{aligned}$$

With the condition that $\underline{x} < \underline{y}$, we have five subcases for the signs of $\underline{x}$ and $\underline{y}$.

Subcase 1.1. Both $\underline{x}$ and $\underline{y}$ are positive. Hence, we get

$$\begin{aligned} X \times Z &= [\underline{x}\underline{z}, \overline{xz}]\,, \\ Y \times Z &= \left[\underline{y}\underline{z}, \overline{yz}\right]. \end{aligned}$$

Since $\underline{x} < \underline{y}$ and $0 < \underline{z}$, it follows, by compatibility of $<$ with real multiplication, that $\underline{x}\underline{z} < \underline{y}\underline{z}$. Therefore, $X \times Z \preceq_{\mathcal{I}} Y \times Z$.

Subcase 1.2. Both $\underline{x}$ and $\underline{y}$ are negative. We get

$$\begin{aligned} X \times Z &= [\underline{x}\overline{z}, \max\{\overline{x}\underline{z}, \overline{xz}\}]\,, \\ Y \times Z &= \left[\underline{y}\overline{z}, \max\{\overline{y}\underline{z}, \overline{yz}\}\right]. \end{aligned}$$

Since $\underline{x} < \underline{y}$ and $0 < \underline{z} \leq \overline{z}$, it follows, by compatibility of $<$ with real multiplication, that $\underline{x}\overline{z} < \underline{y}\overline{z}$, and hence $X \times Z \preceq_{\mathcal{I}} Y \times Z$.

Subcase 1.3. $\underline{x}$ is negative and $\underline{y}$ is positive. Hence, we get

$$\begin{aligned} X \times Z &= [\underline{x}\overline{z}, \max\{\overline{x}\underline{z}, \overline{xz}\}]\,, \\ Y \times Z &= \left[\underline{y}\underline{z}, \overline{yz}\right]. \end{aligned}$$

Since $0 < \underline{z} \leq \overline{z}$, it follows that $\underline{x}\overline{z}$ is negative and $\underline{y}\underline{z}$ is positive. So $\underline{x}\overline{z} < \underline{y}\underline{z}$, and therefore $X \times Z \preceq_{\mathcal{I}} Y \times Z$.

Subcase 1.4. $\underline{x} = 0$ and $\underline{y}$ is positive. We get

$$\begin{aligned} X \times Z &= [0, \max\{\overline{x}\underline{z}, \overline{xz}\}]\,, \\ Y \times Z &= \left[\underline{y}\underline{z}, \overline{yz}\right]. \end{aligned}$$

Since $0 < \underline{z}$, we have $0 < \underline{y}\underline{z}$, and hence $X \times Z \preceq_{\mathcal{I}} Y \times Z$.

Subcase 1.5. $\underline{x}$ is negative and $\underline{y} = 0$. Hence, we get

$$\begin{aligned} X \times Z &= [\underline{x}\overline{z}, \max\{\overline{x}\underline{z}, \overline{x}\overline{z}\}], \\ Y \times Z &= [0, \max\{\overline{y}\underline{z}, \overline{y}\overline{z}\}]. \end{aligned}$$

Since $0 < \underline{z} \leq \overline{z}$, it follows that $\underline{x}\overline{z} < 0$, and therefore $X \times Z \preceq_{\mathcal{I}} Y \times Z$.

Case 2. $\left(\underline{x} = \underline{y} \wedge \overline{x} \leq \overline{y}\right)$ and $0 < \underline{z}$. Then

$$\begin{aligned} X \times Z &= \left[\min\{\underline{y}\underline{z}, \underline{y}\overline{z}\}, \max\{\overline{x}\underline{z}, \overline{x}\overline{z}\}\right], \\ Y \times Z &= \left[\min\{\underline{y}\underline{z}, \underline{y}\overline{z}\}, \max\{\overline{y}\underline{z}, \overline{y}\overline{z}\}\right], \end{aligned}$$

since $0 < \underline{z} \leq \overline{z}$ and $\underline{x} = \underline{y}$. The lower bounds of $X \times Z$ and $Y \times Z$ are the same so we turn to check their upper bounds.

With the condition that $\overline{x} \leq \overline{y}$, we have either $\overline{x} = \overline{y}$, which yields directly that $X \times Z = Y \times Z$, or $\overline{x} < \overline{y}$. We then have five subcases for the signs of $\overline{x}$ and $\overline{y}$.

Subcase 2.1. Both $\overline{x}$ and $\overline{y}$ are positive. Hence, we get

$$\begin{aligned} \max(X \times Z) &= \overline{x}\overline{z}, \\ \max(Y \times Z) &= \overline{y}\overline{z}. \end{aligned}$$

Since $\overline{x} \leq \overline{y}$ and $0 < \underline{z} \leq \overline{z}$, it follows, by compatibility of $\leq$ with real multiplication, that $\overline{x}\overline{z} \leq \overline{y}\overline{z}$, and hence $X \times Z \preceq_{\mathcal{I}} Y \times Z$.

Subcase 2.2. Both $\overline{x}$ and $\overline{y}$ are negative. We get

$$\begin{aligned} \max(X \times Z) &= \overline{x}\underline{z}, \\ \max(Y \times Z) &= \overline{y}\underline{z}. \end{aligned}$$

Since $\overline{x} \leq \overline{y}$ and $0 < \underline{z}$, it follows, by compatibility of $\leq$ with real multiplication, that $\overline{x}\underline{z} \leq \overline{y}\underline{z}$, and hence $X \times Z \preceq_{\mathcal{I}} Y \times Z$.

Subcase 2.3. $\overline{x}$ is negative and $\overline{y}$ is positive. We get

$$\begin{aligned} \max(X \times Z) &= \overline{x}\underline{z}, \\ \max(Y \times Z) &= \overline{y}\overline{z}. \end{aligned}$$

Since $0 < \underline{z} \leq \overline{z}$, it follows that $\overline{x}\underline{z}$ is negative and $\overline{y}\overline{z}$ is positive. So $\overline{x}\underline{z} < \overline{y}\overline{z}$, and therefore $X \times Z \preceq_{\mathcal{I}} Y \times Z$.

Subcase 2.4. $\overline{x} = 0$ and $\overline{y}$ is positive. We get

$$\begin{aligned} \max(X \times Z) &= 0, \\ \max(Y \times Z) &= \overline{yz}. \end{aligned}$$

Since $0 < \underline{z} \leq \overline{z}$, we have $0 < \overline{yz}$, and hence $X \times Z \preceq_{\mathcal{I}} Y \times Z$.

Subcase 2.5. $\overline{x}$ is negative and $\overline{y} = 0$. Hence, we get

$$\begin{aligned} \max(X \times Z) &= \overline{x}\underline{z}, \\ \max(Y \times Z) &= 0. \end{aligned}$$

Since $0 < \underline{z}$, it follows that $\overline{x}\underline{z} < 0$. Therefore $X \times Z \preceq_{\mathcal{I}} Y \times Z$.

Case 3. $\underline{x} < \underline{y}$ and $(0 = \underline{z} \wedge 0 \leq \overline{z})$. Then

$$\begin{aligned} X \times Z &= [\min\{\underline{xz}, \underline{x}\overline{z}, \overline{x}\underline{z}, \overline{xz}\}, \max\{\underline{xz}, \underline{x}\overline{z}, \overline{x}\underline{z}, \overline{xz}\}] \\ &= [\min\{\underline{x}\overline{z}, \overline{xz}\}, \max\{\underline{x}\overline{z}, \overline{xz}\}] \\ &= [\underline{x}\overline{z}, \overline{xz}], \end{aligned}$$

since $0 \leq \overline{z}$. Similarly

$$Y \times Z = [\underline{y}\overline{z}, \overline{yz}].$$

Since $\underline{x} < \underline{y}$ and $0 \leq \overline{z}$, we have $\underline{x}\overline{z} \leq \underline{y}\overline{z}$. We notice that $\underline{x}\overline{z} = \underline{y}\overline{z}$ only when $\overline{z} = 0$, which yields $X \times Z = Y \times Z = [0, 0]$. Hence, $X \times Z \preceq_{\mathcal{I}} Y \times Z$.

