Introduction to Complex Analysis

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Preface

This text is designed for a first course in complex analysis, for beginning graduate students, or well prepared undergraduates, whose background includes multivariable calculus, linear algebra, and advanced calculus. In this course the student will learn that all the basic functions that arise in calculus, first derived as functions of a real variable, such as powers and fractional powers, exponentials and logs, trigonometric functions and their inverses, and also many new functions that the student will meet, are naturally defined for complex arguments. Furthermore, this expanded setting reveals a much richer understanding of such functions.

Care is taken to introduce these basic functions first in real settings. In the opening section on complex power series and exponentials, in Chapter 1, the exponential function is first introduced for real values of its argument, as the solution to a differential equation. This is used to derive its power series, and from there extend it to complex argument. Similarly $\sin t$ and $\cos t$ are first given geometrical definitions, for real angles, and the Euler identity is established based on the geometrical fact that e^{it} is a unit-speed curve on the unit circle, for real t. Then one sees how to define $\sin z$ and $\cos z$ for complex z.

The central objects in complex analysis are functions that are complex-differentiable (i.e., holomorphic). One goal in the early part of the text is to establish an equivalence between being holomorphic and having a convergent power series expansion. Half of this equivalence, namely the holomorphy of convergent power series, is established in Chapter 1.

Chapter 2 starts with two major theoretical results, the Cauchy integral theorem, and its corollary, the Cauchy integral formula. These theorems have a major impact on the entire rest of the text, including the demonstration that if a function $f(z)$ is holomorphic on a disk, then it is given by a convergent power series on that disk. A useful variant of such power series is the Laurent series, for a function holomorphic on an annulus.

The text segues from Laurent series to Fourier series, in Chapter 3, and from there to the Fourier transform and the Laplace transform. These three topics have many applications in analysis, such as constructing harmonic functions, and providing other tools for differential equations. The Laplace transform of a function has the important property of being holomorphic on a half space. It is convenient to have a treatment of the Laplace transform after the Fourier transform, since the Fourier inversion formula serves to motivate and provide a proof of the Laplace inversion formula.

Results on these transforms illuminate the material in Chapter 4. For example, these transforms are a major source of important definite integrals that one cannot evaluate by elementary means, but that are amenable to analysis by residue calculus, a key application of the Cauchy integral theorem. Chapter 4 starts with this, and proceeds to the study of two important special functions, the Gamma function and the Riemann zeta function.

The Gamma function, which is the first “higher” transcendental function, is essentially a Laplace transform. The Riemann zeta function is a basic object of analytic number theory, arising in the study of prime numbers. One sees in Chapter 4 roles of Fourier analysis, residue calculus, and the Gamma function in the study of the zeta function. For example, a relation between Fourier series and the Fourier transform, known as the Poisson summation formula, plays an important role in its study.

In Chapter 5, the text takes a geometrical turn, viewing holomorphic functions as conformal maps. This notion is pursued not only for maps between planar domains, but also for maps to surfaces in $\mathbb{R}^3$. The standard case is the unit sphere S^2, and the associated stereographic projection. The text also considers other surfaces. It constructs conformal maps from planar domains to general surfaces of revolution, deriving for the map a first-order differential equation, nonlinear but separable. These surfaces are discussed as examples of Riemann surfaces. The Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is also discussed as a Riemann surface, conformally equivalent to S^2. One sees the group of linear fractional transformations as a group of conformal automorphisms of $\widehat{\mathbb{C}}$, and certain subgroups as groups of conformal automorphisms of the unit disk and of the upper half plane.

We also bring in the notion of normal families, to prove the Riemann mapping theorem. Application of this theorem to a special domain, together with a reflection argument, shows that there is a holomorphic covering of $\mathbb{C} \setminus \{0, 1\}$ by the unit disk. This leads to key results of Picard and Montel, and applications to the behavior of iterations of holomorphic maps $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, and the Julia sets that arise.

The treatment of Riemann surfaces includes some differential geometric material. In an appendix to Chapter 5, we introduce the concept of a metric tensor, and show how it is associated to a surface in Euclidean space, and how the metric tensor behaves under smooth mappings, and in particular how this behavior characterizes conformal mappings. We discuss the notion of metric tensors beyond the setting of metrics induced on surfaces in Euclidean space. In particular, we introduce a special metric on the unit disk, called the Poincaré metric, which has the property of being invariant under all conformal automorphisms of the disk. We show how the geometry of the Poincaré metric leads to another proof of Picard's theorem, and also provides a different perspective on the proof of the Riemann mapping theorem.

The text next examines elliptic functions, in Chapter 6. These are doubly periodic functions on $\mathbb{C}$, holomorphic except at poles (that is, meromorphic). Such a function can be regarded as a meromorphic function on the torus $\mathbb{T}_\Lambda = \mathbb{C}/\Lambda$, where $\Lambda \subset \mathbb{C}$ is a lattice. A prime example is the Weierstrass function $\wp_\Lambda(z)$, defined by a double series. Analysis shows that $\wp'_\Lambda(z)^2$ is a cubic polynomial in $\wp_\Lambda(z)$, so the Weierstrass function inverts an elliptic integral. Elliptic integrals arise in many situations in geometry and mechanics, including arclengths of ellipses and pendulum problems, to mention two basic cases. The analysis of general elliptic integrals leads to the problem of finding the lattice whose associated elliptic functions are related to these integrals. This is the Abel inversion problem. Section 34 of the text tackles this problem by constructing the Riemann surface associated to $\sqrt{p(z)}$, where $p(z)$ is a cubic or quartic polynomial.

Early in this text, the exponential function was defined by a differential equation and given a power series solution, and these two characterizations were used to develop its properties. Coming full circle, we devote Chapter 7 to other classes of differential equations and their solutions. We first study a special class of functions known as Bessel functions, characterized as solutions to Bessel equations. Part of the central importance of these functions arises from their role in producing solutions to partial differential equations in several variables, as explained in an appendix. The Bessel functions for real values of their arguments arise as solutions to wave equations, and for imaginary values of their arguments they arise as solutions to diffusion equations. Thus it is very useful that they can be understood as holomorphic functions of a complex variable. Next, Chapter 7 deals with more general differential equations on a complex domain. Results include constructing solutions as convergent power series and the analytic continuation of such solutions to larger domains. General results here are used to put the Bessel equations in a larger context. This includes a study of equations with "regular singular points." Other classes of equations with regular singular points are presented, particularly hypergeometric equations.

The text ends with a short collection of appendices. Some of these survey background material that the reader might have seen in an advanced calculus course, including material on convergence and compactness, and differential calculus of several variables. Others develop tools that prove useful in the text, the Laplace asymptotic method, the Stieltjes integral, and results on Abelian and Tauberian theorems. The last appendix shows how to solve cubic and quartic equations via radicals, and introduces a special function, called the Bring radical, to treat quintic equations. (In §36 the Bring radical is shown to be given in terms of a generalized hypergeometric function.)

As indicated in the discussion above, while the first goal of this text is to present the beautiful theory of functions of a complex variable, we have the further objective of placing this study within a broader mathematical framework. Examples of how this text differs from many others in the area include the following.

1) A greater emphasis on Fourier analysis, both as an application of basic results in complex analysis and as a tool of more general applicability in analysis. We see the use of Fourier series in the study of harmonic functions. We see the influence of the Fourier transform on the study of the Laplace transform, and then the Laplace transform as a tool in the study of differential equations.

2) The use of geometrical techniques in complex analysis. This clarifies the study of conformal maps, extends the usual study to more general surfaces, and shows how geometrical concepts are effective in classical problems, from the Riemann mapping theorem to Picard's theorem. An appendix discusses applications of the Poincaré metric on the disk.

3) Connections with differential equations. The use of techniques of complex analysis to study differential equations is a strong point of this text. This important area is frequently neglected in complex analysis texts, and the treatments one sees in many differential equations texts are often confined to solutions for real variables, and may furthermore lack a complete analysis of crucial convergence issues. Material here also provides a more detailed study than one usually sees of significant examples, such as Bessel functions.

4) Special functions. In addition to material on the gamma function and the Riemann zeta function, the text has a detailed study of elliptic functions and Bessel functions, and also material on Airy functions, Legendre functions, and hypergeometric functions.

We follow this introduction with a record of some standard notation that will be used throughout this text.

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Some Basic Notation

$\mathbb{R}$ is the set of real numbers.

$\mathbb{C}$ is the set of complex numbers.

$\mathbb{Z}$ is the set of integers.

$\mathbb{Z}^+$ is the set of integers ≥ 0.

$\mathbb{N}$ is the set of integers ≥ 1 (the "natural numbers").

$x \in \mathbb{R}$ means x is an element of $\mathbb{R}$, i.e., x is a real number.

(a, b) denotes the set of $x \in \mathbb{R}$ such that $a < x < b$.

$[a, b]$ denotes the set of $x \in \mathbb{R}$ such that $a \leq x \leq b$.

$\{x \in \mathbb{R} : a \leq x \leq b\}$ denotes the set of x in $\mathbb{R}$ such that $a \leq x \leq b$.

$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ and $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$.

$\overline{z} = x - iy$ if $z = x + iy \in \mathbb{C},\ x, y \in \mathbb{R}$.

$\overline{\Omega}$ denotes the closure of the set Ω.

$f : A \to B$ denotes that the function f takes points in the set A to points in B. One also says f maps A to B.

$x \to x_0$ means the variable x tends to the limit x_0.

$f(x) = O(x)$ means $f(x)/x$ is bounded. Similarly $g(\varepsilon) = O(\varepsilon^k)$ means $g(\varepsilon)/\varepsilon^k$ is bounded.

$f(x) = o(x)$ as $x \to 0$ (resp., $x \to \infty$) means $f(x)/x \to 0$ as x tends to the specified limit.

$S = \sup\limits_n |a_n|$ means S is the smallest real number that satisfies $S \geq |a_n|$ for all n. If there is no such real number then we take $S = +\infty$.

$\limsup\limits_{k\to\infty} |a_k| = \lim\limits_{n\to\infty} \big(\sup\limits_{k \geq n} |a_k|\big)$.

Chapter 1. Basic calculus in the complex domain

This first chapter introduces the complex numbers and begins to develop results on the basic elementary functions of calculus, first defined for real arguments, and then extended to functions of a complex variable.

An introductory §0 defines the algebraic operations on complex numbers, say $z = x + iy$ and $w = u + iv$, discusses the magnitude $|z|$ of z, defines convergence of infinite sequences and series, and derives some basic facts about power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \tag{1.0.1}$$

such as the fact that if this converges for $z = z_0$, then it converges absolutely for $|z| < R = |z_0|$, to a continuous function. It is also shown that, for $z = t$ real,

$$f'(t) = \sum_{k\geq 1} k a_k t^{k-1}, \quad \text{for} \quad -R < t < R. \tag{1.0.2}$$

Here we allow $a_k \in \mathbb{C}$. As an application, we consider the differential equation

$$\frac{dx}{dt} = x, \quad x(0) = 1, \tag{1.0.3}$$

and deduce from (1.0.2) that a solution is given by $x(t) = \sum_{k\geq 0} t^k/k!$. Having this, we define the exponential function

$$e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k, \tag{1.0.4}$$

and use these observations to deduce that, whenever $a \in \mathbb{C}$,

$$\frac{d}{dt} e^{at} = a e^{at}. \tag{1.0.5}$$

We use this differential equation to derive further properties of the exponential function.

While §0 develops calculus for complex valued functions of a real variable, §1 introduces calculus for complex valued functions of a complex variable. We define the notion of complex differentiability. Given an open set $\Omega \subset \mathbb{C}$, we say a function $f : \Omega \to \mathbb{C}$ is holomorphic on Ω provided it is complex differentiable, with derivative $f'(z)$, and f' is continuous on Ω. Writing $f(z) = u(z) + iv(z)$, we discuss the Cauchy-Riemann equations for u and v. We also introduce the path integral and provide some versions of the fundamental theorem of calculus in the complex setting. (More definitive results will be given in Chapter 2.)

Typically the functions we study are defined on a subset $\Omega \subset \mathbb{C}$ that is open. That is, if $z_0 \in \Omega$, there exists $\varepsilon > 0$ such that $z \in \Omega$ whenever $|z - z_0| < \varepsilon$. Other terms we use for a nonempty open set are "domain" and "region." These terms are used synonymously in this text.

In §2 we return to convergent power series and show they produce holomorphic functions. We extend results of §0 from functions of a real variable to functions of a complex variable. Section 3 returns to the exponential function e^z, defined above. We extend (1.0.5) to

$$\frac{d}{dz}e^{az} = ae^{az}. \tag{1.0.6}$$

We show that $t \mapsto e^t$ maps $\mathbb{R}$ one-to-one and onto $(0,\infty)$, and define the logarithm on $(0,\infty)$, as its inverse:

$$x = e^t \Longleftrightarrow t = \log x. \tag{1.0.7}$$

We also examine the behavior of $\gamma(t) = e^{it}$, for $t \in \mathbb{R}$, showing that this is a unit-speed curve tracing out the unit circle. From this we deduce Euler's formula,

$$e^{it} = \cos t + i \sin t. \tag{1.0.8}$$

This leads to a direct, self-contained treatment of the trigonometric functions.

In §4 we discuss inverses to holomorphic functions. In particular, we extend the logarithm from $(0,\infty)$ to $\mathbb{C} \setminus (-\infty, 0]$, as a holomorphic function. We define fractional powers

$$z^a = e^{a \log z}, \quad a \in \mathbb{C}, \quad z \in \mathbb{C} \setminus (-\infty, 0], \tag{1.0.9}$$

and investigate their basic properties. We also discuss inverse trigonometric functions in the complex plane.

The number π arises in §3 as half the length of the unit circle, or equivalently the smallest positive number satisfying

$$e^{\pi i} = -1. \tag{1.0.10}$$

This is possibly the most intriguing number in mathematics. It will appear many times over the course of this text. One of our goals will be to obtain accurate numerical approximations to π. In another vein, in Appendix I we show that π^2 is irrational.

0. Complex numbers, power series, and exponentials

A complex number has the form

$$z = x + iy, \tag{0.1}$$

where x and y are real numbers. These numbers make up the complex plane, which is just the xy-plane with the real line forming the horizontal axis and the real multiples of i forming the vertical axis. See Figure 0.1. We write

$$x = \operatorname{Re} z, \quad y = \operatorname{Im} z. \tag{0.2}$$

We write $x, y \in \mathbb{R}$ and $z \in \mathbb{C}$. We identify $x \in \mathbb{R}$ with $x + i0 \in \mathbb{C}$. If also $w = u + iv$ with $u, v \in \mathbb{R}$, we have addition and multiplication, given by

$$\begin{aligned} z + w &= (x + u) + i(y + v), \\ zw &= (xu - yv) + i(xv + yu), \end{aligned} \tag{0.3}$$

the latter rule containing the identity

$$i^2 = -1. \tag{0.4}$$

One readily verifies the commutative laws

$$z + w = w + z, \quad zw = wz, \tag{0.5}$$

the associative laws (with also $c \in \mathbb{C}$)

$$z + (w + c) = (z + w) + c, \quad z(wc) = (zw)c, \tag{0.6}$$

and the distributive law

$$c(z + w) = cz + cw, \tag{0.7}$$

as following from their counterparts for real numbers. If $c \neq 0$, we can perform division by c,

$$\frac{z}{c} = w \Longleftrightarrow z = wc.$$

See (0.13) for a neat formula.

For $z = x+iy$, we define $|z|$ to be the distance of z from the origin 0, via the Pythagorean theorem:

$$|z| = \sqrt{x^2 + y^2}. \tag{0.8}$$

Note that

$$|z|^2 = z\,\overline{z}, \quad \text{where } \ \overline{z} = x - iy, \tag{0.9}$$

is called the complex conjugate of z. One readily checks that

$$z + \overline{z} = 2 \ \mathrm{Re}\, z, \quad z - \overline{z} = 2i \ \mathrm{Im}\, z, \tag{0.10}$$

and

$$\overline{z + w} = \overline{z} + \overline{w}, \quad \overline{zw} = \overline{z}\,\overline{w}. \tag{0.11}$$

Hence $|zw|^2 = zw\overline{z}\,\overline{w} = |z|^2|w|^2$, so

$$|zw| = |z| \cdot |w|. \tag{0.12}$$

We also have, for $c \neq 0$,

$$\frac{z}{c} = \frac{1}{|c|^2}\, z\overline{c}. \tag{0.13}$$

The following result is known as the triangle inequality, as Figure 0.2 suggests.

Proposition 0.1. *Given* $z, w \in \mathbb{C}$,

$$|z + w| \le |z| + |w|. \tag{0.14}$$

Proof. We compare the squares of the two sides:

$$\begin{aligned} |z + w|^2 &= (z + w)(\overline{z} + \overline{w}) \\ &= z\overline{z} + w\overline{w} + z\overline{w} + w\overline{z} \\ &= |z|^2 + |w|^2 + 2 \ \mathrm{Re}(z\overline{w}), \end{aligned} \tag{0.15}$$

while

$$\begin{aligned} (|z| + |w|)^2 &= |z|^2 + |w|^2 + 2|z| \cdot |w| \\ &= |z|^2 + |w|^2 + 2|z\overline{w}|. \end{aligned} \tag{0.16}$$

Thus (0.14) follows from the inequality $\mathrm{Re}(z\overline{w}) \le |z\overline{w}|$, which in turn is immediate from the definition (0.8). (For any $\zeta \in \mathbb{C}$, $\ \mathrm{Re}\,\zeta \le |\zeta|$.)

We can define convergence of a sequence (z_n) in $\mathbb{C}$ as follows. We say

$$z_n \to z \ \text{ if and only if } \ |z_n - z| \to 0, \tag{0.17}$$

the latter notion involving convergence of a sequence of real numbers. Clearly if $z_n = x_n + iy_n$ and $z = x + iy$, with $x_n, y_n, x, y \in \mathbb{R}$, then

$$z_n \to z \quad \text{if and only if} \quad x_n \to x \quad \text{and} \quad y_n \to y. \tag{0.18}$$

One readily verifies that

$$z_n \to z, \ w_n \to w \Longrightarrow z_n + w_n \to z + w \quad \text{and} \quad z_n w_n \to zw, \tag{0.19}$$

as a consequence of their counterparts for sequences of real numbers.

A related notion is that a sequence (z_n) in $\mathbb{C}$ is *Cauchy* if and only if

$$|z_n - z_m| \longrightarrow 0 \quad \text{as} \quad m, n \to \infty. \tag{0.19A}$$

As in (0.18), this holds if and only if (x_n) and (y_n) are Cauchy in $\mathbb{R}$. The following is an important fact.

(0.19B) Each Cauchy sequence in $\mathbb{C}$ converges.

This follows from the fact that

(0.19C) each Cauchy sequence in $\mathbb{R}$ converges.

A detailed presentation of the field $\mathbb{R}$ of real numbers, including a proof of (0.19C), is given in Chapter 1 of [T0].

We can define the notion of convergence of an infinite series

$$\sum_{k=0}^{\infty} z_k \tag{0.20}$$

as follows. For each $n \in \mathbb{Z}^+$, set

$$s_n = \sum_{k=0}^{n} z_k. \tag{0.21}$$

Then (0.20) converges if and only if the sequence (s_n) converges:

$$s_n \to w \Longrightarrow \sum_{k=0}^{\infty} z_k = w. \tag{0.22}$$

Note that

$$\begin{aligned} |s_{n+m} - s_n| &= \Big| \sum_{k=n+1}^{n+m} z_k \Big| \\ &\le \sum_{k=n+1}^{n+m} |z_k|. \end{aligned} \tag{0.23}$$

Using this, we can establish the following.

Lemma 0.1A. *Assume that*

(0.24)
$$\sum_{k=0}^{\infty} |z_k| < \infty,$$

i.e., there exists $A < \infty$ such that

(0.24A)
$$\sum_{k=0}^{N} |z_k| \le A, \quad \forall\, N.$$

Then the sequence (s_n) given by (0.21) is Cauchy, hence the series (0.20) is convergent.

Proof. If (s_n) is not Cauchy, there exist $a > 0,\ n_\nu \nearrow \infty$, and $m_\nu > 0$ such that

$$|s_{n_\nu + m_\nu} - s_{n_\nu}| \ge a.$$

Passing to a subsequence, one can assume that $n_{\nu+1} > n_\nu + m_\nu$. Then (0.23) implies

$$\sum_{k=0}^{m_\nu + n_\nu} |z_k| \ge \nu a, \quad \forall\, \nu,$$

contradicting (0.24A).

If (0.24) holds, we say the series (0.20) is *absolutely convergent.*

An important class of infinite series is the class of power series

(0.25)
$$\sum_{k=0}^{\infty} a_k z^k,$$

with $a_k \in \mathbb{C}$. Note that if $z_1 \neq 0$ and (0.25) converges for $z = z_1$, then there exists $C < \infty$ such that

(0.25A)
$$|a_k z_1^k| \le C, \quad \forall\, k.$$

Hence, if $|z| \le r|z_1|,\ r < 1$, we have

(0.26)
$$\sum_{k=0}^{\infty} |a_k z^k| \le C \sum_{k=0}^{\infty} r^k = \frac{C}{1-r} < \infty,$$

the last identity being the classical geometric series computation. See Exercise 3 below. This yields the following.

Proposition 0.2. *If (0.25) converges for some $z_1 \neq 0$, then either this series is absolutely convergent for all $z \in \mathbb{C}$, or there is some $R \in (0,\infty)$ such that the series is absolutely convergent for $|z| < R$ and divergent for $|z| > R$.*

We call R the radius of convergence of (0.25). In case of convergence for all z, we say the radius of convergence is infinite. If $R > 0$ and (0.25) converges for $|z| < R$, it defines a function

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad z \in D_R, \tag{0.27}$$

on the disk of radius R centered at the origin,

$$D_R = \{z \in \mathbb{C} : |z| < R\}. \tag{0.28}$$

Proposition 0.3. *If the series (0.27) converges in D_R, then f is continuous on D_R, i.e., given $z_n, z \in D_R$,*

$$z_n \to z \Longrightarrow f(z_n) \to f(z). \tag{0.29}$$

Proof. For each $z \in D_R$, there exists $S < R$ such that $z \in D_S$, so it suffices to show that f is continuous on D_S whenever $0 < S < R$. Pick T such that $S < T < R$. We know that there exists $C < \infty$ such that $|a_k T^k| \le C$ for all k. Hence

$$z \in D_S \Longrightarrow |a_k z^k| \le C\Big(\frac{S}{T}\Big)^k. \tag{0.30}$$

For each N, write

$$\begin{gathered} f(z) = S_N(z) + R_N(z), \\ S_N(z) = \sum_{k=0}^{N} a_k z^k, \quad R_N(z) = \sum_{k=N+1}^{\infty} a_k z^k. \end{gathered} \tag{0.31}$$

Each $S_N(z)$ is a polynomial in z, and it follows readily from (0.19) that S_N is continuous. Meanwhile,

$$z \in D_S \Rightarrow |R_N(z)| \le \sum_{k=N+1}^{\infty} |a_k z^k| \le C \sum_{k=N+1}^{\infty} \Big(\frac{S}{T}\Big)^k = C\varepsilon_N, \tag{0.32}$$

and $\varepsilon_N \to 0$ as $N \to \infty$, independently of $z \in D_S$. Continuity of f on D_S follows, as a consequence of the next lemma.

Lemma 0.3A. *Let $S_N : D_S \to \mathbb{C}$ be continuous functions. Assume $f : D_S \to \mathbb{C}$ and $S_N \to f$ uniformly on D_S, i.e.,*

$$|S_N(z) - f(z)| \le \delta_N, \ \forall\, z \in D_S, \quad \delta_N \to 0, \ \text{as } N \to \infty. \tag{0.32A}$$

Then f is continuous on D_S.

Proof. Let $z_n \to z$ in D_S. We need to show that, given $\varepsilon > 0$, there exists $M = M(\varepsilon) < \infty$ such that

$$|f(z) - f(z_n)| \le \varepsilon, \quad \forall\, n \ge M.$$

To get this, pick N such that (0.32A) holds with $\delta_N = \varepsilon/3$. Now use continuity of S_N, to deduce that there exists M such that

$$|S_N(z) - S_N(z_n)| \le \frac{\varepsilon}{3}, \quad \forall\, n \ge M.$$

It follows that, for $n \ge M$,

$$\begin{aligned} |f(z) - f(z_n)| &\le |f(z) - S_N(z)| + |S_N(z) - S_N(z_n)| + |S_N(z_n) - f(z_n)| \\ &\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}, \end{aligned}$$

as desired.

Remark. The estimate (0.32) says the series (0.27) converges *uniformly* on D_S, for each $S < R$.

A major consequence of material developed in §§1–5 will be that a function on D_R is given by a convergent power series (0.27) if and only if f has the property of being holomorphic on D_R (a property that is defined in §1). We will be doing differential and integral calculus on such functions. In this preliminary section, we restrict z to be real, and do some calculus, starting with the following.

Proposition 0.4. *Assume $a_k \in \mathbb{C}$ and*

$$f(t) = \sum_{k=0}^{\infty} a_k t^k \tag{0.33}$$

converges for real t satisfying $|t| < R$. Then f is differentiable on the interval $-R < t < R$, and

$$f'(t) = \sum_{k=1}^{\infty} k a_k t^{k-1}, \tag{0.34}$$

the latter series being absolutely convergent for $|t| < R$.

We first check absolute convergence of the series (0.34). Let $S < T < R$. Convergence of (0.33) implies there exists $C < \infty$ such that

$$|a_k| T^k \le C, \quad \forall\, k. \tag{0.35}$$

Hence, if $|t| \le S$,

$$|k a_k t^{k-1}| \le \frac{C}{S}\, k \left(\frac{S}{T}\right)^k. \tag{0.35A}$$

The absolute convergence of

$$\sum_{k=0}^{\infty} k r^k, \quad \text{for} \ \ |r| < 1, \tag{0.36}$$

follows from the ratio test. (See Exercises 4–5 below.) Hence

$$g(t) = \sum_{k=1}^{\infty} k a_k t^{k-1} \tag{0.37}$$

is continuous on $(-R, R)$. To show that $f'(t) = g(t)$, by the fundamental theorem of calculus, it is equivalent to show

$$\int_0^t g(s)\, ds = f(t) - f(0). \tag{0.38}$$

The following result implies this.

Proposition 0.5. *Assume $b_k \in \mathbb{C}$ and*

$$g(t) = \sum_{k=0}^{\infty} b_k t^k \tag{0.39}$$

converges for real t, satisfying $|t| < R$. Then, for $|t| < R$,

$$\int_0^t g(s)\, ds = \sum_{k=0}^{\infty} \frac{b_k}{k+1} t^{k+1}, \tag{0.40}$$

the series being absolutely convergent for $|t| < R$.

Proof. Since, for $|t| < R$,

$$\left| \frac{b_k}{k+1} t^{k+1} \right| \le R |b_k t^k|, \tag{0.41}$$

convergence of the series in (0.40) is clear. Next, parallel to (0.31), write

$$
\begin{aligned}
&g(t) = S_N(t) + R_N(t), \\
&S_N(t) = \sum_{k=0}^{N} b_k t^k, \quad R_N(t) = \sum_{k=N+1}^{\infty} b_k t^k.
\end{aligned} \tag{0.42}
$$

Parallel to (0.32), if we pick $S < R$, we have

$$
|t| \le S \Rightarrow |R_N(t)| \le C\varepsilon_N \to 0 \ \text{ as } \ N \to \infty, \tag{0.43}
$$

so

$$
\int_0^t g(s)\, ds = \sum_{k=0}^{N} \frac{b_k}{k+1} t^{k+1} + \int_0^t R_N(s)\, ds, \tag{0.44}
$$

and

$$
\Bigl| \int_0^t R_N(s)\, ds \Bigr| \le \int_0^t |R_N(s)|\, ds \le CR\varepsilon_N, \tag{0.45}
$$

for $|t| \le S$. This gives (0.40).

We use Proposition 0.4 to solve some basic differential equations, starting with

$$
f'(t) = f(t), \quad f(0) = 1. \tag{0.46}
$$

We look for a solution as a power series, of the form (0.33). If there is a solution of this form, (0.34) requires

$$
a_0 = 1, \quad a_{k+1} = \frac{a_k}{k+1}, \tag{0.47}
$$

i.e., $a_k = 1/k!$, where $k! = k(k-1) \cdots 2 \cdot 1$. We deduce that (0.46) is solved by

$$
f(t) = e^t = \sum_{k=0}^{\infty} \frac{1}{k!} t^k, \quad t \in \mathbb{R}. \tag{0.48}
$$

This defines the exponential function e^t. Convergence for all t follows from the ratio test. (Cf. Exercise 4 below.) More generally, we define

$$
e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k, \quad z \in \mathbb{C}. \tag{0.49}
$$

Again the ratio test shows that this series is absolutely convergent for all $z \in \mathbb{C}$. Another application of Proposition 0.4 shows that

$$e^{at} = \sum_{k=0}^{\infty} \frac{a^k}{k!} t^k \tag{0.50}$$

solves

$$\frac{d}{dt} e^{at} = a e^{at}, \tag{0.51}$$

whenever $a \in \mathbb{C}$.

We claim that e^{at} is the only solution to

$$f'(t) = a f(t), \quad f(0) = 1. \tag{0.52}$$

To see this, compute the derivative of $e^{-at} f(t)$:

$$\frac{d}{dt}\big(e^{-at} f(t)\big) = -a e^{-at} f(t) + e^{-at} a f(t) = 0, \tag{0.53}$$

where we use the product rule, (0.51) (with a replaced by $-a$), and (0.52). Thus $e^{-at} f(t)$ is independent of t. Evaluating at $t = 0$ gives

$$e^{-at} f(t) = 1, \quad \forall\, t \in \mathbb{R}, \tag{0.54}$$

whenever $f(t)$ solves (0.52). Since e^{at} solves (0.52), we have $e^{-at} e^{at} = 1$, hence

$$e^{-at} = \frac{1}{e^{at}}, \quad \forall\, t \in \mathbb{R},\ a \in \mathbb{C}. \tag{0.55}$$

Thus multiplying both sides of (0.54) by e^{at} gives the asserted uniqueness:

$$f(t) = e^{at}, \quad \forall\, t \in \mathbb{R}. \tag{0.56}$$

We can draw further useful conclusions by applying d/dt to products of exponentials. Let $a, b \in \mathbb{C}$. Then

$$\begin{aligned} &\frac{d}{dt}\big(e^{-at} e^{-bt} e^{(a+b)t}\big) \\ &= -a e^{-at} e^{-bt} e^{(a+b)t} - b e^{-at} e^{-bt} e^{(a+b)t} + (a+b) e^{-at} e^{-bt} e^{(a+b)t} \\ &= 0, \end{aligned} \tag{0.57}$$

so again we are differentiating a function that is independent of t. Evaluation at $t = 0$ gives

$$e^{-at} e^{-bt} e^{(a+b)t} = 1, \quad \forall\, t \in \mathbb{R}. \tag{0.58}$$

Using (0.55), we get

$$e^{(a+b)t} = e^{at}e^{bt}, \quad \forall\, t \in \mathbb{R},\ a, b \in \mathbb{C}, \tag{0.59}$$

or, setting $t = 1$,

$$e^{a+b} = e^a e^b, \quad \forall\, a, b \in \mathbb{C}. \tag{0.60}$$

We will resume study of the exponential function in §3, and derive further important properties.

Exercises

1. Supplement (0.19) with the following result. Assume there exists $A > 0$ such that $|z_n| \geq A$ for all n. Then

$$z_n \to z \Longrightarrow \frac{1}{z_n} \to \frac{1}{z}. \tag{0.61}$$

2. Show that

$$|z| < 1 \Longrightarrow z^k \to 0, \quad \text{as} \quad k \to \infty. \tag{0.62}$$

Hint. Deduce (0.62) from the assertion

$$0 < s < 1 \Longrightarrow ks^k \ \text{ is bounded, for } \ k \in \mathbb{N}. \tag{0.63}$$

Note that this is equivalent to

$$a > 0 \Longrightarrow \frac{k}{(1+a)^k} \ \text{ is bounded, for } \ k \in \mathbb{N}. \tag{0.64}$$

Show that

$$(1+a)^k = (1+a)\cdots(1+a) \geq 1 + ka, \quad \forall\, a > 0,\ k \in \mathbb{N}. \tag{0.65}$$

Use this to prove (0.64), hence (0.63), hence (0.62).

3. Letting $s_n = \sum_{k=0}^n r^k$, write the series for rs_n and show that

$$(1-r)s_n = 1 - r^{n+1}, \quad \text{hence} \quad s_n = \frac{1 - r^{n+1}}{1 - r}. \tag{0.66}$$

Deduce that

$$0 < r < 1 \Longrightarrow s_n \to \frac{1}{1-r}, \quad \text{as} \quad n \to \infty, \tag{0.67}$$

as stated in (0.26). More generally, show that

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, \quad \text{for} \quad |z| < 1. \tag{0.67A}$$

4. This exercise discusses the ratio test, mentioned in connection with the infinite series (0.36) and (0.49). Consider the infinite series

$$\sum_{k=0}^{\infty} a_k, \quad a_k \in \mathbb{C}. \tag{0.68}$$

Assume there exists $r < 1$ and $N < \infty$ such that

$$k \ge N \Longrightarrow \left| \frac{a_{k+1}}{a_k} \right| \le r. \tag{0.69}$$

Show that

$$\sum_{k=0}^{\infty} |a_k| < \infty. \tag{0.70}$$

Hint. Show that

$$\sum_{k=N}^{\infty} |a_k| \le |a_N| \sum_{\ell=0}^{\infty} r^\ell = \frac{|a_N|}{1-r}. \tag{0.71}$$

5. In case

$$a_k = \frac{z^k}{k!}, \tag{0.72}$$

show that for each $z \in \mathbb{C}$, there exists $N < \infty$ such that (0.69) holds, with $r = 1/2$. Also show that the ratio test applies to

$$a_k = k z^k, \quad |z| < 1. \tag{0.73}$$

6. This exercise discusses the integral test for absolute convergence of an infinite series,

which goes as follows. Let f be a positive, monotonically decreasing, continuous function on $[0,\infty)$, and suppose $|a_k| = f(k)$. Then

$$\sum_{k=0}^{\infty} |a_k| < \infty \Longleftrightarrow \int_0^{\infty} f(t)\, dt < \infty.$$

Prove this.
Hint. Use

$$\sum_{k=1}^{N} |a_k| \le \int_0^{N} f(t)\, dt \le \sum_{k=0}^{N-1} |a_k|.$$

7. Use the integral test to show that, if $a > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{n^a} < \infty \Longleftrightarrow a > 1.$$

8. This exercise deals with alternating series. Assume $b_k \searrow 0$. Show that

$$\sum_{k=0}^{\infty} (-1)^k b_k \quad \text{is convergent,}$$

be showing that, for $m, n \ge 0$,

$$\Big| \sum_{k=n}^{n+m} (-1)^k b_k \Big| \le b_n.$$

9. Show that $\sum_{k=1}^{\infty} (-1)^k/k$ is convergent, but not absolutely convergent.

10. Show that if $f, g : (a,b) \to \mathbb{C}$ are differentiable, then

$$\frac{d}{dt}\big(f(t)g(t)\big) = f'(t)g(t) + f(t)g'(t). \tag{0.74}$$

Note the use of this identity in (0.53) and (0.57).

11. Use the results of Exercise 10 to show, by induction on k, that

$$\frac{d}{dt} t^k = k t^{k-1}, \quad k = 1, 2, 3, \dots, \tag{0.75}$$

hence

$$\int_0^t s^k\, ds = \frac{1}{k+1} t^{k+1}, \quad k = 0, 1, 2, \dots. \tag{0.76}$$

Note the use of these identities in (0.44), leading to many of the identities in (0.34)–(0.51).

12. Consider

$$\varphi(z) = \frac{z-1}{z+1}.$$

Show that

$$\varphi : \mathbb{C} \setminus \{-1\} \longrightarrow \mathbb{C} \setminus \{1\}$$

is continuous, one-to-one, and onto. Show that, if $\Omega = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and $D = \{z \in \mathbb{C} : |z| < 1\}$, then

$$\varphi : \Omega \to D \quad \text{and} \quad \varphi : \partial\Omega \to \partial D \setminus \{1\}$$

are one-to-one and onto.
REMARK. This exercise is relevant to material in Exercise 7 of §4. The result here is a special case of important results on linear fractional transformations, discussed at length in §20.

1. Holomorphic functions, derivatives, and path integrals

Let $\Omega \subset \mathbb{C}$ be open, i.e., if $z_0 \in \Omega$, there exists $\varepsilon > 0$ such that $D_\varepsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$ is contained in Ω. Let $f : \Omega \to \mathbb{C}$. If $z \in \Omega$, we say f is complex-differentiable at z, with derivative $f'(z) = a$, if and only if

$$\lim_{h \to 0} \frac{1}{h}[f(z+h) - f(z)] = a. \tag{1.1}$$

Here, $h = h_1 + ih_2$, with $h_1, h_2 \in \mathbb{R}$, and $h \to 0$ means $h_1 \to 0$ and $h_2 \to 0$. Note that

$$\lim_{h_1 \to 0} \frac{1}{h_1}[f(z+h_1) - f(z)] = \frac{\partial f}{\partial x}(z), \tag{1.2}$$

and

$$\lim_{h_2 \to 0} \frac{1}{ih_2}[f(z+ih_2) - f(z)] = \frac{1}{i}\frac{\partial f}{\partial y}(z), \tag{1.2A}$$

provided these limits exist. Another notation we use when f is complex differentiable at z is

$$\frac{df}{dz}(z) = f'(z). \tag{1.3}$$

As a first set of examples, we have

$$\begin{aligned} f(z) = z &\Longrightarrow \frac{1}{h}[f(z+h) - f(z)] = 1, \\ f(z) = \overline{z} &\Longrightarrow \frac{1}{h}[f(z+h) - f(z)] = \frac{\overline{h}}{h}. \end{aligned} \tag{1.4}$$

In the first case, the limit exists and we have $f'(z) = 1$ for all z. In the second case, the limit does not exist. The function $f(z) = \overline{z}$ is not complex-differentiable.

Definition. A function $f : \Omega \to \mathbb{C}$ is *holomorphic* if and only if it is complex-differentiable and f' is continuous on Ω. Another term applied to such a function f is that it is *complex analytic.*

Adding the hypothesis that f' is continuous makes for a convenient presentation of the basic results. In §9 it will be shown that every complex differentiable function has this additional property.

So far, we have seen that $f_1(z) = z$ is holomorphic. We produce more examples of holomorphic functions. For starters, we claim that $f_k(z) = z^k$ is holomorphic on $\mathbb{C}$ for each $k \in \mathbb{Z}^+$, and

$$\frac{d}{dz} z^k = kz^{k-1}. \tag{1.5}$$

One way to see this is inductively, via the following result.

Proposition 1.1. *If f and g are holomorphic on Ω, so is fg, and*

$$\frac{d}{dz}(fg)(z) = f'(z)g(z) + f(z)g'(z). \tag{1.6}$$

Proof. Just as in beginning calculus, we have

$$\begin{aligned} &\frac{1}{h}[f(z+h)g(z+h) - f(z)g(z)] \\ &= \frac{1}{h}[f(z+h)g(z+h) - f(z)g(z+h) + f(z)g(z+h) - f(z)g(z)] \\ &= \frac{1}{h}[f(z+h) - f(z)]g(z+h) + f(z)\cdot\frac{1}{h}[g(z+h) - g(z)]. \end{aligned} \tag{1.7}$$

The first term in the last line tends to $f'(z)g(z)$, and the second term tends to $f(z)g'(z)$, as $h \to 0$. This gives (1.6). If f' and g' are continuous, the right side of (1.6) is also continuous, so fg is holomorphic.

It is even easier to see that the sum of two holomorphic functions is holomorphic, and

$$\frac{d}{dz}(f(z) + g(z)) = f'(z) + g'(z), \tag{1.8}$$

Hence every polynomial $p(z) = a_n z^n + \cdots + a_1 z + a_0$ is holomorphic on $\mathbb{C}$.

We next show that $f_{-1}(z) = 1/z$ is holomorphic on $\mathbb{C} \setminus 0$, with

$$\frac{d}{dz}\frac{1}{z} = -\frac{1}{z^2}. \tag{1.9}$$

In fact,

$$\frac{1}{h}\Big[\frac{1}{z+h} - \frac{1}{z}\Big] = -\frac{1}{h}\,\frac{h}{z(z+h)} = -\frac{1}{z(z+h)}, \tag{1.10}$$

which tends to $-1/z^2$ as $h \to 0$, if $z \neq 0$, and this gives (1.9). Continuity on $\mathbb{C} \setminus 0$ is readily established. From here, we can apply Proposition 1.1 inductively and see that z^k is holomorphic on $\mathbb{C} \setminus 0$ for $k = -2, -3, \dots$, and (1.5) holds on $\mathbb{C} \setminus 0$ for such k.

Next, recall the exponential function

$$e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k, \tag{1.11}$$

Introduced in §0. We claim that e^z is holomorphic on $\mathbb{C}$ and

$$\frac{d}{dz} e^z = e^z. \tag{1.12}$$

To see this, we use the identity (0.60), which implies

$$e^{z+h} = e^z e^h. \tag{1.13}$$

Hence

$$\frac{1}{h}[e^{z+h} - e^z] = e^z \frac{e^h - 1}{h}. \tag{1.14}$$

Now (1.11) implies

$$\frac{e^h - 1}{h} = \sum_{k=1}^{\infty} \frac{1}{k!} h^{k-1} = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} h^k, \tag{1.15}$$

and hence (thanks to Proposition 0.3)

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1. \tag{1.16}$$

This gives (1.12). See Exercise 13 below for a second proof of (1.12). A third proof will follow from the results of §2.

We next establish a "chain rule" for holomorphic functions. In preparation for this, we note that the definition (1.1) of complex differentiability is equivalent to the condition that, for h sufficiently small,

$$f(z+h) = f(z) + ah + r(z,h), \tag{1.17}$$

with

$$\lim_{h\to 0} \frac{r(z,h)}{h} = 0, \tag{1.18}$$

i.e., $r(z,h) \to 0$ faster than h. We write

$$r(z,h) = o(|h|). \tag{1.18}$$

Here is the chain rule.

Proposition 1.2. *Let $\Omega, \mathcal{O} \subset \mathbb{C}$ be open. If $f : \Omega \to \mathbb{C}$ and $g : \mathcal{O} \to \Omega$ are holomorphic, then $f \circ g : \mathcal{O} \to \mathbb{C}$, given by*

$$f \circ g(z) = f(g(z)), \tag{1.19}$$

is holomorphic, and

$$\frac{d}{dz} f(g(z)) = f'(g(z)) g'(z). \tag{1.20}$$

Proof. Since g is holomorphic,

$$g(z+h) = g(z) + g'(z)h + r(z,h), \tag{1.21}$$

with $r(z,h) = o(|h|)$. Hence

$$\begin{aligned} f(g(z+h)) &= f(g(z) + g'(z)h + r(z,h)) \\ &= f(g(z)) + f'(g(z))(g'(z)h + r(z,h)) + r_2(z,h) \\ &= f(g(z)) + f'(g(z))g'(z)h + r_3(z,h), \end{aligned} \tag{1.22}$$

with $r_2(z,h) = o(|h|)$, because f is holomorphic, and then

$$r_3(z,h) = f'(g(z))r(z,h) + r_2(z,h) = o(|h|). \tag{1.23}$$

This implies $f \circ g$ is complex-differentiable and gives (1.20). Since the right side of (1.20) is continuous, $f \circ g$ is seen to be holomorphic.

Combining Proposition 1.2 with (1.9), we have the following.

Proposition 1.3. *If $f : \Omega \to \mathbb{C}$ is holomorphic, then $1/f$ is holomorphic on $\Omega \setminus S$, where*

$$S = \{z \in \Omega : f(z) = 0\}, \tag{1.24}$$

and, on $\Omega \setminus S$,

$$\frac{d}{dz}\frac{1}{f(z)} = -\frac{f'(z)}{f(z)^2}. \tag{1.25}$$

We can also combine Proposition 1.2 with (1.12) and get

$$\frac{d}{dz}e^{f(z)} = f'(z)e^{f(z)}. \tag{1.26}$$

We next examine implications of (1.2)–(1.3). The following is immediate.

Proposition 1.4. *If $f : \Omega \to \mathbb{C}$ is holomorphic, then*

$$\frac{\partial f}{\partial x} \quad \textit{and} \quad \frac{\partial f}{\partial y} \quad \textit{exist, and are continuous on} \quad \Omega, \tag{1.27}$$

and

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y} \tag{1.28}$$

on Ω, each side of (1.28) being equal to f' on Ω.

When (1.27) holds, one says f is of class C^1 and writes $f \in C^1(\Omega)$. As shown in Appendix B, if $f \in C^1(\Omega)$, it is $\mathbb{R}$-differentiable, i.e.,

$$f((x+h_1)+i(y+h_2)) = f(x+iy) + Ah_1 + Bh_2 + r(z,h), \tag{1.29}$$

with $z = x+iy,\ h = h_1 + ih_2,\ r(z,h) = o(|h|)$, and

$$A = \frac{\partial f}{\partial x}(z), \quad B = \frac{\partial f}{\partial y}(z). \tag{1.30}$$

This has the form (1.17), with $a \in \mathbb{C}$, if and only if

$$a(h_1 + ih_2) = Ah_1 + Bh_2, \tag{1.31}$$

for all $h_1, h_2 \in \mathbb{R}$, which holds if and only if

$$A = \frac{1}{i}B = a, \tag{1.32}$$

leading back to (1.28). This gives the following converse to Proposition 1.4.

Proposition 1.5. *If $f : \Omega \to \mathbb{C}$ is C^1 and (1.28) holds, then f is holomorphic.*

The equation (1.28) is called the Cauchy-Riemann equation. Here is an alternative presentation. Write

$$f(z) = u(z) + iv(z), \quad u = \operatorname{Re} f,\ v = \operatorname{Im} f. \tag{1.33}$$

Then (1.28) is equivalent to the system of equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{1.34}$$

To pursue this a little further, we change perspective, and regard f as a map from an open subset Ω of $\mathbb{R}^2$ into $\mathbb{R}^2$. We represent an element of $\mathbb{R}^2$ as a column vector. Objects on $\mathbb{C}$ and on $\mathbb{R}^2$ correspond as follows.

$$\begin{array}{ll} \text{On } \mathbb{C} & \text{On } \mathbb{R}^2 \\ z = x + iy & z = \begin{pmatrix} x \\ y \end{pmatrix} \\ f = u + iv & f = \begin{pmatrix} u \\ v \end{pmatrix} \\ h = h_1 + ih_2 & h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \end{array} \tag{1.35}$$

As discussed in Appendix B, a map $f : \Omega \to \mathbb{R}^2$ is differentiable at $z \in \Omega$ if and only if there exists a 2×2 matrix L such that

$$f(z+h) = f(z) + Lh + R(z,h), \quad R(z,h) = o(|h|). \tag{1.36}$$

If such L exists, then $L = Df(z)$, with

$$Df(z) = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix}. \tag{1.37}$$

The Cauchy-Riemann equations specify that

$$Df = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \quad \alpha = \frac{\partial u}{\partial x}, \quad \beta = \frac{\partial v}{\partial x}. \tag{1.38}$$

Now the map $z \mapsto iz$ is a linear transformation on $\mathbb{C} \approx \mathbb{R}^2$, whose 2×2 matrix representation is given by

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.39}$$

Note that, if $L = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$, then

$$JL = \begin{pmatrix} -\beta & -\delta \\ \alpha & \gamma \end{pmatrix}, \quad LJ = \begin{pmatrix} \gamma & -\alpha \\ \delta & -\beta \end{pmatrix}, \tag{1.40}$$

so $JL = LJ$ if and only if $\alpha = \delta$ and $\beta = -\gamma$. (When $JL = LJ$, we say J and L *commute.*) When $L = Df(z)$, this gives (1.38), proving the following.

Proposition 1.6. *If $f \in C^1(\Omega)$, then f is holomorphic if and only if, for each $z \in \Omega$,*

$$Df(z) \quad \text{and} \quad J \quad \text{commute.} \tag{1.41}$$

REMARK. Given that J is the matrix representation of multiplication by i on $\mathbb{C} \approx \mathbb{R}^2$, the content of (1.41) is that the $\mathbb{R}$-linear transformation $Df(z)$ is actually *complex linear.*

In the calculus of functions of a real variable, the interaction of derivatives and integrals, via the fundamental theorem of calculus, plays a central role. We recall the statement.

