13.012 Hydrodynamics for Ocean Engineers

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Introduction to basic principles of fluid mechanics

1. Flow Descriptions

1. Lagrangian:

In rigid body mechanics the motion of a body is described in terms of the body's position in time. This body can be translating and possibly rotating, but not deforming. This description, following a particle in time, is a *Lagrangian* description.

$$\vec{V} = u\,\hat{i} + v\,\hat{j} + w\,\hat{z} \qquad (4.1)$$

Thus we can describe a particle located at point $\overline{x_o} = (x_o, y_o, z_o)$ for some time $t = t_o$, such that

$$\vec{V} = \frac{\partial(x - x_o)}{\partial t} \qquad (4.2)$$

and

$$\vec{a} = \frac{\partial \vec{V}}{\partial t}\,. \qquad (4.3)$$

1. Eulerian:

In a fluid there are many particles and, unlike rigid bodies, parcels of fluid and tend to deform continuously as they move. In order to fully describe the flow we must account for these deformations. Thus it is useful to use the *Eulerian* description, or control volume approach, and describe the flow at every fixed point in space (x, y, z) as a function of time, t.

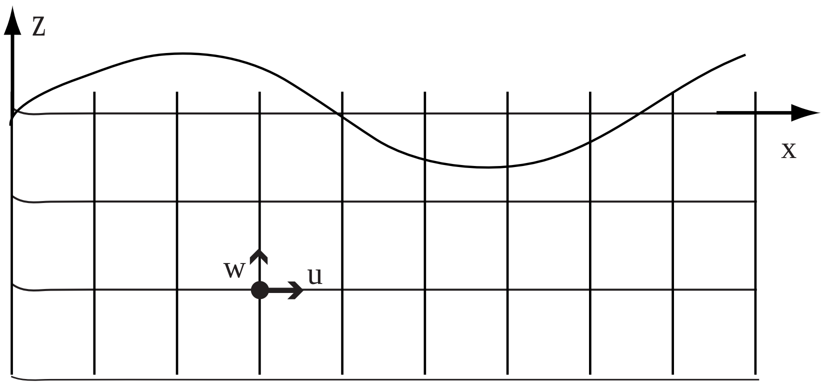


Figure 1: An *Eulerian* description gives a velocity vector at every point in x,y,z as a function of time.

Eulerian velocity field at any time, t, at any position, $p(x, y, z, t)$, such that velocity is a function of the position vector and time: $\vec{V}(\vec{x}, t)$.

$$\text{eg: } \vec{V}(\vec{x}, t) = 6tx^2\hat{i} + 3zy\hat{j} + 10xyt^2\hat{z}$$

2. *Description of Motion:*

Streamlines: Line everywhere tangent to velocity (Eulerian) (No velocity exists perpendicular to the streamline!)

Streaklines: instantaneous loci of all fluid particles that pass through a given point x_o.

Particle Pathlines: Trajectory of fluid particles ("more" lagrangian)

In ***steady flow*** stream, streak, and pathlines are identical!! (Steady flow has no time dependence.)

2. Governing Laws

The governing laws of fluid motion can be derived in multiple forms using a simple control volume approach. This is equivalent to a "fluidic black box" where all we know is what is going in and coming out of the volume (mass, momentum, energy, work, etc). The control volume (CV) can be fixed or move with the fluid. For simplicity it is often ideal to fix the CV. For most of this class the CV will be fixed. In an ideal situation we will pick the control volume that makes our lives easier mathematically.

When analyzing a control volume problem there are three laws that MUST be followed:

1. Conservation of Mass
2. Conservation of Momentum
3. Conservation of Energy

1. Conservation of Mass:

Basic fluid mechanics laws dictate that mass is conserved within a *control volume* for constant density fluids. *Thus the total mass entering the control volume must equal the total mass exiting the control volume.* Using figure 2 we can write a 2D mass balance equation for the fluid entering and exiting the control volume $\delta x \delta z$.

