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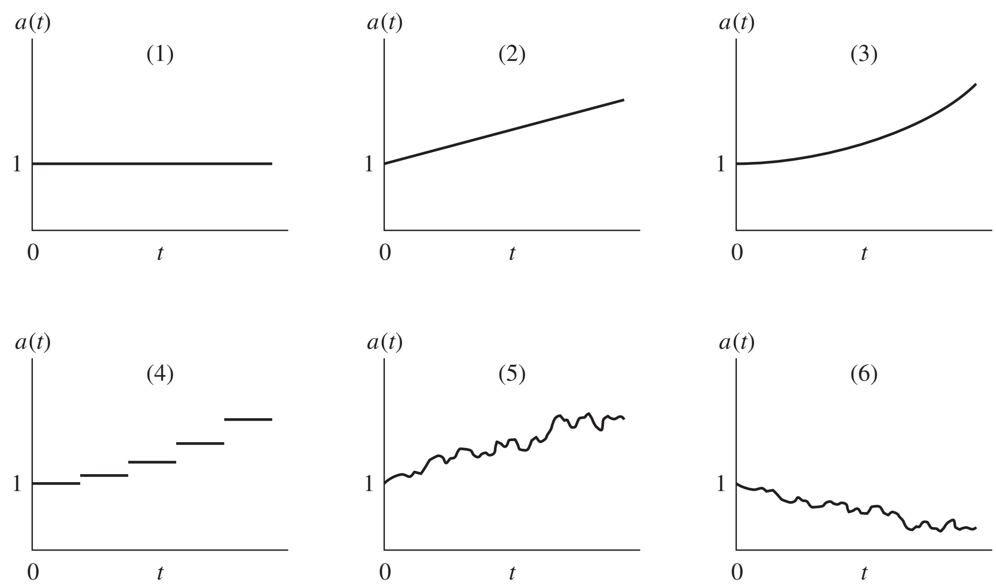
The Measurement of Interest

§ 1a. Basic Concepts

§ 1a(i) Accumulation Function and Effective Rate of Interest

Imagine a fund growing at interest. It would be very convenient to have a function representing the amount in the fund at any time t. The function $a(t)$ is defined as the accumulated value (AV) of the fund at time t of an initial investment of \$1.00 at time 0. $a(t)$ is called the "accumulation function."

Consider the following accumulation functions. Can you think of any real-life situations where you might encounter them?



(1) This is the accumulation function for money put in a piggy bank or under the mattress. It might also represent "a friendly loan from my father-in-law" or a checking account where you get no interest.

(2) This is so-called "simple interest," where the accumulation function is linear. But we will see that "simple interest" isn't so simple at all.

(3) This accumulation function is an exponential. As we shall see, this is referred to as "compound interest," where the fund earns interest on the interest at a constant rate.

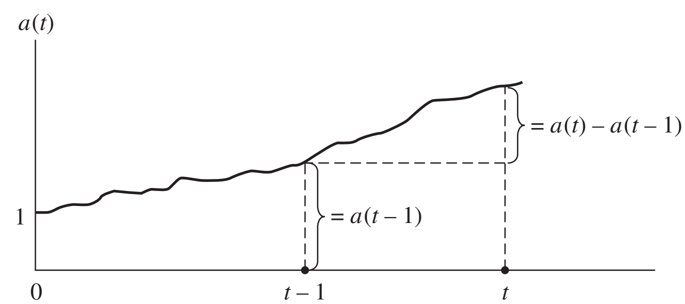
(4) This is the accumulation function for an account where you are credited with interest only at the end of each interest period.

(5) This is an accumulation function for a fund that grows at a varying rate. An investment in stocks is often given as an example.

(6) This is the stock you bought last month.

What are the properties of an accumulation function? By definition, $a(0) \equiv 1$. Other than that, anything goes. But the accumulation functions that we will generally deal with in this course will also have the properties of being (1) continuous and (2) increasing.

Suppose we want to measure the rate of growth of a fund in, say, the t^{th} year. Let's say that the accumulation function looks like the following graph in the t^{th} year. (Remember that the t^{th} year runs from time $(t-1)$ to time t, just as the first year runs from time 0 to time 1.)



The *amount* of growth in the t^{th} year (i.e., the interest earned) is $a(t) - a(t-1)$. The *rate* of growth (based on the amount in the fund at the beginning of the year) is:

$$\frac{a(t) - a(t-1)}{a(t-1)}$$

(Note that the amount in the fund at the beginning of the t^{th} year is the same as the amount in the fund at the end of the $(t-1)^{st}$ year, namely, $a(t-1)$, as long as no new investments are made.) This rate of growth is called the "effective rate of interest" and has the symbol i_t. So we have:

$$i_t = \frac{a(t) - a(t-1)}{a(t-1)}$$

We will also define an "amount function," $A(t)$, as the AV at time t of k invested at time 0 (rather than 1 invested at time 0). Obviously, $A(t) = ka(t)$. Mathematically, we need only one of these two functions, but having both can be handy. Given the definition of $A(t)$, the effective rate of interest can also be defined as:

$$i_t = \frac{A(t) - A(t-1)}{A(t-1)}$$

because substituting $A(t) = ka(t)$ and $A(t-1) = ka(t-1)$, the k's cancel and we're right back to the original definition.

Using this version of the definition of i_t, go ahead and solve for $A(t)$. You should get the following:

$$A(t) = A(t-1) + i_t A(t-1)$$

or $A(t) = (1 + i_t)A(t-1)$

How would you explain this?

The answer is as follows: The fund at the end of the t^{th} year is equal to the fund at the beginning of the year plus the interest earned during the year.

Note that the interest earned during the year is the effective rate of interest multiplied by the fund at the beginning of the year. This is consistent with the definition of the effective rate of interest.

EXAMPLE 1 (ADAPT)
You are given the following amount function: $A(t) = t^2 + 100$. Determine the amount of interest earned in the 5th year.

(A) 7 (B) 8 (C) 9 (D) 10 (E) 11

SOLUTION
Interest earned in 5th year $= A(5) - A(4) = 125 - 116 = 9$. ANS. **(C)**

EXAMPLE 2 (ADAPT)
You are given the following amount function: $A(t) = t^2 - 5t + 20$. Determine the effective rate of interest in the 5th year.

(A) 0.5% (B) 5% (C) 20% (D) 25% (E) 30%

SOLUTION
$i_5 = [A(5) - A(4)]/A(4) = (20 - 16)/16 = 25\%$ ANS. **(D)**

EXAMPLE 3 (ADAPT)
You are given the following amount function: $A(t) = t^2 + 100$. Calculate $a(10)$.

(A) 1.3 (B) 1.9 (C) 2.0 (D) 2.3 (E) 3.5

SOLUTION

$$A(t) = k \cdot a(t)$$

$$A(0) = 100 = k \cdot a(0)$$

By definition, $a(0) = 1$, so $k = 100$

$$a(t) = t^2/100 + 1$$

$$a(10) = 10^2/100 + 1 = 2 \qquad \text{ANS. } \textbf{(C)}$$

§ 1a(ii) Simple Interest

Let's consider so-called "simple interest." What are the effective rates of interest for years 1, 2, 3, . . . , t? Under simple interest, the accumulation function is linear. We will use i as the linear constant:

$$a(t) = 1 + it$$

We have the following by definition of i_t:

$$i_1 = \frac{a(1) - a(0)}{a(0)} = \frac{(1+i) - 1}{1} = i$$

$$i_2 = \frac{a(2) - a(1)}{a(1)} = \frac{(1+2i) - (1+i)}{1+i} = \frac{i}{1+i}$$

$$i_3 = \frac{a(3) - a(2)}{a(2)} = \frac{(1+3i) - (1+2i)}{1+2i} = \frac{i}{1+2i}$$

$$\dots$$

$$i_t = \frac{a(t) - a(t-1)}{a(t-1)} = \frac{(1+it) - [1 + i(t-1)]}{1 + i(t-1)} = \frac{i}{1 + i(t-1)}$$

For each year, the numerator (the interest earned) is a constant i, since $a(t)$ is linear. The denominator (the amount in the fund at the beginning of the year) is increasing, which means that the effective rate of interest decreases with time.