Theorem 1.7. *If $f \in C^1([a,b])$, then*

$$\int_a^b f'(t)\,dt = f(b) - f(a). \tag{1.42}$$

Furthermore, if $g \in C([a,b])$, then, for $a < t < b$,

$$\frac{d}{dt}\int_a^t g(s)\,ds = g(t). \tag{1.43}$$

In the study of holomorphic functions on an open set $\Omega \subset \mathbb{C}$, the partner of d/dz is the integral over a curve (i.e., path integral), which we now discuss.

A C^1 curve (or path) in Ω is a C^1 map

$$\gamma : [a,b] \longrightarrow \Omega,$$

where $[a,b] = \{t \in \mathbb{R} : a \le t \le b\}$. If $f : \Omega \to \mathbb{C}$ is continuous, we define

$$\int\limits_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt, \tag{1.44}$$

the right side being the standard integral of a continuous function, as studied in beginning calculus (except that here the integrand is complex valued). More generally, if $f, g : \Omega \to \mathbb{C}$ are continuous and $\gamma = \gamma_1 + i\gamma_2$, with γ_j real valued, we set

$$\int\limits_\gamma f(z)\,dx + g(z)\,dy = \int_a^b \big[f(\gamma(t))\gamma_1'(t) + g(\gamma(t))\gamma_2'(t)\big]\,dt. \tag{1.44A}$$

Then (1.44) is the special case $g = if$ (with $dz = dx + i\,dy$). A more general notion is that of a *piecewise smooth* path (or curve) in Ω. This is a continuous path $\gamma : [a,b] \to \Omega$ with the property that there is a finite partition

$$a = a_0 < a_1 < \cdots < a_N = b$$

such that each piece $\gamma_j : [a_j, a_{j+1}] \to \Omega$ is smooth of class C^k ($k \ge 1$) with limits $\gamma_j^{(\ell)}$ existing at the endpoints of $[a_j, a_{j+1}]$. In such a case, we set

$$\begin{aligned} \int\limits_\gamma f(z)\,dz &= \sum_{j=0}^{N-1} \int\limits_{\gamma_j} f(z)\,dz, \\ \int\limits_\gamma f(z)\,dx + g(z)\,dy &= \sum_{j=0}^{N-1} \int\limits_{\gamma_j} f(z)\,dx + g(z)\,dy, \end{aligned} \tag{1.44B}$$

with the integrals over eac piece γ_j defined as above.

The following result is a counterpart to (1.42).

Proposition 1.8. *If f is holomorphic on Ω, and $\gamma : [a,b] \to \mathbb{C}$ is a C^1 path, then*

$$\int_\gamma f'(z)\, dz = f(\gamma(b)) - f(\gamma(a)). \tag{1.45}$$

The proof will use the following chain rule.

Proposition 1.9. *If $f : \Omega \to \mathbb{C}$ is holomorphic and $\gamma : [a,b] \to \Omega$ is C^1, then, for $a < t < b$,*

$$\frac{d}{dt} f(\gamma(t)) = f'(\gamma(t))\gamma'(t). \tag{1.46}$$

The proof of Proposition 1.9 is essentially the same as that of Proposition 1.2. To address Proposition 1.8, we have

$$\begin{aligned} \int_\gamma f'(z)\, dz &= \int_a^b f'(\gamma(t))\gamma'(t)\, dt \\ &= \int_a^b \frac{d}{dt} f(\gamma(t))\, dt \\ &= f(\gamma(b)) - f(\gamma(a)), \end{aligned} \tag{1.47}$$

the second identity by (1.46) and the third by (1.42). This gives Proposition 1.8.

The second half of Theorem 1.7 involves producing an *antiderivative* of a given function g. In the complex context, we have the following.

Definition. A holomorphic function $g : \Omega \to \mathbb{C}$ is said to have an antiderivative f on Ω provided $f : \Omega \to \mathbb{C}$ is holomorphic and $f' = g$.

Calculations done above show that g has an antiderivative f in the following cases:

$$\begin{aligned} g(z) &= z^k, \quad f(z) = \frac{1}{k+1} z^{k+1}, \quad k \neq -1, \\ g(z) &= e^z, \quad f(z) = e^z. \end{aligned} \tag{1.48}$$

A function g holomorphic on an open set Ω might not have an antiderivative f on all of Ω. In cases where it does, Proposition 1.8 implies

$$\int_\gamma g(z)\, dz = 0 \tag{1.49}$$

for any closed path γ in Ω, i.e., any C^1 path $\gamma : [a, b] \to \Omega$ such that $\gamma(a) = \gamma(b)$. In §3, we will see that if γ is the unit circle centered at the origin,

$$\int_\gamma \frac{1}{z}\,dz = 2\pi i, \tag{1.50}$$

so $1/z$, which is holomorphic on $\mathbb{C} \setminus 0$, does not have an antiderivative on $\mathbb{C} \setminus 0$. In §§3 and 4, we will construct $\log z$ as an antiderivative of $1/z$ on the smaller domain $\mathbb{C} \setminus (-\infty, 0]$.

We next show that each holomorphic function $g : \Omega \to \mathbb{C}$ has an antiderivative for a significant class of open sets $\Omega \subset \mathbb{C}$, namely sets with the following property.

(1.51) There exists $a + ib \in \Omega$ such that whenever $x + iy \in \Omega$, the vertical line from $a + ib$ to $a + iy$ and the horizontal line from $a + iy$ to $x + iy$ belong to Ω.

(Here $a, b, x, y \in \mathbb{R}$.) See Fig. 1.1.

Proposition 1.10. *If $\Omega \subset \mathbb{C}$ is an open set satisfying (1.51) and $g : \Omega \to \mathbb{C}$ is holomorphic, then there exists a holomorphic $f : \Omega \to \mathbb{C}$ such that $f' = g$.*

Proof. Take $a + ib \in \Omega$ as in (1.51), and set, for $z = x + iy \in \Omega$,

$$f(z) = i \int_b^y g(a + is)\,ds + \int_a^x g(t + iy)\,dt. \tag{1.52}$$

Theorem 1.7 readily gives

$$\frac{\partial f}{\partial x}(z) = g(z). \tag{1.53}$$

We also have

$$\frac{\partial f}{\partial y}(z) = ig(a + iy) + \int_a^x \frac{\partial g}{\partial y}(t + iy)\,dt, \tag{1.54}$$

and applying the Cauchy-Riemann equation $\partial g/\partial y = i\partial g/\partial x$ gives

$$\begin{aligned} \frac{1}{i}\frac{\partial f}{\partial y} &= g(a + iy) + \int_a^x \frac{\partial g}{\partial t}(t + iy)\,dt \\ &= g(a + iy) + [g(x + iy) - g(a + iy)] \\ &= g(z). \end{aligned} \tag{1.55}$$

Comparing (1.54) and (1.55), we have the Cauchy-Riemann equations for f, and Proposition 1.10 follows.

Examples of open sets satisfying (1.51) include disks and rectangles, while $\mathbb{C}\setminus 0$ does not satisfy (1.51), as one can see by taking $a+ib=-1,\ x+iy=1$.

In §7 we extend the conclusion of Proposition 1.10 to a larger class of open sets $\Omega\subset\mathbb{C}$, called simply connected. See Exercise 8 of §7.

Exercises

1. Let $f,g\in C^1(\Omega)$, not necessarily holomorphic. Show that

$$\begin{aligned}\frac{\partial}{\partial x}(f(z)g(z)) &= f_x(z)g(z)+f(z)g_x(z),\\ \frac{\partial}{\partial y}(f(z)g(z)) &= f_y(z)g(z)+f(z)g_y(z),\end{aligned}\tag{1.56}$$

on Ω, where $f_x=\partial f/\partial x$, etc.

2. In the setting of Exercise 1, show that, on $\{z\in\Omega : g(z)\neq 0\}$,

$$\frac{\partial}{\partial x}\frac{1}{g(z)}=-\frac{g_x(z)}{g(z)^2},\quad \frac{\partial}{\partial y}\frac{1}{g(z)}=-\frac{g_y(z)}{g(z)^2}.\tag{1.57}$$

Derive formulas for

$$\frac{\partial}{\partial x}\frac{f(z)}{g(z)}\quad\text{and}\quad\frac{\partial}{\partial y}\frac{f(z)}{g(z)}.$$

3. In (a)–(d), compute $\partial f/\partial x$ and $\partial f/\partial y$. Determine whether f is holomorphic (and on what domain). If it is holomorphic, specify $f'(z)$.

(a) $$f(z)=\frac{z+1}{z^2+1},$$

(b) $$f(z)=\frac{\overline{z}+1}{z^2+1},$$

(c) $$f(z)=e^{1/z},$$

(d) $$f(z)=e^{-|z|^2}.$$

4. Find the antiderivative of each of the following functions.

(a) $$f(z)=\frac{1}{(z+3)^2},$$

(b) $$f(z)=ze^{z^2},$$

(c) $$f(z)=z^2+e^z,$$

5. Let $\gamma : [-1, 1] \to \mathbb{C}$ be given by

$$\gamma(t) = t + it^2.$$

Compute $\int_\gamma f(z)\,dz$ in the following cases.

(a) $$f(z) = z,$$
(b) $$f(z) = \overline{z},$$
(c) $$f(z) = \frac{1}{(z+5)^2},$$
(d) $$f(z) = e^z.$$

6. Do Exercise 5 with

$$\gamma(t) = t^4 + it^2.$$

7. Recall the definition (1.44) for $\int_\gamma f(z)\,dz$ when $f \in C(\Omega)$ and $\gamma : [a, b] \to \Omega$ is a C^1 curve. Suppose $s : [a, b] \to [\alpha, \beta]$ is C^1, with C^1 inverse, such that $s(a) = \alpha,\ s(b) = \beta$. Set $\sigma(s(t)) = \gamma(t)$. Show that

$$\int\limits_\gamma f(z)\,dz = \int\limits_\sigma f(\zeta)\,d\zeta,$$

so path integrals are invariant under change of parametrization.

In the following exercises, let

$$\Delta_h f(z) = \frac{1}{h}(f(z+h) - f(z)).$$

8. Show that $\Delta_h z^{-1} \to -z^{-2}$ uniformly on $\{z \in \mathbb{C} : |z| > \varepsilon\}$, for each $\varepsilon > 0$.
Hint. Use (1.10).

9. Let $\Omega \subset \mathbb{C}$ be open and assume $K \subset \Omega$ is compact. Assume $f, g \in C(\Omega)$ and

$$\Delta_h f(z) \to f'(z),\ \Delta_h g(z) \to g'(z), \quad \text{uniformly on } K.$$

Show that

$$\Delta_h(f(z)g(z)) \longrightarrow f'(z)g(z) + f(z)g'(z), \quad \text{uniformly on} \quad K.$$

Hint. Write

$$\Delta_h(fg)(z) = \Delta_h f(z) \cdot g(z+h) + f(z)\Delta_h g(z).$$

10. Show that, for each $\varepsilon > 0,\ A < \infty$,

$$\Delta_h z^{-n} \longrightarrow -nz^{-(n+1)}$$

uniformly on $\{z \in \mathbb{C} : \varepsilon < |z| \le A\}$.
Hint. Use induction.

11. Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

and assume this converges for $|z| < R$, so, by Propositions 0.2–0.3, f is continuous on D_R. Show that f is complex-differentiable at 0 and $f'(0) = a_1$.
Hint. Write

$$f(z) = a_0 + a_1 z + z^2 \sum_{k=0}^{\infty} a_{k+2} z^k,$$

and show that Propositions 0.2–0.3 apply to $g(z) = \sum_{k=0}^{\infty} a_{k+2} z^k$, for $|z| < R$.

12. Using (1.34)–(1.38), show that if $f : \Omega \to \mathbb{C}$ is a C^1 map, each of the following identities is equivalent to the condition that f be holomorphic on Ω:

$$\begin{aligned} Df &= \frac{\partial u}{\partial x} I + \frac{\partial v}{\partial x} J, \\ Df &= \frac{\partial v}{\partial y} I - \frac{\partial u}{\partial y} J. \end{aligned}$$

13. For another approach to showing that e^z is holomorphic and $(d/dz)e^z = e^z$, use (0.60) to write $e^z = e^x e^{iy}$, with $z = x + iy$. Then use (0.51) to show that

$$\frac{\partial}{\partial x} e^{x+iy} = e^x e^{iy}, \quad \text{and} \quad \frac{\partial}{\partial y} e^{x+iy} = i e^x e^{iy},$$

so the Cauchy-Riemann equation (1.28) holds, and Proposition 1.5 applies.

14. Let $\mathcal{O}, \Omega$ be open in $\mathbb{C}$. Assume $u : \mathcal{O} \to \Omega$ is C^1 (but not necessarily holomorphic), and $f : \Omega \to \mathbb{C}$ is holomorphic. Show that

$$\frac{\partial}{\partial x} f \circ u(z) = f'(u(z)) \frac{\partial u}{\partial x}(z), \quad \frac{\partial}{\partial y} f \circ u(z) = f'(u(z)) \frac{\partial u}{\partial y}(z).$$

2. Holomorphic functions defined by power series

A power series has the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n. \tag{2.1}$$

Recall from §0 that to such a series there is associated a radius of convergence $R \in [0, \infty]$, with the property that the series converges absolutely whenever $|z - z_0| < R$ (if $R > 0$), and diverges whenever $|z - z_0| > R$ (if $R < \infty$). We begin this section by identifying R as follows:

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}. \tag{2.2}$$

This is established in the following result, which reviews and complements Propositions 0.2–0.3.

Proposition 2.1. *The series (2.1) converges whenever $|z-z_0| < R$ and diverges whenever $|z - z_0| > R$, where R is given by (2.2). If $R > 0$, the series converges uniformly on $\{z : |z - z_0| \le R'\}$, for each $R' < R$. Thus, when $R > 0$, the series (2.1) defines a continuous function*

$$f : D_R(z_0) \longrightarrow \mathbb{C}, \tag{2.3}$$

where

$$D_R(z_0) = \{z \in \mathbb{C} : |z - z_0| < R\}. \tag{2.4}$$

Proof. If $R' < R$, then there exists $N \in \mathbb{Z}^+$ such that

$$n \ge N \Longrightarrow |a_n|^{1/n} < \frac{1}{R'} \Longrightarrow |a_n|(R')^n < 1.$$

Thus

$$|z - z_0| < R' < R \Longrightarrow |a_n(z - z_0)^n| \le \left|\frac{z - z_0}{R'}\right|^n, \tag{2.5}$$

for $n \ge N$, so (2.1) is dominated by a convergent geometrical series in $D_{R'}(z_0)$.

For the converse, we argue as follows. Suppose $R'' > R$, so infinitely many $|a_n|^{1/n} \ge 1/R''$, hence infinitely many $|a_n|(R'')^n \ge 1$. Then

$$|z - z_0| \ge R'' > R \Longrightarrow \text{ infinitely many } |a_n(z - z_0)^n| \ge \left|\frac{z - z_0}{R''}\right|^n \ge 1,$$

forcing divergence for $|z - z_0| > R$.

The assertions about uniform convergence and continuity follow as in Proposition 0.3.

The following result, which extends Proposition 0.4 from the real to the complex domain, is central to the study of holomorphic functions. A converse will be established in §5, as a consequence of the Cauchy integral formula.

Proposition 2.2. *If $R > 0$, the function defined by (2.1) is holomorphic on $D_R(z_0)$, with derivative given by*

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}. \tag{2.6}$$

Proof. Absolute convergence of (2.6) on $D_R(z_0)$ follows as in the proof of Proposition 0.4. Alternatively (cf. Exercise 3 below), we have

$$\lim_{n\to\infty} n^{1/n} = 1 \Longrightarrow \limsup_{n\to\infty} |n a_n|^{1/n} = \limsup_{n\to\infty} |a_n|^{1/n}, \tag{2.7}$$

so the power series on the right side of (2.6) converges locally uniformly on $D_R(z_0)$, defining a continuous function $g : D_R(z_0) \to \mathbb{C}$. It remains to show that $f'(z) = g(z)$.

To see this, consider

$$f_k(z) = \sum_{n=0}^{k} a_n (z - z_0)^n, \quad g_k(z) = \sum_{n=1}^{k} n a_n (z - z_0)^{n-1}. \tag{2.8}$$

We have $f_k \to f$ and $g_k \to g$ locally uniformly on $D_R(z_0)$. By (1.12) we have $f_k'(z) = g_k(z)$. Hence it follows from Proposition 1.8 that, for $z \in D_R(z_0)$,

$$f_k(z) = a_0 + \int_{\sigma_z} g_k(\zeta)\, d\zeta, \tag{2.9}$$

for any path $\sigma_z : [a, b] \to D_R(z_0)$ such that $\sigma_z(a) = z_0$ and $\sigma_z(b) = z$. Making use of the locally uniform convergence, we can pass to the limit in (2.9), to get

$$f(z) = a_0 + \int_{\sigma_z} g(\zeta)\, d\zeta. \tag{2.10}$$

Taking σ_z to approach z horizontally, we have (with $z = x + iy,\ z_0 = x_0 + iy_0$)

$$f(z) = a_0 + \int_{y_0}^{y} g(x_0 + it)\, i\, dt + \int_{x_0}^{x} g(t + iy)\, dt,$$

and hence

$$\frac{\partial f}{\partial x}(z) = g(z), \tag{2.11}$$

while taking σ_z to approach z vertically yields

$$f(z) = a_0 + \int_{x_0}^{x} g(t + iy_0)\, dt + \int_{y_0}^{y} g(x + it)\, i\, dt,$$

and hence

$$\frac{\partial f}{\partial y}(z) = ig(z). \tag{2.12}$$

Thus $f \in C^1(D_R(z_0))$ and it satisfies the Cauchy-Riemann equation, so f is holomorphic and $f'(z) = g(z)$, as asserted.

REMARK. A second proof of Proposition 2.2, not involving integral calculus, is given below, following the treatment of Proposition 2.6.

It is useful to note that we can multiply power series with radius of convergence $R > 0$. In fact, there is the following more general result on products of absolutely convergent series.

Proposition 2.3. *Given absolutely convergent series*

$$A = \sum_{n=0}^{\infty} \alpha_n, \quad B = \sum_{n=0}^{\infty} \beta_n, \tag{2.13}$$

we have the absolutely convergent series

$$AB = \sum_{n=0}^{\infty} \gamma_n, \quad \gamma_n = \sum_{j=0}^{n} \alpha_j \beta_{n-j}. \tag{2.14}$$

Proof. Take $A_k = \sum_{n=0}^{k} \alpha_n,\ B_k = \sum_{n=0}^{k} \beta_n$. Then

$$A_k B_k = \sum_{n=0}^{k} \gamma_n + R_k \tag{2.15}$$

with

$$R_k = \sum_{(m,n)\in\sigma(k)} \alpha_m \beta_n, \quad \sigma(k) = \{(m,n) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : m,n \le k, m+n > k\}. \tag{2.16}$$

Hence

$$\begin{aligned} |R_k| &\le \sum_{m \le k/2} \sum_{k/2 \le n \le k} |\alpha_m|\,|\beta_n| + \sum_{k/2 \le m \le k} \sum_{n \le k} |\alpha_m|\,|\beta_n| \\ &\le \overline{A} \sum_{n \ge k/2} |\beta_n| + \overline{B} \sum_{m \ge k/2} |\alpha_m|, \end{aligned} \tag{2.17}$$

where

$$\overline{A} = \sum_{n=0}^{\infty} |\alpha_n| < \infty, \quad \overline{B} = \sum_{n=0}^{\infty} |\beta_n| < \infty. \tag{2.18}$$

It follows that $R_k \to 0$ as $k \to \infty$. Thus the left side of (2.15) converges to AB and the right side to $\sum_{n=0}^{\infty} \gamma_n$. The absolute convergence of (2.14) follows by applying the same argument with α_n replaced by $|\alpha_n|$ and β_n replaced by $|\beta_n|$.

Corollary 2.4. *Suppose the following power series converge for $|z| < R$:*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n. \tag{2.19}$$

Then, for $|z| < R$,

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n z^n, \quad c_n = \sum_{j=0}^{n} a_j b_{n-j}. \tag{2.20}$$

The following result, which is related to Proposition 2.3, has a similar proof.

Proposition 2.5. *If $a_{jk} \in \mathbb{C}$ and $\sum_{j,k} |a_{jk}| < \infty$, then $\sum_j a_{jk}$ is absolutely convergent for each k, $\sum_k a_{jk}$ is absolutely convergent for each j, and*

$$\sum_j \Bigl(\sum_k a_{jk}\Bigr) = \sum_k \Bigl(\sum_j a_{jk}\Bigr) = \sum_{j,k} a_{jk}. \tag{2.21}$$

Proof. Clearly the hypothesis implies $\sum_j |a_{jk}| < \infty$ for each k and $\sum_k |a_{jk}| < \infty$ for each j. It also implies that there exists $B < \infty$ such that

$$S_N = \sum_{j=0}^{N} \sum_{k=0}^{N} |a_{jk}| \le B, \quad \forall\, N.$$

Now S_N is bounded and monotone, so there exists a limit, $S_N \nearrow A < \infty$ as $N \nearrow \infty$. It follows that, for each $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that

$$\sum_{(j,k)\in\mathcal{C}(N)} |a_{jk}| < \varepsilon, \quad \mathcal{C}(N) = \{(j,k) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : j > N \text{ or } k > N\}.$$

Now, whenever $M, K \ge N$,

$$\Bigl|\sum_{j=0}^{M} \Bigl(\sum_{k=0}^{K} a_{jk}\Bigr) - \sum_{j=0}^{N}\sum_{k=0}^{N} a_{jk}\Bigr| \le \sum_{(j,k)\in\mathcal{C}(N)} |a_{jk}|,$$

so

$$\Bigl|\sum_{j=0}^{M} \Bigl(\sum_{k=0}^{\infty} a_{jk}\Bigr) - \sum_{j=0}^{N}\sum_{k=0}^{N} a_{jk}\Bigr| \le \sum_{(j,k)\in\mathcal{C}(N)} |a_{jk}|,$$

and hence

$$\Bigl|\sum_{j=0}^{\infty} \Bigl(\sum_{k=0}^{\infty} a_{jk}\Bigr) - \sum_{j=0}^{N}\sum_{k=0}^{N} a_{jk}\Bigr| \le \sum_{(j,k)\in\mathcal{C}(N)} |a_{jk}|.$$

We have a similar result with the roles of j and k reversed, and clearly the two finite sums agree. It follows that

$$\Bigl|\sum_{j=0}^{\infty} \Bigl(\sum_{k=0}^{\infty} a_{jk}\Bigr) - \sum_{k=0}^{\infty} \Bigl(\sum_{j=0}^{\infty} a_{jk}\Bigr)\Bigr| < 2\varepsilon, \quad \forall\, \varepsilon > 0,$$

yielding (2.21).

Using Proposition 2.5, we demonstrate the following.

Proposition 2.6. *If (2.1) has a radius of convergence $R > 0$, and $z_1 \in D_R(z_0)$, then $f(z)$ has a convergent power series about z_1:*

$$f(z) = \sum_{k=0}^{\infty} b_k (z - z_1)^k, \quad \textit{for} \ \ |z - z_1| < R - |z_1 - z_0|. \tag{2.22}$$

The proof of Proposition 2.6 will not use Proposition 2.2, and we can use this result to obtain a second proof of Proposition 2.2. Shrawan Kumar showed the author this argument.

Proof of Proposition 2.6. There is no loss in generality in taking $z_0 = 0$, which we will do here, for notational simplicity. Setting $f_{z_1}(\zeta) = f(z_1 + \zeta)$, we have from (2.1)

$$\begin{aligned} f_{z_1}(\zeta) &= \sum_{n=0}^{\infty} a_n (\zeta + z_1)^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_n \binom{n}{k} \zeta^k z_1^{n-k}, \end{aligned} \tag{2.23}$$

the second identity by the binomial formula (cf. (2.34) below). Now,

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} |a_n| \binom{n}{k} |\zeta|^k |z_1|^{n-k} = \sum_{n=0}^{\infty} |a_n| (|\zeta| + |z_1|)^n < \infty, \tag{2.24}$$

provided $|\zeta| + |z_1| < R$, which is the hypothesis in (2.22) (with $z_0 = 0$). Hence Proposition 2.5 gives

$$f_{z_1}(\zeta) = \sum_{k=0}^{\infty} \Bigl(\sum_{n=k}^{\infty} a_n \binom{n}{k} z_1^{n-k} \Bigr) \zeta^k. \tag{2.25}$$

Hence (2.22) holds, with

$$b_k = \sum_{n=k}^{\infty} a_n \binom{n}{k} z_1^{n-k}. \tag{2.26}$$

This proves Proposition 2.6. Note in particular that

$$b_1 = \sum_{n=1}^{\infty} n a_n z_1^{n-1}. \tag{2.27}$$

Second proof of Proposition 2.2. The result (2.22) implies f is complex differentiable at each $z_1 \in D_R(z_0)$. In fact, the formula (2.22) yields

$$f(z_1 + h) = b_0 + b_1 h + \sum_{k=2}^{\infty} b_k h^k,$$

and hence

$$f'(z_1) = \lim_{h \to 0} \frac{f(z_1 + h) - f(z_1)}{h} = b_1,$$

and the computation (2.27) translates to (2.6), with $z = z_1$.

Remark. Propositions 2.2 and 2.6 are special cases of the following more general results, which will be established in §5.

Proposition 2.7. *Let $\Omega \subset \mathbb{C}$ be open, and assume $f_k : \Omega \to \mathbb{C}$ are holomorphic. If $f_k \to f$ locally uniformly on Ω, then f is holomorphic on Ω and*

$$f_k' \longrightarrow f' \quad \text{locally uniformly on} \ \ \Omega.$$

Proposition 2.8. *If $f : \Omega \to \mathbb{C}$ is holomorphic, $z_1 \in \Omega$, and $D_S(z_1) \subset \Omega$, then f is given by a convergent power series of the form (2.22) on $D_S(z_1)$.*

Both of these propositions follow from the Cauchy integral formula. See Proposition 5.10 and Theorem 5.5. These results consequently provide alternative proofs of Propositions 2.2 and 2.6.

Exercises

1. Determine the radius of convergence R for each of the following series. If $0 < R < \infty$, examine when convergence holds at points on $|z| = R$.

(a)
$$f_1(z) = \sum_{n=0}^{\infty} z^n$$

(b)
$$f_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$$

(c)
$$f_3(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

(d)
$$f_4(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

(e)
$$f_5(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^n}$$

(f)
$$f_6(z) = \sum_{n=0}^{\infty} \frac{z^{2^n}}{2^n}$$

2. Using Proposition 2.2, find the power series of the derivatives of each of the functions in Exercise 1. Show that
$$\begin{aligned} f_2'(z) &= \frac{1}{1-z}, \\ z f_3'(z) &= f_2(z), \\ f_4'(z) &= f_4(z). \end{aligned}$$
Also, evaluate
$$g(z) = \sum_{n=1}^{\infty} n z^n.$$

3. Show that if the power series (2.1) has radius of convergence $R > 0$, then f'', f''', ... are holomorphic on $D_R(z_0)$ and

$$f^{(n)}(z_0) = n!\, a_n. \tag{2.28}$$

Here we set $f^{(n)}(z) = f'(z)$ for $n = 1$, and inductively $f^{(n+1)}(z) = (d/dz) f^{(n)}(z)$.

4. Given $a > 0$, show that for $n \ge 1$

$$(1 + a)^n \ge 1 + na. \tag{2.29}$$

(Cf. Exercise 2 of §0.) Use (2.29) to show that

$$\limsup_{n\to\infty} n^{1/n} \le 1, \tag{2.30}$$

and hence

$$\lim_{n\to\infty} n^{1/n} = 1, \tag{2.31}$$

a result used in (2.7).
Hint. To get (2.30), deduce from (2.29) that $n^{1/n} \le (1 + a)/a^{1/n}$. Then show that, for each $a > 0$,

$$\lim_{n\to\infty} a^{1/n} = 1. \tag{2.32}$$

For another proof of (2.31), see Exercise 4 of §4.

5. The following is a version of the binomial formula. If $a \in \mathbb{C},\ n \in \mathbb{N}$,

$$(1+a)^n = \sum_{k=0}^{n} \binom{n}{k} a^k, \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{2.33}$$

Another version is

$$(z+w)^n = \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}. \tag{2.34}$$

Verify this identity and show that (2.33) implies (2.29) when $a > 0$.
Hint. To verify (2.34), expand

$$(z+w)^n = (z+w)\cdots(z+w) \tag{2.35}$$

as a sum of monomials and count the number of terms equal to $z^k w^{n-k}$. Use the fact that

$$\binom{n}{k} = \text{ number of combinations of } n \text{ objects, taken } k \text{ at a time.} \tag{2.36}$$

6. As a special case of Exercise 3, note that, given a polynomial

$$p(z) = a_n z^n + \cdots + a_1 z + a_0, \tag{2.37}$$

we have

$$p^{(k)}(0) = k!\, a_k, \quad 0 \le k \le n. \tag{2.38}$$

Apply this to

$$p_n(z) = (1+z)^n. \tag{2.39}$$

Compute $p_n^{(k)}(z)$, using (1.5), then compute $p^{(k)}(0)$, and use this to give another proof of (2.33), i.e.,

$$p_n(z) = \sum_{k=0}^{n} \binom{n}{k} z^k, \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{2.40}$$

3. Exponential and trigonometric functions: Euler's formula

Recall from §§0 and 1 that we define the exponential function by its power series:

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^j}{j!} + \cdots = \sum_{j=0}^{\infty} \frac{z^j}{j!}. \tag{3.1}$$

By the ratio test this converges for all z, to a continuous function on $\mathbb{C}$. Furthermore, the exponential function is holomorphic on $\mathbb{C}$. This function satisfies

$$\frac{d}{dz} e^z = e^z, \quad e^0 = 1. \tag{3.2}$$

One derivation of this was given in §1. Alternatively, (3.2) can be established by differentiating term by term the series (3.1) to get (by Proposition 2.2)

$$\begin{aligned}\frac{d}{dz} e^z &= \sum_{j=1}^{\infty} \frac{1}{(j-1)!} z^{j-1} = \sum_{k=0}^{\infty} \frac{1}{k!} z^k \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} z^k = e^z.\end{aligned} \tag{3.3}$$

The property (3.2) uniquely characterizes e^z. It implies

$$\frac{d^j}{dz^j} e^z = e^z, \quad j = 1, 2, 3, \dots. \tag{3.4}$$

By (2.21), any function $f(z)$ that is the sum of a convergent power series about $z = 0$ has the form

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} z^j, \tag{3.5}$$

which for a function satisfying (3.2) and (3.4) leads to (3.1). A simple extension of (3.2) is

$$\frac{d}{dz} e^{az} = a\, e^{az}. \tag{3.6}$$

Note how this also extends (0.51).

As shown in (0.60), the exponential function satisfies the fundamental identity

$$e^z e^w = e^{z+w}, \quad \forall\, z, w \in \mathbb{C}. \tag{3.7}$$

For an alternative proof, we can expand the left side of (3.7) into a double series:

$$e^z e^w = \sum_{j=0}^{\infty} \frac{z^j}{j!} \sum_{k=0}^{\infty} \frac{w^k}{k!} = \sum_{j,k=0}^{\infty} \frac{z^j w^k}{j!k!}. \tag{3.8}$$

We compare this with

$$e^{z+w} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!}, \tag{3.9}$$

using the binomial formula (cf. (2.34))

$$(z+w)^n = \sum_{j=0}^{n} \binom{n}{j} z^j w^{n-j}, \quad \binom{n}{j} = \frac{n!}{j!(n-j)!}. \tag{3.10}$$

Setting $k = n - j$, we have

$$e^{z+w} = \sum_{n=0}^{\infty} \sum_{j+k=n;j,k\ge 0} \frac{1}{n!} \frac{n!}{j!k!} z^j w^k = \sum_{j,k=0}^{\infty} \frac{z^j w^k}{j!k!}. \tag{3.11}$$

See (2.14) for the last identity. Comparing (3.8) and (3.11) again gives the identity (3.7).

We next record some properties of $\exp(t) = e^t$ for real t. The power series (3.1) clearly gives $e^t > 0$ for $t \ge 0$. Since $e^{-t} = 1/e^t$, we see that $e^t > 0$ for all $t \in \mathbb{R}$. Since $de^t/dt = e^t > 0$, the function is monotone increasing in t, and since $d^2e^t/dt^2 = e^t > 0$, this function is convex. Note that

$$e^t > 1 + t, \quad \text{for} \quad t > 0. \tag{3.12}$$

Hence

$$\lim_{t\to+\infty} e^t = +\infty. \tag{3.13}$$

Since $e^{-t} = 1/e^t$,

$$\lim_{t\to-\infty} e^t = 0. \tag{3.14}$$

As a consequence,

$$\exp : \mathbb{R} \longrightarrow (0, \infty) \tag{3.15}$$

is smooth and one-to-one and onto, with positive derivative, so the inverse function theorem of one-variable calculus applies. There is a smooth inverse

$$L : (0, \infty) \longrightarrow \mathbb{R}. \tag{3.16}$$

We call this inverse the natural logarithm:

$$\log x = L(x). \tag{3.17}$$

See Figures 3.1 and 3.2 for graphs of $x = e^t$ and $t = \log x$.

Applying d/dt to

$$L(e^t) = t \tag{3.18}$$

gives

$$L'(e^t)e^t = 1, \quad \text{hence} \;\; L'(e^t) = \frac{1}{e^t}, \tag{3.19}$$

i.e.,

$$\frac{d}{dx} \log x = \frac{1}{x}. \tag{3.20}$$

Since $\log 1 = 0$, we get

$$\log x = \int_1^x \frac{dy}{y}. \tag{3.21}$$

An immediate consequence of (3.7) (for $z, w \in \mathbb{R}$) is the identity

$$\log xy = \log x + \log y, \quad x, y \in (0, \infty), \tag{3.22}$$

which can also be deduced from (3.21).

We next show how to extend the logarithm into the complex domain, defining $\log z$ for $z \in \mathbb{C} \setminus \mathbb{R}^-$, where $\mathbb{R}^- = (-\infty, 0]$, using Proposition 1.10. In fact, the hypothesis (1.51) holds for $\Omega = \mathbb{C} \setminus \mathbb{R}^-$, with $a + ib = 1$, so each holomorphic function g on $\mathbb{C} \setminus \mathbb{R}^-$ has a holomorphic anti-derivative. In particular, $1/z$ has an anti-derivative, and this yields

$$\log : \mathbb{C} \setminus \mathbb{R}^- \to \mathbb{C}, \quad \frac{d}{dz} \log z = \frac{1}{z}, \quad \log 1 = 0. \tag{3.22A}$$

By Proposition 1.8, we have

$$\log z = \int_1^z \frac{d\zeta}{\zeta}, \tag{3.22B}$$

the integral taken along any path in $\mathbb{C} \setminus \mathbb{R}^-$ from 1 to z. Comparison with (3.21) shows that this function restricted to $(0, \infty)$ coincides with log as defined in (3.17). In §4 we display log in (3.22A) as the inverse of the exponential function $\exp(z) = e^z$ on the domain $\Sigma = \{x + iy : x \in \mathbb{R}, y \in (-\pi, \pi)\}$, making use of some results that will be derived next. (See also Exercises 13–15 at the end of this section.)

We move next to a study of e^z for purely imaginary z, i.e., of

$$\gamma(t) = e^{it}, \quad t \in \mathbb{R}. \tag{3.23}$$

This traces out a curve in the complex plane, and we want to understand which curve it is. Let us set

$$e^{it} = c(t) + is(t), \tag{3.24}$$

with $c(t)$ and $s(t)$ real valued. First we calculate $|e^{it}|^2 = c(t)^2 + s(t)^2$. For $x, y \in \mathbb{R}$,

$$z = x + iy \Longrightarrow \overline{z} = x - iy \Longrightarrow z\overline{z} = x^2 + y^2 = |z|^2. \tag{3.25}$$

It is elementary that

$$\begin{aligned} z, w \in \mathbb{C} \Longrightarrow \overline{zw} = \overline{z}\,\overline{w} \Longrightarrow \overline{z^n} = \overline{z}^n, \\ \text{and} \quad \overline{z+w} = \overline{z} + \overline{w}. \end{aligned} \tag{3.26}$$

Hence

$$\overline{e^z} = \sum_{k=0}^{\infty} \frac{\overline{z}^k}{k!} = e^{\overline{z}}. \tag{3.27}$$

In particular,

$$t \in \mathbb{R} \Longrightarrow |e^{it}|^2 = e^{it}e^{-it} = 1. \tag{3.28}$$

Hence $t \mapsto \gamma(t) = e^{it}$ has image in the unit circle centered at the origin in $\mathbb{C}$. Also

$$\gamma'(t) = ie^{it} \Longrightarrow |\gamma'(t)| \equiv 1, \tag{3.29}$$

so $\gamma(t)$ moves at unit speed on the unit circle. We have

$$\gamma(0) = 1, \quad \gamma'(0) = i. \tag{3.30}$$

Thus, for t between 0 and the circumference of the unit circle, the arc from $\gamma(0)$ to $\gamma(t)$ is an arc on the unit circle, pictured in Figure 3.3, of length

$$\ell(t) = \int_0^t |\gamma'(s)|\, ds = t. \tag{3.31}$$

Standard definitions from trigonometry say that the line segments from 0 to 1 and from 0 to $\gamma(t)$ meet at angle whose measurement in radians is equal to the length of the arc of the unit circle from 1 to $\gamma(t)$, i.e., to $\ell(t)$. The cosine of this angle is defined to be the

x-coordinate of $\gamma(t)$ and the sine of the angle is defined to be the y-coordinate of $\gamma(t)$. Hence the computation (3.31) gives

$$c(t) = \cos t, \quad s(t) = \sin t. \tag{3.32}$$

Thus (3.24) becomes

$$e^{it} = \cos t + i \sin t, \tag{3.33}$$

an identity known as Euler's formula. The identity

$$\frac{d}{dt} e^{it} = ie^{it}, \tag{3.34}$$

applied to (3.33), yields

$$\frac{d}{dt} \cos t = -\sin t, \quad \frac{d}{dt} \sin t = \cos t. \tag{3.35}$$

We can use (3.7) to derive formulas for sin and cos of the sum of two angles. Indeed, comparing

$$e^{i(s+t)} = \cos(s+t) + i \sin(s+t) \tag{3.36}$$

with

$$e^{is} e^{it} = (\cos s + i \sin s)(\cos t + i \sin t) \tag{3.37}$$

gives

$$\begin{aligned} \cos(s+t) &= (\cos s)(\cos t) - (\sin s)(\sin t), \\ \sin(s+t) &= (\sin s)(\cos t) + (\cos s)(\sin t). \end{aligned} \tag{3.38}$$

Derivations of the formulas (3.35) for the derivative of $\cos t$ and $\sin t$ given in first semester calculus courses typically make use of (3.38) and further limiting arguments, which we do not need with the approach used here.

A standard definition of the number π is half the length of the unit circle. Hence π is the smallest positive number such that $\gamma(2\pi) = 1$. We also have

$$\gamma(\pi) = -1, \quad \gamma\Big(\frac{\pi}{2}\Big) = i. \tag{3.39}$$

Furthermore, consideration of Fig. 3.4 shows that

$$\gamma\Big(\frac{\pi}{3}\Big) = \frac{1}{2} + \frac{\sqrt{3}}{2} i, \quad \gamma\Big(\frac{\pi}{6}\Big) = \frac{\sqrt{3}}{2} + \frac{1}{2} i. \tag{3.40}$$

We now show how to compute an accurate approximation to π.

This formula will arise by comparing two ways to compute the length of an arc of a circle. So consider the length of $\gamma(t)$ over $0 \le t \le \varphi$. By (3.29) we know it is equal to φ. Suppose $0 < \varphi < \pi/2$ and parametrize this segment of the circle by

$$\sigma(s) = (\sqrt{1-s^2}, s), \quad 0 \le s \le \tau = \sin \varphi. \tag{3.41}$$

Then we know the length is also given by

$$\ell = \int_0^\tau |\sigma'(s)|\, ds = \int_0^\tau \frac{ds}{\sqrt{1-s^2}}. \tag{3.42}$$

Comparing these two length calculations, we have

$$\int_0^\tau \frac{ds}{\sqrt{1-s^2}} = \varphi, \quad \sin \varphi = \tau, \tag{3.43}$$

when $0 < \varphi < \pi/2$. As another way to see this, note that the substitution $s = \sin \theta$ gives, by (3.35), $ds = \cos \theta \, d\theta$, while $\sqrt{1-s^2} = \cos \theta$ by (3.28), which implies

$$\cos^2 t + \sin^2 t = 1. \tag{3.44}$$

Thus

$$\int_0^\tau \frac{ds}{\sqrt{1-s^2}} = \int_0^\varphi d\theta = \varphi, \tag{3.45}$$

again verifying (3.43).

In particular, using $\sin(\pi/6) = 1/2$, from (3.40), we deduce that

$$\frac{\pi}{6} = \int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}. \tag{3.46}$$

One can produce a power series for $(1-y)^{-1/2}$ and substitute $y = x^2$. (For more on this, see the exercises at the end of §5.) Integrating the resulting series term by term, one obtains

$$\frac{\pi}{6} = \sum_{n=0}^\infty \frac{a_n}{2n+1} \Big(\frac{1}{2}\Big)^{2n+1}, \tag{3.47}$$

where the numbers a_n are defined inductively by

$$a_0 = 1, \quad a_{n+1} = \frac{2n+1}{2n+2} a_n. \tag{3.48}$$

Using a calculator, one can sum this series over $0 \le n \le 20$ and show that

$$\pi = 3.141592653589\cdots. \tag{3.49}$$

We leave the verification of (3.47)–(3.49) as an exercise, once one gets to Exercises 1–2 of §5.

Exercises

Here's another way to demonstrate the formula (3.35) for the derivatives of $\sin t$ and $\cos t$.

1. Suppose you *define* $\cos t$ and $\sin t$ so that $\gamma(t) = (\cos t, \sin t)$ is a unit-speed parametrization of the unit circle centered at the origin, satisfying $\gamma(0) = (1,0)$, $\gamma'(0) = (0,1)$, (as we did in (3.32)). Show directly (without using (3.35)) that

$$\gamma'(t) = (-\sin t, \cos t),$$

and hence deduce (3.35). (*Hint.* Differentiate $\gamma(t)\cdot\gamma(t) = 1$ to deduce that, for each t, $\gamma'(t) \perp \gamma(t)$. Meanwhile, $|\gamma(t)| = |\gamma'(t)| = 1$.)

2. It follows from (3.33) and its companion $e^{-it} = \cos t - i \sin t$ that

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i} \tag{3.50}$$

for $z = t \in \mathbb{R}$. We *define* $\cos z$ and $\sin z$ as holomorphic functions on $\mathbb{C}$ by these identities. Show that they yield the series expansions

$$\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}, \quad \sin z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}. \tag{3.51}$$

3. Extend the identities (3.35) and (3.38) to complex arguments. In particular, for $z \in \mathbb{C}$, we have

$$\begin{aligned} \cos(z + \tfrac{\pi}{2}) &= -\sin z, \quad \cos(z+\pi) = -\cos z, \quad \cos(z + 2\pi) = \cos z, \\ \sin(z + \tfrac{\pi}{2}) &= \cos z, \quad \sin(z+\pi) = -\sin z, \quad \sin(z+2\pi) = \sin z. \end{aligned} \tag{3.52}$$

4. We define

$$\cosh y = \frac{e^y + e^{-y}}{2}, \quad \sinh y = \frac{e^y - e^{-y}}{2}. \tag{3.53}$$

Show that $\cos iy = \cosh y$, $\sin iy = i \sinh y$, and hence

$$\begin{aligned} \cos(x+iy) &= \cos x \, \cosh y - i \, \sin x \, \sinh y, \\ \sin(x+iy) &= \sin x \, \cosh y + i \, \cos x \, \sinh y. \end{aligned} \tag{3.54}$$

5. Define $\tan z$ for $z \neq (k+1/2)\pi$ and $\cot z$ for $z \neq k\pi,\ k \in \mathbb{Z}$, by

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}. \tag{3.55}$$

Show that

$$\tan(z + \tfrac{\pi}{2}) = -\cot z, \quad \tan(z+\pi) = \tan z, \tag{3.56}$$

and

$$\frac{d}{dz}\tan z = \frac{1}{\cos^2 z} = 1 + \tan^2 z. \tag{3.57}$$

6. For each of the following functions $g(z)$, find a holomorphic function $f(z)$ such that $f'(z) = g(z)$.

a) $g(z) = z^k e^z, \quad k \in \mathbb{Z}^+$.
b) $g(z) = e^{az}\,\cos bz$.

7. Concerning the identities in (3.40), verify algebraically that

$$\Bigl(\frac{1}{2} + \frac{\sqrt{3}}{2} i\Bigr)^3 = -1.$$

Then use $e^{\pi i/6} = e^{\pi i/2} e^{-\pi i/3}$ to deduce the stated identity for $\gamma(\pi/6) = e^{\pi i/6}$.

8. Let γ be the unit circle centered at the origin in $\mathbb{C}$, going counterclockwise. Show that

$$\int_\gamma \frac{1}{z}\,dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,dt = 2\pi i,$$

as stated in (1.50).

9. One sets

$$\sec z = \frac{1}{\cos z},$$

so (3.57) yields $(d/dz)\tan z = \sec^2 z$. Where is $\sec z$ holomorphic? Show that

$$\frac{d}{dz}\sec z = \sec z\,\tan z.$$

10. Show that

$$1 + \tan^2 z = \sec^2 z.$$

11. Show that

$$\frac{d}{dz}(\sec z + \tan z) = \sec z\,(\sec z + \tan z),$$

and

$$\begin{aligned}\frac{d}{dz}(\sec z\ \tan z) &= \sec z\ \tan^2 z + \sec^3 z \\ &= 2\sec^3 z - \sec z.\end{aligned}$$

(*Hint.* Use Exercise 10 for the last identity.)

12. The identities (3.53) serve to define $\cosh y$ and $\sinh y$ for $y \in \mathbb{C}$, not merely for $y \in \mathbb{R}$. Show that

$$\frac{d}{dz}\cosh z = \sinh z, \quad \frac{d}{dz}\sinh z = \cosh z,$$

and

$$\cosh^2 z - \sinh^2 z = 1.$$

The next exercises present log, defined on $\mathbb{C} \setminus \mathbb{R}^-$ as in (3.22A), as the inverse function to the exponential function $\exp(z) = e^z$, by a string of reasoning different from what we will use in §4. In particular, here we avoid having to appeal to the inverse function theorem, Theorem 4.2. Set

$$\Sigma = \{x + iy : x \in \mathbb{R},\ y \in (-\pi, \pi)\}. \tag{3.58}$$

13. Using $e^{x+iy} = e^x e^{iy}$ and the properties of $\exp : \mathbb{R} \to (0, \infty)$ and of $\gamma(y) = e^{iy}$ established in (3.15) and (3.33), show that

$$\exp : \Sigma \longrightarrow \mathbb{C} \setminus \mathbb{R}^- \ \text{ is one-to-one and onto.} \tag{3.59}$$

14. Show that

$$\log(e^z) = z \quad \forall\, z \in \Sigma. \tag{3.60}$$

Hint. Apply the chain rule to compute $g'(z)$ for $g(z) = \log(e^z)$. Note that $g(0) = 0$.

15. Deduce from Exercises 13 and 14 that

$$\log : \mathbb{C} \setminus \mathbb{R}^- \longrightarrow \Sigma$$

is one-to-one and onto, and is the inverse to exp in (3.59).

4. Square roots, logs, and other inverse functions

We recall the Inverse Function Theorem for functions of real variables.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be open and let $f : \Omega \to \mathbb{R}^n$ be a C^1 map. Take $p \in \Omega$ and assume $Df(p)$ is an invertible linear transformation on $\mathbb{R}^n$. Then there exists a neighborhood $\mathcal{O}$ of p and a neighborhood U of $q = f(p)$ such that $f : \mathcal{O} \to U$ is one-to-one and onto, the inverse $g = f^{-1} : U \to \mathcal{O}$ is C^1, and, for $x \in \mathcal{O}$, $y = f(x)$,*

$$Dg(y) = Df(x)^{-1}. \tag{4.1}$$

A proof of this is given in Appendix B. This result has the following consequence, which is the Inverse Function Theorem for holomorphic functions.

