FOUNDATIONS OF EMBEDDED CONTACT HOMOLOGY

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1. Introduction

This paper serves as an introduction to the concepts and main results of contact homology in the case of embedded curves of a three dimensional contact manifold. We will begin with preliminary definitions and examples of symplectic and contact manifolds and their associated objects, and then proceed into defining and overviewing the basics of embedded contact homology. The primary reference used will be Hutchings's *Notes on Embedded Contact Homology* [1] as well as his 2008 lectures at MSRI.

The fundamental objects of study for Embedded Contact Homology (ECH) are Reeb orbits on contact manifolds, so we define:

Definition 1.1. A contact manifold (Y, λ) is a smooth, $2n+1$ dimensional manifold together with a 1-form λ such that $\lambda \wedge (d\lambda)^n \neq 0$.

Definition 1.2. The Reeb vector field R associated to a contact manifold (Y, λ) is the vector field on Y defined by $d\lambda(R, \cdot) = 0$ and $\lambda(R) = 1$. A Reeb orbit is a closed orbit of R. If γ is a Reeb orbit, we denote γ^k to be its k-fold iterate.

The contact structure on a contact manifold is the hyperplane field $\xi = \ker(\lambda)$. In a sense, this is the more fundamental object of a contact manifold, and the 1-form λ is just a tidy (non-unique) means of defining it. The non-integrability condition that $\lambda \wedge (d\lambda)^n \neq 0$ is equivalent to the hyperplane configurations being generic at every point in Y.

Example 1.3. The most basic example of a contact manifold is $\mathbb{R}^{2n+1}$. Let $(x_1, ..., x_n, y_1, ..., y_n.z)$ be coordinates on $\mathbb{R}^{2n+1}$; then there is a natural contact form which we can write explictly:

$$\lambda = dz + \sum_{i=1}^{n} x_i dy_i$$

Notice:

$$d\lambda = \sum_{i=1}^{n} dx_i \wedge dy_i$$

We then see that $\lambda \wedge (d\lambda)^n$ is the standard volume form on $\mathbb{R}^{2n+1}$.

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Every contact manifold can be transformed into a symplectic manifold in several natural ways. The one we will use is called the *symplectization*. Given (Y, λ), the manifold $Y \times \mathbb{R}$ has an induced symplectic structure $\omega = d(e^t \lambda)$, where t is the coordinate given to the $\mathbb{R}$ axis. The exponential coefficient is only special in the sense that it is a nowhere vanishing function with nowhere vanishing derivative, which ensures $\omega \wedge \omega \neq 0$.

We define the symplecic action functional:

$$\mathcal{A} : C^\infty(S^1, Y) \to \mathbb{R}$$

$$\mathcal{A}(\gamma) = \int_\gamma \lambda$$

Where $C^\infty(S^1, Y)$ is the free loop space on Y. It can be shown that the critical points of $\mathcal{A}$ are Reeb orbits. The gradient vector field $\nabla \mathcal{A}$ then defines flow "lines" between Reeb orbits. These flows can be realized as cylinders in the syplectization $Y \times \mathbb{R}$. The basis for contact homology is to in some way count these cylinders.

Another aspect of the Reeb orbits that we will use is the properties of their linearized return maps. For any orbit α, we can consider the return map on the contact structure ξ. Namely at a point $x \in \alpha$, we can define a map in a neighborhood of x in ξ_x that takes a point to the next point on its path following the vector flow (see Figure 1).

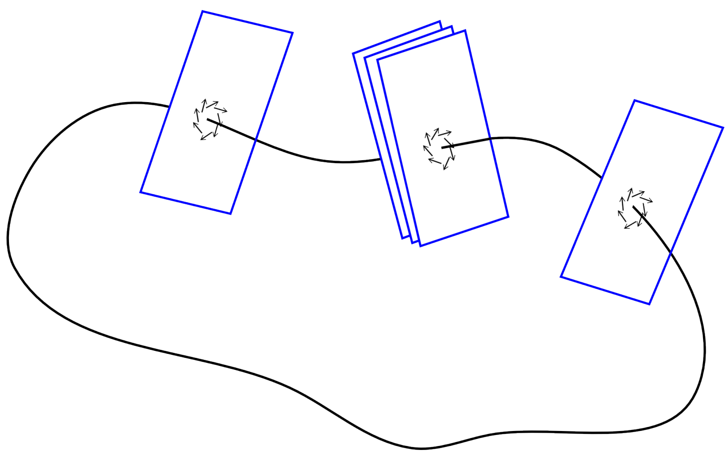

FIGURE 1. Return maps of α on the contact structure ξ.

