R-TRIVIAL SIMPLE VOTING GAMES

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1. Introduction

There are several examples of things that surround us in day to day life that are actually mathematical objects we can study, for example, Rubik's cubes, traffic lights, sudoku boards, and even the way we make decisions. We make decisions on a daily basis: do we go get coffee or not, what should we have for dinner, etc. Based on the article "On the structure of minimal winning coalitions in simple voting games" by Maria Axenovich and Sonali Roy [Axe10], we will talk about a specific mathematical structure related to how we make decisions. Another thing we often see in mathematics is the classification or collection of certain structures under broader structural classifications. Here we will show that all examples of a specific type of voting rule, satisfying three properties, can be classified as a more specific type of voting rule. This type of classification is often useful for seeing relationships between objects, showing that a classification is unique, and talking about objects. In this paper we will define and understand, sometimes with the help of examples, terms necessary to understand and work with the voting structure we are considering, as well as to understand all propositions and theorems. We will first seek to understand and prove five propositions that are necessary for proving the main theorem. Then we will develop an understanding of the main theorem and provide a proof.

2. Definitions

In this section we provide the definitions and notation necessary for understanding the propositions we will use in the proof of our main theorem as well as the statement of this theorem itself. We begin by defining the specific type of voting that we would like to consider.

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Definition 2.1. [Axe10] A **simple voting game,** $\mathcal{F}$, is a collection of subsets of a finite set of voters, N, satisfying the following properties.

(1) $\emptyset \notin \mathcal{F}$

(2) $N \in \mathcal{F}$

(3) if $X \subset X'$ and $X \in \mathcal{F}$ then $X' \in \mathcal{F}$.

We will often refer to a simple voting game as an SVG. The sets in $\mathcal{F}$ are called **winning coalitions**.

In less technical language, a simple voting game is a set of rules defined for voting so that if no one votes to pass an item the item does not pass, if all voters vote to pass an item it does pass, and the collections are monotone - meaning that if an item is passing with certain voters voting yes, the addition of more 'yes' voters will not cause the item to fail. Now that we have defined a simple voting game we develop related language and notation that will be helpful when talking about an SVG.

To help understand this definition even further, we provide two different examples.

Example 2.2. The first example that we provide is one that we often refer to as a majority type simple voting game. This is because the rule for a measure to pass is simply that either a majority or a certain number of voters vote for the measure to pass, regardless of who those voters are. Consider the situation where five friends need to decide whether they will go get coffee or not. They decide that if at least 3 of them want to go get coffee, then they will. Then if x_i where $i \in \{1, 2, 3, 4, 5\}$ represents each friend, the simple voting game for this situation follows.

$$\mathcal{F} = \{\{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_2, x_5\}, \{x_1, x_3, x_4\}, \{x_1, x_3, x_5\}, \{x_1, x_4, x_5\},$$
$$\{x_2, x_3, x_4\}, \{x_2, x_3, x_5\}, \{x_2, x_4, x_5\}, \{x_3, x_4, x_5\}, \{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_3, x_5\},$$
$$\{x_1, x_2, x_4, x_5\}, \{x_1, x_3, x_4, x_5\}, \{x_2, x_3, x_4, x_5\}, \{x_1, x_2, x_3, x_4, x_5\}\}$$

Example 2.3. The next example that we provide is maybe a more realistic or more common type of example. In this example the number of voters voting to pass a measure is not as important as who the voters are. Consider an event committee comprised of 5 members, of these 5 members there is a committee chair as well as two assistant committee chairs. For this event committee the rules for decision making say that a measure passes if at least the chair or both sub-committee chairs vote for a measure. Thus, if X represents the committee chair, Y_1 and Y_2 represent the two sub-committee chairs, and x_1, x_2 represents the other two committee members, the simple voting game for this situation follows.

$$\mathcal{F} = \{\{X\}, \{Y_1, Y_2\}, \{X, x_1\}, \{X, x_2\}, \{X, x_1, x_2\}, \{X, Y_1\},$$
$$\{X, Y_1, x_1\}, \{X, Y_1, x_2\}, \{X, Y_1, x_1, x_2\}, \{X, Y_2\}, \{X, Y_2, x_1\}, \{X, Y_2, x_2\},$$
$$\{X, Y_2, x_1, x_2\}, \{X, Y_1, Y_2\}, \{X, Y_1, Y_2, x_1\}, \{X, Y_1, Y_2, x_2\}\}$$
$$\{Y_1, Y_2, x_1\}, \{Y_1, Y_2, x_2\}, \{Y_1, Y_2, x_1, x_2\}, \{X, Y_1, Y_2, x_1, x_2\}\}$$

Definition 2.4. [Axe10] A coalition $X \in \mathcal{F}$ is called a **minimal winning coalition** if no proper subset of X is winning. The set of these coalitions is denoted $\mathcal{F}^{min}$. If N is the set of all voters, an element $i \in N$ is called a **dummy** if it is not an element of any set in $\mathcal{F}^{min}$. A voter is called a **non-dummy** otherwise. For an SVG $\mathcal{F}$, we define the **support($\mathcal{F}$)** to be the set of non-dummies.

Again, to help clarify these definitions we will provide examples. We will provide examples for each of these definitions by refering to both Example 2.2 and Example 2.3.

Example 2.5. Here we go back to Example 2.2, in which there are five friends deciding whether to go for coffee. In this example,

$$\mathcal{F}^{min} = \{\{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_2, x_5\}, \{x_1, x_3, x_4\}, \{x_1, x_3, x_5\}, \{x_1, x_4, x_5\},$$
$$\{x_2, x_3, x_4\}, \{x_2, x_3, x_5\}, \{x_2, x_4, x_5\}, \{x_3, x_4, x_5\}\}.$$

In this case since all five friends are in at least one winning coalition, support($\mathcal{F}$) $=$ $\{x_1, x_2, x_3, x_4, x_5\}$, and there are no dummy voters.

Example 2.6. Similarly in Example 2.3, involving the decision rule for an event committee, $\mathcal{F}^{min} = \{\{X\}, \{Y_1, Y_2\}\}$. Thus, clearly support($\mathcal{F}$) $= \{X, Y_1, Y_2\}$ and x_1 and x_2 are dummy voters.

We will now define properties that simple voting games may have. These properties will be assumptions of our main theorem.

Definition 2.7. [Axe10] An SVG is called **proper** if for any $X, X' \in \mathcal{F}$, $X \cap X' \neq \emptyset$.

Definition 2.8. [Axe10] The **Coleman Index** is a measure of power that can be calculated for an SVG $\mathcal{F}$ on a set N. This index is denoted $C(\mathcal{F})$ and is calculated

$$C(\mathcal{F}) = \frac{|\mathcal{F}|}{2^{|N|}}.$$

An SVG is **maximal** if $C(\mathcal{F})$ is maximum, that is if $C(\mathcal{F}) = 1/2$.

Looking at the way that the Coleman Index is calculated one can see that the numerator of the fraction is simply the number of winning coalitions and the denominator is simply the total number of coalitions possible (including a coalition with no voters, $\emptyset$). The Coleman Index is the probability that a measure will pass for the voting game. A high Coleman Index would mean that out of all of the possible coalitions, most of them would be winning given the specific decision rule. If several of the possible coalitions are winning, this means that there are multiple ways that members can 'vote' and the measure will pass. Thus a high Coleman Index would mean that given that SVG it would be relatively easy for a measure to pass. Similarly, a low Coleman Index would mean that very few of the possible coalitions could be winning, meaning that it would be difficult for a measure to pass.