So we see that "simple interest" isn't simple at all. A linear accumulation function implies a decreasing effective rate of interest. (In fact, it decreases hyperbolically!)

In financial transactions, simple interest is often used for fractions of an interest period for convenience. (It is a bit confusing to use i as the linear constant, since it is the effective rate of interest for the first year only. It would be better to use another constant, as in $a(t) = 1 + kt$, but the general practice is to use i.)

EXAMPLE 1 (ADAPT)

Elizabeth and Josh both invest 10,000 in a bank over the period from $t = 0$ to $t = x$ at an annual rate of $i\%$, where $i > 0$. Elizabeth's bank credits interest on a simple interest basis.

Josh's bank credits interest on a compound interest basis. Which of the following statements is true?

(A) Elizabeth's savings will always be greater than Josh's savings.
(B) Elizabeth's savings will always be less than Josh's savings.
(C) The value of i must be specified to determine which account is greater.
(D) The value of x must be specified to determine which account is greater.
(E) The correct answer is not given by (A), (B), (C) or (D).

SOLUTION

Elizabeth's amount function at time x is $A(x) = 10,000(1 + ix)$.

Josh's amount function at time x is $A(x) = 10,000(1 + i)^x$.

Note that Elizabeth's amount function is a straight line, while Josh's is an exponential. The straight line is above the exponential until they intersect at $x = 1$. Thus, given that $i > 0$:

Elizabeth's account will be greater than Josh's for $x < 1$.

The accounts will be equal for $x = 1$.

Josh's account will be greater than Elizabeth's for $x > 1$. ANS. **(D)**

§ 1a(iii) The Accumulation Function in Terms of i_t

Suppose we are given that the effective rate of interest is 5% in the first year and 6% in the second year. We invest \$1.00 at time 0. How much is in the fund at the end of one year? Obviously, \$1.05. How much is in the fund at the end of two years? We start with \$1.05 at the beginning of the second year and we earn interest on it at 6%, so:

$$a(2) = 1.05 + .06(1.05)$$

or $a(2) = (1.05)(1.06)$

In symbols:

$$a(2) = (1 + i_1)(1 + i_2)$$

If we continue this process for t years, we have:

$$a(t) = (1 + i_1)(1 + i_2) \ldots (1 + i_t)$$

or using the symbol $\prod$ for product:

$$a(t)=\prod_{j=1}^{t}(1+i_j)$$

Note that each factor in this chain multiplication has the effect of (a) bringing the fund at the beginning of the year to the end of the year without interest (this is the "1" in $(1+i_j)$) and (b) adding interest on this amount at the effective rate (this is the "i_j" in $(1+i_j)$).

§ 1a(iv) Compound Interest

Let's take the case where i_j is a constant i. The chain product collapses into:

$$a(t)=(1+i)^t$$

This special case is called "compound interest." (Any chain multiplication, even where the i_j's are not constant, involves earning interest on the previous interest, or "compounding." But you will often see the term "compound interest" used for the case of a constant effective rate.)

§ 1a(v) Present Value

So far, we've been talking about the AV of a fund, i.e., how much is in the fund after t years, if we invest a given amount today. Consider the "opposite" question: How much should we invest today in order to have a given amount, say \$1.00, at the end of t years? The amount that we should invest is called the ***present value*** (PV) of \$1.00 due in t years.

Why would we be interested in present values? Isn't the real-life question, "How much is in a fund at a time t if we invest a given amount today?"

Suppose that in your personal financial life, you need to save or invest money that will grow to a specified amount in a specified number of years. For example, you may want to start saving now to buy a car for \$20,000 in two years, or to make a down payment of \$25,000 on a home in 3 years, or to pay a year's college expenses of \$30,000 in 8 years. All of these questions involve finding the PV, i.e., the amount invested today that will grow to the desired amount in the desired time.

Also, in your work as an actuary, you will often have to determine how much someone should pay now in order to receive future benefits, such as death benefits under a life insurance policy, or lifetime income under an annuity contract or pension plan. (The payments, or "investments," and the benefits, or "accumulated values," may be paid in installments in many cases. Also, these examples involve contingencies, i.e., probabilities of future events. For this exam, we are only concerned with payments that are certain to be made.) So you can see that the concept of present value is *very important* in actuarial work.

Consider the general accumulation function $a(t)$. We want to determine how much to invest today in order to have \$1.00 in t years. We will designate this amount as (PV). Since we require that (PV) grows to \$1.00 in t years, we have:

$$(PV)a(t)=1$$

$$(PV)=\frac{1}{a(t)}$$

The function $\frac{1}{a(t)}$ is called the "discount function." As we will see in the next section, "discounting" a future amount means the same thing as finding its present value. In particular, if $a(t)=(1+i)^t$ (the compound interest case), we have:

$$(PV) = \frac{1}{a(t)} = \frac{1}{(1+i)^t} = (1+i)^{-t}$$

Because the term $(1+i)^{-1}$ comes up so often (i.e., the PV of $1.00 due in a year), a special symbol has been invented for it, namely, v. So we have:

$$v = \frac{1}{1+i} = (1+i)^{-1}$$

(PV) of $1.00 due in t years$= v^t = (1+i)^{-t}$.

EXAMPLE 1

What deposit made today will provide for a payment of $1,000 in 1 year and $2,000 in 3 years, if the effective rate of interest is 7.5%?

SOLUTION

$$PV = 1,000v + 2,000v^3 \text{ at } 7.5\% = \$2,540.15$$

$$(v = 1.075^{-1})$$

Calculator Notes

It is essential that you use a financial calculator to get numerical results. As we noted in the Introduction, Calculator Notes for the BA II Plus will appear at appropriate points in this manual. If you are using another calculator, do the problems in these notes anyway, and check that you got the same answers.

Calculator Notes #1 begins on the following page.

Calculator Notes #1: Formatting; Present Values and Future Values

"Calculator Notes" will appear at appropriate points in this manual. These notes will cover the functions of the Texas Instruments BA II Plus calculator that are essential for Exam FM/2. (As noted in the Introduction, the BA-35 was discontinued by TI in 2005, although it is still an authorized calculator for the actuarial exams.)

These notes will not cover functions that you don't need for the exam (for example, the trig, statistical or depreciation functions), or that are extremely unlikely to come up on the exam (for example, the $\boxed{\text{CF}}$, $\boxed{\text{NPV}}$ and $\boxed{\text{IRR}}$ keys for handling irregular cash flows).

In addition to the Calculator Notes in this manual, you should print out the official Study Note for the BA II Plus written by Sam Broverman and available on the SOA or CAS website. Also, you should have the TI guidebook as a general reference. If you don't have it, you can download it from the TI website.

Primary and Secondary Functions

The calculator has primary and secondary functions. The primary functions are printed on the keys themselves, for example, $\boxed{\times}$ for "multiply." The secondary functions are shown in yellow *above* the keys. For example, the secondary function of the $\boxed{\times}$ key is x!.

The secondary functions are accessed by pressing the $\boxed{\text{2nd}}$ key (which is the only yellow key) and then the desired key. For example, to compute 5!, use the following sequence of keystrokes: 5 $\boxed{\text{2nd}}$ [x!]. You should get 120.

Note that we have indicated the primary function $\boxed{\times}$ by using a box and the secondary function [x!] by using brackets.

Formatting the Calculator

The calculator was pre-set to certain defaults when it left the factory, but these settings may have been changed since then. Let's format the calculator so that we can all start from the same place.

To access the Format mode, press $\boxed{\text{2nd}}$ [FORMAT]. (The [FORMAT] key is the middle key in the bottom row.)

You can scroll the various formats by pressing the $\boxed{\downarrow}$ or $\boxed{\uparrow}$ keys in the top row. Scrolling down, note that the formats are "DEC," "DEG," "US" (for dates), "US" (for number separation), "Chn," and back to "DEC." We are only interested in a couple of these formats for Exam FM/2.

