Introducing solving equations: Teachers and the one-variable-first curriculum

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As teachers and curriculum developers work to introduce students to complex mathematical notions, starting is always difficult. Starting places can be too abstract or formal, and as a result divorced from students' experience (as critics argued with New Math set theoretic definitions of number). Or, starting places can be too specific and concrete. In such a case, students must abstract the essential aspects of a concept from examples that have many other aspects and they may often include unintended aspects of the example in their definition (as suggested by Vinner, 1983 in his introduction of the term concept image, in addition to concept definition). While a powerful and concrete example can serve as an important conceptual anchor, as researchers have found (e.g., Vinner and Dreyfus, 1989 with respect to students and functions and Even, 1993 with respect to preservice teachers and functions) such concept images can later become pedagogical obstacles (in the sense of Sierpinska, 1992). Students may have to disconnect a particular image from a concept in order to develop a broader sense of a concept, one that matches its definition and not a particular example (Lakatos' 1976 rational reconstruction of the history of the Descartes-Euler conjecture finds a similar dynamic in the work of mathematicians).

In this paper, we will use the pedagogical challenge of starting places, as well as the notions of concept image, concept definition, and pedagogical obstacle to explore a seemingly humble concept (if it can be called a concept!), the equation, as it is usually taught in introductory algebra, and correlated actions, like solving. This exploration involved interviews of teachers in a US high school and observations of their teaching.

I. A school algebra perspective on solving equations and systems of equations

Equations appear throughout mathematics in different guises. However, it is unclear what equations are. They are sometimes thought of as mathematical objects worthy of exploration in their own right (e.g., the now-defunct Theory of Equations). At other times, they are representations of other mathematical objects (e.g., as signaled by the locution “the equation of a function” or “equation of a line”). A common school textbook definition avoids these subtleties by offering a criterion for recognizing an equation when one sees one: an equation is a string of symbols with an equal sign in it.

Mathematics educators have studied equations as they appear in a range of school mathematics contexts. Researchers interested in students’ transition from arithmetic to algebra have expressed concern about ways in which the equal sign in arithmetic is a call for carrying an action on the expression on the left in order to know what to write for the expression on the right, while in algebra students are asked to act on both sides of the equation. Pedagogical interventions based on this concern (e.g., Herscovics and Kieran, 1980) are premised on the notion that overcoming a particular view of the equal sign will help students be more successful in algebra. In this paper, we will make a similar argument within the kinds of equations that students and teachers typically see in school algebra.

Other mathematics educators involved in writing curricula for school algebra have noted the variety of uses of equations in the school curriculum. For example, Zalman Usiskin (1988) suggests that in the US school curriculum equations are treated in five different ways: as formulae, equations to solve, identities, properties of numbers, or the equation of a function (p.9). While we make similar observations to Usiskin, we will focus on examples that all fall within a single type in his framework, equations to solve, what Freudenthal (1983) calls interrogative algebraic sentences (p. 310) and for which he invents a new notation.

In particular, in this paper, we will focus primarily on issues involving equations in one and two variables, and peripherally on systems of equations involving two variables. We will focus on issues that arise in traditional US curricular approaches to developing students' flexible understandings of equations and solving in the context of these equations and systems of equations.

The equations that students in the US meet in introductory algebra (Algebra I or its equivalent) are usually equations in one or two variables, first equations in one variable and then equations in two. These equations have an equal sign, may or may not represent a function, they may or may not admit to closed form solution, and for the most part, with the exception of units on quadratic equations and absolute value, they are linear.

In making this short list of characteristics of equations, we suggest that equations in one and two variables, as treated by the curriculum, are quite different one from another. As others have argued with respect to arithmetic and algebra equations (e.g., Nathan & Koedinger, 2000), we would like to suggest that these differences matter for teaching and learning.

Most curricula introduce students to equations in algebra by starting with linear equations in one variable (Some texts organize these into one-step or two-step equations, others organize them by the types of operations used to solve them, e.g, McConnell, et al., 1990. By way of contrast, functions-based perspectives begin with functions, a subset of equations in two variables, before introducing equations in one variable to solve.). Students are asked to solve these equations. In the context of linear equations in one variable, this task can have three related and as of yet indistinguishable goals. Students are taught procedures to solve equations by operating on both sides of the equation to (i) isolate the variable. They are then taught to check their solution by substituting the solution back into the original equation. This check procedure builds on the notion that solving the equation is meant to (ii) find value(s) of the variable for which the equation will be a true statement.