Theorem 4.2. *Let $\Omega \subset \mathbb{C}$ be open and let $f : \Omega \to \mathbb{C}$ be holomorphic. Take $p \in \Omega$ and assume $f'(p) \neq 0$. Then there exists a neighborhood $\mathcal{O}$ of p and a neighborhood U of $q = f(p)$ such that $f : \mathcal{O} \to U$ is one-to-one and onto, the inverse $g = f^{-1} : U \to \mathcal{O}$ is holomorphic, and, for $z \in \mathcal{O}$, $w = f(z)$,*

$$g'(w) = \frac{1}{f'(z)}. \tag{4.2}$$

Proof. If we check that g is holomorphic, then (4.2) follows from the chain rule, Proposition 1.2, applied to

$$g(f(z)) = z.$$

We know g is C^1. By Proposition 1.6, g is holomorphic on U if and only if, for each $w \in U$, $Dg(w)$ commutes with J, given by (1.39). Also Proposition 1.6 implies $Df(z)$ commutes with J. To finish, we need merely remark that if A is an invertible 2×2 matrix,

$$AJ = JA \Longleftrightarrow A^{-1}J = JA^{-1}. \tag{4.3}$$

As a first example, consider the function $\mathrm{Sq}(z) = z^2$. Note that we can use polar coordinates, $(x, y) = (r \cos\theta, r \sin\theta)$, or equivalently $z = re^{i\theta}$, obtaining $z^2 = r^2 e^{2i\theta}$. This shows that Sq maps the right half-plane

$$H = \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\} \tag{4.4}$$

bijectively onto $\mathbb{C} \setminus \mathbb{R}^-$ (where $\mathbb{R}^- = (-\infty, 0]$). Since $\mathrm{Sq}'(z) = 2z$ vanishes only at $z = 0$, we see that we have a holomorphic inverse

$$\mathrm{Sqrt} : \mathbb{C} \setminus \mathbb{R}^- \longrightarrow H, \tag{4.5}$$

given by

$$\mathrm{Sqrt}(re^{i\theta}) = r^{1/2}e^{i\theta/2}, \quad r > 0, \ -\pi < \theta < \pi. \tag{4.6}$$

We also write

$$\sqrt{z} = z^{1/2} = \mathrm{Sqrt}(z). \tag{4.7}$$

We will occasionally find it useful to extend Sqrt to

$$\mathrm{Sqrt} : \mathbb{C} \longrightarrow \mathbb{C},$$

defining Sqrt on $(-\infty, 0]$ by $\mathrm{Sqrt}(-x) = i\sqrt{x}$, for $x \in [0, \infty)$. This extended map is discontinuous on $(-\infty, 0)$.

We can define other non-integral powers of z on $\mathbb{C} \setminus \mathbb{R}^-$. Before doing so, we take a look at log, the inverse function to the exponential function, $\exp(z) = e^z$. Consider the strip

$$\Sigma = \{x + iy : x \in \mathbb{R}, \ -\pi < y < \pi\}. \tag{4.8}$$

Since $e^{x+iy} = e^x e^{iy}$, we see that we have a bijective map

$$\exp : \Sigma \longrightarrow \mathbb{C} \setminus \mathbb{R}^-. \tag{4.9}$$

Note that $de^z/dz = e^z$ is nowhere vanishing, so (4.9) has a holomorphic inverse we denote log:

$$\log : \mathbb{C} \setminus \mathbb{R}^- \longrightarrow \Sigma. \tag{4.10}$$

Note that

$$\log 1 = 0. \tag{4.11}$$

Applying (4.2) we have

$$\frac{d}{dz}e^z = e^z \Longrightarrow \frac{d}{dz} \log z = \frac{1}{z}. \tag{4.12}$$

Thus, applying Proposition 1.8, we have

$$\log z = \int_1^z \frac{1}{\zeta} \, d\zeta, \tag{4.13}$$

where the integral is taken along any path from 1 to z in $\mathbb{C} \setminus \mathbb{R}^-$. Comparison with (3.22B) shows that the function log produced here coincides with the function arising in (3.22A). (This result also follows from Exercises 13–15 of §3.)

Now, given any $a \in \mathbb{C}$, we can define

$$z^a = \mathrm{Pow}_a(z), \quad \mathrm{Pow}_a : \mathbb{C} \setminus \mathbb{R}^- \to \mathbb{C} \tag{4.14}$$

by

$$z^a = e^{a \log z}. \tag{4.15}$$

The identity $e^{u+v} = e^u e^v$ then gives

$$z^{a+b} = z^a z^b, \quad a, b \in \mathbb{C}, \ z \in \mathbb{C} \setminus \mathbb{R}^-. \tag{4.16}$$

In particular, for $n \in \mathbb{Z}, n \neq 0$,

$$(z^{1/n})^n = z. \tag{4.17}$$

Making use of (4.12) and the chain rule (1.20), we see that

$$\frac{d}{dz} z^a = a\, z^{a-1}. \tag{4.18}$$

While Theorem 4.1 and Corollary 4.2 are local in nature, the following result can provide global inverses, in some important cases.

Proposition 4.3. *Suppose $\Omega \subset \mathbb{C}$ is convex. Assume f is holomorphic in Ω and there exists $a \in \mathbb{C}$ such that*

$$\mathrm{Re}\, af' > 0 \quad \text{on} \quad \Omega.$$

Then f maps Ω one-to-one onto its image $f(\Omega)$.

Proof. Consider distinct points $z_0, z_1 \in \Omega$. The convexity implies the line $\sigma(t) = (1-t)z_0 + tz_1$ is contained in Ω, for $0 \le t \le 1$. By Proposition 1.8, we have

$$a \frac{f(z_1) - f(z_0)}{z_1 - z_0} = \int_0^1 af'\big((1-t)z_0 + tz_1\big)\, dt, \tag{4.19}$$

which has positive real part and hence is not zero.

As an example, consider the strip

$$\widetilde{\Sigma} = \Big\{x + iy : -\frac{\pi}{2} < x < \frac{\pi}{2},\ y \in \mathbb{R}\Big\}. \tag{4.20}$$

Take $f(z) = \sin z$, so $f'(z) = \cos z$. It follows from (3.54) that

$$\mathrm{Re}\, \cos z = \cos x \, \cosh y > 0, \quad \text{for} \quad z \in \widetilde{\Sigma}, \tag{4.21}$$

so f maps $\widetilde{\Sigma}$ one-to-one onto its image. Note that

$$\sin z = g(e^{iz}), \quad \text{where} \quad g(\zeta) = \frac{1}{2i}\Big(\zeta - \frac{1}{\zeta}\Big), \tag{4.22}$$

and the image of $\widetilde{\Sigma}$ under $z \mapsto e^{iz}$ is the right half plane H, given by (4.4). Below we will show that the image of H under g is

$$\mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}. \tag{4.23}$$

It then follows that sin maps $\widetilde{\Sigma}$ one-to-one onto the set (4.23). The inverse function is denoted $\sin^{-1}$:

$$\sin^{-1} : \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\} \longrightarrow \widetilde{\Sigma}. \tag{4.24}$$

We have $\sin^2 z \in \mathbb{C} \setminus [1, \infty)$ for $z \in \widetilde{\Sigma}$, and it follows that

$$\cos z = (1 - \sin^2 z)^{1/2}, \quad z \in \widetilde{\Sigma}. \tag{4.25}$$

Hence, by (4.2), $g(z) = \sin^{-1} z$ satisfies

$$g'(z) = (1 - z^2)^{-1/2}, \quad z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}, \tag{4.26}$$

and hence, by Proposition 1.8,

$$\sin^{-1} z = \int_0^z (1 - \zeta^2)^{-1/2} \, d\zeta, \tag{4.27}$$

where the integral is taken along any path from 0 to z in $\mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}$. Compare this identity with (3.43), which treats the special case of real $z \in (-1, 1)$.

It remains to prove the asserted mapping property of g, given in (4.22). We rephrase the result for

$$h(\zeta) = g(i\zeta) = \frac{1}{2}\Big(\zeta + \frac{1}{\zeta}\Big). \tag{4.28}$$

Proposition 4.4. *The function h given by (4.28) maps both the upper half plane $U = \{\zeta : \operatorname{Im} \zeta > 0\}$ and the lower half plane $U^* = \{\zeta : \operatorname{Im} \zeta < 0\}$, one-to-one onto*

$$\mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}.$$

Proof. Note that $h : \mathbb{C} \setminus 0 \to \mathbb{C}$, and

$$h\Big(\frac{1}{\zeta}\Big) = h(\zeta). \tag{4.29}$$

Taking $w \in \mathbb{C}$, we want to solve $h(\zeta) = w$ for ζ. This is equivalent to

$$\zeta^2 - 2w\zeta + 1 = 0, \tag{4.30}$$

with solutions

$$\zeta = w \pm \sqrt{w^2 - 1}. \tag{4.31}$$

Thus, for each $w \in \mathbb{C}$, there are two solutions, except for $w = \pm 1$, with single solutions $h(-1) = -1$, $h(1) = 1$. If we examine $h(x)$ for $x \in \mathbb{R} \setminus 0$ (see Fig. 4.1), we see that h maps $\mathbb{R} \setminus 0$ onto $(-\infty, -1] \cup [1, \infty)$, 2-to-1 except at $x = \pm 1$. It follows that, given $\zeta \in \mathbb{C} \setminus 0$, $h(\zeta) = w$ belongs to $(-\infty, -1] \cup [1, \infty)$ if and only if $\zeta \in \mathbb{R}$. If w belongs to the set (4.23), i.e., if $w \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}$, then $h(\zeta) = w$ has two solutions, both in $\mathbb{C} \setminus \mathbb{R}$, and by (4.29) they are reciprocals of each other. Now

$$\frac{1}{\zeta} = \frac{\overline{\zeta}}{|\zeta|^2}, \tag{4.32}$$

so, given $\zeta \in \mathbb{C} \setminus \mathbb{R}$, we have $\zeta \in U \Leftrightarrow 1/\zeta \in U^*$. This proves Proposition 4.4.

Exercises

1. Show that, for $|z| < 1$,

$$\log(1 + z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}. \tag{4.33}$$

Hint. Use (4.13) to write

$$\log(1 + z) = \int_0^z \frac{1}{1 + \zeta} \, d\zeta,$$

and plug in the power series for $1/(1 + \zeta)$.

2. Using Exercise 1 (plus further arguments), show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \log 2. \tag{4.34}$$

Hint. Using properties of alternating series, show that, for $r \in (0, 1)$,

$$\sum_{n=1}^{N} \frac{(-1)^{n-1}}{n} r^n = \log(1 + r) + \varepsilon_N(r), \quad |\varepsilon_N(r)| \le \frac{r^{N+1}}{N + 1}. \tag{4.34A}$$

Then let $r \to 1$ in (4.34A).

3. Take $x \in (0, \infty)$. Show that
$$\lim_{x\to\infty} \frac{\log x}{x} = 0.$$
Hint. If $x = e^y$, $(\log x)/x = ye^{-y}$.

4. Using Exercise 3, show that
$$\lim_{x\to+\infty} x^{1/x} = 1.$$
Note that this contains the result (2.31).
Hint. $x^{1/x} = e^{(\log x)/x}$.

5. Write the Euler identity as
$$e^{iw} = \sqrt{1-z^2} + iz, \quad z = \sin w,$$
for w near 0. Deduce that
$$\sin^{-1} z = \frac{1}{i} \log\bigl(\sqrt{1-z^2} + iz\bigr), \quad |z| < 1. \tag{4.35}$$
Does this extend to $\mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}$?

6. Compute the following quantities:
 a) $i^{1/2}$,
 b) $i^{1/3}$,
 c) i^i.

7. Show that $\tan z$ maps the strip $\widetilde{\Sigma}$ given by (4.20) diffeomorphically onto
$$\mathbb{C} \setminus \{(-\infty, -1]i \cup [1, \infty)i\}. \tag{4.36}$$
Hint. Consult Fig. 4.2. To get the last step, it helps to show that
$$R(z) = \frac{1-z}{1+z}$$
has the following properties:
(a) $R : \mathbb{C} \setminus \{-1\} \to \mathbb{C} \setminus \{-1\}$, and $R(z) = w \Leftrightarrow z = R(w)$,
(b) $R : \mathbb{R} \setminus \{-1\} \to \mathbb{R} \setminus \{-1\}$,
(c) $R : (0, \infty) \to (-1, 1)$,
and in each case the map is one-to-one and onto.

8. Making use of (3.57), show that on the region (4.36) we have

$$\tan^{-1} z = \int_0^z \frac{d\zeta}{1+\zeta^2}, \tag{4.37}$$

where $\tan^{-1}$ is the inverse of tan in Exercise 7, and the integral is over any path from 0 to z within the region (4.36). Writing $(1+\zeta^2)^{-1}$ as a geometric series and integrating term by term, show that

$$\tan^{-1} z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{2k+1}. \quad \text{for} \quad |z| < 1. \tag{4.37A}$$

Use the fact that

$$\tan \frac{\pi}{6} = \frac{\sqrt{3}}{4} \tag{4.37B}$$

to obtain

$$\frac{\pi}{6} = \frac{\sqrt{3}}{4} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \Bigl(\frac{3}{16}\Bigr)^k. \tag{4.37C}$$

Compare the series (3.47). See Exercise 29 below for an approach to evaluating $\sqrt{3}$. How many terms in (4.37C) do you need to obtain 16 digits of accuracy?

9. Show that

$$\frac{d}{dz} \log \sec z = \tan z.$$

On what domain in $\mathbb{C}$ does this hold?

10. Using Exercise 11 of §3, compute

$$\frac{d}{dz} \log(\sec z + \tan z),$$

and find the antiderivatives of

$$\sec z \quad \text{and} \quad \sec^3 z.$$

On what domains do the resulting formulas work?

11. Consider the integral

$$I(u) = \int_0^u \frac{dt}{\sqrt{1+t^2}}, \tag{4.38}$$

with $u \in \mathbb{R}$.

(a) Use the change of variable $t = \sinh v$ to show that

$$I(\sinh v) = v.$$

(b) Use the change of variable $t = \tan x$ to show that

$$I(\tan x) = \int_0^x \sec s\, ds, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

(c) Deduce from (a) and (b) that

$$\int_0^x \sec s\, ds = \sinh^{-1}(\tan x), \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

(d) Use the last identity to show that

$$\cosh\Bigl(\int_0^x \sec s\, ds\Bigr) = \sec x.$$

Compare the last results with conclusions of Exercise 10.

12. As a variant of (4.20)–(4.23), show that $z \mapsto \sin z$ maps

$$\Bigl\{x + iy : -\frac{\pi}{2} < x < \frac{\pi}{2},\ y > 0\Bigr\}$$

one-to-one and onto the upper half plane $\{z : \operatorname{Im} z > 0\}$.

The next exercises construct the holomorphic inverse to sin on the set

$$\Omega = \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}, \tag{4.39}$$

satisfying (4.27), in a manner that avoids appeal to Theorem 4.2. (Compare the construction of log as an inverse to exp in Exercises 13–15 of §3.) We will use the result that

$$\sin : \widetilde{\Sigma} \longrightarrow \Omega \ \text{ is one-to-one and onto,} \tag{4.40}$$

with $\widetilde{\Sigma}$ as in (4.20), established above via Proposition 4.4.

13. Show that, for Ω as in (4.39),

$$z \in \Omega \Rightarrow z^2 \in \mathbb{C} \setminus [1, \infty) \Rightarrow 1 - z^2 \in \mathbb{C} \setminus (-\infty, 0],$$

and deduce that the function $f(z) = (1-z^2)^{-1/2}$ is holomorphic on Ω.

14. Show that the set Ω in (4.39) satisfies the hypotheses of Proposition 1.10, with $a+ib = 0$. Deduce that the function $f(z) = (1-z^2)^{-1/2}$ has a holomorphic anti-derivative G on Ω:

$$G : \Omega \to \mathbb{C}, \quad G'(z) = (1-z^2)^{-1/2}, \quad G(0) = 0. \tag{4.41}$$

Deduce from Proposition 1.8 that

$$G(z) = \int_0^z (1-\zeta^2)^{-1/2}\, d\zeta, \quad z \in \Omega, \tag{4.42}$$

the integral taken along any path in Ω from 0 to z.

15. Show that

$$G(\sin z) = z, \quad \forall\, z \in \widetilde{\Sigma}. \tag{4.43}$$

Hint. Apply the chain rule to $f(z) = G(\sin z)$, making use of (4.25), to show that $f'(z) = 1$ for all $z \in \widetilde{\Sigma}$.

16. Use (4.40) and Exercise 15 to show that

$$G : \Omega \longrightarrow \widetilde{\Sigma} \ \text{ is one-to-one and onto,} \tag{4.44}$$

and is the holomorphic inverse to sin in (4.40).

17. Expanding on Proposition 4.4, show that the function h, given by (4.28), has the following properties:
(a) $h : \mathbb{C} \setminus 0 \to \mathbb{C}$ is onto,
(b) $h : \mathbb{C} \setminus \{0, 1, -1\} \to \mathbb{C} \setminus \{1, -1\}$ is two-to-one and onto,
(c) $h : \mathbb{R} \setminus 0 \to \mathbb{R} \setminus (-1, 1)$ is onto, and is two-to-one, except at $x = \pm 1$.

18. Given $a \in \mathbb{C} \setminus \mathbb{R}^-$, set

$$E_a(z) = a^z = e^{z \log a}, \quad z \in \mathbb{C}.$$

Show that E_a is holomorphic in z and compute $E_a'(z)$.

19. Deduce from Exercise 1 that

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{n} = \lambda(z), \quad \text{for } |z| < 1,$$

where

$$\lambda(z) = -\frac{\log(1-z)}{z}, \quad \text{for } 0 < |z| < 1,$$
$$1 \qquad \text{for } \ z = 0.$$

Use this to show that

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2} = \int_0^z \frac{1}{\zeta} \log \frac{1}{1-\zeta}\, d\zeta,$$

for $|z| < 1$, where the integral is taken along any path from 0 to z in the unit disk $D_1(0)$.

20. Show that

$$\lim_{k\to\infty} \sum_{\ell=-k}^{k} \frac{1}{k+i\ell} = \int_{-1}^{1} \frac{dz}{1+iz} = \frac{\pi}{2}.$$

21. Show that, for $x > 0$,

$$\log \frac{x+i}{x-i} = 2i \tan^{-1} \frac{1}{x}$$

22. Show that, for $x > 0$,

$$\int_0^x \log t\, dt = x \log x - x.$$

23. Show that, for $x > 0,\ a \in \mathbb{C} \setminus (-\infty, 0]$,

$$\int_0^x \log(t+a)\, dt = (x+a)\log(x+a) - x - a\log a.$$

Show that

$$\int_0^x \log(t^2+1)\, dt = x\log(x^2+1) + i\log\frac{x+i}{x-i} - 2x + \pi.$$

Compute

$$\int_0^x \log\Bigl(1 + \frac{1}{t^2}\Bigr)\, dt,$$

and show that

$$\int_0^\infty \log\Bigl(1 + \frac{1}{t^2}\Bigr)\, dt = \pi.$$

24. Take the following path to explicitly finding the real and imaginary parts of a solution to

$$z^2 = a + ib,$$

given $a + ib \notin \mathbb{R}^-$. Namely, with $x = \operatorname{Re} z,\ y = \operatorname{Im} z$, we have

$$x^2 - y^2 = a, \quad 2xy = b,$$

and also

$$x^2 + y^2 = \rho = \sqrt{a^2 + b^2},$$

hence

$$x = \sqrt{\frac{\rho + a}{2}}, \quad y = \frac{b}{2x}.$$

Exercises 25–27 deal with a method of solving

$$z^3 = \alpha = 1 + \beta, \tag{4.45}$$

given a restriction on the size of $\beta \in \mathbb{C}$.

25. As a first approximation to a solution to (4.45), take $\zeta_0 = 1 + \beta/3$, so

$$\begin{aligned}\zeta_0^3 &= 1 + \beta + \frac{\beta^2}{3} + \frac{\beta^3}{27} \\ &= \alpha + \frac{\beta^2}{3}\Bigl(1 + \frac{\beta}{9}\Bigr).\end{aligned}$$

Now set

$$z = \Bigl(1 + \frac{\beta}{3}\Bigr) z_1^{-1},$$

and show that solving (4.45) is equivalent to solving

$$\begin{aligned}z_1^3 &= 1 + \frac{1}{\alpha}\Bigl(\frac{\beta^2}{3} + \frac{\beta^3}{27}\Bigr) \\ &= 1 + \beta_1 \\ &= \alpha_1,\end{aligned} \tag{4.46}$$

where the last 2 identities define β_1 and α_1.

26. Apply the procedure of Exercise 25 to the task of solving $z_1^3 = \alpha_1 = 1 + \beta_1$. Set

$$z_1 = \Bigl(1 + \frac{\beta_1}{3}\Bigr) z_2^{-1},$$

so solving (4.46) is equivalent to solving

$$\begin{aligned}z_2^3 &= 1 + \frac{1}{\alpha_1}\Bigl(\frac{\beta_1^2}{3} + \frac{\beta_1^3}{27}\Bigr) \\ &= 1 + \beta_2 \\ &= \alpha_2.\end{aligned}$$

Continue this process, obtaining that (4.45) is equivalent to

$$z = \Big(1 + \frac{\beta}{3}\Big)\Big(1 + \frac{\beta_1}{3}\Big)^{-1} \cdots \Big(1 + \frac{\beta_k}{3}\Big)^{\sigma(k)} z_{k+1}^{-1}, \tag{4.47}$$

with $\beta(k) = (-1)^k$ and z_{k+1} solving

$$\begin{aligned} z_{k+1}^3 &= 1 + \frac{1}{\alpha_k}\,\frac{\beta_k^2}{3}\Big(1 + \frac{\beta_k}{9}\Big) \\ &= 1 + \beta_{k+1} \\ &= \alpha_{k+1}. \end{aligned}$$

27. With β_k as in (4.48), show that

$$\begin{aligned} |\beta_k| \le \frac{1}{2} \Rightarrow |\alpha_k| &\ge \frac{1}{2} \\ \Rightarrow |\beta_{k+1} &\le \frac{19}{27}|\beta_k|^2, \end{aligned}$$

and

$$\begin{aligned} |\beta_k| \le \frac{1}{4} \Rightarrow |\alpha_k| &\ge \frac{3}{4} \\ \Rightarrow |\beta_{k+1}| &\le \frac{1}{2}|\beta_k|^2. \end{aligned}$$

Use this to establish the following.

Assertion. Assume

$$|\beta| \le \frac{1}{2}, \quad \alpha = 1 + \beta,$$

and form the sequence $\beta_1, \dots, \beta_k, \dots$, as in Exercises 25–26, i.e.,

$$\begin{aligned} \beta_1 &= \frac{1}{\alpha}\,\frac{\beta^2}{3}\Big(1 + \frac{\beta}{9}\Big), \quad \alpha_1 = 1 + \beta_1, \\ \beta_{k+1} &= \frac{1}{\alpha_k}\,\frac{\beta_k^2}{3}\Big(1 + \frac{\beta_k}{9}\Big), \quad \alpha_{k+1} = 1 + \beta_{k+1}. \end{aligned}$$

Then the sequence

$$\zeta_k = \Big(1 + \frac{\beta}{3}\Big)\Big(1 + \frac{\beta_1}{3}\Big)^{-1} \cdots \Big(1 + \frac{\beta_k}{3}\Big)^{\sigma(k)}$$

converges as $k \to \infty$ to $\alpha^{1/3}$.

Note that an equivalent description of ζ_k is

$$\begin{aligned} \zeta_0 &= 1 + \frac{\beta}{3}, \\ \zeta_k &= \zeta_{k-1}\Big(1 + \frac{\beta_k}{3}\Big)^{\sigma(k)}. \end{aligned}$$

28. Take $k \in \mathbb{N},\ k \geq 2$. Extend the method of Exercises 25–27 to approximate a solution to

$$z^k = \alpha = 1 + \beta,$$

given a restriction on the size of β.

Exercises 29–33 are recommended for the reader able to use a computer or calculator for numerical work.

29. Here is an alternative method of approximating $\sqrt{\alpha}$, given $\alpha \in \mathbb{C} \setminus (-\infty, 0]$. Suppose you have an approximation ζ_k,

$$\zeta_k - \sqrt{\alpha} = \delta_k.$$

Square this to obtain $\zeta_k^2 + \alpha - 2\zeta_k\sqrt{\alpha} = \delta_k^2$, hence

$$\sqrt{\alpha} = \frac{\alpha + \zeta_k^2}{2\zeta_k} - \frac{\delta_k^2}{2\zeta_k}.$$

Hence

$$\zeta_{k+1} = \frac{\alpha + \zeta_k^2}{2\zeta_k}$$

is an improved approximation, as long as $|\delta_k| < 2|\zeta_k|$. One can iterate this. Try it on

$$\sqrt{2} \approx \frac{7}{5}, \quad \sqrt{3} \approx \frac{7}{4}, \quad \sqrt{5} \approx \frac{9}{4}.$$

How many iterations does it take to approximate these quantities to 16 digits of accuracy?

30. Given that

$$\begin{aligned} e^{\pi i/6} &= \frac{\sqrt{3}}{2} + \frac{i}{2}, \\ e^{\pi i/4} &= \frac{1+i}{\sqrt{2}}, \qquad \text{and} \\ e^{\pi i/12} &= e^{\pi i/4} e^{-\pi i/6}, \end{aligned}$$

use the method of Exercise 27 to obtain an accurate numerical evaluation of

$$e^{\pi i/36} = \cos\frac{\pi}{36} + i \sin\frac{\pi}{36}.$$

How many iterations does it take to approximate this quantity to 16 digits of accuracy?

31. Peek at (Q.32) to see that

$$\cos\frac{2\pi}{5} = \frac{\sqrt{5} - 1}{4}.$$

Use this to obtain an accurate numerical evaluation of

$$e^{2\pi i/5} = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5},$$

and from this of

$$\begin{aligned} e^{\pi i/5} &= -e^{-4\pi i/5}, \quad \text{and} \\ e^{\pi i/30} &= e^{\pi i/5}e^{-\pi i/6}. \end{aligned}$$

Then use the method of Exercise 27 to obtain an accurate numerical evaluation of

$$e^{\pi i/90} = \cos\frac{\pi}{90} + i\sin\frac{\pi}{90}.$$

32. Apply Exercise 28 or 29 to $\alpha = e^{\pi i/90}$, to give an accurate numerical evaluation of

$$\begin{aligned} e^{\pi i/180} &= \cos\frac{\pi}{180} + i\sin\frac{\pi}{180} \\ &= \cos 1^\circ + i\sin 1^\circ. \end{aligned}$$

If you use Exercise 29, take $\zeta_1 = (1+\alpha)/2$. Alternatively, apply Exercise 28, with $k = 6$, to $\alpha = e^{\pi i/30}$.
Using these calculations, evaluate

$$\tan\frac{\pi}{180} = \delta. \tag{4.49}$$

33. Substitute δ in (4.49) into

$$\tan^{-1} x = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}, \quad |x| < 1, \tag{4.50}$$

from (4.37A), to obtain

$$\begin{aligned} \frac{\pi}{180} &= \sum_{k=0}^{\infty}(-1)^k\frac{\delta^{2k+1}}{2k+1} \\ &= \delta - \frac{\delta^3}{3} + \frac{\delta^5}{5} - \cdots. \end{aligned} \tag{4.51}$$

Show that

$$\delta - \frac{\delta^3}{3} < \frac{\pi}{180} < \delta.$$

How many terms in the series (4.51) are needed to obtain π to 16 digits of accuracy, assuming the evaluation of δ is sufficiently accurate?

Similarly, apply (4.50) to the following quantities, whose evaluations are available from the results of Exercises 30–31.

$$\begin{aligned} \tan\frac{\pi}{12} &= \delta_1 = 2 - \sqrt{3}, \\ \tan\frac{\pi}{30} &= \delta_2, \\ \tan\frac{\pi}{36} &= \delta_3. \end{aligned}$$

Compare the approach to evaluating π described in Exercise 8.

I. π^2 is Irrational

The following proof that π^2 is irrational follows a classic argument of I. Niven, [Niv]. The idea is to consider

$$I_n = \int_0^\pi \varphi_n(x) \sin x \, dx, \quad \varphi_n(x) = \frac{1}{n!} x^n (\pi - x)^n. \tag{I.1}$$

Clearly $I_n > 0$ for each $n \in \mathbb{N}$, and $I_n \to 0$ very fast, faster than geometrically. The next key fact is that I_n is a polynomial of degree n in π^2 with integer coefficients:

$$I_n = \sum_{k=0}^{n} c_{nk} \pi^{2k}, \quad c_{nk} \in \mathbb{Z}. \tag{I.2}$$

Given this it follows readily that π^2 is irrational. In fact, if $\pi^2 = a/b, \ a, b \in \mathbb{N}$, then

$$\sum_{k=0}^{n} c_{nk} a^{2k} b^{2n-2k} = b^{2n} I_n. \tag{I.3}$$

But the left side of (I.3) is an integer for each n, while by the estimate on (I.1) mentioned above the right side belongs to the interval $(0, 1)$ for large n, yielding a contradiction. It remains to establish (I.2).

A method of computing the integral in (I.1), which works for any polynomial $\varphi_n(x)$) is the following. One looks for an antiderivative of the form

$$G_n(x) \sin x - F_n(x) \cos x, \tag{I.4}$$

where F_n and G_n are polynomials. One needs

$$G_n(x) = F_n'(x), \quad G_n'(x) + F_n(x) = \varphi_n(x), \tag{I.5}$$

hence

$$F_n''(x) + F_n(x) = \varphi_n(x). \tag{I.6}$$

One can exploit the nilpotence of ∂_x^2 on the space of polynomials of degree $\leq 2n$ and set

$$\begin{aligned} F_n(x) &= (I + \partial_x^2)^{-1} \varphi_n(x) \\ &= \sum_{k=0}^{n} (-1)^k \varphi_n^{(2k)}(x). \end{aligned} \tag{I.7}$$

Then

$$\text{(I.8)} \qquad \frac{d}{dx}\Big(F_n'(x)\sin x - F_n(x)\cos x\Big) = \varphi_n(x)\sin x.$$

Integrating (I.8) over $x \in [0,\pi]$ gives

$$\text{(I.9)} \qquad \int_0^\pi \varphi_n(x)\sin x\,dx = F_n(0) + F_n(\pi) = 2F_n(0),$$

the last identity holding for $\varphi_n(x)$ as in (I.1) because then $\varphi_n(\pi - x) = \varphi_n(x)$ and hence $F_n(\pi - x) = F_n(x)$. For the first identity in (I.9), we use the defining property that $\sin\pi = 0$ while $\cos\pi = -1$.

In light of (I.7), to prove (I.2) it suffices to establish an analogous property for $\varphi_n^{(2k)}(0)$. Comparing the binomial formula and Taylor's formula for $\varphi_n(x)$:

$$\text{(I.10)} \qquad \begin{aligned} \varphi_n(x) &= \frac{1}{n!}\sum_{\ell=0}^{n}(-1)^\ell \binom{n}{\ell}\pi^{n-\ell}x^{n+\ell}, \quad \text{and} \\ \varphi_n(x) &= \sum_{k=0}^{2n}\frac{1}{k!}\varphi_n^{(k)}(0)x^k, \end{aligned}$$

we see that

$$\text{(I.11)} \qquad k = n + \ell \Rightarrow \varphi_n^{(k)}(0) = (-1)^\ell \frac{(n+\ell)!}{n!}\binom{n}{\ell}\pi^{n-\ell},$$

so

$$\text{(I.12)} \qquad 2k = n + \ell \Rightarrow \varphi_n^{(2k)}(0) = (-1)^n \frac{(n+\ell)!}{n!}\binom{n}{\ell}\pi^{2(k-\ell)}.$$

Of course $\varphi_n^{(2k)}(0) = 0$ for $2k < n$. Clearly the multiple of $\pi^{2(k-\ell)}$ in (I.12) is an integer. In fact,

$$\text{(I.13)} \qquad \begin{aligned} \frac{(n+\ell)!}{n!}\binom{n}{\ell} &= \frac{(n+\ell)!}{n!}\,\frac{n!}{\ell!(n-\ell)!} \\ &= \frac{(n+\ell)!}{n!\ell!}\,\frac{n!}{(n-\ell)!} \\ &= \binom{n+\ell}{n} n(n-1)\cdots(n-\ell+1). \end{aligned}$$

Thus (I.2) is established, and the proof that π^2 is irrational is complete.

Chapter 2. Going deeper – the Cauchy integral theorem and consequences

Chapter 1 was devoted to elementary properties of holomorphic functions on open sets (also referred to as "domains," or "regions") in $\mathbb{C}$. Here we develop deeper tools for this study. In §5 we introduce a major theoretical tool of complex analysis, the Cauchy integral theorem. We provide a couple of proofs, one using Green's theorem and one based simply on the chain rule and the fundamental theorem of calculus. Cauchy's integral theorem leads to Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{f(\zeta)}{\zeta - z}\, d\zeta. \tag{2.0.1}$$

In turn, this result leads to the general development of holomorphic functions on a domain $\Omega \subset \mathbb{C}$ in power series about any $p \in \Omega$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-p)^n, \tag{2.0.2}$$

convergent in any disk centered at p and contained in Ω.

Results of §5 are applied in §6 to prove a maximum principle for holomorphic functions, and also a result called Liouville's theorem, stating that a holomorphic function on $\mathbb{C}$ that is bounded must be constant. We show that each of these results imply the fundamental theorem of algebra, that every non-constant polynomial $p(z)$ must vanish somewhere in $\mathbb{C}$.

In §7 we discuss harmonic functions on planar regions, and their relationship to holomorphic functions. A function $u \in C^2(\Omega)$ is harmonic provided $\Delta u = 0$, where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{2.0.3}$$

is the Laplace operator. We show that if u is harmonic and real valued, then u is locally the real part of a holomorphic function, and this holds globally if Ω is simply connected. This connection leads to maximum modulus theorems and Liouville theorems for harmonic functions.

In §8 we establish Morera's theorem, a sort of converse to Cauchy's integral theorem. We apply Morera's theorem to a result known as the Schwarz reflection principle, a device to extend a holomorphic function f from a domain in the upper half plane to its reflacted image in the lower half plane, if f is real valued on the real axis. We also use Morera's theorem to prove Goursat's theorem, to the effect that the C^1 hypothesis can be dropped in the characterization of holomorphic functions.

Section 9 considers infinite products, a frequently useful alternative to infinite series as a device for exhibiting a holomorphic function. Particularly incisive uses for infinite products will arise in sections 18 and 19 of Chapter 4. Section 9 concentrates on elementary results for infinite products. Deeper results of J. Hadamard are covered in an appendix to Chapter 4.

In §10 we obtain a uniqueness result: two holomorphic functions on a connected domain are identical if they agree on more than a discrete set. We then introduce the notion of analytic continuation, taking a function holomorphic on one domain and extending it to a larger domain. A number of processes arise to do this. One is Schwarz reflection. Another is analytic continuation along a curve, which will play a key role in the study of differential equations in Chapter 7.

Section 11 studies holomorphic functions with isolated singularities. A key result is Riemann's removable singularity theorem: if f is holomorphic on $\Omega\setminus\{p\}$ and bounded, then it can be extended to be holomorphic on all of Ω. Another possibility is that $|f(z)| \to \infty$ as $z \to p$. Then we say f has a pole at p. A function that is holomorphic on an open set Ω except for a discrete set of poles is said to be meromorphic on Ω. If p is neither a removable singularity nor a pole of f, it is called an essential singularity of f. An example is

$$f(z) = e^{1/z}, \tag{2.0.4}$$

wiht an essential singularity at $z = 0$. The relatively wild behavior of f near an essential singularity is described by the Casorati-Weierstrass theorem (which will be strengthened in Chapter 5 by Picard's theorem).

In §12 we consider Laurent series,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-p)^n, \tag{2.0.5}$$

which holds for a function f holomorphic on an annulus

$$\mathcal{A} = \{z \in \mathbb{C} : r_0 < |z-p| < r_1\}. \tag{2.0.6}$$

Here $0 \le r_0 < r_1 \le \infty$. In case $r_0 = 0$, f has an isolated singularity at p, and the type of this singularity is reflected in the behavior of the coefficients a_n for $n < 0$.

This chapter ends with several appendices. In Appendix C we treat Green's theorem, which is used in our first proof of the Cauchy integral theorem. Appendix F gives another proof of the fundamental theorem of algebra. This proof is more elementary than that given in §6, in that it does not use results that are consequences of the Cauchy integral theorem (on the other hand, it is longer). We also have a general treatment of absolutely convergent series in Appendix L, which complements results regarding operations on power series that have arisen in this chapter and in Chapter 1.

5. The Cauchy integral theorem and the Cauchy integral formula

The Cauchy integral theorem is of fundamental importance in the study of holomorphic functions on domains in $\mathbb{C}$. Our first proof will derive it from Green's theorem, which we now state.

Theorem 5.1. *If Ω is a bounded region in $\mathbb{R}^2$ with piecewise smooth boundary, and f and g belong to $C^1(\overline{\Omega})$, then*

$$\iint_{\Omega} \Bigl(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\Bigr) dx\, dy = \int_{\partial\Omega} (f\, dx + g\, dy). \tag{5.1}$$

We define some terminology here. In Theorem 5.1, Ω is a nonempty open set in $\mathbb{R}^2$, with closure $\overline{\Omega}$ and boundary $\partial\Omega = \overline{\Omega} \setminus \Omega$. (As mentioned in the Introduction, we use the terms "region" and also "domain" as alternative labels for "nonempty open set.") We assume $\partial\Omega$ is a finite disjoint union of simple closed curves $\gamma_j : [0,1] \to \mathbb{R}^2$, so $\gamma_j(0) = \gamma_j(1)$. We assume each curve γ_j is continuous and piecewise C^1, as defined in §1. Also we assume Ω lies on one side of γ_j. Furthermore, if $\gamma_j(t)$ is differentiable at t_0, we assume $\gamma_j'(t_0) \in \mathbb{R}^2$ is nonzero and that the vector $J\gamma_j'(t_0)$ points *into* Ω. Here J is counterclockwise rotation by $90°$, given by (1.39). This defines an *orientation* on $\partial\Omega$. We have

$$\int_{\partial\Omega} (f\, dx + g\, dy) = \sum_j \int_{\gamma_j} (f\, dx + g\, dy).$$

To say $f \in C^1(\overline{\Omega})$ is to say f is continuous on $\overline{\Omega}$ and smooth of class C^1 on Ω, and furthermore that $\partial f/\partial x$ and $\partial f/\partial y$ extend continuously from Ω to $\overline{\Omega}$. See Appendix C for further discussion of Green's theorem.

We will apply Green's theorem to the line integral

$$\int_{\partial\Omega} f\, dz = \int_{\partial\Omega} f(dx + i\, dy). \tag{5.2}$$

Clearly (5.1) applies to complex-valued functions, and if we set $g = if$, we get

$$\int_{\partial\Omega} f\, dz = \iint_{\Omega} \Bigl(i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\Bigr)\, dx\, dy. \tag{5.3}$$

Whenever f is holomorphic, the integrand on the right side of (5.3) vanishes, so we have the following result, known as Cauchy's integral theorem:

Theorem 5.2. *If $f \in C^1(\overline{\Omega})$ is holomorphic on Ω, then*

$$\int_{\partial\Omega} f(z)\, dz = 0. \tag{5.4}$$

Until further notice, we assume Ω is a bounded region in $\mathbb{C}$, with piecewise smooth boundary. Using (5.4), we can establish Cauchy's integral formula:

Theorem 5.3. *If $f \in C^1(\overline{\Omega})$ is holomorphic and $z_0 \in \Omega$, then*

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z - z_0}\, dz. \tag{5.5}$$

Proof. Note that $g(z) = f(z)/(z - z_0)$ is holomorphic on $\Omega \setminus \{z_0\}$. Let D_r be the open disk of radius r centered at z_0. Pick r so small that $\overline{D}_r \subset \Omega$. Then (5.4) implies

$$\int_{\partial\Omega} \frac{f(z)}{z - z_0}\, dz = \int_{\partial D_r} \frac{f(z)}{z - z_0}\, dz. \tag{5.6}$$

To evaluate the integral on the right, parametrize the curve ∂D_r by $\gamma(\theta) = z_0 + re^{i\theta}$. Hence $dz = ire^{i\theta}\, d\theta$, so the integral on the right is equal to

$$\int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}\, d\theta = i \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta. \tag{5.7}$$

As $r \to 0$, this tends in the limit to $2\pi i f(z_0)$, so (5.5) is established.

Note that, when (5.5) is applied to $\Omega = D_r$, the disk of radius r centered at z_0, the computation (5.7) yields

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta = \frac{1}{\ell(\partial D_r)} \int_{\partial D_r} f(z)\, ds(z), \tag{5.8}$$

when f is holomorphic and C^1 on D_r, and $\ell(\partial D_r) = 2\pi r$ is the length of the circle ∂D_r. This is a *mean value property*. We will develop this further in §§6–7.

Let us rewrite (5.5) as

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \tag{5.9}$$

for $z \in \Omega$. We can differentiate the right side with respect to z, obtaining

$$f'(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta, \tag{5.10}$$

for $z \in \Omega$, and we can continue, obtaining (for $f^{(n)}(z)$ defined inductively, as done below (2.28))

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int\limits_{\partial\Omega} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta. \tag{5.11}$$

In more detail, with

$$\Delta_h f(z) = \frac{1}{h}(f(z+h) - f(z)),$$

(5.9) gives

$$\Delta_h f(z) = \frac{1}{2\pi i} \int\limits_{\partial\Omega} \Delta_h\Big(\frac{1}{\zeta - z}\Big) f(\zeta) \, d\zeta.$$

Now, as $h \to 0$, given $z \notin \partial\Omega$,

$$\Delta_h(\zeta - z)^{-1} \longrightarrow (\zeta - z)^{-2}, \quad \text{uniformly on } \ \partial\Omega,$$

by Exercise 8 of §1, and this gives (5.10). The formula (5.11) follows inductively, using

$$\Delta_h f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \int\limits_{\partial\Omega} \Delta_h\Big(\frac{1}{(\zeta - z)^n}\Big) f(\zeta) \, d\zeta,$$

and applying Exercise 10 of §1.

Here is one consequence of (5.10)–(5.11).

Corollary 5.4. *Whenever f is holomorphic on an open set $\Omega \subset \mathbb{C}$, we have*

$$f \in C^\infty(\Omega). \tag{5.12}$$

Suppose $f \in C^1(\overline{\Omega})$ is holomorphic, $z_0 \in D_r \subset \Omega$, where D_r is the disk of radius r centered at z_0, and suppose $z \in D_r$. Then Theorem 5.3 implies

$$f(z) = \frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)} \, d\zeta. \tag{5.13}$$

We have the infinite series expansion

$$\frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \Big(\frac{z - z_0}{\zeta - z_0}\Big)^n, \tag{5.14}$$

valid as long as $|z - z_0| < |\zeta - z_0|$. Hence, given $|z - z_0| < r$, this series is uniformly convergent for $\zeta \in \partial\Omega$, and we have

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int\limits_{\partial\Omega} \frac{f(\zeta)}{\zeta - z_0} \Big(\frac{z - z_0}{\zeta - z_0}\Big)^n \, d\zeta. \tag{5.15}$$

This establishes the following key result.

Theorem 5.5. *If $f \in C^1(\overline{\Omega})$ is holomorphic, then for $z \in D_r(z_0) \subset \Omega, f(z)$ has the convergent power series expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{5.16}$$

with

$$a_n = \frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta = \frac{f^{(n)}(z_0)}{n!}. \tag{5.17}$$

REMARK. The second identity in (5.17) follows from (5.11). Alternatively, once we have (5.16), it also follows from (2.28) that a_n is equal to the last quantity in (5.17).

Next we use the Cauchy integral theorem to produce an integral formula for the inverse of a holomorphic map.

Proposition 5.6. *Suppose f is holomorphic and one-to-one on a neighborhood of $\overline{\Omega}$, the closure of a piecewise smoothly bounded domain $\Omega \subset \mathbb{C}$. Set $g = f^{-1} : f(\Omega) \to \Omega$. Then*

$$g(w) = \frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{z f'(z)}{f(z) - w}\, dz, \quad \forall\, w \in f(\Omega). \tag{5.18}$$

Proof. Set $\zeta = g(w)$, so $h(z) = f(z) - w$ has one zero in $\overline{\Omega}$, at $z = \zeta$, and $h'(\zeta) \neq 0$. (Cf. Exercise 8 below.) Then the right side of (5.18) is equal to

$$\frac{1}{2\pi i} \int\limits_{\partial\Omega} z\, \frac{h'(z)}{h(z)}\, dz = \frac{1}{2\pi i} \int\limits_{\partial\Omega} z\Big(\frac{1}{z - \zeta} + \frac{\varphi'(z)}{\varphi(z)}\Big) dz = \zeta, \tag{5.19}$$

where we set $h(z) = (z - \zeta)\varphi(z)$ with φ holomorphic and nonvanishing on a neighborhood of $\overline{\Omega}$.

Having discussed fundamental consequences of Cauchy's theorem, we return to the theorem itself, and give three more proofs. We begin with the following result, closely related though not quite identical, to Theorem 5.2. We give a proof using not Green's theorem but simply the chain rule, the fundamental theorem of calculus, and the equality of mixed partial derivatives for C^2 functions of two real variables.

Proposition 5.7. *Let $f \in C^1(\Omega)$ be holomorphic. Let γ_s be a smooth family of smooth (class C^2) closed curves in Ω. Then*

$$\int\limits_{\gamma_s} f(z)\, dz = A \tag{5.20}$$

is independent of s.

To set things up, say $\gamma_s(t) = \gamma(s,t)$ is C^2 for $a \le s \le b, t \in \mathbb{R}$, and periodic of period 1 in t. Denote the left side of (5.20) by $\psi(s)$:

$$\psi(s) = \int_0^1 f\bigl(\gamma(s,t)\bigr)\, \partial_t\gamma(s,t)\, dt. \tag{5.21}$$

Hence

$$\psi'(s) = \int_0^1 \bigl[f'\bigl(\gamma(s,t)\bigr)\, \partial_s\gamma(s,t)\, \partial_t\gamma(s,t) + f\bigl(\gamma(s,t)\bigr)\, \partial_s\partial_t\gamma(s,t)\bigr]\, dt. \tag{5.22}$$

We compare this with

$$\begin{aligned} &\int_0^1 \frac{\partial}{\partial t}\bigl[f\bigl(\gamma(s,t)\bigr)\, \partial_s\gamma(s,t)\bigr]\, dt \\ &\quad = \int_0^1 \bigl[f'\bigl(\gamma(s,t)\bigr)\, \partial_t\gamma(s,t)\, \partial_s\gamma(s,t) + f\bigl(\gamma(s,t)\bigr)\, \partial_t\partial_s\gamma(s,t)\bigr]\, dt. \end{aligned} \tag{5.23}$$

Using the identity $\partial_s\partial_t\gamma(s,t) = \partial_t\partial_s\gamma(s,t)$ and the identity

$$f'(\gamma)\partial_s\gamma\, \partial_t\gamma = f'(\gamma)\partial_t\gamma\, \partial_s\gamma, \tag{5.24}$$

we see that the right sides of (5.22) and (5.23) are equal. But the fundamental theorem of calculus implies the left side of (5.23) is equal to

$$f\bigl(\gamma(s,1)\bigr)\, \partial_s\gamma(s,1) - f\bigl(\gamma(s,0)\bigr)\, \partial_s\gamma(s,0) = 0. \tag{5.25}$$

Thus $\psi'(s) = 0$ for all s, and the proposition is proven.

For a variant of Proposition 5.7, see Exercise 13.

Our third proof of Cauchy's theorem establishes a result slightly weaker than Theorem 5.1, namely the following.

Proposition 5.8. *With $\overline{\Omega}$ as in Theorem 5.1, assume $\overline{\Omega} \subset \mathcal{O}$, open in $\mathbb{C}$, and f is holomorphic on $\mathcal{O}$. Then (5.4) holds.*

The proof will not use Green's thorem. Instead, it is based on Propositions 1.8 and 1.10. By Proposition 1.8, if f had a holomorphic antiderivative on $\mathcal{O}$, then we'd have $\int_\gamma f(z)\, dz = 0$ for each closed path γ in $\mathcal{O}$. Since $\partial\Omega$ is a union of such closed paths, this would give (5.4). As we have seen, f might not have an antiderivative on $\mathcal{O}$, though Proposition 1.10 does give conditions guaranteeing the existence of an antiderivative. The next strategy is to chop some neighborhood of $\overline{\Omega}$ in $\mathcal{O}$ into sets to which Proposition 1.10 applies.

To carry this out, tile the plane $\mathbb{C}$ with closed squares R_{jk} of equal size, with sides parallel to the coordinate axes, and check whether the following property holds:

$$\text{If } R_{jk} \text{ intersects } \overline{\Omega}, \text{ then } R_{jk} \subset \mathcal{O}. \tag{5.26}$$

See Fig. 5.1 for an example of part of such a tiling. If (5.26) fails, produce a new tiling by dividing each square into four equal subsquares, and check (5.26) again. Eventually, for example when the squares have diameters less than $\operatorname{dist}(\overline{\Omega}, \partial\mathcal{O})$, which is positive, (5.26) must hold.

