Introduction to Classical and Quantum Integrability

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Prospects in Fields, Strings and Related Topics

Summary

1. From classical to quantum - The non-linear Schrödinger eq.

2. Classical and quantum R-matrices. Exact S-matrices

3. Bethe-ansatz and thermodynamics. Massless flows

4. Quantisation of the Kadomtsev-Petviashvili (KP) equation

with K. Kozlowski and E. Sklyanin

1. From classical to quantum integrability

Based on AT, arXiv:1606.02946

and on

- Babelon, Bernard, Talon, "Introduction to classical integrable systems" Cambridge U. Press, 2003
- Faddeev, Takhtajan, "Hamiltonian methods in the theory of solitons", Springer, 1987
- Dunajski, "Integrable systems", U. of Cambridge lecture notes, 2012 [available online]
- Beisert, "Introduction to integrability", ETH lecture notes, 2017 [available online]
- Bombardelli *et al.*, "An integrability primer for the gauge-gravity correspondence: an introduction", 2016, arxiv:1606.02945

The Chronicles

- Exact solutions to Newton's equations hard to come by

Kepler problem exactly solved by Newton himself

- 1800s Liouville *integrability* for Hamilt. systems $\longrightarrow$ *quadratures*.
- 1900s systematic method of the *classical inverse scattering*

Gardner, Green, Kruskal and Miura in 1967 solved Korteweg-deVries (KdV) equation of fluid mechanics

- Quantum mechanical version of the inverse scattering method 1970s by Leningrad – St. Petersburg school

connection to Drinfeld and Jimbo's theory of quantum groups

$\rightarrow$ single math. framework for integrable QFT (Zamolodchikov^2) and lattice spin systems (Baxter).

- Nowadays integrability in different areas of maths and math-phys

L. Faddeev once wrote in 1996...

"One can ask, what is good in $1+1$ models, when our spacetime is $3+1$ dimensional. There are several particular answers to this question.

1. The toy models in $1+1$ dimension can teach us about the realistic field-theoretical models in a nonperturbative way. Indeed such phenomena as renormalisation, asymptotic freedom, dimensional transmutation (i.e. the appearance of mass via the regularisation parameters) hold in integrable models and can be described exactly.
2. There are numerous physical applications of the $1+1$ dimensional models in the condensed matter physics.
3. The formalism of integrable models showed several times to be useful in the modern string theory, in which the world-sheet is 2 dimensional anyhow. In particular the conformal field theory models are special massless limits of integrable models.
4. The theory of integrable models teaches us about new phenomena, which were not appreciated in the previous developments of Quantum Field Theory, especially in connection with the mass spectrum.
5. I cannot help mentioning that working with the integrable models is a delightful pastime. They proved also to be very successful tool for the educational purposes.

These reasons were sufficient for me to continue to work in this domain for the last 25 years (including 10 years of classical solitonic systems) and teach quite a few followers, often referred to as Leningrad - St. Petersburg school."

Liouville's theorem

Consider Hamilt. system with $2d$-dim. phase space and $H = H(q_\mu, p_\mu)$

$$(q_\mu, p_\mu) \qquad \mu = 1, ..., d$$

Poisson brackets

$$\{q_\mu, q_\nu\} = \{p_\mu, p_\nu\} = 0 \qquad \{q_\mu, p_\nu\} = \delta_{\mu\nu} \qquad \forall\ \mu, \nu = 1, ..., d$$

Liouville integrable if $\exists$ d indep. integrals of motion $F_\nu(q_\mu, p_\mu)$ globally defined and *in involution*:

$$\{F_\mu, F_\nu\} = 0 \qquad \forall\ \mu, \nu = 1, ..., d$$

- pointwise linear independence of the set of gradients ∇F_μ
- we take the Hamiltonian to be F_1

Theorem (Liouville)

E.o.m.s of a Liouville-integrable system can be solved "by quadratures"

"finite number of algebraic operations and integrations of known functions"

Upshot: $\exists$ canonical transf.

$$\omega = dp_\mu \wedge dx^\mu = dI_\mu \wedge d\theta^\mu$$

canonical 2-form

s.t. - all new momenta I_μ are integrals of motion and $H = H(I_\mu)$

- time-evolution of new coordinates is simply linear

$$\frac{d\theta_\mu}{dt} = \frac{\partial H}{\partial I_\mu} = \text{constant in time}$$

$\rightarrow$ solution by straightforward integration (performing one *quadrature*)

Typically mfld of const. I_μ is d-torus (Liouville-Arnold theorem) param. by $\theta_\mu \in [0, 2\pi)$

Motion occurs on such torus

Action-Angle variables

Typical example: harmonic oscillator

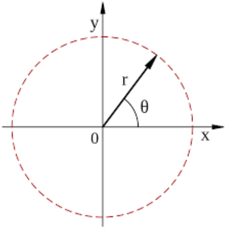
$$H = \frac{1}{2}(p^2 + q^2) \qquad \text{conserved}$$

The appropriate coordinate θ depends linearly on time:

$$q = \sqrt{2E}\cos\theta = \sqrt{2E}\cos(t+\phi)$$

for an initial phase ϕ and energy $E = r$.

Action-angle variables are the polar coordinates in the $(x, y) = (q, p)$ plane



One can have more than d integrals of motion (not all in involution)

$\longrightarrow$ superintegrable system

Maximally superintegrable system $\longrightarrow$ $2d - 1$ integrals of motion

Closed orbits - *e.g.* Kepler problem

But how can we find such integrals of motion?

Let us focus on algebraic methods $\longrightarrow$

Lax pairs

Suppose $\exists$ matrices L, M (*Lax pair*) s.t. e.o.m.s equivalent to:

$$\frac{dL}{dt} = [M, L]$$

These (not all indep.) quantites are all conserved:

$$I_n \equiv trL^n \qquad \frac{dI_n}{dt} = \sum_{i=0}^{n-1} tr\, L^i\,[M, L]L^{n-1+i} = 0 \qquad \forall\ n$$

Lax pair not unique:

$$L \longrightarrow g\, L\, g^{-1} \qquad M \longrightarrow g\, M\, g^{-1} + \frac{dg}{dt}\, g^{-1}$$

We regard L and M as elements of some matrix algebra $\mathfrak{g}$. Define

$$X_1 \equiv X \otimes 1, \qquad X_2 \equiv 1 \otimes X \qquad \in \mathfrak{g} \otimes \mathfrak{g}$$

Harmonic osc. example

L, M may depend on "spectral" parameter λ

Theorem

The eigenvalues of L are in involution iff $\exists\ r_{12} \in \mathfrak{g} \otimes \mathfrak{g}$ s.t

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2]$$

$r_{21} = \Pi \circ r_{12}$ $\quad \Pi$ *permutation operator on* $\mathfrak{g} \otimes \mathfrak{g}$

Jacobi identity requires

$$[L_1, [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] + \{L_2, r_{13}\} - \{L_3, r_{12}\}] + \text{cyclic} = 0$$

Harmonic osc. example

If r_{12} constant indep. on dynamical variables $\rightarrow$

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

is sufficient condition

Classical Yang-Baxter equation (CYBE)

If we require antisymmetry

$$r_{12} = -r_{21}$$

then r_{12} called constant *"classical r-matrix"*

Borel example $\longrightarrow$

Field-theory Lax pair

Suppose $\exists$ L, M s.t Euler-Lagrange equations are equiv. to

$$\frac{\partial L}{\partial t} - \frac{\partial M}{\partial x} = [M, L]$$

Such field theories are *'classically integrable"*

harder to assess nr. of d.o.f.