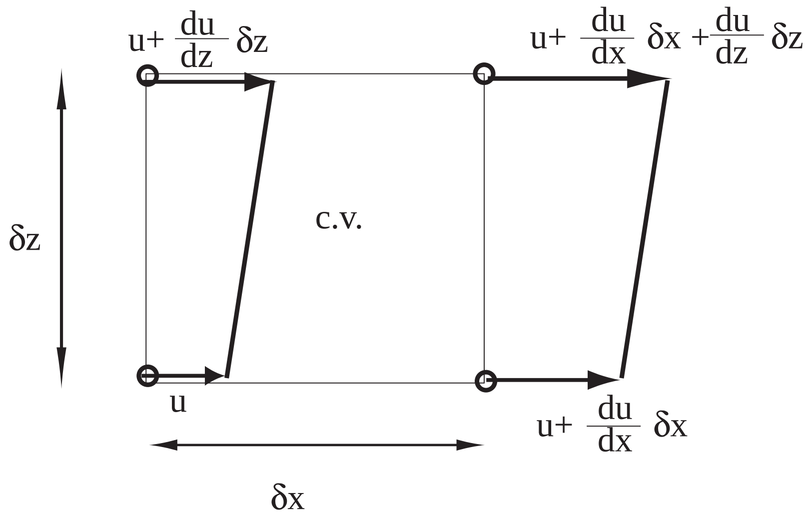


Figure 2. Control volume $\delta x \delta z$.

In the x-direction the mass balance equation is shown in equation 1,

$$\left\{u+\frac{1}{2}\frac{\partial u}{\partial z}\delta z\right\}\delta z-\left\{u+\frac{\partial u}{\partial x}\delta x+\frac{1}{2}\frac{\partial u}{\partial z}\delta z\right\}\delta z=-\left\{\frac{\partial u}{\partial x}\delta x\right\}\delta z. \tag{4.4}$$

The first term on the LHS of eq. 1 represents the mass entering the control volume and the second term (on the LHS) the fluid exiting the control volume. A similar equation can be written for the mass balance in the z direction resulting in a net vertical fluid flux of

$$-\frac{\partial w}{\partial z}\delta x\delta z. \tag{4.5}$$

For an incompressible fluid, the sum of these two balances must be zero for mass to be conserved. Therefore

$$-\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)\delta x\delta z=0. \tag{4.6}$$

This equation is valid for all δx and δz and can be simplified to arrive at the two-dimensional equation for conservation of mass:

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0}. \tag{4.7}$$

Similarly in three-dimensions, the equation for mass conservation can be written as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{4.8}$$

Recalling the gradient operator from vector calculus: $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$, we can abbreviate equation 5 as

$$\nabla \cdot \mathbf{V} = 0. \tag{4.9}$$

2. Conservation of Momentum:

Newton's second law is simply the *law of conservation of momentum*.

It states that *the time rate of change of momentum of a system of particles is equal to the sum of external forces acting on that body*.

$$\Sigma \mathbf{F_i} = \frac{d}{dt}\{M\mathbf{V}\} \tag{4.10}$$

where $M = \rho \delta x \delta z$ is the mass of the fluid parcel (in two dimensions, ie mass per unit length) and $M\mathbf{V}$ is the linear momentum of the system ($\mathbf{V}$ is the velocity vector). Since the fluid density is constant, the time-rate of change of linear momentum can be written as

$$\frac{d}{dt}\{M\mathbf{V}\} = \rho \delta x \delta z \frac{d\mathbf{V}}{dt}. \tag{4.11}$$

The rate of change of velocity of the fluid parcel can be found, for small δt, as

$$\frac{d\mathbf{V}}{dt} = \frac{lim}{\delta t \to 0} \frac{1}{\delta t} \left\{ \mathbf{V}(x + \delta x_p, z + \delta z_p, t + \delta t_p) - \mathbf{V}(x, z, t) \right\} \tag{4.12}$$

We can substitute, $\delta x_p = u\delta t$, and, $\delta z_p = w\delta t$, into equation 9 and cancel terms to arrive at a more familiar form of the momentum equation.

The total derivative of the velocity is written as:

$$\frac{D\mathbf{V}}{Dt} = \frac{\partial \mathbf{V}}{\partial t} + \frac{\partial \mathbf{V}}{\partial x} u + \frac{\partial \mathbf{V}}{\partial z} w \tag{4.13}$$

which can be simplified using the vector identity,

$$\mathbf{V} \cdot \nabla = u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \tag{4.14}$$

The total (material) derivative of the velocity is the sum of the conventional acceleration, $\frac{\partial V}{\partial t}$, and the advection term, $(\mathbf{V} \cdot \nabla)\mathbf{V}$:

$$\frac{D\mathbf{V}}{Dt} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V}. \tag{4.15}$$

Finally, the momentum equation, from equation 7, can be rewritten in two dimensions as

$$\Sigma \mathbf{F_i} = \rho \frac{D\mathbf{V}}{Dt} \delta x \delta z . \tag{4.16}$$

3. Forces

The LHS of equation 7 is the *sum of the forces acting on the control volume*. Contributions from gravity and pressure both play a role in this term as well as any applied external forces.