If $R_{jk} \cap \overline{\Omega} \neq \emptyset$, denote this intersection by Ω_{jk}. We have $\overline{\Omega} = \cup_{j,k}\Omega_{jk}$, and furthermore

$$\int\limits_{\partial\Omega} f(z)\,dz = \sum_{j,k} \int\limits_{\partial\Omega_{jk}} f(z)\,dz, \tag{5.27}$$

the integrals over those parts of $\partial\Omega_{jk}$ not in $\partial\Omega$ cancelling out. Now Proposition 1.10 implies f has a holomorphic antiderivative on each $R_{jk} \subset \mathcal{O}$, and then Proposition 1.8 implies

$$\int\limits_{\partial\Omega_{jk}} f(z)\,dz = 0, \tag{5.28}$$

so (5.4) follows.

Finally, we relax the hypotheses on f, for a certain class of domains Ω. To specify the class, we say an open set $\mathcal{O} \subset \mathbb{C}$ is *star shaped* if there exists $p \in \mathcal{O}$ such that

$$0 < a < 1,\ p + z \in \mathcal{O} \Longrightarrow p + az \in \mathcal{O}. \tag{5.29}$$

Theorem 5.9. *Let Ω be a bounded domain with piecewise smooth boundary. Assume $\overline{\Omega}$ can be partitioned into a finite number of piecewise smoothly bounded domains $\overline{\Omega}_j$, $1 \le j \le K$, such that each Ω_j is star shaped. Assume $f \in C(\overline{\Omega})$ and that f is holomorphic on Ω. Then (5.4) holds.*

Proof. Since

$$\int\limits_{\partial\Omega} f(z)\,dz = \sum_{j=1}^{K} \int\limits_{\partial\Omega_j} f(z)\,dz, \tag{5.30}$$

it suffices to prove the result when Ω itself is star shaped, so (5.29) holds with $\mathcal{O} = \Omega$, $p \in \Omega$. Given $f \in C(\overline{\Omega})$, holomorphic on Ω, define

$$f_a : \overline{\Omega} \longrightarrow \mathbb{C}, \quad f_a(p + z) = f(p + az), \quad 0 < a < 1.$$

Then Proposition 5.8 (or Theorem 5.2) implies

$$\int\limits_{\partial\Omega} f_a(z)\,dz = 0 \text{ for each } a < 1. \tag{5.31}$$

On the other hand, since f is continuous on $\overline{\Omega}$, $f_a \to f$ uniformly on $\overline{\Omega}$ (and in particular on $\partial\Omega$) as $a \nearrow 1$, so (5.4) follows from (5.31) in the limit as $a \nearrow 1$.

We conclude this section with the following consequence of (5.9)–(5.10), which yields, among other things, a far reaching extension of Proposition 2.2.

Proposition 5.10. *Let $\Omega \subset \mathbb{C}$ be an open set and let $f_\nu : \Omega \to \mathbb{C}$ be holomorphic. Assume $f_\nu \to f$ locally uniformly (i.e., uniformly on each compact subset of Ω). Then $f : \Omega \to \mathbb{C}$ is holomorphic and*

$$f'_\nu \longrightarrow f', \quad \textit{locally uniformly on } \ \Omega. \tag{5.32}$$

Proof. Given a compact set $K \subset \Omega$, pick a smoothly bounded $\mathcal{O}$ such that $K \subset \mathcal{O} \subset \overline{\mathcal{O}} \subset \Omega$. Then, by (5.9)–(5.10),

$$\begin{aligned} f_\nu(z) &= \frac{1}{2\pi i} \int\limits_{\partial\mathcal{O}} \frac{f_\nu(\zeta)}{\zeta - z}\, d\zeta, \\ f'_\nu(z) &= \frac{1}{2\pi i} \int\limits_{\partial\mathcal{O}} \frac{f_\nu(\zeta)}{(\zeta - z)^2}\, d\zeta, \end{aligned} \tag{5.33}$$

for all $z \in \mathcal{O}$. Since $f_\nu \to f$ uniformly on $\partial\mathcal{O}$, we have, for each $n \in \mathbb{N}$,

$$\frac{f_\nu(\zeta)}{(\zeta - z)^n} \longrightarrow \frac{f(\zeta)}{(\zeta - z)^n}, \quad \text{uniformly for } \ \zeta \in \partial\mathcal{O}, \ z \in K. \tag{5.34}$$

Thus letting $\nu \to \infty$, we have

$$f(z) = \frac{1}{2\pi i} \int\limits_{\partial\mathcal{O}} \frac{f(\zeta)}{\zeta - z},\ d\zeta, \quad \forall\, z \in \mathcal{O}, \tag{5.35}$$

so f is holomorphic on $\mathcal{O}$, and

$$f'(z) = \frac{1}{2\pi i} \int\limits_{\partial\mathcal{O}} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta. \tag{5.36}$$

In light of (5.34), with $n = 2$, this gives (5.32).

Exercises

1. Show that, for $|z| < 1, \ \gamma \in \mathbb{C}$,

$$(1 + z)^\gamma = \sum_{n=0}^{\infty} a_n(\gamma) z^n, \tag{5.37}$$

where $a_0(\gamma) = 1, \ a_1(\gamma) = \gamma$, and, for $n \geq 2$,

$$a_n(\gamma) = \frac{\gamma(\gamma - 1) \cdots (\gamma - n + 1)}{n!}. \tag{5.38}$$

Hint. Use (4.18) to compute $f^{(n)}(0)$ when $f(z) = (1+z)^\gamma$.

2. Deduce from Exercise 1 that, for $|z| < 1$,

$$(1 - z^2)^\gamma = \sum_{n=0}^{\infty} (-1)^n a_n(\gamma) z^{2n}, \tag{5.39}$$

with $a_n(\gamma)$ as above. Take $\gamma = -1/2$ and verify that (3.46) yields (3.47).
Hint. Replace z by $-z^2$ in (5.37).

3. Find the coefficients a_k in the power series

$$\frac{1}{z^2+1} = \sum_{k=0}^{\infty} a_k (z-1)^k.$$

What is the radius of convergence?
Hint. Write the left side as

$$\frac{1}{2i}\Bigl(\frac{1}{z+i} - \frac{1}{z-i}\Bigr).$$

If $c \in \mathbb{C},\ c \neq 1$, write

$$\frac{1}{z-c} = \frac{1}{(z-1)-(c-1)} = -\frac{1}{c-1}\,\frac{1}{1-\frac{z-1}{c-1}},$$

and use the geometric series.

4. Suppose f is holomorphic on a disk centered at 0 and satisfies

$$f'(z) = af(z),$$

for some $a \in \mathbb{C}$. Prove that $f(z) = Ke^{az}$ for some $K \in \mathbb{C}$.
Hint. Find the coefficients in the power series for f about 0. *Alternative.* Apply d/dz to $e^{-az}f(z)$.

5. Suppose $f : \mathbb{C} \to \mathbb{C}$ is a function that is not identically zero. Assume that f is complex-differentiable at the origin, with $f'(0) = a$. Assume that

$$f(z+w) = f(z)f(w)$$

for all $z, w \in \mathbb{C}$. Prove that $f(z) = e^{az}$.
Hint. Begin by showing that f is complex-differentiable on all of $\mathbb{C}$.

6. Suppose $f : \Omega \to \mathbb{C}$ is holomorphic, $p \in \Omega$, and $f(p) = 0$. Show that $g(z) = f(z)/(z-p)$, defined at p as $g(p) = f'(p)$, is holomorphic. More generally, if $f(p) = \cdots = f^{(k-1)}(p) = 0$

and $f^{(k)}(p) \neq 0$, show that $f(z) = (z-p)^k g(z)$ with g holomorphic on Ω and $g(p) \neq 0$.
Hint. Consider the power series of f about p.

7. For f as in Exercise 6, show that on some neighborhood of p we can write $f(z) = [(z-p)h(z)]^k$, for some nonvanishing holomorphic function h.

8. Suppose $f : \Omega \to \mathbb{C}$ is holomorphic and one-to-one. Show that $f'(p) \neq 0$ for all $p \in \Omega$.
Hint. If $f'(p) = 0$, then apply Exercise 7 (to $f(z) - f(p)$), with some $k \geq 2$. Apply Theorem 4.2 to the function $G(z) = (z-p)h(z)$.
Reconsider this problem when you get to §11, and again in §17.

9. Assume $f : \Omega \to \mathbb{C}$ is holomorphic, $\overline{D_r(z_0)} \subset \Omega$, and $|f(z)| \leq M$ for $z \in D_r(z_0)$. Show that

$$\frac{|f^{(n)}(z_0)|}{n!} \leq \frac{M}{r^n}. \tag{5.40}$$

Hint. Use (5.11), with $\partial\Omega$ replaced by $\partial D_r(z_0)$.
These inequalities are known as Cauchy's inequalities. Applications of Cauchy's inequalities can be found in Exercise 9 of §6 and Exercise 1 of §11.

A connected open set $\Omega \subset \mathbb{C}$ is said to be simply connected if each smooth closed curve γ in Ω is part of a smooth family of closed curves γ_s, $0 \leq s \leq 1$, in Ω, such that $\gamma_1 = \gamma$ and $\gamma_0(t)$ has a single point as image.

10. Show that if $\Omega \subset \mathbb{C}$ is open and simply connected, γ is a smooth closed curve in Ω, and f is holomorphic on Ω, then $\int_\gamma f(z)\,dz = 0$.

11. Take $\Omega = \{z \in \mathbb{C} : 0 < |z| < 2\}$, and let $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$. Calculate $\int_\gamma dz/z$ and deduce that Ω is not simply connnected.

12. Show that if $\Omega \subset \mathbb{C}$ is open and convex, then it is simply connected.

13. Modify the proof of Proposition 5.7 to establish the following.

Proposition 5.7A. *Let $\Omega \subset \mathbb{C}$ be open and connected. Take $p, q \in \Omega$ and let γ_s be a smooth family of curves $\gamma_s : [0,1] \to \Omega$ such that $\gamma_s(0) \equiv p$ and $\gamma_s(1) \equiv q$. Let f be holomorphic on Ω. Then*

$$\int_{\gamma_s} f(z)\,dz = A$$

is independent of s.

For more on this, see Exercise 8 of §7.

6. The maximum principle, Liouville's theorem, and the fundamental theorem of algebra

Here we will apply results of §5 and derive some useful consequences. We start with the mean value property (5.8), i.e.,

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta, \tag{6.1}$$

valid whenever

$$\overline{D_r(z_0)} = \{z \in \mathbb{C} : |z - z_0| \le r\} \subset \Omega, \tag{6.2}$$

provided f is holomorphic on an open set $\Omega \subset \mathbb{C}$. Note that, in such a case,

$$\begin{aligned} \iint\limits_{D_r(z_0)} f(z)\, dx\, dy &= \int_0^{2\pi} \int_0^r f(z_0 + se^{i\theta}) s\, ds\, d\theta \\ &= \pi r^2 f(z_0), \end{aligned} \tag{6.3}$$

or

$$f(z_0) = \frac{1}{A_r} \iint\limits_{D_r(z_0)} f(z)\, dx\, dy, \tag{6.4}$$

where $A_r = \pi r^2$ is the area of the disk $D_r(z_0)$. This is another form of the mean value property. We use it to prove the following result, known as the maximum principle for holomorphic functions.

Proposition 6.1. *Let $\Omega \subset \mathbb{C}$ be a connected, open set. If f is holomorphic on Ω, then, given $z_0 \in \Omega$,*

$$|f(z_0)| = \sup_{z \in \Omega} |f(z)| \Longrightarrow f \quad \text{is constant on} \quad \Omega. \tag{6.5}$$

If, in addition, Ω is bounded and $f \in C(\overline{\Omega})$, then

$$\sup_{z \in \overline{\Omega}} |f(z)| = \sup_{z \in \partial\Omega} |f(z)|. \tag{6.6}$$

Proof. In the latter context, $|f|$ must assume a maximum at some point in $\overline{\Omega}$ (cf. Proposition A.14). Hence it suffices to prove (6.5).

Thus, assume there exists $z_0 \in \Omega$ such that the hypotheses of (6.5) hold. Set

$$\mathcal{O} = \{\zeta \in \Omega : f(\zeta) = f(z_0)\}. \tag{6.7}$$

We have $z_0 \in \mathcal{O}$. Continuity of f on Ω implies $\mathcal{O}$ is a closed subset of Ω. Now, if $\zeta_0 \in \mathcal{O}$, there is a disk of radius ρ, $\overline{D_\rho(\zeta_0)} \subset \Omega$, and, parallel to (6.9),

$$f(\zeta_0) = \frac{1}{A_\rho} \iint\limits_{D_\rho(\zeta_0)} f(z)\, dx\, dy. \tag{6.8}$$

The fact that $|f(\zeta_0)| \ge |f(z)|$ for all $z \in D_\rho(\zeta_0)$ forces

$$f(\zeta_0) = f(z), \quad \forall\, z \in D_\rho(\zeta_0). \tag{6.9}$$

(See Exercise 13 below.) Hence $\mathcal{O}$ is an open subset of Ω, as well as a nonempty closed subset. As explained in Appendix A, the hypothesis that Ω is connected then implies $\mathcal{O} = \Omega$. This completes the proof.

One useful consequence of Proposition 6.1 is the following result, known as the Schwarz lemma.

Proposition 6.2. *Suppose f is holomorphic on the unit disk $D_1(0)$. Assume $|f(z)| \le 1$ for $|z| < 1$, and $f(0) = 0$. Then*

$$|f(z)| \le |z|. \tag{6.10}$$

Furthermore, equality holds in (6.10), for some $z \in D_1(0) \setminus 0$, if and only if $f(z) = cz$ for some constant c of absolute value 1.

Proof. The hypotheses imply that $g(z) = f(z)/z$ is holomorphic on $D_1(0)$ (cf. §5, Exercise 6), and that $|g(z)| \le 1/a$ on the circle $\{z : |z| = a\}$, for each $a \in (0,1)$. Hence the maximum principle implies $|g(z)| \le 1/a$ on $D_a(0)$. Letting $a \nearrow 1$, we obtain $|g(z)| \le 1$ on $D_1(a)$, which implies (6.10).

If $|f(z_0)| = |z_0|$ at some point in $D_1(0)$, then $|g(z_0)| = 1$, so Proposition 6.1 implies $g \equiv c$, hence $f(z) \equiv cz$.

Important applications of the Schwarz lemma can be found in §20 (Proposition 20.2) and Appendix E (Proposition E.5, which leads to proofs given there of Picard's big theorem and the Riemann mapping theorem).

The next result is known as Liouville's theorem. It deals with functions holomorphic on all of $\mathbb{C}$, also known as *entire functions*.

Proposition 6.3. *If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded, then f is constant.*

Proof. Given $z \in \mathbb{C}$, we have by (5.10), for each $R \in (0, \infty)$,

$$f'(z) = \frac{1}{2\pi i} \int\limits_{\partial D_R(z)} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta, \tag{6.11}$$

where

$$\partial D_R(z) = \{\zeta \in \mathbb{C} : |\zeta - z| = R\}. \tag{6.12}$$

Parametrizing $\partial D_R(z)$ by $\zeta(t) = z + Re^{it}$, $0 \le t \le 2\pi$, we have

$$\begin{aligned} f'(z) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z + Re^{it})}{R^2 e^{2it}}\, iRe^{it}\, dt \\ &= \frac{1}{2\pi R} \int_0^{2\pi} f(z + Re^{it}) e^{-it}\, dt, \end{aligned} \tag{6.13}$$

hence, if $|f(z)| \le M$ for all $z \in \mathbb{C}$,

$$|f'(z)| \le \frac{M}{R}. \tag{6.14}$$

Compare the case $n = 1$ of (5.32). Since (6.13) holds for all $R < \infty$, we obtain

$$f'(z) = 0, \quad \forall\, z \in \mathbb{C}, \tag{6.15}$$

which implies f is constant.

Second proof of Proposition 6.3. With $f(0) = a$, set

$$g(z) = \begin{cases} \dfrac{f(z) - a}{z} & \text{for } z \ne 0, \\ f'(0) & \text{for } z = 0. \end{cases}$$

Then $g : \mathbb{C} \to \mathbb{C}$ is holomorphic (cf. §5, Exercise 6). The hypothesis $|f(z)| \le M$ for all z implies

$$|g(z)| \le \frac{M + |a|}{R} \quad \text{for} \quad |z| = R,$$

so the maximum principle implies $|g(z)| \le (M + |a|)/R$ on $D_R(0)$, and letting $R \to \infty$ gives $g \equiv 0$, hence $f \equiv a$.

We are now in a position to prove the following result, known as the fundamental theorem of algebra.

Theorem 6.4. *If $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial of degree $n \ge 1$ ($a_n \ne 0$), then $p(z)$ must vanish somewhere in $\mathbb{C}$.*

Proof. Consider

$$f(z) = \frac{1}{p(z)}. \tag{6.16}$$

If $p(z)$ does not vanish anywhere on $\mathbb{C}$, then $f(z)$ is holomorphic on all of $\mathbb{C}$. On the other hand, when $z \neq 0$,

$$f(z) = \frac{1}{z^n} \frac{1}{a_n + a_{n-1}z^{-1} + \cdots + a_0 z^{-n}}, \tag{6.17}$$

so

$$|f(z)| \to 0, \quad \text{as} \quad |z| \to \infty. \tag{6.18}$$

Thus f is bounded on $\mathbb{C}$, if $p(z)$ has no roots. By Proposition 6.3, $f(z)$ must be constant, which is impossible, so $p(z)$ must have a complex root.

Alternatively, having (6.18), we can apply the maximum principle. Applied to $f(z)$ on $D_R(0)$, it gives $|f(z)| \leq \sup_{|\zeta|=R} |f(\zeta)|$ for $|z| \leq R$, and (6.18) then forces f to be identically 0, which is impossible.

See Appendix F for an "elementary" proof of the fundamental theorem of algebra, i.e., a proof that does not make use of consequences of the Cauchy integral theorem.

Exercises

1. Establish the following improvement of Liouville's theorem.
Proposition. Assume $f : \mathbb{C} \to \mathbb{C}$ is holomorphic, and that there exist $w_0 \in \mathbb{C}$ and $a > 0$ such that

$$|f(z) - w_0| \geq a, \quad \forall\, z \in \mathbb{C}.$$

Then f is constant.
Hint. Consider

$$g(z) = \frac{1}{f(z) - w_0}.$$

2. Let $\mathcal{A} \subset \mathbb{C}$ be an open annulus, with two boundary components, γ_0 and γ_1. Assume $f \in C(\overline{\mathcal{A}})$ is holomorphic in $\mathcal{A}$. Show that one cannot have

$$\operatorname{Re} f < 0 \quad \text{on} \quad \gamma_0 \quad \text{and} \quad \operatorname{Re} f > 0 \quad \text{on} \quad \gamma_1. \tag{6.19}$$

Hint. Assume (6.19) holds. Then $K = \{z \in \overline{\mathcal{A}} : \operatorname{Re} f(z) = 0\}$ is a nonempty compact subset of $\overline{\mathcal{A}}$ (disjoint from $\partial\mathcal{A}$), and $J = \{f(z) : z \in K\}$ is a nonempty compact subset of the imaginary axis. Pick $ib \in J$ so that b is maximal. Show that, for sufficiently small $\delta > 0$,

$$g_\delta(z) = \frac{1}{i(b+\delta) - f(z)},$$

which is holomorphic on $\mathcal{A}$, would have to have an interior maximum, which is not allowed.

3. Show that the following calculation leads to another proof of the mean value property for f holomorphic on $\Omega \subset \mathbb{C}$, when $D_R(p) \subset \Omega$.

$$\psi(r) = \frac{1}{2\pi} \int_0^{2\pi} f(p + re^{i\theta})\, d\theta \tag{6.20}$$

satisfies

$$\begin{aligned} \psi'(r) &= \frac{1}{2\pi} \int_0^{2\pi} f'(p + re^{i\theta}) e^{i\theta}\, d\theta \\ &= \frac{1}{2\pi i r} \int_0^{2\pi} \frac{d}{d\theta} f(p + re^{i\theta})\, d\theta. \end{aligned} \tag{6.21}$$

4. Let $\Omega = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$. Assume f is bounded and continuous on $\overline{\Omega}$ and holomorphic on Ω. Show that

$$\sup_{\Omega} |f| = \sup_{\partial\Omega} |f|.$$

Hint. For $\varepsilon > 0$, consider $f_\varepsilon(z) = f(z) e^{\varepsilon z^2}$.
Relax the hypothesis that f is bounded.

For Exercises 5–8, suppose we have a polynomial $p(z)$, of the form

$$p(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0. \tag{6.22}$$

5. Show that there exist $r_k \in \mathbb{C},\ 1 \le k \le n$, such that

$$p(z) = (z - r_1) \cdots (z - r_n). \tag{6.23}$$

6. Show that

$$\frac{p'(z)}{p(z)} = \frac{1}{z - r_1} + \cdots + \frac{1}{z - r_n}. \tag{6.24}$$

7. Suppose each root r_k of $p(z)$ belongs to the right half-plane $H = \{z : \text{Re } z > 0\}$. Show that

$$\text{Re } z < 0 \Longrightarrow \text{Re } \frac{p'(z)}{p(z)} < 0 \Longrightarrow \frac{p'(z)}{p(z)} \neq 0. \tag{6.25}$$

8. Show that the set of zeros of $p'(z)$ is contained in the convex hull of the set of zeros of $p(z)$.
Hint. Given a closed set $S \subset \mathbb{R}^n$, the convex hull of S is the intersection of all the half-spaces containing S.

9. Suppose $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and satisfies an estimate

$$|f(z)| \le C(1+|z|)^{n-1},$$

for some $n \in \mathbb{Z}^+$. Show that $f(z)$ is a polynomial in z of degree $\le n-1$.
Hint. Apply (5.40) with $\Omega = D_R(z)$ and let $R \to \infty$ to show that $f^{(n)}(z) = 0$ for all z.

10. Show that if $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and not constant, then its range $f(\mathbb{C})$ is dense in $\mathbb{C}$.
Hint. If $f(\mathbb{C})$ omits a neighborhood of $p \in \mathbb{C}$, consider the holomorphic function $g(z) = 1/(f(z)-p)$.

11. Consider the functions

$$f(z) = e^z - z, \quad f'(z) = e^z - 1.$$

Show that all the zeros of f are contained in $\{z : \text{Re } z > 0\}$ while all the zeros of f' lie on the imaginary axis. (Contrast this with the result of Exercise 7.)
Hint for the first part. What is the image of $\{z \in \mathbb{C} : \operatorname{Re} z \le 0\}$ under the map $\exp \circ \exp$?
Remark. Results of §29 imply that $e^z - z$ has infinitely many zeros.

12. Let $\Omega \subset \mathbb{C}$ be open and connected, and $f : \Omega \to \mathbb{C}$ holomorphic. Assume there exists $z_0 \in \Omega$ such that

$$|f(z_0)| = \min_{z\in\Omega} |f(z)|.$$

Show that either $f(z_0) = 0$ or f is constant.

13. Supplement the proof of Proposition 6.1 with details on why (6.9) holds.
One approach. Say $\sup_{z\in\Omega} |f(z)| = |f(\zeta_0)| = Be^{i\alpha}$, $B > 0$, $\alpha \in \mathbb{R}$. Set $g(z) = e^{-i\alpha} f(z)$, so $|g(z)| \equiv |f(z)|$ and $g(\zeta_0) = B$. Parallel to (6.8),

$$0 = g(\zeta_0) - B = \frac{1}{A_\rho} \iint\limits_{D_\rho(\zeta_0)} \operatorname{Re}[g(z) - B]\, dx\, dy.$$

Note that $\operatorname{Re}[g(z) - B] \le 0$, and deduce that

$$\operatorname{Re}[g(z) - B] \equiv 0 \ \text{ on } \ D_\rho(\zeta_0).$$

Thus, on $D_\rho(\zeta_0)$, $g(z) = B + i\gamma(z)$, with $\gamma(z)$ real valued, hence $|g(z)|^2 = B^2 + |\gamma(z)|^2$. Show that this forces $\gamma(z) \equiv 0$ on $D_\rho(\zeta_0)$.

14. Do Exercise 4 with Ω replaced by

$$\Omega = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}.$$

Hint. For $\varepsilon > 0$, consider $f_\varepsilon(z) = f(z)/(1+\varepsilon z)$.

15. Let $D = \{z \in \mathbb{C} : |z| < 1\}$. Assume f is bounded on D, continuous on $\overline{D} \setminus \{1\}$, and holomorphic on D. Show that $\sup_D |f| = \sup_{\partial D \setminus \{1\}} |f|$.
Hint. Use $\varphi(z) = (z-1)/(z+1)$ to map Ω in Exercise 14 to D, and consider $f \circ \varphi$.

7. Harmonic functions on planar regions

We can write the Cauchy-Riemann equation (1.28) as $(\partial/\partial x + i\partial/\partial y)f = 0$. Applying $\partial/\partial x - i\partial/\partial y$ to this gives

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0, \tag{7.1}$$

on an open set $\Omega \subset \mathbb{C}$, whenever f is holomorphic on Ω. In general, a C^2 solution to (7.1) on such Ω is called a harmonic function. More generally, if $\mathcal{O}$ is an open set in $\mathbb{R}^n$, a function $f \in C^2(\mathcal{O})$ is said to be harmonic on $\mathcal{O}$ if $\Delta f = 0$ on $\mathcal{O}$, where

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}. \tag{7.2}$$

Here we restrict attention to the planar case, $n = 2$. Material on harmonic functions in higher dimensions can be found in many books on partial differential equations, for example [T2] (particularly in Chapters 3 and 5), and also in Advanced Calculus texts, such as [T] (see §10).

If $f = u + iv$ is holomorphic, with $u = \operatorname{Re} f,\ v = \operatorname{Im} f$, (7.1) implies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0, \tag{7.3}$$

so the real and imaginary parts of a function holomorphic on a region Ω are both harmonic on Ω. Our first task in this section will be to show that many (though not all) domains $\Omega \subset \mathbb{C}$ have the property that if $u \in C^2(\Omega)$ is real valued and harmonic, then there exists a real valued function $v \in C^2(\Omega)$ such that $f = u + iv$ is holomorphic on Ω. One says v is a harmonic conjugate to u. Such a property is equivalent to the form (1.34) of the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{7.4}$$

To set up a construction of harmonic conjugates, we fix some notation. Given $\alpha = a + ib,\ z = x + iy$ $(a, b, x, y \in \mathbb{R})$, let $\gamma_{\alpha z}$ denote the path from α to z consisting of the vertical line segment from $a + ib$ to $a + iy$, followed by the horizontal line segment from $a + iy$ to $x + iy$. Let $\sigma_{\alpha z}$ denote the path from α to z consisting of the horizontal line segment from $a + ib$ to $x + ib$, followed by the vertical line segment from $x + ib$ to $x + iy$. Also, let $R_{\alpha z}$ denote the rectangle bounded by these four line segments. See Fig. 7.1. Here is a first construction of harmonic conjugates.

Proposition 7.1. *Let $\Omega \subset \mathbb{C}$ be open, $\alpha = a + ib \in \Omega$, and assume the following property holds:*

$$\text{If also } z \in \Omega, \text{ then } R_{\alpha z} \subset \Omega. \tag{7.5}$$

Let $u \in C^2(\Omega)$ be harmonic. Then u has a harmonic conjugate $v \in C^2(\Omega)$.

Proof. For $z \in \Omega$, set

$$\begin{aligned} v(z) &= \int\limits_{\gamma_{\alpha z}} \Bigl(-\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy\Bigr) \\ &= \int_b^y \frac{\partial u}{\partial x}(a, s)\,ds - \int_a^x \frac{\partial u}{\partial y}(t, y)\,dt. \end{aligned} \tag{7.6}$$

Also set

$$\begin{aligned} \tilde{v}(z) &= \int\limits_{\sigma_{\alpha z}} \Bigl(-\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy\Bigr) \\ &= -\int_a^x \frac{\partial u}{\partial y}(t, b)\,dt + \int_b^y \frac{\partial u}{\partial x}(x, s)\,ds. \end{aligned} \tag{7.7}$$

Straightforward applications of the fundamental theorem of calculus yield

$$\frac{\partial v}{\partial x}(z) = -\frac{\partial u}{\partial y}(z), \tag{7.8}$$

and

$$\frac{\partial \tilde{v}}{\partial y}(z) = \frac{\partial u}{\partial x}(z). \tag{7.9}$$

Furthermore, since $R_{\alpha z} \subset \Omega$, we have

$$\begin{aligned} \tilde{v}(z) - v(z) &= \int\limits_{\partial R_{\alpha z}} \Bigl(-\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy\Bigr) \\ &= \iint\limits_{R_{\alpha z}} \Bigl(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Bigr)\,dx\,dy \\ &= 0, \end{aligned} \tag{7.10}$$

the second identity by Green's theorem, (5.1), and the third because $\Delta u = 0$ on Ω. Hence (7.8)–(7.9) give the Cauchy-Riemann equations (7.4), proving Proposition 7.1.

The next result, whose proof is similar to that of Proposition 1.10, simultaneously extends the scope of Proposition 7.1 and avoids the use of Green's theorem.

Proposition 7.2. *Let $\Omega \subset \mathbb{C}$ be open, $\alpha = a + ib \in \Omega$, and assume the following property holds.*

$$\text{If also } z \in \Omega, \text{ then } \gamma_{\alpha z} \subset \Omega. \tag{7.11}$$

Let $u \in C^2(\Omega)$ be harmonic. Then u has a harmonic conjugate $v \in C^2(\Omega)$.

Proof. As in the proof of Proposition 7.1, define v on Ω by (7.6). We again have (7.8). It remains to compute $\partial v/\partial y$. Applying $\partial/\partial y$ to (7.6) gives

$$\begin{aligned}
\frac{\partial v}{\partial y}(z) &= \frac{\partial u}{\partial x}(a, y) - \int_a^x \frac{\partial^2 u}{\partial y^2}(t, y)\, dt \\
&= \frac{\partial u}{\partial x}(a, y) + \int_a^x \frac{\partial^2 u}{\partial t^2}(t, y)\, dt \\
&= \frac{\partial u}{\partial x}(a, y) + \frac{\partial u}{\partial x}(t, y)\Big|_{t=a}^{x} \\
&= \frac{\partial u}{\partial x}(z),
\end{aligned} \tag{7.12}$$

the second identity because u is harmonic and the third by the fundamental theorem of calculus. This again establishes the Cauchy-Riemann equations, and completes the proof of Proposition 7.2.

Later in this section, we will establish the existence of harmonic conjugates for a larger class of domains. However, the results given above are good enough to yield some important information on harmonic functions, which we now look into. The following is the mean value property for harmonic functions.

Proposition 7.3. *If $u \in C^2(\Omega)$ is harmonic, $z_0 \in \Omega$, and $\overline{D_r(z_0)} \subset \Omega$, then*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta. \tag{7.13}$$

Proof. We can assume u is real valued. Take $\rho > r$ such that $\overline{D_\rho(z_0)} \subset \Omega$. By Proposition 7.1, u has a harmonic conjugate v on $D_\rho(z_0)$, so $f = u + iv$ is holomorphic on $D_\rho(z_0)$. By (5.8),

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta. \tag{7.14}$$

Taking the real part gives (7.13).

As in (6.3), we also have, under the hypotheses of Proposition 7.3,

$$\begin{aligned}
\iint\limits_{D_r(z_0)} u(z)\, dx\, dy &= \int_0^{2\pi} \int_0^r u(z_0 + se^{i\theta}) s\, ds\, d\theta \\
&= \pi r^2 u(z_0),
\end{aligned} \tag{7.15}$$

hence

$$u(z_0) = \frac{1}{A_r} \iint\limits_{D_r(z_0)} u(z)\, dx\, dy. \tag{7.16}$$

With this, we can establish a maximum principle for harmonic functions.

Proposition 7.4. *Let $\Omega \subset \mathbb{C}$ be a connected, open set. If $u : \Omega \to \mathbb{R}$ is harmonic on Ω, then, given $z_0 \in \Omega$,*

$$u(z_0) = \sup_{z\in\Omega} u(z) \Longrightarrow u \quad \textit{is constant on} \quad \Omega. \tag{7.17}$$

If, addition, Ω is bounded and $u \in C(\overline{\Omega})$, then

$$\sup_{z\in\overline{\Omega}} u(z) = \sup_{z\in\partial\Omega} u(z). \tag{7.18}$$

Proof. Essentially the same as the proof of Proposition 6.1.

Next, we establish Liouville's theorem for harmonic functions on $\mathbb{C}$.

Proposition 7.5. *If $u \in C^2(\mathbb{C})$ is bounded and harmonic on all of $\mathbb{C}$, then u is constant.*

Proof. Pick any two points $p, q \in \mathbb{C}$. We have, for all $r > 0$,

$$u(p) - u(q) = \frac{1}{A_r} \Big[\iint\limits_{D_r(p)} u(z)\, dx\, dy - \iint\limits_{D_r(q)} u(z)\, dx\, dy \Big], \tag{7.19}$$

where, as before, $A_r = \pi r^2$. Hence

$$|u(p) - u(q)| \le \frac{1}{\pi r^2} \iint\limits_{\Delta(p,q,r)} |u(z)|\, dx\, dy, \tag{7.20}$$

where

$$\begin{aligned} \Delta(p,q,r) &= D_r(p) \triangle D_r(q) \\ &= \big(D_r(p) \setminus D_r(q)\big) \cup \big(D_r(q) \setminus D_r(p)\big). \end{aligned} \tag{7.21}$$

Note that if $a = |p - q|$, then $\Delta(p,q,r) \subset D_{r+a}(p) \setminus D_{r-a}(p)$, so

$$\text{Area}(\Delta(p,q,r)) \le \pi[(r+a)^2 - (r-a)^2] = 4\pi a r. \tag{7.22}$$

It follows that, if $|u(z)| \le M$ for all $z \in \mathbb{C}$, then

$$|u(p) - u(q)| \le \frac{4M|p-q|}{r}, \quad \forall\, r < \infty, \tag{7.23}$$

and taking $r \to \infty$ gives $u(p) - u(q) = 0$, so u is constant.

Second proof of Proposition 7.5. Take u as in the statement of Proposition 7.5. By Proposition 7.1, u has a harmonic conjugate v, so $f(z) = u(z) + iv(z)$ is holomorphic on $\mathbb{C}$, and, for some $M < \infty$,

$$|\operatorname{Re} f(z)| \le M, \quad \forall\, z \in \mathbb{C}.$$

In such a case, $\operatorname{Re}(f(z) + M + 1) \ge 1$ for all z, so

$$g(z) = \frac{1}{f(z) + M + 1}$$

is holomorphic on $\mathbb{C}$ and $|g(z)| \le 1$ for all z, so Proposition 6.3 implies g is constant, which implies f is constant. (Compare Exercise 1 in §6.)

We return to the question of when does a harmonic function on a domain $\Omega \subset \mathbb{C}$ have a harmonic conjugate. We start with a definition. Let $\Omega \subset \mathbb{C}$ be a connected, open set. We say Ω is a simply connected domain if the following property holds. Given $p, q \in \Omega$ and a pair of smooth paths

$$\gamma_0, \gamma_1 : [0,1] \longrightarrow \Omega, \quad \gamma_j(0) = p,\ \gamma_j(1) = q, \tag{7.24}$$

there is a smooth family γ_s of paths, such that

$$\gamma_s : [0,1] \longrightarrow \Omega, \quad \gamma_s(0) = p,\ \gamma_s(1) = q,\ \forall\, s \in [0,1]. \tag{7.25}$$

Compare material in Exercises 10–13 of §5. The definition given there looks a little different, but it is equivalent to the one given here. We also write $\gamma_s(t) = \gamma(s,t)$, $\gamma : [0,1] \times [0,1] \to \Omega$. We will prove the following.

Proposition 7.6. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain. Then each harmonic function $u \in C^2(\Omega)$ has a harmonic conjugate.*

The proof starts like that of Proposition 7.1, picking $\alpha \in \Omega$ and setting

$$v(z) = \int\limits_{\gamma_{\alpha z}} \Big(-\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy\Big), \tag{7.26}$$

except this time, $\gamma_{\alpha z}$ denotes an *arbitrary* piecewise smooth path from α to z. The crux of the proof is to show that (7.26) is independent of the choice of path from α to z. If this known, we can simply write

$$v(z) = \int_\alpha^z \Big(-\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy\Big), \tag{7.27}$$

and proceed as follows. Given $z \in \Omega$, take $r > 0$ such that $D_r(z) \subset \Omega$. With $z = x + iy$, pick $\xi + iy, x + i\eta \in D_r(z)$ ($x, y, \xi, \eta \in \mathbb{R}$). We have

$$v(z) = \int_\alpha^{\xi+iy} \Big(-\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy\Big) + \int_{\xi+iy}^{x+iy} \Big(-\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy\Big), \tag{7.28}$$

where we go from $\xi + iy$ to $z = x + iy$ on a horizontal line segment, and a calculation parallel to (7.8) gives

$$\frac{\partial v}{\partial x}(z) = -\frac{\partial u}{\partial y}(z). \tag{7.29}$$

We also have

$$v(z) = \int_{\alpha}^{x+i\eta} \Big(-\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy\Big) + \int_{x+i\eta}^{x+iy} \Big(-\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy\Big), \tag{7.30}$$

where we go from $x + i\eta$ to $z = x + iy$ on a vertical line segment, and a calculation parallel to (7.9) gives

$$\frac{\partial v}{\partial y}(z) = \frac{\partial u}{\partial x}(z), \tag{7.31}$$

thus establishing that v is a harmonic conjugate of u.

It remains to prove the asserted path independence of (7.26). This is a special case of the following result.

Lemma 7.7. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain, and pick $p, q \in \Omega$. Assume $F_1,\ F_2 \in C^1(\Omega)$ satisfy*

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}. \tag{7.32}$$

Then

$$\int\limits_{\gamma_{pq}} F_1\,dx + F_2\,dy \tag{7.33}$$

is independent of the choice of path from p to q.

To see how this applies to the path independence in (7.26), which has the form (7.33) with

$$F_1 = -\frac{\partial u}{\partial y}, \quad F_2 = \frac{\partial u}{\partial x}, \tag{7.34}$$

note that in this case we have

$$\frac{\partial F_1}{\partial y} = -\frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial^2 u}{\partial x^2}, \tag{7.35}$$

so (7.32) is equivalent to the assertion that u is harmonic on Ω.

In view of the definition of a simply connected domain, the following result implies Lemma 7.7.

Lemma 7.8. *Let $\Omega \subset \mathbb{C}$ be open and connected, and pick $p, q \in \Omega$. Let $F_1, F_2 \in C^1(\Omega)$ satisfy (7.32). Then, if $\gamma_s : [0,1] \to \Omega,\ 0 \le s \le 1$, is a smooth family of paths from p to q,*

$$\int\limits_{\gamma_s} F_1\,dx + F_2\,dy \tag{7.36}$$

is independent of s.

To prove Lemma 7.8, it is convenient to introduce some notation. Set

$$x_1 = x, \quad x_2 = y, \tag{7.37}$$

so (7.32) yields

$$\frac{\partial F_k}{\partial x_j} = \frac{\partial F_j}{\partial x_k}, \tag{7.38}$$

for all $j, k \in \{1, 2\}$. Also, represent elements of $\mathbb{R}^2 \approx \mathbb{C}$ as vectors:

$$F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \quad \gamma' = \begin{pmatrix} \gamma_1' \\ \gamma_2' \end{pmatrix}, \quad F \cdot \gamma' = \sum_j F_j \gamma_j'. \tag{7.39}$$

Then (7.36) takes the form

$$\int_0^1 F(\gamma_s(t)) \cdot \gamma_s'(t)\,dt. \tag{7.40}$$

We are assuming

$$\begin{gathered} \gamma_s(t) = \gamma(s,t), \quad \gamma : [0,1] \times [0,1] \to \Omega, \\ \gamma(s,0) = p, \quad \gamma(s,1) = q, \end{gathered}$$

and the claim is that (7.40) is independent of s, provided (7.38) holds.

To see this independence, we compute the s-derivative of (7.40), i.e., of

$$\begin{aligned} \psi(s) &= \int_0^1 F(\gamma(s,t)) \cdot \frac{\partial \gamma}{\partial t}(s,t)\,dt \\ &= \int_0^1 \sum_j F_j(\gamma(s,t)) \frac{\partial \gamma_j}{\partial t}(s,t)\,dt. \end{aligned} \tag{7.42}$$

The s-derivative of the integrand in (7.42) is obtained via the product rule and the chain rule. Thus

$$\begin{aligned} \psi'(s) = &\int_0^1 \sum_{j,k} \frac{\partial F_j}{\partial x_k}(\gamma(s,t)) \frac{\partial}{\partial s}\gamma_k(s,t)\, \frac{\partial}{\partial t}\gamma_j(s,t)\,dt \\ &+ \int_0^1 \sum_j F_j(\gamma(s,t))\, \frac{\partial}{\partial s}\frac{\partial}{\partial t}\gamma_j(s,t)\,dt. \end{aligned} \tag{7.43}$$

Compare (7.43) with

$$
\begin{aligned}
\omega(s) &= \int_0^1 \frac{\partial}{\partial t}\Bigl[F(\gamma_s(t))\cdot \frac{\partial}{\partial s}\gamma_s(t)\Bigr]\,dt \\
&= \int_0^1 \frac{\partial}{\partial t}\sum_k F_k(\gamma(s,t))\frac{\partial \gamma_k}{\partial s}(s,t)\,dt \\
&= \int_0^1 \sum_{j,k} \frac{\partial F_k}{\partial x_j}(\gamma(s,t))\frac{\partial \gamma_j}{\partial t}(s,t)\frac{\partial \gamma_k}{\partial t}(s,t)\,dt \\
&\quad + \int_0^1 \sum_j F_j(\gamma(s,t))\frac{\partial}{\partial t}\frac{\partial}{\partial s}\gamma_j(s,t)\,dt,
\end{aligned}
\tag{7.44}
$$

where we have relabeled the index of summation in the last sum. Noting that

$$
\frac{\partial}{\partial s}\frac{\partial}{\partial t}\gamma_j(s,t) = \frac{\partial}{\partial t}\frac{\partial}{\partial s}\gamma_j(s,t),
$$

we see that

$$
\omega(s) = \psi'(s), \quad \text{provided that (7.38) holds.} \tag{7.45}
$$

However, the fundamental theorem of calculus implies

$$
\omega(s) \equiv 0, \quad \text{provided} \quad \frac{\partial}{\partial s}\gamma_s(0) \equiv \frac{\partial}{\partial s}\gamma_s(1) \equiv 0,
$$

which holds provided $\gamma_s(0) \equiv p$ and $\gamma_s(1) \equiv q$. This proves Lemma 7.8.

With this done, the proof of Proposition 7.6 is complete.

We next describe a useful method of taking one harmonic function and producing others.

Proposition 7.9. *Let $\mathcal{O}, \Omega \subset \mathbb{C}$ be open sets. Let $u \in C^2(\Omega)$ be harmonic. If $g : \mathcal{O} \to \Omega$ is holomorphic, then $u \circ g$ is harmonic on $\mathcal{O}$.*

Proof. It suffices to treat the case of real valued u. Also, it suffices to show that $u \circ g$ is harmonic on a neighborhood of each point $p \in \mathcal{O}$. Let $q = g(p) \in \Omega$ and pick $\rho > 0$ such that $D_\rho(q) \subset \Omega$. Then $\mathcal{O}_p = g^{-1}(D_\rho(q))$ is a neighborhood of p in $\mathcal{O}$, and $u \circ g \in C^2(\mathcal{O}_p)$. By Proposition 7.1, there is a holomorphic function $f : D_\rho(q) \to \mathbb{C}$ such that $u = \operatorname{Re} f$. Hence $u \circ g = \operatorname{Re} f \circ g$ on $\mathcal{O}_p$. But $f \circ g$ is holomorphic on $\mathcal{O}_p$, by Proposition 1.2, so we have the desired result.

Exercises

1. Modify the second proof of Proposition 7.5 to establish the following.

Proposition 7.5A. *If $u \in C^2(\mathbb{C})$ is harmonic on $\mathbb{C}$, real valued, and bounded from below, then u is constant.*

Hint. Let $u = \operatorname{Re} f$, $f : \mathbb{C} \to \mathbb{C}$, holomorphic. See Exercise 1 of §6.

Alternative. Modify the argument involving (7.19)–(7.23), used in the *first* proof of Proposition 7.5, to show that $u(p) - u(q) \le 0$, and then reverse the roles of p and q.

2. Let $u : \Omega \to \mathbb{R}$ be harmonic. Assume u has a harmonic conjugate v, so $f(z) = u(z) + iv(z)$ is holomorphic. Show that, if $\gamma : [a, b] \to \Omega$ is a piecewise smooth path,

$$\frac{1}{i} \int_\gamma f'(z)\, dz = \int_\gamma \Bigl(-\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy\Bigr) + i \int_\gamma \Bigl(-\frac{\partial v}{\partial y}\, dx + \frac{\partial v}{\partial x}\, dy\Bigr). \tag{7.46}$$

Deduce that the integral on the right side of (7.46) must vanish whenever γ is a closed curve in Ω.
Hint. Write $f'(z)\, dz = (u_x + iv_x)(dx + idy)$, separate the left side of (7.46) into real and imaginary parts, and use the Cauchy-Riemann equations to obtain the right side of (7.46).

3. Let $\Omega \subset \mathbb{C}$ be a connected, open set and $u : \Omega \to \mathbb{R}$ be harmonic. Show that u has a harmonic conjugate $v \in C^2(\Omega)$ if and only if

$$\int_\gamma \Bigl(-\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy\Bigr) = 0, \tag{7.47}$$

for every smooth closed curve γ in Ω.
Hint. For the "only if" part, use (7.46). For the converse, consider the role of (7.26) in the proof of Proposition 7.6.

Recall from §4 the holomorphic function $\log : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$, characterized by

$$z = re^{i\theta},\ r > 0,\ -\pi < \theta < \pi \Longrightarrow \log z = \log r + i\theta.$$

In particular, if we define $\operatorname{Arg}(z)$ to be θ, then $\log|z|$ is harmonic on $\mathbb{C} \setminus (-\infty, 0]$, with harmonic conjugate $\operatorname{Arg}(z)$.

4. Show directly that $\log|z|$ is harmonic on $\mathbb{C} \setminus 0$.

5. Show that $\log|z|$ does not have a harmonic conjugate on $\mathbb{C} \setminus 0$.
One approach: Apply Exercise 3.
Hint. Show that if $\gamma(t) = re^{it}$, $0 \le t \le 2\pi$, and $u(z) = \psi(|z|^2)$, then

$$\int_\gamma \Bigl(-\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy\Bigr) = 4\pi r^2 \psi'(r^2). \tag{7.48}$$

6. Let $\Omega \subset \mathbb{C}$ be open and connected, $u : \Omega \to \mathbb{R}$ harmonic. If v and $\tilde{v}$ are both harmonic conjugates of u on Ω, show that $v - \tilde{v}$ is constant.
Hint. Use the Cauchy-Riemann equations to show that if $g : \Omega \to \mathbb{R}$ is holomorphic, then g is constant.

7. Use Exercise 6 to get another solution to Exercise 5, via the results on $\mathrm{Arg}(z)$.

8. Let $\Omega \subset \mathbb{C}$ be simply connected. Pick $p \in \Omega$. Given g holomorphic on Ω, set

$$f(z) = \int_{\gamma_{pz}} g(\zeta)\, d\zeta, \quad z \in \Omega,$$

with γ_{pz} a smooth path in Ω from p to z. Show that this integral is independent of the choice of such a path, so $f(z)$ is well defined. Show that f is holomorphic on Ω and

$$f'(z) = g(z), \quad \forall\, z \in \Omega.$$

Hint. For the independence, see Exercise 13 of §5. For the rest, adapt the argument used to prove Proposition 7.6, to show that

$$\frac{\partial f}{\partial x}(z) = g(z), \quad \text{and} \quad \frac{1}{i}\frac{\partial f}{\partial y}(z) = g(z).$$

9. Let $\Omega \subset \mathbb{C}$ be a bounded domain with piecewise smooth boundary. Let $F_1, F_2 \in C^1(\overline{\Omega})$ satisfy (7.32). Use Green's theorem to show that

$$\int_{\partial\Omega} F_1\, dx + F_2\, dy = 0.$$

Compare this conclusion with that of Lemma 7.7. Compare the relation between these two results with the relation of Proposition 5.7 to Theorem 5.2.