Define *monodromy matrix*

$$T(\lambda) = P \exp \left[\int_a^b L(x, t, \lambda) dx \right] \qquad \longleftarrow$$

$$\begin{aligned}
\partial_t\, T &= \int_a^b dx\, P \exp \left[\int_x^b L(x', t, \lambda) dx' \right] [\partial_t\, L(x, t, \lambda)]\ P \exp \left[\int_a^x L(x', t, \lambda) dx' \right] \\
&= \int_a^b dx\, P \exp \left[\int_x^b L(x', t, \lambda) dx' \right] \left(\frac{\partial M}{dx} + [M, L] \right) P \exp \left[\int_a^x L(x', t, \lambda) dx' \right] \\
&= \int_a^b dx\, \partial_x \left(P \exp \left[\int_x^b L(x', t, \lambda) dx' \right] M\ \ P \exp \left[\int_a^x L(x', t, \lambda) dx' \right] \right) \\
&= M(b)\, T - T\, M(a)
\end{aligned}$$

Periodic b.c.

$$\partial_t\, T = [M(a), T]$$

This implies that the trace of T, called the *transfer matrix*

$$\mathfrak{t} \equiv trT$$

is conserved *for all* λ *spectral parameter*

By Taylor-expanding in $\lambda \rightarrow$ family of conserved charges

Suppose $\exists$ r-matrix s.t.

$$\{L_1(x,t,\lambda), L_2(y,t,\mu)\} = [r_{12}(\lambda-\mu), L_1(x,t,\lambda) + L_2(y,t,\mu)]\,\delta(x-y)$$

with the r-matrix $r_{12}(\lambda-\mu)$ indep. of the fields and antisymmetric

Theorem (Sklyanin Exchange Relations)

$$\{T_1(\lambda), T_2(\mu)\} = [r_{12}(\lambda-\mu), T_1(\lambda)T_2(\mu)]$$

This means

$$[\mathfrak{t}(\lambda), \mathfrak{t}(\mu)] = 0 \qquad \text{by applying } tr \otimes tr$$

All charges are in involution (by Taylor expansion)

The Poisson brackets are called *"ultra-local"* - no derivatives of Dirac delta

Example: Sklyanin's treatment of Non-linear Schrödinger equation

[Sklyanin, "Quantum version of the method of inverse scattering problem", 1980]

Non-relativistic $1+1$ dimensional field theory with Hamiltonian

$$H = \int_{-\infty}^{\infty} dx \left(\left| \frac{\partial \psi}{\partial x} \right|^2 + \kappa |\psi|^4 \right), \qquad \psi(x) \in \mathbb{C} \qquad \text{assume } \kappa > 0$$

$$\{\psi(x), \psi^*(y)\} = i\delta(x-y) \qquad i\frac{\partial \psi}{\partial t} = \{H, \psi\} = -\frac{\partial^2 \psi}{\partial x^2} + 2\kappa |\psi|^2 \psi \quad \text{(hence the name)}$$

Lax pair

$$L = \begin{pmatrix} -i\frac{u}{2} & i\kappa\psi^* \\ -i\psi & i\frac{u}{2} \end{pmatrix} \qquad M = \begin{pmatrix} i\frac{u^2}{2} + i\kappa|\psi|^2 & \kappa\frac{\partial \psi^*}{\partial x} - i\kappa u\psi^* \\ \frac{\partial \psi}{\partial x} + iu\psi & -i\frac{u^2}{2} - i\kappa|\psi|^2 \end{pmatrix}$$

Monodromy matrix

$$T(u) = \begin{pmatrix} a(u) & \kappa\, b^*(u) \\ b(u) & a^*(u) \end{pmatrix}$$

take u real and then anal. cont. + call extrema $s_\pm$ for no confusion

$$a(u) = e^{-i\frac{u}{2}(s_+ - s_-)} \left[1 + \sum_{n=1}^{\infty} \kappa^n \int_{s_+ > \xi_n > \eta_n > \xi_{n-1} \dots > \eta_1 > s_-} d\xi_1 \dots d\xi_n \, d\eta_1 \dots d\eta_n \right.$$
$$\left. e^{iu(\xi_1 + \dots + \xi_n - \eta_1 - \dots - \eta_n)} \psi^*(\xi_1) \dots \psi^*(\xi_n)\, \psi(\eta_1) \dots \psi(\eta_n) \right]$$

$$b(u) = -i\, e^{i\frac{u}{2}(s_+ + s_-)} \sum_{n=0}^{\infty} \kappa^n \int_{s_+ > \eta_{n+1} > \xi_n > \eta_n > \xi_{n-1} \dots > \eta_1 > s_-} d\xi_1 \dots d\xi_n \, d\eta_1 \dots d\eta_{n+1}$$
$$e^{iu(\xi_1 + \dots + \xi_n - \eta_1 - \dots - \eta_{n+1})} \psi^*(\xi_1) \dots \psi^*(\xi_n)\, \psi(\eta_1) \dots \psi(\eta_{n+1})$$

Taking $s_\pm \to \pm\infty$ we have local charges

$$\mathfrak{I}_1 = \int_{-\infty}^{\infty} dx|\psi|^2 \qquad \mathfrak{I}_2 = \frac{i}{2}\int_{-\infty}^{\infty} dx\left(\frac{\partial\psi^*}{\partial x}\psi - \psi^*\frac{\partial\psi}{\partial x}\right)$$

$$\mathfrak{I}_3 = H \qquad \mathfrak{I}_m = \int_{-\infty}^{\infty} dx\,\psi^*\chi_m \qquad \chi_{m+1} = -i\frac{d\chi_m}{dx} + \kappa\psi^*\sum_{s=1}^{m-1}\chi_s\chi_{m-s}$$

Upon quantisation, the first charge corresponds to the particle number, the second one to the momentum, the third one to the Hamiltonian

Sklyanin shows $\exists\ r$ s.t.

$$\{T_1(\lambda), T_2(\mu)\} = [r_{12}(\lambda-\mu), T_1(\lambda)T_2(\mu)]$$

Action-angle type variables are those for which "the Hamiltonian H can be written as a quadratic form (and the equations of motion, correspondingly, become linear)." (E. Sklyanin)

$\longrightarrow$ Sklyanin's separation of variables

[classical and quantum - open problems]

Classical r-matrix for NLS

$$\{T_1(\lambda), T_2(\mu)\} = [r_{12}(\lambda-\mu), T_1(\lambda)T_2(\mu)]$$

$$r(\lambda-\mu) = -\frac{\kappa}{\lambda-\mu}P$$

P is the permutation matrix $\quad P|u\rangle \otimes |v\rangle = |v\rangle \otimes |u\rangle$

Prototypical solution of classical Yang-Baxter equation

$$[r_{12}(u_1-u_2), r_{13}(u_1-u_3)] + [r_{12}(u_1-u_2), r_{23}(u_2-u_3)] + [r_{13}(u_1-u_3), r_{23}(u_2-u_3)] = 0$$

As we will now show...

... exact quantisation will be a mere matter of algebra

Belavin-Drinfeld classifications theorems

[Belavin,Drinfeld, 1981]   there is no smaller font than this one

Theorem (Belavin Drinfeld I)

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, and $r = r(u_1 - u_2) \in \mathfrak{g} \otimes \mathfrak{g}$ a solution of the (spectral-parameter dependent) classical Yang-Baxter equation. Furthermore, assume one of the following three equivalent conditions:

- *(i) r has at least one pole in the complex plane $u \equiv u_1 - u_2$, and there is no Lie subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ such that $r \in \mathfrak{g}' \otimes \mathfrak{g}'$ for any u,*
- *(ii) $r(u)$ has a simple pole at the origin, with residue proportional to the tensor Casimir $\sum_a t_a \otimes t_a$, with t_a a basis in $\mathfrak{g}$ orthonormal with respect to a chosen nondegenerate invariant bilinear form,*
- *(iii) the determinant of the matrix $r_{ab}(u)$ obtained from $r(u) = \sum_{ab} r_{ab}(u)\, t_a \otimes t_b$ does not vanish identically.*

Under these assumptions, $r_{12}(u) = -r_{21}(-u)$ where $r_{21}(u) = \Pi \circ r_{12}(u) = \sum_{ab} r_{ab}(u)\, t_b \otimes t_a$, and $r(u)$ can be extended meromorphically to the entire u-plane. All the poles of $r(u)$ are simple, and they form a lattice Γ. One has three possible equivalence classes of solutions: "elliptic" - when Γ is a two-dimensional lattice -, "trigonometric" - when Γ is a one-dimensional array -, or "rational"- when $\Gamma = \{0\}$-, respectively.