1. Force on a fluid volume due to gravity:

$$\mathbf{F_g} = -(\rho g \delta x \delta z)\hat{k} \tag{4.17}$$

2. Pressure Forces:

$$F_P = P \cdot A. \tag{4.18}$$

Pressure force in x-direction:

$$\mathbf{F_{Px}} = \left(p + \frac{1}{2}\frac{\partial p}{\partial z}\delta z \right)\delta z - \left(p + \frac{1}{2}\frac{\partial p}{\partial z}\delta z + \frac{\partial p}{\partial x}\delta x \right)\delta z = -\frac{\partial p}{\partial x}\delta x \delta z \tag{4.19}$$

Pressure force in z-direction:

$$\mathbf{F_{Pz}} = -\frac{\partial p}{\partial z}\delta x \delta z$$

Total pressure force in two dimensions:

$$F_P = -(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial z})\delta x \delta z = -\nabla p \delta x \delta z. \tag{4.20}$$

4. Euler Equation

Substituting relations 14 and 17 for the gravity and pressure forces acting on the body, into the momentum equation 13 we arrive at

$$\rho \left\{ \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} \right\} \delta x \delta z = (-\rho g \delta x \delta z)\hat{k} - \nabla p \delta x \delta z \tag{4.21}$$

for any $\delta x, \delta z$. The final result is the Euler equation in vector form:

$$\rho\left\{\frac{\partial \mathbf{V}}{\partial t}+(\mathbf{V}\cdot\nabla)\mathbf{V}\right\}=-\rho g\hat{k}-\nabla p. \tag{4.22}$$

We can further manipulate this equation with the vector identity

$$(\mathbf{V}\cdot\nabla)\mathbf{V}=\tfrac{1}{2}\nabla(\mathbf{V}\cdot\mathbf{V}), \tag{4.23}$$

such that the Euler equation becomes

$$\rho\left\{\frac{\partial \mathbf{V}}{\partial t}+\frac{1}{2}\nabla(\mathbf{V}\cdot\mathbf{V})\right\}=-\rho g\hat{k}-\nabla p. \tag{4.24}$$

5. Bernoulli's Equation:

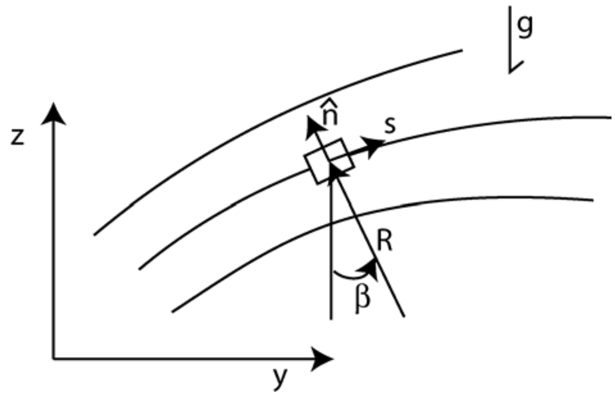
Application of *Newton's Second Law along a streamline*:

$$p_1+\tfrac{1}{2}\rho V_1^2+\rho g z_1 = p_2+\tfrac{1}{2}\rho V_2^2+\rho g z_2 = C \tag{4.25}$$

Assuming the following conditions:

1) Points 1 and 2 are on the same streamline!
2) Fluid density is constant
3) Flow is steady: $\frac{dV}{dt}=0$ (no time dependence or turbulence)
4) Fluid is "inviscid" or can be approximated as inviscid. No frictional effects
5) No Work Added!

We can derive this through a Lagrangian derivation:



Looking at a small elemental volume along a streamline $d\forall = dndsdx$ (dx is the depth into the paper).

Fluid weight in the (-z) direction.:

$$\rho g\ dn\ ds\ dx \tag{4.26}$$

Component of weight acting in the s-direction:

$$-\rho g \sin\beta\ dn\ ds\ dx \tag{4.27}$$

Where $\sin\beta = \dfrac{dz}{ds}$ so that the weight in the s-direction is:

$$-\rho g\ \frac{dz}{ds} dn\ ds\ dx\ . \tag{4.28}$$

The force due to pressure in the s-direction is found similarly:

$$F_s = \left(p - \frac{\partial p}{\partial s} \right) dn\ dx - \left(p + \frac{\partial p}{\partial s}\frac{ds}{2} \right) dn\ dx = \left(-\frac{\partial p}{\partial s} - \rho g \frac{\partial z}{\partial s} \right) dn\ ds\ dx \tag{4.29}$$

The force accelerates the fluid along the streamline such that the rate of change in momentum, per unit volume, is

$$\rho \left(\frac{V + \dfrac{\partial V}{\partial s} ds - V}{dt} \right) = \rho V \frac{\partial V}{\partial s} \tag{4.30}$$

where $V = \frac{\partial s}{\partial t}$.