10. Let $\Omega \subset \mathbb{C}$ be open, and $\gamma_{pq} : [a, b] \to \Omega$ a path from p to q. Show that if $v \in C^1(\Omega)$, then

$$v(q) - v(p) = \int_{\gamma_{pq}} \Big(\frac{\partial v}{\partial x}\, dx + \frac{\partial v}{\partial y}\, dy \Big). \tag{7.49}$$

Relate this to the use of (7.6), (7.7), and (7.26).
Hint. Compute $(d/dt) v(\gamma(t))$.

11. Show that (7.49) implies Proposition 1.8.

12. Work out an "index free" proof of Lemma 7.8 along the following lines. With $\psi(s)$ as in (7.42), i.e.,

$$\psi(s) = \int_0^1 \frac{\partial \gamma}{\partial t}(s,t) \cdot F(\gamma(s,t))\, dt,$$

show that

$$\begin{aligned} \psi'(s) = & \int_0^1 \frac{\partial \gamma}{\partial t}(s,t) \cdot DF(\gamma(s,t)) \frac{\partial \gamma}{\partial s}(s,t)\, dt \\ & + \int_0^1 \frac{\partial}{\partial s} \frac{\partial}{\partial t} \gamma(s,t) \cdot F(\gamma(s,t))\, dt. \end{aligned}$$

With $\omega(s)$ as in (7.44), show that

$$\begin{aligned} \omega(s) = & \int_0^1 \frac{\partial \gamma}{\partial s}(s,t) \cdot DF(\gamma(s,t)) \frac{\partial \gamma}{\partial t}(s,t)\, dt \\ & + \int_0^1 \frac{\partial}{\partial t} \frac{\partial}{\partial s} \gamma(s,t) \cdot F(\gamma(s,t))\, dt. \end{aligned}$$

Conclude that $\omega(s) = \psi'(s)$ provided

$$DF(z) = DF(z)^t, \quad \forall\, z \in \Omega,$$

and compare this condition with (7.38).

8. Morera's theorem, the Schwarz reflection principle, and Goursat's theorem

Let Ω be a connected open set in $\mathbb{C}$. We have seen that if $f : \Omega \to \mathbb{C}$ is holomorphic, i.e., $f \in C^1(\Omega)$ and f is complex-differentiable, then the Cauchy integral theorem and the Cauchy integral formula hold for f, and hence $f \in C^\infty(\Omega)$ and f' is also holomorphic. Here we will establish a converse of the Cauchy integral theorem, known as Morera's theorem.

Theorem 8.1. *Assume $g : \Omega \to \mathbb{C}$ is continuous and*

$$\int_\gamma g(z)\, dz = 0 \tag{8.1}$$

whenever $\gamma = \partial R$ and $R \subset \Omega$ is a rectangle (with sides parallel to the real and imaginary axes). Then g is holomorphic.

Proof. Since the property of being holomorphic is local, there is no loss of generality in assuming Ω is a rectangle. Fix $\alpha = a + ib \in \Omega$. Given $z = x + iy \in \Omega$, let $\gamma_{\alpha z}$ and $\sigma_{\alpha z}$ be the piecewise linear paths from α to z described below (7.4); cf. Fig. 7.1. That is, $\gamma_{\alpha z}$ goes vertically from $a + ib$ to $a + iy$, then horizontally from $a + iy$ to $x + iy$, and $\sigma_{\alpha z}$ goes horizontally from $a + ib$ to $x + ib$, then vertically from $x + ib$ to $x + iy$. Now set

$$f(z) = \int_{\gamma_{\alpha z}} g(\zeta)\, d\zeta = i \int_b^y g(a + is)\, ds + \int_a^x g(t + iy)\, dt. \tag{8.2}$$

By (8.1), we also have

$$f(z) = \int_{\sigma_{\alpha z}} g(\zeta)\, d\zeta = \int_a^x g(s + ib)\, ds + i \int_b^y g(x + it)\, dt. \tag{8.2A}$$

Applying $\partial/\partial x$ to (8.2) gives (as in (1.53))

$$\frac{\partial f}{\partial x}(z) = g(z). \tag{8.3}$$

Similarly, applying $\partial/\partial y$ to (8.2A) gives

$$\frac{\partial f}{\partial y}(z) = ig(z). \tag{8.4}$$

This shows that $f : \Omega \to \mathbb{C}$ is C^1 and satisfies the Cauchy-Riemann equations. Hence f is holomorphic and $f'(z) = g(z)$. Thus g is holomorphic, as asserted.

Morera's theorem helps prove an important result known as the Schwarz reflection principle, which we now discuss. Assume $\Omega \subset \mathbb{C}$ is an open set that is symmetric about the real axis, i.e.,

$$z \in \Omega \Longrightarrow \overline{z} \in \Omega. \tag{8.5}$$

Say $L = \Omega \cap \mathbb{R}$, and set $\Omega^\pm = \{z \in \Omega : \pm \operatorname{Im} z > 0\}$.

Proposition 8.2. *In the set-up above, assume $f : \Omega^+ \cup L \to \mathbb{C}$ is continuous, holomorphic in Ω^+, and real valued on L. Define $g : \Omega \to \mathbb{C}$ by*

$$
\begin{aligned}
g(z) = f(z), &\quad z \in \Omega^+ \cup L, \\
\overline{f(\overline{z})}, &\quad z \in \Omega^-.
\end{aligned}
\tag{8.6}
$$

Then g is holomorphic on Ω.

Proof. It is readily verified that g is C^1 on Ω^- and satisfies the Cauchy-Riemann equation there, so g is holomorphic on $\Omega \setminus L$. Also it is clear that g is continuous on Ω. To see that g is holomorphic on all of Ω, we show that g satisfies (8.1) whenever $\gamma = \partial R$ and $R \subset \Omega$ is a (closed) rectangle. If $R \subset \Omega^+$ or $R \subset \Omega^-$ this is clear. If $R \subset \Omega^+ \cup L$, it follows by the continuity of g and a limiting argument (see Theorem 5.9); similarly we treat the case $R \subset \Omega^- \cup L$. Finally, if R intersects both Ω^+ and Ω^-, then we set $R = R^+ \cup R^-$ with $R^\pm = \Omega^\pm \cup L$, and note that

$$
\int\limits_{\partial R} g(z)\, dz = \int\limits_{\partial R^+} g(z)\, dz + \int\limits_{\partial R^-} g(z)\, dz,
$$

to finish the proof.

Remark. For a stronger version of the Schwarz reflection principle, see Proposition 13.9.

We next apply Morera's theorem to the proof of a result of E. Goursat, described as follows. If $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$, we have defined f to be holomorphic provided $f \in C^1(\Omega)$ and f is complex-differentiable. Goursat's theorem states that the hypothesis $f \in C^1(\Omega)$ can be dispensed with.

Theorem 8.3. *If $f : \Omega \to \mathbb{C}$ is complex-differentiable at each point of Ω, then f is holomorphic, so $f \in C^1(\Omega)$, and in fact $f \in C^\infty(\Omega)$.*

Proof. We will show that the hypothesis yields

$$
\int\limits_{\partial R} f(z)\, dz = 0 \tag{8.7}
$$

for every rectangle $R \subset \Omega$. The conclusion then follows from Morera's theorem.

Given a rectangle $R \subset \Omega$, set $a = \int_{\partial R} f(z)\, dz$. Divide R into 4 equal rectangles. The integral of $f(z)\, dz$ over their boundaries sums to a. Hence one of them (call it R_1) must have the property that

$$
\Bigl| \int\limits_{\partial R_1} f(z)\, dz \Bigr| \ge \frac{|a|}{4}. \tag{8.8}
$$

Divide R_1 into four equal rectangles. One of them (call it R_2) must have the property that

$$\Bigl|\int_{\partial R_2} f(z)\,dz\Bigr| \ge 4^{-2}\,|a|. \tag{8.9}$$

Continue, obtaining nested rectangles R_k, with perimeter ∂R_k of length $2^{-k}\ell(\partial R) = 2^{-k}b$, such that

$$\Bigl|\int_{\partial R_k} f(z)\,dz\Bigr| \ge 4^{-k}\,|a|. \tag{8.10}$$

The rectangles R_k shrink to a point; call it p. Since f is complex-differentiable at p, we have

$$f(z) = f(p) + f'(p)(z-p) + \Phi(z), \tag{8.11}$$

with

$$|\Phi(z)| = o(|z-p|). \tag{8.12}$$

In particular,

$$\sup_{z\in\partial R_k} \frac{|\Phi(z)|}{|z-p|} = \delta_k \to 0, \quad \text{as} \quad k\to\infty. \tag{8.13}$$

Now it is directly verifiable, e.g., via (1.45), that

$$\int_{\partial R_k} dz = 0, \qquad \int_{\partial R_k} z\,dz = 0. \tag{8.14}$$

Hence, with δ_k as in (8.13),

$$\Bigl|\int_{\partial R_k} f(z)\,dz\Bigr| = \Bigl|\int_{\partial R_k} \Phi(z)\,dz\Bigr| \le C\delta_k\cdot 2^{-k}\cdot 2^{-k}, \tag{8.15}$$

since $|z-p| \le C\,2^{-k}$ for $z\in\partial R_k$, and the length of ∂R_k is $\le C\,2^{-k}$. Comparing (8.10) and (8.15), we see that $|a| \le C\delta_k$ for all k, and hence $a=0$. This establishes (8.7) and hence proves Goursat's theorem.

Exercises

1. Let $\Omega \subset \mathbb{C}$ be a connected domain. Suppose γ is a smooth curve in Ω, and $\Omega \setminus \gamma$ has two connected pieces, say $\Omega_\pm$. Assume g is continuous on Ω, and holomorphic on Ω_+ and on Ω_-. Show that g is holomorphic on Ω.
Hint. Verify the hypotheses of Morera's theorem.

2. Suppose f is holomorphic on the semidisk $|z| < 1$, $\mathrm{Im}\, z > 0$, continuous on its closure, and real valued on the semicircle $|z| = 1$, $\mathrm{Im}\, z > 0$. Show that setting

$$\begin{aligned} g(z) = f(z), &\qquad |z| \le 1, \ \mathrm{Im}\, z > 0, \\ \overline{f(1/\overline{z})}, &\qquad |z| > 1, \ \mathrm{Im}\, z > 0, \end{aligned}$$

defines a holomorphic function on the upper half-plane $\mathrm{Im}\, z > 0$.

3. Take $a > 1$. Suppose that f is holomorphic (and nowhere vanishing) on the annulus $1 < |z| < a$, continuous on its closure, and $|f| = 1$ on the circle $|z| = 1$. Show that setting

$$\begin{aligned} g(z) = f(z), &\qquad 1 \le |z| \le a, \\ \frac{1}{\overline{f(1/\overline{z})}}, &\qquad 1/a \le |z| < 1, \end{aligned}$$

defines a holomorphic function on the annulus $1/a < |z| < a$.

4. The proof of Proposition 8.2 used the assertion that if $f : \Omega^+ \to \mathbb{C}$ is holomorphic and $g(z) = \overline{f(\overline{z})}$, then g is holomorphic on $\Omega^- = \{z \in \mathbb{C} : \overline{z} \in \Omega^+\}$. Prove this.
Hint. Given $z_0 \in \Omega^+$, write $f(z) = \sum_{k\ge 0} a_k (z - z_0)^k$ on a neighborhood of z_0 in Ω^+. Then produce a power series for g about $\overline{z}_0$.

5. Given $f : \Omega \to \mathbb{C}$, we say f is *antiholomorphic* if $\overline{f}$ is holomorphic, where $\overline{f}(z) = \overline{f(z)}$. Let $g : \mathcal{O} \to \Omega$, with $\mathcal{O}$ and Ω open in $\mathbb{C}$. Prove the following:
(a) f holomorphic, g antiholomorphic $\Longrightarrow$ $f \circ g$ antiholomorphic.
(b) f antiholomorphic, g holomorphic $\Longrightarrow$ $f \circ g$ antiholomorphic.
(c) f antiholomorphic, g antiholomorphic $\Longrightarrow$ $f \circ g$ holomorphic.

6. Let $\Omega \subset \mathbb{C}$ be open. Assume $f_k : \Omega \to \mathbb{C}$ are holomorphic and $f_k \to f$ uniformly on Ω. Show that f is holomorphic on Ω.

7. Define $f : \mathbb{R} \to \mathbb{C}$ by

$$\begin{aligned} f(x) = x^2 \sin \frac{1}{x}, &\quad x \ne 0, \\ 0, &\quad x = 0. \end{aligned}$$

Show that f is differentiable on $\mathbb{R}$, but f' is not continuous on $\mathbb{R}$. Contrast this with Theorem 8.3.

8. Define $f : \mathbb{C} \setminus 0 \to \mathbb{C}$ by

$$f(z) = z^2 \sin \frac{1}{z}, \quad z \neq 0.$$

Show that f is not bounded on $\{z \in \mathbb{C} : 0 < |z| < 1\}$.

9. Infinite products

In previous sections, we have seen the usefulness of infinite series representations, particularly power series, as a tool in the analysis of holomorphic functions. Here we take up infinite products, as another useful tool. We start with infinite products of numbers, before proceeding to infinite products of functions.

We first look at infinite products of the form

$$\prod_{k=1}^{\infty}(1+a_k). \tag{9.1}$$

Disregarding cases where one or more factors $1+a_k$ vanish, the convergence of $\prod_{k=1}^{M}(1+a_k)$ as $M \to \infty$ amounts to the convergence

$$\lim_{M\to\infty} \prod_{k=M}^{N}(1+a_k) = 1, \quad \text{uniformly in} \;\; N > M. \tag{9.2}$$

In particular, we require $a_k \to 0$ as $k \to \infty$. To investigate when (9.2) happens, write

$$\begin{aligned} \prod_{k=M}^{N}(1+a_k) &= (1+a_M)(1+a_{M+1})\cdots(1+a_N) \\ &= 1 + \sum_j a_j + \sum_{j_1<j_2} a_{j_1}a_{j_2} + \cdots + a_M\cdots a_N, \end{aligned} \tag{9.3}$$

where, e.g., $M \le j_1 < j_2 \le N$. Hence

$$\begin{aligned} \Bigl|\prod_{k=M}^{N}(1+a_k) - 1\Bigr| &\le \sum_j |a_j| + \sum_{j_1<j_2} |a_{j_1}a_{j_2}| + \cdots + |a_M\cdots a_N| \\ &= \prod_{k=M}^{N}(1+|a_k|) - 1 \\ &= b_{MN}, \end{aligned} \tag{9.4}$$

the last identity defining b_{MN}. Our task is to investigate when $b_{MN} \to 0$ as $M \to \infty$, uniformly in $N > M$. To do this, we note that

$$\begin{aligned} \log(1+b_{MN}) &= \log \prod_{k=M}^{N}(1+|a_k|) \\ &= \sum_{k=M}^{N} \log(1+|a_k|), \end{aligned} \tag{9.5}$$

and use the facts

$$
\begin{aligned}
x \geq 0 &\Longrightarrow \log(1+x) \leq x, \\
0 \leq x \leq 1 &\Longrightarrow \log(1+x) \geq \frac{x}{2}.
\end{aligned} \tag{9.6}
$$

Assuming $a_k \to 0$ and taking M so large that $k \geq M \Rightarrow |a_k| \leq 1/2$, we have

$$
\frac{1}{2} \sum_{k=M}^{N} |a_k| \leq \log(1 + b_{MN}) \leq \sum_{k=M}^{N} |a_k|, \tag{9.7}
$$

and hence

$$
\lim_{M\to\infty} b_{MN} = 0, \quad \text{uniformly in } N > M \Longleftrightarrow \sum_k |a_k| < \infty. \tag{9.8}
$$

Consequently,

$$
\begin{aligned}
\sum_k |a_k| < \infty &\Longrightarrow \prod_{k=1}^{\infty} (1 + |a_k|) \quad \text{converges} \\
&\Longrightarrow \prod_{k=1}^{\infty} (1 + a_k) \quad \text{converges.}
\end{aligned} \tag{9.9}
$$

Another consequence of (9.8) is the following:

$$
\text{If } 1 + a_k \neq 0 \text{ for all } k, \text{ then } \sum |a_k| < \infty \Rightarrow \prod_{k=1}^{\infty} (1 + a_k) \neq 0. \tag{9.10}
$$

See Exercise 3 below for more on this.

We can replace the sequence (a_k) of complex numbers by a sequence (f_k) of holomorphic functions, and deduce from the estimates above the following.

Proposition 9.1. *Let $f_k : \Omega \to \mathbb{C}$ be holomorphic. Assume that for each compact set $K \subset \Omega$ there exist $M_k(K)$ such that*

$$
\sup_{z\in K} |f_k(z)| \leq M_k(K), \quad \textit{and} \quad \sum_k M_k(K) < \infty. \tag{9.11}
$$

Then we have a convergent infinite product

$$
\prod_{k=1}^{\infty} (1 + f_k(z)) = F(z). \tag{9.12}
$$

In fact,

$$\prod_{k=1}^{n}(1+f_k(z)) \longrightarrow F(z), \quad as \quad n \to \infty, \tag{9.13}$$

uniformly on compact subsets of Ω. *Thus* F *is holomorphic on* Ω. *If* $z_0 \in \Omega$ *and* $1+f_k(z_0) \neq 0$ *for all* k, *then* $F(z_0) \neq 0$.

Now assume $f_k, g_k : \Omega \to \mathbb{C}$ are both holomorphic and assume, in addition to (9.11), that $\sup_K |g_k| \le M_k(K)$. Thus one has a convergent inner product

$$\prod_{k=1}^{\infty}(1+g_k(z)) = G(z), \tag{9.14}$$

with G holomorphic on Ω. Note that

$$\begin{gathered}(1+f_k(z))(1+g_k(z)) = 1+h_k(z), \\ h_k(z) = f_k(z)+g_k(z)+f_k(z)g_k(z),\end{gathered} \tag{9.15}$$

and

$$\sup_{z\in K} |h_k(z)| \le 2M_k(K) + M_k(K)^2 \le C(K)M_k(K), \tag{9.16}$$

where $C(K) = 2+\max_k M_k(K)$. It follows that Proposition 9.1 also applies to $h_k(z)$, and we have the convergent infinite product

$$\prod_{k=1}^{\infty}(1+h_k(z)) = H(z), \tag{9.17}$$

with H holomorphic on Ω. Since we clearly have, for $n \in \mathbb{N}$,

$$\prod_{k=1}^{n}(1+f_k(z))(1+g_k(z)) = \prod_{k=}^{n}(1+f_k(z)) \cdot \prod_{k=1}^{n}(1+g_k(z)), \tag{9.18}$$

we have

$$H(z) = F(z)G(z). \tag{9.19}$$

We present some examples to which Proposition 9.1 applies. We start with

$$S(z) = z\prod_{k=1}^{\infty}\Bigl(1-\frac{z^2}{k^2}\Bigr), \tag{9.20}$$

to which (9.11) applies with $f_k(z) = -z^2/k^2$ and $M_k(K) = R^2/k^2$ for $K = \{z \in \mathbb{C} : |z| \le R\}$. By Proposition 9.1, S is holomorphic on all of $\mathbb{C}$, and

$$S(z) = 0 \Longleftrightarrow z \in \mathbb{Z}. \tag{9.21}$$

Also, all the zeros of $S(z)$ are simple.

We next seek an infinite product representation of a function that is holomorphic on $\mathbb{C}$ and whose zeros are precisely the elements of $\mathbb{N} = \{1, 2, 3, \dots\}$, all simple. We might want to try $\prod_{k \ge 1}(1 - z/k)$, but the hypothesis (9.11) fails for $f_k(z) = -z/k$. We fix this by taking

$$G(z) = \prod_{k=1}^{\infty} \Bigl(1 - \frac{z}{k}\Bigr) e^{z/k}, \tag{9.22}$$

which has the form (9.12) with

$$1 + f_k(z) = \Bigl(1 - \frac{z}{k}\Bigr) e^{z/k}. \tag{9.23}$$

To see that (9.11) applies, note that

$$e^w = 1 + w + R(w), \quad |w| \le 1 \Rightarrow |R(w)| \le C|w|^2. \tag{9.24}$$

Hence

$$\begin{aligned} \Bigl(1 - \frac{z}{k}\Bigr) e^{z/k} &= \Bigl(1 - \frac{z}{k}\Bigr)\Bigl(1 + \frac{z}{k} + R\Bigl(\frac{z}{k}\Bigr)\Bigr) \\ &= 1 - \frac{z^2}{k^2} + \Bigl(1 - \frac{z}{k}\Bigr) R\Bigl(\frac{z}{k}\Bigr). \end{aligned} \tag{9.25}$$

Hence (9.23) holds with

$$f_k(z) = -\frac{z^2}{k^2} + \Bigl(1 - \frac{z}{k}\Bigr) R\Bigl(\frac{z}{k}\Bigr), \tag{9.26}$$

so

$$|f_k(z)| \le C\Bigl|\frac{z}{k}\Bigr|^2 \quad \text{for} \quad k \ge |z|, \tag{9.27}$$

which yields (9.11). Hence $G(z)$ in (9.22) is holomorphic on $\mathbb{C}$, and

$$G(z) = 0 \Longleftrightarrow z \in \mathbb{N}, \tag{9.28}$$

and all the zeros of $G(z)$ are simple. Note that $S(z)$ in (9.20) satisfies

$$S(z) = zG(z)G(-z), \tag{9.29}$$

as one sees by applying (9.18).

Returning to $S(z)$, defined by (9.20), we see that a familiar function that satisfies (9.21) is $\sin \pi z$. Noting that $\lim_{z \to 0} S(z)/z = 1$, we are tempted to compare $S(z)$ to

$$s(z) = \frac{1}{\pi} \sin \pi z. \tag{9.30}$$

We assert that $S(z) = s(z)$. To begin the demonstration, we note that the identity $\sin(z - \pi) = -\sin z$ is equivalent to $s(z - 1) = -s(z)$, and claim that $S(z)$ satisfies the same identity.

Lemma 9.2. *For $S(z)$ as in (9.20),*

$$S(z-1) = -S(z). \tag{9.31}$$

Proof. We have $S(z) = \lim_{n\to\infty} S_n(z)$, where

$$\begin{aligned} S_n(z) &= z \prod_{k=1}^{n} \Big(1 - \frac{z^2}{k^2}\Big) \\ &= z \prod_{k=1}^{n} \Big(1 - \frac{z}{k}\Big)\Big(1 + \frac{z}{k}\Big) \\ &= z \prod_{k=1}^{n} \frac{k-z}{k} \cdot \frac{k+z}{k} \\ &= \frac{(-1)^n}{(n!)^2} (z-n)(z-n+1)\cdots z(z+1)\cdots(z+n-1)(z+n). \end{aligned} \tag{9.32}$$

Replacing z by $z-1$ yields the identity

$$S_n(z-1) = \frac{z-n-1}{z+n} S_n(z), \tag{9.33}$$

and letting $n \to \infty$ yields (9.31).

To proceed, we form

$$f(z) = \frac{1}{S(z)} - \frac{1}{s(z)}, \tag{9.34}$$

which is holomorphic on $\mathbb{C} \setminus \mathbb{Z}$ and satisfies

$$f(z-1) = -f(z). \tag{9.35}$$

Furthermore, we have, for $z \in \mathbb{C}$,

$$S(z) = zH(z), \quad s(z) = zh(z), \tag{9.36}$$

with H and h holomorphic on $\mathbb{C}$ and $H(0) = h(0) = 1$. Hence, on some neighborhood $\mathcal{O}$ of 0,

$$\frac{1}{H(z)} = 1 + zA(z), \quad \frac{1}{h(z)} = 1 + za(z), \tag{9.37}$$

with A and a holomorphic on $\mathcal{O}$. Consequently, on $\mathcal{O} \setminus 0$,

$$\frac{1}{S(z)} - \frac{1}{s(z)} = \frac{1}{z}(1 + zA(z)) - \frac{1}{z}(1 + za(z)) = A(z) - a(z). \tag{9.38}$$

It follows that we can set $f(0) = A(0) - a(0)$ and have f holomorphic on a neighborhood of 0. Using (9.35), we can set $f(-k) = (-1)^k[A(0 - a(0)]$, for each $k \in \mathbb{Z}$, and we get

$$f : \mathbb{C} \longrightarrow \mathbb{C}, \quad \text{holomorphic}, \tag{9.39}$$

such that (9.34) holds on $\mathbb{C} \setminus \mathbb{Z}$. (In language to be introduced in §11, one says the singularities of (9.34) at points of $\mathbb{Z}$ are *removable*.) The following result will allow us to show that $f \equiv 0$ on $\mathbb{C}$.

Lemma 9.3. *We have*

$$f(z) \longrightarrow 0, \quad as \quad |z| \to \infty, \tag{9.40}$$

uniformly on the set

$$\{z \in \mathbb{C} : 0 \le \operatorname{Re} z \le 1\}. \tag{9.41}$$

Proof. It suffices to show that

$$|S(z)|,\ |s(z)| \longrightarrow \infty, \quad \text{as} \quad |z| \to \infty, \tag{9.42}$$

uniformly on the set (9.41). Since

$$\sin(x+iy) = \frac{1}{2i}(e^{-y+ix} - e^{y-ix}), \tag{9.43}$$

this is clear for $s(z)$. As for $S(z)$, given by (9.20), we have

$$\Bigl|1 - \frac{z^2}{k^2}\Bigr| \ge 1 + \frac{y^2 - x^2}{k^2} \ge 1 + \frac{y^2-1}{k^2}, \quad \text{for} \quad |x| \le 1, \tag{9.44}$$

with $z = x + iy$, so

$$|\operatorname{Re} z| \le 1,\ |\operatorname{Im} z| \ge 1 \Longrightarrow |S(z)| \ge |z|. \tag{9.45}$$

This gives (9.42).

Having Lemma 9.3, we deduce from $f(z-1) = -f(z)$ that f is bounded on $\mathbb{C}$, hence constant, and (9.40) implies the constant is 0. This concludes the proof that $S(z) \equiv s(z)$, which we formalize:

Proposition 9.4. *For $z \in \mathbb{C}$,*

$$\sin \pi z = \pi z \prod_{k=1}^{\infty} \Bigl(1 - \frac{z^2}{k^2}\Bigr). \tag{9.46}$$

We will see other derivations of (9.46) in (18.21) and (S.36).

In parallel with (9.31), it is also of interest to relate $G(z-1)$ to $G(z)$, for $G(z)$ given by (9.22), that is,

$$G(z) = \lim_{n\to\infty} G_n(z), \quad G_n(z) = \prod_{k=1}^{n} \Bigl(1 - \frac{z}{k}\Bigr) e^{z/k}. \tag{9.47}$$

Calculations parallel to (9.32)–(9.33) give

$$
\begin{aligned}
G_n(z-1) &= \frac{G_n(z)}{z-1}(z-n-1)\exp\Bigl(-\sum_{k=1}^{n}\frac{1}{k}\Bigr) \\
&= -\frac{G_n(z)}{z-1}\,\frac{n+1-z}{n+1}\,e^{-\gamma_n},
\end{aligned}
\tag{9.48}
$$

where

$$
\gamma_n = \sum_{k=1}^{n}\frac{1}{k} - \log(n+1). \tag{9.49}
$$

As $n \to \infty$, there is a limit

$$
\gamma_n \longrightarrow \gamma, \tag{9.50}
$$

and passing to the limit $n \to \infty$ in (9.48) yields

$$
G(z-1) = -\frac{G(z)}{z-1}\,e^{-\gamma}. \tag{9.51}
$$

The number γ in (9.50) is called *Euler's constant*. It is discussed at length in §18 (and in Appendix J), where $G(z)$ is related to the Euler gamma function (cf. (18.19)).

The factor $(1-z)e^z$ that appears (evaluated at z/k) in (9.22) is the first in a sequence of factors that figure in more general infinite product expansions. We set $E(z,0) = 1-z$, and, for $p \in \mathbb{N}$,

$$
E(z,p) = (1-z)\exp\sum_{k=1}^{p}\frac{z^k}{k}. \tag{9.52}
$$

Noting that

$$
\log\frac{1}{1-z} = \sum_{k=1}^{\infty}\frac{z^k}{k}, \tag{9.53}
$$

for $|z| < 1$, we see that

$$
\log E(z,p) = -\sum_{k=p+1}^{\infty}\frac{z^k}{k}, \tag{9.54}
$$

for $|z| < 1$, hence

$$
E(z,p) - 1 = O(z^{p+1}). \tag{9.55}
$$

In this notation, we can write (9.22) as

$$G(z)=\prod_{k=1}^{\infty}E\Bigl(\frac{z}{k},1\Bigr). \tag{9.56}$$

Note that, if instead we take the product over $k\in\mathbb{Z}\setminus 0$, the exponential factors cancel, and

$$\prod_{k\in\mathbb{Z}\setminus 0}E\Bigl(\frac{z}{k},1\Bigr)=\prod_{k=1}^{\infty}\Bigl(1-\frac{z^2}{k^2}\Bigr), \tag{9.57}$$

bringing us back to (9.20) (and to (9.29)).

We now look at some examples where $E(z,2)$ plays a role. Pick two numbers $\alpha,\beta\in\mathbb{C}$, linearly independent over $\mathbb{R}$, and consider the lattice

$$\Lambda=\{m\alpha+n\beta:m,n\in\mathbb{Z}\}\subset\mathbb{C}. \tag{9.58}$$

We seek a holomorphic function on $\mathbb{C}$ whose zeros are precisely the points of Λ (and are all simple). The infinite product

$$z\prod_{\omega\in\Lambda\setminus 0}\Bigl(1-\frac{z}{\omega}\Bigr) \tag{9.59}$$

is not convergent, nor do we get convergence upon replacing the factors by $E(z/\omega,1)$. Instead, we consider

$$H(z)=z\prod_{\omega\in\Lambda\setminus 0}E\Bigl(\frac{z}{\omega},2\Bigr). \tag{9.60}$$

By (9.55),

$$E\Bigl(\frac{z}{\omega},2\Bigr)=1+f_\omega(z),\quad |f_\omega(z)|\le C\frac{|z|^3}{|\omega|^3}, \tag{9.61}$$

and one can verify (e.g., via the integral test) that

$$\sum_{\omega\in\Lambda\setminus 0}|\omega|^{-3}<\infty, \tag{9.62}$$

so (9.60) is a convergent infinite product, defining such a holomorphic function as described above.

We note an alternative formula for (9.60) (paralleling (9.57)), when

$$\Lambda=\mathbb{Z}^2=\{m+ni:m,n\in\mathbb{Z}\}. \tag{9.63}$$

This latice has the property that $\omega \in \Lambda \Rightarrow i\omega \in \Lambda$. Now, for $z \in \mathbb{C}$,

$$\begin{aligned} &E(z,2)E(iz,2)E(-z,2)E(-iz,2) \\ &= (1-z^4)e^{z+z^2/2}e^{iz-z^2/2}e^{-z+z^2/2}e^{-iz-z^2/2} \\ &= 1-z^4. \end{aligned} \tag{9.64}$$

Hence, for

$$H(z) = z \prod_{\omega \in \mathbb{Z}^2 \setminus 0} E\Big(\frac{z}{\omega}, 2\Big), \tag{9.65}$$

if we set

$$\Lambda^+ = \{m + ni : m \geq 0, n > 0\}, \tag{9.66}$$

then

$$\begin{aligned} H(z) &= z \prod_{\omega \in \Lambda^+} E\Big(\frac{z}{\omega}, 2\Big)E\Big(\frac{z}{i\omega}, 2\Big)E\Big(\frac{z}{-\omega}, 2\Big)E\Big(\frac{z}{-i\omega}, 2\Big) \\ &= z \prod_{\omega \in \Lambda^+} \Big(1 - \frac{z^4}{\omega^4}\Big), \end{aligned} \tag{9.67}$$

an infinite product whose convergence follows directly from

$$\sum_{\omega \in \Lambda^+} |\omega|^{-4} < \infty. \tag{9.68}$$

We have

$$H(z) = \lim_{k \to \infty} H_k(z), \quad H_k(z) = z \prod_{0 \leq m \leq k, 0 < n \leq k} \Big(1 - \frac{z^4}{\omega_{mn}^4}\Big), \tag{9.69}$$

where $\omega_{mn} = m + in$, and recalling the factorization of $1 - z^4$, we can rewrite this as

$$H_k(z) = z \prod_{|m|,|n| \leq k, m+in \neq 0} \Big(1 - \frac{z}{\omega_{mn}}\Big). \tag{9.70}$$

Proceeding beyond the use of $E(z, 2)$, we have the following.

Proposition 9.5. *Let z_k be a sequence in $\mathbb{C} \setminus 0$. Assume that $|z_k| \to \infty$, and furthermore that*

$$\sum_k |z_k|^{-(p+1)} < \infty, \tag{9.71}$$

for some $p \in \mathbb{Z}^+$. *Then*

$$f(z) = \prod_k E\Big(\frac{z}{z_k}, p\Big) \tag{9.72}$$

converges locally uniformly to a holomorphic function on $\mathbb{C}$. *The seros of* f *are precisely the points* z_k. *The multiplicity of a zero* w *of* f *is equal to the number of labels* k *such that* $z_k = w$.

The proof is again a straightforward consequence of (9.55), together with Proposition 9.1.

Holomorphic functions of the form (19.72) are said to be of finite order. We will return to this class. For now, we look at more general entire holomorphic functions on $\mathbb{C}$. For this study, it is convenient to replace (9.55) by the following refinement.

Proposition 9.6. *The function* $E(z, p)$, *defined in (9.52), satisfies*

$$|E(z,p) - 1| \le |z|^{p+1}, \quad \text{for} \ \ |z| \le 1. \tag{9.73}$$

Proof. To start, a calculation gives

$$\frac{d}{dz}(1 - E(z,p)) = -\frac{d}{dz}E(z,p) = z^p \exp\Big(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\Big). \tag{9.74}$$

This function has a power series whose coefficients are all ≥ 0. Hence the same holds for the holomorphic function

$$\varphi(z) = \frac{1 - E(z,p)}{z^{p+1}}. \tag{9.75}$$

Thus

$$\sup_{|z|\le 1} |\varphi(z)| = \varphi(1) = 1, \tag{9.76}$$

and we have (9.73).

With this, we prepare to obtain a result called the Weierstrass product formula.

Lemma 9.7. *Let* (z_k) *be a sequence in* $\mathbb{C} \setminus 0$. *Assume* $|z_k| \to \infty$, *and furthermore assume* $p_k \in \mathbb{Z}^+$ *has the property that*

$$\sum_{k\ge 1} \Big|\frac{r}{z_k}\Big|^{p_k+1} < \infty, \quad \forall\, r > 0. \tag{9.77}$$

Then the product

$$f(z) = \prod_{k\ge 1} E\Big(\frac{z}{z_k}, p_k\Big) \tag{9.78}$$

converges locally uniformly on $\mathbb{C}$ to a holomorphic function whose zeros (counting multiplicities) are given by the sequence z_k.

Proof. By (9.73),

$$\left|E\Bigl(\frac{z}{z_k},p_k\Bigr)-1\right|\le\left|\frac{z}{z_k}\right|^{p_k+1},\quad \text{if } |z|\le|z_k|. \tag{9.79}$$

By hypothesis, for $R<\infty$, $R\le|z_k|$ for all but finitely many k. The result then follows from Proposition 9.1.

The product on the right side of (9.78) is called a Weierstrass product. The following is the Weierstrass product theorem.

Proposition 9.8. *If (z_k) is any sequence in $\mathbb{C}$ such that $|z_k|\to\infty$, then there exists an entire function f with $\{z_k\}$ as its set of zeros, counted with multiplicity.*

Proof. Only finitely many z_k are zero (say m of them). Separate them out, so $|z_k|>0$ for $k\ge m+1$. If $R>0$, then there exists $K>m$ such that $|z_k|>2R$ for each $k>K$. Hence the series

$$\sum_{k=m+1}^{\infty}\left|\frac{z}{z_k}\right|^k \tag{9.80}$$

converges uniformly on $\{z:|z|\le R\}$. Thus the hypotheses of Lemma 9.7 hold. The resulting Weierstrass product

$$\prod_{k\ge m+1}E\Bigl(\frac{z}{z_k},p_k\Bigr) \tag{9.81}$$

converges, with $p_k=k-1$, and we obtain the advertised function as z^m times this Weierstrass product.

With these results in hand, we establish the following result, known as the Weierstrass factorization theorem for entire functions.

Proposition 9.9. *Let $f:\mathbb{C}\to\mathbb{C}$ be entire and not identically zero. Let f have a zero of order m at $z=0$, and let (z_k) be the sequence of other zeros of f, counted with multiplicity. Then there exist $p_k\in\mathbb{N}$ and a holomorphic function $h:\mathbb{C}\to\mathbb{C}$ such that*

$$f(z)=e^{h(z)}z^m\prod_{k\ge1}E\Bigl(\frac{z}{z_k},p_k\Bigr). \tag{9.82}$$

Here, p_k can be any sequence for which (9.77) holds.

Proof. The conclusion of Proposition 9.8 yields a holomorphic function $g(z)=z^m\prod_{k\ge1}E(z/z_k,p_k)$ with the required set of zeros. Then f/g is entire and nowhere vanishing, so we know that this has the form $e^{h(z)}$ for an entire function h.

We return to the setting of Proposition 9.5. More precisely, we assume (z_k) is a sequence in $\mathbb{C} \setminus 0$ and

$$\sum_k |z_k|^{-\rho} < \infty, \tag{9.83}$$

for some $\rho < \infty$. Say $p \le \rho < p+1$, with $p \in \mathbb{Z}^+$. We want to obtain a global estimate on the holomorphic function (9.72). This requires an estimate on the factors on the right side of (9.72), given as follows.

Lemma 9.10. *For $p \in \mathbb{Z}^+$ and $p \le \rho < p+1$,*

$$|E(z,p)| \le e^{A|z|^\rho}, \tag{9.84}$$

with $A = A_{p,\rho} < \infty$.

Proof. For $w \in \mathbb{C}$, $|w| < 1/2$, we have $|\log(1+w)| \le 2|w|$. This together with (9.73) gives $|\log E(z,p)| \le 2|z|^{p+1} \le 2|z|^\rho$, and hence (9.84) holds in this range for any $A \ge 2$.

If $|z| > 1/2$ and $k \le p$, then $|z|^k \le 2^{\rho-k}|z|^\rho$, so

$$\begin{aligned} \log|E(z,p)| &= \log|1-z| + \sum_{k=1}^p \frac{\operatorname{Re} z^k}{k} \\ &\le |z| + \sum_{k=1}^p |z|^k \\ &\le (p+1)2^\rho |z|^\rho, \end{aligned} \tag{9.85}$$

which gives (9.82) with $A = (p+1)2^\rho$, in this range.

Here is our global estimate:

Proposition 9.11. *If (z_k) is a sequence in $\mathbb{C} \setminus 0$ satisfying (9.83), and $p \le \rho < p+1$, then there exists $B < \infty$ such that the product*

$$f(z) = \prod_{k \ge 1} E\Big(\frac{z}{z_k}, p\Big) \tag{9.86}$$

satisfies

$$|f(z)| \le e^{B|z|^\rho}. \tag{9.87}$$

Proof. By (9.84),

$$|f(z)| \le \prod_{k \ge 1} e^{A|z/z_k|^\rho} = e^{B|z|^\rho}, \tag{9.88}$$

with

$$B = A \sum_{k \geq 1} |z_k|^{-\rho}. \tag{9.89}$$

An entire function f that satisfies an estimate of the form

$$|f(z)| \leq C e^{B|z|^\rho}, \quad \forall\, z \in \mathbb{C}, \tag{9.90}$$

for some $B, C, \rho < \infty$, is called an entire function of *finite order*. The infimum of all ρ for which such an estimate holds is called the order of f. Proposition 9.11 has a deep converse, called Hadamard's factorization theorem:

Theorem 9.12. *Assume f is an entire function of order σ, and $p \leq \sigma < p+1$ ($p \in \mathbb{Z}^+$). Then f can be written as*

$$f(z) = z^m e^{q(z)} \prod_{k \geq 1} E\Bigl(\frac{z}{z_k}, p\Bigr), \tag{9.91}$$

where $q(z)$ is a polynomial of degree at most p, and (z_k) is the set of zeros of f in $\mathbb{C} \setminus 0$, counted with multiplicity, whose associated series (9.83) converges for all $\rho > \sigma$.

Hadamard established this result as a tool for his famous proof of the prime number theorem. Other proofs of the prime number theorem, such as the one given in §19 of this text, do not use this factorization theorem, but it is a beautiful and powerful result in its own right. We provide a proof of Theorem 9.12 in Appendix S.

Exercises

1. Using (9.6), show that if $0 \le a_k \le 1$, for $k \ge M$, then

$$\frac{1}{2}\sum_{k=M}^{\infty} a_k \le \log \prod_{k=M}^{\infty} (1+a_k) \le \sum_{k=M}^{\infty} a_k.$$

2. Complement (9.6) with the estimates

$$x \le \log \frac{1}{1-x} \le 2x, \quad \text{for} \ \ 0 \le x \le \frac{1}{2}.$$

Use this to show that, if $0 \le a_k \le 1/2$ for $k \ge M$, then

$$\sum_{k=M}^{\infty} a_k \le \log \prod_{k=M}^{\infty} (1-a_k)^{-1} \le 2\sum_{k=M}^{\infty} a_k.$$

3. Show that, if $z \in \mathbb{C}$ and $|z| \le 1/2$, then

$$\Bigl|\log \frac{1}{1-z}\Bigr| \le 2|z|.$$

Use this to show that, if $a_k \in \mathbb{C}$ and $|a_k| \le 1/2$ for $k \ge M$, then

$$\Bigl|\log \prod_{k=M}^{\infty} (1-a_k)^{-1}\Bigr| \le 2\sum_{k=M}^{\infty} |a_k|.$$

4. Take γ_n as in (9.49). Show that $\gamma_n \nearrow$ and $0 < \gamma_n < 1$. Deduce that $\gamma = \lim_{n\to\infty} \gamma_n$ exists, as asserted in (9.50).

5. Show that

$$\prod_{n=1}^{\infty}\Bigl(1 - \frac{1}{4n^2}\Bigr) = \frac{2}{\pi}. \tag{9.A}$$

Hint. Take $z = 1/2$ in (9.46).

6. Show that, for all $z \in \mathbb{C}$,

$$\cos \frac{\pi z}{2} = \prod_{\text{odd}\, n\ge 1}\Bigl(1 - \frac{z^2}{n^2}\Bigr). \tag{9.B}$$

Hint. Use $\cos \pi z/2 = -\sin((\pi/2)(z-1))$ and (9.46) to obtain

$$\cos \frac{\pi z}{2} = \frac{\pi}{2}(1-z) \prod_{n=1}^{\infty} \Bigl(1 - \frac{(z-1)^2}{4n^2}\Bigr). \tag{9.C}$$

Use $(1-u^2) = (1-u)(1+u)$ to write the general factor in this infinite product as

$$\begin{aligned} &\Bigl(1 + \frac{1}{2n} - \frac{z}{2n}\Bigr)\Bigl(1 - \frac{1}{2n} + \frac{z}{2n}\Bigr) \\ &\qquad = \Bigl(1 - \frac{1}{4n^2}\Bigr)\Bigl(1 - \frac{z}{2n+1}\Bigr)\Bigl(1 + \frac{z}{2n-1}\Bigr), \end{aligned} \tag{9.D}$$

and obtain from (9.C) that

$$\cos \frac{\pi z}{2} = \frac{\pi}{2} \prod_{n=1}^{\infty} \Bigl(1 - \frac{1}{4n^2}\Bigr) \cdot \prod_{\text{odd}\, n \ge 1} \Bigl(1 - \frac{z}{n}\Bigr)\Bigl(1 + \frac{z}{n}\Bigr).$$

Deduce (9.B) from this and (9.A).

7. Show that

$$\frac{\sin \pi z}{\pi z} = \cos \frac{\pi z}{2} \cdot \cos \frac{\pi z}{4} \cdot \cos \frac{\pi z}{8} \cdots . \tag{9.E}$$

Hint. Make use of (9.46) and (9.B).

8. Take H and H_k as in (9.65)–(9.70). Show that

$$\begin{aligned} H_k(z) &= \alpha_k \prod_{|m|,|n| \le k} (z - \omega_{mn}), \\ \alpha_k &= \prod_{|m|,|n| \le k,\, m+in \ne 0} (-\omega_{mn})^{-1}. \end{aligned}$$

Using this, examine $H_k(z-1)$, and show that

$$H(z-1) = -e^{-\pi(z-1/2)} H(z).$$

Make a parallel computation for $H(z-i)$.
Hint. Modify the path from (9.32) to (9.33). Exercise 20 of §4 might be useful.

10. Uniqueness and analytic continuation

It is a central fact that a function holomorphic on a connected open set $\Omega \subset \mathbb{C}$ is uniquely determined by its values on any set S with an accumulation point in Ω, i.e., a point $p \in \Omega$ with the property that for all $\varepsilon > 0$, the disk $D_\varepsilon(p)$ contains infinitely many points in S. Phrased another way, the result is:

Proposition 10.1. *Let $\Omega \subset \mathbb{C}$ be open and connected, and let $f : \Omega \to \mathbb{C}$ be holomorphic. If $f = 0$ on a set $S \subset \Omega$ and S has an accumulation point $p \in \Omega$, then $f = 0$ on Ω.*

Proof. There exists $R > 0$ such that the disk $D_R(p) \subset \Omega$ and f has a convergent power series on $D_R(p)$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-p)^n. \tag{10.1}$$

If all $a_n = 0$, then $f = 0$ on $D_R(p)$. Otherwise, say a_j is the first nonzero coefficient, and write

$$f(z) = (z-p)^j g(z), \quad g(z) = \sum_{n=0}^{\infty} a_{j+n} (z-p)^n. \tag{10.2}$$

Now $g(p) = a_j \neq 0$, so there is a neighborhood U of p on which g is nonvanishing. Hence $f(z) \neq 0$ for $z \in U \setminus p$, contradicting the hypothesis that p is an accumulation point of S.

This shows that if $S^\#$ is the set of accumulation points in Ω of the zeros of f, then $S^\#$ is open. It is elementary that $S^\#$ is closed, so if Ω is connected and $S^\# \neq \emptyset$, then $S^\# = \Omega$, which implies f is identically zero.

To illustrate the use of Proposition 10.1, we consider the following Gaussian integral:

$$G(z) = \int_{-\infty}^{\infty} e^{-t^2+tz} \, dt. \tag{10.3}$$

It is easy to see that the integral is absolutely convergent for each $z \in \mathbb{C}$ and defines a continuous function of z. Furthermore, if γ is any closed curve on $\mathbb{C}$ (such as $\gamma = \partial R$ for some rectangle $R \subset \mathbb{C}$) then we can interchange order of integrals to get

$$\int_\gamma G(z) \, dz = \int_{-\infty}^{\infty} \int_\gamma e^{-t^2+tz} \, dz \, dt = 0, \tag{10.4}$$

the last identity by Cauchy's integral theorem. Then Morera's theorem implies that G is holomorphic on $\mathbb{C}$.

For $z = x$ *real* we can calculate (10.3) via elementary calculus. Completing the square in the exponent gives

$$G(x) = e^{x^2/4} \int_{-\infty}^{\infty} e^{-(t-x/2)^2}\, dt = e^{x^2/4} \int_{-\infty}^{\infty} e^{-t^2}\, dt. \tag{10.5}$$

To evaluate the remaining integral, which we denote I, we write

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-s^2-t^2}\, ds\, dt = \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r\, dr\, d\theta = \pi, \tag{10.6}$$

where $\int_0^\infty e^{-r^2} r\, dr$ is evaluated via the change of variable $\rho = r^2$. Thus $I = \sqrt{\pi}$, so

$$G(x) = \sqrt{\pi}\, e^{x^2/4}, \quad x \in \mathbb{R}. \tag{10.7}$$

Now we assert that

$$G(z) = \sqrt{\pi}\, e^{z^2/4}, \quad z \in \mathbb{C}, \tag{10.8}$$

since both sides are holomorphic on $\mathbb{C}$ and coincide on $\mathbb{R}$. In particuler, $G(iy) = \sqrt{\pi} e^{-y^2/4}$, for $y \in \mathbb{R}$, so we have

$$\int_{-\infty}^{\infty} e^{-t^2+ity}\, dt = \sqrt{\pi} e^{-y^2/4}, \quad y \in \mathbb{R}. \tag{10.9}$$

We next prove the following

Proposition 10.2. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain. Assume $f : \Omega \to \mathbb{C}$ is holomorphic and nowhere vanishing. Then there exists a holomorphic function g on Ω such that*

$$e^{g(z)} = f(z), \quad \forall\, z \in \Omega. \tag{10.10}$$

Proof. We may as well assume f is not constant. Take $p \in \Omega$ such that $f(p) \notin \mathbb{R}^- = (-\infty, 0]$. Let $\mathcal{O} \subset \Omega$ be a neighborhood of p such that

$$f : \mathcal{O} \longrightarrow \mathbb{C} \setminus \mathbb{R}^-. \tag{10.11}$$

We can define g on $\mathcal{O}$ by

$$g(z) = \log f(z), \quad z \in \mathcal{O}, \tag{10.12}$$

where $\log : \mathbb{C} \setminus \mathbb{R}^- \to \mathbb{C}$ is as in §4. Applying d/dz and using the chain rule gives

$$g'(z) = \frac{f'(z)}{f(z)}, \quad z \in \mathcal{O}. \tag{10.13}$$

Consequently

$$g(z) = \log f(p) + \int_p^z \frac{f'(\zeta)}{f(\zeta)}\, d\zeta, \tag{10.14}$$

for $z \in \mathcal{O}$, where we integrate along a path from p to z within $\mathcal{O}$. Now if f is nowhere vanishing, then f'/f is holomorphic on Ω, and if Ω is simply connected, then the integral on the right side is well defined for all $z \in \Omega$ and is independent of the choice of path from p to z, within Ω, and defines a holomorphic function on Ω. (See Exercise 13 of §5 and Exercise 8 of §7.)