The assumption of difference-form is not too restrictive:

Theorem (Belavin Drinfeld II)

Assume the hypothesis of Belavin-Drinfeld I theorem to hold (appropriately adapted) but $r = r(u_1, u_2)$ **not** *to be of difference form, with the classical Yang-Baxter equation being*

$$[r_{12}(u_1, u_2), r_{13}(u_1, u_3)] + [r_{12}(u_1, u_2), r_{23}(u_2, u_3)] + [r_{13}(u_1, u_3), r_{23}(u_2, u_3)] \;=\; 0$$

Now the three statements (i), (ii) and (iii) are no longer equivalent, and we will only retain (ii). Then, there exists a transformation which reduces r to a difference form.

If time $\longrightarrow$ proof

Quantisation by "Quantum group"

Completing the classical algebraic structure to a quantum group

$\leftrightarrow$

going from r to a solution to the *quantum Yang-Baxter Equation*

$$R_{12}\, R_{13}\, R_{23} \;=\; R_{23}\, R_{13}\, R_{12}\,, \qquad R_{ij} \sim 1 \otimes 1 + i\, \hbar\, r_{ij} + \mathcal{O}(\hbar^2)$$

and quantise Sklyanin relations by postulating "RTT" relations

$$R(u-u')\,\widehat{T}_1(u)\,\widehat{T}_2(u') \;=\; \widehat{T}_2(u')\,\widehat{T}_1(u) R(u-u') \qquad \widehat{T}(u) = T(u) + \mathcal{O}(\hbar)$$

for NLS coincides with normal ordering prescription of $\psi, \psi^\dagger$ on finite interval with periodic b.c.

The classical limit is literally

$$\{A, B\} = \lim_{\hbar \to 0} \frac{i[A, B]}{\hbar}$$

Integrability manifest: tower of commuting charges by tracing RTT relations

In the NLS case: $\quad R(\lambda - \mu) = \mathbb{1} \otimes \mathbb{1} - \frac{i\hbar\kappa}{\lambda - \mu}\, P$

The three cases of the Belavin-Drinfeld theorem correspond to how a classical r-matrix (resp. classical Lie bi-algebra) is *quantised into* (*e.g. the small* $\hbar$ *limit of*) one of the possible quantum R-matrices (resp. quantum groups):

- rational $\longrightarrow$ quantise to Yangians
- trigonometric $\longrightarrow$ quantise to trigonometric quantum groups (Jimbo-Drinfeld)
- elliptic $\longrightarrow$ quantise to elliptic quantum groups (Sklyanin, Felder)

[Etingof-Kazhdan]

Spin chains follow the pattern: *XXX, XXZ, XYZ*

or, Heisenberg, 6-vertex model - Sine Gordon, 8-vertex model - Baxter

Example: from rational classical r-matrix to Yangians

$r = \frac{T^a \otimes T_a}{u_1 - u_2}$ prototypical solution of CYBE (“Yang’s solution”)

T^a generate Lie algebra $\mathfrak{g}$ - contraction done with Killing form

Seeking more abstract rewriting:

$r = \sum_{n=0}^{\infty} T^a \otimes T_a u_1^{-n-1} u_2^n$ if $|u_1| < |u_2|$ which means

$r = \sum_{n=0}^{\infty} T^a_{-n-1} \otimes T_{a,n}$ if we define

$T^a_n = u^n T^a$ (*) and keep track of the spaces 1 and 2

These implies that the classical r-matrix is abstractly written in terms of an ∞ dimensional algebra

$[T^a_n, T^b_m] = i f^{ab}_c T^c_{m+n}$ “loop” algebra, particular case of Kac-Moody

Moreover, one can prove that r solves CYBE only using algebra comm. rels

of which (*) is merely a particular rep

[Exercise for the reader: proof of this statement]

Mathematicians then tell us that loop algebra quantise to Yangians in the sense of quantum groups, with Sklyanin’s R for the NLS as R-matrix! circle is closed

MESSAGE we are learning here:

ALGEBRA is universal $\rightarrow$ REPS are various physical realisations

It pays off to look for universal structures behind our formulae

"The same equations have the same solutions" - R. Feynman

What is the "quantum" R-matrix mathematically?

Algebraic setting $\longrightarrow$ Hopf algebras

$R : V_1 \otimes V_2 \longrightarrow V_1 \otimes V_2$ R is called universal R-matrix

V_i *carries a representation of algebra* A *- endowed with multiplication and unit*

We will assume A to be a Lie (super-)algebra rather its universal envelop

Additional structure: coproduct

$$\Delta \ : \ A \longrightarrow A \otimes A$$

$$[\Delta(a), \Delta(b)] = \Delta([a, b]) \text{ (homomorphism)}$$

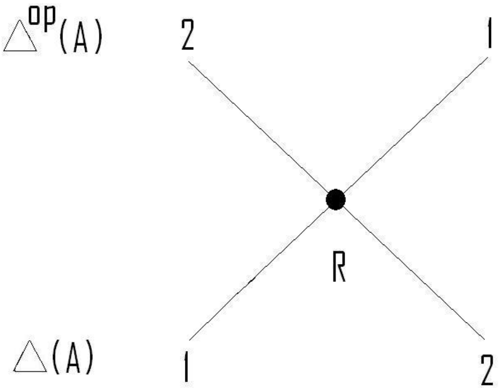
even $\Delta(a)\Delta(b) = \Delta(ab)$, $a, b, \in A$

Universal R-matrix

renders Hopf algebra "quasi-co-commutative"

$$(P\Delta)\,R \;=\; R\,\Delta$$

P (graded) permutation $\quad$ $P\Delta$ 'opposite' coproduct Δ^{op} ('out')



Lie algebras normally have ‘trivial’ coproduct (“co-commutative”)

$$\Delta^{op}(Q) = \Delta(Q) = Q \otimes 1 + 1 \otimes Q \qquad \forall\, Q \in A$$

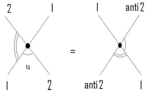
non trivial $\rightarrow$ quantum groups

Hopf algebra: coproduct + extra algebraic structures
e.g. antipode, counit + list of axioms

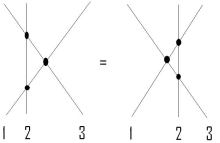
If time $\longrightarrow$ Hopf algebra axioms

The Yangian is an ∞-dim non-abelian Hopf algebra

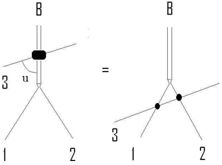
{ *books* } [Chari-Pressley ’94; Kassel ’95; Etingof-Schiffmann ’98]



$$(\Sigma \otimes 1)R = R^{-1} = (1 \otimes \Sigma^{-1})R$$ quasi-triangular - Drinfeld's theorem



$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$ Yang-Baxter



$$(\Delta \otimes 1)R = R_{13}\, R_{23}\,; \quad (1 \otimes \Delta)R = R_{13}\, R_{12}$$ bootstrap

Example: $U_q\left[(sl(2)\right]$

$$[h, e^{\pm}] = \pm 2e^{\pm} \qquad [e^+, e^-] = \frac{q^h - q^{-h}}{q - q^{-1}}$$

$$\Delta(h) = h \otimes 1 + 1 \otimes h \qquad \Delta(e^{\pm}) = e^{\pm} \otimes q^{\frac{h}{2}} + q^{-\frac{h}{2}} \otimes e^{\pm}$$

$$R = q^{\frac{h\otimes h}{2}} \sum_{n \geq 0} \frac{(1 - q^{-2})^n}{[n]!} q^{\frac{n(n-1)}{2}} \left(q^{\frac{h}{2}} e^+\right)^n \otimes \left(q^{-\frac{h}{2}} e^-\right)^n$$

$$[n]! = [n][n-1]... \qquad [n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

In the fundamental rep

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad e^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

R becomes 4×4 matrix

Compute antipode

How does this help us finding the exact quantum spectrum?