Case 4. $(\underline{x} = \underline{y} \wedge \overline{x} \leq \overline{y})$ and $(0 = \underline{z} \wedge 0 \leq \overline{z})$. Then

$$\begin{aligned} X \times Z &= [\min\{\underline{xz}, \underline{x}\overline{z}, \overline{x}\underline{z}, \overline{xz}\}, \max\{\underline{xz}, \underline{x}\overline{z}, \overline{x}\underline{z}, \overline{xz}\}] \\ &= [\min\{\underline{x}\overline{z}, \overline{xz}\}, \max\{\underline{x}\overline{z}, \overline{xz}\}] \\ &= [\underline{x}\overline{z}, \overline{xz}], \end{aligned}$$

since $0 \leq \overline{z}$. Similarly

$$Y \times Z = [\underline{y}\overline{z}, \overline{yz}].$$

Since $\underline{x} = \underline{y}$, it follows that $\underline{x}\overline{z} = \underline{y}\overline{z}$. Also, since $\overline{x} \leq \overline{y}$ and $0 \leq \overline{z}$, we get $\overline{xz} \leq \overline{yz}$. Hence, $X \times Z \preceq_{\mathcal{I}} Y \times Z$, and, by definition 6.21, $\preceq_{\mathcal{I}}$ thus is compatible with multiplication in $[\mathbb{R}]$.

It is therefore shown that $\preceq_{\mathcal{I}}$ is compatible with the algebraic system of classical interval arithmetic. ■

Thus, if we endow the classical interval algebra with the *compatible* total ordering $\preceq_{\mathcal{I}}$, then we have the *totally-ordered nondistributive abelian semiring*, $\langle [\mathbb{R}]; +_{\mathrm{c}}, \times_{\mathrm{c}}; [0,0], [1,1]; \preceq_{\mathcal{I}} \rangle$.

The following examples make this clear.

Example 6.2 *For three given interval numbers* $[-1, 2]$, $[2, 3]$, *and* $[1, 2]$, *we have*

(i) $[-1, 2] \preceq_{\mathcal{I}} [2, 3]$,

(ii) $[0, 4] = [-1, 2] + [1, 2] \preceq_{\mathcal{I}} [2, 3] + [1, 2] = [3, 5]$,

(iii) $[-2, 4] = [-1, 2] \times [1, 2] \preceq_{\mathcal{I}} [2, 3] \times [1, 2] = [2, 6]$.

Geometrically, the compatibility of the ordering $\preceq_{\mathcal{I}}$ with classical interval addition and multiplication can be conceived as shown in Figures 6.2 and 6.3 respectively.

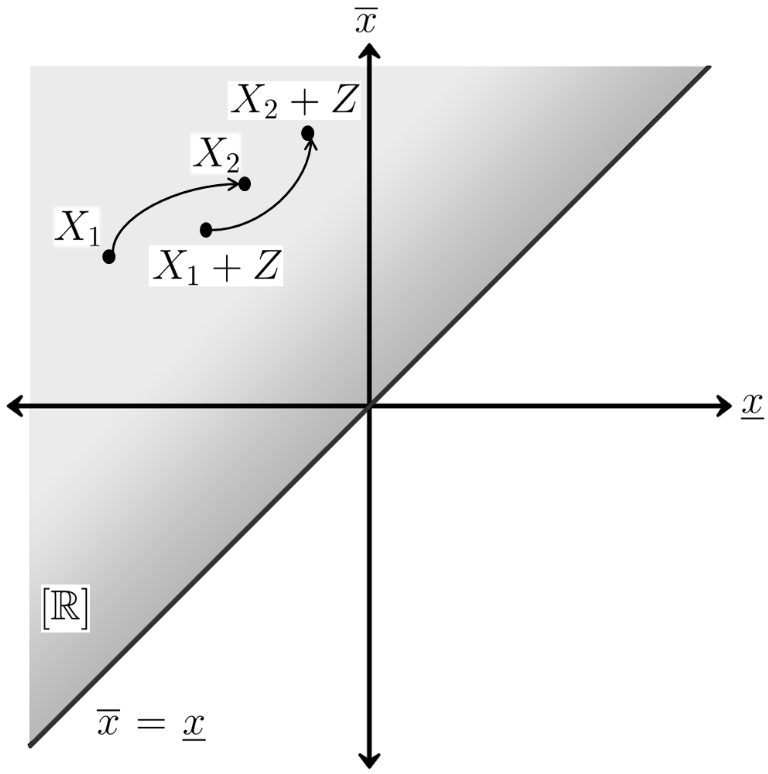

Figure 6.2: Compatibility of $\preceq_{\mathcal{I}}$ with interval addition.

By means of theorem 6.6, the preceding theorem yields the following result for the converse relation $\succeq_{\mathcal{I}}$.

Corollary 6.2 *The converse ordering* $\succeq_{\mathcal{I}}$ *is compatible with both classical interval addition and multiplication.*

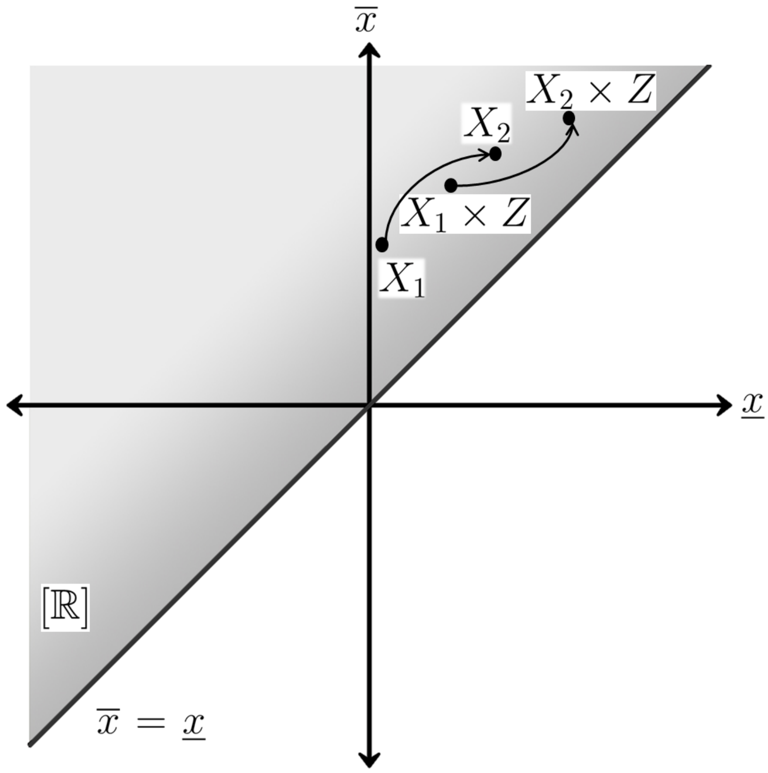

Figure 6.3: Compatibility of $\preceq_{\mathcal{I}}$ with interval multiplication.

As it is the case with the strict ordering $<_{\mathbb{R}}$ on the set of real numbers, the following theorem asserts that the set of interval numbers cannot be well-ordered by the strict relation $\prec_{\mathcal{I}}$.

Theorem 6.19 (Well-Orderness of $\prec_{\mathcal{I}}$). *The strict relation $\prec_{\mathcal{I}}$ is not a well-ordering on the set $[\mathbb{R}]$ of interval numbers.*

Proof. The proof is immediate from theorem 6.14, by the fact that the set $\mathbb{R}$ is not well-ordered by the relation $<_{\mathbb{R}}$. ■

The next theorem asserts that the *density* property holds for the set $[\mathbb{R}]$ of interval numbers relative to the strict ordering $\prec_{\mathcal{I}}$.

Theorem 6.20 (Density of $\prec_{\mathcal{I}}$). *The set $[\mathbb{R}]$ of interval numbers is dense with respect to the strict ordering $\prec_{\mathcal{I}}$. That is*

$$(\forall X, Y \in [\mathbb{R}])\,(X \prec_{\mathcal{I}} Y \Rightarrow (\exists Z \in [\mathbb{R}])(X \prec_{\mathcal{I}} Z \wedge Z \prec_{\mathcal{I}} Y)).$$

Proof. Let $X = [\underline{x}, \overline{x}]$ and $Y = \left[\underline{y}, \overline{y}\right]$ be any two elements of $[\mathbb{R}]$ such that $X \prec_{\mathcal{I}} Y$. By theorem 6.15, we have

$$X \prec_{\mathcal{I}} Y \Leftrightarrow \underline{x} < \underline{y} \vee \left(\underline{x} = \underline{y} \wedge \overline{x} < \overline{y}\right).$$

We then have two possible cases.