Definition 2.9. [Axe10] An SVG is **swap robust** if for any two winning coalitions X, Y and $x \in X \setminus Y$, $y \in Y \setminus X$, either $(X \setminus \{x\}) \cup \{y\}$ or $(Y \setminus \{y\}) \cup \{x\}$ is a winning coalition.

Here we will provide three different examples that will show that these properties are independent, for example, if a simple voting game is proper that will not automatically imply that the simple voting game is also swap robust or maximal.

Example 2.10. In this first example, $\mathcal{F}$ is a simple voting game that is maximal but not proper and not swap robust.

$$\mathcal{F} = \{\{x_1, x_2\}, \{x_1, x_5\}, \{x_4, x_5\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_2, x_5\},$$
$$\{x_1, x_5, x_3\}, \{x_1, x_5, x_4\}, \{x_4, x_5, x_2\}, \{x_4, x_5, x_3\}, \{x_1, x_2, x_3, x_4\},$$
$$\{x_1, x_2, x_3, x_5\}, \{x_1, x_2, x_4, x_5\}\{x_1, x_3, x_4, x_5\}, \{x_2, x_3, x_4, x_5\}, \{x_1 x_2 x_3 x_4 x_5\}\}$$

We notice here that we have 16 elements in $\mathcal{F}$ and there are five voters. Thus calculating the Coleman Index we have; $C(\mathcal{F}) = \frac{|\mathcal{F}|}{2^{|N|}} = \frac{16}{2^5} = \frac{16}{32} = \frac{1}{2}$ thus clearly this SVG is maximal. Also, if we look at $\mathcal{F}^{min}$ for this simple voting game, $\{\{x_1, x_2\}, \{x_1, x_5\}, \{x_4, x_5\}\}$ we can clearly see that this simple voting game is neither swap robust nor proper.

Example 2.11. Here we provide the following simple voting game,

$$\mathcal{F} = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_1, x_5\}, \{x_2, x_3\}, \{x_2, x_4\},$$
$$\{x_2, x_5\}, \{x_3, x_4\}, \{x_3, x_5\}, \{x_4, x_5\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_2, x_5\}, \{x_1, x_3, x_4\},$$
$$\{x_1, x_3, x_5\}, \{x_1, x_4, x_5\}, \{x_2, x_3, x_4\}, \{x_2, x_3, x_5\}, \{x_2, x_4, x_5\}, \{x_3, x_4, x_5\}, \{x_1, x_2, x_3, x_4\},$$
$$\{x_1, x_2, x_3, x_5\}, \{x_1, x_2, x_4, x_5\}, \{x_1, x_3, x_4, x_5\}, \{x_2, x_3, x_4, x_5\}, \{x_1, x_2, x_3, x_4, x_5\}\}.$$

This simple voting game is swap robust but not proper and not maximal. Looking at $\mathcal{F}^{min}$ in this simple voting game it is very clear that this simple voting game is not proper. Also, since there are 31 winning coalitions in $\mathcal{F}$ and 5 voters,$C(\mathcal{F}) = \frac{31}{32}$ thus this SVG is not maximal.

Example 2.12. Finally we provide a simple voting game that is proper but not swap robust and not maximal.

$$\mathcal{F} = \{\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_3, x_4, x_5\}, \{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_3, x_5\},$$
$$\{x_1, x_3, x_4, x_5\}, \{x_2, x_3, x_4, x_5\}, \{x_1, x_2, x_3, x_4, x_5\}\}\}$$

This simple voting game is proper but not maximal as $C(\mathcal{F}) = \frac{8}{32}$. Also by looking at the two winning coalitions $\{x_1, x_2, x_3\}$ and $\{x_2, x_3, x_4\}$ we can see that if we switch voters 1 and 4 we have the two coalitions $\{x_4 x_2 x_3\}$ and $\{x_2 x_3 x_1\}$. Neither of these two coalitions are in $\mathcal{F}$, thus neither of them are winning coalitions, therefore this simple voting game is not swap robust.

Definition 2.13. [Axe10] An SVG is **r-trivial** if $\exists N'$ such that $|N'| = 2r - 1$ and $\mathcal{F}^{min} = \{X \in \mathcal{P}(N') \mid |X| = r\}$.

To make this more clear let us understand what would need to be shown to show an SVG, $\mathcal{F}$, is r-trivial. We would need to show that there is a set X in $\mathcal{F}$ that has $2r-1$ elements, as well as show that the elements in $\mathcal{F}^{min}$ are all possible subsets of X consisting of r elements.

We have seen that thinking about simple voting games in different ways can be helpful to prove different results. We will now define the necessary terminology to discuss a simple voting game as the *set system* that it is. We will also define a *hypergraph* and related terminology, specifically colorings, as the relationship between a simple voting game and a hypergraph coloring is the basis for the majority of the proofs in this paper.

Definition 2.14. [Axe10] A **set system on a set** N is a collection of subsets of N. We say that a set system is r**-uniform** if all sets in the system have size r. A set system is called an **intersecting system** if given any two sets, X, Y in the system, $X \cap Y \neq \emptyset$. A **Sperner system** is a set system such that for any two sets X, Y in the system, $X \nsubseteq Y$. In a set system $\mathcal{F}$, the **up-set($\mathcal{F}$)** is the collection of all sets $S \in \mathcal{P}(N)$ such that $X \subseteq S$ where $X \in \mathcal{F}$

Clearly we can see that simple voting games are set systems on a set of voters. If we look at Example 2.2 we see that all sets in $\mathcal{F}^{min}$ have size 3, thus $\mathcal{F}^{min}$ is 3-uniform. Also, $\mathcal{F}^{min}$ is clearly an intersecting and Sperner system. However, looking at Example 2.3, we see that the two sets in $\mathcal{F}^{min}$ are not the same size, thus we can not say for any r that $\mathcal{F}^{min}$ is r-uniform. Also, clearly looking at $\mathcal{F}^{min}$ we can see that it is not an intersecting system but that it is a Sperner system. For a simple voting game $\mathcal{F}$, $\mathcal{F} = \text{upset}(\mathcal{F}^{min})$.

Definition 2.15. [Axe10] Let $\mathcal{F}$ be a set system on N. The pair (N,$\mathcal{F}$) is called a **hypergraph** on vertex set N with edge-set $\mathcal{F}$. We call the elements of N **vertices** and elements of $\mathcal{F}$ **edges**. An **isolated vertex** is a vertex that is not contained in any edge. A **proper coloring of a hypergraph** is an assignment of colors to the vertices of a hypergraph so that each edge of the hypergraph has at least two distinctly colored vertices. We say that a hypergraph is ***k*−colorable** if there exists a proper coloring with k colors. A hypergraph that is **k-chromatic** is one in which the hypergraph is k-colorable but not $(k-1)$-colorable.

Example 2.16. Here, we show an example of a hypergraph that could represent Example 2.2. In this hypergraph each of the five friends, the voters, are represented by a vertex and the winning coalitions are the edges of the hypergraph. The edges in the hypergraph are ovals around the three voters in a given coalition, that is the three vertices in that edge.

