# Finite Element Method of Analysis

## Introduction

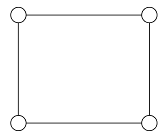
- Engineers model physical phenomena.
- Analytical descriptions of physical phenomena and processes are called mathematical models.
  - Developed using assumptions on the process.
  - Often characterized by differential and/or integral equations.
- Numerical methods are typically used to solve engineering mathematical models – referred to as numerical simulation.

## Finite Element Method

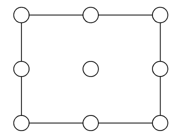
- Finite element method (FEM) is a numerical procedure for solving mathematical models numerically.
- FEM uses discretization (nodes and elements) to model the engineering system, i.e., subdivide the problem system into small components or pieces called elements and the elements are comprised of nodes.
- Approximations are introduced over each element to represent the behavior of the unknown variables.

## FEM – con't

- Different types of elements are available.
- Accuracy of the finite element approximation is improved by using more elements to approximate the engineering system and/or elements that involve more nodes to define the unknown function(s) variation over the element, e.g.,

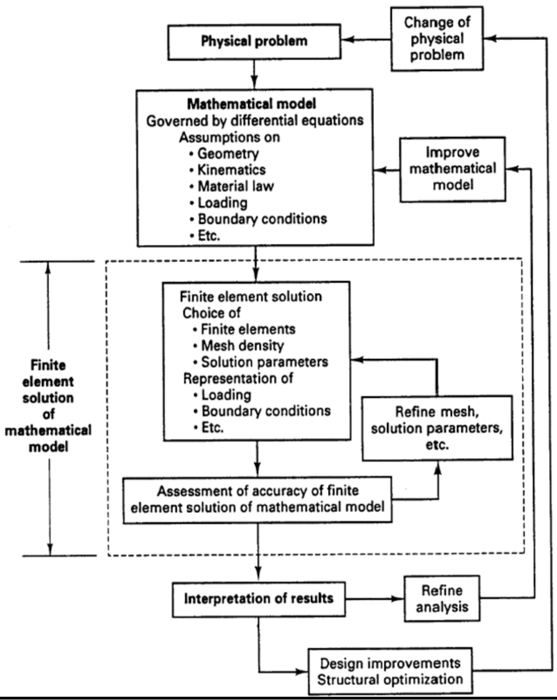


Q4 Bilinear Element



Q9 Biquadratic Element

# FEA Process




# Sample FE Meshes

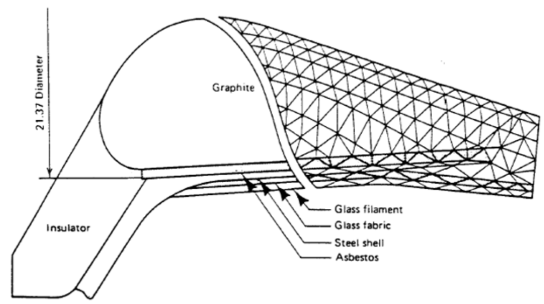



**Figure 1.1-4.** Cross section of a multimaterial rocket nozzle, showing construction (left portion) and possible finite element mesh (right portion). This problem was solved in the early days of finite element technology [10.1].

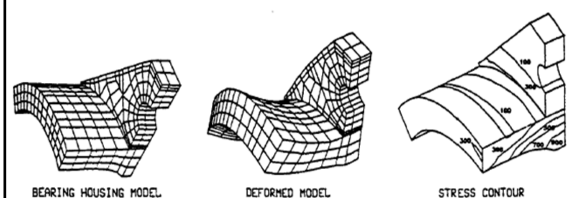



**Figure 1.1-7.** Finite element mesh and computed deformations and stresses in a portion of a bearing housing. (*Courtesy of Algor Interactive Systems Inc., Pittsburgh, Pennsylvania.*)

# Basic Components of FEA

1) Definition of the geometric form of the elements.
2) Representation of the assumed modes of behavior; generally polynomial interpolation.
3) Construction of the algebraic equations that correspond to the governing equations of the problem.

# FEM Summary

1. Divide the structure or continuum into finite elements. Mesh generation programs, call preprocessors, help the user in doing this work.

2. Formulation the properties of each element. In stress analysis, this means determining nodal loads associated with all element deformation states that are allowed. In heat transfer, it means determining nodal heat fluxes associated with all element temperature fields.