Work with linear equations in one variable can support a view of the variable as an as of yet unknown number. In "most" cases, there is only one such number. The notion of a solution set, and solving as (iii) representing the solution set to an equation, is implicit until attention changes from linear equations to quadratic equations or to inequalities, or when linear equations with no solution or infinitely many solutions arise.

The standard school algebra curriculum takes a very narrow approach to equations in one variable. Students, and teachers, rarely see equations of one variable that cannot be solved by isolating the variable, equations like: $3(x - 2) + 2 = 4x - 4(x-3)$, $4(x-1) - x = 3x - 4$, $2^x = x^2$, or $|x-2| = |x+1| + 3$. While such examples drive a wedge between the "isolate the variable" and "find values that make the statement true" or the "describe the solution set" meanings of solve, they do not regularly appear in classrooms.

Meaning of solve	Equations like: $3x + 2 = 4x - 7$
Isolate the variable	Taught to students
Find member(s) of the solution set	Implicit part of checking
Represent the solution set	Implicit, until more than one solution

When the curriculum moves to equations in two variables, there is a large transition for students. A linear equation in two variables defines a function, explicitly or implicitly. Equations in two variables challenge the as of yet unknown number view of literal symbols like x and y. And, solving changes. Building on the "isolate a variable" meaning of solve, linear equations in two variables can be solved for a particular variable, but there is a disconnect between isolating the variable and finding member(s) of the solution set. With linear equations in one variable, isolating the variable had resulted in finding the (unique) solution, except in "special cases"; whereas, with equations in two variables, isolating the variable may facilitate finding members of the solution set, but it is just a first step toward that end. Isolate the variable is now signified by "solve for", and finding members of the solution set now requires a different set of actions, one must choose a value for one variable and then solve the resulting equation in one variable.

The "represent the solution set" meaning of solve is more available, though it is easy to overlook. Any equation in two variables represents a solution set that can be graphed on the Cartesian plane. Isolating a variable may make the task of graphing the equation easier, but even without doing any manipulation of the equation, the equation defines the solution set (as it did as well earlier with equations in one variable). While all of these different meanings of solve are still at play with equations in two variables, the command "solve" is generally not used; instead it is replaced by *solve for*, *create a table of* (x, y) *values*, and *graph*. Similarly, checking a *solution* becomes checking that an ordered pair *satisfies* the equation. This shift in language potentially obscures the connections among these actions and with the concept of solution.

Finally, when the curriculum turns to a short unit on systems of equations, solve takes yet another turn. With systems of linear equations, the notion of isolating the variable becomes a step in one solution procedure, no longer a defining meaning for the operation of solving. While the notion of solving for a member of the solution set takes a back seat with equations in two variables, this meaning returns to the forefront with the move to systems of two linear equations in two variables. Students find the coordinated values of x and y that satisfy both equations. With systems of two linear equations, the notion of representing the solution set can be lost again, as it was with the linear equation in one variable, though it can be resurrected by examining systems involving non-linear equations, or systems with no solutions, or infinitely many, or systems of inequalities.

Our point in cataloguing the meanings of solve as one moves through the curriculum is to emphasize the transitions required of students as they move through a standard curricular approach to school algebra (A functions-based approach would have a different, but related, set of transitions.). We were interested in seeing how teachers think about these issues and how their thinking plays out in their interactions with their students.

Meaning of solve	Equations like: $3x + 2 = 4x - 7$	Equations like: $9x + 3y = 12$	Systems like: $\begin{cases} 2x+3y=7 \\ x-y=-4 \end{cases}$
Isolate the variable	Taught to students.	Becomes "solve for".	A step in possible solution procedures.
Find member(s) of the solution set	Implicit part of checking.	Not part of "solve for." Signified by "create a table".	Stated goal. Part of checking.
Represent the solution set	Implicit, until more than one solution.	More readily available. Signified by "graph".	Implicit, until more than one solution.

Before turning to examples of teachers grappling with their own understandings of these transitions and how to communicate with their students, we give a little background on the teachers and our interaction with them.