Hence (10.14) gives a well defined holomorphic function on Ω. From (10.12), we have

$$e^{g(z)} = f(z), \quad \forall\, z \in \mathcal{O}, \tag{10.15}$$

and then Proposition 10.1 implies (10.10).

It is standard to denote the function produced above by $\log f(z)$, so

$$\log f(z) = \log f(p) + \int_p^z \frac{f'(\zeta)}{f(\zeta)}\, d\zeta, \tag{10.16}$$

under the hypotheses of Proposition 10.2. One says that $\log f(z)$ is extended from $\mathcal{O}$ to Ω by "analytic continuation." In the setting of Proposition 10.2, we can set

$$\sqrt{f(z)} = e^{(1/2)\log f(z)}, \quad z \in \Omega, \tag{10.17}$$

and, more generally, for $a \in \mathbb{C}$,

$$f(z)^a = e^{a \log f(z)}, \quad z \in \Omega, \tag{10.18}$$

These functions are hence analytically continued from $\mathcal{O}$ (where they have a standard definition from §4) to Ω.

Generally speaking, analytic continuation is a process of taking a holomorphic function f on some domain $\mathcal{O} \subset \mathbb{C}$ and extending it to a holomorphic function on a larger domain, $\Omega \supset \mathcal{O}$. (It would also be reasonable to call this process "holomorphic continuation," but this latter terminology seems not to be so popular.)

A number of means have been devised to produce analytic continuations of various functions. One is to use the Schwarz reflection principle, established in §8. Sometimes one can iterate this reflection construction. An important example of such an iterated reflection argument appears in §26, regarding a holomorphic covering of $\mathbb{C} \setminus \{0, 1\}$ by the unit disk.

In other cases we can analytically continue a function by establishing a functional equation. For example, as we will see in §18, we can define

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \quad \mathrm{Re}\, z > 0, \tag{10.19}$$

holomorphic in the right half plane, establish the identity

$$\Gamma(z+1) = z\Gamma(z) \tag{10.20}$$

for such z, and use this to analytically continue $\Gamma(z)$ to $\mathbb{C} \setminus \{0, -1, -2, \dots\}$. A more sophisticated functional equation is seen in §19 to lead to an analytic continuation of the Riemann zeta function

$$\zeta(z) = \sum_{k=1}^{\infty} k^{-z}, \quad \operatorname{Re} z > 1, \tag{10.21}$$

to $\mathbb{C} \setminus \{1\}$.

Another very important technique involves analytic continuation along a curve. The set-up is the following. Let $\Omega \subset \mathbb{C}$ be open and connected. Take $z_0, z_1 \in \Omega$, and suppose we have a continuous path

$$\gamma : [a,b] \longrightarrow \Omega, \quad \gamma(a) = z_0,\ \gamma(b) = z_1. \tag{10.22}$$

Assue f is holomorphic on a neighborhood of z_0. We say a chain along γ is a partition of $[a,b]$,

$$a = a_0 < a_1 < \cdots < a_{n+1} = b, \tag{10.23}$$

together with a collection of open, convex sets D_j satisfying

$$\gamma([a_j, a_{j+1}]) \subset D_j \subset \Omega, \quad \text{for } \ 0 \le j \le n. \tag{10.24}$$

Given $f = f_{D_0}$, holomorphic on D_0, we say an analytic continuation of f along this chain is a collection

$$\begin{aligned} &f_{D_j} : D_j \longrightarrow \mathbb{C} \ \text{ is holomorphic, such that} \\ &f_{D_j} = f_{D_{j+1}} \text{ on } D_j \cap D_{j+1}, \text{ for } 0 \le j \le n-1. \end{aligned} \tag{10.25}$$

We make the following definition. Let $\mathcal{O}_0$ and $\mathcal{O}_1$ be open convex neighborhoods of z_0 and z_1 in Ω, and let γ be a continuous path from z_0 to z_1, as in (10.22). Let f be holomorphic on $\mathcal{O}_0$. Assume there exists a chain $\{D_0, \dots, D_n\}$ along γ such that $D_0 = \mathcal{O}_0$ and $D_n = \mathcal{O}_1$, and an analytic continuation of f along this chain, as in (10.25). We set

$$f_\gamma : \mathcal{O}_1 \longrightarrow \mathbb{C}, \tag{10.26}$$

equal to f_{D_n} on $D_n = \mathcal{O}_1$. One has the following uniqueness result.

Proposition 10.3. *Given that $f : \mathcal{O}_0 \to \mathbb{C}$ is holomorphic, and that there is an analytic continuation of f along γ, as described above, the function f_γ in (10.28) is independent of the choice of chain along γ for which such an analytic continuation exists.*

Supplementing Proposition 10.3, we have the following key uniqueness result, known as the *monodromy theorem.* To formulate it, suppose we have a continuous family of curves

$$\gamma_s : [a,b] \longrightarrow \Omega, \quad \gamma_s(a) \equiv z_0,\ \gamma_s(b) \equiv z_1, \quad 0 \le s \le 1, \tag{10.27}$$

so $\gamma(s,t) = \gamma_s(t)$ defines a continuous map $\gamma : [0,1] \times [a,b] \to \Omega$. As above, assume $\mathcal{O}_0$ and $\mathcal{O}_1$ are convex open neighborhoods of z_0 and z_1 in Ω, and that f is holomorphic on $\mathcal{O}_0$.

Proposition 10.4. *Assume that for each $s \in [0,1]$ there is a chain along γ_s from $\mathcal{O}_0$ to $\mathcal{O}_1$ and an analytic continuation of f along the chain, producing*

$$f_{\gamma_s} : \mathcal{O}_1 \longrightarrow \mathbb{C}. \tag{10.28}$$

Then f_{γ_s} is independent of $s \in [0,1]$.

Propositions 10.3–10.4 apply to situations where one can analytically continue a function f, first known to be holomorphic on a neighborhood $\mathcal{O}_0$ of z_0. One of the most important class of cases where such analytic continuation can be done involves the solution of linear differential equations with holomorphic coefficients on a donain $\Omega \subset \mathbb{C}$. This is treated in §36, where one can find proofs of Propositions 10.3–10.4 (given there as Propositions 36.2–36.3).

We end this section with the following result, which is elementary but which pushes the concept of analytic continuation a bit. To state it, let $I = (a,b) \subset \mathbb{R}$ be an interval, and define a function

$$f : (a,b) \longrightarrow \mathbb{C} \tag{10.29}$$

to be *real analytic* provided that for each $x_0 \in (a,b)$, there exists $\varepsilon(x_0) > 0$ such that for $|x - x_0| < \varepsilon(x_0)$, $f(x)$ is given by a convergent power series

$$f(x) = \sum_{k=0}^{\infty} a_k(x_0)(x - x_0)^k. \tag{10.30}$$

With the next result, we analytically continue f to a complex neighborhood of I.

Proposition 10.5. *Let $f : (a,b) \to \mathbb{C}$ be real analytic. Then there is a neighborhood U of (a,b) in $\mathbb{C}$ and a holomorphic function*

$$F : U \longrightarrow \mathbb{C} \quad \textit{such that} \quad F = f \ \textit{on} \ (a,b). \tag{10.31}$$

Proof. For each $x_0 \in (a,b)$, let $U_{x_0} = D_{\varepsilon(x_0)}(x_0)$ be a disk in $\mathbb{C}$ centered at x_0 such that (10.30) holds for $x \in \mathbb{R} \cap D_{\varepsilon(x_0)}(x_0)$. We set

$$F_{x_0}(z) = \sum_{k=0}^{\infty} a_k(x_0)(z - x_0)^k, \quad z \in D_{\varepsilon(x_0)}(x_0), \tag{10.32}$$

and propose to take

$$\begin{aligned} U &= \bigcup_{x_0 \in (a,b)} D_{\varepsilon(x_0)}(x_0), \\ F &: U \longrightarrow \mathbb{C}, \quad F = F_{x_0} \ \text{ on } \ D_{\varepsilon(x_0)}(x_0). \end{aligned} \tag{10.33}$$

Now U, defined in (10.33), is an open subset of $\mathbb{C}$ containing (a,b). Clearly each F_{x_0} is holomorphic on $D_{\varepsilon(x_0)}(x_0)$. To show that F is well defined by (10.33), it suffices to show that, for $x_0, x_1 \in (a,b)$,

$$\begin{aligned} &D_{\varepsilon(x_0)}(x_0) \cap D_{\varepsilon(x_1)}(x_1) \neq \emptyset \\ &\Longrightarrow F_{x_0} = F_{x_1} \quad \text{on} \quad D_{\varepsilon(x_0)}(x_0) \cap D_{\varepsilon(x_1)}(x_1). \end{aligned} \tag{10.34}$$

In fact, $F_{x_0} = F_{x_1} = f$ on $D_{\varepsilon(x_0)}(x_0) \cap D_{\varepsilon(x_1)}(x_1) \cap \mathbb{R}$, so (10.34) follows from Proposition 10.1. This completes the proof of Proposition 10.5.

Exercises

1. Suppose $\Omega \subset \mathbb{C}$ is a connected region that is symmetric about the real axis, i.e., $z \in \Omega \Rightarrow \overline{z} \in \Omega$. If f is holomorphic on Ω and real valued on $\Omega \cap \mathbb{R}$, show that

$$f(z) = \overline{f(\overline{z})}. \tag{10.35}$$

Hint. Both sides are holomorphic. How are they related on $\Omega \cap \mathbb{R}$?

1A. Set $z^* = -\overline{z}$ and note that $z \mapsto z^*$ is reflection about the imaginary axis, just as $z \mapsto \overline{z}$ is reflection about the real axis. Suppose $\Omega \subset \mathbb{C}$ is a connected region that is symmetric about the imaginary axis, i.e., $z \in \Omega \Leftrightarrow z^* \in \Omega$. If f is holomorphic on Ω and is purely imaginary on $\Omega \cap i\mathbb{R}$, show that

$$f(z) = f(z^*)^*. \tag{10.36}$$

Hint. Show that $f(z^*)^* = -\overline{f(-\overline{z})}$ is holomorphic in z.

2. Let D be the unit disk centered at the origin. Assume $f : D \to \mathbb{C}$ is holomorphic and that f is real on $D \cap \mathbb{R}$ and purely imaginary on $D \cap i\mathbb{R}$. Show that f is *odd*, i.e., $f(z) = -f(-z)$.
Hint. Show that (10.35) and (10.36) both hold.

3. Let Ω be a simply connected domain in $\mathbb{C}$ and f a holomorphic function on Ω with the property that $f : \Omega \to \mathbb{C} \setminus \{-1, 1\}$. Assume f is not constant, and take $p \in \Omega$ such that $f(p) \notin \mathbb{C} \setminus \{(-\infty, 1] \cup [1, \infty)\}$, the set where $\sin^{-1}$ is defined; cf. (4.23)–(4.27). Show that $\sin^{-1} f(z)$ can be analytically continued from a small neighborhood of p to all of Ω, and

$$\sin^{-1} f(z) = \sin^{-1} f(p) + \int_p^z \frac{f'(\xi)}{\sqrt{1 - f(\xi)^2}}\, d\xi.$$

4. In the setting of Proposition 10.2, if $g(z)$ is given by (10.14), show directly that

$$\frac{d}{dz}\Big(e^{-g(z)} f(z)\Big) = 0,$$

and deduce that (10.10) holds, without appealing to Proposition 10.1.

5. Consider

$$I(a) = \int_0^\infty e^{-at^2}\, dt, \quad \operatorname{Re} a > 0.$$

Show that I is holomorphic in $\{a \in \mathbb{C} : \operatorname{Re} a > 0\}$. Show that

$$I(a) = \frac{\sqrt{\pi}}{2} a^{-1/2}.$$

Hint. Use a change of variable to evaluate $I(a)$ for $a \in (0, \infty)$.

6. Evaluate

$$\int_0^\infty e^{-bt^2} \cos t^2\, dt, \quad b > 0.$$

Make the evaluation explicit at $b = 1$.
Hint. Evaluate $I(b - i)$. Consult Exercise 20 of §4.

7. Show that

$$\lim_{b \searrow 0} I(b - i) = \frac{\sqrt{\pi}}{2} e^{\pi i/4}.$$

8. Show that

$$\lim_{R\to\infty} \int_0^R \cos t^2\, dt = I_c, \quad \lim_{R\to\infty} \int_0^R \sin t^2\, dt = I_s$$

exist. Given this result, use the Abelian theorem, Proposition R.6, to show that

$$\lim_{b \searrow 0} \int_0^\infty e^{-bt^2} \cos t^2\, dt = I_c, \quad \lim_{b \searrow 0} \int_0^\infty e^{-bt^2} \sin t^2\, dt = I_s.$$

Then use the result of Exercise 7 to show that

$$I_c + iI_s = \frac{\sqrt{\pi}}{2} e^{\pi i/4}, \quad \text{i.e.,} \quad I_c = I_s = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

REMARK. The quantities I_c and I_s are called *Fresnel integrals.*

9. Completing the square as in (10.5), show that

$$\int_0^\infty e^{-t^2 - \xi t}\, dt = e^{\xi^2/4} \Bigl[\frac{\sqrt{\pi}}{2} - \int_0^{\xi/2} e^{-x^2}\, dx \Bigr],$$

for $\xi > 0$. Extend this to an identity between functions holomorphic in $\xi \in \mathbb{C}$, replacing the last integral by

$$\int_0^{\xi/2} e^{-z^2}\, dz.$$

Deduce that, for $\xi \in \mathbb{R}$,

$$\int_0^\infty e^{-t^2 - i\xi t}\, dt = e^{-\xi^2/4}\Big[\frac{\sqrt{\pi}}{2} - i\int_0^{\xi/2} e^{y^2}\, dy\Big],$$

and hence

$$\int_0^\infty e^{-t^2}\sin \xi t\, dt = e^{-\xi^2/4}\int_0^{\xi/2} e^{y^2}\, dy.$$

11. Singularities

The function $1/z$ is holomorphic on $\mathbb{C} \setminus 0$ but it has a *singularity* at $z = 0$. Here we will make a further study of singularities of holomorphic functions.

A point $p \in \mathbb{C}$ is said to be an isolated singularity of f if there is a neighborhood U of p such that f is holomorphic on $U \setminus p$. The singularity is said to be *removable* if there exists a holomorphic function $\tilde{f}$ on U such that $\tilde{f} = f$ on $U \setminus p$. Clearly 0 is not a removable singularity for $1/z$, since this function is not bounded on any set $D_\varepsilon(0) \setminus 0$. This turns out to be the only obstruction to removability, as shown in the following result, known as the removable singularities theorem.

Theorem 11.1. *If $p \in \Omega$ and f is holomorphic on $\Omega \setminus p$ and bounded, then p is a removable singularity.*

Proof. Consider the function $g : \Omega \to \mathbb{C}$ defined by

$$\begin{aligned} g(z) &= (z-p)^2 f(z), \quad z \in \Omega \setminus p, \\ g(p) &= 0. \end{aligned} \tag{11.1}$$

That f is bounded implies g is continuous on Ω. Furthermore, g is seen to be complex-differentiable at each point of Ω:

$$\begin{aligned} g'(z) &= 2(z-p)f(z) + (z-p)^2 f'(z), \quad z \in \Omega \setminus p, \\ g'(p) &= 0. \end{aligned} \tag{11.2}$$

Thus (by Goursat's theorem) g is holomorphic on Ω, so on a neighborhood U of p it has a convergent power series:

$$g(z) = \sum_{n=0}^{\infty} a_n (z-p)^n, \quad z \in U. \tag{11.3}$$

Since $g(p) = g'(p) = 0$, $a_0 = a_1 = 0$, and we can write

$$g(z) = (z-p)^2 h(z), \quad h(z) = \sum_{n=0}^{\infty} a_{2+n}(z-p)^n, \quad z \in U. \tag{11.4}$$

Comparison with (11.1) shows that $h(z) = f(z)$ on $U \setminus p$, so setting

$$\tilde{f}(z) = f(z), \quad z \in \Omega \setminus p, \quad \tilde{f}(p) = h(p) \tag{11.5}$$

defines a holomorphic function on Ω and removes the singularity.

By definition an isolated singularity p of a holomorphic function f is a *pole* if $|f(z)| \to \infty$ as $z \to p$. Say f is holomorphic on $\Omega \setminus p$ with pole at p. Then there is a disk U centered at p such that $|f(z)| \geq 1$ on $U \setminus p$. Hence $g(z) = 1/f(z)$ is holomorphic on $U \setminus p$ and $g(z) \to 0$ as $z \to p$. Thus p is a removable singularity for g. Let us also denote by g the holomorphic extension, with $g(p) = 0$. Thus g has a convergent power series expansion valid on U:

$$g(z) = \sum_{n=k}^{\infty} a_n (z-p)^n, \tag{11.6}$$

where we have picked a_k as the first nonzero coefficient in the power series. Hence

$$g(z) = (z-p)^k h(z), \quad h(p) = a_k \neq 0, \tag{11.7}$$

with h holomorphic on U. This establishes the following.

Proposition 11.2. *If f is holomorphic on $\Omega \setminus p$ with a pole at p, then there exists $k \in \mathbb{Z}^+$ such that*

$$f(z) = (z-p)^{-k} F(z) \tag{11.8}$$

on $\Omega \setminus p$, with F holomorphic on Ω and $F(p) \neq 0$.

If Proposition 11.2 works with $k = 1$, we say f has a *simple pole* at p.

An isolated singularity of a function that is not removable and not a pole is called an *essential singularity.* An example is

$$f(z) = e^{1/z}, \tag{11.9}$$

for which 0 is an essential singularity. The following result is known as the Casorati-Weierstrass theorem.

Proposition 11.3. *Suppose $f : \Omega \setminus p \to \mathbb{C}$ has an essential singularity at p. Then, for any neighborhood U of p in Ω, the image of $U \setminus p$ under f is dense in $\mathbb{C}$.*

Proof. Suppose that, for some neighborhood U of p, the image of $U \setminus p$ under f omits a neighborhood of $w_0 \in \mathbb{C}$. Replacing $f(z)$ by $f(z) - w_0$, we may as well suppose $w_0 = 0$. Then $g(z) = 1/f(z)$ is holomorphic and bounded on $U \setminus p$, so p is a removable singularity for g, which hence has a holomorphic extension $\tilde{g}$. If $\tilde{g}(p) \neq 0$, then p is removable for f. If $\tilde{g}(p) = 0$, then p is a pole of f.

Remark. There is a strengthening of Proposition 11.3, due to E. Picard, which we will treat in §28.

A function holomorphic on Ω except for a set of poles is said to be *meromorphic* on Ω. For example,

$$\tan z = \frac{\sin z}{\cos z}$$

is meromorphic on $\mathbb{C}$, with poles at $\{(k + 1/2)\pi : k \in \mathbb{Z}\}$.

Here we have discussed isolated singularities. In §4 we have seen examples of functions, such as $z^{1/2}$ and $\log z$, with singularities of a different nature, called branch singularities.

Another useful consequence of the removable singularities theorem is the following characterization of polynomials.

Proposition 11.4. *If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and $|f(z)| \to \infty$ as $|z| \to \infty$, then $f(z)$ is a polynomial.*

Proof. Consider $g : \mathbb{C} \setminus 0 \to \mathbb{C}$, defined by

$$g(z) = f\Bigl(\frac{1}{z}\Bigr). \tag{11.10}$$

The hypothesis on f implies $|g(z)| \to \infty$ as $z \to 0$, so g has a pole at 0. By Proposition 11.2, we can write

$$g(z) = z^{-k} G(z) \tag{11.11}$$

on $\mathbb{C} \setminus 0$, for some $k \in \mathbb{Z}^+$, with G holomorphic on $\mathbb{C}$ and $G(0) \neq 0$. Then write

$$G(z) = \sum_{j=0}^{k-1} g_j z^j + z^k h(z), \tag{11.12}$$

with $h(z)$ holomorphic on $\mathbb{C}$. Then

$$g(z) = \sum_{j=0}^{k-1} g_j z^{j-k} + h(z),$$

so

$$f(z) = \sum_{j=0}^{k-1} g_j z^{k-j} + h\Bigl(\frac{1}{z}\Bigr), \quad \text{for} \quad z \neq 0.$$

It follows that

$$f(z) - \sum_{j=0}^{k-1} g_j z^{k-j} \tag{11.13}$$

is holomorphic on $\mathbb{C}$ and, as $|z| \to \infty$, this tends to the finite limit $h(0)$. Hence, by Liouville's theorem, this difference is constant, so $f(z)$ is a polynomial.

Exercises

1. Show that 0 is a removable singularity for each of the following functions.

$$\begin{array}{cc} \dfrac{\sin z}{z}, & \dfrac{1 - \cos z}{z^2} \\ \dfrac{z}{1 - e^z}, & \dfrac{\sin(\tan z)}{\tan(\sin z)}. \end{array}$$

2. In the setting of Theorem 11.1, suppose $|z-p| = r$ and $\overline{D_r(z)} \subset \Omega$. If $|f| \le M$ on $\Omega \setminus p$, show that

$$|f'(z)| \le \frac{M}{r}, \tag{11.14}$$

and hence, in (11.2),

$$|(z-p)^2 f'(z)| \le M\,|z-p|. \tag{11.15}$$

Use this to show directly that the hypotheses of Theorem 11.1 imply $g \in C^1(\Omega)$, avoiding the need for Goursat's theorem in the proof.
Hint. Use (5.10) to prove (11.14). In (5.10), replace Ω by $D_s(z)$ with $s < r$, and then let $s \nearrow r$.

3. For yet another approach to the proof of Theorem 11.1, define $h : \Omega \to \mathbb{C}$ by

$$\begin{aligned} h(z) &= (z-p) f(z), \quad z \in \Omega \setminus p, \\ h(p) &= 0. \end{aligned} \tag{11.16}$$

Show that $h : \Omega \to \mathbb{C}$ is continuous. Show that $\int_{\partial R} h(z)\,dz = 0$ for each closed rectangle $R \subset \Omega$, and deduce that h is holomorphic on Ω. Use a power series argument parallel to that of (11.3)–(11.4) to finish the proof of Theorem 11.1.

4. Suppose $\Omega, \mathcal{O} \subset \mathbb{C}$ are open and $f : \Omega \to \mathcal{O}$ is holomorphic and a homeomorphism. Show that $f'(p) \neq 0$ for all $p \in \Omega$.
Hint. Apply the removable singularities theorem to $f^{-1} : \mathcal{O} \to \Omega$.
Compare the different approach suggested in Exercise 8 of §5.

5. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic and assume f is not a polynomial. (We say f is a transcendental function.) Show that $g : \mathbb{C} \setminus 0 \to \mathbb{C}$, defined by $g(z) = f(1/z)$, has an essential singularity at 0. Apply the Casorati-Weierstrass theorem to g, and interpret the conclusion in terms of the behavior of f.
Remark. (Parallel to that after Proposition 11.3) For an improvement of this conclusion, see §28.

12. Laurent series

There is a generalization of the power series expansion, which works for functions holomorphic in an annulus, rather than a disk. Let

$$\mathcal{A} = \{z \in \mathbb{C} : r_0 < |z - z_0| < r_1\}. \tag{12.1}$$

be such an annulus. For now assume $0 < r_0 < r_1 < \infty$. Let γ_j be the counter-clockwise circles $\{|z - z_0| = r_j\}$, so $\partial\mathcal{A} = \gamma_1 - \gamma_0$. If $f \in C^1(\overline{\mathcal{A}})$ is holomorphic in $\mathcal{A}$, the Cauchy integral formula gives

$$f(z) = \frac{1}{2\pi i} \int\limits_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int\limits_{\gamma_0} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \tag{12.2}$$

for $z \in \mathcal{A}$. For such a z, we write

$$\begin{aligned} \frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} &= \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}, \quad \zeta \in \gamma_1, \\ &\quad - \frac{1}{z - z_0} \frac{1}{1 - \frac{\zeta - z_0}{z - z_0}}, \quad \zeta \in \gamma_0, \end{aligned}$$

and use the fact that

$$\begin{aligned} |z - z_0| &< |\zeta - z_0|, \quad \text{for } \zeta \in \gamma_1, \\ &> |\zeta - z_0|, \quad \text{for } \zeta \in \gamma_0, \end{aligned}$$

to write

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \Bigl(\frac{z - z_0}{\zeta - z_0}\Bigr)^n, \quad \zeta \in \gamma_1, \tag{12.3}$$

and

$$\frac{1}{\zeta - z} = -\frac{1}{z - z_0} \sum_{m=0}^{\infty} \Bigl(\frac{\zeta - z_0}{z - z_0}\Bigr)^m, \quad \zeta \in \gamma_0. \tag{12.4}$$

Plugging these expansions into (12.2) yields

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad z \in \mathcal{A}, \tag{12.5}$$

with

$$a_n = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta, \quad n \geq 0, \tag{12.6}$$

and

$$a_n = \frac{1}{2\pi i} \int_{\gamma_0} f(\zeta)(\zeta - z_0)^m\, d\zeta, \quad n = -m - 1 < 0. \tag{12.7}$$

Now in (12.6) and (12.7) γ_0 and γ_1 can be replaced by any circle in $\mathcal{A}$ concentric with these. Using this observation, we can prove the following.

Proposition 12.1. *Given $0 \leq r_0 < r_1 \leq \infty$, let $\mathcal{A}$ be the annulus (12.1). If $f : \mathcal{A} \to \mathbb{C}$ is holomorphic, then it is given by the absolutely convergent series (12.5), with*

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta, \quad n \in \mathbb{Z}, \tag{12.8}$$

where γ is any (counter-clockwise oriented) circle centered at z_0, of radius $r \in (r_0, r_1)$.

Proof. The preceding argument gives this result on every annulus

$$\mathcal{A}^b = \{z \in \mathbb{C} : r_0' < |z - z_0| < r_1'\}$$

for $r_0 < r_0' < r_1' < r_1$, which suffices.

Of particular interest is the case $r_0 = 0$, dealing with an isolated singularity at z_0.

Proposition 12.2. *Suppose f is holomorphic on $D_R(z_0) \setminus z_0$, with Laurent expansion (12.5). Then f has a pole at z_0 if and only if $a_n = 0$ for all but finitely many $n < 0$ (and $a_n \neq 0$ for some $n < 0$). Hence f has an essential singularity at z_0 if and only if $a_n \neq 0$ for infinitely many $n < 0$.*

Proof. If z_0 is a pole, the stated conclusion about the Laurent series expansion follows from Proposition 11.2. The converse is elementary.

We work out Laurent series expansions for the function

$$f(z) = \frac{1}{z - 1}, \tag{12.9}$$

on the regions

$$\{z : |z| < 1\} \quad \text{and} \quad \{z : |z| > 1\}. \tag{12.10}$$

On the first region, we have

$$f(z) = -\frac{1}{1-z} = -\sum_{k=0}^{\infty} z^k, \tag{12.11}$$

and on the second region,

$$f(z) = \frac{1}{z}\,\frac{1}{1-\frac{1}{z}} = \frac{1}{z}\sum_{k=0}^{\infty}\Bigl(\frac{1}{z}\Bigr)^k = \sum_{k=-\infty}^{-1} z^k. \tag{12.12}$$

Similarly, for

$$g(z) = \frac{1}{z-2} \tag{12.13}$$

on the regions

$$\{z : |z| < 2\} \quad \text{and} \quad \{z : |z| > 2\}, \tag{12.14}$$

on the first region we have

$$g(z) = -\frac{1}{2}\,\frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\sum_{k=0}^{\infty}\Bigl(\frac{z}{2}\Bigr)^k, \tag{12.15}$$

and on the second region,

$$g(z) = \frac{1}{z}\,\frac{1}{1-\frac{2}{z}} = \frac{1}{z}\sum_{k=0}^{\infty}\Bigl(\frac{2}{z}\Bigr)^k = \sum_{k=-\infty}^{-1} 2^{-k-1} z^k. \tag{12.16}$$

The next result can be regarded as an extension of Proposition 2.2.

Proposition 12.3. *Assume $f(z)$ is given by the series (12.5), converging for $z \in \mathcal{A}$, i.e., for $r_0 < |z - z_0| < r_1$. Then f is holomorphic on $\mathcal{A}$, and*

$$f'(z) = \sum_{n=-\infty}^{\infty} n a_n (z - z_0)^{n-1}, \quad \forall\, z \in \mathcal{A}. \tag{12.17}$$

Proof. Arguments parallel to those used for Proposition 0.3 show that the series (12.5) converges absolutely and uniformly on $r_0' \le |z - z_0| \le r_1'$, whenever $r_0 < r_0' < r_1' < r_1$. Hence, with

$$f_\nu(z) = \sum_{n=-\nu}^{\nu} a_n (z - z_0)^n, \tag{12.18}$$

we have $f_\nu \to f$ locally uniformly on $\mathcal{A}$. That the limit f is holomorphic on $\mathcal{A}$, with derivative given by (12.17), follows from Proposition 5.10.

We next consider products of holomorphic functions on $\mathcal{A}$, say fg, where f is given by (12.5) and

$$g(z) = \sum_{n=-\infty}^{\infty} b_n (z - z_0)^n, \quad z \in \mathcal{A}. \tag{12.19}$$

Both (12.5) and (12.19) are absolutely convergent, locally uniformly on $\mathcal{A}$. The analysis here is parallel to that of (2.19)–(2.20). Extending Proposition 2.3, we have the following.

Proposition 12.4. *Given absolutely convergent series*

$$A = \sum_{n=-\infty}^{\infty} \alpha_n, \quad B = \sum_{n=-\infty}^{\infty} \beta_n, \tag{12.20}$$

we have the absolutely convergent series

$$AB = \sum_{n=-\infty}^{\infty} \gamma_n, \quad \gamma_n = \sum_{k=-\infty}^{\infty} \alpha_k \beta_{n-k}. \tag{12.21}$$

Indeed, we have

$$\begin{aligned} AB = \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \alpha_k \beta_\ell &= \sum_{n=-\infty}^{\infty} \sum_{k+\ell=n} \alpha_k \beta_\ell \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \alpha_k \beta_{n-k}. \end{aligned} \tag{12.22}$$

See Appendix L, particularly Proposition L.8, in conjunction with Corollary L.5, for general results of this nature.

Proposition 12.4 readily yields the following.

Proposition 12.5. *Given f, g holomorphic on $\mathcal{A}$, as in (12.5) and (12.19), we have*

$$f(z)g(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n, \quad c_n = \sum_{k=-\infty}^{\infty} a_k b_{n-k}. \tag{12.23}$$

Exercises

Consider the Laurent series

$$e^{z+1/z} = \sum_{n=-\infty}^{\infty} a_n z^n. \tag{12.24}$$

1. Where does (12.24) converge?

2. Show that
$$a_n = \frac{1}{\pi} \int_0^\pi e^{2\cos t} \cos nt \, dt.$$

3. Show that, for $k \geq 0$,

$$a_k = a_{-k} = \sum_{m=0}^{\infty} \frac{1}{m!(m+k)!}.$$

Hint. For Exercise 2, use (12.7); for Exercise 3 multiply the series for e^z and $e^{1/z}$.

4. Consider the function

$$f(z) = \frac{1}{(z-1)(z-2)}.$$

Give its Laurent series about $z = 0$:

a) on $\{z : |z| < 1\}$,
b) on $\{z : 1 < |z| < 2\}$,
c) on $\{z : 2 < |z| < \infty\}$.

Hint. Use the calculations (12.9)–(12.16).

5. Let $\Omega \subset \mathbb{C}$ be open, $z_0 \in \Omega$ and let f be holomorphic on $\Omega \setminus z_0$ and bounded. By Proposition 12.1, we have a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

valid on $\{z : 0 < |z - z_0| < b\}$ for some $b > 0$. Use (12.8), letting γ shrink, to show that $a_n = 0$ for each $n < 0$, thus obtaining another proof of the removable singularities theorem (Theorem 11.1).

6. Show that, for $|z|$ sufficiently small,

$$\frac{1}{2}\frac{e^z+1}{e^z-1} = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^{2k-1}. \tag{12.25}$$

Rewrite the left side as

$$F(z) = \frac{1}{2}\frac{\cosh z/2}{\sinh z/2},$$

and show that

$$F'(z) = \frac{1}{4} - F(z)^2.$$

Using (2.19)–(2.20), write out the Laurent expansion for $F(z)^2$, in terms of that for $F(z)$ given above. Comparing terms in the expansions of $F'(z)$ and $1/4 - F(z)^2$, show that

$$a_1 = \frac{1}{2},$$

and, for $k \geq 2$,

$$a_k = -\frac{1}{2k+1} \sum_{\ell=1}^{k-1} a_\ell a_{k-\ell}.$$

One often writes

$$a_k = (-1)^{k-1} \frac{B_k}{(2k)!},$$

and B_k are called the *Bernoulli numbers.* For more on these numbers, see the exercises at the end of §30.

7. As an alternative for Exercise 6, rewrite (12.25) as

$$\frac{z}{2}\Bigl(2 + \sum_{n=1}^{\infty} \frac{z^n}{n!}\Bigr) = \Bigl(\sum_{\ell=1}^{\infty} \frac{z^\ell}{\ell!}\Bigr)\Bigl(1 + \sum_{k=1}^{\infty} a_k z^{2k}\Bigr),$$

multiply out using (2.19)–(2.20), and solve for the coefficients a_k.

8. As another alternative, note that

$$\frac{1}{2}\,\frac{e^z+1}{e^z-1} = \frac{1}{e^z-1} + \frac{1}{2},$$

and deduce that (12.25) is equivalent to

$$z = \Bigl(\sum_{\ell=1}^{\infty} \frac{z^\ell}{\ell!}\Bigr)\Bigl(1 - \frac{z}{2} + \sum_{k=1}^{\infty} a_k z^{2k}\Bigr).$$

Use this to solve for the coefficients a_k.

C. Green's theorem

Here we prove Green's theorem, which was used in one approach to the Cauchy integral theorem in §5.

Theorem C.1. *If Ω is a bounded region in $\mathbb{R}^2$ with piecewise smooth boundary, and $f, g \in C^1(\overline{\Omega})$, then*

$$\iint\limits_{\Omega} \Bigl(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\Bigr)\, dx\, dy = \int\limits_{\partial\Omega} (f\, dx + g\, dy). \tag{C.1}$$

We recall some terms used above. The set $\Omega \subset \mathbb{R}^2$ is a nonempty open set, contained in some finite disk $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < A^2\}$, with closure $\overline{\Omega}$ and boundary $\partial\Omega = \overline{\Omega} \setminus \Omega$. We assume $\partial\Omega$ is a finite disjoint union of simple closed curves $\gamma_j : [0, 1] \to \mathbb{R}^2$, with $\gamma_j(0) = \gamma_j(1)$. We assume each curve γ_j is continuous and piecewise C^1, as defined in §2. Also we assume Ω lies on one side of γ_j. Furthermore, if $\gamma_j(t)$ is differentiable at t_0, we assume $\gamma_j'(t_0) \in \mathbb{R}^2$ is nonzero and that the vector $J\gamma_j'(t_0)$ points into Ω, where J is counterclockwise rotation by 90° (see (1.39)). This defines an orientation on $\partial\Omega$. To say $f \in C^1(\overline{\Omega})$ is to say f is continuous on $\overline{\Omega}$ and smooth of class C^1 on Ω, and furthermore that $\partial f/\partial x$ and $\partial f/\partial y$ extend continuously from Ω to $\overline{\Omega}$.

The identity (C.1) combines two separate identities, namely

$$\int\limits_{\partial\Omega} f\, dx = -\iint\limits_{\Omega} \frac{\partial f}{\partial y}\, dx\, dy, \tag{C.2}$$

and

$$\int\limits_{\partial\Omega} g\, dy = \iint\limits_{\Omega} \frac{\partial g}{\partial x}\, dx\, dy. \tag{C.3}$$

We will first prove (C.2) for regions Ω of the form depicted in Fig. C.1 (which we will call type I), then prove (C.3) for regions of the form depicted in Fig. C.2 (which we call type II), and then discuss more general regions.

If Ω is type I (cf. Fig. C.1), then

$$\int\limits_{\partial\Omega} f\, dx = \int_a^b f(x, \psi_0(x))\, dx - \int_a^b f(x, \psi_1(x))\, dx. \tag{C.4}$$

Now the fundamental theorem of calculus gives

$$f(x, \psi_1(x)) - f(x, \psi_0(x)) = \int_{\psi_0(x)}^{\psi_1(x)} \frac{\partial f}{\partial y}(x, y)\, dy, \tag{C.5}$$

so the right side of (C.4) is equal to

$$-\int_a^b \int_{\psi_0(x)}^{\psi_1(x)} \frac{\partial f}{\partial y}(x,y)\,dy\,dx = -\iint\limits_{\Omega} \frac{\partial f}{\partial y}\,dx\,dy, \tag{C.6}$$

and we have (C.2). Similarly, if Ω is type II (cf. Fig. C.2), then

$$\begin{aligned}\int\limits_{\partial\Omega} g\,dy &= \int_c^d g(\varphi_1(y))\,dy - \int_c^d g(\varphi_0(y),y)\,dy \\ &= \int_c^d \int_{\varphi_0(y)}^{\varphi_1(y)} \frac{\partial g}{\partial x}(x,y),dx\,dy \\ &= \iint\limits_{\Omega} \frac{\partial g}{\partial x}\,dx\,dy,\end{aligned} \tag{C.7}$$

and we have (C.3).

Figure C.3 depicts a region Ω that is type I but not type II. The argument above gives (C.2) in this case, but we need a further argument to get (C.3). As indicated in the figure, we divide Ω into two pieces, Ω_1 and Ω_2, and observe that Ω_1 and Ω_2 are each of type II. Hence, given $g \in C^1(\overline{\Omega})$,

$$\int\limits_{\partial\Omega_j} g\,dy = \iint\limits_{\Omega_j} \frac{\partial g}{\partial y}\,dx\,dy, \tag{C.8}$$

for each j. Now

$$\sum_j \iint\limits_{\Omega_j} \frac{\partial g}{\partial y}\,dx\,dy = \iint\limits_{\Omega} \frac{\partial g}{\partial y}\,dx\,dy. \tag{C.9}$$

On the other hand, if we sum the integrals $\int_{\partial\Omega_j} g\,dy$, we get an integral over $\partial\Omega$ plus two integrals over the interface between Ω_1 and Ω_2. However, the latter two integrals cancel out, since they traverse the same path except in opposite directions. Hence

$$\sum_j \int\limits_{\partial\Omega_j} g\,dy = \int\limits_{\partial\Omega} g\,dy, \tag{C.10}$$

and (C.3) follows.

A garden variety piecewise smoothly bounded domain Ω might not be of type I or type II, but typically can be divided into domains Ω_j of type I, as depicted in Fig. C.4. For each such Ω, we have

$$\int\limits_{\partial\Omega_j} f\,dx = -\iint\limits_{\Omega_j} \frac{\partial f}{\partial y}\,dx\,dy, \tag{C.11}$$

and summing over j yields (C.2). Meanwhile, one can typically divide Ω into domains of type II, as depicted in Fig. C.5, get (C.8) on each such domain, and sum over j to get (C.3).

It is possible to concoct piecewise smoothly bounded domains that would require an infinite number of divisions to yield subdomins of type I (or of type II). In such a case a limiting argument can be used to establish (C.1). We will not dscuss the details here. Arguments applying to general domains can be found in [T3], in Appendix G for domains with C^2 boundary, and in Appendix I for domains substantially rougher than treated here.

Of use for the proof of Cauchy's integral theorem in §5 is the special case $f = -iu,\ g = u$ of (C.1), which yields

$$\begin{aligned} \iint_{\Omega} \Bigl(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\Bigr)\, dx\, dy &= -i \int_{\partial\Omega} u\,(dx + i\, dy) \\ &= -i \int_{\partial\Omega} u\, dz. \end{aligned} \tag{C.12}$$

When $u \in C^1(\overline{\Omega})$ is holomorphic in Ω, the integrand on the left side of (C.12) vanishes. Another special case arises by taking $f = iu,\ g = u$. Then we get

$$\begin{aligned} \iint_{\Omega} \Bigl(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\Bigr)\, dx\, dy &= i \int_{\partial\Omega} u\,(dx - i\, dy) \\ &= i \int_{\partial\Omega} u\, d\bar{z}. \end{aligned} \tag{C.13}$$

In case u is holomorphic and $u'(z) = v(z)$, the integrand on the left side of (C.13) is $2v(z)$. Such an identity (together with a further limiting argument) is useful in Exercise 7 of §30.

F. The fundamental theorem of algebra (elementary proof)

Here we provide a proof of the fundamental theorem of algebra that is quite a bit different from that given in §6. The proof here is more "elementary," in the sense that it does not make use of consequences of the Cauchy integral theorem. On the other hand, it is a little longer than the proof in §6. Here it is.

Theorem F.1. *If $p(z)$ is a nonconstant polynomial (with complex coefficients), then $p(z)$ must have a complex root.*

Proof. We have, for some $n \geq 1,\ a_n \neq 0$,

$$\begin{aligned} p(z) &= a_n z^n + \cdots + a_1 z + a_0 \\ &= a_n z^n \big(1 + O(z^{-1})\big), \quad |z| \to \infty, \end{aligned} \tag{F.1}$$

which implies

$$\lim_{|z|\to\infty} |p(z)| = \infty. \tag{F.2}$$

Picking $R \in (0, \infty)$ such that

$$\inf_{|z|\geq R} |p(z)| > |p(0)|, \tag{F.3}$$

we deduce that

$$\inf_{|z|\leq R} |p(z)| = \inf_{z\in\mathbb{C}} |p(z)|. \tag{F.4}$$

Since $D_R = \{z : |z| \leq R\}$ is compact and p is continuous, there exists $z_0 \in D_R$ such that

$$|p(z_0)| = \inf_{z\in\mathbb{C}} |p(z)|. \tag{F.5}$$

The theorem hence follows from:

Lemma F.2. *If $p(z)$ is a nonconstant polynomial and (F.5) holds, then $p(z_0) = 0$.*

Proof. Suppose to the contrary that

$$p(z_0) = a \neq 0. \tag{F.6}$$

We can write

$$p(z_0 + \zeta) = a + q(\zeta), \tag{F.7}$$

where $q(\zeta)$ is a (nonconstant) polynomial in ζ, satisfying $q(0) = 0$. Hence, for some $k \geq 1$ and $b \neq 0$, we have $q(\zeta) = b\zeta^k + \cdots + b_n\zeta^n$, i.e.,

$$q(\zeta) = b\zeta^k + O(\zeta^{k+1}), \quad \zeta \to 0, \tag{F.8}$$

so, uniformly on $S^1 = \{\omega : |\omega| = 1\}$

$$p(z_0 + \varepsilon\omega) = a + b\omega^k\varepsilon^k + O(\varepsilon^{k+1}), \quad \varepsilon \searrow 0. \tag{F.9}$$

Pick $\omega \in S^1$ such that

$$\frac{b}{|b|}\omega^k = -\frac{a}{|a|}, \tag{F.10}$$

which is possible since $a \neq 0$ and $b \neq 0$. Then

$$p(z_0 + \varepsilon\omega) = a\Big(1 - \Big|\frac{b}{a}\Big|\varepsilon^k\Big) + O(\varepsilon^{k+1}), \tag{F.11}$$

which contradicts (F.5) for $\varepsilon > 0$ small enough. Thus (F.6) is impossible. This proves Lemma F.2, hence Theorem F.1.

L. Absolutely convergent series

Here we produce some results on absolutely convergent infinite series, along the lines of those that arose in §§0, 2, and 12, but in a more general setting. Rather than looking at a series as a sum of a_k for $k \in \mathbb{N}$, we find it convenient to let Z be a countably infinite set, and take a function

$$f : Z \longrightarrow \mathbb{C}. \tag{L.1}$$

We say f is absolutely summable, and write $f \in \ell^1(Z)$, provided there exists $M < \infty$ such that

$$\sum_{k\in F} |f(k)| \le M, \quad \text{for each finite set} \ \ F \subset Z. \tag{L.2}$$

In notation used in §0, we would have $f(k)$ denoted by $f_k,\ k \in \mathbb{N}$ (or maybe $k \in \mathbb{Z}^+$), but we use the notation $f(k)$ here. If $f \in \ell^1(Z)$, we say the series

$$\sum_{k\in Z} f(k) \quad \text{is absolutely convergent.} \tag{L.3}$$

Also we would like to write the characterization (L.2) as

$$\sum_{k\in Z} |f(k)| < \infty. \tag{L.4}$$

Of course, implicit in (L.3)–(L.4) is that $\sum_{k\in Z} f(k)$ and $\sum_{k\in Z} |f(k)|$ are well defined elements of $\mathbb{C}$ and $[0,\infty)$, respectively. We will see shortly that this is the case.

To start, we note that, by hypothesis (L.2), if $f \in \ell^1(Z)$, the quantity

$$M(f) = \sup\Big\{ \sum_{k\in F} |f(k)| : F \subset Z \text{ finite} \Big\} \tag{L.5}$$

is well defined, and $M(f) \le M$. Hence, given $\varepsilon > 0$, there is a finite set $K_\varepsilon(f) \subset Z$ such that

$$\sum_{k\in K_\varepsilon(f)} |f(k)| \ge M(f) - \varepsilon. \tag{L.6}$$

These observations yield the following.

Lemma L.1. *If $f \in \ell^1(Z)$, then*

$$F \subset Z \setminus K_\varepsilon(f) \text{ finite} \Longrightarrow \sum_{k\in F} |f(k)| \le \varepsilon. \tag{L.7}$$

This leads to:

Corollary L.2. *If $f \in \ell^1(Z)$ and $A, B \supset K_\varepsilon(f)$ are finite, then*

$$\Bigl|\sum_{k\in A} f(k) - \sum_{k\in B} f(k)\Bigr| \le 2\varepsilon. \tag{L.8}$$

To proceed, we bring in the following notion. Given subsets $F_\nu \subset Z$ ($\nu \in \mathbb{N}$), we say $F_\nu \to Z$ provided that, if $F \subset Z$ is finite, there exists $N = N(F) < \infty$ such that $\nu \ge N \Rightarrow F_\nu \supset F$. Since Z is countable, we see that there exist sequences $F_\nu \to Z$ such that each F_ν is finite.

Proposition L.3. *Take $f \in \ell^1(Z, \mathbb{R}^n)$. Assume $F_\nu \subset Z$ are finite and $F_\nu \to Z$. Then there exists $S_Z(f) \in \mathbb{R}^n$ such that*

$$\lim_{\nu\to\infty} \sum_{k\in F_\nu} f(k) = S_Z(f). \tag{L.9}$$

Furthermore, the limit $S_Z(f)$ is independent of the choice of finite $F_\nu \to Z$.

Proof. By Corollary L.2, the sequence $S_\nu(f) = \sum_{k\in F_\nu} f(k)$ is a Cauchy sequence in $\mathbb{C}$, so it converges to a limit we call $S_Z(f)$. As for the independence of the choice, note that if also F'_ν are finite and $F'_\nu \to Z$, we can interlace F_ν and F'_ν.

Given Proposition L.3, we set

$$\sum_{k\in Z} f(k) = S_Z(f), \quad \text{for} \ \ f \in \ell^1(Z, \mathbb{R}^n). \tag{L.10}$$

Note in particular that, if $f \in \ell^1(Z)$, then $|f| \in \ell^1(Z)$, and

$$\sum_{k\in Z} |f(k)| = M(f), \tag{L.11}$$

defined in (L.6). (These two results illuminate (L.3)–(L.4).)

Remark. Proposition L.3 contains Lemma 0.1A. It is stronger than that result, in that it makes clear that the order of summation is irrelevant.