The linearization of the map near x we denote as P_α, which is a symplectic linear map with respect to $d\lambda$. Then we classify the orbit α based on the spectrum of P_α:

- α is nondegenerate if $1 \notin \text{spec}(P_\alpha)$.
- α is elliptic if the eigenvalues of P_α have norm 1 in $\mathbb{C}$.
- α is hyperbolic if the eigenvalues are real.

2. J-holomorphic curves and Currents

To generalize the idea of flow lines of $\nabla \mathcal{A}$, we define J holomorphic curves between Reeb orbits in the space $Y \times \mathbb{R}$, which will be the building blocks of ECH.

Definition 2.1. An almost complex structure on an even dimensional manifold X is an endomorphism of the tangent bundle J such that $J^2 = -1$.

In the case of a symplectization $X = Y \times \mathbb{R}$, we define an *admissible almost complex structure* J to be one that is $\mathbb{R}$ invariant, $J(\partial_t) = R$, J sends the contact structure ξ to itself, and $d\lambda(v, Jv) \geq 0$ for all $v \in \xi$. Any time we mention J, we assume it is generic on the space of admissible almost-symplectic structures on $Y \times \mathbb{R}$.

Definition 2.2. A J holomorphic (or pseudoholomorphic) curve us a map u from a Riemann surface (Σ, j) to X that is compatible with the almost complex structure, namely $du \circ j = J \circ du$. We say a curve is somewhere injective if there exists $z \in \Sigma$ such that $u^{-1}(u(z)) = \{z\}$ and du_z is injective.

We only consider J-holomorphic curves up to biholomorphism of the domains. As mentioned, we are concering ourselves with J holomorphic curves between Reeb orbits on $X = Y \times \mathbb{R}$, so we must define precisely what we mean by this. We define an orbit set α to be a collection of pairs (α_i, m_i), where α_i is a Reeb orbit and m_i is the covering multiplicity of the orbit. A J-holomorphic curve between orbit sets $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_j, m_j)\}$ is one for which Σ has a puncture for each orbit. We also require that for each i, there is a component of $u(\Sigma)$ that is asymptotic at $+\infty$ to a m_i-fold cover of $\mathbb{R} \times \alpha_i$, and similarly for the β_j at $-\infty$ (Figure 2).

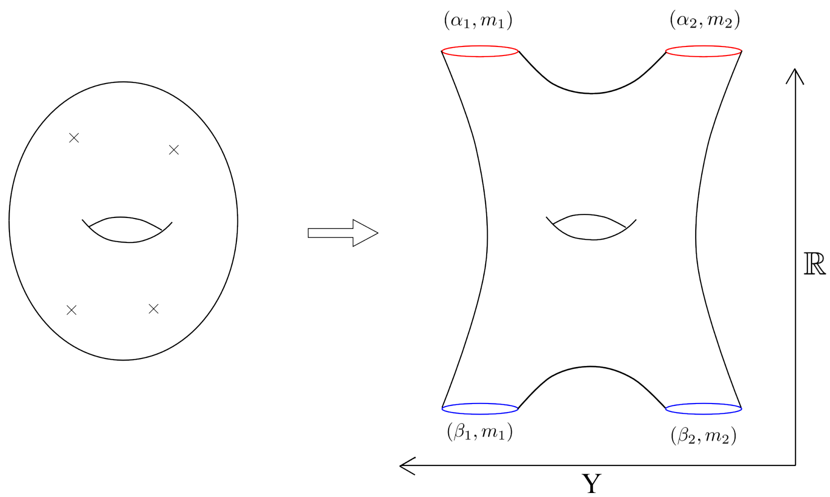

FIGURE 2. A J-holomorphic curve in $Y \times \mathbb{R}$ with two orbits at $\pm\infty$.