II. Background

We would like to juxtapose our analysis of equations and solving in the school algebra, one-variable-equations-first, way of introducing algebra with interviews and observations of teachers in a US high school. We precede examination of the thinking and actions of these teachers with some background information that situates them in their local context and indicates how we interacted with them.

The teachers we interviewed and observed are from one high school in a district that ranks among the 20 largest school districts in the US. Like many schools that are described as urban schools in the US, this high school has a predominantly minority population (96%), high turnover among teachers (only 39% of classes are taught by a "highly qualified" teacher), a high rate of poverty among the families of students (51%

on free/reduced meals), a high rate of student mobility (24% in, 17% out), poor student achievement on exams (a passing rate of 16% on a state Algebra 1 test in 2004), and a low graduation rate (65%).

The teachers we interviewed and observed were teaching introductory school algebra (Algebra I) to high school students (mostly grade 9, age 14 and above). Students in the state must pass high school algebra and geometry courses in order to graduate, as well as take end of course exams for both courses. Beginning with students entering high school in Fall 2005, students will be required to pass the algebra end of course exam for graduation.

The Text

The adopted textbook in the school district is Prentice Hall's *Algebra: Tools for a Changing World* (Bellman, Bragg, Chapin, Gardella, Hall, Handlin, & Manfre, 1998). This text combines a fairly standard approach to solving equations, as we described earlier, with an earlier than standard introduction of functions of one variable and their defining equations.

In this text, in Chapter 1, equations are defined (p. 11) as indicating that two expressions are equal. The first equations that students meet are *function rules*, defined as equations (of two variables) that describe functions. For example, on page 11, the first example of an equation is $t = s - 2$, but the example of an equation in the glossary at the back of the book is $x + 5 = 3x - 7$. The remainder of the first chapter then deals with order of operations, arithmetic with integers, properties of real numbers, experimental probability, and introductions to matrices and spreadsheets. Chapter 2 returns to functions and their representations.

After the material on functions of one variable, chapter 3 focuses on equations of one variable. It begins with a balance scale analogy for equations and describes solving an equation containing a variable as finding "the value (or values) of the variable that make the equation true" (p. 108). *Solutions* are defined as these values. The chapter contains six sections that develop techniques for solving linear equations in one variable by "get[ting] the variable alone on one side of the equal sign" (p. 108) using inverse operations. Students first solve one-step equations, then two-step equations, then equations that contain like terms on one side, parentheses, fractions, and percents.

In Chapter 4, students meet linear equations in one variable with a variable on both sides of the equal sign. In the main body of the text, the technique for solving linear equations in one variable with a variable on both sides involves applying the properties of equality "to get terms with variables on the same side of the equation" (p. 164), initially aided by the use of algebra tiles. "Special types of equations", those having no solution and those that are identities, are introduced and defined. In the homework exercises, students are asked to write *identity* or *no solution* when they encounter such equations. At the end of the exercises for this section, there is a Self-Assessment journal prompt: "Summarize what you know about solving equations with variables on both sides by writing a list of steps for solving this type of equation" (p. 168). Following, there is a Technology page that provides a list of steps and practice exercises for using a graphing calculator to solve an equation with the variable on both sides by graphing and finding the x-coordinate of the point of intersection (p. 169). This chapter also contains one section on solving literal equations for a given variable.

In Chapter 5 students take a closer look at linear equations in two variables, with an emphasis on graphing linear equations and writing equations for lines in both slope-intercept and standard form. Then, in Chapter 6 they meet systems of linear equations in two variables.

Our Interview and Classroom Observation

Our interview was designed to explore how teachers think about what an equation is and how to teach students about equations. With these goals in mind, we focused on understanding:

- how teachers conceptualize "x" in their work with students,
- whether they believe that it is reasonable in an algebra class to conceptualize equations as questions about or comparisons of functions,
- how they distinguish for themselves between solving in the context of equations of one and two variables,
- how they conceptualize the role in an algebra class of equations in one variable in which the variable cannot be isolated,
- how they talk with students about expressions and equations, and
- how these issues interconnect and interrelate in their thinking about the design of instruction in a year-long course.