Algebraic Bethe ansatz

compactify with periodic b.c. [Faddeev "How algebraic Bethe...", 1996; Levkovich-Maslyuk, arXiv:1606.02950]

Entries of $T \to$ quantum operators $\qquad T(u) = \begin{pmatrix} A(u) & \kappa B^\dagger(u) \\ B(u) & A^\dagger(u) \end{pmatrix}$

normal ordering prescription

RTT relations imply in particular

$$A(u)B^\dagger(v) = \left(1 + \frac{i\kappa}{u-v}\right)B^\dagger(v)A(u) - \frac{i\kappa}{u-v}B^\dagger(u)A(v)$$

$$[B(u), B(v)] = 0$$

The operator $B^\dagger$ looks like creation op.: postulate $\exists$ vacuum

$$|0\rangle \qquad \text{s.t} \qquad B(u)|0\rangle = 0 \qquad \text{"no particles"}$$

one can see $A(u)|0\rangle = e^{-i\frac{L}{2}}|0\rangle$

$\longrightarrow$ vacuum is eigenstate of $A + A^\dagger$ - the transfer matrix (trace)

Then, $|u_1, ..., u_M\rangle = B^\dagger(u_1)...B^\dagger(u_M)|0\rangle$ eigenstate of $A + A^\dagger$ iff Bethe eq.s

In turn, transfer matrix contains Hamiltonian - but also all commuting charges simultaneously diag. on these states

States look like *magnons*: *e.g.* $M = 2$

$$\int dx_1 dx_2 \left[\theta(x_2 < x_1) + S(u_1 - u_2)\,\theta(x_1 < x_2)\right] e^{iu_1x_1 + iu_2x_2}\, \psi^\dagger(x_1)\psi^\dagger(x_2)|0\rangle$$

$S = \frac{u-v+i\kappa}{u-v-i\kappa}$ with u, v satisfying certain algebraic condition $\longrightarrow$

Using field comm rels, directly proven to be energy eigenstate

$$[\psi(x), \psi^\dagger(y)] = \delta(x-y) \qquad Energy = u_1^2 + u_2^2$$

Similarly, directly resumming perturbation theory

Lippman-Schwinger-type eq. for 2-body wave function

[see Thacker '81]

Point is:

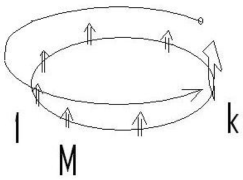
perturb. you create with $\psi^\dagger$, but exact eigenstates you create with $B^\dagger$

States are M travelling particles subject to Bethe Equations

exactly as Bethe would have written in 1931 on physical grounds - now called *coordinate BA*

$$e^{i p_k L} = \prod_{\substack{j=1 \\ j \neq k}}^{M} S_{kj}$$

$$S_{kj} = S(p_k, p_j)$$



M-magnon state

S-matrix $\quad S = \frac{u - v + i\kappa}{u - v - i\kappa} \quad$ momentum $\quad p = u$

M-body eff. Schrödinger problem isom. to collection of M-particle Bose gas with delta-function potential

The key algebraic steps are then taken and applied *mutatis mutandis* in general

The R-matrix actually encodes everything

or, *following onto Zamolodchikov's footsteps...*

...the last vestiges of the S-matrix program

before QFT ever was

EXACT S-MATRICES

{*for reviews*} [P. Dorey '98; Bombardelli '15]

2D integrable massive S-matrices

- No particle production/annihilation

- Equality of initial and final sets of momenta

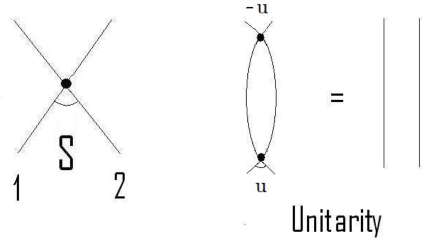
- Factorisation: $S_{M \longrightarrow M} = \prod S_{2 \longrightarrow 2}$

  (all info in 2-body processes)

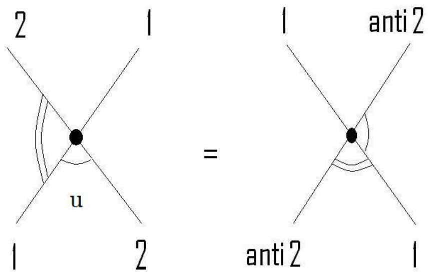
See how the Bethe equations are particle-preserving and factorised ⟵

Extrapolate from relativistic case

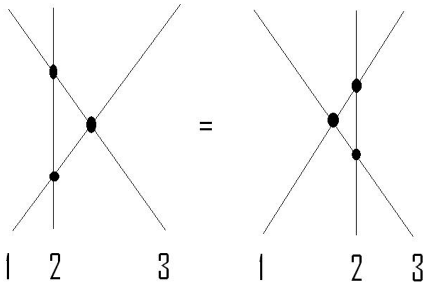
$$S_{2 \longrightarrow 2} = {\color{red} S(\mathrm{u}_1 - \mathrm{u}_2)} \equiv S(\mathrm{u}) \qquad [E_i = m_i \cosh \mathrm{u}_i ,\; p_i = m_i \sinh \mathrm{u}_i]$$



$$S_{12}(\mathrm{u}) \; S_{21}(-\mathrm{u}) = 1$$



Crossing symmetry $\quad S_{12}(\mathrm{u}) = S_{\bar{2}1}(i\pi - \mathrm{u})$



Yang-Baxter Equation (YBE) $S_{12}\, S_{13}\, S_{23} = S_{23}\, S_{13}\, S_{12}$

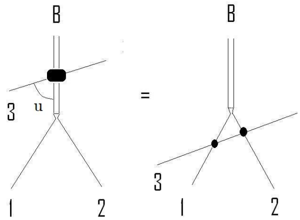
- S-matrix real analytic: $S(s^*) = S^*(s)$

Mandelstam $s = 2m^2(1 + \cosh \mathrm{u})$ for equal masses

- S-matrix simple poles $\leftrightarrow$ bound states

  They occur at imaginary spatial momentum $\leftrightarrow$ wave function decays at spatial infinity

$$0 < s < 4m^2 \quad \leftrightarrow \quad \mathrm{u} = i\vartheta \quad \leftrightarrow \quad p = m \sinh \mathrm{u} = im \sin \vartheta$$



Bootstrap

$$S_{B3}(\mathrm{u}) \;=\; \sum_c |R_c|^{-\frac{1}{2}} P_c S_{23}(u - iu_{12}^B) S_{13}(u + iu_{12}^B) \sum_b |R_b|^{\frac{1}{2}} P_b$$

if residue of S-matrix at the pole is $\sum_a R_a P_a$

[from Karowski '79]

E.g. $R = \frac{u}{u-1}\left(1 \otimes 1 - \frac{P}{u}\right)$ with $P =$ *permutation*: only pole at $u = 1$ with residue $1 - P$

$\longrightarrow$ projector onto the anti-symmetric rep ($=$ bound state rep)