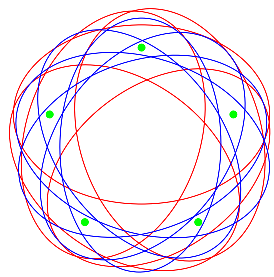
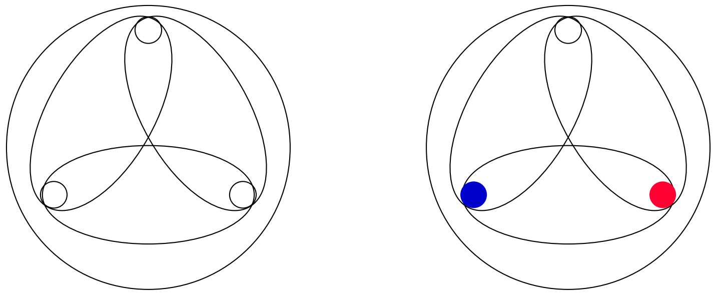


Figure 2.1. Hypergraph of Example 2.2

Example 2.17. Figure 2.2a is an example of a hypergraph in which there are four vertices in the vertex set and four edges in the edge set. This would mean that if this represented a simple voting game, we would have four voters and four winning coalitions. Two of the four edges have 2 vertices each, one edge has three vertices, and the last edge has four vertices.

Figure 2.2b is an example of a properly colored hypergraph. Each edge has at least two distinctly colored vertices. We can see that this hypergraph was properly colored with 3 colors; red, white, and blue, thus its is 3-colorable. We can also see that the vertex positioned towards the top of the figure is currently colored white, if it were to be colored either red or blue, it would not be a proper coloring as that would create an edge with vertices of only one color, which would not be a proper coloring. Hence, this hypergraph is not 2-colorable and is thus 3-chromatic.

Here we define necessary terms for understanding relationships between different vertices of hypergraphs.



(A) Hypergraph (B) 3-Chromatic Hypergraph

Figure 2.2. Hypergraph and Hypergraph Colorings

Definition 2.18. [Axe10] A vertex $x \in N$ is said to have **smaller than or equal influence** with a vertex $y \in N$ if for each edge E containing x but not y, $E' = \{E \setminus \{x\}\} \cup \{y\}$ is an edge. We denote this $x \leq y$. A vertex $x \in N$ has **smaller influence** than a vertex $y \in N$ if $x \leq y$ and there exists a set $E'' \in \mathcal{F}$ containing y and not x so that $\{E'' \setminus y\} \cup \{x\} \notin \mathcal{F}$. If two vertices x,$y \in N$ are such that $x \leq y$ and $y \leq x$ we say that x and y are **twins** and write $x \sim y$. Clearly this means that x and y have equal influence.

We note here that this definition is very important to this project. In the proof of the main theorem we see that it is possible to order the set of voters in a useful and meaningful way, crucial to making arguments and providing justification in the proof. Since it is such an important definition, we will now provide an example to help clarify these definitions. Then, we will prove that $\sim$ is an equivalence relation.

Example 2.19. An example we will use to help understand these definitions is the procedure for amending the Canadian constitution [TP08]. In this procedure, for an amendment to pass, 7 provinces must agree to the amendment and 50 % of the population must be represented. As a consequence, Quebec or Ontario must be in favor. Table 1 includes information about the provinces and the percent of the population residing in those provinces. Note that the numbers in this table are rounded, clearly Prince Edward Island represents some percentage of the population. However, since this is simply an example to show how influence relationships are determined between voters we will leave the numbers in the rounded condition. This allows for ease of calculation as well as a few interesting cases. Using this information we can construct examples that show some provinces have smaller influence than others and that some provinces are actually twins.

Province	**Percentage of Population**
Prince Edward Island	0 %
Newfoundland	2 %
New Brunswick	2 %
Nova Scotia	3 %
Saskatchewan	3 %
Manitoba	4 %
Alberta	11 %
British Columbia	13 %
Quebec	23 %
Ontario	39 %

TABLE 1. Information for Amending the Canadian Constitution [SC12]

Given the procedure for amending the Canadian constitution there are two general claims we can make about the comparative influence of respective provinces. We will state and prove those claims here.

Claim 2.20. *Given any two provinces x and y, where x represents a larger percentage of total population than y, $x \geq y$.*

Proof. Assume that there is a winning coalition meaning that there is a group of seven provinces representing at least 50% of the population. Also assume that province y is one of these seven provinces but province x is not. Clearly, if we replace province y with province x we will still have seven provinces in the coalition. However, since x represents a larger percentage of the population than y, the coalition will still represent at least 50% of the population. Therefore, the new coalition is also a winning coalition, implying that $x \geq y$. □

Claim 2.21. *Given any two provinces x and y where x and y both represent the same percentage of the population, $x \geq y$ and $x \leq y$ or $x \sim y$*

Proof. Assume that there is a winning coalition containing province x but not province y. If province x was removed and province y was added to the coalition, the new coalition would clearly be winning. This is because there would be the same number of provinces represented, which since the previous coalition was winning, must be at least seven. Also, the percentage of population represented should be the same in the new coalition as the old, which must be at least 50%. Thus, $y \geq x$ Similarly, by relabeling, we can see $x \geq y$. Hence, $x \sim y$. □

Now, we will show how the provinces are related in terms of influence.

Claim 2.22. *The provinces are related in terms of influence in the following way:*

(1) *Ontario* $>$ *Quebec*
(2) *Quebec* $\sim$ *British Columbia*
(3) *British Columbia* $>$ *Alberta*
(4) *Alberta* $>$ *Manitoba*
(5) *Manitoba* $\sim$ *Saskatchewan*
(6) *Saskatchewan* $\sim$ *Nova Scotia*
(7) *Nova Scotia* $\sim$ *New Brunswick*
(8) *New Brunswick* $\sim$ *Newfoundland*
(9) *Newfoundland* $>$ *Prince Edward Island.*

Proof. (1) **Ontario > Quebec**

By Claim 2.20 we already know that Quebec $\leq$ Ontario. In order to show that Quebec has strictly less influence than Ontario, we need to show that there exists a winning coalition containing Ontario but not Quebec, so that if Ontario is removed and Quebec is added, this new coalition is no longer winning. **Coalition 1**: Ontario, Alberta, Manitoba, Saskatchewan, Nova Scotia, New Brunswick, Newfoundland We note that this coalition is winning because it has 7 provinces and represents 64% of the population. Now, replacing Ontario with Quebec we obtain Coalition 2. **Coalition 2**: Quebec, Alberta, Manitoba, Saskatchewan, Nova Scotia, New Brunswick, Newfoundland This new coalition is not winning because even though it does have 7 provinces, it only represents 48% of the population. Thus Ontario > Quebec.

(2) **Quebec ∼ British Columbia**

By Claim 2.20 we already know that British Columbia ≤ Quebec. We now start with the smallest winning coalition by percentage of population represented. In several of these examples we will start with smallest winning coalition. This is because in order to show a ≤ relationship between two voters we need to prove that a swap of the provinces can be made in all winning coalitions with the given conditions and the coalition remain winning. Thus, if we start with the smallest winning coalition and can show that removing a province and replacing it with a province that represents a smaller percentage of population results in a winning coalition, the same will be true for other winning coalitions with equal or bigger initial population percentage. Essentially, proving the relationship for the smallest case will in turn prove the relationship for all cases.