Scroll to "DEC." This controls the number of decimal places shown in the display. The factory default is two decimal places but you can set the calculator to show from 0 to 9 places. (The calculator uses 13 places internally, regardless of what is shown in the display.)

Let's say we want the display to show 4 decimal places. While "DEC" is in the display, press 4 $\boxed{\text{ENTER}}$. Let's check that this has worked. Leave the [FORMAT] mode by pressing $\boxed{\text{2nd}}$ [QUIT]. (The [QUIT] key is in the upper left-hand corner.) This puts us in the standard calculator mode. Now multiply two numbers, such as 2.6835 $\boxed{\times}$ 5.625 $\boxed{=}$. The answer shows to 4 decimals (rounded) as 15.0947.

Leave your calculator set to display 4 decimals. This is enough accuracy for most questions on the exam. (Remember that internally, 13 places are carried.) But you can set it for fewer or more places if you wish. It's a matter of personal preference: After you've used the calculator for awhile, you can decide for yourself.

Let's return to the Format mode by pressing 2nd [FORMAT] again. Now scroll down to "DEG." This controls whether angles are expressed in degrees or radians, which doesn't concern us for this exam. So we scroll down to "US 12-31-1990." This controls how dates are shown (i.e., whether month or day comes first). We're not going to be using dates very much, so scroll down to "US 1,000.0000." This controls how numbers are separated. For example, in US format, "one thousand" with 4 decimals is shown as 1,000.0000. In European format, commas become decimal points and decimal points become commas, so "one thousand" is shown as 1.000,0000. You should set the calculator in US format. In case it's in European format (shown as "EUR"), you can change it as follows: 2nd [SET]. (The [SET] key is secondary to the ENTER key.) If you keep pressing 2nd [SET], the display will alternate between "US" and "EUR." Leave it at "US."

Finally, scroll down to "Chn." This controls the calculation method, i.e., either "Chain" or "AOS" (Algebraic Operating System). To alternate between "Chn" and "AOS," press 2nd [SET]. Leave the setting at "Chn."

What's the difference between the Chain and AOS calculation methods? The Chain method is probably what you are most familiar with in a calculator. For example, if you press 2 + 5 × 3 =, you expect the answer to be 21, i.e., the calculator first adds 5 to 2 and then multiples the result by 3. The AOS method follows the hierarchy of mathematical operations: multiplication and division are completed before addition and subtraction. Thus, for the same keystrokes in the AOS mode, the answer will be 17: The calculator multiples 5 by 3 first and then adds 2. You should leave the calculator in "Chn" mode.

You now have completed formatting the calculator. Press 2nd [QUIT] to return to standard calculator mode.

Accumulated Values and Present Values

1. What is the accumulated value of $1,537 at the end of 8 years at an effective annual interest rate of 7%?

There are two ways of computing this:

a. Using the regular keys

The exponential key y^x is located in the 4th column, 5 keys down. You'll be using it a lot, so get to know where it is.

The answer to this question is $1{,}537(1.07)^8$. Use these keystrokes:

1.07 y^x 8 × 1,537 =

The display should show 2,640.8522.

Note that it is not necessary to press = after entering 1.07 y^x 8. When you press ×, the exponentiation is automatically done and the result is ready to be multiplied by the next value you enter. (Some guides include this equal sign in the keystrokes. This doesn't affect the final result, but it's not necessary.)

b. Using the TVM keys

"TVM" stands for "time value of money." This simply means that $1.00 in your greedy little hand today is worth more than $1.00 paid to you at a future time. (See the first page of Chapter 2 of this manual.)

The TVM keys are the keys in the 3rd row: N, I/Y, PV, PMT and FV. For simple accumulation and present value problems, it really isn't necessary to use these keys: It's easy and direct to use the y^x key, as we did above. However, the TVM keys are very important when it comes to handling annuities, loans, mortgages, bonds, etc. (covered in Chapter 3 and later), so we may as well get familiar with them now.

N is the number of years (or periods).
I/Y is the interest rate per year (or per period).
PV is the present value.
PMT is the amount of each payment of an annuity. (We won't use this key until Chapter 3.)
FV is the future value. (This is the same as what we have been calling the accumulated value, or AV.)

There are two very important "hidden" settings that affect the TVM calculations. Press 2nd [P/Y] ([P/Y] is secondary to the I/Y key) and the display will show "P/Y = ." The default is 12, so this may be the setting in your display. "P/Y" stands for the number of payments per year. Now press the arrow key ↓. "C/Y = " will show in the display. "C/Y" stands for the number of times that interest is compounded per year. The default is also 12.

We strongly recommend that you set P/Y equal to 1 and always keep it at 1.[1]

This is because it's very easy to forget what value is in the P/Y register. If you know that it's always set to 1, you are much less likely to make calculation errors. (This will become clearer when we get to Chapter 4.)

This is how to set P/Y equal to 1: Press 2nd [P/Y], then press 1 ENTER. C/Y will also automatically be set to 1. (You can check this by pressing ↓.)

To get back to the standard calculation mode, press 2nd [QUIT].

Trap Alert!

When you take the actual exam, the proctor will probably reset the calculator to its factory defaults by pressing 2nd [RESET]. (The [RESET] key is secondary to the +/− key.) If you want to keep [P/Y] = [C/Y] = 1, you will have to set them again as described above. Also, resetting the calculator will set two decimal places to show in the display. If you want to display a different number of places, you will have to set that as well.

Now we're ready to do problem 1. We have to enter the values of N, I/Y and PV, and then compute FV. But first make sure that you clear any current values in the TVM registers by pressing 2nd [CLR TVM]. ([CLR TVM] is secondary to the FV key. Pressing it will *not* change the settings of P/Y and C/Y.) Then use these keystrokes:

8 N 7 I/Y 1,537 +/− PV CPT FV

The answer should appear as 2,640.8522, which agrees with the previous answer. (Pressing the +/− key, which is in the bottom row, changes the sign of the value in the display. We will explain why we enter −1,537 as the PV in the comments below.)

Some important comments:

- The data can be entered in any order in the TVM keys. For example, we could have entered the interest rate first, then the number of years and then the PV. If we then press CPT FV, we would get the same answer as above.
- The effective rate of interest is entered as a percent in I/Y, so 7% is entered as 7, not .07. Similarly, a rate like 3.4562% would be entered as 3.4562.
- After we enter the data, we compute the unknown by pressing CPT and then the value that we want, in this case FV.

[1] C/Y will automatically be set to whatever P/Y is set to. It can also be set independently of P/Y.

- Here is an explanation of why we enter the PV as a negative amount: We will take the point of view of a depositor, a lender, etc., i.e., someone who has a cash flow *out* in the amount of 1,537 at time 0. At the end of 8 years, we expect to receive a cash flow *in*, as a withdrawal from the bank, the repayment of the loan by the borrower, etc. The calculator expects us to enter a cash flow *out* as a negative value. When we press CPT FV, the calculator computes the cash flow *in* that we receive at the end of 8 years as a positive value.

  Of course, we could reverse roles. If we take the point of view of the bank, the borrower, etc., 1,537 is the amount we *receive* at time 0, i.e., it is a cash flow *in,* and we would enter it as a positive value. When we press CPT FV, the calculator computes FV as a negative amount, since it is a cash flow *out*, i.e., the amount the bank, the borrower, etc., repays the investor.

  It really doesn't matter which point of view we take, as long as we are consistent within the same problem. (In Calculator Notes, we will sometimes take one point of view and sometimes the other. If you happen to take the other point of view, our answers will have opposite signs.)
- Sometimes you may forget what values are in the TVM registers. You can check a register by pressing RCL and then the register you want. Try this now.
- If there is an incorrect value in a register, simply enter the correct value in the display and then press the register key.
- To check the value currently in the P/Y register, press RCL 2nd [P/Y]. For the value of C/Y, press ↓.
- It's essential to clear the TVM registers before you start a new problem. You do this by pressing 2nd [CLR TVM]. *Even if you turn the calculator off, the "garbage" that you left in TVM will still be there when you turn the calculator back on! Try it and see! So you must clear these registers for each new problem.*

Now that you have done this problem using the TVM keys, and have read all of the comments, we wouldn't blame you for asking, "Why go through all of this when we could simply use the y^x key?" The answer is that, as we noted before, you will be using TVM *big time* starting with Chapter 3, so you may as well get used to it now. With a little practice, using TVM will become second nature to you. Also, some problems, like finding an unknown interest rate or an unknown time are easier to do using the TVM keys.