DRESSING FACTOR

$$S(\mathrm{u}) = \Phi(\mathrm{u})\,\widehat{S}(\mathrm{u})$$

$\widehat{S}(u)$ acts as 1 on highest weight state

- Dressing factor $\Phi(\mathrm{u})$ not fixed by symmetry, matrix $\widehat{S}(\mathrm{u})$ yes
- Dressing factor $\Phi(\mathrm{u})$ constrained by crossing $S_{12}(\mathrm{u}) = S^{-1}_{1\bar{2}}(\mathrm{u} - i\pi)$
- Dressing factor $\Phi(\mathrm{u})$ essential for pole structure

Ex: Sine-Gordon at special value of coupling $\beta^2 = \frac{16\pi}{3} \longrightarrow$ repulsive regime: ∞ poles but none in physical region

$$\Phi(u) = \prod_{\ell=1}^{\infty} \frac{\Gamma^2(\ell-\tau)\,\Gamma(\frac{1}{2}+\ell+\tau)\,\Gamma(-\frac{1}{2}+\ell+\tau)}{\Gamma^2(\ell+\tau)\,\Gamma(\frac{1}{2}+\ell-\tau)\,\Gamma(-\frac{1}{2}+\ell-\tau)} \qquad \tau \equiv \frac{u}{2\pi i}$$

even though S is tanh and sech, and crossing implies $\Phi(u)\Phi(u+i\pi) = i\tanh\frac{u}{2}$

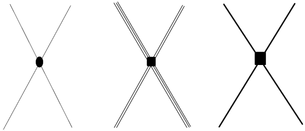
Universal R-matrix is abstract object which generates all S-matrices by projecting into irreps

$$R \longrightarrow_{fundam \otimes fundam} \quad S_{fund1, fund2}$$

$$\longrightarrow_{b.state \otimes b.state} \quad S_{bound1, bound2}$$

$$\longrightarrow_{inf.dim \otimes inf.dim} \quad S_{inf.1, inf.2}$$

Inclusive of (minimal) dressing factors



The universal R-matrix for a class of (super-)algebras and their subalgebras controls both the monodromy-matrix exchange relations and generates the physical S-matrices

The universal R-matrix is dictated by the (usually infinite-dimensional) quantum-group symmetry of the problem, and can be written using the generators of the associated quantum group

The procedure to find the universal R-matrix is based on Drinfeld's double and the associated Drinfeld's theorem, and explicit formulas have been given by Khoroshkin and Tolstoy

Indeed algebraic Bethe ansatz solves integrable spin-chains as well

Insert exercise on supersymmetric chain - if time permits

Research in string theory and AdS/CFT uses integrability $\longrightarrow$ adapting the standard framework to the more exotic environments

- incredibly rich representation theory
- non-standard quantum groups
- rich set of new boundary conditions

[Beisert *et al.* arXiv:1012.3982]

but we always go back to Faddeev, Zamolodchikov, etc. - and to all those nice papers with no archive version

Relation to CFT

$\longrightarrow$ massless integrability

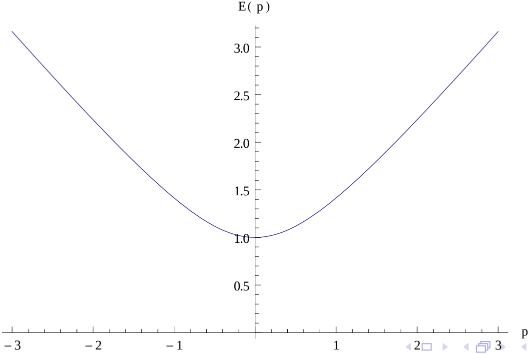
Bazhanov-Lukyanov-Zamolodchikov; Fendley-Saleur-Zamolodchikov

MASSLESS INTEGRABILITY: relativistic

[Zamolodchikov-Zamolodchikov '92, Fendley-Saleur-Zamolodchikov '93]

$$E = m \cosh u \qquad p = m \sinh u \qquad E^2 - p^2 = m^2$$

Lorentz boost $u \to u + b$: the two branches are connected



SEND $m \to 0$ $E = \frac{m}{2}\left(e^u + e^{-u}\right)$ $p = \frac{m}{2}\left(e^u - e^{-u}\right)$

$$u = u_0 + \nu$$

$$\frac{m}{2}\, e^{|u_0|} = M = \text{finite} \qquad u_0 \to \pm\infty \qquad (\text{boost to } |v| = c)$$

ν fixed

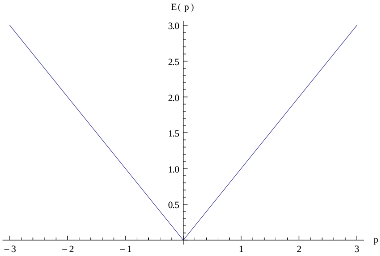
Two branches of $E = \sqrt{p^2}$

- *right moving* $u_0 \to +\infty$

$$E = Me^{\nu_+} \qquad p = Me^{\nu_+} \qquad E = p \qquad \nu_+ \in (-\infty, \infty)$$

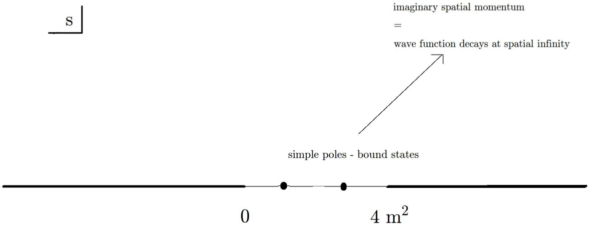
- *left moving* $u_0 \to -\infty$

$$E = Me^{-\nu_-} \qquad p = -Me^{-\nu_-} \qquad E = -p \qquad \nu_- \in (-\infty, \infty)$$

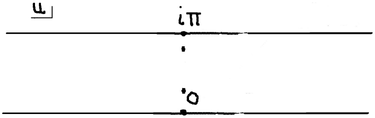


- "Perturbative" intuition for left-left (right-right) scattering fails
- Left vs right: standard framework of integrability needs adaptation
- Still, $\exists$ notion of Yang-Baxter: the 4 limiting S-matrices are solutions

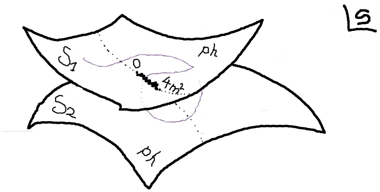
Analyticity: Massive case $s = 2m^2[1 + \cosh u]$ $u \equiv u_1 - u_2$



review [P. Dorey '98]



Switching branch cuts (if allowed): consider ℓr, $r\ell$ [Zamolodchikov '91]



$$S(s) = S_1(s) \quad \text{if } \Im s > 0, \qquad S(s) = S_2(s) \quad \text{if } \Im s < 0$$

Braiding Unitarity: $S_1(s)\, S_2(s) = 1$ Crossing: $S_1(s) = S_2(4m^2 - s)$

Now send $m \to 0$: shrinking of bound-state region

$$S_1(s) = S_2(-s) \qquad S_i(s)\, S_i(-s) = 1 \text{ (crossing-unitarity)} \qquad \forall \;\; i = 1, 2$$

Two touching sheets with algebraic condition. No bound states

Interpolating massless flows

Left-right, right-left S-matrices do depend on M: $|u_0| \to -\log m + \log 2M$

$$u_1 - u_2 = |u_0| + \nu_{1,+} - (-|u_0| + \nu_{2,-}) = 2|u_0| + \nu_{1,+} - \nu_{2,-}$$