Our next goal is to establish the following result, known as a *dominated convergence theorem.*

Proposition L.4. *For $\nu \in \mathbb{N}$, let $f_\nu \in \ell^1(Z)$, and let $g \in \ell^1(Z)$. Assume*

$$|f_\nu(k)| \le g(k), \quad \forall\, \nu \in \mathbb{N},\ k \in Z, \tag{L.12}$$

and

$$\lim_{\nu\to\infty} f_\nu(k) = f(k), \quad \forall\, k \in Z. \tag{L.13}$$

Then $f \in \ell^1(Z)$ and

$$\lim_{\nu\to\infty} \sum_{k\in Z} f_\nu(k) = \sum_{k\in Z} f(k). \tag{L.14}$$

Proof. We have $\sum_{k\in Z} |g(k)| = \sum_{k\in Z} g(k) = M < \infty$. Parallel to (L.6)–(L.7), for each $\varepsilon > 0$, we can take a finite set $K_\varepsilon(g) \subset Z$ such that $\sum_{k\in K_\varepsilon(g)} g(k) \ge M - \varepsilon$, and hence

$$\begin{aligned} F \subset Z \setminus K_\varepsilon(g) \ \text{ finite } &\Longrightarrow \sum_{k\in F} g(k) \le \varepsilon \\ &\Longrightarrow \sum_{k\in F} |f_\nu(k)| \le \varepsilon, \quad \forall\, \nu \in \mathbb{N}, \end{aligned} \tag{L.15}$$

the last implication by (L.12). In light of Proposition L.3, we can restate this conclusion as

$$\sum_{k\in Z\setminus K_\varepsilon(g)} |f_\nu(k)| \le \varepsilon, \quad \forall\, \nu \in \mathbb{N}. \tag{L.16}$$

Bringing in (L.13), we also have

$$\sum_{k\in F} |f(k)| \le \varepsilon, \quad \text{for each finite } F \subset Z \setminus K_\varepsilon(g), \tag{L.17}$$

and hence

$$\sum_{k\in Z\setminus K_\varepsilon(g)} |f(k)| \le \varepsilon. \tag{L.18}$$

On the other hand, since $K_\varepsilon(g)$ is finite,

$$\lim_{\nu\to\infty} \sum_{k\in K_\varepsilon(g)} f_\nu(k) = \sum_{k\in K_\varepsilon(g)} f(k). \tag{L.19}$$

It follows that

$$\begin{aligned} &\limsup_{\nu\to\infty} |S_Z(f_\nu) - S_Z(f)| \\ &\le \limsup_{\nu\to\infty} |S_{K_\varepsilon(g)}(f_\nu) - S_{K_\varepsilon(g)}(f)| + \limsup_{\nu\to\infty} |S_{Z\setminus K_\varepsilon(g)}(f_\nu) - S_{Z\setminus K_\varepsilon(g)}(f)| \\ &\le 2\varepsilon, \end{aligned} \tag{L.20}$$

for each $\varepsilon > 0$, hence

$$\limsup_{\nu\to\infty} |S_Z(f_\nu) - S_Z(f)| = 0, \tag{L.21}$$

which is equivalent to (L.14).

Here is one simple but basic application of Proposition L.4.

Corollary L.5. *Assume $f \in \ell^1(Z)$. For $\nu \in \mathbb{N}$, let $F_\nu \subset Z$ and assume $F_\nu \to Z$. One need not assume that F_ν is finite. Then*

$$\lim_{\nu\to\infty} \sum_{k\in F_\nu} f(k) = \sum_{k\in Z} f(k). \tag{L.22}$$

Proof. Apply Proposition L.4 with $g(k) = |f(k)|$ and $f_\nu(k) = \chi_{F_\nu}(k) f(k)$.

The following result recovers Proposition 2.5.

Proposition L.6. *Let Y and Z be countable sets, and assume $f \in \ell^1(Y \times Z)$, so*

$$\sum_{(j,k)\in Y\times Z} |f(j,k)| = M < \infty. \tag{L.23}$$

Then, for each $j \in Y$,

$$\sum_{k\in Z} f(j,k) = g(j) \tag{L.24}$$

is absolutely convergent,

$$g \in \ell^1(Y), \tag{L.25}$$

and

$$\sum_{(j,k)\in Y\times Z} f(j,k) = \sum_{j\in Y} g(j). \tag{L.26}$$

hence

$$\sum_{(j,k)\in Y\times Z} f(j,k) = \sum_{j\in Y} \Bigl(\sum_{k\in Z} f(j,k) \Bigr). \tag{L.27}$$

Proof. Since $\sum_{k\in Z} |f(j,k)|$ is dominated by (L.23), the absolute convergence in (L.24) is clear. Next, if $A \subset Y$ is finite, then

$$\sum_{j\in A} |g(j)| \le \sum_{j\in A} \sum_{k\in Z} |f(j,k)| \le M, \tag{L.28}$$

so $g \in \ell^1(Y, \mathbb{R}^n)$. Furthermore, if $A_\nu \subset Y$ are finite, then

$$\sum_{j\in A_\nu} g(j) = \sum_{(j,k)\in F_\nu} f(j,k), \quad F_\nu = A_\nu \times Z, \tag{L.29}$$

and $A_\nu \to Y \Rightarrow F_\nu \to Y \times Z$, so (L.26) follows from Corollary L.5.

We next examine implications for multiplying two absolutely convergent series, extending Proposition 2.3.

Proposition L.7. *Let Y and Z be countable sets, and assume $f \in \ell^1(Y)$, $g \in \ell^1(Z)$. Define*

$$f \times g : Y \times Z \longrightarrow \mathbb{C}, \quad (f \times g)(j,k) = f(j)g(k). \tag{L.30}$$

Then

$$f \times g \in \ell^1(Y \times Z). \tag{L.31}$$

Proof. Given a finite set $F \subset Y \times Z$, there exist finite $A \subset Y$ and $B \subset Z$ such that $F \subset A \times B$. Then

$$\begin{aligned} \sum_{(j,k)\in F} |f(j)g(k)| &\le \sum_{(j,k)\in A\times B} |f(j)g(k)| \\ &= \sum_{j\in A} |f(j)| \sum_{k\in B} |g(k)| \\ &\le M(f)M(g), \end{aligned} \tag{L.32}$$

where $M(f) = \sum_{j\in Y} |f(j)|$ and $M(g) = \sum_{k\in Z} |g(k)|$.

We can apply Proposition L.6 to $f \times g$ to deduce:

Proposition L.8. *In the setting of Proposition L.7,*

$$\sum_{(j,k)\in Y\times Z} f(j)g(k) = \Bigl(\sum_{j\in Y} f(j)\Bigr)\Bigl(\sum_{k\in Z} g(k)\Bigr). \tag{L.33}$$

In case $Y = Z = \mathbb{N}$, we can then apply Proposition L.3, with Z replaced by $\mathbb{N} \times \mathbb{N}$, f replaced by $f \times g$, and

$$F_\nu = \{(j,k) \in \mathbb{N} \times \mathbb{N} : j + k \le \nu\}, \tag{L.34}$$

and recover Proposition 2.3, including (2.14).

In case $Y = Z = \mathbb{Z}$, we can instead apply Corollary L.5, with Z replaced by $\mathbb{Z} \times \mathbb{Z}$, f replaced by $f \times g$, and

$$F_\nu = \{(j,k) \in \mathbb{Z} \times \mathbb{Z} : |j + k| \le \nu\}, \tag{L.35}$$

and recover Proposition 12.4.

Fourier analysis is an area of mathematics that is co-equal to the area of complex analysis. These two areas interface with each other in numerous ways, and these interactions magnify the power of each area. Thus it is very natural to bring in Fourier analysis in a complex analysis text.

In one variable, the two main facets of Fourier analysis are Fourier series,

$$f(\theta) = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}, \quad \theta \in \mathbb{R}/(2\pi\mathbb{Z}), \tag{3.0.1}$$

and the Fourier transform

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix\xi}\, dx, \quad \xi \in \mathbb{R}. \tag{3.0.2}$$

Major results include Fourier inversion formulas, to the effect that (3.0.1) holds with

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-ik\theta}\, d\theta, \tag{3.0.3}$$

and that in the situation of (3.0.2),

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi}\, d\xi, \tag{3.0.4}$$

with appropriate hypotheses on f in each case. We note that the Fourier series (3.0.1) can be regarded as the limiting case of the Laurent series of a holomorphic function on the annulus $\mathcal{A} = \{z \in \mathbb{C} : 1 - \varepsilon < |z| < 1 + \varepsilon\}$, in the limit $\varepsilon \searrow 0$.

Fourier series is studied in §13, and the Fourier transform in §14. In §15 we introduce the Laplace transform,

$$\mathcal{L}f(s) = \int_0^{\infty} f(t) e^{-st}\, dt, \tag{3.0.5}$$

defined and holomorphic on $\{s \in \mathbb{C} : \operatorname{Re} s > A\}$, under the hypothesis that

$$\int_0^{\infty} |f(t)| e^{-at}\, dt < \infty, \quad \text{for} \quad a > A. \tag{3.0.6}$$

We use results from §14 to establish a Laplace inversion formula. A variant is the Mellin transform,

$$\mathcal{M}f(z) = \int_0^{\infty} f(t) t^{z-1}\, dt. \tag{3.0.7}$$

We will see these transforms arise in subsequent chapters, in various ways. For one, evaluation of such transforms often leads to non-elementary integrals, which can nevertheless be evaluated by residue calculus, developed in §16 of Chapter 4. For another, these transforms have intimate connections with important special functions. We will see this on display in Chapter 4, first for the Gamma function, then for the Riemann zeta function.

There are several appendices at the end of this chapter. Appendix H considers inner product spaces, with emphasis on the inner product

$$(f,g)=\int f(x)\overline{g(x)}\,dx \tag{3.0.8}$$

on functions. It is shown that taking

$$\|f\|^2=(f,f) \tag{3.0.9}$$

defines a *norm*. In particular, there is the triangle inequality

$$\|f+g\|\le\|f\|+\|g\|. \tag{3.0.10}$$

This is deduced from the Cauchy inequality

$$|(f,g)|\le\|f\|\cdot\|g\|. \tag{3.0.11}$$

These results hold in an abstract setting of vector spaces equipped with an inner product. They are significant for Fourier analysis for functions f and g that are square integrable, for which (3.0.8) is germane.

Appendix N discusses the matrix exponential

$$e^{tA}=\sum_{k=0}^{\infty}\frac{t^k}{k!}A^k, \tag{3.0.12}$$

for an $n\times n$ matrix A. This is of use in the analysis in §15 of the Laplace transform approach to solutions of systems of differential equations, and the equivalence of this result to an identity known as Duhamel's formula.

This chapter is capped by Appendix G, which derives two important approximation results. One, due to Weierstrass, says any continuous function on an interval $[a,b]\subset\mathbb{R}$ is a uniform limit of polynomials. The other, due to Runge, says that if $K\subset\mathbb{C}$ is compact and if f is holomorphic on a neighborhood of K, then, on K, f is a limit of rational functions. These results could have been pushed to Chapter 2, but the arguments involved seem natural in light of material developed in this chapter. This appendix also contains an important extension of the Weierstrass approximation result, known as the Stone-Weierstrass theorem. This will play a useful role in the proof of Karamata's Tauberian theorem, in Appendix R.

13. Fourier series and the Poisson integral

Given an integrable function $f : S^1 \to \mathbb{C}$, we desire to write

$$f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\, e^{ik\theta}, \tag{13.1}$$

for some coefficients $\hat{f}(k) \in \mathbb{C}$. We identify S^1 with $\mathbb{R}/(2\pi\mathbb{Z})$. If (13.1) is absolutely convergent, we can multiply both sides by $e^{-in\theta}$ and integrate. A change in order of summation and integration is then justified, and using

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\ell\theta}\, d\theta &= 0 \quad \text{if } \ \ell \neq 0, \\ &\ 1 \quad \text{if } \ \ell = 0, \end{aligned} \tag{13.2}$$

we see that

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-ik\theta}\, d\theta. \tag{13.3}$$

The series on the right side of (13.1) is called the Fourier series of f.

If $\hat{f}$ is given by (13.3), and if (13.1) holds, it is called the Fourier inversion formula. To examine whether (13.1) holds. we first sneak up on the sum on the right side. For $0 < r < 1$, set

$$J_r f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}(k) r^{|k|} e^{ik\theta}. \tag{13.4}$$

Note that

$$|\hat{f}(k)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|\, d\theta, \tag{13.5}$$

so whenever the right side of (13.5) is finite we see that the series (13.4) is absolutely convergent for each $r \in [0, 1)$. Furthermore, we can substitute (13.3) for $\hat{f}$ in (13.4) and change order of summation and integration, to obtain

$$J_r f(\theta) = \int_{S^1} f(\theta') p_r(\theta - \theta')\, d\theta', \tag{13.6}$$

where

$$p_r(\theta) = p(r,\theta) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} r^{|k|} e^{ik\theta} \tag{13.7}$$
$$= \frac{1}{2\pi} \Big[1 + \sum_{k=1}^{\infty} (r^k e^{ik\theta} + r^k e^{-ik\theta}) \Big],$$

and summing these geometrical series yields

$$p(r,\theta) = \frac{1}{2\pi} \frac{1-r^2}{1-2r\cos\theta + r^2}. \tag{13.8}$$

Let us examine $p(r,\theta)$. It is clear that the numerator and denominator on the right side of (13.8) are positive, so $p(r,\theta) > 0$ for each $r \in [0,1),\ \theta \in S^1$. As $r \nearrow 1$, the numerator tends to 0; as $r \nearrow 1$, the denominator tends to a nonzero limit, except at $\theta = 0$. Since we have

$$\int_{S^1} p(r,\theta)\, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum r^{|k|} e^{ik\theta}\, d\theta = 1, \tag{13.9}$$

we see that, for r close to 1, $p(r,\theta)$ as a function of θ is highly peaked near $\theta = 0$ and small elsewhere, as in Fig. 13.1. Given these facts, the following is an exercise in real analysis.

Proposition 13.1. *If $f \in C(S^1)$, then*

$$J_r f \to f \quad \textit{uniformly on} \quad S^1 \quad \textit{as} \quad r \nearrow 1. \tag{13.10}$$

Proof. We can rewrite (13.6) as

$$J_r f(\theta) = \int_{-\pi}^{\pi} f(\theta - \theta') p_r(\theta')\, d\theta'. \tag{13.11}$$

The results (13.8)–(13.9) imply that for each $\delta \in (0,\pi)$,

$$\int_{|\theta'| \le \delta} p_r(\theta')\, d\theta' = 1 - \varepsilon(r,\delta), \tag{13.12}$$

with $\varepsilon(r,\delta) \to 0$ as $r \nearrow 1$. Now, we break (13.11) into three pieces:

$$\begin{aligned} J_r f(\theta) &= f(\theta) \int_{-\delta}^{\delta} p_r(\theta')\, d\theta' \\ &\quad + \int_{-\delta}^{\delta} [f(\theta-\theta') - f(\theta)] p_r(\theta')\, d\theta' \\ &\quad + \int_{\delta \le |\theta'| \le \pi} f(\theta - \theta') p_r(\theta')\, d\theta' \\ &= I + II + III. \end{aligned} \tag{13.13}$$

We have

$$
\begin{aligned}
I &= f(\theta)(1-\varepsilon(r,\delta)), \\
|II| &\le \sup_{|\theta'|\le\delta} |f(\theta-\theta')-f(\theta)|, \\
|III| &\le \varepsilon(r,\delta)\,\sup|f|.
\end{aligned}
\tag{13.14}
$$

These estimates yield (13.10).

From (13.10) the following is an elementary consequence.

Proposition 13.2. *Assume $f \in C(S^1)$. If the Fourier coefficients $\hat{f}(k)$ form a summable series, i.e., if*

$$
\sum_{k=-\infty}^{\infty} |\hat{f}(k)| < \infty,
\tag{13.15}
$$

then the identity (13.1) holds for each $\theta \in S^1$.

Proof. What is to be shown is that if $\sum_k |a_k| < \infty$, then

$$
\sum_k a_k = S \Longrightarrow \lim_{r\nearrow 1} \sum_k r^{|k|} a_k = S.
\tag{13.15A}
$$

To get this, let $\varepsilon > 0$ and pick N such that

$$
\sum_{|k|>N} |a_k| < \varepsilon.
$$

Then

$$
S_N = \sum_{k=-N}^{N} a_k \Longrightarrow |S - S_N| < \varepsilon,
$$

and

$$
\Bigl|\sum_{|k|>N} r^{|k|} a_k\Bigr| < \varepsilon, \quad \forall\, r \in (0,1).
$$

Since clearly

$$
\lim_{r\nearrow 1} \sum_{k=-N}^{N} r^{|k|} a_k = \sum_{k=-N}^{N} a_k,
$$

the conclusion in (13.15A) follows.

Remark. A stronger result, due to Abel, is that the implication (13.15A) holds without the requirement of absolute convergence. This is treated in Appendix R.

Note that if (13.15) holds, then the right side of (13.1) is absolutely and uniformly convergent, and its sum is continuous on S^1.

We seek conditions on f that imply (13.15). Integration by parts for $f \in C^1(S^1)$ gives, for $k \neq 0$,

$$
\begin{aligned}
\hat{f}(k) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, \frac{i}{k}\, \frac{\partial}{\partial\theta}(e^{-ik\theta})\, d\theta \\
&= \frac{1}{2\pi i k} \int_{-\pi}^{\pi} f'(\theta) e^{-ik\theta}\, d\theta,
\end{aligned}
\tag{13.16}
$$

hence

$$
|\hat{f}(k)| \le \frac{1}{2\pi|k|} \int_{-\pi}^{\pi} |f'(\theta)|\, d\theta. \tag{13.17}
$$

If $f \in C^2(S^1)$, we can integrate by parts a second time, and get

$$
\hat{f}(k) = -\frac{1}{2\pi k^2} \int_{-\pi}^{\pi} f''(\theta) e^{-ik\theta}\, d\theta, \tag{13.18}
$$

hence

$$
|\hat{f}(k)| \le \frac{1}{2\pi k^2} \int_{-\pi}^{\pi} |f''(\theta)|\, d\theta.
$$

In concert with (13.5), we have

$$
|\hat{f}(k)| \le \frac{1}{2\pi(k^2+1)} \int_{S^1} [|f''(\theta)| + |f(\theta)|]\, d\theta. \tag{13.19}
$$

Hence

$$
f \in C^2(S^1) \Longrightarrow \sum |\hat{f}(k)| < \infty. \tag{13.20}
$$

We will produce successive sharpenings of (13.20) below. We start with an interesting example. Consider

$$
f(\theta) = |\theta|, \quad -\pi \le \theta \le \pi, \tag{13.21}
$$

continued to be periodic of period 2π. This defines a Lipschitz function on S^1, whose Fourier coefficients are

$$
\begin{aligned}
\hat{f}(k) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\theta|\, e^{-ik\theta}\, d\theta \\
&= -\bigl[1 - (-1)^k\bigr] \frac{1}{\pi k^2},
\end{aligned}
\tag{13.22}
$$

for $k \neq 0$, while $\hat{f}(0) = \pi/2$. It is clear this forms a summable series, so Proposition 13.2 implies that, for $-\pi \le \theta \le \pi$,

$$
\begin{aligned}
|\theta| &= \frac{\pi}{2} - \sum_{k\,\mathrm{odd}} \frac{2}{\pi k^2} e^{ik\theta} \\
&= \frac{\pi}{2} - \frac{4}{\pi} \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^2} \cos(2\ell+1)\theta.
\end{aligned} \tag{13.23}
$$

We note that evaluating this at $\theta = 0$ yields the identity

$$\sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^2} = \frac{\pi^2}{8}. \tag{13.24}$$

Writing

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \tag{13.25}$$

as a sum of two series, one for $k \ge 1$ odd and one for $k \ge 2$ even, yields an evaluation of (13.25). (See Exercise 1 below.)

We see from (13.23) that if f is given by (13.21), then $\hat{f}(k)$ satisfies

$$|\hat{f}(k)| \le \frac{C}{k^2+1}. \tag{13.26}$$

This is a special case of the following generalization of (13.20).

Proposition 13.3. *Let f be continuous and piecewise C^2 on S^1. Then (13.26) holds.*

Proof. Here we are assuming f is C^2 on $S^1 \setminus \{p_1, \dots, p_\ell\}$, and f' and f'' have limits at each of the endpoints of the associated intervals in S^1, but f is not assumed to be differentiable at the endpoints p_ℓ. We can write f as a sum of functions f_ν, each of which is Lipschitz on S^1, C^2 on $S^1 \setminus p_\nu$, and f'_ν and f''_ν have limits as one approaches p_ν from either side. It suffices to show that each $\hat{f}_\nu(k)$ satisfies (13.26). Now $g(\theta) = f_\nu(\theta + p_\nu - \pi)$ is singular only at $\theta = \pi$, and $\hat{g}(k) = \hat{f}_\nu(k) e^{ik(p_\nu - \pi)}$, so it suffices to prove Proposition 13.3 when f has a singularity only at $\theta = \pi$. In other words, $f \in C^2([-\pi, \pi])$, and $f(-\pi) = f(\pi)$.

In this case, we still have (13.16), since the endpoint contributions from integration by parts still cancel. A second integration by parts gives, in place of (13.18),

$$
\begin{aligned}
\hat{f}(k) &= \frac{1}{2\pi i k} \int_{-\pi}^{\pi} f'(\theta) \, \frac{i}{k} \, \frac{\partial}{\partial \theta} (e^{-ik\theta}) \, d\theta \\
&= -\frac{1}{2\pi k^2} \Big[\int_{-\pi}^{\pi} f''(\theta) e^{-ik\theta} \, d\theta + f'(\pi) - f'(-\pi) \Big],
\end{aligned} \tag{13.27}
$$

which yields (13.26).

Given $f \in C(S^1)$, let us say

$$f \in \mathcal{A}(S^1) \Longleftrightarrow \sum |\hat{f}(k)| < \infty. \tag{13.28}$$

Proposition 13.2 applies to elements of $\mathcal{A}(S^1)$, and Proposition 13.3 gives a sufficient condition for a function to belong to $\mathcal{A}(S^1)$. A more general sufficient condition will be given in Proposition 13.6.

We next make use of (13.2) to produce results on $\int_{S^1} |f(\theta)|^2 \, d\theta$, starting with the following.

Proposition 13.4. *Given $f \in \mathcal{A}(S^1)$,*

$$\sum |\hat{f}(k)|^2 = \frac{1}{2\pi} \int\limits_{S^1} |f(\theta)|^2 \, d\theta. \tag{13.29}$$

More generally, if also $g \in \mathcal{A}(S^1)$,

$$\sum \hat{f}(k)\overline{\hat{g}(k)} = \frac{1}{2\pi} \int\limits_{S^1} f(\theta)\overline{g(\theta)} \, d\theta. \tag{13.30}$$

Proof. Switching order of summation and integration and using (13.2), we have

$$\begin{aligned} \frac{1}{2\pi} \int\limits_{S^1} f(\theta)\overline{g(\theta)} \, d\theta &= \frac{1}{2\pi} \int\limits_{S^1} \sum_{j,k} \hat{f}(j)\overline{\hat{g}(k)} e^{-i(j-k)\theta} \, d\theta \\ &= \sum_k \hat{f}(k)\overline{\hat{g}(k)}, \end{aligned} \tag{13.31}$$

giving (13.30). Taking $g = f$ gives (13.29).

We will extend the scope of Proposition 13.4 below. Closely tied to this is the issue of convergence of $S_N f$ to f as $N \to \infty$, where

$$S_N f(\theta) = \sum_{|k| \le N} \hat{f}(k) e^{ik\theta}. \tag{13.32}$$

Clearly $f \in \mathcal{A}(S^1) \Rightarrow S_N f \to f$ uniformly on S^1 as $N \to \infty$. Here, we are interested in convergence in L^2-norm, where

$$\|f\|_{L^2}^2 = \frac{1}{2\pi} \int\limits_{S^1} |f(\theta)|^2 \, d\theta. \tag{13.33}$$

Given f and $|f|^2$ integrable on S^1 (we say f is square integrable), this defines a "norm," satisfying the following result, called the triangle inequality:

$$\|f+g\|_{L^2} \le \|f\|_{L^2} + \|g\|_{L^2}. \tag{13.34}$$

See Appendix H for details on this. Behind these results is the fact that

$$\|f\|_{L^2}^2 = (f,f)_{L^2}, \tag{13.35}$$

where, when f and g are square integrable, we define the inner product

$$(f,g)_{L^2} = \frac{1}{2\pi} \int\limits_{S^1} f(\theta)\overline{g(\theta)}\, d\theta. \tag{13.36}$$

Thus the content of (13.29) is that

$$\sum |\hat{f}(k)|^2 = \|f\|_{L^2}^2, \tag{13.37}$$

and that of (13.30) is that

$$\sum \hat{f}(k)\overline{\hat{g}(k)} = (f,g)_{L^2}. \tag{13.38}$$

The left side of (13.37) is the square norm of the sequence $(\hat{f}(k))$ in ℓ^2. Generally, a sequence (a_k) $(k \in \mathbb{Z})$ belongs to ℓ^2 if and only if

$$\|(a_k)\|_{\ell^2}^2 = \sum |a_k|^2 < \infty. \tag{13.39}$$

There is an associated inner product

$$((a_k),(b_k)) = \sum a_k \overline{b_k}. \tag{13.40}$$

As in (13.34), one has (see Appendix H)

$$\|(a_k)+(b_k)\|_{\ell^2} \le \|(a_k)\|_{\ell^2} + \|(b_k)\|_{\ell^2}. \tag{13.41}$$

As for the notion of L^2-norm convergence, we say

$$f_\nu \to f \quad \text{in} \quad L^2 \Longleftrightarrow \|f-f_\nu\|_{L^2} \to 0. \tag{13.42}$$

There is a similar notion of convergence in ℓ^2. Clearly

$$\|f-f_\nu\|_{L^2} \le \sup_\theta |f(\theta)-f_\nu(\theta)|. \tag{13.43}$$

In view of the uniform convergence $S_N f \to f$ for $f \in \mathcal{A}(S^1)$ noted above, we have

$$f \in \mathcal{A}(S^1) \Longrightarrow S_N f \to f \text{ in } L^2, \text{ as } N \to \infty. \tag{13.44}$$

The triangle inequality implies

$$\Big| \|f\|_{L^2} - \|S_N f\|_{L^2} \Big| \le \|f - S_N f\|_{L^2}, \tag{13.45}$$

and clearly (by Proposition 13.4)

$$\|S_N f\|_{L^2}^2 = \sum_{k=-N}^{N} |\hat{f}(k)|^2, \tag{13.46}$$

so

$$\|f - S_N f\|_{L^2} \to 0 \text{ as } N \to \infty \Longrightarrow \|f\|_{L^2}^2 = \sum |\hat{f}(k)|^2. \tag{13.47}$$

We now consider more general square integrable functions f on S^1. With $\hat{f}(k)$ and $S_N f$ defined by (13.3) and (13.32), we define $R_N f$ by

$$f = S_N f + R_N f. \tag{13.48}$$

Note that $\int_{S^1} f(\theta) e^{-ik\theta}\, d\theta = \int_{S^1} S_N f(\theta) e^{-ik\theta}\, d\theta$ for $|k| \le N$, hence

$$(f, S_N f)_{L^2} = (S_N f, S_N f)_{L^2}, \tag{13.49}$$

and hence

$$(S_N f, R_N f)_{L^2} = 0. \tag{13.50}$$

Consequently,

$$\begin{aligned} \|f\|_{L^2}^2 &= (S_N f + R_N f, S_N f + R_N f)_{L^2} \\ &= \|S_N f\|_{L^2}^2 + \|R_N f\|_{L^2}^2. \end{aligned} \tag{13.51}$$

In particular,

$$\|S_N f\|_{L^2} \le \|f\|_{L^2}. \tag{13.52}$$

We are now in a position to prove the following.

Lemma 13.5. *Let f, f_ν be square integrable on S^1. Assume*

$$\lim_{\nu\to\infty} \|f - f_\nu\|_{L^2} = 0, \tag{13.53}$$

and, for each ν,

$$\lim_{N\to\infty} \|f_\nu - S_N f_\nu\|_{L^2} = 0. \tag{13.54}$$

Then

$$\lim_{N\to\infty} \|f - S_N f\|_{L^2} = 0. \tag{13.55}$$

In such a case, (13.47) holds.

Proof. Writing $f - S_N f = (f - f_\nu) + (f_\nu - S_N f_\nu) + S_N(f_\nu - f)$, and using the triangle inequality, we have, for each ν,

$$\|f - S_N f\|_{L^2} \le \|f - f_\nu\|_{L^2} + \|f_\nu - S_N f_\nu\|_{L^2} + \|S_N(f_\nu - f)\|_{L^2}. \tag{13.56}$$

Taking $N \to \infty$ and using (13.52), we have

$$\limsup_{N\to\infty} \|f - S_N f\|_{L^2} \le 2\|f - f_\nu\|_{L^2}, \tag{13.57}$$

for each ν. Then (13.53) yields the desired conclusion (13.55).

Given $f \in C(S^1)$, we have seen that $J_r f \to f$ uniformly (hence in L^2-norm) as $r \nearrow 1$. Also, $J_r f \in \mathcal{A}(S^1)$ for each $r \in (0,1)$. Thus (13.44) and Lemma 13.5 yield the following.

$$\begin{aligned} f \in C(S^1) \Longrightarrow S_N f \to f \ \text{ in } \ L^2, \ \text{and} \\ \sum |\hat{f}(k)|^2 = \|f\|_{L^2}^2. \end{aligned} \tag{13.58}$$

Lemma 13.5 also applies to many discontinuous functions. Consider, for example

$$\begin{aligned} f(\theta) = 0 \quad &\text{for} \quad -\pi < \theta < 0, \\ 1 \quad &\text{for} \ \ 0 < \theta < \pi. \end{aligned} \tag{13.59}$$

We can set, for $\nu \in \mathbb{N}$,

$$\begin{aligned} f_\nu(\theta) = 0 \quad &\text{for} \quad -\pi \le \theta \le 0, \\ \nu\theta \quad &\text{for} \ \ 0 \le \theta \le \frac{1}{\nu}, \\ 1 \quad &\text{for} \ \ \frac{1}{\nu} \le \theta \le \pi - \frac{1}{\nu} \\ \nu(\pi - \theta) \ &\text{for} \ \ \pi - \frac{1}{\nu} \le \theta \le \pi. \end{aligned} \tag{13.60}$$

Then each $f_\nu \in C(S^1)$. (In fact, $f_\nu \in \mathcal{A}(S^1)$, by Proposition 13.3.). Also, one can check that $\|f - f_\nu\|_{L^2}^2 \le 2/\nu$. Thus the conclusion in (13.58) holds for f given by (13.59).

More generally, any piecewise continuous function on S^1 is an L^2 limit of continuous functions, so the conclusion of (13.58) holds for them. To go further, let us recall the class of Riemann integrable functions. (Details can be found in Chapter 4, §2 of [T0] or in §0 of [T].) A function $f : S^1 \to \mathbb{R}$ is Riemann integrable provided f is bounded (say $|f| \le M$) and, for each $\delta > 0$, there exist piecewise constant functions g_δ and h_δ on S^1 such that

$$g_\delta \le f \le h_\delta, \quad \text{and} \quad \int\limits_{S^1} \big(h_\delta(\theta) - g_\delta(\theta)\big)\, d\theta < \delta. \tag{13.61}$$

Then

$$\int\limits_{S^1} f(\theta)\, d\theta = \lim_{\delta\to 0} \int\limits_{S^1} g_\delta(\theta)\, d\theta = \lim_{\delta\to 0} \int\limits_{S^1} h_\delta(\theta)\, d\theta. \tag{13.62}$$

Note that we can assume $|h_\delta|,\ |g_\delta| < M+1$, and so

$$\begin{aligned} \frac{1}{2\pi} \int\limits_{S^1} |f(\theta) - g_\delta(\theta)|^2\, d\theta &\le \frac{M+1}{\pi} \int\limits_{S^1} |h_\delta(\theta) - g_\delta(\theta)|\, d\theta \\ &< \frac{M+1}{\pi}\delta, \end{aligned} \tag{13.63}$$

so $g_\delta \to f$ in L^2-norm. A function $f : S^1 \to \mathbb{C}$ is Riemann integrable provided its real and imaginary parts are. In such a case, there are also piecewise constant functions $f_\nu \to f$ in L^2-norm, so

$$\begin{aligned} f \text{ Riemann interable on } S^1 \Longrightarrow S_N f \to f \ \text{ in } \ L^2, \text{ and} \\ \sum |\hat{f}(k)|^2 = \|f\|_{L^2}^2. \end{aligned} \tag{13.64}$$

This is not the end of the story. There are unbounded functions on S^1 that are square integrable, such as

$$f(\theta) = |\theta|^{-\alpha} \ \text{ on } \ [-\pi,\pi], \quad 0 < \alpha < \frac{1}{2}. \tag{13.65}$$

In such a case, one can take $f_\nu(\theta) = \min(f(\theta), \nu),\ \nu \in \mathbb{N}$. Then each f_ν is continuous and $\|f - f_\nu\|_{L^2} \to 0$ as $\nu \to \infty$. Hence the conclusion of (13.64) holds for such f.

The ultimate theory of functions for which the result

$$S_N f \longrightarrow f \ \text{ in } \ L^2\text{-norm} \tag{13.66}$$

holds was produced by H. Lebesgue in what is now known as the theory of Lebesgue measure and integration. There is the notion of measurability of a function $f : S^1 \to$

$\mathbb{C}$. One says $f \in L^2(S^1)$ provided f is measurable and $\int_{S^1} |f(\theta)|^2 \, d\theta < \infty$, the integral here being the Lebesgue integral. Actually, $L^2(S^1)$ consists of equivalence classes of such functions, where $f_1 \sim f_2$ if and only if $\int |f_1(\theta) - f_2(\theta)|^2 \, d\theta = 0$. With ℓ^2 as in (13.39), it is then the case that

$$\mathcal{F} : L^2(S^1) \longrightarrow \ell^2, \tag{13.67}$$

given by

$$(\mathcal{F}f)(k) = \hat{f}(k), \tag{13.68}$$

is one-to-one and onto, with

$$\sum |\hat{f}(k)|^2 = \|f\|_{L^2}^2, \quad \forall \, f \in L^2(S^1), \tag{13.69}$$

and

$$S_N f \longrightarrow f \ \text{ in } \ L^2, \quad \forall \, f \in L^2(S^1). \tag{13.70}$$

For the reader who has not seen Lebesgue integration, we refer to books on the subject (eg., [T3]) for more information.

We mention two key propositions which, together with the arguments given above, establish these results. The fact that $\mathcal{F}f \in \ell^2$ for all $f \in L^2(S^1)$ and (13.69)–(13.70) hold follows via Lemma 13.5 from the following.

Proposition A. *Given $f \in L^2(S^1)$, there exist $f_\nu \in C(S^1)$ such that $f_\nu \to f$ in L^2.*

As for the surjectivity of $\mathcal{F}$ in (13.67), note that, given $(a_k) \in \ell^2$, the sequence

$$f_\nu(\theta) = \sum_{|k| \le \nu} a_k e^{ik\theta}$$

satisfies, for $\mu > \nu$,

$$\|f_\mu - f_\nu\|_{L^2}^2 = \sum_{\nu < |k| \le \mu} |a_k|^2 \to 0 \ \text{ as } \ \nu \to \infty.$$

That is to say, (f_ν) is a Cauchy sequence in $L^2(S^1)$. Surjectivity follows from the fact that Cauchy sequences in $L^2(S^1)$ always converge to a limit:

Proposition B. *If (f_ν) is a Cauchy sequence in $L^2(S^1)$, there exists $f \in L^2(S^1)$ such that $f_\nu \to f$ in L^2-norm.*

Proofs of these results can be found in the standard texts on measure theory and integration, such as [T3].

We now establish a sufficient condition for a function f to belong to $\mathcal{A}(S^1)$, more general than that in Proposition 13.3.

Proposition 13.6. *If f is a continuous, piecewise C^1 function on S^1, then $\sum |\hat{f}(k)| < \infty$.*

Proof. As in the proof of Proposition 13.3, we can reduce the problem to the case $f \in C^1([-\pi, \pi])$, $f(-\pi) = f(\pi)$. In such a case, with $g = f' \in C([-\pi, \pi])$, the integration by parts argument (13.16) gives

$$\hat{f}(k) = \frac{1}{ik} \hat{g}(k), \quad k \neq 0. \tag{13.71}$$

By (13.64),

$$\sum |\hat{g}(k)|^2 = \|g\|_{L^2}^2. \tag{13.72}$$

Also, by Cauchy's inequality (cf. Appendix H),

$$\begin{aligned} \sum_{k \neq 0} |\hat{f}(k)| &\leq \Bigl(\sum_{k \neq 0} \frac{1}{k^2}\Bigr)^{1/2} \Bigl(\sum_{k \neq 0} |\hat{g}(k)|^2\Bigr)^{1/2} \\ &\leq C \|g\|_{L^2}. \end{aligned} \tag{13.73}$$

This completes the proof.

There is a great deal more that can be said about convergence of Fourier series. For example, material presented in the appendix to the next section has an analogue for Fourier series. We also mention Chapter 5, §4 in [T0]. For further results, one can consult treatments of Fourier analysis such as Chapter 3 of [T2].

Fourier series connects with the theory of harmonic functions, as follows. Taking $z = re^{i\theta}$ in the unit disk, we can write (13.4) as

$$J_r f(\theta) = \sum_{k=0}^{\infty} \hat{f}(k) z^k + \sum_{k=1}^{\infty} \hat{f}(-k) \overline{z}^k. \tag{13.74}$$

We write this as

$$(\mathrm{PI}\ f)(z) = (\mathrm{PI}_+\ f)(z) + (\mathrm{PI}_-\ f)(z), \tag{13.75}$$

a sum of a holomorphic function and a conjugate-holomorphic function on the unit disk D. Thus the left side is a *harmonic* function, called the Poisson integral of f.

Given $f \in C(S^1)$, PI f is the unique function in $C^2(D) \cap C(\overline{D})$ equal to f on $\partial D = S^1$ (uniqueness being a consequence of Proposition 7.4). Using (13.6)–(13.8), we can write the following Poisson integral formula:

$$\mathrm{PI}\ f(z) = \frac{1 - |z|^2}{2\pi} \int_{S^1} \frac{f(w)}{|w - z|^2}\, ds(w), \tag{13.76}$$

the integral being with respect to arclength on S^1. To see this, note that if $w = e^{i\theta'}$ and $z = re^{i\theta}$, then $ds(w) = d\theta'$ and

$$\begin{aligned} |w - z|^2 &= (e^{i\theta'} - re^{i\theta})(e^{-i\theta'} - re^{-i\theta}) \\ &= 1 - r(e^{i(\theta - \theta')} + e^{-i(\theta - \theta')}) + r^2. \end{aligned}$$

Since solutions to $\Delta u = 0$ remain solutions upon translation and dilation of coordinates, we have the following result.

Proposition 13.7. *If $D \subset \mathbb{C}$ is an open disk and $f \in C(\partial D)$ is given, there exists a unique $u \in C(\overline{D}) \cap C^2(D)$ satisfying*

$$\Delta u = 0 \quad \text{on} \quad D, \quad u\big|_{\partial D} = f. \tag{13.77}$$

We call (13.77) the Dirichlet boundary problem.

Now we make use of Proposition 13.7 to improve the version of the Schwarz reflection principle given in Proposition 8.2. As in the discussion of the Schwarz reflection principle in §8, we assume $\Omega \subset \mathbb{C}$ is a connected, open set that is symmetric with respect to the real axis, so $z \in \Omega \Rightarrow \overline{z} \in \Omega$. We set $\Omega^{\pm} = \{z \in \Omega : \pm \operatorname{Im} z > 0\}$ and $L = \Omega \cap \mathbb{R}$.

Proposition 13.8. *Assume $u : \Omega^+ \cup L \to \mathbb{C}$ is continuous, harmonic on Ω^+, and $u = 0$ on L. Define $v : \Omega \to \mathbb{C}$ by*

$$\begin{aligned} v(z) = {}& u(z), \quad z \in \Omega^+ \cup L, \\ & -u(\overline{z}), \quad z \in \Omega^-. \end{aligned} \tag{13.78}$$

Then v is harmonic on Ω.

Proof. It is readily verified that v is harmonic in $\Omega^+ \cup \Omega^-$ and continuous on Ω. We need to show that v is harmonic on a neighborhood of each point $p \in L$. Let $D = D_r(p)$ be a disk centered at p such that $\overline{D} \subset \Omega$. Let $f \in C(\partial D)$ be given by $f = v|_{\partial D}$. Let $w \in C^2(D) \cap C(\overline{D})$ be the unique harmonic function on D equal to f on ∂D.

Since f is odd with respect to reflection about the real axis, so is w, so $w = 0$ on $\overline{D} \cap \mathbb{R}$. Thus both v and w are harmonic on $D^+ = D \cap \{\operatorname{Im} z > 0\}$, and continuous on $\overline{D}^+$, and agree on ∂D^+, so the maximum principle implies $w = v$ on $\overline{D}^+$. Similarly $w = v$ on $\overline{D}^-$, and this gives the desired harmonicity of v.

Using Proposition 13.8, we establish the following stronger version of Proposition 8.2, the Schwarz reflection principle, weakening the hypothesis that f is continuous on $\Omega^+ \cup L$ to the hypothesis that $\operatorname{Im} f$ is continuous on $\Omega^+ \cup L$ (and vanishes on L). While this improvement may seem a small thing, it can be quite useful, as we will see in §24.

Proposition 13.9. *Let Ω be as in Proposition 13.8, and assume $f : \Omega^+ \to \mathbb{C}$ is holomorphic. Assume $\operatorname{Im} f$ extends continuously to $\Omega^+ \cup L$ and vanishes on L. Define $g : \Omega^+ \cup \Omega^-$ by*

$$\begin{aligned} g(z) = {}& f(z), \quad z \in \Omega^+, \\ & \overline{f(\overline{z})}, \quad z \in \Omega^-. \end{aligned} \tag{13.79}$$

Then g extends to a holomorphic function on Ω.

Proof. It suffices to prove this under the additional assumption that Ω is a disk. We apply Proposition 13.8 to $u(z) = \operatorname{Im} f(z)$ on Ω^+, 0 on L, obtaining a harmonic extension

$v : \Omega \to \mathbb{R}$. By Proposition 7.1, v has a harmonic conjugate $w : \Omega \to \mathbb{R}$, so $v + iw$ is holomorphic, and hence $h : \Omega \to \mathbb{C}$, given by

$$h(z) = -w(z) + iv(z), \tag{13.80}$$

is holomorphic. Now $\operatorname{Im} h = \operatorname{Im} f$ on Ω^+, so $g - h$ is real valued on Ω^+, so, being holomorphic, it must be constant. Thus, altering w by a real constant, we have

$$h(z) = g(z), \quad z \in \Omega^+. \tag{13.81}$$

Also, $\operatorname{Im} h(z) = v(z) = 0$ on L, so (cf. Exercise 1 in §10)

$$h(z) = \overline{h(\overline{z})}, \quad \forall\, z \in \Omega. \tag{13.82}$$

It follows from this and (13.79) that

$$h(z) = g(z), \quad \forall\, z \in \Omega^+ \cup \Omega^-, \tag{13.83}$$

so h is the desired holomorphic extension.

Exercises

1. Verify the evaluation of the integral in (13.22). Use the evaluation of (13.23) at $\theta = 0$ (as done in (13.24)) to show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \tag{13.84}$$

2. Compute $\hat{f}(k)$ when

$$\begin{aligned} f(\theta) = 1 \quad &\text{for} \quad 0 < \theta < \pi, \\ 0 \quad &\text{for} \quad -\pi < \theta < 0. \end{aligned}$$

Then use (13.64) to obtain another proof of (13.84).

3. Apply (13.29) when $f(\theta)$ is given by (13.21). Use this to show that

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}.$$

4. Give the details for (13.76), as a consequence of (13.6) and (13.8).

5. Suppose f is holomorphic on an annulus Ω containing the unit circle S^1, with Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n.$$

Show that

$$a_n = \hat{g}(n), \quad g = f\big|_{S^1}.$$

Compare this with (12.8), with $z_0 = 0$ and γ the unit circle S^1.

Exercises 6–8 deal with the convolution of functions on S^1, defined by

$$f * g(\theta) = \frac{1}{2\pi} \int\limits_{S^1} f(\varphi) g(\theta - \varphi) \, d\varphi.$$

6. Show that

$$h = f * g \Longrightarrow \hat{h}(k) = \hat{f}(k)\hat{g}(k).$$

7. Show that

$$f, g \in L^2(S^1), \quad h = f * g \Longrightarrow \sum_{k=-\infty}^{\infty} |\hat{h}(k)| < \infty.$$

8. Let χ be the characteristic function of $[-\pi/2, \pi/2]$, regarded as an element of $L^2(S^1)$. Compute $\hat{\chi}(k)$ and $\chi * \chi(\theta)$. Relate these computations to (13.21)–(13.23).

9. Show that a formula equivalent to (13.76) is

$$\text{PI}\, f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big(\text{Re}\, \frac{e^{i\theta} + z}{e^{i\theta} - z}\Big) f(\theta) \, d\theta. \tag{13.85}$$

(We abuse notation by confounding $f(\theta)$ and $f(e^{i\theta})$, identifying S^1 and $\mathbb{R}/(2\pi\mathbb{Z})$.)

10. Give the details for the improvement of Proposition 8.2 mentioned right after the proof of Proposition 13.8. Proposition 7.6 may come in handy.

11. Given $f(\theta) = \sum_k a_k e^{ik\theta}$, show that f is real valued on S^1 if and only if

$$\overline{a}_k = a_{-k}, \quad \forall\, k \in \mathbb{Z}.$$

12. Let $f \in C(S^1)$ be real valued. Show that $\text{PI}\, f$ and $(1/i)\,\text{PI}\, g$ are harmonic conjugates, provided

$$\begin{aligned} \hat{g}(k) = \hat{f}(k) \quad & \text{for} \quad k > 0, \\ -\hat{f}(-k) \quad & \text{for} \quad k < 0. \end{aligned}$$

13. Using Exercise 1 and (13.15A), in concert with Exercise 19 of §4, show that

$$\begin{aligned}\frac{\pi^2}{6} &= \int_0^1 \frac{1}{x} \log \frac{1}{1-x}\,dx \\ &= \int_0^\infty \frac{t}{e^t - 1}\,dt.\end{aligned}$$

14. Let Ω be symmetric about $\mathbb{R}$, as in Proposition 13.8. Suppose f is holomorphic and nowhere vanishing on Ω^+ and $|f(z)| \to 1$ as $z \to L$. Show that f extends to be holomorphic on Ω, with $|f(z)| = 1$ for $z \in L$.
Hint. Consider the harmonic function $u(z) = \log |f(z)| = \operatorname{Re} \log f(z)$.

15. Establish the variant of Exercise 14 (and strengthening of Exercise 3 from §8). Take $a > 1$. Suppose f is holomorphic and nowhere vanishing on the annulus $1 < |z| < a$ and that $|f(z)| \to 1$ as $|z| \to 1$. Show that f extends to a holomorphic function g on $1/a < |z| < a$, satisfying $g(z) = f(z)$ for $1 < |z| < a$ and

$$g(z) = \frac{1}{\overline{f(1/\overline{z})}}, \quad \frac{1}{a} < |z| < 1.$$

14. Fourier transforms

Take a function f that is integrable on $\mathbb{R}$, so

$$\|f\|_{L^1} = \int_{-\infty}^{\infty} |f(x)|\,dx < \infty. \tag{14.1}$$

We define the Fourier transform of f to be

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ix\xi}\,dx, \quad \xi \in \mathbb{R}. \tag{14.2}$$

Similarly, we set

$$\mathcal{F}^*f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{ix\xi}\,dx, \quad \xi \in \mathbb{R}, \tag{14.2A}$$

and ultimately plan to identify $\mathcal{F}^*$ as the inverse Fourier transform.

Clearly the Fourier transform of an integrable function is bounded:

$$|\hat{f}(\xi)| \le \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|\,dx. \tag{14.3}$$

We also have continuity.