Now let's continue using the calculator to solve problems.

2. What is the PV of $1,250 due in 8 years at an effective annual interest rate of 5.23%?

a. Using the regular keys

1.0523 y^x 8 +/− × 1,250 =

The answer is $831.37 to the nearest $.01.

b. Using the TVM keys

Make sure that you have kept P/Y and C/Y equal to 1, as we recommended.

2nd [CLR TVM] (to clear the TVM registers)

8 N 5.23 I/Y 1,250 FV CPT PV

The answer is −831.37. If you had entered 1,250 as a negative, the answer would show as a positive value.

Unknown interest rate

3. $10,250 accumulates to $23,237 in 13 years. What is the effective annual interest rate?

a. Using the regular keys

The equation of value for this problem is:

$10{,}250(1+i)^{13} = 23{,}237$

Solving for i:

$i = \left(\frac{23{,}237}{10{,}250}\right)^{1/13} - 1$

Note: The calculator has 10 storage registers numbered 0 to 9. To store a value that appears in the display, let's say in register 0, simply press $\boxed{\text{STO}}$ 0. The value will be stored in register 0 and will also continue to be shown in the display. If you want to recall this value later, press $\boxed{\text{RCL}}$ 0 and it will be shown in the display. It will also remain in the register.

We can use the following keystrokes:

13 $\boxed{\text{1/x}}$ $\boxed{\text{STO}}$ 0 23,237 $\boxed{\div}$ 10,250 $\boxed{y^x}$ $\boxed{\text{RCL}}$ 0 $\boxed{-}$ 1 $\boxed{=}$

The answer is .0650 or 6.50%.

Note that we computed 1/13 as the reciprocal of 13, stored it, and later recalled it to use as the exponent in the $\boxed{y^x}$ operation.

b. Using the TVM keys

$\boxed{\text{2nd}}$ [CLR TVM] *(From now on, we will not show this step. We'll assume that you do it at the beginning of each new problem.)*

13 $\boxed{\text{N}}$ 10,250 $\boxed{+/-}$ $\boxed{\text{PV}}$ 23,237 $\boxed{\text{FV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{I/Y}}$

The answer is 6.50% to 2 decimals, as before.

Unknown time

4. At an effective rate of interest of 8% per annum, the present value of \$100,000 due in X years is \$65,322. Determine X.

a. Using the regular keys

The equation of value is $65{,}322(1.08)^X = 100{,}000$. Solving for X:

$1.08^X = \frac{100{,}000}{65{,}322}$

$X = \frac{\ln\left(\frac{100{,}000}{65{,}322}\right)}{\ln 1.08}$

The natural log key $\boxed{\text{LN}}$ is located in the first column of keys. We can use the following keystrokes:

1.08 $\boxed{\text{LN}}$ $\boxed{\text{STO}}$ 0 100,000 $\boxed{\div}$ 65,322 $\boxed{=}$ $\boxed{\text{LN}}$ $\boxed{\div}$ $\boxed{\text{RCL}}$ 0 $\boxed{=}$

The answer is 5.5332 years.

b. Using the TVM keys

8 $\boxed{\text{I/Y}}$ 65,322 $\boxed{+/-}$ $\boxed{\text{PV}}$ 100,000 $\boxed{\text{FV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{N}}$

The answer is $N = 5.5332$ years.

5. How long does it take money to double at 5% effective?

a. Using the regular keys

$1.05^x = 2$

$x = \frac{\ln 2}{\ln 1.05}$

1.05 [LN] [STO] 0 2 [LN] [÷] [RCL] 0 [=]

The answer is 14.2067 years.

b. Using the TVM keys

5 [I/Y] 1 [+/−] [PV] 2 [FV] [CPT] [N]

The answer is N = 14.2067 years.

Summary of Concepts and Formulas in Sections 1a(i) to 1a(v)

(1) The accumulation function $a(t)$ is the AV at time t of \$1.00 invested at time 0.

(2) The effective rate of interest in the t^{th} year is based on the amount in the fund at the beginning of the year:

$$i_t = \frac{a(t) - a(t-1)}{a(t-1)}$$

(3) Under "simple interest":

$$a(t) = 1 + it$$

$$i_t = \frac{i}{1 + i(t-1)}$$

(4) In terms of effective rates of interest:

$$a(t) = (1 + i_1)(1 + i_2) \cdots (1 + i_t)$$

(5) Under "compound interest," the effective rate of interest is a constant in all years:

$$i_t = i \quad \text{and} \quad a(t) = (1+i)^t$$

(6) (a) For any accumulation function $a(t)$, the present value of \$1.00 due in t years is:

$$PV = \frac{1}{a(t)}$$

(b) Under compound interest:

$$PV = \frac{1}{(1+i)^t} = (1+i)^{-t} = v^t$$

Past Exam Questions on Sections 1a(i) to 1a(v)

Note: Only the *first five* questions on this section are included below.

1. Sally has two IRA's. IRA #1 earns interest at 8% effective annually and IRA #2 earns interest at 10% effective annually. She has not made any contributions since January 1, 1985, when the amount in IRA #1 was twice the amount in IRA #2. The sum of the two accounts on January 1, 1993 was $75,000. Determine how much was in IRA #2 on January 1, 1985. [CAS 5/93 #6]

(A) <$12,750 (B) $12,750/$13,000 (C) $13,000/$13,250 (D) $13,250/$13,500 (E) ≥ $13,500

2. Money accumulates in a fund at an effective annual interest rate of i during the first 5 years, and at an effective annual interest rate of $2i$ thereafter.

A deposit of 1 is made into the fund at time 0. It accumulates to 3.09 at the end of 10 years and to 13.62 at the end of 20 years.

What is the value of the deposit at the end of 7 years? [SOA 5/93 #2]

(A) 1.90 (B) 1.98 (C) 2.06 (D) 2.14 (E) 2.23

3. At an effective annual interest rate of i, $i > 0$, each of the following two sets of payments has present value K:

(i) A payment of 121 immediately and another payment of 121 at the end of one year.
(ii) A payment of 144 at the end of two years and another payment of 144 at the end of three years.

Calculate K. [SOA 11/92 #2]

(A) 237 (B) 232 (C) 227 (D) 222 (E) 217

4. A business permits its customers to pay with a credit card or to receive a percentage discount r for paying cash.

For credit card purchases, the business receives 95% of the purchase price one-half month later.

At an effective annual rate of 12%, the two payment methods are equivalent.

Determine r. [SOA 11/92 #3]

(A) 4.55 (B) 4.85 (C) 5.15 (D) 5.45 (E) 5.75

5. At an annual effective interest rate of i, $i > 0$, the following are all equal:

(i) the present value of 10000 at the end of 6 years;
(ii) the sum of the present values of 6000 at the end of year t and 56000 at the end of year $2t$; and
(iii) 5000 immediately.