Definition 2.3. A J-holomorphic current between orbit sets α and β is a finite set of pairs $\mathcal{C} = \{(C_k, d_k)\}$, where the C_k are distinct irreducible somewhere injective J-holomorphic curves in $Y \times \mathbb{R}$ and the d_k are positive integers, such that the positive ends of the C_k curves are at covers of the Reeb orbits α_i, the sum over k of d_k times the total covering multiplicity of all ends of C_k at covers of α_i is m_i, and similarly for the negative ends.

Embedded Contact Homology will count homology classes of J-holomorphic currents between a pair of orbit sets. We denote $\mathcal{M}(\alpha, \beta)$ to be the Moduli space of J-holomorphic currents between α and β.

3. Motivating ECH: The Gromov invariant

We briefly retreat to the case of any closed, symplectic 4 manifold (X, ω) to state an important result due to Taubes that will serve as a motivating example for ECH. First we choose an almost complex structure J that is compatible with ω in the sense that $\omega(v, Jv) \geq 0$ for all $v \in T_x(X)$ (also called ω-tame). For any J-holomorphic curve $u : (C, j) \to X$, we define the Fredholm index:

$$\text{ind}(C) := -\chi(C) + 2\langle C_1(TX), [C]\rangle \tag{3.1}$$

Where $\chi(C)$ is the Euler characteristic of (the domain of) C, $C_1(TX)$ is the first Chern class of the tangent bundle (thought of as an element of the second cohomology group), $[C]$ is the homology class of C, and $\langle\ ,\ \rangle$ denotes the pairing of cohomology and homology groups via Poincaré duality.

Proposition 3.2. *If C is not multiply covered, the Fredholm index ind(C) is the dimension of the moduli space of curves near C.*

Our use of the word "near" here is with respect to deformations of C. This is quantified in § 2.3 of [1], along with a proof that this moduli space is in fact a manifold. A second index that can be defined for J-holomorphic curves is the Seiberg-Witten index:

$$I(C) := \langle C_1(TX), [C]\rangle + C \cdot C \tag{3.3}$$

Where $C \cdot C$ denotes the self intersection number of C (which we assume to be embedded).

Remark 3.4. It is now that our restriction to dimension 4 symplectic manifolds comes into play. There are two useful properties of J-holomorphic curves in four dimensions that are important. The first is positivity of intersections. Namely, if C_1 and C_2 are distinct somewhere injective J-holomorphic curves, then ther intersection points are finite, isolated, and of positive multiplicity. The second is the adjunction formula:

$$\langle C_1(TX), [C]\rangle = \chi(C) + C \cdot C - 2\delta(C) \tag{3.5}$$

Where C is somewhere injective and $\delta(C)$ is a weighted count of singularities of C in X (points where it is not locally an embedding).

Combining equations (3.1), (3.3), and (3.5), we observe:

$$\text{ind}(C) = I(C) - 2\delta(C)$$

This shows that the maximum attainable value of ind(C) is the Seiberg-Witten index, with equality happening when $\delta(C) = 0 \iff C$ embedded. These are useful equations in defining Taubes's Gromov invariant on X. Roughly speaking, this fixes a homology class $A \in H_2(X)$ and assigns to it a count $Gr(X, \omega, A) \in \mathbb{Z}$ of "admisible" holomorphic curves with the same homology class in X after choosing J generically. An important case happens when $I(A) = 0$, in which case the curves are embedded and disjoint. For a more detailed treatment, see [1] §2.5.

The important result concerning the Gromov invariant, due to Taubes, is its relationship to the Seiberg Witten invariant of X. This is an invariant that counts solutions to the Seiberg-Witten equations via the map $SW(X) : \text{Spin}^c(X) \to \mathbb{Z}$. Through an identification of the spaces $H_2(X)$ and $\text{Spin}^c(X)$, Taubes proves that:

Theorem 3.6 (Taubes). *If the dimension of the maximal positive definite subspace of $H_2(X;\mathbb{R})$ is greater than 1, then*

$$SW(X) = Gr(X, \omega, \cdot)$$

4. Defining ECH

Returning our focus to symplectizations, we can't immediately apply the Gromov invariant to $Y \times \mathbb{R}$ because its compactification has boundary, whereas we assumed X was closed above. Taking inspiration from Taubes's result, Embedded Contact Homology tries to construct a similar invariant on a symplectization $Y \times \mathbb{R}$ which similarly agrees with Seiberg Witten Floer homology, which is constructed from solutions to the Seiberg Witten equations. That is, we wish to define a chain complex associated to a contact manifold (Y, λ) with almost complex structure J whose homology is isomorphic to the S-W Floer homology $\hat{H}M^*(Y)$.