Coalition 1, containing Quebec and not British Columbia. **Coalition 1**: Ontario, Quebec, Saskatchewan, Nova Scotia, New Brunswick, Newfoundland, and Prince Edward Island. At first glance it may not be clear why these provinces make up the smallest winning coalition. We will explain the logic for this coalition here, it can then be understood that a similar logic has been applied to determine the smallest winning coalitions in the following examples. For our proof we need to begin with a coalition that contains Quebec but not British Columbia. For such a coalition to be winning it must also contain Ontario in order to reach the required representation of at least 50% of the population. Ontario and Quebec together represent more than 50% of the population, thus to compile the rest of the coalition we simply choose the smallest 5 remaining provinces to satisfy the requirement of a winning coalition having 7 provinces represented.

This coalition has seven provinces and represents 72% of the population. Now replacing Quebec and adding British Columbia, we have Coalition 2. **Coalition**

2: Ontario, British Columbia, Saskatchewan, Nova Scotia, New Brunswick, Newfoundland, and Prince Edward Island. This coalition again has seven provinces and represents 62% of the population, thus it is still winning. This means that Quebec $\leq$ British Columbia. Hence, since British Columbia $\leq$ Quebec and Quebec $\leq$ British Columbia; by definition Quebec $\sim$ British Columbia.

(3) **British Columbia > Alberta**

By Claim 2.20 we already know that Alberta $\leq$ British Columbia. We now start with the smallest winning coalition, Coalition 1, containing British Columbia and not Alberta. **Coalition 1**: Quebec, British Columbia, Manitoba, Saskatchewan, Nova Scotia, New Brunswick, and Newfoundland. This coalition has seven provinces and represents 50% of the population. Now replacing British Columbia and adding Alberta, we have Coalition 2. **Coalition 2**: Quebec, Alberta, Manitoba, Saskatchewan, Nova Scotia, New Brunswick, and Newfoundland. This coalition again has seven provinces and represents 48% of the population, thus it is no longer winning. This means that British Columbia > Alberta.

(4) **Alberta > Manitoba**

By Claim 2.20 we already know that Manitoba $\leq$ Alberta. We now start with the smallest winning coalition, Coalition 1, containing Alberta and not Manitoba. **Coalition 1**: Quebec, British Columbia, Alberta, Nova Scotia, New Brunswick, Newfoundland, and Prince Edward Island. This coalition has seven provinces and represents 54% of the population. Now replacing Alberta and adding Manitoba, we have Coalition 2. **Coalition 2**: Quebec, British Columbia, Manitoba, Nova Scotia, New Brunswick, Newfoundland, and Prince Edward Island. This coalition again has seven provinces and represents 47% of the population, thus it is no longer winning. This means that Alberta > Manitoba.

(5) **Manitoba $\sim$ Saskatchewan**

By Claim 2.20 we already know that Saskatchewan $\leq$ Manitoba. We now start with the smallest winning coalition, Coalition 1, containing Manitoba and not Saskatchewan. **Coalition 1**: Quebec, British Columbia, Alberta, Manitoba, New Brunswick, Newfoundland, and Prince Edward Island. This coalition has seven provinces and represents 55% of the population. Now replacing Manitoba and adding Saskatchewan, we have Coalition 2. **Coalition 2**: Quebec, British Columbia, Alberta, Saskatchewan, New Brunswick, Newfoundland, and Prince Edward Island. This coalition again has seven provinces and represents 54% of the population, thus it is still winning. This means that Manitoba $\leq$ Saskatchewan. Hence, since Saskatchewan $\leq$ Manitoba and Manitoba $\leq$ Saskatchewan; by definition Manitoba $\sim$ Saskatchewan.

(6) **Saskatchewan $\sim$ Nova Scotia**

This is given by Claim 2.21.

(7) **Nova Scotia $\sim$ New Brunswick**

By Claim 2.20 we already know that New Brunswick $\leq$ Nova Scotia. We now start with the smallest winning coalition, Coalition 1, containing Nova Scotia and not New Brunswick . **Coalition 1**: Quebec, British Columbia, Alberta, Saskatchewan, Nova Scotia, Newfoundland, and Prince Edward Island. This coalition has seven provinces and represents 55% of the population. Now replacing Nova Scotia and adding New Brunswick, we have Coalition 2. **Coalition 2**: Quebec, British Columbia, Alberta, Saskatchewan, New Brunswick, Newfoundland, and Prince Edward Island. This coalition again has seven provinces and represents 54% of the population, thus it is still winning. This means that Nova Scotia $\leq$ New Brunswick. Hence, since New Brunswick $\leq$ Nova Scotia and Nova Scotia $\leq$ New Brunswick; by definition Nova Scotia $\sim$ New Brunswick.

(8) **New Brunswick $\sim$ Newfoundland**

This is also given by Claim 2.21.

(9) **Newfoundland $>$ Prince Edward Island**

By Claim 2.20 we already know that Prince Edward Island $\leq$ Newfoundland. We now start with the smallest winning coalition, Coalition 1, containing Newfoundland

and not Prince Edward Island . **Coalition 1**: Quebec, British Columbia, Manitoba,Saskatchewan, Nova Scotia, New Brunswick, and Newfoundland. This coalition has seven provinces and represents 50% of the population. Now replacing Newfoundland and adding Prince Edward Island, we have Coalition 2. **Coalition 2**: Quebec, British Columbia, Manitoba,Saskatchewan, Nova Scotia, New Brunswick, and Prince Edward Island. This coalition again has seven provinces and represents 48% of the population, thus it is no longer winning. This means that Newfoundland $>$ Prince Edward Island.

□

Claim 2.23. $\sim$ *is an equivalence relation.*

Proof. To show that $\sim$ is an equivalence relation, we must show that it is reflexive, symmetric, and transitive.

First we note that the reflexive case is true vacuously.

Now, to show that $\sim$ is symmetric, we must show that if $x \sim y$ then $y \sim x$. Assume that $x \sim y$. This means that $x \leq y$ and $y \leq x$. Since, $y \leq x$ and $x \leq y$, $y \sim x$. Thus, $\sim$ is symmetric.

Finally, we will show that $\sim$ is transitive. To do so we need to show that if $x \sim y$ and $y \sim z$, then $x \sim z$. Assume that $x \sim y$ and $y \sim z$. This means that $x \leq y$ and $y \leq x$ as well as $y \leq z$ and $z \leq y$. We will first show that $x \leq z$. Consider a coalition E such that $x \in E$ and $z \notin E$. There are then two cases for y, $y \in E$ or $y \notin E$.

Case 1: $y \notin E$

If we consider the coalition $E' = (E \backslash \{x\}) \cup \{y\}$ it is clear that this coalition is winning since $x \leq y$. Now consider the coalition $E'' = (E' \backslash \{y\}) \cup \{z\}$, this coalition is also clearly winning since $y \leq z$. Notice however that E'' is simply the coalition $(E \backslash \{x\}) \cup \{z\}$. Thus since this coalition is winning, it is clear then that $x \leq z$

Case 2: $y \in E$

Consider the coalition $E' = (E \backslash \{y\}) \cup \{z\}$ it is clear that this coalition is winning since $y \leq z$. Now consider the coalition $E'' = (E' \backslash \{x\}) \cup \{y\}$, this coalition is also clearly winning since $x \leq y$. Notice however that E'' is simply the coalition $(E \backslash \{x\}) \cup \{z\}$. Thus since this coalition is winning, it is clear then that $x \leq z$

Hence, if $x \sim y$ and $y \sim z$ then $x \leq z$. By a similar argument since $y \leq x$ and $z \leq y$, $z \leq x$.