Case 1. $\underline{x} < \underline{y}$. Then, by density of $\mathbb{R}$ with respect to its usual strict ordering, $<$, there exists some real number, say $\underline{z}$, such that $\underline{x} < \underline{z}$ and $\underline{z} < \underline{y}$. Now, let $\overline{z}$ be any chosen element of $\mathbb{R}$ with $\underline{z} \leq \overline{z}$. Hence, the interval number $Z = [\underline{z}, \overline{z}]$ is an element of $[\mathbb{R}]$ which satisfies

$$X \prec_{\mathcal{I}} Z \wedge Z \prec_{\mathcal{I}} Y.$$

Case 2. $\underline{x} = \underline{y} \wedge \overline{x} < \overline{y}$. Again, by $<$-density of $\mathbb{R}$, there exists some real number, say $\overline{z}$, such that $\overline{x} < \overline{z}$ and $\overline{z} < \overline{y}$. Let $\underline{z}$ be an element of $\mathbb{R}$ with $\underline{z} = \underline{x} = \underline{y}$. Hence, the interval number $Z = [\underline{z}, \overline{z}]$ is an element of $[\mathbb{R}]$ which satisfies

$$X \prec_{\mathcal{I}} Z \wedge Z \prec_{\mathcal{I}} Y.$$

Therefore, the set $[\mathbb{R}]$ of interval numbers is $\prec_{\mathcal{I}}$-dense. ■

Some examples follow.

Example 6.3 *The following instances show the density of* $[\mathbb{R}]$ *with respect to the relation* $\prec_{\mathcal{I}}$.

(i) Let $X = [1, 2]$ *and* $Y = [2, 3]$. *Then* $X \prec_{\mathcal{I}} Y$. *Take* $Z = [1.5, 4]$. *Then* $X \prec_{\mathcal{I}} Z \wedge Z \prec_{\mathcal{I}} Y$.

(ii) Let $X = [1, 3]$ *and* $Y = [1, 5]$. *Then* $X \prec_{\mathcal{I}} Y$. *Take* $Z = [1, 4]$. *Then* $X \prec_{\mathcal{I}} Z \wedge Z \prec_{\mathcal{I}} Y$.

Next we prove that the set $[\mathbb{R}]$ of interval numbers has a *weak* form of the *Archimedean property* with respect to the ordering $\preceq_{\mathcal{I}}$.

Theorem 6.21 (Weak Archimedeanity of $\preceq_{\mathcal{I}}$). *The set* $[\mathbb{R}]$ *of interval numbers is weakly Archimedean with respect to the ordering* $\preceq_{\mathcal{I}}$. *That is, the relation* $\preceq_{\mathcal{I}}$ *does not satisfy the condition*

(i) $(\forall X, Y \in [\mathbb{R}])\,([0, 0] \preceq_{\mathcal{I}} X \wedge [0, 0] \neq X \Rightarrow (\exists n \in \mathbb{Z}^+)\,(Y \preceq_{\mathcal{I}} nX))$,

but satisfies the condition

(ii) $(\forall X, Y \in [\mathbb{R}])\,([0, 0] \preceq_{\mathcal{I}} X \wedge 0 \notin X \Rightarrow (\exists n \in \mathbb{Z}^+)\,(Y \preceq_{\mathcal{I}} nX))$.

Proof. For *(i)*, it suffices to give a counterexample. Let $X = [0,1]$ and $Y = [1,2]$. Then

$$[0,0] \preceq_{\mathcal{I}} X \wedge [0,0] \neq X,$$

while, for all $n \in \mathbb{Z}^+$, the following does hold

$$n \times [0,1] = [0,n] \preceq_{\mathcal{I}} [1,2]\,.$$

That is $nX \preceq_{\mathcal{I}} Y$. Hence, (i) does not hold for $\preceq_{\mathcal{I}}$ in $[\mathbb{R}]$.

For *(ii)*, let $X = [\underline{x}, \overline{x}]$ and $Y = \left[\underline{y}, \overline{y}\right]$ be any two elements of $[\mathbb{R}]$. Let the formula

$$[0,0] \preceq_{\mathcal{I}} X \wedge 0 \notin X,$$

be true. By means of definition 6.35, we get

$$[0,0] \preceq_{\mathcal{I}} X \wedge 0 \notin X \Leftrightarrow 0 < \underline{x}.$$

Now, we need to find some $n \in \mathbb{Z}^+$ such that

$$\left[\underline{y}, \overline{y}\right] \preceq_{\mathcal{I}} n \times [\underline{x}, \overline{x}] = [n\underline{x}, n\overline{x}]\,.$$

From definition 6.35, this is equivalent to

$$\underline{y} < n\underline{x} \vee \left(\underline{y} = n\underline{x} \wedge \overline{y} \leq n\overline{x}\right).$$

By $\leq$-Archimedeanity of $\mathbb{R}$, with $\underline{x} > 0$, we could always find such number $n \in \mathbb{Z}^+$ such that $\underline{y} < n\underline{x}$. Hence, we could find $n \in \mathbb{Z}^+$ such that $Y \preceq_{\mathcal{I}} nX$, and therefore the set $[\mathbb{R}]$ of interval numbers is weakly $\preceq_{\mathcal{I}}$-Archimedean. ■

In the proof of the preceding theorem, we proved the Archimedean statement with $0 \notin X$ instead of $0 \neq X$, hence the name "*weak Archimedeanity*". However, in this sense, the weakly Archimedean property for interval numbers relative to the ordering $\preceq_{\mathcal{I}}$ implies, as a special case, the Archimedean property for real numbers relative to the ordering $\leq_{\mathbb{R}}$, for the case when X and Y are point intervals.

Examples are shown below.

Example 6.4 *The following instances show the weak Archimedeanity of $[\mathbb{R}]$ with respect to the relation $\preceq_{\mathcal{I}}$.*

(i) Let $X = [2,4]$ and $Y = [-1,3]$. Then we could find some $n \in \mathbb{Z}^+$ such that $Y \preceq_{\mathcal{I}} nX$ by solving

$$-1 < 2n,$$

which yields $n = 1$ or $n > 1$.

(ii) *Let $X = [2, 4]$ and $Y = [2, 5]$. Then we could find some $n \in \mathbb{Z}^+$ such that $Y \preceq_{\mathcal{I}} nX$ by solving*

$$2 < 2n,$$

which yields $n = 2$ or $n > 2$.

In contrast to the case with the ordering $<_{\mathbb{R}}$ on the set of real numbers, the following theorem proves that the set of interval numbers does not satisfy the *Dedekind completeness* property with respect to the strict relation $\prec_{\mathcal{I}}$.

Theorem 6.22 (Dedekind Completeness of $\prec_{\mathcal{I}}$). *The set $[\mathbb{R}]$ of interval numbers is not Dedekind complete with respect to the strict ordering $\prec_{\mathcal{I}}$. That is, there is some nonempty subset of $[\mathbb{R}]$ which has a $\prec_{\mathcal{I}}$-upper bound in $[\mathbb{R}]$ and does not have a $\prec_{\mathcal{I}}$-supremum in $[\mathbb{R}]$.*

Proof. To prove the theorem, it suffices to give a counterexample.

Consider a subset $\mathcal{K}$ of $[\mathbb{R}]$ defined by

$$\mathcal{K} = \{\left[\underline{k}, \overline{k}\right] \in [\mathbb{R}] | \underline{k} = l \wedge n \leq \overline{k}\},$$

with $l < 0 < n$. Clearly, the set $\mathcal{K}$ has $\prec_{\mathcal{I}}$-upper bounds in $[\mathbb{R}]$.

An interval number $[\underline{z}, \overline{z}]$ is a $\prec_{\mathcal{I}}$-supremum of $\mathcal{K}$ in $[\mathbb{R}]$ iff for all $\left[\underline{k}, \overline{k}\right]$ in $\mathcal{K}$, $\underline{z} = l$ and $\overline{z} > \overline{k}$, which cannot hold.

Thus $\mathcal{K}$ does not have a $\prec_{\mathcal{I}}$-supremum in $[\mathbb{R}]$, and therefore the set $[\mathbb{R}]$ of interval numbers is not Dedekind $\prec_{\mathcal{I}}$-complete. ■

The graphical representation of the set $\mathcal{K}$, defined in the proof of the preceding theorem, is shown in Figure 6.4.

However, it can be easily shown that the ordering $\prec_{\mathcal{I}}$ is Dedekind complete in the set

$$[\mathbb{R}^*] = \{X \in \wp(\mathbb{R}^*) \,|\, (\exists \underline{x} \in \mathbb{R}^*) (\exists \overline{x} \in \mathbb{R}^*) (\underline{x} \leq \overline{x} \wedge X = [\underline{x}, \overline{x}])\},$$

of *extended intervals*[17], where $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$ is the set of *extended real numbers*. This result follows from the fact that the set $\mathbb{R}^*$, unlike the set $\mathbb{R}$ of

[17] For further details on *extended intervals* (*complete intervals*, or *Kulisch intervals*), the reader may consult, e.g., [Kulisch2008a] and [Kulisch2008b].