Calculate the present value of a payment of 8000 at the end of year $t + 3$ using the same annual effective interest rate. [SOA 11/90 #4]

(A) 1330 (B) 1415 (C) 1600 (D) 1775 (E) 2000

Solutions to Past Exam Questions on Sections 1a(i) to 1a(v)

1. $2X(1.08)^8 + X(1.1)^8 = 75{,}000$

$$X = \frac{75{,}000}{(2)(1.08)^8 + 1.1^8} = \$12{,}830 \qquad \text{ANS. } \mathbf{(B)}$$

2. $(1+i)^5(1+2i)^5 = 3.09$

$(1+i)^5(1+2i)^{15} = 13.62$

Dividing the second equation by the first:

$$(1+2i)^{10} = \frac{13.62}{3.09} = 4.407767$$

$$i = 7.9952\%$$

$$X = (1.079952^5)(1.159904^2) = 1.98 \qquad \text{ANS. } \mathbf{(B)}$$

3. $121(1+v) = 144(v^2+v^3)$

$= 144v^2(1+v)$

$$\therefore\ 121 = 144v^2, \quad v^2 = \left(\frac{11}{12}\right)^2, \quad v = \frac{11}{12}$$

$$K = 121 + 121\left(\frac{11}{12}\right) = 231.9 \qquad \text{ANS. } \mathbf{(B)}$$

4. Assume that the purchase price is \$1.00, since the price will cancel out anyway. If cash is paid, the business receives $(1-r)$ now. If payment is by credit card, the business receives .95 one-half a month later. The PV of this payment is $.95v^{1/24}$ at 12% effective. Setting these PV's equal:

$$1 - r = .95v^{\frac{1}{24}}$$

$$r = 1 - .95v^{\frac{1}{24}} = 5.45\% \qquad \text{ANS. } \mathbf{(D)}$$

5. $(\text{i}) = 10{,}000v^6$

$(\text{ii}) = 6{,}000v^t + 56{,}000v^{2t}$

$(\text{iii}) = 5{,}000$

We are given that (i) = (ii) = (iii). Setting (ii) = (iii) and simplifying:

$$56v^{2t} + 6v^t - 5 = 0$$

This is a quadratic in v^t, which can be seen more clearly by setting $X = v^t$:

$$56X^2 + 6X - 5 = 0$$

This can be solved for X by using the quadratic formula or by factoring:

$$(14X+5)(4X-1) = 0$$

Taking the positive root:

$$4X - 1 = 0$$

$$X = 0.25 = v^t$$

Setting (i) = (iii):

$$10{,}000v^6 = 5{,}000$$
$$v^6 = 0.5, \text{ so } v^3 = 0.5^{1/2}$$

We want $8{,}000v^{t+3} = 8{,}000v^t v^3 = (8{,}000)(0.25)(0.5^{1/2}) = 1{,}414.$ ANS. **(B)**

3

Basic Annuities

§ 3g. The $a_{\overline{2n}|} \,/\, a_{\overline{n}|}$ Trick (and Variations)

Trick Alert!

One type of problem that has appeared on past exams a number of times can be solved by using the quotient $a_{\overline{2n}|}/a_{\overline{n}|}$, or variations on this theme. The following example illustrates the method:

EXAMPLE 1

A "plain vanilla" version of this type of problem is as follows: Given that $a_{\overline{n}|} = 10$ and $a_{\overline{2n}|} = 15$, determine i.

SOLUTION

A few lines further down, we will show that $a_{\overline{2n}|}/a_{\overline{n}|} = 1 + v^n$. Using this result, we have:

$$\frac{a_{\overline{2n}|}}{a_{\overline{n}|}} = \frac{15}{10} = 1.5 = 1 + v^n$$

This gives $v^n = 0.5$. All that remains is to plug v^n into the formula for $a_{\overline{n}|}$:

$$a_{\overline{n}|} = \frac{1 - v^n}{i} = \frac{1 - 0.5}{i} = 10$$

Thus, $i = 5\%$.

It is easy to show that $a_{\overline{2n}|}/a_{\overline{n}|} = 1 + v^n$. First, a purely algebraic approach:

$$\frac{a_{\overline{2n}|}}{a_{\overline{n}|}} = \frac{\frac{1-v^{2n}}{i}}{\frac{1-v^n}{i}} = \frac{1 - v^{2n}}{1 - v^n}$$

Now, $1 - v^{2n}$ is the difference between two squares, which can be factored into

$(1 + v^n)(1 - v^n)$. Substituting in the above:

$$\frac{a_{\overline{2n}|}}{a_{\overline{n}|}} = \frac{(1 + v^n)(1 - v^n)}{1 - v^n} = 1 + v^n$$

Another way to derive this result is by a general reasoning approach. Think of $a_{\overline{2n}|}$ as consisting of two annuities: (1) an immediate annuity for n years, followed by (2) a deferred annuity for another n years. The relationship is:

$$a_{\overline{2n}|} = a_{\overline{n}|} + v^n a_{\overline{n}|} = a_{\overline{n}|}(1 + v^n)$$

Hence $a_{\overline{2n}|}/a_{\overline{n}|} = 1 + v^n$.

It would be a good idea to know both of these approaches. It will help you to remember the above relationship and similar ones.

A couple of additional points:

(1) In Section 4e, we will show that "double dots cancel." This means that the quotient $\ddot{a}_{\overline{2n}|}/\ddot{a}_{\overline{n}|}$ is the same as the quotient $a_{\overline{2n}|}/a_{\overline{n}|}$:

$$\frac{\ddot{a}_{\overline{2n}|}}{\ddot{a}_{\overline{n}|}} = \frac{a_{\overline{2n}|}}{a_{\overline{n}|}} = 1 + v^n$$

(More advanced Chapter 4 annuities with "upper m's" in their symbols also have the same quotient. See Section 4e.)

(2) Formulas can also be derived for other annuities, for example, $a_{\overline{3n}|}/a_{\overline{n}|}$. Think of $a_{\overline{3n}|}$ as consisting of 3 annuities: an immediate annuity for n years, followed by a deferred annuity for n years, followed by another deferred annuity for n years. You should then be able to visualize the following relationship:

$$a_{\overline{3n}|} = a_{\overline{n}|} + v^n a_{\overline{n}|} + v^{2n} a_{\overline{n}|}$$
$$= a_{\overline{n}|}(1 + v^n + v^{2n})$$

so that $\frac{a_{\overline{3n}|}}{a_{\overline{n}|}} = 1 + v^n + v^{2n}$.

EXAMPLE 2 (ADAPT)

You are given that $a_{\overline{n}|} = 3.982$ and $a_{\overline{3n}|} = 8.507$. Determine i.

(A) 7.70% (B) 7.90% (C) 8.10% (D) 8.30% (E) 8.50%

SOLUTION

$$a_{\overline{3n}|} = a_{\overline{n}|}(1 + v^n + v^{2n})$$
$$8.507 = 3.982(1 + v^n + v^{2n})$$

We get the following quadratic in v^n:

$$v^{2n} + v^n - 1.1364 = 0$$

Solving for the positive root of v^n, we get:

$$v^n = .6775$$

Substitute this in $a_{\overline{n}|}$:

$$a_{\overline{n}|} = (1 - v^n)/i$$
$$3.982 = (1 - .6775)/i$$
$$i = (1 - .6775)/3.982 = .081$$

ANS. **(C)**

Summary of Concepts and Formulas in Section 3g

(1) $$\frac{a_{\overline{2n}|}}{a_{\overline{n}|}} = 1 + v^n$$

(2) This can be derived algebraically or by using the fact that the PV of a $2n$-year annuity is the sum of the PV of an n-year annuity and the PV of an n-year deferred n-year annuity:

$$\begin{aligned} a_{\overline{2n}|} &= a_{\overline{n}|} + v^n a_{\overline{n}|} \\ &= (1 + v^n) a_{\overline{n}|} \end{aligned}$$

(3) Similar formulas can be derived for $3n$-year annuities, etc. For example:

$$\begin{aligned} a_{\overline{3n}|} &= a_{\overline{n}|} + v^n a_{\overline{n}|} + v^{2n} a_{\overline{n}|} \\ &= (1 + v^n + v^{2n}) a_{\overline{n}|} \end{aligned}$$

(4) These formulas can be used to solve for i and n; for example, when numerical values are given for $a_{\overline{n}|}$ and $a_{\overline{2n}|}$.

Past Exam Questions on Section 3g

Note: Only the *first five* questions on this section are included below.

1. At an annual effective interest rate of i, $i > 0$, both of the following annuities have a present value of X:

(i) a 20-year annuity-immediate with annual payments of 55
(ii) a 30-year annuity-immediate with annual payments that pays 30 per year for the first 10 years, 60 per year for the second 10 years, and 90 per year for the final 10 years

Calculate X. [5/03 #33]

(A) 575 (B) 585 (C) 595 (D) 605 (E) 615

2. Dottie receives payments of X at the end of each year for n years. The present value of her annuity is 493.

Sam receives payments of $3X$ at the end of each year for $2n$ years. The present value of his annuity is 2748.