Thus if $x \sim y$ and $y \sim z$, $x \sim z$.

Therefore, since $\sim$ is reflexive, symmetric, and transitive; it is an equivalence relation. $\square$

Definition 2.24. [Axe10] In a swap robust family $\mathcal{F}$, for $E, E' \in \mathcal{F}$, we say that $E \preceq E'$ if for some $x \in E$ and $y \notin E$, and $x \leq y$, $E' = (E \backslash \{x\}) \cup \{y\}$. We also say that E and E' are in **shift order** and write $E \leq_s E'$ if there is a sequence of sets $E = E_1 \preceq E_2 \preceq ... \preceq E_m = E'$.

Example 2.25. Here, using the information above about the procedure for amending the Canadian constitution, we have constructed an example that shows two sets in shift order. Consider the following three sets:

E = {Ontario, Quebec, British Columbia, Alberta, Nova Scotia, New Brunswick, Newfoundland } - 7 provinces, 93% population

E^* = {Ontario, Quebec, British Columbia, Alberta, Manitoba, Nova Scotia, New Brunswick} - 7 provinces, 95% population

E' = {Quebec, British Columbia, Alberta, Manitoba, Saskatchewan, Nova Scotia} - 7 provinces, 96% population

Note that all three of these sets would be winning coalitions via the requirements for amending the Canadian constitution. Since Newfoundland is in E and Manitoba is not in E and Newfoundland $\leq$ Manitoba; $E \preceq E^*$. Similarly, since New Brunswick is in E^* and Saskatchewan is not in E^* and New Brunswick $\leq$ Saskatchewan; $E^* \preceq E'$. Thus we have $E \preceq E^* \preceq E'$, which implies that $E \leq_s E'$.

Definition 2.26. [Sib08] A relation R on a set X is **antisymmetric** iff for all $x, y \in X$, if xRy and yRx, then $x = y$. A relation R on a set X is a **partial order** iff it is reflexive, transitive, and antisymmetric. A partial order, $\trianglelefteq$ on a set X is a **total order** iff for all $a, b \in X$, we have $a \trianglelefteq b$ or $b \trianglelefteq a$.

Example 2.27. Here we provide an example of two relations that are partial orders, one that is a total order and one that is not. The first relation we present is the familiar operator $\subseteq$ on the power set of some set X. Since this is a familiar relation it is easy to see that this in fact a partial order. However, this relation is not a total order. To see this consider $\subseteq$ on $\mathcal{P}(S)$ where $S = \{1, 2, 3\}$. Note that $\{1\}, \{2\} \in \mathcal{P}(S)$ but $\{1\} \nsubseteq \{2\}$ and $\{2\} \nsubseteq \{1\}$. Thus, it isn't a total order. If we now consider the familiar relation $\leq$ on a $\mathbb{Z}$, again we can easily see that it is a partial order. Also, for any two integers a, b clearly $a \leq b$ or $b \leq a$, thus $\leq$ is a total order.

3. Theorems

Now that we have discussed the necessary definitions we state the propositions we will use to prove our main result and then finally state this result, Theorem 3.6.

Proposition 3.1. [Axe10] *Each intersecting family of subsets of* $[n] = \{1, 2, \ldots, n\}$ *has at most* 2^{n-1} *members. It has exactly* 2^{n-1} *members if and only if for any* $X \subseteq [n]$ *either* X *or* X^c *is in the family.*

Proof. We will begin by showing that each intersecting family of subsets of $[n]$ has at most 2^{n-1} members. Let $\mathcal{F}$ be an intersecting family of subsets of $[n]$. Let $S \in \mathcal{F}$, then since $\mathcal{F}$ is intersecting $S^c \notin \mathcal{F}$. Define $\mathcal{F}^c = \{S^c : S \in \mathcal{F}\}$. Then $\mathcal{F} \cup \mathcal{F}^c \subseteq 2^{[n]}$. Note that $2^{[n]}$ is the power set of the set $[n]$. By definition, we know that $\mathcal{F} \cap \mathcal{F}^c = \emptyset$. As a direct result of the way we have constructed $\mathcal{F}^c$, $|\mathcal{F}| = |\mathcal{F}^c|$. Thus we have that $|\mathcal{F}| \leq |2^{[n]}|/2 \Rightarrow |\mathcal{F}| \leq 2^n/2 = 2^{n-1}$. Hence, $\mathcal{F}$ has at most 2^{n-1} members.

Now we will show that if $\mathcal{F}$ has exactly 2^{n-1} members, then for any $X \subseteq [n]$ either X or X^c is in the family. Assume that $\mathcal{F}$ has exactly 2^{n-1} members. Thus, $\mathcal{F}^c$ has 2^{n-1} members. This then means that $|\mathcal{F} \cup \mathcal{F}^c| = 2^n$ which implies that $\mathcal{F} \cup \mathcal{F}^c = 2^{[n]}$. Now, for any $K \subseteq [n]$, $K \in \mathcal{F} \cup \mathcal{F}^c$, meaning $K \in \mathcal{F}$ or $K \in \mathcal{F}^c$. If $K \in \mathcal{F}$ we are done and if $K \in \mathcal{F}^c$ by definition $K^c \in \mathcal{F}$. Thus if $\mathcal{F}$ has exactly 2^{n-1} members, then for $K \subseteq [n]$ either K or K^c is in the family. Since K was arbitrary, this proves our proposition for any set $X \subseteq [n]$.

Finally we will show that if for any $X \subseteq [n]$ either X or X^c is in the family then $\mathcal{F}$ has exactly 2^{n-1} members. We will do this by the contrapositive. That is we will show that if $\mathcal{F}$ has less than 2^{n-1} members then there exists a set $X \subseteq [n]$ such that neither X nor X^c is in the family. Assume that $\mathcal{F}$ has less than 2^{n-1} members, then $\mathcal{F}^c$ has less than 2^{n-1} members. This means that $|\mathcal{F} \cup \mathcal{F}^c| < 2^n$ which implies that $\mathcal{F} \cup \mathcal{F}^c \subset 2^{[n]}$. Since this union is only a subset, there are subsets of $[n]$ that are not in $\mathcal{F} \subsetneq \mathcal{F}^c$. Call one of these subsets J, then $J \notin \mathcal{F}$ and $J \notin \mathcal{F}^c$. Thus we have shown that such a set X as desired exists. Therefore, if for a set $X \subseteq [n]$ either X or X^c is in the family then $\mathcal{F}$ has exactly 2^{n-1} members. $\square$

Proposition 3.2. [Axe10] *An intersecting Sperner system $\mathcal{F}$ of subset of $[n]$ has size 2^{n-1} if and only if $\mathcal{F}^{min}$ corresponds to a 3-chromatic hypergraph.*