After a few introductory background questions, we began by asking teachers how they explain to beginning algebra students what an equation is. Then we presented the first major item: a card sort task. Teachers were presented with the following nine equations on nine index cards and asked to discuss them in relation to their conception of equations, to compare and contrast them, to order the cards in the ideal order in which they would want students to meet these examples, and to explain the reasoning behind their choices.

- $y = 3x - 4$
- $3x - 4 = 12$
- $9x - 3y = 12$
- $f(x) = 3x - 4$
- $4(x - 1) - x = 3x - 4$
- $3(x - 2) + 2 = 4x - 4(x - 3)$
- $g(x, y) = 3x - y$
- $y = 3x^2 + x - 2$
- $3x^2 + x - 2 = 0$

Subsequent items included tasks that asked teachers :

- to compare and contrast the following three tasks:
  1. Solve $3x - 4 = 12$.
  2. If $y = 3x - 4$, what value of x makes $y = 12$?
  3. If $f(x) = 3x - 4$, what value of x makes $f(x) = 12$?

  then, to respond to two student solutions: a numerical solution based on the use of tables and a solution that involved operations on both sides of an equation.
- how they would talk with students about how to solve an equation like: $4(x-1) - x = 3x - 4$.
- how they would solve equations, like:

  $$\sqrt{(3x-4)^2} = 3x - 4$$

  $$2^x = x^2$$

  and, whether they would think that equations like this belong in an algebra class.
- how they think about solving $9x - 3y = 12$.
- and, how they might respond to students who, when asked to simplify, set an expression equal to zero and solved it.

The responses to these tasks provide rich glimpses of how the teachers understand what an equation is and how they imagine one might gradually introduce students to the complexities involved in this mathematical construct.

The following fall we observed three of the seven teachers we had interviewed while they were teaching a unit on solving linear equations in one variable (Chapter 3 in their text). This is when the textbook first introduces solving equations. For some of the students, it may have been their first experience with solving equations. At the time of our observations, both teachers had been teaching the solving of linear equations for about one week.

In the next section, we choose to focus on the two teachers for whom we have both an interview and a classroom observation. Ms. Alley was in her third year of teaching at the time of the observation, and she had taught Algebra 1 each year. She had also taught pre-algebra and pre-calculus. Ms. Alley has an undergraduate degree in mathematics and considers herself a mathematician. Ms. Lewis was in her sixth year of teaching, and she primarily teaches Algebra 1; although some years she taught pre-algebra as well. Ms. Lewis changed careers to become a math teacher; her undergraduate degree was in finance.

Both teachers explained that a primary reason why they are assigned to teach Algebra 1 every year is because they both have a reputation as effective classroom managers. It became evident during our conversations that both teachers care deeply about their students and have given a lot of thought to how to best prepare their Algebra 1 students for future mathematics courses and success on high stakes exams.

III. Ms. Alley and Ms. Lewis

On Equations: What are They? What is Their Place in the School Algebra Curriculum?

Early in the interview, we asked the teachers how they explain to students what an equation is. An excerpt from Ms. Lewis' response follows:

> Now that's a good question. Have I ever explained what an equation is? Now, when we set up an equation from a word problem, I mean, I can tell them where everything goes, but – what is an equation? Um. Hmm. Let me think about that. Good question. Um. <pause> Well we solve an equation to get a value for the variable. So an equation would be… hmm. Now I understand the concept of it, but putting it into words, okay, let me think about that. Can you help me out? … I mean, I know what an equation is, I know what the ultimate goal is… of an equation: to find a value for the variable that would make the equation true. … But they understand that I solve for the particular variable that will make this equation true; but, to give them a definition, I mean, that would really be hard to think of what a definition would be."

Ms. Alley gave examples of equations: $x + 5 = 7$ and $x - 5 = 12$. She explained that she first gives students examples with blanks instead of x; then, she tries to have students extend their informal solving to the formal procedures. When presented with the nine equations on index cards in the first task, Ms. Alley distinguished equations one would "solve as equations" (those containing one variable) and "equations that are functions" (those containing two variables).