## FEM Summary – con't

3. Assemble elements to obtain the finite element model of the structure or continuum.
4. Apply the known loads: nodal forces and/or moments in stress analysis; nodal heat fluxes in heat transfer.
5. In stress analysis, specify how the structure is supported. This step involves specifying the known nodal displacements, which are often zero. In heat transfer, impose all known values of nodal temperature.

## FEM Summary – con't

6. Solve simultaneous linear algebraic equations to determine nodal degrees of freedom (dof) – displacements for stress analysis and temperature for heat transfer.
7. (a) In stress analysis, calculate element strains for the nodal dof and the element displacement interpolation field so that the element stresses can be calculated from the element strains.

## FEM Summary – con't

7. (b) In heat transfer analysis, calculate element heat fluxes from the nodal temperatures and the element temperature interpolation field.
8. Output interpretation programs, call postprocessors, help the user sort the output and display it in graphical form.

## Euler-Bernoulli Beam Element

Governing Equation:

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2w(x)}{dx^2}\right) + c_f\,w(x) = q(x)$$

$$\text{for } x \in \Omega$$

Variables:

w = transverse displacement

E = elastic modulus

I = moment of inertia

$c_f$ = foundation modulus

q(x) = distributed load

$\Omega$ = domain ($0 \le x \le L$)

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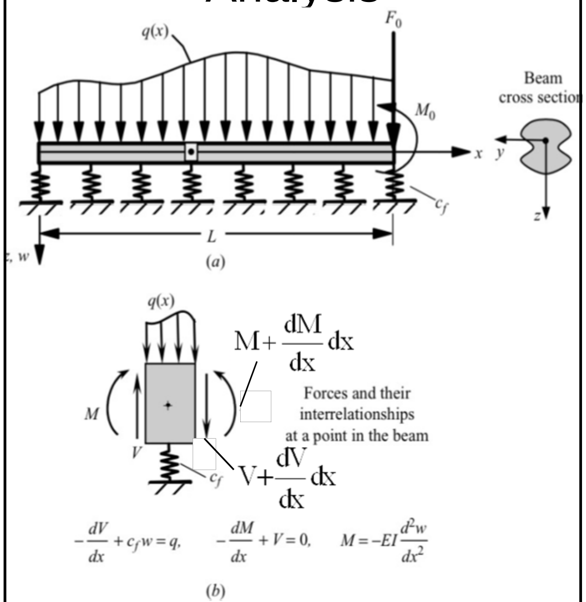
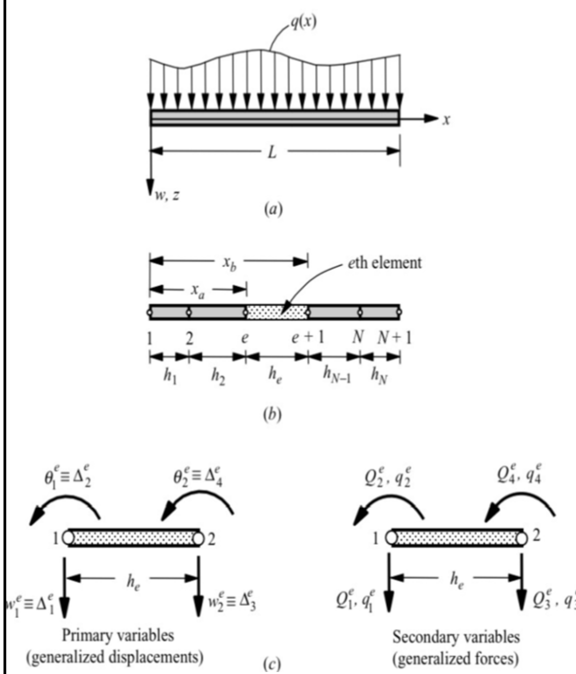
# Beam Finite Element Analysis




# Weighted Residual Form

$$\int_{x_e}^{x_{e+1}} v\left[\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) + c_f\, w - q\right]dx = 0$$

$v(x) \equiv$ weighting function

## First Integration by Parts

$$0 = \int_{x_e}^{x_{e+1}}\left[-\frac{dv}{dx}\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) + v\, c_f(x)\, w - vq\right]dx$$

$$+\left[v\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\right]_{x_e}^{x_{e+1}}$$