Proof. Assume that $\mathcal{F}$ is an intersecting Sperner system of subsets of $[n]$. We will first show that if $\mathcal{F}$ has size 2^{n-1} then $\mathcal{F}^{min}$ corresponds to a 3-chromatic hypergraph. Assume that $\mathcal{F}$ has size 2^{n-1}. Let $A \in \mathcal{F}^{min}$ and let $x \in A$. We will now color $\mathcal{F}$ in the following way; $A \backslash \{x\}$ is red, vertex $\{x\}$ is green, and the remaining vertices of $\mathcal{F}$ are blue. We know that $\mathcal{F}$ is intersecting, thus each set in $\mathcal{F}$ intersects by at least one vertex. This means that each set in $\mathcal{F}$ has a red vertex or green vertex since it must intersect the set A by either a red vertex or by x. We know that $\mathcal{F}^{min}$ is Sperner by definition, thus each set except for A has a blue vertex. This must be the case, because if it was entirely red or red and green, since the set A is the only set with these colored vertices, these sets would be subsets of A which can not be the case since $\mathcal{F}^{min}$ is Sperner. Thus, having red, green, and blue vertices, $\mathcal{F}^{min}$ is 3-colorable. We are trying to show that $\mathcal{F}^{min}$ is 3-chromatic, which would mean that $\mathcal{F}^{min}$ is not 2-colorable. To show that this is the case, we will assume by way of contradiction that

$\mathcal{F}^{min}$ is 2-colorable. In a proper 2-coloring of $\mathcal{F}^{min}$ we will let R be the set of red vertices and the set $B = R^c$ be the set of blue vertices. Since $\mathcal{F}$ has size 2^{n-1}, by Proposition 3.1 either R or B must be in $\mathcal{F}$. Assume that $R \in \mathcal{F}$, this means that some subset of R, r' is in $\mathcal{F}^{min}$. This is a contradiction because $\mathcal{F}^{min}$ is intersecting, thus the sets in $\mathcal{F}^{min}$ must contain both blue and red vertices, however R' has only red vertices. Thus $R \notin \mathcal{F}$. Thus by Proposition 3.1, $B \in \mathcal{F}$. Thus, $B' \subseteq B \in \mathcal{F}^{min}$. Again, this is a contradiction since B' contains only blue vertices. Therefore, we have a contradiction and $\mathcal{F}^{min}$ is not 2-colorable. Since $\mathcal{F}^{min}$ is 3-colorable, this means that $\mathcal{F}^{min}$ is 3-chromatic, as desired.

We will now show if $\mathcal{F}^{min}$ corresponds to an 3-chromatic hypergraph then $\mathcal{F}$ has size 2^{n-1}. Let $\mathcal{F}'$ be a Sperner, 3-chromatic, intersecting hypergraph and $\mathcal{F} = \text{upset}(\mathcal{F}')$. By Proposition 3.1, if we show that for each set $A \in 2^{[n]}$ either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$ this will prove that $\mathcal{F}$ has size 2^{n-1}. Thus, we assume by way of contradiction that there exists a set A such that $A \notin \mathcal{F}$ and $A^c \notin \mathcal{F}$. We now color the vertices of A red and the vertices of A^c blue. Such a coloring is possible since neither A of A^c are in $\mathcal{F}$. Consider a set S in $\mathcal{F}'$. This set cannot be a subset of A because this would then imply that $A \subseteq \mathcal{F}$ since $\mathcal{F} = \text{upset}(\mathcal{F}')$. However, $A \subseteq \mathcal{F}$ is a contradiction. A similar argument leads to the contradiction $A^c \subseteq \mathcal{F}$. Thus, if S is not a subset of either A or A^c and these sets together comprise all of the vertices, S must have at least one vertex from each set. This then means that S has both red and blue vertices. Since S was an arbitrary set in $\mathcal{F}'$, this means each set in $\mathcal{F}'$ has both red and blue vertices. Therefore, $\mathcal{F}'$ is 2-colorable. Yet, $\mathcal{F}'$ being 2-colorable is a contradiction since our initial assumption was that $\mathcal{F}'$ was 3-chromatic.

Therefore, $\mathcal{F}$ has size 2^{n-1} if and only if $\mathcal{F}^{min}$ corresponds to a 3-chromatic hypergraph.

□

This second proposition allows us to define the relationship between a simple voting game, $\mathcal{F}$, and a 3-chromatic hypergraph. The following three propositions provide the necessary tools to use this relationship to prove our main theorem. More specifically, they allow us to show that every case other than the desired result would lead to $\mathcal{F}$ corresponding to a 2-chromatic hypergraph which by the first proposition is a contradiction.

Proposition 3.3. [Axe10] *If $\mathcal{F}$ is an intersecting 3-chromatic family with no isolated vertices then for any $E \in \mathcal{F}$ and any $x \in E$, there is $E' \in \mathcal{F}$ such that $E' \cap E = \{x\}$. Also, for any two vertices x, y, $x \neq y$, there is $E \in \mathcal{F}$ such that $x \in E$ and $y \notin E$.*

Proof. Assume that $\mathcal{F}$ is an intersecting 3-chromatic family with no isolated vertices. Let $E \in \mathcal{F}$ and $x \in E$. We need to show that there exists an $E' \in \mathcal{F}$ such that $E \cap E' = \{x\}$. Color the set $E \backslash \{x\}$ red and the other vertices in $\mathcal{F}$ blue. We have just colored all of the vertices in $\mathcal{F}$ with two colors and we know that $\mathcal{F}$ is 3-chromatic and thus not 2-colorable. This means that the coloring we have proposed is not a proper coloring thus there exists some monochromatic edge. We will call this edge E^*. E^* cannot be red, as that would imply that $E^* \subseteq E$ which cannot be the case since we have assumed that $\mathcal{F}$ is Sperner. This then means that E^* is blue. We have also assumed that $\mathcal{F}$ is an intersecting family, thus $E \cap E^* \neq \emptyset$. Clearly, the vertices in this intersection must be blue, hence $E \cap E^*$ must be $\{x\}$. Thus, for any set $E \in \mathcal{F}$, E^* is the desired set E'.

We will now show that for any two vertices x, y, $x \neq y$, there is $S \in \mathcal{F}$ such that $x \in S$ and $y \notin S$. By the first part of this proposition, we know that there are two sets in $\mathcal{F}$ such that their intersection is x; let these sets be S' and S''. Then we know that there are three cases, $y \in S'$, $y \in S''$, or y is in neither S' or S''. For the first case, if $y \in S'$, since the intersection of S' and S'' is only x, $y \notin S''$. Similarly for the second case, if $y \in S''$,$y \notin S'$. Finally, if y is in neither S' or S'', either of these two sets meet the desired condition. Thus, we have shown for all three cases that there exists some set S where $x \in S$ and $y \notin S$. □

Proposition 3.4. [Axe10] *Vertices of a swap robust SVG $\mathcal{F}$ can be ordered $x_1, x_2, \ldots, x_n$ such that $x_i \leq x_j$ for all $1 \leq i < j \leq n$.*

Proof. Showing that $\leq$ is a total order on $\mathcal{F}$, up to $\sim$-equivalence, will suffice to show that vertices of $\mathcal{F}$ can be ordered in such a way. Thus we will show that $\mathcal{F}$ is a partial order, up to $\sim$-equivalence, and is in fact actually a total order, up to $\sim$-equivalence .