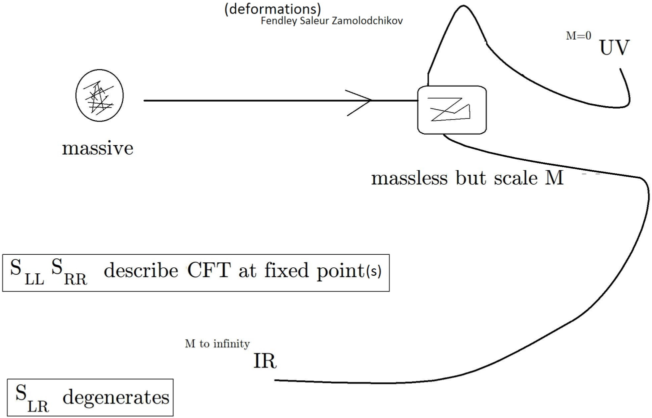
Left-left and right-right S-matrices do not (formally retain standard features):

$$u_1 - u_2 = |u_0| + \nu_{1,+} - (|u_0| + \nu_{2,+}) = \nu_{1,+} - \nu_{2,+}$$

$M \to \infty$ (MR in TBA) $\longrightarrow$ IR fixed-point CFT via $\ell\ell$ and rr S-matrices

Massive integrable $\longrightarrow$ Massless non scale-invariant $\longrightarrow$ IR CFT

cf. Bazhanov-Lukyanov-Zamolodchikov: characterising CFT as integrable FT


(deformations)
Fendley Saleur Zamolodchikov
M=0
UV
massive
massless but scale M
S LL S RR describe CFT at fixed point(s)
M to infinity
IR
S LR degenerates

Thermodynamic Bethe ansatz in a nutshell

Put theory on a torus Zamolodchikov 1990

spatial circle radius R - time circle radius L

at large L, partition function $Z \sim e^{-E(R)L}$

where $E(R)$ is the ground state energy on a finite circle

Relativistic invariance (double Wick rot): same theory on spatial L, time R

Partition function is the same, but now at large space with periodic time

$\longrightarrow$ can use S-matrix and Bethe equations

$Z = tr\, e^{-RH}$ temperature $\beta \sim \frac{1}{R}$

Minimising free energy with the Bethe eq as constraint gives TBA equations

solution gives original ground state energy $E(R)$

Dialing dimensionless param MR moves along flow. Towards IR $E(R) \sim -\frac{\pi c}{6R}$ with c CFT central charge

Example of massless flow: Tricritical to critical Ising

Zamolodchikov 1991

$S_1(s) = \frac{iM^2 - s}{iM^2 + s}$ $\quad S_2(s) = S_1(-s) = \frac{1}{S_1(s)}$ respectively, crossing and br. unit.

$S_1(s^*) = S_2^*(s)$ real analyticity (assume M is real)

No poles in physical region (massless particles form no bound states)

Physical unitarity: S is a pure phase for real momenta

$E_1 = p_1 = \frac{M}{2} e_1^{\theta}$ $\quad E_2 = -p_2 = \frac{M}{2} e_2^{\theta}$ $\quad s = M^2 e^{\theta}$ $\quad \theta = \theta_1 - \theta_2$ (rel inv - diff form)

$S = \frac{i - e^{\theta}}{i + e^{\theta}} = -\tanh\left[\frac{\theta}{2} - i\frac{\pi}{4}\right]$ $\quad S(\theta) = \frac{1}{S(\theta + i\pi)} = \frac{1}{S(-\theta)}$

- TBA reveals

$MR \to 0$ $\quad c_{UV} = \frac{7}{10}$ (tricritical Ising model)

$MR \to \infty$ $\quad c_{IR} = \frac{1}{2}$ (critical Ising model)

$c_{UV} > c_{IR}$ (Zamolodchikov's theorem)

IR theory: free 2D Majorana fermion (S-matrix $\to 1$ at $M \to \infty$)

$$S_{eff} = \frac{1}{2\pi}\int d^2x(\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}) - \frac{1}{\pi^2 M^2}\int d^2x(\psi\partial\psi)(\bar{\psi}\bar{\partial}\bar{\psi}) + \ldots$$

Yangian interlude - if time permits

Principal chiral model prelude - if time permits

Quantisation of the Kadomtsev-Petviashvili Equation

with Karol Kozlowski (Lyon) and Evgeny Sklyanin (York)

based on Teor. Mat. Fiz. 192 (2017) 259 - 1607.07685

Why this section?

- (First and foremost: chance to plug in some of my work...)
- The KP equation is a universal integrable system of crucial importance
- Practise what we have learnt on a difficult problem
- Having a go at 3D integrability *"lasciate ogni speranza..."*
- Tell you the things which I have religiously learnt from my collaborators
- Great deal of questions we need young people to tell us the answer to
- Convince you all to start working on it

Classical KP equation

- Konopelchenko, "Introduction to multidimensional integrable equations...", Springer, 1993
- Biondini, Pelinovsky, "Kadomtsev-Petviashvili equation", Scholarpedia, 2008

Kadomtsev-Petviashvili equation

[Kadomtsev, B. B., Petviashvili, V. I. (1970), Sov. Phys. Dokl. 15 (1970) 539]

Non-linear PDE in $2+1D$

$u = u(X, Y, t)$ perturbation profile of long waves with

- small amplitude
- weak dependence on Y (transverse) vs. X (longitudinal) coordinate w.r.t direction of motion

$$\partial_X \left(\partial_t u + u\, \partial_X u + \epsilon^2\, \partial_{XXX} u \right) + \lambda\, \partial_{YY} u = 0$$

Parameters: $\epsilon \in \mathbb{R} \qquad \lambda = \pm 1$

Partial Chronology

- '67 Gardner-Green-Kruskal-Miura $\longrightarrow$ Inverse scattering for KdV
- '70 K-P $\longrightarrow$ Long ion-acoustic waves in plasmas

  Adding transverse dynamics to KdV and studying stability of solitons
- '74 Dryuma $\longrightarrow$ Lax-pair formulation
- '74 Zakharov-Shabat $\longrightarrow$ Inverse scattering for KP

  Rich set of soliton solutions found
- '79 Ablowitz-Segur $\longrightarrow$ Application to water waves
- '86 Fokas-Santini $\longrightarrow$ Bi-Hamiltonian structure

Comments

- Although classically possible $\longrightarrow$ we do not rescale away ϵ^2

  Quantisation introduces $\hbar$ which couples to dimensionful constants

- Two types:
  - $\lambda = -1$ KP-I high surface tension - positive dispersion

    $\longrightarrow$ Lump solitons
  - $\lambda = 1$ KP-II small surface tension - negative dispersion

    $\longrightarrow$ Resonant multi-solitons and Web structures

- Both types have Line solitons $\longrightarrow$ KdV solitons with no Y-dep.
  - KP-I Line solitons unstable
  - KP-II Line solitons stable

- Perfect balance of non-linearity and dispersion $\longrightarrow$ Integrable

- Universal integrable eq. $\longrightarrow$ Universal quantum integrable system?

Preparing for Quantisation

Construct a theory of particles which reduces to KP when $\hbar \to 0$

New var. $u = 2\beta\lambda\phi \quad X = -\lambda\sigma \quad Y = x \quad \gamma = -\lambda\epsilon^2$

Equation becomes

$$\phi_{t\sigma} - \phi_{xx} - 2\beta(\phi\phi_\sigma)_\sigma + \gamma\phi_{\sigma\sigma\sigma} = 0$$

Assume: $\beta > 0$ (if not, $\phi \to -\phi$) $\longrightarrow$ Unitary transf. in quantum case

Trick: Kaluza-Klein compactification of the ocean!

Preserves integrability (but upsets God...)