Both present values are calculated at the same annual effective interest rate.

Determine v^n. [SOA 5/98 #4]

(A) 0.86 (B) 0.87 (C) 0.88 (D) 0.89 (E) 0.90

3. An annuity pays 2 at the end of each year for 18 years.

Another annuity pays 2.5 at the end of each year for 9 years.

At an effective annual interest rate of i, $0 < i < 1$, the present values of both annuities are equal.

Calculate i. [SOA 11/92 #5]

(A) 14% (B) 17% (C) 20% (D) 23% (E) 26%

4. You are given:

(i) $a_{\overline{n}|} = 10.00$; and
(ii) $a_{\overline{3n}|} = 24.40$.

Determine $a_{\overline{4n}|}$. [SOA 11/90 #2]

(A) 28.74 (B) 29.00 (C) 29.26 (D) 29.52 (E) 29.78

5. Eric receives 12000 from a life insurance policy. He uses the fund to purchase two different annuities, each costing 6000.

The first annuity is a 24-year annuity-immediate paying K per year to himself. The second annuity is an 8-year annuity-immediate paying $2K$ per year to his son.

Both annuities are based on an annual effective interest rate of i, $i > 0$.

Determine i. [SOA 11/90 #7]

(A) 6.0% (B) 6.2% (C) 6.4% (D) 6.6% (E) 6.8%

Solutions to Past Exam Questions on Section 3g

1.

$$X = 55a_{\overline{20|}} = 30a_{\overline{10|}} + 60v^{10}a_{\overline{10|}} + 90v^{20}a_{\overline{10|}}$$

$$= a_{\overline{10|}}\left(30 + 60v^{10} + 90v^{20}\right)$$

Dividing by $a_{\overline{10|}}$, we have:

$$55\frac{a_{\overline{20|}}}{a_{\overline{10|}}} = 55\left(1 + v^{10}\right) = 30 + 60v^{10} + 90v^{20}$$

$$90v^{20} + 5v^{10} - 25 = 0$$

This is a quadratic in v^{10}:

$$v^{10} = \frac{-5 + \sqrt{25 - (4)(90)(-25)}}{180} \text{ (positive root)}$$

$$= \frac{-5 + \sqrt{9025}}{180} = \frac{90}{180} = 0.5$$

$$\therefore\ i = 7.177346\%$$

$$X = 55a_{\overline{20|}i} = 574.97$$ ANS. **(A)**

2.

Dottie: $Xa_{\overline{n|}} = 493$

Sam: $3Xa_{\overline{2n|}} = 2{,}748$

Dividing: $\dfrac{3Xa_{\overline{2n|}}}{Xa_{\overline{n|}}} = 3\left(1 + v^n\right) = \dfrac{2{,}748}{493}$

$$v^n = \frac{2{,}748}{(3)(493)} - 1 = .858$$ ANS. **(A)**

3.

$$2a_{\overline{18|}} = 2.5a_{\overline{9|}}$$

$$\frac{2a_{\overline{18|}}}{a_{\overline{9|}}} = 2\left(1 + v^9\right) = 2.5$$

$$v^9 = \frac{2.5}{2} - 1 = .25,\ \ i = 16.65\%$$ ANS. **(B)**

4. Dividing (ii) by (i):

$$\frac{a_{\overline{3n|}}}{a_{\overline{n|}}} = 1 + v^n + v^{2n} = \frac{24.40}{10} = 2.44$$

$$v^{2n} + v^n - 1.44 = 0,\ \ v^n = \frac{-1 + \sqrt{1 - (4)(-1.44)}}{2} = 0.8$$

$$a_{\overline{4n|}} = a_{\overline{n|}} + v^n a_{\overline{3n|}} = 10 + (0.8)(24.4) = 29.52$$ ANS. **(D)**

5. $Ka_{\overline{24|}} = 2Ka_{\overline{8|}}$; Dividing:

$$\frac{a_{\overline{24|}}}{a_{\overline{8|}}} = 2 = 1 + v^8 + v^{16}$$

$$v^{16} + v^8 - 1 = 0, \ \ v^8 = \frac{-1 + \sqrt{1 - (4)(-1)}}{2}$$

$$= \frac{\sqrt{5} - 1}{2} = .618034$$

$i = 6.2\%$ ANS. **(B)**

11

Forward Contracts

§ 11a. What is a Forward Contract?

Let's say that George and Sue want to speculate on the price of Stock X, which does not pay dividends. Sue thinks that the price will be higher than $104 in 6 months; George thinks that the price will be lower than $104. They make a contract that says that Sue is obligated to buy from George, and George is obligated to sell to Sue, 100 shares of this stock in 6 months at a price of $104, regardless of the actual market price of the stock at that time.

Obviously, if the actual market price in 6 months turns out to be more than $104 per share, say $114, Sue would be a winner. George would be obligated to sell to her at $104 per share, while the stock that Sue gets would be worth $114. She would make a profit of $10 per share. George would be a loser; he would have to buy the stock for $114 a share and sell it to Sue for only $104. He would lose $10 per share.

Of course, if the market price in 6 months turns out to be less than $104, say $94 per share, George would be a winner. He would buy the stock for only $94 and Sue would be obligated to buy it from George for $104. George would make a profit of $10 per share, while Sue would lose $10 per share.

This type of contract is called a ***forward contract***. You can see that a forward contract has the following key elements:

- The contract obligates one party to sell and the other party to buy a specified quantity of an asset. The asset on which the contract is based is called the *underlying asset*.
- The contract specifies the date on which the sale will take place. This date is called the ***expiration date***. (The contract may also specify the time, place, manner of delivery, etc., if appropriate.)
- The contract specifies the price that will be paid on the expiration date. This price is called the ***forward price***.

Under a forward contract, neither party pays anything to the other at the outset. Their obligation is to buy or sell the underlying asset on the expiration date.

§ 11b. The Long and Short of It

The party under the forward contract who is obligated to *buy* (Sue in the above example) is said to have a ***long position***. So we can call the contract from the buyer's point of view a ***long forward***. As we have seen, the long forward makes money if the price of the underlying asset *goes up*. (Generally speaking, we say that someone has a long position in an asset if they would make money if the price of the asset goes up.)

The party under the forward contract who is obligated to *sell* (George in the above example) is said to have a ***short position***. So we can call the contract from the seller's point of view a ***short forward***. The short forward makes money if the price of the underlying asset *goes down*. (Generally speaking, we say that someone has a short position in an asset if they would make money if the price of the asset goes down.)

§ 11c. The Payoff

By the ***payoff***, we mean the value of the contract to one of the parties on a particular date. In the above example, Sue's payoff at the end of 6 months would be \$10 if the price of the stock were \$114 at that time. George's payoff would be −\$10 in this case. If the price of the stock in 6 months were \$94, these payoffs would be reversed.

If you have any doubts about what the payoff is for any party on a particular date, think of it this way: The payoff is the amount that party would have if he/she completely cashed out. For example, Sue is obligated to buy a share of stock from George for \$104 in 6 months, regardless of its actual value at that time. If the price of the stock were actually \$114, Sue could cash out by selling the share for \$114, and she would be left with her payoff of \$10. On the other hand, to fulfill his obligation, George would have to buy a share for \$114 on the open market and sell it to Sue for \$104. His payoff would be −\$10.

This way of determining the payoff (the "cash out" approach) is pretty obvious in this example, but it can be helpful in more complex situations like those we will cover in later sections.

Let's formalize the computation of the payoff under a forward contract. The actual market price of an asset on a particular date is called the ***spot price*** on that date. So we can express the payoffs as follows:

$$\text{Payoff to long forward} = \text{spot price at expiration} - \text{forward price}$$

$$\text{Payoff to short forward} = \text{forward price} - \text{spot price at expiration}$$

(Remember that the forward price is the price specified in the contract at which the buyer *must* buy and the seller *must* sell the underlying asset.)