Assume that $\mathcal{F}$ is a swap robust family and that x, y and z are vertices in $\mathcal{F}$. In order to show that $\leq$ is a proper order on $\mathcal{F}$ we must show that $\leq$ is reflexive, transitive, and antisymmetric. Reflexive and transitive are clear from the proof of Claim 2.23 (that $\sim$ is an equivalence relation). We know that $\leq$ is not antisymmetric because it is possible for two distinct voters, say l and m to be twins. In this case $l \leq m$ and $m \leq l$ but l and m are distinct voters. However, given such conditions $l \sim m$ and have equal influence. This is why we will say that, $\leq$ is a partial order up to $\sim$-equivalence. Now, we will show that $\leq$ is a total order up to $\sim$-equivalence. To show this, we must show that for all $x, y \in \mathcal{F}$, $x \leq y$ or $y \leq x$. This is true because $\mathcal{F}$ is swap robust. We will prove this by way of contradiction. Assume that $x \nleq y$ and that $y \nleq x$. This means that there are winning coalitions X and Y such that $x \in X, y \notin X, y \in Y, x \notin Y$. Also, $X \backslash \{x\} \cup \{y\} \notin \mathcal{F}$ because if the coalition was in $\mathcal{F}$, $x \leq y$. Also, $Y \backslash \{y\} \cup \{x\} \notin \mathcal{F}$ because if the coalition was in $\mathcal{F}$, $y \leq x$. However, this means that neither of the previously winning coalitions are still winning after the swap of voters x and y, but this contradicts the fact that $\mathcal{F}$ is swap robust. Therefore, for any $x, y \in \mathcal{F}$ $x \leq y$ or $y \leq x$. This proves that $\leq$ is a total order, up to $\sim$-equivalence, on $\mathcal{F}$. □

Proposition 3.5. [Axe10]

Assume that $\mathcal{F}$ is a proper, maximal, swap robust SVG such that $\mathcal{F}^{min}$ is r-uniform.

i) Let X = support($\mathcal{F}$) and |support($\mathcal{F}$)| $\geq 2r$. By proposition 3.4, order the vertices of X, $x_1, x_2, \ldots x_{n'}$ so that $x_i \leq x_j$ for all $1 \leq i < j \leq n'$. Let $X' = \{x_1, x_2, \ldots, x_{n'-2r+2}\}$ Then for any x,$y \in X', x \sim y$.

ii) Again, by the influence relation, let $X'' = \{x_{n'}, x_{n'-1}, \ldots, x_{n'-r+2}\}$ where $x_{n'} \geq x_{n'-1} \geq \ldots \geq x_{n'-r+2}$. For any $E \in \mathcal{F}^{min}, E \cap X'' \neq \emptyset$. In other words, any set in $\mathcal{F}^{min}$ will contain at least one of the $(r-1)$ most influential voters.

Proof. Assume that $\mathcal{F}$ is a proper, maximal, swap robust SVG such that $\mathcal{F}^{min}$ is r-uniform.

i) Let x be a voter and two winning coalitions, E' and E'', be such that E', $E'' \in \mathcal{F}^{min}$ and $E' \cap E'' = \{x\}$. We know that such an E' and E'' exist by Proposition 3.3. Now consider $y \in E' \cup E''$; such a y is possible because $|X| = n' \geq 2r$ and $|E' \cup E''| = 2r-1$. Let $E \in \mathcal{F}^{min}$ so that $y \in E$ and $x \notin E$, again such a coalition E exist by Proposition 3.3. Since $\mathcal{F}$ is a swap robust simple voting game, if we consider x, y, E, and E'', either $(E \setminus \{y\}) \cup \{x\} \in \mathcal{F}^{min}$ or $(E'' \setminus \{x\}) \cup \{y\} \in \mathcal{F}^{min}$. However, we know that $((E'' \setminus \{x\}) \cup \{y\}) \cap E' = \emptyset$, thus

$(E'' \setminus \{x\}) \cup \{y\} \notin \mathcal{F}^{min}$ since $\mathcal{F}$ is proper. Thus we know that $(E \setminus \{y\}) \cup \{x\} \in \mathcal{F}^{min}$ which implies that $y \leq x$.

We know that for any x, since y was an arbitrary element from $X \setminus \{E' \cup E''\}$ and $|X \setminus \{E' \cup E''\}| = n' - (2r - 1) = n' - 2r + 1$, there are at least $n' - 2r + 1$ voters with smaller or equal influence as x. If all of the voters in X were twins, then clearly the subset X' would contain voters that were all twins and the proposition would then be proven, thus we will assume that not all voters in X are twins. Recall that voters in X are ordered by influence in which x_1 has smaller than or equal influence than x_2 which has smaller than or equal influence than x_3 and so on. Thus consider the smallest index j in which x_1 has smaller influence than x_j, $x_1 < x_j$, and thus voters x_1 to x_{j-1} are all twins. Thus there are $j - 2$ voters y in which $y \leq x_{j-1}$. Previously, we have noted that there are at least $n' - 2r + 1$ voters y such that $y \leq x_{j-1}$, thus since we know that there are $j - 2$ such voters, $n' - 2r + 1 \leq j - 2$. This implies that $j \geq n' - 2r + 3$. Hence $\{x_1, \ldots, x_{n'-2r+2}\} \subseteq \{x_1, \ldots, x_{j-1}\}$ are twins, as desired.

ii) By the influence relation, let $X' = \{x_1, x_2, \ldots, x_{n'-2r+2}\}$ and $X'' = \{x_{n'}, x_{n'-1}, \ldots, x_{n'-r+2}\}$ where $x_{n'} \geq x_{n'-1} \geq \ldots \geq x_{n'-r+2} \geq \ldots \geq x_{n'-2r+2} \geq \ldots \geq x_1$. Also, $\{x_1, x_2, \ldots, x_{n'}\} \in X = \text{support}(\mathcal{F})$, $|\text{support}(\mathcal{F})| \geq 2r$. It is helpful to think about these elements of X in order of influence. We order them here from largest influence to smallest, meaning x_1 has the smallest influence; that is x_1 is the least influential voter of those in X.

$$\underbrace{x_{n'}, x_{n'-1}, \ldots, x_{n'-r+2}}_{X''}, \ldots, \underbrace{x_{n'-2r+2}, \ldots, x_2, x_1}_{X'}$$

We note here that $|X'| = n' - 2r + 2$ and $|X''| = r - 1$. Let $\mathcal{F}$ be a proper, maximal, swap robust simple voting game with elements in $\mathcal{F}^{min}$ having size r. Let $E \in \mathcal{F}^{min}$ and assume by way of contradiction that $E \cap X'' = \emptyset$. Let $A = E \cap X'$. We can see that A is nonempty because if it were empty that would mean that $E \cap (X' \cup X'') = \emptyset$ and this would imply that $|E| = |X - X' - X''| = n' - (n' - 2r + 2) - (r - 1) = r - 1$. However, since $E \in \mathcal{F}^{min}$, it must be that $|E| = r$ thus A must be nonempty.