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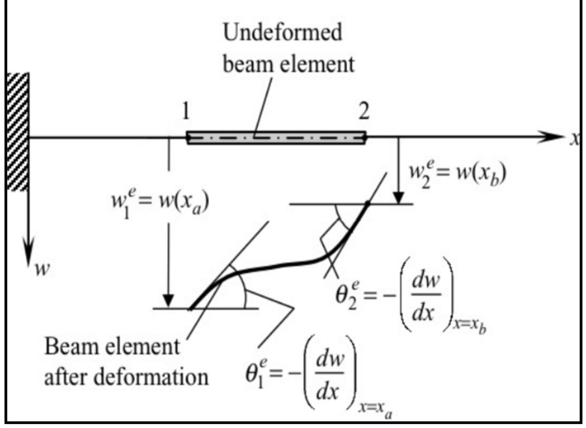
## Symmetric Weak Form

$$0 = \int_{x_e}^{x_{e+1}}\left[\frac{d^2v}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) + v\, c_f\, w - vq\right]dx$$

$$+\left[v\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\right]_{x_e}^{x_{e+1}} - \left[\frac{dv}{dx}\left(EI\frac{d^2w}{dx^2}\right)\right]_{x_e}^{x_{e+1}}$$



Natural Boundary Conditions:

$$Q_1^e \equiv \left[\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\right]_{x_e} = -V(x_e)$$

$$Q_2^e \equiv \left[\left(EI\frac{d^2w}{dx^2}\right)\right]_{x_e} = -M(x_e)$$

$$Q_3^e \equiv -\left[\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\right]_{x_{e+1}} = V(x_{e+1})$$

$$Q_4^e \equiv -\left[\left(EI\frac{d^2w}{dx^2}\right)\right]_{x_{e+1}} = M(x_{e+1})$$

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$$0 = \int_{x_e}^{x_{e+1}} \left[\frac{d^2v}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) + v\,c_f\,w - vq\right]dx$$
$$- v(x_e)Q_1^e + \left.\frac{dv}{dx}\right|_{x_e} Q_2^e$$
$$- v(x_{e+1})Q_3^e + \left.\frac{dv}{dx}\right|_{x_{e+1}} Q_4^e$$

Essential Boundary Conditions:

$$w(x_e) = w_1^e, \quad -\left.\frac{dw}{dx}\right|_{x_e} = \theta_1^e$$

$$w(x_{e+1}) = w_2^e, \quad -\left.\frac{dw}{dx}\right|_{x_{e+1}} = \theta_2^e$$

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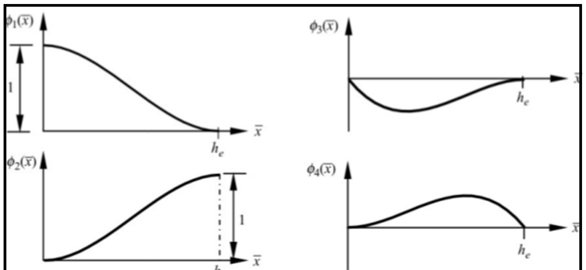

Hermite cubic shape functions:

$$\phi_1^e(x) = 1 - 3\left(\frac{\bar{x}}{h_e}\right)^2 + 2\left(\frac{\bar{x}}{h_e}\right)^3$$

$$\phi_2^e(x) = -\bar{x}\left(1 - \frac{\bar{x}}{h_e}\right)^2$$

$$\phi_3^e(x) = 3\left(\frac{\bar{x}}{h_e}\right)^2 - 2\left(\frac{\bar{x}}{h_e}\right)^3$$

$$\phi_4^e(x) = -\bar{x}\left[\left(\frac{\bar{x}}{h_e}\right)^2 - \frac{\bar{x}}{h_e}\right]$$

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# Finite Element Calculations