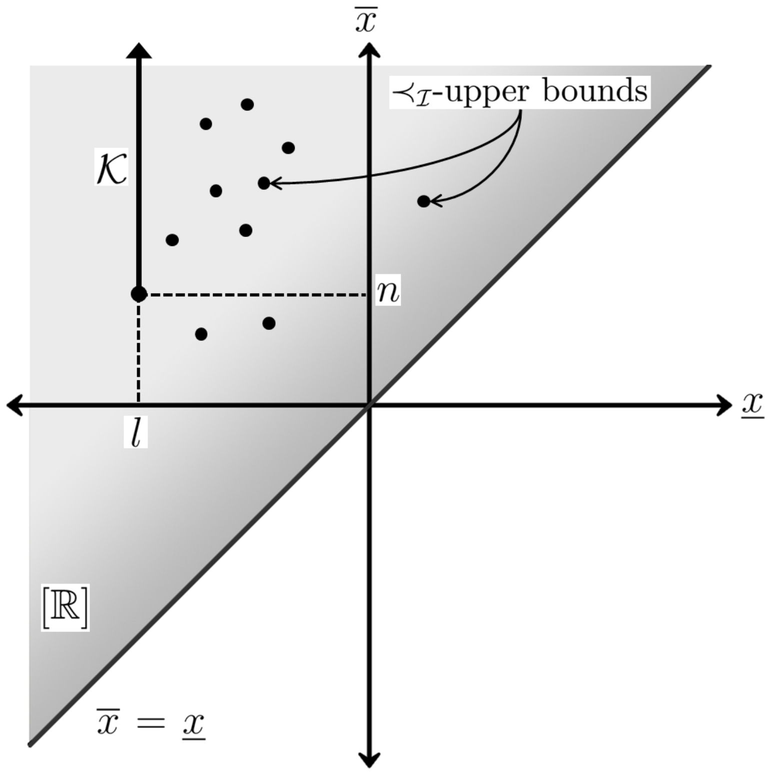

Figure 6.4: A subset $\mathcal{K}$ of interval numbers with $\prec_{\mathcal{I}}$-upper bounds, but with no $\prec_{\mathcal{I}}$-supremum.

ordinary real numbers, is an *order-complete set* (or a *complete lattice*)[18].

An important result with profound consequences is that the ordering relation $\preceq_{\mathcal{I}}$ induces a *distributive*, and hence *modular*, *lattice* structure for interval numbers. These are proved in the next theorem and its corollary.

Theorem 6.23 (Latticity of $\preceq_{\mathcal{I}}$). *The set* $[\mathbb{R}]$ *of interval numbers is a distributive lattice with respect to the ordering* $\preceq_{\mathcal{I}}$.

Proof. According to theorem 6.4, if a relation $\Re$ is a total ordering on a set $\mathcal{S}$, then $\mathcal{S}$ is a distributive $\Re$-lattice. By theorem 6.17, the relation $\preceq_{\mathcal{I}}$ is a non-strict total ordering on $[\mathbb{R}]$, and therefore the set $[\mathbb{R}]$ of interval numbers is a distributive $\preceq_{\mathcal{I}}$-lattice. ∎

By theorem 6.5, the modularity of the $\preceq_{\mathcal{I}}$-lattice is immediate.

[18] *Order completeness* is a generalization of the notion of Dedekind completeness. A set $\mathcal{S}$ is *order-complete* (or a *complete lattice*) with respect to an ordering relation $\Re$ iff every subset of $\mathcal{S}$ has an $\Re$-infimum and an $\Re$-supremum in $\mathcal{S}$. If $\mathcal{S}$ is an order-complete set, then the Cartesian power $\mathcal{S}^{\langle n \rangle}$ of $\mathcal{S}$ is order-complete, and hence Dedekind complete, with respect to the lexicographical ordering (see, e.g., [Blyth2010] and [Pouzet1981]).

Corollary 6.3 *The set* $[\mathbb{R}]$ *of interval numbers is a modular lattice with respect to the ordering* $\preceq_{\mathcal{I}}$.

Finally, we are led to the following three theorems, which assert that the total ordering $\preceq_{\mathcal{I}}$ is an extension of the ordering $\leq_{\mathbb{R}}$ on the reals, Moore's partial ordering $<_{\mathrm{M}}$ on $[\mathbb{R}]$, and Kulisch's partial ordering $\leq_{\mathrm{K}}$ on $[\mathbb{R}]$.

Theorem 6.24 *The ordering* $\preceq_{\mathcal{I}}$ *extends the ordering* $\leq_{\mathbb{R}}$ *on the set* $\mathbb{R}$ *of real numbers. That is*

$$(\forall\,[x,x]\,,[y,y]\in[\mathbb{R}])\,([x,x]\preceq_{\mathcal{I}}[y,y]\Leftrightarrow x\leq_{\mathbb{R}}y)\,.$$

Proof. By means of definition 6.35, for any two point intervals $[x,x]$ and $[y,y]$, we have

$$[x,x]\preceq_{\mathcal{I}}[y,y]\Leftrightarrow x<_{\mathbb{R}}y\vee(x=y\wedge x\leq_{\mathbb{R}}y)\Leftrightarrow x\leq_{\mathbb{R}}y,$$

and therefore the ordering $\preceq_{\mathcal{I}}$ extends the usual ordering $\leq_{\mathbb{R}}$ on the set $\mathbb{R}$ of real numbers. ■

Theorem 6.25 *The ordering* $\preceq_{\mathcal{I}}$ *is a superset of Moore's partial ordering* $<_{\mathrm{M}}$ *on the set* $[\mathbb{R}]$ *of interval numbers. That is*

$$\left(\forall\,[\underline{x},\overline{x}]\,,\left[\underline{y},\overline{y}\right]\in[\mathbb{R}]\right)\left([\underline{x},\overline{x}]<_{\mathrm{M}}\left[\underline{y},\overline{y}\right]\Rightarrow[\underline{x},\overline{x}]\preceq_{\mathcal{I}}\left[\underline{y},\overline{y}\right]\right),$$

or, equivalently

$$\left(\forall\,[\underline{x},\overline{x}]\,,\left[\underline{y},\overline{y}\right]\in[\mathbb{R}]\right)\left([\underline{x},\overline{x}]\not\preceq_{\mathcal{I}}\left[\underline{y},\overline{y}\right]\Rightarrow[\underline{x},\overline{x}]\not<_{\mathrm{M}}\left[\underline{y},\overline{y}\right]\right).$$

Proof. By means of definition 6.26, for any two interval numbers $[\underline{x},\overline{x}]$ and $\left[\underline{y},\overline{y}\right]$, we have

$$[\underline{x},\overline{x}]<_{\mathrm{M}}\left[\underline{y},\overline{y}\right]\Leftrightarrow\overline{x}<_{\mathbb{R}}\underline{y}.$$

But, $\overline{x}<_{\mathbb{R}}\underline{y}$, in conjunction with the fact that $\underline{x}\leq_{\mathbb{R}}\overline{x}$ and $\underline{y}\leq_{\mathbb{R}}\overline{y}$, obviously implies $\left(\underline{x}<_{\mathbb{R}}\underline{y}\right)\wedge(\overline{x}<_{\mathbb{R}}\overline{y})$, which, in turn, by definition 6.35, yields $[\underline{x},\overline{x}]\preceq_{\mathcal{I}}\left[\underline{y},\overline{y}\right]$. Therefore, the ordering $\preceq_{\mathcal{I}}$ is a superset of Moore's ordering $<_{\mathrm{M}}$ on the set $[\mathbb{R}]$ of interval numbers. ■

Theorem 6.26 *The ordering* $\preceq_{\mathcal{I}}$ *is a superset of Kulisch's partial ordering* $\leq_{\mathrm{K}}$ *on the set* $[\mathbb{R}]$ *of interval numbers. That is*

$$\left(\forall\,[\underline{x},\overline{x}]\,,\left[\underline{y},\overline{y}\right]\in[\mathbb{R}]\right)\left([\underline{x},\overline{x}]\leq_{\mathrm{K}}\left[\underline{y},\overline{y}\right]\Rightarrow[\underline{x},\overline{x}]\preceq_{\mathcal{I}}\left[\underline{y},\overline{y}\right]\right),$$

or, equivalently

$$\left(\forall \left[\underline{x},\overline{x}\right],\left[\underline{y},\overline{y}\right]\in[\mathbb{R}]\right)\left(\left[\underline{x},\overline{x}\right]\not\preceq_{\mathcal{I}}\left[\underline{y},\overline{y}\right]\Rightarrow\left[\underline{x},\overline{x}\right]\not\leq_{\mathrm{K}}\left[\underline{y},\overline{y}\right]\right).$$