Ms. Lewis made many more distinctions: equations with variables on both sides, two-step equations, equations in slope-intercept form, and two-variable equations. For Ms. Lewis, functions are distinct from equations; although she used *equation* to describe linear equations in two variables (i.e. Ms. Lewis distinguishes $f(x) = 3x - 4$ as a *function*, but describes $y = 3x - 4$ as an *equation*). In ordering the equations, she considered the amount and type of solving involved—equations move earlier in pedagogical order the more solving is involved and the easier it is to isolate the variable. For example, an equation of the form $ax^2 + b = c$ is "like a two-step [linear] equation, and then you just take the square root at the end; so equations of that sort will occur fairly early." This may suggest that an equation of the form $ax + b = c$, a "two-step equation", is her concept image of equation because that becomes the basis for "moving up" other equations.

Ms. Alley's primary focus in describing and explaining equations was also *solving*; however, she indicated that she wants students "to understand how to write a function first—what it is to put something in and get something out—so they understand that there is a solution." Despite this statement early in the interview, there was little evidence later in the interview, and in the classroom observation, of an impact from the early introduction to functions on the treatment of linear equations in one variable.

Both teachers agreed that non-routine equations do not warrant much attention in an introductory algebra course. Ms. Lewis was not at all comfortable with the idea of including equations for which isolating the variable does not solve the equation in the school algebra curriculum; except perhaps for honors or gifted/talented students, after they have "mastered" techniques for solving. When probed, she was not at all concerned that a very tiny portion of all possible equations can be solved by known symbol manipulation techniques. Similarly, when asked about the importance of providing opportunities for students to grapple with making sense of linear equations that are identities or contradictions, Ms. Alley responded, "Usually these equations are far and few between. So, you might get one or two on a test, or three or four on a homework assignment. So it's just for them to know: hey, if I get 'something = something', it means either 'no solution' or 'infinite solutions'."

On Solving: What's the Name of the Game? Isolate the Variable!

Both teachers emphasized how important it is for algebra students to learn how to solve equations. When asked about her primary goals for her Algebra I classes, Ms. Lewis responded, "Definitely solving equations I would say." She further explained, "…what's more important for them: to understand it, or just to be able to punch it in their calculator and write something that they really don't understand? … I know the teacher who teaches Trig and Algebra 2/Trig and the Trig Analysis; he's an old teacher. Ok, so you know how the older teachers are – solve solve solve. ... So for my kids to be successful when they go to him, they need to know how to solve." During her interview, Ms. Alley commented, "You can only make solving equations so exciting; whereas, that's the basis of everything you learn after it, and you have to know it."

Both Ms. Alley and Ms. Lewis commented on the difficulty students have with solving (via symbol manipulation), even with "simple" equations. In their attempts to help their students manage this difficulty, they both place a heavy emphasis on the phrase "isolate the variable" as articulating the goal of solving when they first introduce students to solving in the context of linear equations in one variable.

When we observed Ms. Alley teaching a lesson on solving linear equations in one variable, she repeatedly asked the class, "What's the name of the game?", to which the students responded in chorus, "Isolate the variable!" During an observation of a lesson at the same point in the algebra curriculum, Ms. Lewis explained to her class, "The number that is on the same side as the variable is the one we want to get rid of…so we can isolate the variable…'cause remember: that's the goal."

All three teachers we observed used language about removing “zero pairs”, a term for *additive inverses* that is intended to be student-friendly, as part of the procedure to isolate the variable. This language is introduced in the textbook when explaining how to use algebra tiles to model solving a linear equation in one variable.

One might wonder what drives the teachers’ decisions to emphasize solving as isolating the variable, rather than as finding the value or values of the variable that make the statement true. The teachers’ responses to two different hypothetical students’ solutions to questions that could be answered by solving the equation $3x - 4 = 12$ provide some insight. Sam’s solution involved operating on both sides of the equation to isolate x, while Karim’s solution involved creating a table of sequential $(x, 3x - 4)$ values to approximate the value of x for which $3x - 4$ equals 12. Both teachers expressed a preference for Sam’s solution.

Ms. Alley commented, “It isn’t that I wouldn’t accept it [Karim’s solution], I’m saying I would be less inclined to accept it.” She further explained that she viewed Karim’s solution as “guesstimating”; she sees rounding, Sam’s method for approximating the value of x, as better estimation. When asked how her view would change if Karim used proportional reasoning to interpolate a precise value for x, she said that that would make his solution completely acceptable, but she would never teach this method because “No student would ever be able to see it—like, you have to give the students ways that they’ll be able to see, and after teaching for 2 years, they would never see it that way. They don’t think critically.” Karim’s solution builds upon the early experience with functions Ms. Alley advocated at the start of her interview; however, her comments here suggest that not only does she not use thinking about functions in her teaching of solving equations in one variable, she does not believe that it is feasible to do so.