$x \in \mathbb{R} \quad \sigma \in [0, 2\pi]$ periodic b.c. $\longrightarrow$ reduces to 1+1 D problem (for now)

$1+1$ D integrable field theory with KK tower

Assume $\phi \to 0$ suff. fast as $x \to \pm\infty$ and $\frac{1}{2\pi}\int_0^{2\pi} \phi\, d\sigma = 0$

First few conserved charges

$$\sigma-\text{transl} \qquad h_0 \equiv \mathcal{M} = \frac{1}{2}\phi^2 \qquad M = \int_0^{2\pi} \frac{d\sigma}{2\pi} \int_{-\infty}^{\infty} dx\, \mathcal{M}$$

$$x-\text{transl} \qquad h_1 \equiv \mathcal{P} = \frac{1}{2}\left(\partial_\sigma^{-1}\phi\right)\partial_x\phi \qquad P = \int_0^{2\pi} \frac{d\sigma}{2\pi} \int_{-\infty}^{\infty} dx\, \mathcal{P}$$

$$t-\text{transl} \qquad h_2 \equiv \mathcal{H} = \frac{1}{2}\left(\partial_\sigma^{-2}\phi\right)\partial_x^2\phi + \frac{\beta}{3}\phi^3 + \frac{\gamma}{2}\left(\partial_\sigma\phi\right)^2$$

'84 Case: higher charges $h_p = \frac{1}{2}\left(\partial_\sigma^{-p}\phi\right)\partial_x^p\phi + \mathcal{O}(\beta) + \mathcal{O}(\gamma) \qquad p \in \mathbb{N}$

Notice: $\partial_\sigma^{-1} \equiv \frac{1}{2}\int_0^{\sigma} \frac{d\sigma}{2\pi} - \frac{1}{2}\int_\sigma^{2\pi} \frac{d\sigma}{2\pi}$

Poisson structure

KP equation recovered by $\partial_t \phi = \{\phi, H\}$ followed by ∂_σ

to cancel some ∂_σ^{-1}

with Poisson brackets ('86 Lipovskii)

$$\{\phi(\sigma, x), \phi(\tilde{\sigma}, y)\} = 2\pi\, \delta(x-y)\, \delta'(\sigma - \tilde{\sigma})$$

Non-ultralocal in $\sigma \in (0, 2\pi)$ Known problem for integrability $\longrightarrow$

- Galilei symmetry $x \to x + 2vt$ $\sigma \to \sigma + vx + v^2 t$

$\longrightarrow$ boost: $\int_0^\sigma \frac{d\sigma}{2\pi} \int_{-\infty}^{\infty} dx\, x\, h_0$ s.t. $\{B, H_p\} = p\, H_{p-1}$

where $H_p = \int_0^\sigma \frac{d\sigma}{2\pi} \int_{-\infty}^{\infty} dx\, h_p$

Quantisation

Canonically Quantise

$$\phi \in \mathbb{R} \qquad \longrightarrow \qquad \phi^\dagger = \phi \qquad \longrightarrow \qquad \phi = \sum_{n\in\mathbb{Z}} a_n(x)\, e^{-in\sigma}$$

with

$$a_n^\dagger(x) = a_{-n}(x)$$

Moreover $\quad a_0(x) = 0 \quad$ average of ϕ is central w.r.t. P.B. and $= 0$

$$\{.,.\} \longrightarrow \frac{[.,.]}{i\hbar} \qquad \text{followed by } \hbar \to 1$$

Redefining the modes

$$\psi_n(x) \equiv n^{-\frac{1}{2}}\, a_n(x) \qquad n = 1, 2, \ldots$$

$$[\psi_m(x), \psi_n^\dagger(y)] = \delta_{mn}\, \delta(x - y)$$ non-ultralocality reabsorbed

Second Quantisation

Postulate vacuum and creators/annihilators

$$|0\rangle \qquad \text{s.t.} \qquad \psi_m(x)|0\rangle = 0 \qquad \forall\ x \in \mathbb{R} \quad m = 1, 2, ...$$

Lowest-weigth (Fock) rep. of Heisenberg algebra

$$\text{span}\{\psi^\dagger_{m_1}(x_1)...\psi^\dagger_{m_N}(x_N)|0\rangle \mid m_i \in \mathbb{N}, x_i \in \mathbb{R}, \quad i = 1, .2, ..., N\}$$

Commuting creators $\longrightarrow$ bosons

Conserved quantities quantised by

- plugin field expansion
- normal order - creators to the left

Conserved Charges

Total Mass

$$M = \int_0^{2\pi} \frac{d\sigma}{2\pi} \int_{-\infty}^{\infty} dx\, \frac{1}{2}\phi^2 \quad \longrightarrow \quad \sum_{m=1}^{\infty} m \int_{-\infty}^{\infty} dx\, \psi_m^\dagger(x)\psi_m(x)$$

Total Momentum

$$P = \int_0^{2\pi} \frac{d\sigma}{2\pi} \int_{-\infty}^{\infty} dx\, \frac{1}{2}(\partial_\sigma^{-1}\phi)\partial_x\phi \quad \longrightarrow \quad -i\sum_{m=1}^{\infty} \int_{-\infty}^{\infty} dx\, \psi_m^\dagger(x)\partial_x\psi_m(x)$$

Total Energy

$$H = \int_0^{2\pi} \frac{d\sigma}{2\pi} \int_{-\infty}^{\infty} dx \left[\frac{1}{2}(\partial_\sigma^{-2}\phi)\,\partial_x^2\phi + \frac{\beta}{3}\phi^3 + \frac{\gamma}{2}(\partial_\sigma\phi)^2\right] \quad \longrightarrow$$

Quantum Hamiltonian

$\longrightarrow$

$$H = -\sum_{m=1}^{\infty} \frac{1}{m} \int_{-\infty}^{\infty} dx\, \psi_m^\dagger(x) \partial_x^2 \psi_m(x) + \sum_{m=1}^{\infty} \gamma_m \int_{-\infty}^{\infty} dx\, \psi_m^\dagger(x) \psi_m(x)$$
$$+ \sum_{m_1, m_2=1}^{\infty} \beta_{m_1 m_2} \int_{-\infty}^{\infty} dx \left[\psi_{m_1+m_2}^\dagger(x) \psi_{m_1}(x) \psi_{m_2}(x) + \psi_{m_1}^\dagger(x) \psi_{m_2}^\dagger(x) \psi_{m_1+m_2}(x) \right]$$

$\beta_{m_1 m_2} = \beta \sqrt{m_1 m_2 (m_1 + m_2)}$ $\qquad \gamma_m = \gamma m^3$ $\qquad$ real parameters

Hermitean Hamiltonian composed of

- (Non-relativistic) kinetic term
- (Non-relativistic) zero-point energy
- Total-mass preserving three-particle interactions

Number of particles is not conserved

Comments

Assume $\beta > 0$

If not

$$U^\dagger \psi_m U = -\psi_m \quad U^\dagger \psi_m^\dagger U = -\psi_m^\dagger \quad U = \exp\left[i\pi \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} dx\, \psi_m^\dagger(x)\psi_m(x) \right]$$

If $\beta = 0$ $\longrightarrow$ Free particles of mass m and zero-point energy γm^3

Two possibilities:

- $\gamma < 0$ KP-I $\longrightarrow$ positive zero-point energy
- $\gamma > 0$ KP-II $\longrightarrow$ Unbounded below, but conservation laws
- $\gamma = 0$ $\longrightarrow$ Dispersionless KP

Boost $\sum_{m=1}^{\infty} m \int_{-\infty}^{\infty} dx\, x\, \psi_m^\dagger(x)\psi_m(x)$

Bethe Ansatz and S-matrix

Spectrum of the Hamiltonian

Organise states by total mass

- $M = 0$

$$|\Phi_0\rangle = |0\rangle \qquad E_0 = 0$$

- $M = 1$

$$|\Phi_1\rangle = \int_{-\infty}^{\infty} dx\, f_1|_x^1\, \psi_1^\dagger(x)|0\rangle \qquad f_1|_x^1 \equiv e^{ipx} \qquad E_1 = p^2 + \gamma$$

$f_1|_x^1$ wave function $\quad f_{\text{nr of particles}}|_{\text{locations}}^{\text{masses}}$