Proof. According to definitions 6.27 and 6.35, for any two interval numbers $[\underline{x},\overline{x}]$ and $\left[\underline{y},\overline{y}\right]$, we have

$$\begin{aligned}
\left[\underline{x},\overline{x}\right]\leq_{\mathrm{K}}\left[\underline{y},\overline{y}\right] &\Leftrightarrow \left(\underline{x}\leq_{\mathbb{R}}\underline{y}\right)\wedge\left(\overline{x}\leq_{\mathbb{R}}\overline{y}\right)\\
&\Rightarrow \left(\underline{x}<_{\mathbb{R}}\underline{y}\vee\underline{x}=\underline{y}\right)\wedge\left(\overline{x}\leq_{\mathbb{R}}\overline{y}\right)\\
&\Rightarrow \left(\underline{x}<_{\mathbb{R}}\underline{y}\wedge\overline{x}\leq_{\mathbb{R}}\overline{y}\right)\vee\left(\underline{x}=\underline{y}\wedge\overline{x}\leq_{\mathbb{R}}\overline{y}\right)\\
&\Rightarrow \left(\left[\underline{x},\overline{x}\right]\preceq_{\mathcal{I}}\left[\underline{y},\overline{y}\right]\right)\vee\left(\left[\underline{x},\overline{x}\right]\preceq_{\mathcal{I}}\left[\underline{y},\overline{y}\right]\right)\\
&\Rightarrow \left(\left[\underline{x},\overline{x}\right]\preceq_{\mathcal{I}}\left[\underline{y},\overline{y}\right]\right),
\end{aligned}$$

and thus the ordering $\preceq_{\mathcal{I}}$ is a superset of Kulisch's partial ordering $\leq_{\mathrm{K}}$ on the set $[\mathbb{R}]$ of interval numbers. ■

The two important results established in the preceding two theorems can be combined in the single statement $(<_{\mathrm{M}}\cup\ \leq_{\mathrm{K}})\subset\ \preceq_{\mathcal{I}}$, that is

$$(\forall X,Y\in[\mathbb{R}])\left(X<_{\mathrm{M}}Y\vee X<_{\mathrm{K}}Y\Rightarrow X\preceq_{\mathcal{I}}Y\right),$$

or, equivalently

$$(\forall X,Y\in[\mathbb{R}])\left(X\not\preceq_{\mathcal{I}}Y\Rightarrow X\not<_{\mathrm{M}}Y\wedge X\not<_{\mathrm{K}}Y\right).$$

In other words, for two interval numbers X and Y, if the relation $X\preceq_{\mathcal{I}}Y$ is not true, then *none* of the relations $X<_{\mathrm{M}}Y$ and $X<_{\mathrm{K}}Y$ can be true.

Finally, let us remark that the results obtained in this chapter give an insight into some further consequences. As examples of these consequences we mention the following:

- By means of the results that the ordering $\preceq_{\mathcal{I}}$ is *total* and *compatible* with the interval algebraic operations, *intervals of intervals* (*higher-order intervals*) are definable as

  $$\left[\underline{X},\overline{X}\right]=\{X\in[\mathbb{R}]|\underline{X}\preceq_{\mathcal{I}}X\preceq_{\mathcal{I}}\overline{X}\},$$

  and their algebra can be constructed in the standard way.

- By virtue of the *totality* of the ordering $\preceq_{\mathcal{I}}$, an *interval distributive* $\preceq_{\mathcal{I}}$-

lattice algebra $\langle [\mathbb{R}]\,; +, \times; \curlyvee, \curlywedge \rangle$ is definable such that

$$(\forall X, Y \in [\mathbb{R}])\,(X \curlyvee Y = Y \Leftrightarrow\ \preceq_{\mathcal{I}}\text{-sup}\,(X, Y) = Y)\,,$$
$$(\forall X, Y \in [\mathbb{R}])\,(X \curlywedge Y = X \Leftrightarrow\ \preceq_{\mathcal{I}}\text{-inf}\,(X, Y) = X)\,,$$

and its fundamental properties are derivable.

- By virtue of the *totality* of the ordering $\preceq_{\mathcal{I}}$, we can define the *open* higher-order intervals

$$\begin{aligned}]\underline{X}, \overline{X}[\ &= \ \{X \in [\mathbb{R}] | \underline{X} \prec_{\mathcal{I}} X \prec_{\mathcal{I}} \overline{X}\}, \\ ]-\infty, \overline{X}[\ &= \ \{X \in [\mathbb{R}] | X \prec_{\mathcal{I}} \overline{X}\}, \\ ]\underline{X}, \infty[\ &= \ \{X \in [\mathbb{R}] | \underline{X} \prec_{\mathcal{I}} X\}, \\ ]-\infty, \infty[\ &= \ [\mathbb{R}], \end{aligned}$$

and consequently an *order topology* for interval numbers is definable and its properties can be investigated in the standard way.

Chapter 7

Concluding Remarks

It is not once nor twice but times without number that the same ideas make their appearance in the world.

–Aristotle (384 BC–322 BC)

This final chapter offers a concluding view to the future of interval computations, brief selections of some application areas of interval arithmetic, and an overview of the current research topics in the interval theory. After this concluding look ahead, section 7.4 provides a summary of contributions of this thesis and outlines several directions and perspectives for future research.

7.1 A View to the Future of Interval Computations

What is the future of interval computations? Fortunately, computers are getting faster and most existing *parallel processors* provide a tremendous computing power. So, with little extra hardware, it is very possible to make interval computations as fast as floating-point computations. The interval theory is likely to have more widespread applications in the future, for many reasons:

- Interval algorithms are *naturally parallel*, because they progress by deleting regions where solutions are proved not to exist. Intervals provide the *only known* general algorithms that achieve *linear* speedup, as the number of processors increases in parallel computing systems.
- Interval arithmetic is, arguably, the *best* and *most efficient* way to safely translate ever-increasing computer speed into mission-critical problem solutions that are otherwise impractical or impossible to obtain.

- By using interval algorithms to solve nonlinear problems, more accurate mathematical models of physical phenomena become practical.
- With interval arithmetic, it is possible to automatically perform rigorous error analysis, and solve nonlinear problems that were previously thought to be impossible to solve.
- Speed is not all-important anymore. We can get worthwhile accurate results by sacrificing some speed.

7.2 More Scientific Applications of Interval Arithmetic

Despite the persisting interval dependency problem and other implementation drawbacks, interval methods are becoming a mainstream. Interval arithmetic is still more accurate and reliable than floating-point arithmetic and traditional numerical approximation methods. Today, the interval theory has numerous applications in scientific and engineering disciplines that deal intensely with *uncertain data*.

Brief selections of some application areas of interval arithmetic are listed below (For further details, see, e.g., [Hansen2003], [Jaulin2001], and [Moore2009]).

- Electrical Engineering. Interval computations, besides providing validated results, are hundreds of times faster than a Monte Carlo method for solving AC network equations. Moreover, interval computations are applied in quality control in the manufacture of radioelectric devices.
- Control Theory. Interval computations are used to analyze Hurwitz stability in control theory applications.
- Remote Sensing and GISs. Interval computations are used to bound errors in decisions based on remote sensing. Furthermore, interval methods are used in sensitivity analysis in geographic information systems (GISs).
- Quality Control. Interval computations are used for quality control in manufacturing processes in which the factors fluctuate within bounds.
- Economics: Linear interval methods are used in modeling economic systems to determine the effects of uncertainties in input parameters, and to include the effects of forecast uncertainties.
- Experimental Physics. Interval computations are used to handle the gathered uncertain data about observed physical phenomena.

- Computational Physics. Interval algorithms are used to solve physical problems that arise from experimental and theoretical physics.
- Dynamical and Chaotic Systems: Interval computations are used to verify that computed numerical solutions to chaotic dynamical systems are close to the actual solutions. Furthermore, cell-mapping methods based on interval arithmetic are used to robustly visualize strange attractors (SAs) in discrete chaotic systems.
- Computer Graphics and Computational Geometry: Interval computations are used to handle many problems in computer graphics and computational geometry. Operations on geometric objects such as rendering, surface intersection, and hidden line removal require robustness in nonlinear equation solvers that can be provided by interval computations. A set of tools and techniques, based on interval arithmetic and affine geometry, has been developed to improve robustness in such operations.