We will first define the ECH chain complex, then we will develop more general versions of the Fredholm and SW indices used above as well as overview the relative adjunction formula in the case of a symplectization. We will then discuss a lemma of partition conditions wich allows us to demonstrate that the differential map on ECH is well defined.

4.1. **The ECH Chain Complex.** Let Y be a closed, oriended 3 manifold with contact form λ, and assume the Reeb orbits on Y are nondegenerate. Fix $\Gamma \in H_1(Y)$ and define $ECH(Y, \lambda, \Gamma)$ to be the homology of the chain complex $C_\bullet$ that is freely generated by a finite set of pairs $\alpha = \{(\alpha_i, m_i)\}$, where:

- The α_i are distinct embedded Reeb orbits,
- The m_i are positive integers,
- $\sum_i m_i[\alpha_i] = \Gamma$, and
- If α_i is hyperbolic, then $m_i = 1$.

To define the differential map d on $C_\bullet$, we choose J a generic admissible almost complex structure. Then we must define a reasonable index I (analagous to the SW index) which we use to count $I = 1$ curves between orbit sets. These counts serve as the weights $M_{\alpha,\beta}$ in the following definition of the differential map:

$$d(\alpha) = \sum_\beta M_{\alpha,\beta} \beta$$

We will now detail the choice of index I as well as its relationship to the Fredholm index and the relative adjunction formula.

4.2. **The Fredholm Index.** Fix a J-holomorphic curve C between α and β, and let $q_{i,k}$ denote the multiplicities of the positive ends of C at α_i. Here, k is the index counting the ends of C and i is the index keeping track of the Reeb orbits themselves (which could be multiply covered). Similarly, let $q_{j,k}$ denote the multiplicities at the negative ends at β_j. Then the Fredholm index is:

$$\text{ind}(C) = -\chi(C) + 2C_1(\xi|_C, \tau) + \sum_{i,k} cz_\tau(\alpha_i^{q_{i,k}}) - \sum_{jk} cz(\beta_j^{q_{j,k}}) \tag{4.1}$$

Here, τ is a choice of trivialization on the orbits α_i and β_j. This trivialization allows us to compute a well-defined first Chern class of ξ over C with respect to τ, which is denoted as $C_1(\xi|_C, \tau)$ above. The cz_τ terms are the Conley-Zehnder indices of the orbits, which we define now.

Let γ be a Reeb orbit (not necessarily embedded), and τ a trivialization of $\xi|_\gamma$, define $cz_\tau(\gamma)$ as follows:

- If γ is hyperbolic, then the linearized return map P_γ has real eigenspaces, which it rotates by $n\pi$ (with respect to the trivialization). Then we let $cz_\tau(\gamma) = n$.
- If γ is elliptic, then the linearized flow rotates its eigenspaces by some angle $2\pi\theta$ with respect to the trivialization. Then we let $cz_\tau(\gamma) = 2\lfloor\theta\rfloor + 1$.

It can be shown that ind(C) is independent of the choice of trivialization, and in fact:

Theorem 4.2. *If J is generic, and C is not multiply covered, then the space $\mathcal{M}(\alpha, \beta)$ of curves near C is a manifold of dimension ind(C).*

The proof of this theorem comes in two parts. The first is demonstrating that $\mathcal{M}(\alpha, \beta)$ is a manifold of of dimension the index of a deformation operator defined in § 2.3 of [1]. The second is showing that for generic J, the Fredholm index coincides with the index of the deformation operator.