By Proposition 3.5 part (i), we know that elements of X' are twins, thus they all have the same influence. Because of this, we can reorder the elements of X' without changing the meaning of these elements being ordered in terms of influence. Hence, we will reorder the elements of X' so that $A = \{x_1, \ldots, x_i\}$. We will now let voter $v \in X \setminus (E \cup X'')$. This means that voter v is not in E, X'', or A. We note here that we can say such a voter v exists because $|X| \geq 2r$ and $|E \cup X''| = r + (r-1) = 2r-1$. Clearly, even if $|X| = 2r$, $X \setminus (E \cup X'')$ would have at least one element since $2r - (2r-1) = 1$. Now that we have established the existence of voter v we define $Q = \{v\} \cup X''$. We can see that every element in Q except for v has greater influence than elements in E; we know this because X'' is the set of most influential voters in X and that $E \cap X'' = \emptyset$. Similarly, we also know that v is the least influential voter in Q because $\{v\} \cap X'' = \emptyset$. Also, since $\{v\} \cap A = \emptyset$ and A contains the least influential voters $x_i \leq v$ where $x_i \in A$. Thus if we start with the set $E = E_1$ and swap voter v for a voter $x_i \in A$ we will obtain a set E_2 in which $E_1 \preceq E_2$. Now, by continuing to swap voters in E now with voters in X'' we will continually obtain sets E_i with at least one voter having larger than or equal influence than a voter in E_{i-1}. We can make these swaps $r-1$ times since E has r elements and one of them has already been swapped for voter v, thus leaving $r-1$ elements to swap with the $r-1$ elements of X''. Thus we can construct a sequence $E = E_1 \preceq E_2 \preceq \ldots \preceq E_r = Q$, meaning that $E \leq_s Q$. Thus since $E \in \mathcal{F}^{min}$, $Q \in \mathcal{F}^{min}$. Thus since $\mathcal{F}$ is proper, $E \cap Q$ is nonempty. However, since $\{v\} \cap E = \emptyset$ and $X'' \cap E = \emptyset$ and $Q = \{v\} \cup X''$, $E \cap Q = \emptyset$. Since this is a contradiction it must be that for any $E \in \mathcal{F}^{min}$, $E \cap X'' \neq \emptyset$.

□

We now state the theorem that we will prove for simple voting games.

Theorem 3.6. [Axe10] *If $\mathcal{F}$ is a proper, maximal swap robust SVG such that $\mathcal{F}^{min}$ is r-uniform, then $\mathcal{F}$ is r-trivial.*

Proof. Assume that $\mathcal{F}$ is a proper, maximal, swap robust SVG such that $\mathcal{F}^{min}$ is r-uniform. Since $\mathcal{F}^{min}$ is r-uniform, we know that all sets in $\mathcal{F}^{min}$ have size r. We are trying to show that $\mathcal{F}$ is r-trivial; to do so we must show that the sets in $fmin$ are in fact all possible r element sets of some other set with cardinality $2r-1$. To begin, assume that $r=1$, then clearly $\mathcal{F}^{min}$ consists of one singleton set. If $r=1$ then $2r-1=2(1)-1=1$. Since the singleton set is a subset of itself, it is clearly an element of a 1 element set. Thus if $r=1$, $\mathcal{F}$ would be the collection of all sets containing this singleton set and satisfies the definition of being r-trivial. This is a trivial case, thus we will consider when $r \geq 2$. Assume $r \geq 2$. Now we will let X be the support($\mathcal{F}$) and $n' = |\text{support}(\mathcal{F})|$. Since we are trying to show that sets in $\mathcal{F}^{min}$ are in fact all possible r element sets of some other set with cardinality $2r-1$, and we know that the sets in $\mathcal{F}^{min}$ are all composed of elements from support($\mathcal{F}$) we have four cases. These cases are that $n' < 2r-1$, $n' = 2r-1$ but $\mathcal{F}^{min}$ is incomplete, meaning that $fmin$ is not composed of *all possible* r element sets of X, $n' > 2r-1$, or $n' = 2r-1$ and $\mathcal{F}^{min}$ is complete. We will show that the first three cases lead to a contradiction.

Case 1: Assume by way of contradiction that $n' < 2r-1$. In this case we can thus say that $n' \leq 2r-2$. If this is the case the largest that n' can be is $2r-2 = 2(r-1)$ Thus we can think of splitting n' into two groups, each of size no bigger than $r-1$. Then, color one of the groupings red and one of the groupings blue. We now recall that elements of $\mathcal{F}^{min}$ must be members of support($\mathcal{F}$) and that sets in $\mathcal{F}^{min}$ have size r, thus clearly any set in $\mathcal{F}^{min}$ would have both blue and red members. Now, if we considered a hypergraph of $\mathcal{F}^{min}$ where the edge set is $\mathcal{F}^{min}$ and the vertex set is X, this hypergraph would be two colorable. However, by Proposition 3.2 , we know that $\mathcal{F}^{min}$ should correspond to a 3 - chromatic hypergraph, thus $\mathcal{F}^{min}$ being two colorable is a contradiction. Hence it cannot be the case that $n' \leq 2r-1$.

Case 2: Assume by way of contradiction that $n' = 2r-1$ but $\mathcal{F}^{min}$ is not complete. If $\mathcal{F}^{min}$ is not complete, this means that $\mathcal{F}^{min}$ is not comprised of all possible r sized sets of X. Thus there exists a subset of X with cardinality r that is not in $\mathcal{F}^{min}$. Color this set blue. Then the remaining elements of X have cardinality $2r-1-r=r-1$. Color these $r-1$ remaining elements red. Again, since $\mathcal{F}^{min}$ consists of sets of size r, each set in $\mathcal{F}^{min}$ must then have both blue and red elements. Thus if we consider the hypergraph of $\mathcal{F}^{min}$ where the edge set is $\mathcal{F}^{min}$ and the vertex set is X, this hypergraph would be two colorable. Once again, by Proposition 3.2, we know that $\mathcal{F}^{min}$ should correspond to a 3 - chromatic hyporgraph, thus $\mathcal{F}^{min}$ being two colorable is a contradiction. Therefore, it cannot be that $n' = 2r - 1$ and $\mathcal{F}^{min}$ is incomplete.

Case 3: Assume by way of contradiction that $n' > 2r-1$. In this case we can say that $n' \geq 2r$. We will now order the elements of X by the influence relation, allowable by proposition 2; we will order them $x_1, x_2, ..., x_{n'}$, where $x_i \leq x_j$ for all $1 \leq q < j \leq n'$. Define the set $X'' = \{x_{n'}, x_{n'-1}, ..., x_{n'-r+2}\}$. Now consider any element E in $\mathcal{F}^{min}$. Then by Proposition 3.5 we know that $E \cap X'' \neq \emptyset$. Thus we begin by coloring elements in X'' red and all other elements of X blue. We first notice that no element in $\mathcal{F}^{min}$ can be all red, because $|X''| = r - 1$ and we have assumed that elements in $\mathcal{F}^{min}$ have size r. We also know that no element in $\mathcal{F}^{min}$ can be all blue since $E \cap X'' \neq \emptyset$, meaning each element in $\mathcal{F}^{min}$ will contain at least one element of X'', a red element. Thus $\mathcal{F}^{min}$ is 2 colorable, which again by Proposition 3.2 is a contradiction. Thus, it cannot be the case that $n' > 2r - 1$.

Therefore, since all other cases lead to a contradiction, the only possible case is that $n' = 2r - 1$ and $\mathcal{F}^{min}$ is complete. Thus, by definition, this shows that $\mathcal{F}$ is r-trivial as desired.

□

This concludes our study on this topic; however we believe that there are questions interesting for further study. In this paper we have shown that simple voting games that are swap robust, proper, maximal, and have r-uniform minimal winning coalitions are r-trivial. Further study on this topic could involve removing one of the four conditions on the simple voting games to observe where the proofs break down and to see if a different result could be obtained. Another method would be to relax some of the properties; for instance one could look at what would happen if the Coleman Index was not maximal, i.e. not $\frac{1}{2}$, but maybe it was $\frac{1}{3}$ or $\frac{2}{3}$. One could even look at imposing restrictions based on a different power index.

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