7.3 Current Research Trends in Interval Arithmetic

In the early stages, classical interval arithmetic was preoccupied with the effect of rounding errors on the accuracy of expression evaluation. Later, it was realized that the interval theory has the potential of going beyond expression evaluation and on to solving problems that are inaccessible to conventional approaches. For example, interval analysis has been used, by Thomas Hales of the university of Michigan, in computational parts of a proof of Kepler's conjecture on the densest packing of spheres. Making use of classical interval mathematics, Hales introduced a *definitive proof* of a conjecture that perplexed mathematicians for nearly 400 years (see, e.g., [Hales2005] and [Moore2009]).

Current and future research efforts in interval mathematics can be classified mainly into three categories; *interval standardizations*, *interval implementations*, and *generalizations* of the mathematical theory of intervals. A considerable scientific effort is put into developing special methods and algorithms that try to overcome the difficulties imposed by the algebraic disadvantages of the classical interval arithmetic structure. Various proposals for possible alternate theories of interval arithmetic were introduced to reduce the dependency effect or to enrich the algebraic structure of interval numbers (For further details, see, e.g., [Gardenyes1985], [Hansen1975], [Hayes2009], [Kulisch2008a], [Lodwick1999], and [Markov1995]).

Now, the subject of interval mathematics concerns numerous scientific journals and publishers. The journal "*Interval Computations*" started as a joint

Soviet-Western enterprise in 1991, and continues as the journal "*Reliable Computing*". Besides that, "*Computing*" commonly publishes material related to interval computations, as well as the journal "*Global Optimization*". Traditional numerical analysis journals such as "*BIT*", the "*SIAM Journal on Numerical Analysis*", the "*SIAM Journal on Scientific and Statistical Computing*", and the "*ACM Transactions on Mathematical Software*" contain articles on interval computations (For further resources, see, e.g., [Hansen2003], [Jaulin2001], [Kearfott1996], and [Moore2009]).

7.4 Summary of Contributions and Future Work

This research makes the following contributions.

- We *formally* construct the algebraic system of classical interval arithmetic, deduce its fundamental properties, and finally prove that it is a *nondistributive abelian semiring.*
- Although the notion of interval dependency is widely used in the interval literature, no attempt has been made to put on a systematic basis its meaning, that is, to indicate formally the criteria by which it is to be characterized. We *formalize the notion of interval dependency*, along with discussing the algebraic systems of two important alternate theories of interval arithmetic: *modal interval arithmetic*, and *constraint interval arithmetic*. The algebra of modal intervals is studied and then uncovered to be a *nondistributive abelian ring.* We pay great attention, in particular, to studying the foundations of the theory of *constraint intervals*, and deduce some important results about its underlying algebraic system.
- Based on the idea of representing a real closed interval as a *convex* set, along with our formalization of the notion of interval dependency, we present an alternate theory of intervals, namely the "*theory of optimizational intervals*", with a mathematical construction that tries to avoid some of the defects in the current theories of interval arithmetic, to provide a richer interval algebra, and to better account for the notion of interval dependency. We carefully construct the algebraic system of optimizational interval arithmetic, deduce its fundamental properties, and then prove that the optimizational interval theory constitutes a *rich $\mathcal{S}$-field algebra*, which extends the *ordinary field structure* of real numbers.
- After formalizing the *classical theory of complex intervals*, we present a new systematic construction of complex interval arithmetic, based on the

theory of optimizational intervals. We carefully construct the algebraic system of *optimizational complex interval arithmetic*, deduce its fundamental properties, and then prove that optimizational complex interval arithmetic possesses a rich $\mathcal{S}$-field algebra, which extends the field structure of ordinary complex numbers and the $\mathcal{S}$-field of optimizational interval numbers.

- Of all the existing ordering relations for interval numbers, *none is a total ordering*. After providing a quick survey of such partial ordering relations, along with discussing their *compatibility* with the interval algebraic operations, we define an ordering relation $\preceq_{\mathcal{I}}$ on the set $[\mathbb{R}]$ of interval numbers, then we present the proofs that the relation $\preceq_{\mathcal{I}}$ is a *non-strict total ordering* on $[\mathbb{R}]$, *compatible* with interval addition and multiplication, *dense* in $[\mathbb{R}]$, and *weakly Archimedean* in $[\mathbb{R}]$. Furthermore, we prove that the relation $\preceq_{\mathcal{I}}$ induces a *distributive lattice structure* for interval numbers, and that $\preceq_{\mathcal{I}}$ is an extension of the usual ordering $\leq_{\mathbb{R}}$ on the reals, *Moore's partial ordering* $<_{\mathrm{M}}$ on $[\mathbb{R}]$, and *Kulisch's partial ordering* $\leq_{\mathrm{K}}$ on $[\mathbb{R}]$.

The results obtained in this thesis give an insight into some further consequences and propose some directions for future research. As future prospects, we shall attempt to work on some of the following.

- Constructing axiomatizations for the theory of optimizational intervals with the triple interval representation ("*an axiomatic theory of optimizational triple intervals*"), in the language of *standard predicate logic*, and in the language of *dependence logic*.
- Developing proper *machine arithmetics* for optimizational intervals and optimizational complex intervals.
- By means of the results that the ordering $\preceq_{\mathcal{I}}$ is *total* and *compatible* with the interval algebraic operations, *intervals of intervals* (*higher-order intervals*) are definable, and their algebra can be constructed in the standard way. We aim to develop a "*higher-order interval algebra*" and investigate its applications in automated reasoning.
- By virtue of the *totality* of the ordering $\preceq_{\mathcal{I}}$, an *interval distributive $\preceq_{\mathcal{I}}$-lattice algebra* $\langle [\mathbb{R}]\,; +, \times; \curlyvee, \curlywedge \rangle$ is definable. We aim to study this lattice structure and derive its fundamental properties.
- By virtue of the *totality* of the ordering $\preceq_{\mathcal{I}}$, an *order topology* for interval numbers is definable. We aim to investigate the fundamental properties of this topology.

Bibliography

[Alefeld1983] Gotz Alefeld and Jurgen Herzberger; *Introduction to Interval Computation*; First Edition; Academic Press; New York; 1983.

[Allchin2000] Douglas Allchin; *The Epistemology of Error*; Philosophy of Science Association Meeting; 2000.

[Allchin2007] Douglas Allchin; *Thinking about Technology and the Technology of "Thinking About"*; Techné; Volume 11, Number 1; 2007.

[Alolyan2011] Ibraheem Alolyan; *A New Method for Comparing Closed Intervals*; The Australian Journal of Mathematical Analysis and Applications; Volume 8, Issue 1, Article 2; 2011; pp. 1-6.

[Amer1989] Mohamed A. Amer; *First Order Logic with Empty Structures*; Studia Logica; Volume 48, Number 2; 1989; pp. 169-177.

[Archimedes2002] Archimedes of Syracuse; *The Works of Archimedes*; translated with introduction and commentary by Sir Thomas L. Heath; Dover Publications, 2002.

[Armstrong1974] William Ward Armstrong; *Dependency Structures of Data Base Relationships*; Proceedings of IFIP Congress; 1974; pp. 580-583.

[Barnes1975] Donald W. Barnes and. John M. Mack; *An Algebraic Introduction to Mathematical Logic*; First Edition; Springer Verlag; 1975.

[Basu2003] Saugata Basu, Richard Pollack, and Marie-Françoise Roy; *Algorithms in Real Algebraic Geometry*; First Edition; Springer Verlag; 2003.

[Birkhoff1948] Garrett Birkhoff; *Lattice Theory*; Revised Edition; American Mathematical Society; 1948.

[Blyth1994] T. S. Blyth and J. C. Varlet; *Ockham Algebras*; Oxford University Press; 1994.

[Blyth2010] T.S. Blyth; *Lattices and Ordered Algebraic Structures*; Springer Verlag; 2010.

[Boche1963] Raymond E. Boche; *An Operational Interval Arithmetic*; University of Illinois National Electronics Conference; August 1963.

[Boche1966] Raymond E. Boche; *Complex Interval Arithmetic with Some Applications*; Technical Report Number LMSC4-22-66-1; Lockheed General Research Program, 1966.