Continuing with the same example that we have been using, the payoffs for some of the possible outcomes are as follows:

Spot Price of Stock In 6 Months	Payoff	
	Long Forward (Sue)	Short Forward (George)
\$74	−\$30	\$30
84	−20	20
94	−10	10
104	0	0
114	10	−10
124	20	−20
134	30	−30

We can graph the payoffs as shown in Figure 11.1.

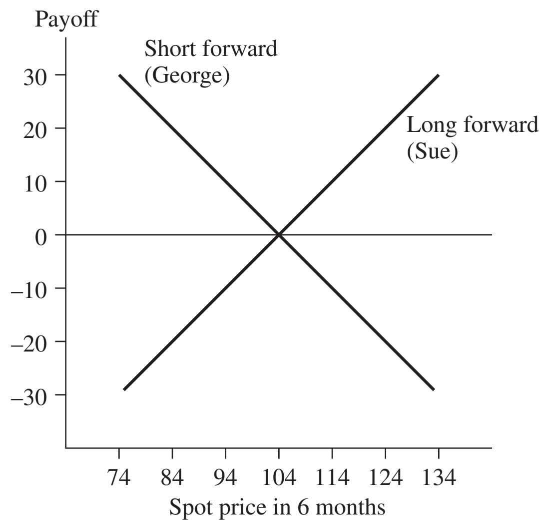


FIGURE 11.1

A few comments:

- Graphs of this kind are widely used in the financial literature, since they give a quick summary of the results
- If the spot price at expiration is \$104, the payoff is 0 to both parties and the graphs intersect at that point.
- For any other spot price, Sue's payoff is the opposite of George's (the long's payoff is the opposite of the short's). Thus, the payoff graphs for the long and the short are mirror images.
- For a "pure" forward contract, the payoff graph is also the ***profit*** graph, since the only cash flows occur on the expiration date. But for other types of contracts, or for combinations of contracts, there may be cash flows at other points in time. In order to determine the profit (or loss) in such cases, all cash flows must be taken into account. Thus, the profit graph may not be the same as the payoff graph in other situations.

EXAMPLE 1

Ed entered into a long forward with a forward price of \$100. Bob entered into a short forward based on a different underlying asset and having the same expiration date, but with a forward price of \$110. Both assets have the same spot price at expiration. Ed's profit on the expiration date is \$20. What is Bob's payoff on the expiration date?

SOLUTION

Payoff and profit are the same under a forward contract.

Ed:

$$\text{Payoff on long forward} = \text{spot price at expiration} - \text{forward price}$$

$$20 = \text{spot price at expiration} - 100$$

$$\text{Spot price at expiration} = \$120$$

Bob:

$$\text{Payoff on short forward} = \text{forward price} - \text{spot price at expiration}$$

$$\text{Payoff on short forward} = 110 - 120 = -\$10$$

EXAMPLE 2

You are given the following information about two forward contracts with expiration dates in 6 months:

	Current Spot Price	Spot Price in 6 Months	Forward Price
Contract A	\$100	\$90	\$105
Contract B	120	145	126

You take a long position under Contract A and a short position under Contract B. What is your total payoff at the end of 6 months?

SOLUTION

The current spot prices have nothing to do with the payoff in 6 months.

$$\begin{aligned}\text{Payoff on long position under Contract A} &= \text{spot price at expiration} - \text{forward price}\\ &= 90 - 105 = -\$15\\ \text{Payoff on short position under Contract B} &= \text{forward price} - \text{spot price at expiration}\\ &= 126 - 145 = -\$19\\ \text{Total payoff} &= -15 - 19 = -\$34\end{aligned}$$

EXAMPLE 3

Which of the following actions result(s) in a short position?

(I) Selling borrowed stock.

(II) Entering into a forward contract as the obligated buyer of the underlying asset

(III) Purchasing stock outright.

SOLUTION

I is the action of a short-seller, who would benefit from a decrease in the price of the stock when s/he purchased it to cover the short. (See Section 10d.)

II and III benefit from an increase in the stock price. They represent long positions.

Therefore, only I results in a short position.

§ 11d. A Forward Contract vs. Immediate Purchase

There are at least two ways of owning a stock 6 months from now:

- Method 1: Buy the stock immediately and hold it for 6 months.
- Method 2: Take a long position in a 6-month forward contract.

Let's say that the current spot price of the stock is \$100 and that the forward price in a forward contract is \$104.[1]

[1] We are going to assume that the forward price for a non-dividend-paying stock is determined as the current spot price of \$100 accumulated with interest at the risk-free rate of interest. In this example, the risk-free rate of interest is assumed to be 4% effective for a 6-month period, so that the forward price is \$104. The basis for determining forward prices will be covered in section 18c.

Under Method 1, we would pay \$100 now and hold the stock for 6 months.

Under Method 2, we would pay the forward price of \$104 in 6 months (regardless of the spot price at that time).

Under either method, we would end up owning the stock in 6 months.

The payoff under Method 1 is simply the spot price in 6 months, since we still own the stock at that time and could sell it for the spot price. (For simplicity, we are going to assume for now that there are no dividends payable on the stock during the 6-month period of ownership.)

Payoff to immediate purchase = spot price in 6 months

By definition, the payoff at any point in time does not take into account any cash flows at other points in time. Thus, the fact that the investor spent \$100 at time 0 to purchase the stock does not affect the amount of the payoff 6 months later.

The payoff under Method 2 is:

Payoff to long forward = spot price in 6 months − forward price

= spot price in 6 months − \$104

Let's put both of these results together on the same graph in Figure 11.2.

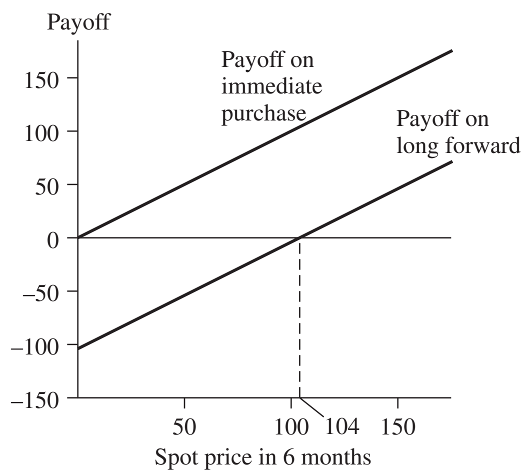


FIGURE 11.2

(Note that the line for the payoff on the long forward intersects the *y*-axis at −104.)

The difference in payoffs between Method 1 and Method 2 is a constant \$104. To take one result from the graph, if the spot price in 6 months is \$104, the payoff under Method 1 is \$104, while the payoff under Method 2 is 0.

Does this mean that we are better off buying the stock outright (Method 1) than we are by entering into a long forward (Method 2)?

That's a pretty silly question: "payoffs" look at the cash position at a point in time, without regard to any cash flows at other points in time. Obviously, to make a fair comparison between the two methods, we must take into account the fact that under Method 1 there was a cash outflow of \$100 to buy the stock at time 0.

The accumulated value of the initial cash outflow of \$100 is \$104 at the end of 6 months, under the assumed 4% risk-free rate for 6 months. (We will follow the textbook and refer to the accumulated value as the *future value,* or FV.) Thus, what we will call the ***net payoff*** or ***profit*** under Method 1 is equal to the spot price in 6 months minus \$104 (FV of original purchase price). This is the same as the profit under Method 2. (The profit is the same as the payoff under a forward contract.) You really end up in the same economic position under either method.

To demonstrate the equivalence of the two methods in a concrete way, we could arrange things so that their cash flows are identical. We could do this in one of two ways:

- Alternative 1: Since we are obligated to buy the stock for \$104 in 6 months, we decide to "pre-fund" this obligation by investing \$100 now in a 6-month zero-coupon bond that will mature for \$104. Thus, our cash flow at time 0 is \$100.
- Alternative 2: We decide to borrow \$100 to pay for the stock outright. Thus, our cash flow at time 0 is 0, but we will have a cash outflow of \$104 in 6 months to repay the loan.