4.3. **The ECH index.** Now we will define the ECH index of a curve $C \in \mathcal{M}(\alpha, \beta)$ that is somewhere injective to be:

$$I(C) := C_1(\xi|_C, \tau) + Q_\tau(C) + \sum_i \sum_{k=1}^{m_i} cz_\tau(\alpha_i^k) - \sum_j \sum_{k=1}^{m_j} cz_\tau(\beta_j^k) \tag{4.3}$$

Where $Q_\tau(C)$ is the "relative intersection pairing", which is the symplectization analogue of the intersection number $C \cdot C$ in the Gromov invariant case. Note that the sums in the ECH index are different from those in the Fredholm index. The former only sums Conley-Zhender indices along orbits at the ends of C, whereas the latter sums over all iterates of the orbits up to m_i.

To define $Q_\tau(C)$, we let S be an embedded (except at its boundary) surface in $[-1, 1] \times Y$ (identified as the compactification of $Y \times \mathbb{R}$) with the following properties:

- $\partial S = \sum_i m_i\{1\} \times \alpha_i - \sum_j m_j\{-1\} \times \beta_j$,
- S has the same relative homology class as the compactification of C,
- the projection $\pi : S \to Y$ is an immersion near the boundary,
- and the conormal at the boundary has winding number 0 with respect to the trivialization τ.

To understand the last condition, fix $x \in \alpha_i$, and consider the rays obtained by projecting S to Y as it approaches the boundary point x. The last condition says that these rays do not rotate with respect to the trivialization τ as x moves around α_i (see Figure 3).

Finally, to define $Q_\tau(C)$, we pick S and S' as defined above with different conormal directions on the boundary and we set $Q_\tau(C)$ to be the signed count of intersections of the interiors of S and S':

$$Q_\tau(C) := \#\left(\dot{S} \cap \dot{S}'\right)$$

It can be shown that these boundary conditions make $Q_\tau(C)$ well-defined.

A nice property of I is that only depends on α, β and the relative homology class of C (and in particular, not on the choice of trivialization). For this reason, it is sometimes written as $I(\alpha, \beta, [C])$.

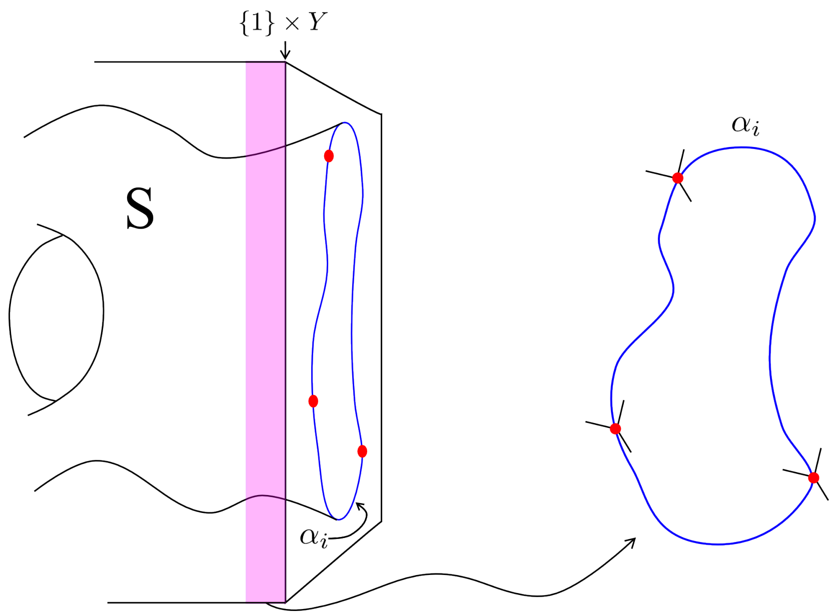

FIGURE 3. The surface S near $\{1\} \times Y$ on a $m_i = 3$ Reeb orbit, with the projection rays not rotating as we move around the orbit.

Theorem 4.4 (Index Inequality)**.** *Let $C \in \mathcal{M}(\alpha, \beta)$ is not multiply covered. Then:*

$$ind(C) \leq I(C) - 2\delta(C)$$

with equality holding if and only if C satisfies a particular set of "partition conditions."