Mass-2 sector

- $M = 2$

$$|\Phi_2\rangle = \int_{x_1<x_2} dx_1\, dx_2\, f_2|^{1\ 1}_{x_1 x_2}\, \psi_1^\dagger(x_1)\, \psi_2^\dagger(x_2)|0\rangle \, + \int_{-\infty}^{\infty} dx\, f_1|^2_x\, \psi_2^\dagger(x)|0\rangle$$

Scattering state

$$f_2|^{1\ 1}_{x_1 x_2} = e^{ip_1 x_2 + ip_2 x_1} + S(p_1, p_2)\, e^{ip_1 x_1 + ip_2 x_2}$$

$$f_1|^2_x = R(p_1, p_2)\, e^{i(p_1+p_2)x} \qquad\qquad E_2 = p_1^2 + p_2^2 + 2\gamma$$

Two-particle S-matrix

$$S(p_1, p_2) = S(p_1 - p_2) = -\frac{P(ip_2 - ip_1)}{P(ip_1 - ip_2)} = S^{-1}(p_2, p_1)$$ Galilei invar - braiding unitar

$$R(p_1, p_2) = \frac{4\sqrt{2}\, i\beta\, (p_2 - p_1)}{P(ip_1 - ip_2))} \qquad P(v) = v^3 + 12\gamma v - 4\beta^2$$ no quadratic term!

particular limit of affine Toda $A_{N\to\infty}$ S-matrix (infinitely many fields)

General solution

Infinite set of auxiliary Schrödinger problems

$$|\Phi_M\rangle = \sum_{N=1}^{M} \frac{1}{N!} \sum_{\vec{m}\in\mathbb{N}^N} \int_{\mathbb{R}^N} dx_1 ... dx_N \, f_N|_{\vec{x}}^{\vec{m}} \prod_{j=1}^{N} \psi_{m_j}^{\dagger}(x_j)\, |0\rangle$$

Solve for wave functions

$$\langle \psi_{m_1}^{\dagger}(x_1) ... \psi_{m_N}^{\dagger}(x_N)|\,(H - E_M)\,|\Phi_M\rangle = 0$$

turns into coupled PDEs:

$$- \sum_{i=1}^{N} \frac{1}{m_i} \partial_{x_i}^2 f_N|_{x_1 \,\dots\, x_N}^{m_1 \dots m_N} + \sum_{i=1}^{N} \gamma_{m_i} f_N|_{x_1 \,\dots\, x_N}^{m_1 \dots m_N}$$

$$+ 2 \sum_{1\le i_1 < i_2 \le N} \beta_{m_{i_1} m_{i_2}} \, f_{N-1}|_{x_1 \dots \cup \dots \cup \dots x_N \;\; x_{i_1}}^{m_1 \dots \cup \dots \cup \dots m_N \;(m_1+m_2)} \, \delta(x_{i_1} - x_{i_2}) \qquad \text{missing } m_{i_1}\; m_{i_2}$$

$$+ \sum_{k=1}^{N} \sum_{n_1, n_2 \in \mathbb{N}^+} \delta_{n_1+n_2, m_k} \, \beta_{n_1 n_2} \, f_{N+1}|_{x_1 \dots \cup \dots x_N \;\; x_k \, x_k}^{m_1 \dots \cup \dots m_N \;\; n_1 \; n_2} \qquad \text{missing } m_k$$

$$= E_M \, f_N|_{x_1 \dots x_N}^{m_1 \dots m_N} \qquad \forall \;\; N = 1, \dots, \infty$$

$$R(p_1, p_2) = \frac{4\sqrt{2}\, i\beta\, (p_2 - p_1)}{P(ip_1 - ip_2))} \qquad P(v) = v^3 + 12\gamma v - 4\beta^2$$

no quadratic term!

Integrability

- Seemingly impossible $\longrightarrow$ but quantisation should preserve integrability

  Two-body problem encodes all the info

- We have conjectured a general Bethe Ansatz :
  - recursively uses two-body data
  - tested with computer algebra up to total mass $M = 8$

- The solution is expressed in terms of compositions (bosonic symmetry)

- Need proof of master combinatoric formula

- Treatment of delta-functions $\longrightarrow$

Delta-functions as boundary conditions

E.g. $M = 2$

$$(-\partial_1^2 - \partial_2^2 + 2\gamma)\, f_2|^{1\,1}_{x_1 x_2} + 2\sqrt{2}\beta\, f_1|^2_{x_1}\, \delta(x_1 - x_2) = E_2\, f_2|^{1\,1}_{x_1 x_2}$$

$$\Big(-\frac{1}{2}\partial_x^2 + 8\gamma\Big)\, f_1|^2_x + \sqrt{2}\beta\, f_2|^{1\,1}_{x\,x} = E_2\, f_1|^2_x$$

Separate domains

$$f_2|^{1\,1}_{x_1 x_2} \equiv f_+(x_1, x_2)\,\Theta(x_1 - x_2) + f_-(x_1, x_2)\,\Theta(x_2 - x_1)$$

plug-in and collect terms

from Θ $\quad (-\partial_1^2 - \partial_2^2 + 2\gamma)\, f_2|^{1\,1}_{x_1 x_2} = E_2\, f_2|^{1\,1}_{x_1 x_2} \qquad x_1 \neq x_2$

from δ' $\quad f_+(x, x) = f_-(x, x)$

from δ $\quad (-\partial_1 + \partial_2) f_+(x, x) + (\partial_1 - \partial_2) f_-(x, x) = \sqrt{2}\beta\, f_1|^2_x$

Bose symmetry $\longrightarrow$ $f_+(x_1, x_2) = f_-(x_2, x_1)$ hence

$$2(\partial_1 - \partial_2) f_+(x, x) = \sqrt{2}\beta\, f_1|^2_x$$

Examples

"Leading" part

$$f_N|^{1\ \dots\ 1}_{x_1 \dots x_N} = \sum_{\sigma \in S_N} \prod_{j<k=1}^{N} P\left(ip_{\sigma(k)} - ip_{\sigma(j)}\right) \exp\left[i \sum_{i=1}^{N} p_{\sigma(i)} x_i\right]$$

E.g. $M = 3$

$$f_3|^{1\ 1\ 1}_{x_1 x_2 x_3} = \sum_{\sigma \in S_3} \prod_{j<k=1}^{3} P\left(ip_{\sigma(k)} - ip_{\sigma(j)}\right) \exp\left[i \sum_{i=1}^{3} p_{\sigma(i)} x_i\right]$$

$$f_2|^{2\ 1}_{x_{12} x_3} = \frac{i}{\sqrt{2}\beta} \sum_{\sigma \in S_3} (-)^{\sigma} \prod_{j<k=1}^{3} P\left(ip_{\sigma(k)} - ip_{\sigma(j)}\right) \left(p_{\sigma(2)} - p_{\sigma(1)}\right) \exp\left[i(p_{\sigma(1)} + p_{\sigma(2)}) x_{12} + ip_{\sigma(3)} x_3\right]$$

$$f_1|^{3}_{x_{123}} = -\frac{1}{\sqrt{2}\beta\ \beta_{12}} \sum_{\sigma \in S_3} (-)^{\sigma} \prod_{j<k=1}^{3} P\left(ip_{\sigma(k)} - ip_{\sigma(j)}\right) \Big[(p_{\sigma(3)} - p_{\sigma(2)})(p_{\sigma(2)} + p_{\sigma(3)} - 2p_{\sigma(1)}) -$$

$$(p_{\sigma(2)} - p_{\sigma(1)})(p_{\sigma(1)} + p_{\sigma(2)} - 2p_{\sigma(3)})\Big] \exp\left[i(p_{\sigma(1)} + p_{\sigma(2)} + p_{\sigma(3)}) x_{123}\right]$$

These are all scattering states for real momenta (continuum spectrum)