So Euler's equation in one dimension along a streamline becomes:

$$\boxed{\rho V \frac{\partial V}{\partial s} + \frac{\partial p}{\partial s} + \rho g \frac{\partial z}{\partial s} = 0}. \tag{4.31}$$

Change in Pressure along a streamline: $dp = \frac{\partial p}{\partial s} ds$

Change in Velocity along a streamline: $dV = \frac{\partial V}{\partial s} ds$

Change in height along a streamline: $dz = \frac{\partial z}{\partial s} ds$

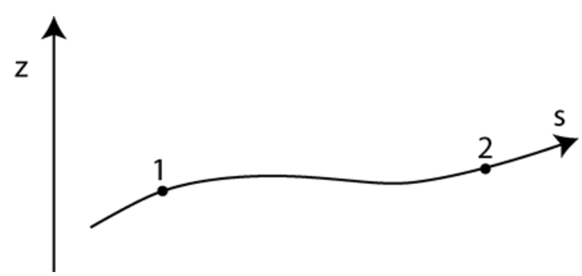
Multiplying equation 23 through by *ds* gives us

$$\rho V\ dV + dp + \rho g\ dz = 0 \tag{4.32}$$

$$\frac{dp}{\rho} + V\ dV + g\ dz = 0 \tag{4.33}$$

If density is constant along the streamline then we can integrate along the streamline to get:

$$\frac{p}{\rho} + \tfrac{1}{2} V^2 + g\ z = C$$



Along a streamline Bernoulli's equation relates pressure, height and velocity at two points:

$$\frac{p_1}{\rho} + \tfrac{1}{2}V_1^2 + g\ z_1 = \frac{p_2}{\rho} + \tfrac{1}{2}V_2^{\ 2} + g\ z_2 = C \tag{4.34}$$

This equation also assumes that NO additional heat or work is added to the system along the streamline.

1. Irrotational flow

For *irrotational flow* the curl of the velocity must be zero.

$$\omega = \nabla \times \mathbf{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = 0 \tag{4.35}$$

$$\omega = \hat{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0. \tag{4.36}$$

For 2D flow this reduces to $\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$.

2. Potential Flow

Define a potential function, $\phi(x,z,t)$, as a continuous function that satisfies the basic laws of fluid mechanics: *conservation of mass and momentum*, assuming incompressible and irrotational flow. Such that

$$u = \frac{\partial \phi}{dx} \text{ and } w = \frac{\partial \phi}{dz} \tag{4.37}$$

which satisfy the irrotational condition.

Since

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\frac{\partial \phi}{\partial x} \ , \tag{4.38}$$

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\frac{\partial \phi}{\partial z} \tag{4.39}$$

and ϕ is continuous we can prove that the potential function also satisfies conservation of mass (derivative in x and z are interchangeable). So

$$\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} . \tag{4.40}$$

3. Unsteady Bernoulli's Equation

The potential function can be substituted into equation 20 resulting in the unsteady Bernoulli Equation.

$$\rho\left\{\frac{\partial}{\partial t}\nabla\phi + \frac{1}{2}\nabla V^2\right\} + \nabla p + \rho g \nabla z = 0 \tag{4.41}$$

$$\nabla\left\{\rho\frac{\partial \phi}{\partial t} + \frac{1}{2}\rho V^2 + p + \rho g z\right\} = 0 \tag{4.42}$$

$$UnsteadyBernoulli \Rightarrow \rho\frac{\partial \phi}{\partial t} + \frac{1}{2}\rho V^2 + p + \rho g z = c(t) \tag{4.43}$$

4. Laplace Equation

Returning to the conservation of mass equation, we can substitute in the relationship between potential and velocity and arrive at a new form for the equation of mass conservation. This equation is the *Laplace Equation* which we will revisit in our discussion on linear waves.

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{4.44}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{4.45}$$

$$LaplaceEquation \Rightarrow \nabla^2\phi = 0$$