We will detail exactly what the partition conditions are later. This inequality is the analagous statement of the equality we found in our discussion of the Gromov invariant. Following similar reasoning, we obtain the corollary:

Corollary 4.5. *If $C \in \mathcal{M}(\alpha, \beta)$ has ECH index $I(C) = 1$, and J is chosen generically, then C is embedded.*

Proof.) This follows immediately from Theorems 4.2 and 4.4 because, if J is generic, then $\text{ind}(C)$ is a dimension, so it is nonnegative. Then having $I(C) = 1$ forces $\delta(C)$ to be zero to keep $\text{ind}(C)$ nonnegative, and so C is embedded. □

The proof of Theorem 4.4 follows from two important formulas. The first is the relative adjunction formula, and the second is the writhe bound. We will briefly describe these in the next two sections, after which we will precisely define the partition conditions referenced in Theorem 4.4.

4.4. **Relative Adjunction Formula.** The adjunction formula used in the Gromov invariant case will be of similar use in ECH. It allows us to compute the relative first Chern class of a somewhere injective curve purely topologically:

$$C_1(\xi|_C, \tau) = \chi(C) + Q_\tau(C) + \omega_\tau(C) - 2\delta(C) \tag{4.6}$$

where $\omega_\tau(C)$ is the *asymptotic writhe* of C. To define this, consider the slice $C \cap (\{T\} \times Y)$ for some $T \in R$. For $T \gg 0$, this slice is a disjoint union of embedded braids ζ_i^+ near each

α_i with m_i strands. We can then use the trivialization τ to identify ζ_i^+ with a link in the solid torus $S^1 \times D^2$. By "flattening" the torus, we can compute the standard writhe of ζ_i^+, which we will denote $\omega_\tau(\zeta_i^+)$, by counting crossings with sign. We can do the same with the braids for $T \ll 0$, which we will denote ζ_i^-. Putting these together, we define the asymptotic writhe of C to be:

$$\omega_\tau(C) := \sum_i \omega_\tau(\zeta_i^+) - \sum_j \omega_\tau(\zeta_i^-)$$

4.5. **Write Bound.** Recall the two summation terms in our expressions for ind(C) and $I(C)$ (equations 4.1 and 4.3). We will more compactly denote them as:

$$CZ_\tau^{\text{ind}}(C) := \sum_{i,k} cz_\tau(\alpha_i^{q_{i,k}}) - \sum_{j,k} cz_\tau(\beta_j^{q_{j,k}})$$

$$CZ_\tau^I(C) := \sum_i \sum_{k=1}^{m_i} cz_\tau(\alpha_i^k) - \sum_j \sum_{k=1}^{m_j} cz_\tau(\beta_j^k)$$

Then, with some work (and indeed this is the main part of the proof of Theorem 4.4), one can show that:

$$\omega_\tau(C) \leq CZ_\tau^I(C) - CZ_\tau^{\text{ind}}(C)$$

with equality when C satisfies the aforementioned "partition conditions." Using this inequality, Theorem 4.4 follows from equations 4.3 and 4.1.

4.6. **Partition Conditions.** Let γ be an embedded Reeb orbit, and let m be a positive integer. We define two paritions $P_\gamma^+(m)$ and $P_\gamma^-(m)$ of the integer m:

- If γ is hyperbolic with positive eigenvalues, then $P_\gamma^+(m) = P_\gamma^-(m) = (1, ..., 1)$,
- If γ is hyperbolic with negative eigenvalues, then $P_\gamma^+(m) = P_\gamma^-(m) = (2, ..., 2, 1)$ if m is odd and $(2, ..., 2)$ if m is even, and
- If γ is elliptic, then it has an associated monodromy angle θ (wich is irrational by our assumption that γ and all of its iterates are nondegenerate). Then the parition for $P_\gamma^+(m)$ is the horizontal cordinates of the maximal concave lattice path between $(0, 0)$ and $(m, \lfloor m\theta \rfloor)$ lying below the $y = \theta x$ line (see Figure 4). The negative partition $P_\gamma^-(m)$ is the horizontal components of the minimal convex lattice path between the origin and $(m, \lceil m\theta \rceil)$ lying above $y = \theta x$.