Similarly, Ms. Lewis could not understand why Karim would use a table to solve $3x - 4 = 12$. She explained that she would never teach students to solve this way; she would teach them by solving (i.e. using symbol manipulation to isolate the variable). She views Karim's solution as using higher order thinking as compared to Sam's, which she calls "basic, basic solving." Ms. Lewis further explained that when two variable equations are introduced, students would create tables and answer related questions about inputs and associated outputs of functions; however, this is not connected to solving one-variable equations.

The comments from both teachers about Sam's and Karim's solutions suggest that the goal of finding the values that make the equation a true statement may receive less emphasis when teaching students to solve linear equations in one variable because such an emphasis may be seen as encouraging guesstimating or other undesirable strategies. Their comments further suggest that they believe that even though they have observed students struggle to solve "simple" linear equations via symbol manipulation, that is the easiest way to help students learn to solve equations. In fact, both teachers asserted through their classroom instruction that this is the *only* way.

During our interview, Ms. Alley commented, "it's just like I tell my kids, there's not one way to do every problem. So, … I think every student finds a way that they like the best, and then uses that." However, during our observation, a student started two or three times to explain that she had solved an equation a different way and arrived at the same solution. Both times Ms. Alley interrupted the student, "This is the *only* way," referring to a very specific procedure for isolating the variable by operating on both sides of the equation.

Similarly, Ms. Lewis presented the following problem to her class when we observed:

Which student solved correctly?

Kendra:	Tony:
$x - 3 = 12$	$x - 3 = 12$
$+3 \quad +3$	$-12 \quad -12$
$x \quad = 15$	$x - 15 = 0$
	$+15 \quad +15$
	$x \quad = 15$

In the discussion that followed, Ms. Lewis indicated that Kendra had solved correctly. According to Ms. Lewis, Tony had solved incorrectly because his first step did not progress him toward the goal of isolating the variable; therefore, Tony required two steps to solve the equation, rather than just one step. While the first step in Tony's solution process may not have been the most strategic, he certainly presents a *correct* solution.

Both teachers emphasized a particular procedure for solving equations in one variable that involves operating on both sides of an equation to isolate the variable in the least number of moves possible. At this point in the curriculum, 'solving' is synonymous with this procedure; other procedures that might find the value(s) of the variable that make an equation a true statement are not considered *solving*. This emphasis on isolating the variable may be an attempt to ease the transition to solving equations in two variables for a particular variable.

On Identities and Contradictions: Assigning a Label

Early in the interviews we asked teachers to consider the equation $4(x - 1) - x = 3x - 4$. We asked them what questions or difficulties they expected students to have and how they would help students to interpret a "solution" of $0 = 0$.

Ms. Lewis explained, "I try to make them think of *the same* as being identity, and *different* as being no solution. … so that's the way that I try to make them understand that … They know that if my variable cancels, that my variable's gone, that it's one of two things. I look at what's left. If I have, … numbers that are same, it's 'identity', or numbers that are different, it's 'no solution'." When asked whether students understand that for an identity, x could be any number, she replied that they do not; "that's higher order thinking."

Ms. Alley explained, "They'll think because it doesn't have 'x = something' that it implies no solution. And that's why … I *show* them on their graphing calculator. I'm like, 'well graph it, and … see what it comes out to when you're talking about functions; … put this as one, and put this as another, and see it's the same line, and that means that it's always true.' So, if they say 'always true' or 'never true'—I try not to say 'no solution' and 'infinite', but that's just the mathematician in me—so, what I should say: 'always true' and 'not true'."

When we described using a graphing calculator to create a table and/or graph of both sides of the equation to help students understand that x could be any number, Ms. Lewis said,

> That could be taking it a step higher, to make them understand that it could be any number. Because I think that's the big picture. … I think I concentrate so much on the little things, that… the big picture just never comes into view. … You made a good point when you said that, I was

> like – oh, that would be wonderful to do – but, would they really understand that? Now while I'm looking at that on a calculator, yeah, I really do. Because, I mean, …when we get to…systems and the equations come out to be the same, they understand that it is one line, which both things are the same. I mean they understand that concept. But I never thought of doing it in that manner and putting it in the calculator. That would be a good idea. That would be a good idea. I never thought of that.