[Bryant1985] Victor Bryant; *Metric Spaces: Iteration and Application*; Cambridge University Press; 1985.

[Burkill1924] John Charles Burkill; *Functions of Intervals*; Proceedings of the London Mathematical Society; Volume 22, 1924; pp. 275-310.

[Cantor1955] Georg Cantor; "*Beitrage zur Begrundung der transfiniten Mengenlehre II*"; Mathematische Annalen; Volume 49; 1897; pp. 207-246; translated with introduction and commentary by Philip E. B. Jourdain as "*Contributions to the Founding of the Theory of Transfinite Numbers*"; Dover Publications; New York; 1955.

[Carnap1958] Rudolf Carnap; *Introduction to Symbolic Logic and its Applications*; Dover Publications; 1958.

[Causey1994] Robert L. Causey; *Logic, Sets, and Recursion*; Jones and Bartlett Publishers; Boston; 1994.

[Chabert2007] Gilles Chabert and Luc Jaulin; *Computing the Pessimism of Inclusion Functions*; Reliable Computing; Volume 13, Number 6; 2007; pp. 489-504.

[Chen2000] Guanrong Chen and Trung Tat Pham; *Introduction to Fuzzy Sets, Fuzzy Logic, and Fuzzy Control Systems*; First Edition; CRC Press; 2000.

[Chu2010] Dominique Chu and David J. Barnes; *Introduction to Modeling for Biosciences*; Springer Verlag; 2010.

[Cleary1987] John G. Cleary; *Logical Arithmetic*; Future Computing Systems; Volume 2, Number 2; 1987; pp. 125-149.

[Cleary1993] John G. Cleary; *Proving the Existence of Solutions in Logical Arithmetic*; Constraint Programming; NATO Advanced Study Institute; Estonia; 1993; pp. 52-63.

[Collins1975] G. E. Collins; *Quantifier Elimination for the Elementary Theory of Real Closed Fields by Cylindrical Algebraic Decomposition*; in "*Automata Theory and Formal Languages 2nd GI Conference Kaiserslautern*"; Lecture Notes in Computer Science; Volume 33; pp. 134-183; Springer Verlag; 1975.

[Dawood2011] Hend Dawood; *Theories of Interval Arithmetic: Mathematical Foundations and Applications*; LAP Lambert Academic Publishing; Saarbrücken; 2011.

[Dedekind1900] Richard Dedekind; *Üeber die von drei Moduln erzeugte Dualgruppe*; Mathematische Annalen; Volume 53, Number 3; 1900; pp. 371-403.

[Dedekind1963] Richard Dedekind; *Essays on the Theory of Numbers: Continuity and Irrational Numbers*; Dover Publications; New York; 1963.

[Devlin1993] Keith Devlin; *The Joy of Sets: Fundamentals of Contemporary Set Theory*; Springer Verlag; 1993.

[Dixon1902] Edward T. Dixon; *On the Notion of Order*; Mind; New Series, Volume 11, Number 44; October, 1902; pp. 527-534.

[Dunn2001] J. Michael Dunn and Gary Hardegree; *Algebraic Methods in Philosophical Logic*; Oxford University Press; 2001.

[Dwyer1951] Paul S. Dwyer; *Linear computations*; Chapman & Hall; New York; 1951.

[Euler2000] Leonhard Euler; *Foundations of Differential Calculus*; Springer Verlag; 2000.

[Fateman2009] Richard J. Fateman; *Interval Arithmetic, Extended Numbers and Computer Algebra Systems*; University of California at Berkeley; 2009.

[Feferman2006] Solomon Feferman; *What Kind of Logic is Independence-Friendly Logic?*; Auxier and Hahn; 2006; pp. 453–469.

[Fomin1999] Sergei Vasilievich Fomin and Andrei Nikolaevich Kolmogorov; *Elements of the Theory of Functions and Functional Analysis*; Dover Publications; 1999.

[Gaganov1985] A. A. Gaganov; *Computational Complexity of the Range of the Polynomial in Several Variables*; Cybernetics; 1985, pp. 418-421.

[Gardenyes1985] E. Gardenyes, H. Mielgo, and A. Trepat; *Modal Intervals: Reason and Ground Semantics*; in "*Interval Mathematics*"; Lecture Notes in Computer Science; Volume 212; pp. 27-35; Springer Verlag; 1985.

[Gardenyes2001] E. Gardenyes, M.A. Sainz, L. Jorba, R. Calm, R. Estela, H. Mielgo, and A. Trepat; *Modal Intervals*; Reliable Computing; Volume 7; 2001; pp. 77-111.

[Gleason1992] Andrew Gleason; *Fundamentals of Abstract Analysis*; CRC Press; 1992.

[Hales2005] Thomas C. Hales; *A Proof of the Kepler Conjecture*; The Annals of Mathematics; Volume 162, Number 3; 2005; pp. 1065-1185.

[Hansen1975] Eldon R. Hansen; *A Generalized Interval Arithmetic*; in "*Interval Mathematics*"; Lecture Notes in Computer Science; Volume 29; pp. 7-18; Springer Verlag; 1975.

[Hansen2003] Eldon R. Hansen and G. William Walster; *Global Optimization Using Interval Analysis: Revised And Expanded*; Second Edition; CRC Press; 2003.

[Hausdorff1978] Felix Hausdorff; "*Mengenlehre*"; Third Edition; Sammlung Göschen, de Gruyter; Berlin; 1937; translated by John R. Aumann as "*Set Theory*"; Third Edition; Chelsea Publishing Company; New York; 1978.

[Hayes2009] Nathan T. Hayes; *Introduction to Modal Intervals*; IEEE 1788 Working Group; March 2009.

[Heijenoort1967] Jean van Heijenoort (ed.); *From Frege to Godel: A Source Book in Mathematical Logic 1879-1931*; Harvard University Press; 1967.

[Hintikka1959] Jaakko Hintikka; *Existential Presuppositions and Existential Commitments*; The Journal of Philosophy; Volume 56, Number 3; 1959; pp. 125-137.

[Hintikka1996] Jaakko Hintikka; *The Principles of Mathematics Revisited*; Cambridge University Press; 1996.

[Hodges1997] Wilfrid Hodges; *A Shorter Model Theory*; Cambridge University Press; 1997.

[Huntington1905a] Edward V. Huntington; *A Set of Postulates for Real Algebra*; Transactions of the American Mathematical Society; Volume 6, Number 1; 1905; pp. 17-41.

[Huntington1905b] Edward V. Huntington; *A Set of Postulates for Ordinary Complex Algebra*; Transactions of the American Mathematical Society; Volume 6, Number 2; 1905; pp. 209-229.

[Ishibuchi1990] Hisao Ishibuchi and Hideo Tanaka; *Multiobjective Programming in Optimization of the Interval Objective Function*; European Journal of Operational Research; Volume 48, Issue 2; 1990; pp. 219–225.

[Jaulin2001] Luc Jaulin et al.; *Applied Interval Analysis*; First Edition; Springer Verlag; 2001.

[Kahraman2008] Cengiz Kahraman; *Fuzzy Engineering Economics with Applications*; Springer Verlag; 2008.

[Kearfott1996] R. Baker Kearfott; *Interval computations: Introduction, uses, and resources*; Euromath Bulletin; Volume 2, Number 1; 1996; pp. 95-112.

[Keene1988] Sonya Keene; *Object-Oriented Programming in Common Lisp: A Programmer's Guide to CLOS*; Addison-Wesley Publishing Company; 1988.

[Kleene1952] Stephen Kleene; *Introduction to Metamathematics*; North-Holland Publishing Company; Amsterdam; 1952.

[Kline1982] Morris Kline; *Mathematics: The Loss of Certainty*; Oxford University Press; 1982.

[Knuth1997] Donald E. Knuth; *The Art of Computer Programming*, Volume 1: "*Fundamental Algorithms*"; Third Edition; Addison-Wesley Professional; 1997.

[Kolev1993] L. V. Kolev; *Interval Methods for Circuit Analysis*; World Scientific Publishing Company; 1993.

[Kreinovich1995] Vladik Kreinovich; *Interval Rational = Algebraic*; ACM SIGNUM Newsletter; Volume 30, Issue 4; New York; 1995; pp. 2-13.

[Kulisch1981] Ulrich W. Kulisch and Willard L. Miranker; *Computer Arithmetic in Theory and Practice*; Academic Press; New York; 1981.

[Kulisch2008a] Ulrich W. Kulisch; *Complete Interval Arithmetic and its Implementation*; in "Numerical Validation in Current Hardware Architectures"; Internationales Begegnungs; 2008.