Substituting

$$w \approx w_h^e = \phi_1^e(\bar{x})\Delta_1^e + \phi_2^e(\bar{x})\Delta_2^e + \phi_3^e(\bar{x})\Delta_3^e + \phi_4^e(\bar{x})\Delta_4^e$$

$$v \approx v_h^e = \phi_1^e(\bar{x})v_1^e + \phi_2^e(\bar{x})v_2^e + \phi_3^e(\bar{x})v_3^e + \phi_4^e(\bar{x})v_4^e$$

into the symmetric weak form leads to

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# Element Equations

$$0 = \sum_{j=1}^{4}\left[\int_0^{h_e}\left(\frac{d^2\phi_i^e}{d\bar{x}^2}\left(EI\frac{d^2\phi_j^e}{d\bar{x}^2}\right) + \phi_i^e c_f \phi_j^e\right)d\bar{x}\right]\Delta_j^e$$

$$-\int_0^{h_e}\phi_i^e\, q\, d\bar{x} - Q_i^e \qquad \text{for } i = 1, 2, 3, 4$$

or

$$\sum_{j=1}^{4} k_{ij}^e \Delta_j^e - F_i^e = 0 \qquad \text{for } i = 1, 2, 3, 4$$

or

$$[k^e]\,\{\Delta^e\} = \{F^e\}$$

⇒ element stiffness equations



$$k_{ij}^e = \int_0^{h_e}\left(EI\frac{d^2\phi_i^e}{d\bar{x}^2}\,\frac{d^2\phi_j^e}{d\bar{x}^2} + c_f\,\phi_i^e\,\phi_j^e\right)d\bar{x}$$

= element stiffness matrix coefficients

$$F_i^e = \int_0^{h_e}\phi_i^e\, q\, d\bar{x} + Q_i^e = q_i^e + Q_i^e$$

= element load vector components

Explicit expressions for $[k^e]$ and $\{F^e\}$ (uniform load) are given in Eqs. (5.2.18), see next slide.



Note that the coefficients $K_{ij}^e$ are symmetric: $K_{ij}^e = K_{ji}^e$. In matrix notation, (5.2.15*b*) can be written as

$$\begin{bmatrix} K_{11}^e & K_{12}^e & K_{13}^e & K_{14}^e \\ K_{21}^e & K_{22}^e & K_{23}^e & K_{24}^e \\ K_{31}^e & K_{32}^e & K_{33}^e & K_{34}^e \\ K_{41}^e & K_{42}^e & K_{43}^e & K_{44}^e \end{bmatrix}\begin{Bmatrix} \Delta_1^e \\ \Delta_2^e \\ \Delta_3^e \\ \Delta_4^e \end{Bmatrix} = \begin{Bmatrix} q_1^e \\ q_2^e \\ q_3^e \\ q_4^e \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix} \qquad (5.2.17)$$

This represents the finite element model of (5.2.1). For the case in which $EI$ and $q$ are constant over an element, the element stiffness matrix $[K^e]$ and force vector $\{F^e\}$ have the following specific forms [see Fig. 5.2.2(*c*) for the element displacement and force degrees of freedom]

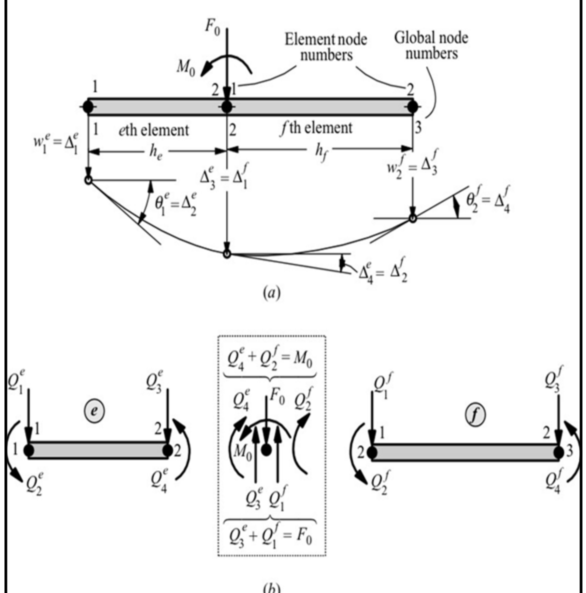
$$[K^e] = \frac{2E_eI_e}{h_e^3}\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2 & 3h_e & h_e^2 \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2 & 3h_e & 2h_e^2 \end{bmatrix} + \frac{c_f^e h_e}{420}\begin{bmatrix} 156 & -22h_e & 54 & 13h_e \\ -22h_e & 4h_e^2 & -13h_e & -3h_e^2 \\ 54 & -13h_e & 156 & 22h_e \\ 13h & -3h_e^2 & 22h_e & 4h_e^2 \end{bmatrix}$$

$$\{F^e\} = \frac{q_e h_e}{12}\begin{Bmatrix} 6 \\ -h_e \\ 6 \\ h_e \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} \qquad (5.2.18)$$