In a later discussion of a system of two linear equations in two variables, Ms. Alley said of the equation $4(x - 1) - x = 3x - 4$, "Usually these equations are far and few between. So, you might get one or two on a test, or three or four on a homework assignment. So it's just for them to know: hey, if I get 'something = something', it means either 'no solution' or 'infinite solutions'. So graphing, I wouldn't really emphasize graphing for [the one-variable equation], where I would for [the system of equations]."

Linear equations in one variable that are identities or contradictions challenge the isolate-the-variable meaning of solve and require thinking about solving as finding the value(s) of the variable that make the equation a true statement in order to make sense of the results obtained by the procedure for isolating the variable. Both teachers, following the approach taken in the textbook, avoid this complexity by teaching students to label these "special cases" as *identity* or *infinite solutions* or *no solution*.

On Solving Equations in Two Variables: Changing the Feel of the Game

When we asked Ms. Lewis how she thought about solving equations like $9x - 3y = 12$, she explained that equations in two variables are first introduced in Chapter 5 of the textbook, where students learn to solve for y and also to find values for x and y. We followed up by asking her if she found it difficult to explain to students what it means to solve for y, since she had earlier defined solving as finding the values of the variable that make the equation true. Even though she focuses on isolating the variable as the name of the game in solving, she responded that this is a difficulty for students and further explained,

> …when you say solve, they're looking for *one* answer, that x is equal to something or y is equal to something. And then when you say 'Solve for y'… So *I* made it a point, I never say 'solve for y'…if I say 'solve', I think that they would understand that I'm looking for *a* value for x and *a* value for y. But…I try not to say 'solve for y'…I would just say…'write this in the slope-intercept form'. I won't say 'solve for y'. Because when you say 'solve for y' they're gonna say '$y = 4$' or…something like that.

Ms. Alley focuses on finding values of x and y when discussing solving linear equations in two variables, and she heavily emphasizes thinking about this process as evaluating functions, whether explicitly or implicitly defined. Referring to the equations $y = 3x - 4$ and $9x - 3y = 12$, Ms. Alley explained, "I would use function boxes where I would have an x and y, an input and output." She contrasted this way of thinking to how she thinks about solving equations like $3x - 4 = 12$, which she would "solve as a two-step equation." She went on to clarify, "Or if I had a system of equations I could solve for an x and a y."

When asked how she explains to students what "solve" means when they're asked to "solve for y", Ms. Alley responded, " …The solving for x and y's, and you're given an input, you have to find an ouput, …that's more of like a function rule, a function box, an in-and-out; whereas solving in that case is you're solving for *a* variable." Several times during the interview she referred to solving an equation in two variables for a particular variable as "transforming" the equation into a function; therefore, Ms. Alley, like Ms. Lewis, may avoid using the language "solve for y".

Even though solving for a given variable in a two-variable equation is procedurally the same as isolating the variable in a one-variable equation, both teachers find that students encounter difficulty because their "answer" isn't "an answer." Even though both teachers characterize solving equations in one variable as isolating a variable, students still are focused on solving as producing a numerical solution.

Conclusions

In this paper, we describe how two teachers grapple with a problem that every teacher has to confront. Every teacher needs to figure out how to roll out equations to students—do they deal with equations in one variable first, or functions in one variable (defined by equations in two variables)? When teaching equations in one variable, do they start with equations of the form $ax + b = cx + d$, or build up to that? And, when do they introduce the idea that not all equations can be solved?

Both Ms. Alley and Ms. Lewis have many strengths; both teachers know the school algebra curriculum well; both teachers revealed a sensitivity to student difficulties in learning to solve equations; and both teachers respond to these observed difficulties through their teaching in the best ways they know how. And they do this within a particularly challenging context in which to teach.

Ms. Alley and Ms. Lewis use similar strategies; they carefully build up from less complicated equations to more complicated equations. And, they trim the complexity of the mathematics to aid their students; for example, they intentionally do not explore equations that are not amenable to closed form solutions.