Looking at the first bullet (forward contract with purchase of bond), we have arranged for the initial cash flow to be \$100, just as it is for the "pure" outright purchase. Now we can properly compare the payoffs in 6 months:

$$\begin{aligned}\text{Payoff on forward contract with bond} &= \text{spot price in 6 months} - \$104(\text{forward price})\\ &\quad + \$104(\text{maturity of bond})\\ &= \text{spot price in 6 months}\\ \text{Payoff on ``pure'' immediate purchase} &= \text{spot price in 6 months}\end{aligned}$$

Note that the payoffs are the same.

Looking at the second bullet (outright purchase with a loan), we have arranged for the initial cash flow to be 0, just as it is for the "pure" forward contract. Now we can properly compare the payoffs in 6 months:

$$\begin{aligned}\text{Payoff on outright purchase with a loan} &= \text{spot price in 6 months} - \$104(\text{repayment of loan})\\ \text{Payoff on ``pure'' forward contract} &= \text{spot price in 6 months} - \$104(\text{forward price})\end{aligned}$$

Again, the payoffs are the same.

We have gone through a bit of work just to show what is probably obvious to most of you anyway: that the financial results are the same whether you buy the stock outright or enter into a long forward.[2] This analysis may seem to be unnecessary, but it will be helpful in understanding some of the material in later sections.

Payoff and Profit Graphs for Long Forward with Bond

We can graph the payoff for a long forward with the purchase of a bond (Alternative 1) as shown in Figure 11.3.

[2] This relies on our assumption that the forward price is set equal to the future value of the current spot price. (\$104 is the FV of \$100.) Stay tuned. Also, as noted before, we're assuming that there are no dividends on the stock purchased outright.

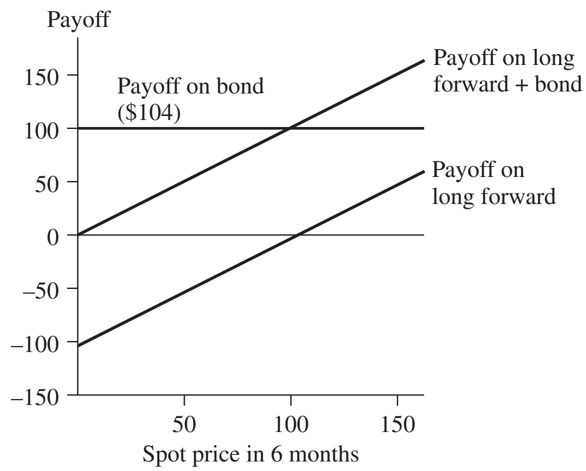


FIGURE 11.3

(Note again that the line for the payoff on the long forward intersects the y-axis at -104.)

As noted before, if we enter into a long forward and purchase a bond for $100 at time 0, the cash outflow of $100 is the same as the cash outflow for an outright purchase. Thus, the payoff graph for a long forward plus bond must be the same as for an outright purchase. (See Figure 11.2.) The payoff is simply the spot price at expiration.

To obtain the profit graph for the long forward plus bond (not shown), we must subtract the FV of the $100 cash outflow at time 0 (the purchase price of the bond), i.e., we must subtract $104 from the payoff. The result is:

Profit on long forward plus bond = spot price at expiration − $104

This is the same profit as on a "pure" long forward. (See long forward graph in Figure 11.1.) This shows that merely adding a bond to the long forward has no effect on the profit, which is what we would expect.

§ 11e. Cash Settlement

Let's say that the spot price at expiration under the forward contract we have been discussing is $114. The party at the long end of the contract would have a payoff (and profit) of $10 and the party at the short end would have a payoff (and profit) of −$10. Theoretically, the short forward would buy the stock for $114 and sell it to the long forward for the forward price of $104, for a loss of $10. The long forward would pay $104 for the stock, and if he/she wanted to realize the profit of $10 in cash, would sell the stock for $114.

All of this buying and selling would incur transaction costs (the bid-ask spread and commissions). The parties may have agreed at the outset to avoid these costs by making any settlements in cash, rather than by actual delivery of the stock. Thus, in the above example, the short would pay the long $10 and that would be the end of it.

If a cash settlement had been agreed to and the spot price at expiration were, say, $74, the long would pay the short $30 in full settlement of the contract.

Summary of Concepts and Formulas in Section 11

(1) ***Forward contract:*** an agreement to enter into a transaction at a pre-specified time and price
   (a) ***Underlying asset:*** the asset on which the agreement is based
      – Parties to forward contract agree to buy / sell the underlying asset
   (b) ***Expiration date:*** date on which transaction will take place
   (c) ***Forward price:*** price at which the transaction will take place
   (d) Forward price is set such that no up-front payment or premium need be paid by either party to the other

(2) Positions in a forward contract
   (a) ***Long:*** party which is obligated to buy the underlying asset
      – Long position benefits from an increase in the price of the underlying asset
   (b) ***Short:*** party which is obligated to sell the underlying asset
      – Short position benefits from a decrease in the price of the underlying asset

(3) ***Payoffs*** to forward contract positions
   (a) ***Long-forward payoff*** =
   {spot price of underlying asset at expiration} − {forward price} $= S_T - F$
   (b) ***Short-forward payoff*** =
   {forward price} − {spot price of underlying asset at expiration} $= F - S_T$

Practice Problems on Section 11

Note: Only the *first five* questions on this section are included below.

1. Max enters into a 6-month long forward contract with a forward price of \$100. Shirley enters into a 6-month short forward contract with a different underlying asset and with a forward price of \$120. The spot price at expiration of both underlying assets is \$130. X is the sum of Max's and Shirley's payoffs. Determine X.

(A) -\$40 (B) -\$20 (C) \$10 (D) \$20 (E) \$40

2. Paul enters into a forward contract with Tim. Paul is obligated to sell the underlying asset to Tim at expiration at the forward price of F. If the spot price at expiration were S, Paul's payoff would be \$10. If the spot price at expiration were 20% higher, Tim's payoff would be \$18. Determine S.

(A) \$40 (B) \$80 (C) \$140 (D) \$150 (E) \$168

3. Which of the following actions would benefit you if the price of a certain stock declines over the next 6 months:

(I) Enter into a 6-month long forward contract and buy a zero-coupon bond that matures to the forward price at the risk-free interest rate.
(II) Enter into a 6-month short forward contract.
(III) Short-sell the stock and close your position in 6 months.

(A) I and II only (B) I and III only (C) II and III only (D) I, II, and III
(E) The correct answer is not given by (A), (B), (C), or (D)

4. Aleshia enters into a long forward contract. If the spot price at expiration were S, her payoff would be $-$\$10. If the spot price at expiration were 20% higher, her payoff would be \$8. Determine S.

(A) \$10 (B) \$40 (C) \$70 (D) \$90 (E) \$100

5. Jason enters into a long forward based on Asset A, with a forward price of \$85. He also enters into a short forward based on Asset B, with a forward price of \$95. At a spot price of S for both assets, his payoffs under the two contracts would be the same. At a spot price of $S + \$8$, his payoff under Contract A would be X. Determine X.

(A) $-$\$5 (B) $-$\$3 (C) \$3 (D) \$5 (E) \$13

Solutions to Practice Problems on Section 11

1. Max's payoff = $130 - 100 = \$30$.
Shirley's payoff = $120 - 130 = -\$10$.
Total payoff = \$20. ANS. **(D)**

Actually, the sum of the payoffs is \$20 for all spot prices at expiration.

2. Paul's payoff at spot price of $S = F - S = \$10$.
Tim's payoff at spot price of $1.2S = 1.2S - F = \$18$.
Solving for S we get $S = \$140$. ANS. **(C)**

3. Action I would lose money if the price declines. (Adding a bond doesn't change the profit.) II would make money if the price declines. So would III. ANS. **(C)**

4. Let F = forward price. We are given:

$$S - F = -\$10$$

$$1.2S - F = \$8$$

Solving, we get $S = \$90$. ($F = \$100$) ANS. **(D)**

5. Payoff under Contract A = $S - 85$
Payoff under Contract B = $95 - S$
Equating these payoffs, we get $S = \$90$. At a spot price of $90 + 8 = \$98$, the payoff under Contract A = $98 - 85 = \$13$. ANS. **(E)**