[Kulisch2008b] Ulrich W. Kulisch; *Computer Arithmetic and Validity: Theory, Implementation, and Applications*; First Edition; Walter de Gruyter; 2008.

[Kuratowski1976] Kazimierz Kuratowski and Andrzej Mostowski; *Set Theory: with an Introduction to Descriptive Set Theory*; Second Revised Edition; North-Holland Publishing Company; Amsterdam; 1976.

[Lambert2003] Karel Lambert; *Free Logic: Selected Essays*; Cambridge University Press; 2003.

[Levi1961] Howard Levi; *Elements of Algebra*; Fourth Edition; American Mathematical Society; 1961.

[Lobach1951] Nikolai Ivanovich Lobachevskii; *Complete works 5*; Moscow-Leningrad; 1951.

[Lodwick1999] Weldon A. Lodwick; *Constrained Interval Arithmetic*; CCM Report 138; University of Colorado at Denver; Denver, USA; February 1999.

[Malcev1971] Anatolij Ivanovic Mal'cev; *The Metamathematics of Algebraic Systems, Collected Papers: 1936-1967*; North-Holland Publishing Company; Amsterdam; 1971.

[Markov1995] Svetoslav M. Markov; *On Directed Interval Arithmetic and its Applications*; Journal of Universal Computer Science; Volume 1, Number 7; 1995; pp. 514-526.

[McCallum2001] S. McCallum; *On Propagation of Equational Constraints in CAD-Based Quantifier Elimination*; Proceedings of the International Symposium on Symbolic and Algebraic Computation; 2001; pp. 223-231.

[Menini2004] Claudia Menini and Freddy Van Oystaeyen; *Abstract Algebra: A Comprehensive Treatment*; First Edition; CRC Press; 2004.

[Montague1974] Richard Montague; *Formal Philosophy: Selected Papers*; edited with introduction by Richmond H. Thomason; Yale University Press; London; 1974.

[Montague1980] Richard Montague, Donald Kalish, and Gary Mar; *Logic: Techniques of Formal Reasoning*; Second Edition; Oxford University Press; New York; 1980.

[Moore1959] Ramon E. Moore; *Automatic Error Analysis in Digital Computation*; Technical Report Number LMSD84821; Lockheed General Research Program; 1959.

[Moore1962] Ramon E. Moore; *Interval Arithmetic and Automatic Error Analysis in Digital Computing*; Ph.D. Thesis; Stanford University; 1962.

[Moore1966] Ramon E. Moore; *Interval Analysis*; Prentice Hall; 1966.

[Moore1979] Ramon E. Moore; *Methods and Applications of Interval Analysis*; SIAM; 1979.

[Moore2009] Ramon E. Moore, R. Baker Kearfott, and Michael J. Cloud; *Introduction to Interval Analysis*; SIAM; 2009.

[Mostowski1951] Andrzej Mostowski; *On the Rules of Proof in the Pure Functional Calculus of the First Order*; The Journal of Symbolic Logic; Volume 16, Number 2; 1951; pp. 107-111.

[Muller2009] Jean-Michel Muller et al.; *Handbook of Floating-Point Arithmetic*; First Edition; Birkhauser Boston; 2009.

[Neumaier1991] Arnold Neumaier; *Interval Methods for Systems of Equations*; Cambridge University Press; 1991.

[Pedrycz2008] Witold Pedrycz (ed.), Andrzej Skowron (ed.), and Vladik Kreinovich (ed.); *Handbook of Granular Computing*; First Edition; Wiley-Interscience; 2008.

[Petkovic1998] Miodrag S. Petkovic, Ljiljana D. Petkovic, L. D. Petkovic; *Complex Interval Arithmetic and Its Applications*; First Edition; Wiley-VCH; 1998.

[Pichler2007] Franz Pichler; *Computer Aided Systems Theory*; Springer Verlag; 2007.

[Pouzet1981] Maurice Pouzet and Ivan Rival; *Which Ordered Sets have a Complete Linear Extension?*; Canadian Journal of Mathematics; Volume 33, Number 5; 1981; pp. 1245–1254.

[Quine1954] Willard Van Orman Quine; *Quantification and the Empty Domain*; The Journal of Symbolic Logic; Volume 19, Number 3; 1954; pp. 177-179.

[Rasiowa1963] Helena Rasiowa and Roman Sikorski; *The Mathematics of Metamathematics*; Panstwowe Wydawnictwo Naukowe; Warszawa; 1963.

[Robert1996] John S. Robertson; *Engineering Mathematics with Maple*; McGraw-Hill; New York; 1996.

[Rokne1984] Jon Rokne and Helmut Ratschek; *Computer Methods for the Range of Functions*; Ellis Horwood Publications; 1984.

[Russell1901] Bertrand Russell; *On the Notion of Order*; Mind; New Series, Volume 10, Number 37; January, 1901; pp. 30-51.

[Russell1910] Bertrand Russell and Alfred North Whitehead; *Principia Mathematica*; 3 Volumes; Cambridge University Press; 1910, 1912, and 1913.

[Sengupta2000] Atanu Sengupta and Tapan Kumar Pal; *On comparing interval numbers*; Proceedings of European Journal of Operational Research; Volume 127, Number 1; 2000; pp. 28-43.

[Sengupta2009] Atanu Sengupta and Tapan Kumar Pal; *Fuzzy Preference Ordering of Interval Numbers in Decision Problems*; Springer Verlag; 2009.

[Shankar1997] N. Shankar; *Metamathematics, Machines, and Gödel's Proof*; Cambridge University Press; Cambridge; 1997.

[Shayer1965] Sidney Shayer; *Interval Arithmetic with Some Applications for Digital Computers*; Technical Report Number LMSD5-13-65-12; Lockheed General Research Program; 1965.

[Shoenfield1967] Joseph Robert Shoenfield; *Mathematical Logic*; Addison-Wesley Publishing Company; 1967.

[Skolem1920] Thoralf Albert Skolem; "*Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze nebst einem Theoreme über dichte Mengen*"; Skrifter utgit av Videnskabsselskapet i Kristiania; I. Matematisk-Naturvidenskabelig Klasse No. 4; 1920, pp. 1–36; translated by Stefan Bauer-Mengelberg as "*Logico-combinatorial investigations on the satisfiability or provability of mathematical propositions: A simplified proof of a theorem by Loewenheim*"; in [Heijenoort1967]; pp. 252–263.

[Smullyan1961] Raymond Smullyan; *Theory of Formal Systems*; Princeton University Press; Princeton; 1961.

[Steen2008] S. W. P. Steen; *Mathematical Logic with Special Reference to the Natural Numbers*; Cambridge University Press; 2008.

[Sunaga1958] Teruo Sunaga; *Theory of an Interval Algebra and its Application to Numerical Analysis*; RAAG Memoirs; Volume 2; 1958; pp. 29-46.

[Suppes1972] Patrick Suppes; *Axiomatic Set Theory*; Dover Publications; New York; 1972.

[Tan2008] Kiat Shi Tan, Willi-Hans Steeb, and Yorick Hardy; *Computer Algebra With Symbolic C++*; World Scientific Publishing Company; 2008.

[Tarski1941] Alfred Tarski; *On the Calculus of Relations*; The Journal of Symbolic Logic; Volume 6, Number 3; 1941; pp. 73-89.

[Tarski1951] Alfred Tarski; *A Decision Method for Elementary Algebra and Geometry*; Second Edition; University of California Press; 1951.

[Tarski1965] Alfred Tarski; *Logic, Semantics, Metamathematics*; translated by J. H. Woodger; Oxford At the Clarendon Press; 1965.

[Tarski1994] Alfred Tarski; *Introduction to Logic and to the Methodology of the Deductive Sciences*; translated by Olaf Helmer; Fourth Edition; Oxford University Press; New York; 1994.

[Vaananen2001] Jouko Vaananen; *Second-Order Logic and Foundations of Mathematics*; Bulletin of Symbolic Logic; Volume 7, Number 4; 2001; pp. 504–520.

[Vaananen2007] Jouko Vaananen; *Dependence Logic: A New Approach to Independence Friendly Logic*; Cambridge University Press; 2007.

[Young1931] Rosalind Cecily Young; *The Algebra of Many-Valued Quantities*; Mathematische Annalen; Volume 104; 1931; pp. 260-290.

[Yu2004] Xing Huo Yu; *Advances in Artificial Intelligence*; Proceedings of 17th Australian Joint Conference on Artificial Intelligence; Australia; 2004.