It can be verified that the generalized force vector in (5.2.18) represents the "statically equivalent" forces and moments at nodes 1 and 2 due to the uniformly distributed load of intensity $q_e$ over the element (see Fig. 5.2.7). For any given function $q(x)$, (5.2.16*b*) provides a straightforward way of computing the components of the generalized force vector $\{q^e\}$

$$q_i^e = \int_{x_e}^{x_{e+1}} \phi_i^e q\, dx \qquad (5.2.19)$$

# Element Assembly



Force Equilibrium:

$$\sum_1 F = 0 = Q_1^e + q_1^e; \quad \sum_1 M = 0 = Q_2^e + q_2^e$$

$$\sum_2 F = 0 = Q_3^e + q_3^e + Q_1^f + q_1^f;$$

$$\sum_2 M = 0 = Q_4^e + q_4^e + Q_2^f + q_2^f$$

$$\sum_3 F = 0 = Q_3^f + q_3^f; \quad \sum_3 M = 0 = Q_4^f + q_4^f$$

Displacement Continuity (5.2.21):

$$\Delta_1^e = U_1; \ \Delta_2^e = U_2$$

$$\Delta_3^e = U_3 = \Delta_1^f; \quad \Delta_4^e = U_4 = \Delta_2^f$$

$$\Delta_3^f = U_5; \ \Delta_4^f = U_6$$

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$$[K]\{U\} = \{F\}$$

$$[K] = \begin{bmatrix} k_{11}^e & k_{12}^e & k_{13}^e & k_{14}^e & 0 & 0 \\ k_{21}^e & k_{22}^e & k_{23}^e & k_{24}^e & 0 & 0 \\ k_{31}^e & k_{32}^e & k_{33}^e + k_{11}^f & k_{34}^e + k_{12}^f & k_{13}^f & k_{14}^f \\ k_{41}^e & k_{42}^e & k_{43}^e + k_{21}^f & k_{44}^e + k_{22}^f & k_{23}^f & k_{24}^f \\ 0 & 0 & k_{31}^f & k_{32}^f & k_{33}^f & k_{34}^f \\ 0 & 0 & k_{41}^f & k_{42}^f & k_{43}^f & k_{44}^f \end{bmatrix}$$

$$\{U\} = \langle U_1 \ U_2 \ U_3 \ U_4 \ U_5 \ U_6 \rangle^T$$

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} = \begin{Bmatrix} q_1^e + Q_1^e \\ q_2^e + Q_2^e \\ q_3^e + Q_3^e + q_1^f + Q_1^f \\ q_4^e + Q_4^e + q_2^f + Q_2^f \\ q_3^f + Q_3^f \\ q_4^f + Q_4^f \end{Bmatrix}$$

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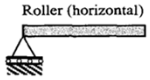
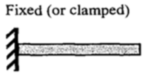
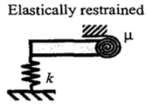
# Boundary Conditions

**Table 5.2.1** Types of commonly used support conditions for beams and fram

| Type of support | Displacement boundary conditions | Force boundary conditions |
|---|---|---|
| Free  | None | All, as specified |
| Pinned  | $u = 0$<br>$w = 0$ | Moment is specified |
| Roller (vertical)  | $u = 0$ | Transverse force and moment are specified |
| Roller (horizontal)  | $w = 0$ | Horizontal force and bending moment are specified |
| Fixed (or clamped)  | $u = 0$<br>$w = 0$<br>$dw/dx = 0$ | None specified |
| Elastically restrained  | $EI(d^2w/dx^2) + \mu\theta = M_0$, $M_0$ specified<br>$EI(d^3w/dx^3) + kw = Q_0$, $Q_0$ specified | |

Finite Element Bending Moment Calculation:

$$M = -EI\frac{d^2w}{dx^2}$$

$$\Rightarrow M^e = -EI\frac{d^2w_h^e}{dx^2}$$

$$= -EI\left\langle \frac{d^2\phi_1^e}{d\bar{x}^2} \ \frac{d^2\phi_2^e}{d\bar{x}^2} \ \frac{d^2\phi_3^e}{d\bar{x}^2} \ \frac{d^2\phi_4^e}{d\bar{x}^2} \right\rangle \begin{Bmatrix} w_1^e \\ \theta_1^e \\ w_2^e \\ \theta_2^e \end{Bmatrix}$$

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