Yet, both teachers continue to find that their students struggle to solve even "simple" equations. Considering the transitions in the curriculum offers some insight into students' difficulties. After a brief introduction to equations that define functions, students spend nearly half of the school year solving linear equations in one variable. After the initial definition of solving as finding the value of the variable that makes the equation true, students are told that solving is operating on both sides of an equation to isolate the variable. Sometimes they are instructed to check their answers and reminded that this involves plugging their answer in for the variable in the (original) equation. When they meet identities or inconsistent equations, students are given labels to remember to assign to particular results when the variable is "lost" or "disappears". Then, late in the year, students are taught to solve equations in two variables for a single variable. Yet, while such solving also involves isolating a variable, this variable is no longer identified with a particular number.

By the time students begin focused study of equations in two variables (in Chapter 5), students have learned that solve means to isolate the variable, and they have observed that this has always, with rare exception, yielded a unique solution. One might expect that early and sustained emphasis on solving as isolating the variable would ease the transition to solving equations in two variables for a particular variable; however, students have come to expect that this process will produce a unique numerical solution. So while the procedure for isolating the variable may seem familiar to students, their teachers must help them transition to the idea that solving, in the isolate the variable sense of the word, will not always yield a numerical solution.

Both teachers recognize the difficulties their students have when facing these transitions. Both are knowledgeable in the school algebra curriculum and its teaching, and they employ their knowledge in finding ways to ease the transition from equations in one variable to equations in two variables for their students. For example, Ms. Alley utilizes a graphical representation to initially help students "see" that some equations in one variable have no solutions and others have infinitely many. In the context of solving equations in two variables for a particular variable, both teachers attempt to alleviate students' discomfort with obtaining results that do not fit what they have come to expect from solving, in the isolate the variable sense of the word, by using more descriptive commands that more clearly indicate the desired form of the result. For example, rather than use the standard instruction "solve for" that appears in their text, Ms. Lewis instructs students to "write this [equation] in the slope-intercept form," and Ms. Alley describes solving in this context as "transform[ing] the equation into a function".

These decisions are places where one can find teachers' knowledge in action in teaching. Teachers must make many choices about how and when to introduce concepts and skills and what to emphasize; teachers have to decide when to introduce complexity and when to remove it. Fully weighing the potential short- and long-term payoffs, as well as tradeoffs, of such instructional decisions requires not only knowledge of the procedures and definitions of algebra, but also knowledge of various representations and the connections among them, of alternative approaches to teaching and learning, and of more advanced mathematics and its requisite skills, concepts, and habits of mind, as well as knowledge of one's students. Perhaps in the context of such work, one can find the kinds of knowledge of mathematics uniquely the province of teachers, rather than mathematicians or engineers.

References

Bellman, A., S. Chavis Bragg, S. Chapin, T. Gardella, B. Hall, W. Handlin, Sr., and Manfre, E. (1998). *Algebra: Tools for a Changing World*. Englewood Cliffs, NJ: Pearson Prentice Hall.

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education 24*(2): 94-116.

Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures*. Dordrecht: D. Reidel Publishing company.

Herscovics, N. and Kieran, C. (1980). Constructing meaning for the concept of equation. *Mathematics Teacher, 73*(8): 572-580.

Lakatos, I. (1976). *Proof and Refutations: The Logic of Mathematical Discovery, ed. J. Worrall and E. Zahar.* Cambridge: Cambridge University.

McConnell, J., S. Brown, S. Eddins, M. Hackwarth, L. Sachs, Woodward., E., J. Flanders, D. Hirschhorn, C. Hynes, L. Polonsky, and Usiskin, Z. (1990). *The University of Chicago School Mathematics Project Algebra*. Glenview, IL: Scott, Foresman and Company.

Nathan, M. and Koedinger, K. (2000) An investigation of teachers' beliefs of students' algebra development. *Cognition and Instruction. 18*(2), 207-235.

Sierpinska, A. (1992). *Understanding in mathematics.* London: Falmer.

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In *The ideas of algebra* (pp. 8-19). A. F. Coxford and A. P. Schulte (Eds.). Reston, VA: National Council of Teachers of Mathematics.

Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematics Education and Science Technology, 14*(3): 293-305.

Vinner, S. and Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*(4): 356-366.