With this definition, we say what it means for a curve C to satisfy the "partition conditions" referenced above. For each i, the curve C induces a partition of m_i. For example, if $\alpha = \{\alpha_1, 4\}$ and C has two ends at $+\infty$, then the paritition of 4 that C induces is either $(1, 3)$ or $(2, 2)$. We say that C satisfies the partition conditions if these induced paritions are equal to our prescribed partitions above for each m_i and m_j. Namely, they are equal to $P_{\alpha_i}^+(m_i)$ on the positive ends and are equal to $P_{\beta_j}^-(m_j)$ on the negative ends.

As a final ingredient, we will state a classification theorem for low ECH index curves which is useful in defining the differential.

Proposition 4.7. *Suppose J is generic, and let α, β be orbit sets and let $C \in \mathcal{M}(\alpha, \beta)$ be a J-holomorphic current. Then:*

(1) $I(C) \geq 0$, with equality holding if and only if C is a union of trivial cylinders or covers thereof.

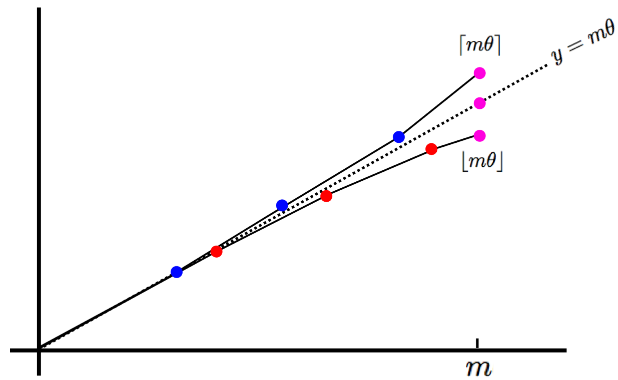

FIGURE 4. Maximal and minimal lattice paths defining the partitions of m. The red dots form the partition $P_\gamma^+(m)$ and the blue dots form the partition $P_\gamma^-(m)$

(2) If $I(C) = 1$, then $C = C_0 \sqcup C_1$, where $I(C_0) = 0$ and C_1 is embedded with $I(C_1) = 1$.

Here "trivial cylinder" means a cylinder of the form $\gamma \times \mathbb{R}$, with γ an embedded Reeb orbit.

5. Defining the differential

Recall our ansatz for the differential:

$$\partial(\alpha) = \sum_\beta M_{\alpha,\beta}\beta$$

for some weighting $M_{\alpha,\beta}$. We expect these weights to count curves between α and β in some way. We can now precisely define it using the language of th ECH index. Define:

$$\mathcal{M}_k(\alpha, \beta) := \{C \in \mathcal{M}(\alpha, \beta) \mid I(C) = k\}$$

We saw that for generic J, the moduli space has dimension $\mathrm{ind}(C)$. In the case of embedded $I(C) = 1$ curves, the dimension of $\mathcal{M}_1(\alpha, \beta)$ will also be 1 by the Index Inequality (4.4). There is also a natural 1 dimensional $\mathbb{R}$ action on $\mathcal{M}_1(\alpha, \beta)$ given by translating the $\mathbb{R}$ coordinate. We then expect the quotient $\mathcal{M}_1(\alpha, \beta)/\mathbb{R}$ to be a finite set of points (provided it is compact). Then we write:

$$\partial(\alpha) = \sum_\beta \#(\mathcal{M}_1(\alpha, \beta)/\mathbb{R})\beta$$

The above argument shows that this is a plausible definition for a differential that is well defined. There are still subtleties to show compactness, which can be shown using the partition conditions and the classification of J-holomorphic currents. See [1] for a full treatment. The harder result, due to Huchings and Taubes, is that $\partial^2 = 0$. The difficulty lies in trying to glue $I = 2$ curves together, when their boundary partitions are different.

As a final word, we state the main connection to Seiberg-Witten theory:

Theorem 5.1. *If Y is connected, there is a canonical isomorphism of graded modules:*

$$ECH_\bullet(Y, \lambda, \Gamma, J) \cong \widehat{HM}^{-\bullet}(Y)$$

Where $\widehat{HM}^{-\bullet}(Y)$ is the Seiber-Witten Floer cohomology.

References

[1] Michael Hutchings. *Lecture notes on Embedded Contatct Homology*, arXiv:1303:5789v2, February 2014.