Proposition 14.1. *If f is integrable on $\mathbb{R}$, then $\hat{f}$ is continuous on $\mathbb{R}$.*

Proof. Given $\varepsilon > 0$, pick $N < \infty$ such that $\int_{|x|>N} |f(x)|\,dx < \varepsilon$. Write $f = f_N + g_N$, where $f_N(x) = f(x)$ for $|x| \le N,\ 0$ for $|x| > N$. Then

$$\hat{f}(\xi) = \hat{f}_N(\xi) + \hat{g}_N(\xi), \tag{14.4}$$

and

$$|\hat{g}_N(\xi)| < \frac{\varepsilon}{\sqrt{2\pi}}, \quad \forall\, \xi. \tag{14.5}$$

Meanwhile, for $\xi, \zeta \in \mathbb{R}$,

$$\hat{f}_N(\xi) - \hat{f}_N(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} f(x)\Big(e^{-ix\xi} - e^{-ix\zeta}\Big)\,dx, \tag{14.6}$$

and

$$\begin{aligned} |e^{-ix\xi} - e^{-ix\zeta}| &\le |\xi - \zeta| \max_{\eta} \Big|\frac{\partial}{\partial\eta} e^{-ix\eta}\Big| \\ &\le |x| \cdot |\xi - \zeta| \\ &\le N|\xi - \zeta|, \end{aligned} \tag{14.6A}$$

for $|x| \le N$, so

$$|\hat{f}_N(\xi) - \hat{f}_N(\zeta)| \le \frac{N}{\sqrt{2\pi}} \|f\|_{L^1} \, |\xi - \zeta|, \tag{14.7}$$

where $\|f\|_{L^1}$ is defined by (14.1). Hence each $\hat{f}_N$ is continuous, and, by (14.5), $\hat{f}$ is a uniform limit of continuous functions, so it is continuous.

The Fourier inversion formula asserts that

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi} \, d\xi, \tag{14.8}$$

in appropriate senses, depending on the nature of f. We approach this in a spirit similar to the Fourier inversion formula (13.1) of the previous section. First we sneak up on (14.8) by inserting a factor of $e^{-\varepsilon\xi^2}$. Set

$$J_\varepsilon f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-\varepsilon\xi^2} e^{ix\xi} \, d\xi. \tag{14.9}$$

Note that, by (14.3), whenever $f \in L^1(\mathbb{R})$, $\hat{f}(\xi) e^{-\varepsilon\xi^2}$ is integrable for each $\varepsilon > 0$. Furthermore, we can plug in (14.2) for $\hat{f}(\xi)$ and switch order of integration, getting

$$\begin{aligned} J_\varepsilon f(x) &= \frac{1}{2\pi} \iint f(y) e^{i(x-y)\xi} e^{-\varepsilon\xi^2} \, dy \, d\xi \\ &= \int_{-\infty}^{\infty} f(y) H_\varepsilon(x-y) \, dy, \end{aligned} \tag{14.10}$$

where

$$H_\varepsilon(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\varepsilon\xi^2 + ix\xi} \, d\xi. \tag{14.11}$$

A change of variable shows that $H_\varepsilon(x) = (1/\sqrt{\varepsilon}) H_1(x/\sqrt{\varepsilon})$, and the computation of $H_1(x)$ is accomplished in §10; we see that $H_1(x) = (1/2\pi) G(ix)$, with $G(z)$ defined by (10.3) and computed in (10.8). We obtain

$$H_\varepsilon(x) = \frac{1}{\sqrt{4\pi\varepsilon}} e^{-x^2/4\varepsilon}. \tag{14.12}$$

The computation $\int e^{-x^2} \, dx = \sqrt{\pi}$ done in (10.6) implies

$$\int_{-\infty}^{\infty} H_\varepsilon(x) \, dx = 1, \quad \forall \, \varepsilon > 0. \tag{14.13}$$

We see that $H_\varepsilon(x)$ is highly peaked near $x = 0$ as $\varepsilon \searrow 0$. An argument parallel to that used to prove Proposition 13.1 then establishes the following.

Proposition 14.2. *Assume f is integrable on $\mathbb{R}$. Then*

$$J_\varepsilon f(x) \to f(x) \quad \textit{whenever} \quad f \quad \textit{is continuous at} \quad x. \tag{14.14}$$

If, in addition, f is continuous on $\mathbb{R}$ and $f(x) \to 0$ as $|x| \to \infty$, then $J_\varepsilon f(x) \to f(x)$ uniformly on $\mathbb{R}$.

From here, parallel to Proposition 13.2, we have:

Corollary 14.3. *Assume f is bounded and continuous on $\mathbb{R}$, and f and $\hat{f}$ are integrable on $\mathbb{R}$. Then (14.8) holds for all $x \in \mathbb{R}$.*

Proof. If $f \in L^1(\mathbb{R})$, then $\hat{f}$ is bounded and continuous. If also $\hat{f} \in L^1(\mathbb{R})$, then $\mathcal{F}^*\hat{f}$ is continuous. Furthermore, arguments similar to those used to prove Proposition 13.2 show that the right side of (14.9) converges to the right side of (14.8) as $\varepsilon \searrow 0$. That is to say,

$$J_\varepsilon f(x) \longrightarrow \mathcal{F}^*\hat{f}(x), \quad \text{as} \quad \varepsilon \to 0. \tag{14.14A}$$

It follows from (14.14) that $f(x) = \mathcal{F}^*\hat{f}(x)$.

Remark. With some more work, one can omit the hypothesis in Corollary 14.3 that f be bounded and continuous, and use (14.14A) to deduce these properties as a conclusion. This sort of reasoning is best carried out in a course on measure theory and integration.

At this point, we take the space to discuss integrable functions and square integrable functions on $\mathbb{R}$. Examples of integrable functions on $\mathbb{R}$ are bounded, piecewise continuous functions satisfying (14.1). More generally, f could be Riemann integrable on each interval $[-N, N]$, and satisfy

$$\lim_{N\to\infty} \int_{-N}^{N} |f(x)|\,dx = \|f\|_{L^1} < \infty, \tag{14.15}$$

where Riemann integrability on $[-N, N]$ has a definition similar to that given in (13.61)–(13.62) for functions on S^1. Still more general is Lebesgue's class, consisting of measurable functions $f : \mathbb{R} \to \mathbb{C}$ satisfying (14.1), where the Lebesgue integral is used. An element of $L^1(\mathbb{R})$ consists of an equivalence class of such functions, where we say $f_1 \sim f_2$ provided $\int_{-\infty}^{\infty} |f_1 - f_2|\,dx = 0$. The quantity $\|f\|_{L^1}$ is called the L^1 norm of f. It satisfies the triangle inequality

$$\|f + g\|_{L^1} \le \|f\|_{L^1} + \|g\|_{L^1}, \tag{14.16}$$

as an easy consequence of the pointwise inequality $|f(x)+g(x)| \le |f(x)|+|g(x)|$ (cf. (0.14)). Thus $L^1(\mathbb{R})$ has the structure of a metric space, with $d(f, g) = \|f - g\|_{L^1}$. We say $f_\nu \to f$ in L^1 if $\|f_\nu - f\|_{L^1} \to 0$. Parallel to Propositions A and B of §13, we have the following.

Proposition A1. *Given $f \in L^1(\mathbb{R})$ and $k \in \mathbb{N}$, there exist $f_\nu \in C_0^k(\mathbb{R})$ such that $f_\nu \to f$ in L^1.*

Here, $C_0^k(\mathbb{R})$ denotes the space of functions with compact support whose derivatives of order $\le k$ exist and are continuous. There is also the following completeness result.

Proposition B1. *If (f_ν) is a Cauchy sequence in $L^1(\mathbb{R})$, there exists $f \in L^1(\mathbb{R})$ such that $f_\nu \to f$ in L^1.*

As in §13, we will also be interested in square integrable functions. A function $f : \mathbb{R} \to \mathbb{C}$ is said to be square integrable if f and $|f|^2$ are integrable on each finite interval $[-N, N]$ and

$$\|f\|_{L^2}^2 = \int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty. \tag{14.17}$$

Taking the square root gives $\|f\|_{L^2}$, called the L^2-norm of f. Parallel to (13.34) and (14.16), there is the triangle inequality

$$\|f + g\|_{L^2} \le \|f\|_{L^2} + \|g\|_{L^2}. \tag{14.18}$$

The proof of (14.18) is not as easy as that of (14.16), but, like (13.34), it follows, via results of Appendix H, from the fact that

$$\|f\|_{L^2}^2 = (f, f)_{L^2}, \tag{14.19}$$

where, for square integrable functions f and g, we define the inner product

$$(f, g)_{L^2} = \int_{-\infty}^{\infty} f(x)\overline{g(x)} \, dx. \tag{14.20}$$

The triangle inequality (14.18) makes $L^2(\mathbb{R})$ a metric space, with distance function $d(f, g) = \|f - g\|_{L^2}$, and we say $f_\nu \to f$ in L^2 if $\|f_\nu - f\|_{L^2} \to 0$. Parallel to Propositions A1 and B1, we have the following.

Proposition A2. *Given $f \in L^2(\mathbb{R})$ and $k \in \mathbb{N}$, there exist $f_\nu \in C_0^k(\mathbb{R})$ such that $f_\nu \to f$ in L^2.*

Proposition B2. *If (f_ν) is a Cauchy sequence in $L^2(\mathbb{R})$, there exists $f \in L^2(\mathbb{R})$ such that $f_\nu \to f$ in L^2.*

As in §13, we refer to books on measure theory, such as [T3], for further material on $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$, including proofs of results stated above.

Somewhat parallel to (13.28), we set

$$\mathcal{A}(\mathbb{R}) = \{f \in L^1(\mathbb{R}) : f \text{ bounded and continuous}, \ \hat{f} \in L^1(\mathbb{R})\}. \tag{14.21}$$

By Corollary 14.3, the Fourier inversion formula (14.8) holds for all $f \in \mathcal{A}(\mathbb{R})$. It also follows that $f \in \mathcal{A}(\mathbb{R}) \Rightarrow \hat{f} \in \mathcal{A}(\mathbb{R})$. Note also that

$$\mathcal{A}(\mathbb{R}) \subset L^2(\mathbb{R}). \tag{14.22}$$

In fact, if $f \in \mathcal{A}(\mathbb{R})$,

$$\begin{aligned} \|f\|_{L^2}^2 &= \int |f(x)|^2 \, dx \\ &\le \sup |f(x)| \cdot \int |f(x)| \, dx \\ &\le \frac{1}{\sqrt{2\pi}} \|\hat{f}\|_{L^1} \|f\|_{L^1}. \end{aligned} \tag{14.23}$$

It is of interest to know when $f \in \mathcal{A}(\mathbb{R})$. We mention one simple result here. Namely, if $f \in C^k(\mathbb{R})$ has compact support (we say $f \in C_0^k(\mathbb{R})$), then integration by parts yields

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^{(j)}(x) e^{-ix\xi} \, dx = (i\xi)^j \hat{f}(\xi), \quad 0 \le j \le k. \tag{14.24}$$

Hence

$$C_0^2(\mathbb{R}) \subset \mathcal{A}(\mathbb{R}). \tag{14.25}$$

While (14.25) is crude, it will give us a start on the L^2-theory of the Fourier transform. Let us denote by $\mathcal{F}$ the map $f \mapsto \hat{f}$, and by $\mathcal{F}^*$ the map you get upon replacing $e^{-ix\xi}$ by $e^{ix\xi}$. Then, with respect to the inner product (14.20), we have, for $f, g \in \mathcal{A}(\mathbb{R})$,

$$(\mathcal{F} f, g) = (f, \mathcal{F}^* g). \tag{14.26}$$

Now combining (14.26) with Corollary 14.3 we have

$$f, g \in \mathcal{A}(\mathbb{R}) \Longrightarrow (\mathcal{F} f, \mathcal{F} g) = (\mathcal{F}^* \mathcal{F} f, g) = (f, g). \tag{14.27}$$

One readily obtains a similar result with $\mathcal{F}$ replaced by $\mathcal{F}^*$. Hence

$$\|\mathcal{F} f\|_{L^2} = \|\mathcal{F}^* f\|_{L^2} = \|f\|_{L^2}, \tag{14.28}$$

for $f, g \in \mathcal{A}(\mathbb{R})$.

The result (14.28) is called the Plancherel identity. Using it, we can extend $\mathcal{F}$ and $\mathcal{F}^*$ to act on $L^2(\mathbb{R})$, obtaining (14.28) and the Fourier inversion formula on $L^2(\mathbb{R})$.

Proposition 14.4. *The maps $\mathcal{F}$ and $\mathcal{F}^*$ have unique continuous linear extensions from*

$$\mathcal{F}, \mathcal{F}^* : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R}) \tag{14.28A}$$

to

$$\mathcal{F}, \mathcal{F}^* : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}), \tag{14.29}$$

and the identities

$$\mathcal{F}^*\mathcal{F}f = f, \quad \mathcal{F}\mathcal{F}^*f = f \tag{14.30}$$

hold for all $f \in L^2(\mathbb{R})$, as does (14.28).

This result can be proven using Propositions A2 and B2, and the inclusion (14.25), which together with Proposition A2 implies that

$$\text{For each } f \in L^2(\mathbb{R}),\ \exists f_\nu \in \mathcal{A}(\mathbb{R}) \text{ such that } f_\nu \to f \text{ in } L^2. \tag{14.31}$$

The argument goes like this. Given $f \in L^2(\mathbb{R})$, take $f_\nu \in \mathcal{A}(\mathbb{R})$ such that $f_\nu \to f$ in L^2. Then $\|f_\mu - f_\nu\|_{L^2} \to 0$ as $\mu, \nu \to \infty$. Now (14.28), applied to $f_\mu - f_\nu \in \mathcal{A}(\mathbb{R})$, gives

$$\|\mathcal{F}f_\mu - \mathcal{F}f_\nu\|_{L^2} = \|f_\mu - f_\nu\|_{L^2} \to 0, \tag{14.32}$$

as $\mu, \nu \to \infty$. Hence $(\mathcal{F}f_\mu)$ is a Cauchy sequence in $L^2(\mathbb{R})$. By Proposition B2, there exists a limit $h \in L^2(\mathbb{R})$; $\mathcal{F}f_\nu \to h$ in L^2. One gets the same element h regardless of the choice of (f_ν) such that (14.31) holds, and so we set $\mathcal{F}f = h$. The same argument applies to $\mathcal{F}^*f_\nu$, which hence converges to $\mathcal{F}^*f$. We have

$$\|\mathcal{F}f_\nu - \mathcal{F}f\|_{L^2},\ \|\mathcal{F}^*f_\nu - \mathcal{F}^*f\|_{L^2} \to 0. \tag{14.33}$$

From here, the result (14.30) and the extension of (14.28) to $L^2(\mathbb{R})$ follow.

Given $f \in L^2(\mathbb{R})$, we have

$$\chi_{[-R,R]}\hat{f} \longrightarrow \hat{f} \ \text{ in } \ L^2, \quad \text{as } \ R \to \infty,$$

so Proposition 14.4 yields the following.

Proposition 14.5. *Define S_R by*

$$S_Rf(x) = \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \hat{f}(\xi)e^{ix\xi}\, d\xi. \tag{14.34}$$

Then

$$f \in L^2(\mathbb{R}) \Longrightarrow S_Rf \to f \quad \textit{in} \quad L^2(\mathbb{R}), \tag{14.35}$$

as $R \to \infty$.

Having Proposition 14.4, we can sharpen (14.25) as follows.

Proposition 14.6. *There is the inclusion*

$$C_0^1(\mathbb{R}) \subset \mathcal{A}(\mathbb{R}). \tag{14.36}$$

Proof. Given $f \in C_0^1(\mathbb{R})$, we can use (14.24) with $j = k = 1$ to get

$$g = f' \Longrightarrow \hat{g}(\xi) = i\xi \hat{f}(\xi). \tag{14.37}$$

Proposition 14.4 implies

$$\|(1 + i\xi)\hat{f}\|_{L^2} = \|f + f'\|_{L^2}. \tag{14.38}$$

Now, parallel to the proof of Proposition 13.6, we have

$$\begin{aligned} \|\hat{f}\|_{L^1} &= \int \bigl|(1+i\xi)^{-1}\bigr| \cdot \bigl|(1+i\xi)\hat{f}(\xi)\bigr|\, d\xi \\ &\le \Bigl\{\int \frac{d\xi}{1+\xi^2}\Bigr\}^{1/2} \Bigl\{\int \bigl|(1+i\xi)\hat{f}(\xi)\bigr|^2\, d\xi\Bigr\}^{1/2} \\ &= \sqrt{\pi}\, \|f + f'\|_{L^2}, \end{aligned} \tag{14.39}$$

the inequality in (14.39) by Cauchy's inequality (cf. (H.18)) and the last identity by (14.37)–(14.38). This proves (14.36).

Remark. Parallel to Proposition 13.6, one can extend Proposition 14.6 to show that if f has compact support, is continuuous, and is piecewise C^1 on $\mathbb{R}$, then $f \in \mathcal{A}(\mathbb{R})$. In conjunction with (14.39), the following is useful for identifying other elements of $\mathcal{A}(\mathbb{R})$.

Proposition 14.7. *Let $f_\nu \in \mathcal{A}(\mathbb{R})$ and $f \in C(\mathbb{R}) \cap L^1(\mathbb{R})$. Assume*

$$f_\nu \to f \ \text{ in } L^1\text{-norm, and } \ \|\hat{f}_\nu\|_{L^1} \le A,$$

for some $A < \infty$. Then $f \in \mathcal{A}(\mathbb{R})$.

Proof. Clearly $\hat{f}_\nu \to \hat{f}$ uniformly on $\mathbb{R}$. Hence, for each $R < \infty$,

$$\int_{-R}^{R} |\hat{f}_\nu(\xi) - \hat{f}(\xi)|\, d\xi \longrightarrow 0, \quad \text{as} \quad \nu \to \infty.$$

Thus

$$\int_{-R}^{R} |\hat{f}(\xi)|\, d\xi \le A, \quad \forall\, R < \infty,$$

and it follows that $\hat{f} \in L^1(\mathbb{R})$, completing the proof.

The interested reader can consult §14A for a still sharper condition guaranteeing that $f \in \mathcal{A}(\mathbb{R})$.

Exercises

In Exercises 1–2, assume $f : \mathbb{R} \to \mathbb{C}$ is a C^2 function satisfying

$$|f^{(j)}(x)| \le C(1+|x|)^{-2}, \quad j \le 2. \tag{14.40}$$

1. Show that

$$|\hat{f}(\xi)| \le \frac{C'}{\xi^2+1}, \quad \xi \in \mathbb{R}. \tag{14.41}$$

Deduce that $f \in \mathcal{A}(\mathbb{R})$.

2. With $\hat{f}$ given as in (14.1), show that

$$\frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx} = \sum_{\ell=-\infty}^{\infty} f(x+2\pi\ell). \tag{14.42}$$

This is known as the Poisson summation formula.
Hint. Let g denote the right side of (14.42), pictured as an element of $C^2(S^1)$. Relate the Fourier series of g (à la §13) to the left side of (14.42).

3. Use $f(x) = e^{-x^2/4t}$ in (14.42) to show that, for $\tau > 0$,

$$\sum_{\ell=-\infty}^{\infty} e^{-\pi\ell^2\tau} = \sqrt{\frac{1}{\tau}} \sum_{k=-\infty}^{\infty} e^{-\pi k^2/\tau}. \tag{14.43}$$

This is a Jacobi identity.
Hint. Use (14.11)–(14.12) to get $\hat{f}(\xi) = \sqrt{2t}\, e^{-t\xi^2}$. Take $t = \pi\tau$, and set $x = 0$ in (14.42).

4. For each of the following functions $f(x)$, compute $\hat{f}(\xi)$.

(a) $$f(x) = e^{-|x|},$$

(b) $$f(x) = \frac{1}{1+x^2},$$

(c) $$f(x) = \chi_{[-1/2,1/2]}(x),$$

(d) $$f(x) = (1-|x|)\chi_{[-1,1]}(x),$$

Here $\chi_I(x)$ is the characteristic function of a set $I \subset \mathbb{R}$. Reconsider the computation of (b) when you get to §16.

5. In each case, (a)–(d), of Exercise 4, record the identity that follows from the Plancherel identity (14.28). In particular, show that

$$\int_{-\infty}^{\infty} \frac{\sin^2 \xi}{\xi^2}\, d\xi = \pi.$$

Exercises 6–8 deal with the convolution of functions on $\mathbb{R}$, defined by

$$f * g(x) = \int_{-\infty}^{\infty} f(y) g(x-y)\, dy. \tag{14.44}$$

6. Show that

$$\|f * g\|_{L^1} \le \|f\|_{L^1} \|g\|_{L^1}, \quad \sup |f * g| \le \|f\|_{L^2} \|g\|_{L^2}.$$

7. Show that

$$\widehat{(f * g)}(\xi) = \sqrt{2\pi}\, \hat{f}(\xi) \hat{g}(\xi).$$

8. Compute $f * f$ when $f(x) = \chi_{[-1/2,1/2]}(x)$, the characteristic function of the interval $[-1/2, 1/2]$. Compare the result of Exercise 7 with the computation of (d) in Exercise 4.

9. Prove the following result, known as the Riemann-Lebesgue lemma.

$$f \in L^1(\mathbb{R}) \Longrightarrow \lim_{|\xi| \to \infty} |\hat{f}(\xi)| = 0. \tag{14.45}$$

Hint. (14.41) gives the desired conclusion for $\hat{f}_\nu$ when $f_\nu \in C_0^2(\mathbb{R})$. Then use Proposition A1 and apply (14.3) to $f - f_\nu$, to get $\hat{f}_\nu \to \hat{f}$ uniformly.

10. Sharpen the result of Exercise 1 as follows, using the reasoning in the proof of Proposition 14.6. Assume $f : \mathbb{R} \to \mathbb{C}$ is a C^1 function satisfying

$$|f^{(j)}(x)| \le C(1 + |x|)^{-2}, \quad j \le 1. \tag{14.45A}$$

Then show that $f \in \mathcal{A}(\mathbb{R})$. More generally, show that $f \in \mathcal{A}(\mathbb{R})$ provided that f is Lipschitz continuous on $\mathbb{R}$, C^1 on $(-\infty, 0]$ and on $[0, \infty)$, and (14.45A) holds for all $x \neq 0$.

14A. More general sufficient condition for $f \in \mathcal{A}(\mathbb{R})$

Here we establish a result substantially sharper than Proposition 14.6. We mention that an analogous result holds for Fourier series. The interested reader can investigate this.

To set things up, given $f \in L^2(\mathbb{R})$, let

$$f_h(x) = f(x + h). \tag{14.46}$$

Our goal here is to prove the following.

Proposition 14.8. *If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and there exists $C < \infty$ such that*

$$\|f - f_h\|_{L^2} \le C h^\alpha, \quad \forall\, h \in [0,1], \tag{14.47}$$

with

$$\alpha > \frac{1}{2}, \tag{14.48}$$

then $f \in \mathcal{A}(\mathbb{R})$.

Proof. A calculation gives

$$\hat{f}_h(\xi) = e^{ih\xi} \hat{f}(\xi), \tag{14.49}$$

so, by the Plancherel identity,

$$\|f - f_h\|_{L^2}^2 = \int_{-\infty}^{\infty} |1 - e^{ih\xi}|^2 \, |\hat{f}(\xi)|^2 \, d\xi. \tag{14.50}$$

Now,

$$\frac{\pi}{2} \le |h\xi| \le \frac{3\pi}{2} \Longrightarrow |1 - e^{ih\xi}|^2 \ge 2, \tag{14.51}$$

so

$$\|f - f_h\|_{L^2}^2 \ge 2 \int\limits_{\frac{\pi}{2} \le |h\xi| \le \frac{3\pi}{2}} |\hat{f}(\xi)|^2 \, d\xi. \tag{14.52}$$

If (14.47) holds, we deduce that, for $h \in (0,1]$,

$$\int\limits_{\frac{2}{h} \le |\xi| \le \frac{4}{h}} |\hat{f}(\xi)|^2 \, d\xi \le C h^{2\alpha}, \tag{14.53}$$

hence (setting $h = 2^{-\ell+1}$), for $\ell \ge 1$,

$$\int\limits_{2^\ell \le |\xi| \le 2^{\ell+1}} |\hat{f}(\xi)|^2 \, d\xi \le C 2^{-2\alpha\ell}. \tag{14.51}$$

Cauchy's inequality gives

$$\begin{aligned} &\int\limits_{2^\ell \le |\xi| \le 2^{\ell+1}} |\hat{f}(\xi)| \, d\xi \\ &\le \Big\{ \int\limits_{2^\ell \le |\xi| \le 2^{\ell+1}} |\hat{f}(\xi)|^2 \, d\xi \Big\}^{1/2} \times \Big\{ \int\limits_{2^\ell \le |\xi| \le 2^{\ell+1}} 1 \, d\xi \Big\}^{1/2} \\ &\le C 2^{-\alpha\ell} \cdot 2^{\ell/2} \\ &= C 2^{-(\alpha - 1/2)\ell}. \end{aligned} \tag{14.55}$$

Summing over $\ell \in \mathbb{N}$ and using (again by Cauchy's inequality)

$$\int\limits_{|\xi|\le 2} |\hat{f}|\, d\xi \le C\|\hat{f}\|_{L^2} = C\|f\|_{L^2}, \tag{14.56}$$

then gives the proof.

To see how close to sharp Proposition 14.8 is, consider

$$\begin{aligned} f(x) = \chi_I(x) = 1 \quad &\text{if} \ \ 0 \le x \le 1, \\ 0 \quad &\text{otherwise}. \end{aligned} \tag{14.57}$$

We have, for $0 \le h \le 1$,

$$\|f - f_h\|_{L^2}^2 = 2h, \tag{14.58}$$

so (14.47) holds, with $\alpha = 1/2$. Since $\mathcal{A}(\mathbb{R}) \subset C(\mathbb{R})$, this function does not belong to $\mathcal{A}(\mathbb{R})$, so the condition (14.48) is about as sharp as it could be.

Remark. Using

$$\int |gh|\, dx \le \sup |g| \int |h|\, dx, \tag{14.59}$$

we have the estimate

$$\|f - f_h\|_{L^2}^2 \le \sup_x |f(x) - f_h(x)| \cdot \|f - f_h\|_{L^1}, \tag{14.60}$$

so, with

$$\|f\|_{BV} = \sup_{0<h\le 1} \|h^{-1}(f - f_h)\|_{L^1}, \quad \|f\|_{C^r} = \sup_{x\in\mathbb{R}, 0<h\le 1} h^{-r}|f(x) - f_h(x)|, \tag{14.61}$$

for $0 < r < 1$, we have

$$\|f - f_h\|_{L^2}^2 \le h^{1+r}\|f\|_{BV}\|f\|_{C^r}, \tag{14.62}$$

which can be applied to the hypothesis (14.47) in Proposition 14.8.

14B. Fourier uniqueness

The Fourier inversion formula established in Corollary 14.3 yields

$$f \in \mathcal{A}(\mathbb{R}), \ \hat{f} = 0 \Longrightarrow f = 0. \tag{14.63}$$

Similarly, Proposition 14.4 yields

$$f \in L^2(\mathbb{R}), \ \hat{f} = 0 \Longrightarrow f = 0. \tag{14.64}$$

We call these Fourier uniqueness results. An extension of (14.63) is the following consequence of Proposition 14.2:

$$f \in L^1(\mathbb{R}) \cap C(\mathbb{R}), \ \hat{f} = 0 \Longrightarrow f = 0. \tag{14.65}$$

Here, we advertize the following strengthening of (14.63).

Proposition 14.9. *We have the implication*

$$f \in L^1(\mathbb{R}),\ \hat{f} = 0 \Longrightarrow f = 0. \tag{14.66}$$

We indicate a proof of this result, starting with the following variant of (14.26). If $f \in L^1(\mathbb{R})$ and also $g \in L^1(\mathbb{R})$, then

$$\begin{aligned} \int_{-\infty}^{\infty} \hat{f}(\xi) g(\xi)\, d\xi &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) e^{-ix\xi} g(\xi)\, dx\, d\xi \\ &= \int_{-\infty}^{\infty} f(x) \hat{g}(x)\, dx, \end{aligned} \tag{14.67}$$

where the second identity uses a change in the order of integration. Thus

$$f \in L^1(\mathbb{R}),\ \hat{f} = 0 \Longrightarrow \int_{-\infty}^{\infty} f(x) h(x)\, dx = 0, \tag{14.68}$$

for all $h = \hat{g},\ g \in L^1(\mathbb{R})$. In particular, (14.64) holds for all $h \in \mathcal{A}(\mathbb{R})$, and so, by (14.25), it holds for all $h \in C_0^2(\mathbb{R})$. The implication

$$f \in L^1(\mathbb{R}),\ \int f(x) h(x)\, dx = 0\ \forall\, h \in C_0^2(\mathbb{R}) \Longrightarrow f = 0 \tag{14.69}$$

is a basic result in a course in measure theory and integration.

15. Laplace transforms and Mellin transforms

Suppose we have a function $f : \mathbb{R}^+ \to \mathbb{C}$ that is integrable on $[0, R]$ for all $R < \infty$ and satisfies

$$\int_0^\infty |f(t)| e^{-at}\, dt < \infty, \quad \forall\, a > A, \tag{15.1}$$

for some $A \in (-\infty, \infty)$. We define the Laplace transform of f by

$$\mathcal{L}f(s) = \int_0^\infty f(t) e^{-st}\, dt, \quad \text{Re } s > A. \tag{15.2}$$

It is clear that this integral is absolutely convergent for each s in the half-plane $H_A = \{z \in \mathbb{C} : \text{Re } z > A\}$ and defines a continuous function $\mathcal{L}f : H_A \to \mathbb{C}$. Also, if γ is a closed curve (e.g., the boundary of a rectangle) in H_A, we can change order of integration to see that

$$\int\limits_\gamma \mathcal{L}f(s)\, ds = \int_0^\infty \int\limits_\gamma f(t) e^{-st}\, ds\, dt = 0. \tag{15.3}$$

Hence Morera's theorem implies $\mathcal{L}f$ is holomorphic on H_A. We have

$$\frac{d}{ds} \mathcal{L}f(s) = \mathcal{L}g(s), \quad g(t) = -t f(t). \tag{15.4}$$

On the other hand, if $f \in C^1([0, \infty))$ and $\int_0^\infty |f'(t)| e^{-at}\, dt < \infty$ for all $a > A$, then we can integrate by parts and get

$$\mathcal{L}f'(s) = s\mathcal{L}f(s) - f(0), \tag{15.5}$$

and a similar hypothesis on higher derivatives of f gives

$$\mathcal{L}f^{(k)}(s) = s^k \mathcal{L}f(s) - s^{k-1} f(0) - \cdots - f^{(k-1)}(0). \tag{15.6}$$

Thus, if f satisfies an ODE of the form

$$c_n f^{(n)}(t) + c_{n-1} f^{(n-1)}(t) + \cdots + c_0 f(t) = g(t) \tag{15.7}$$

for $t \ge 0$, with initial data

$$f(0) = a_0, \ldots, f^{(n-1)}(0) = a_{n-1}, \tag{15.8}$$

and hypotheses yielding (15.6) hold for all $k \le n$, we have

$$p(s)\mathcal{L}f(s) = \mathcal{L}g(s) + q(s), \tag{15.9}$$

with

$$\begin{aligned} p(s) &= c_n s^n + c_{n-1} s^{n-1} + \cdots + c_0, \\ q(s) &= c_n(a_0 s^{n-1} + \cdots + a_{n-1}) + \cdots + c_1 a_0. \end{aligned} \tag{15.10}$$

If all the roots of $p(s)$ satisfy $\mathrm{Re}\, s \le B$, we have

$$\mathcal{L}f(s) = \frac{\mathcal{L}g(s) + q(s)}{p(s)}, \quad s \in H_C,\ C = \max\{A, B\}, \tag{15.11}$$

and we are motivated to seek an inverse Laplace transform.

We can get this by relating the Laplace transform to the Fourier transform. In fact, if (15.1) holds, and if $B > A$, then

$$\mathcal{L}f(B + i\xi) = \sqrt{2\pi}\,\hat{\varphi}(\xi), \quad \xi \in \mathbb{R}, \tag{15.12}$$

with

$$\begin{aligned} \varphi(x) = f(x)e^{-Bx}&, \quad x \ge 0, \\ 0 \quad &, \quad x < 0. \end{aligned} \tag{15.13}$$

In §14 we have seen several senses in which

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\varphi}(\xi) e^{ix\xi}\, d\xi, \tag{15.14}$$

hence giving, for $t > 0$,

$$\begin{aligned} f(t) &= \frac{e^{Bt}}{2\pi} \int_{-\infty}^{\infty} \mathcal{L}f(B + i\xi) e^{i\xi t}\, d\xi \\ &= \frac{1}{2\pi i} \int_\gamma \mathcal{L}f(s) e^{st}\, ds, \end{aligned} \tag{15.15}$$

where γ is the vertical line $\gamma(\xi) = B + i\xi,\ -\infty < \xi < \infty$.

For example, if φ in (15.13) belongs to $L^2(\mathbb{R})$, then (15.15) holds in the sense of Proposition 14.5. If φ belongs to $\mathcal{A}(\mathbb{R})$, then (15.15) holds in the sense of Corollary 14.3. Frequently, f is continuous on $[0, \infty)$ but $f(0) \neq 0$. Then φ in (15.13) has a discontinuity at $x = 0$, so $\varphi \notin \mathcal{A}(\mathbb{R})$. However, sometimes one has $\psi(x) = x\varphi(x)$ in $\mathcal{A}(\mathbb{R})$, which is obtained as in (15.13) by replacing $f(t)$ by $tf(t)$. (See Exercise 7 below.) In light of (15.4), we obtain

$$-tf(t) = \frac{1}{2\pi i} \int_\gamma \frac{d}{ds} \mathcal{L}f(s)\, e^{st}\, ds, \tag{15.16}$$

with an absolutely convergent integral, provided $\psi \in \mathcal{A}(\mathbb{R})$.

Related to such inversion formulas is the following uniqueness result, which, via (15.12)–(15.13), is an immediate consequence of Proposition 14.9.

Proposition 15.1. *If f_1 and f_2 are integrable on $[0,R]$ for all $R<\infty$ and satisfy (15.1), then*

$$\mathcal{L}f_1(s)=\mathcal{L}f_2(s),\quad \forall\, s\in H_A \Longrightarrow f_1=f_2 \quad \text{on} \quad \mathbb{R}^+. \tag{15.17}$$

We can also use material of §10 to deduce that $f_1=f_2$ given $\mathcal{L}f_1(s)=\mathcal{L}f_2(s)$ on a set with an accumulation point in H_A.

We next introduce a transform called the Mellin transform:

$$\mathcal{M}f(z)=\int_0^\infty f(t)t^{z-1}\,dt, \tag{15.18}$$

defined and holomoprhic on $A<\operatorname{Re} z<B$, provided $f(t)t^{x-1}$ is integrable for real $x\in(A,B)$. This is related to the Laplace transform via a change of variable, $t=e^x$:

$$\mathcal{M}f(z)=\int_{-\infty}^\infty f(e^x)e^{zx}\,dx. \tag{15.19}$$

Assuming $0\in(A,B)$, evaluation of these integrals for z on the imaginary axis yields

$$\begin{aligned}\mathcal{M}^\# f(\xi)&=\int_0^\infty f(t)t^{i\xi-1}\,dt\\ &=\int_{-\infty}^\infty f(e^x)e^{ix\xi}\,dx.\end{aligned} \tag{15.20}$$

The Fourier inversion formula and Plancherel formula imply

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^\infty (\mathcal{M}^\# f)(\xi)t^{-i\xi}\,d\xi, \tag{15.21}$$

and

$$\int_{-\infty}^\infty |\mathcal{M}^\# f(\xi)|^2\,d\xi=2\pi\int_0^\infty |f(t)|^2\,\frac{dt}{t}. \tag{15.22}$$

If $0\notin(A,B)$ but $\tau\in(A,B)$, one can evaluate $\mathcal{M}f(z)$ on the vertical axis $z=\tau+i\xi$, and modify (15.20)–(15.22) accordingly.

Exercises

1. Show that the Laplace transform of $f(t)=t^{z-1}$,

$$\mathcal{L}f(s)=\int_0^\infty e^{-st}t^{z-1}\,dt,\quad \operatorname{Re} z>0, \tag{15.23}$$

is given by

$$\mathcal{L}f(s) = \Gamma(z)s^{-z}, \tag{15.24}$$

where $\Gamma(z)$ is the Gamma function:

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,dt. \tag{15.25}$$

For more on this function, see §18.
REMARK. The integral (15.23) has the remarkable property of being simultaneously a Laplace transform and a Mellin transform. It is the simplest integral with this property.

2. Compute the Laplace transforms of the following functions (defined for $t \geq 0$).

(a) e^{at},
(b) $\cosh at$,
(c) $\sinh at$,
(d) $\sin at$,
(e) $t^{z-1}\,e^{at}$.

3. Compute the inverse Laplace transforms of the following functions (defined in appropriate right half-spaces).

(a) $\dfrac{1}{s-a}$,
(b) $\dfrac{s}{s^2-a^2}$,
(c) $\dfrac{a}{s^2-a^2}$,
(d) $\dfrac{a}{s^2+a^2}$,
(e) $\dfrac{1}{\sqrt{s+1}}$.

Reconsider these problems when you read §16.

Exercises 4–6 deal with the convolution of functions on $\mathbb{R}^+$, defined by

$$f * g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau. \tag{15.26}$$

4. Show that (15.26) coincides with the definition (14.44) of convolution, provided $f(t)$ and $g(t)$ vanish for $t < 0$.

5. Show that if $f_z(t) = t^{z-1}$ for $t > 0$, with Re $z > 0$, and if also Re $\zeta > 0$, then

$$f_z * f_\zeta(t) = B(z,\zeta)\, f_{z+\zeta}(t), \tag{15.27}$$

with

$$B(z,\zeta) = \int_0^1 s^{z-1}(1-s)^{\zeta-1}\, ds. \tag{15.28}$$

6. Show that

$$\mathcal{L}(f * g)(s) = \mathcal{L}f(s) \cdot \mathcal{L}g(s). \tag{15.29}$$

See §18 for an identity resulting from applying (15.29) to (15.27).

7. Assume $f \in C^1([0,\infty))$ satisfies

$$|f(x)|,\ |f'(x)| \le Ce^{Ax}, \quad x \ge 0.$$

Take $B > A$ and define $\varphi(x)$ as in (15.13). Show that

$$\psi(x) = x\varphi(x) \Longrightarrow \psi \in \mathcal{A}(\mathbb{R}).$$

Hint. Use the result of Exercise 10 in §14.

The matrix Laplace transform and Duhamel's formula

The matrix Laplace transform allows one to treat $n \times n$ first-order systems of differential equations, of the form

$$f'(t) = Kf(t) + g(t), \quad f(0) = a, \tag{15.30}$$

in a fashion parallel to (15.7)–(15.11). Here K is an $n \times n$ matrix,

$$K \in M(n,\mathbb{C}), \quad a \in \mathbb{C}^n, \quad g(t) \in \mathbb{C}^n, \quad \forall\, t, \tag{15.31}$$

and we seek a solution $f(t)$, taking values in $\mathbb{C}^n$.

A key ingredient in this study is the matrix exponential

$$e^{tK} = \sum_{j=0}^{\infty} \frac{t^j}{j!} K^j, \quad K \in M(n,\mathbb{C}), \tag{15.32}$$

which has a development parallel to that of the exponential (for $K \in \mathbb{C}$) given in §0. In particular, we have

$$\frac{d}{dt} e^{tK} = K e^{tK} = e^{tK} K, \quad e^{(t+\tau)K} = e^{tK} e^{\tau K}. \tag{15.33}$$

See Appendix N for more details, and definitions of some of the concepts used below, such as the norm $\|K\|$ of the matrix K. Using the notation

$$E_K(t) = e^{tK}, \tag{15.34}$$

we have

$$\mathcal{L}E_K(s) = \int_0^\infty e^{tK} e^{-st}\, dt, \tag{15.35}$$

valid whenever

$$\|e^{tK}\| \le c e^{\alpha t}, \quad \operatorname{Re} s > \alpha. \tag{15.36}$$

In particular, the first estimate always holds for $\alpha = \|K\|$.

Turning to (15.30), if we assume g and f have Laplace transforms and apply $\mathcal{L}$ to both sides, then (15.5) continues to apply, and we get

$$s\mathcal{L}f(s) - a = K\mathcal{L}f(s) + \mathcal{L}g(s), \tag{15.37}$$

hence

$$\mathcal{L}f(s) = (sI - K)^{-1}(a + \mathcal{L}g(s)), \tag{15.38}$$

if $\operatorname{Re} s$ is sufficiently large. To solve (15.30), it suffices to identify the right side of (15.38) as the Laplace transform of a function.

To start, we assert that, with $E_K(t) = e^{tK}$,

$$\mathcal{L}E_K(s) = (sI - K)^{-1}, \tag{15.39}$$

whenever s satisfies (15.36). To see this, let $L \in M(n, \mathbb{C})$ and note that the identity $(d/dt)e^{-tL} = -Le^{-tL}$ implies

$$L \int_0^T e^{-tL}\, dt = I - e^{-TL}, \tag{15.40}$$

for each $T \in (0, \infty)$. If L satisfies

$$\|e^{-tL}\| \le c e^{-\delta t}, \quad \forall\, t > 0, \tag{15.41}$$

for some $\delta > 0$, then we can take $T \to \infty$ in (15.40), and deduce that

$$L \int_0^\infty e^{-tL}\, dt = I, \quad \text{i.e.,} \quad \int_0^\infty e^{-tL}\, dt = L^{-1}. \tag{15.42}$$

Clearly (15.41) applies to $L = sI - K$ as long as (15.36) holds, since

$$\|e^{t(K-sI)}\| = e^{-t \operatorname{Re} s} \|e^{tK}\|, \tag{15.43}$$

so we have (15.39). This gives

$$(sI - K)^{-1} a = \mathcal{L}(E_K a)(s). \tag{15.44}$$

Also, by (15.29),

$$(sI - K)^{-1} \mathcal{L}g(s) = \mathcal{L}(E_K * g)(s), \tag{15.45}$$

where

$$\begin{aligned} E_K * g(t) &= \int_0^t E_K(t - \tau) g(\tau)\, d\tau \\ &= \int_0^t e^{(t-\tau)K} g(\tau)\, d\tau. \end{aligned} \tag{15.46}$$

Now (15.38) yields $f(t) = E_K(t)a + E_K * g(t)$. In conclusion, the solution to (15.30) is given by

$$f(t) = e^{tK} a + \int_0^t e^{(t-\tau)K} g(\tau)\, d\tau. \tag{15.47}$$

This is known as Duhamel's formula.

Here is another route to the derivation of Duhamel's formula. We seek the solution to (15.30) in the form

$$f(t) = e^{tK} F(t), \tag{15.48}$$

and find that $F(t)$ satisfies a differential equation that is simpler than (15.30). In fact, differentiating (15.48) and using (15.33) gives

$$\begin{aligned} \frac{df}{dt} &= e^{tK} \Big(\frac{dF}{dt} + KF(t) \Big), \\ Kf(t) + g(t) &= Ke^{tK} F(t) + g(t), \end{aligned} \tag{15.49}$$

hence

$$\frac{dF}{dt} = e^{-tK} g(t), \quad F(0) = a. \tag{15.50}$$

Simply integrating this gives

$$F(t) = a + \int_0^t e^{-\tau K} g(\tau)\, d\tau, \tag{15.51}$$

and applying e^{tK} to both sides (and using the last identity in (15.33)) again gives (15.47).

H. Inner product spaces

On occasion, particularly in §§13–14, we have looked at norms and inner products on spaces of functions, such as $C(S^1)$ and $\mathcal{S}(\mathbb{R})$, which are vector spaces. Generally, a complex vector space V is a set on which there are operations of vector addition:

$$f, g \in V \Longrightarrow f + g \in V, \tag{H.1}$$

and multiplication by an element of $\mathbb{C}$ (called scalar multiplication):

$$a \in \mathbb{C},\ f \in V \Longrightarrow af \in V, \tag{H.2}$$

satisfying the following properties. For vector addition, we have

$$f + g = g + f,\ (f + g) + h = f + (g + h),\ f + 0 = f,\ f + (-f) = 0. \tag{H.3}$$

For multiplication by scalars, we have

$$a(bf) = (ab)f, \quad 1 \cdot f = f. \tag{H.4}$$

Furthermore, we have two distributive laws:

$$a(f + g) = af + ag, \quad (a + b)f = af + bf. \tag{H.5}$$

These properties are readily verified for the function spaces arising in §§13–14.

An inner product on a complex vector space V assigns to elements $f, g \in V$ the quantity $(f, g) \in \mathbb{C}$, in a fashion that obeys the following three rules:

$$\begin{gathered} (a_1 f_1 + a_2 f_2, g) = a_1 (f_1, g) + a_2 (f_2, g), \\ (f, g) = \overline{(g, f)}, \\ (f, f) > 0 \quad \text{unless} \quad f = 0. \end{gathered} \tag{H.6}$$

A vector space equipped with an inner product is called an inner product space. For example,

$$(f, g) = \frac{1}{2\pi} \int\limits_{S^1} f(\theta) \overline{g(\theta)}\, d\theta \tag{H.7}$$

defines an inner product on $C(S^1)$. Similarly,

$$(f, g) = \int_{-\infty}^{\infty} f(x) \overline{g(x)}\, dx \tag{H.8}$$

defines an inner product on $\mathcal{S}(\mathbb{R})$ (defined in §14). As another example, in §13 we defined ℓ^2 to consist of sequences $(a_k)_{k\in\mathbb{Z}}$ such that

$$\sum_{k=-\infty}^{\infty} |a_k|^2 < \infty. \tag{H.9}$$

An inner product on ℓ^2 is given by

$$\big((a_k),(b_k)\big) = \sum_{k=-\infty}^{\infty} a_k\overline{b_k}. \tag{H.10}$$

Given an inner product on V, one says the object $\|f\|$ defined by

$$\|f\| = \sqrt{(f,f)} \tag{H.11}$$

is the *norm* on V associated with the inner product. Generally, a norm on V is a function $f \mapsto \|f\|$ satisfying

$$\|af\| = |a|\cdot\|f\|, \quad a\in\mathbb{C},\ f\in V, \tag{H.12}$$

$$\|f\| > 0 \quad \text{unless} \quad f = 0, \tag{H.13}$$

$$\|f+g\| \le \|f\| + \|g\|. \tag{H.14}$$

The property (H.14) is called the triangle inequality. A vector space equipped with a norm is called a normed vector space. We can define a distance function on such a space by

$$d(f,g) = \|f-g\|. \tag{H.15}$$

Properties (H.12)–(H.14) imply that $d : V\times V \to [0,\infty)$ satisfies the properties in (A.1), making V a metric space.

If $\|f\|$ is given by (H.11), from an inner product satisfying (H.6), it is clear that (H.12)–(H.13) hold, but (H.14) requires a demonstration. Note that

$$\begin{aligned} \|f+g\|^2 &= (f+g,f+g) \\ &= \|f\|^2 + (f,g) + (g,f) + \|g\|^2 \\ &= \|f\|^2 + 2\,\mathrm{Re}(f,g) + \|g\|^2, \end{aligned} \tag{H.16}$$

while

$$(\|f\|+\|g\|)^2 = \|f\|^2 + 2\|f\|\cdot\|g\| + \|g\|^2. \tag{H.17}$$

Thus to establish (H.17) it suffices to prove the following, known as Cauchy's inequality.

Proposition H.1. *For any inner product on a vector space V, with $\|f\|$ defined by (H.11),*

$$|(f,g)| \le \|f\| \cdot \|g\|, \quad \forall\, f,g \in V. \tag{H.18}$$

Proof. We start with

$$0 \le \|f-g\|^2 = \|f\|^2 - 2\,\mathrm{Re}(f,g) + \|g\|^2, \tag{H.19}$$

which implies

$$2\,\mathrm{Re}(f,g) \le \|f\|^2 + \|g\|^2, \quad \forall\, f,g \in V. \tag{H.20}$$

Replacing f by af for arbitrary $a \in \mathbb{C}$ of absolute velue 1 yields $2\,\mathrm{Re}\, a(f,g) \le \|f\|^2 + \|g\|^2$, for all such a, hence

$$2|(f,g)| \le \|f\|^2 + \|g\|^2, \quad \forall\, f,g \in V. \tag{H.21}$$

Replacing f by tf and g by $t^{-1}g$ for arbitrary $t \in (0,\infty)$, we have

$$2|(f,g)| \le t^2\|f\|^2 + t^{-2}\|g\|^2, \quad \forall\, f,g \in V,\ t \in (0,\infty). \tag{H.22}$$

If we take $t^2 = \|g\|/\|f\|$, we obtain the desired inequality (H.18). This assumes f and g are both nonzero, but (H.18) is trivial if f or g is 0.

An inner product space V is called a Hilbert space if it is a complete metric space, i.e., if every Cauchy sequence (f_ν) in V has a limit in V. The space ℓ^2 has this completeness property, but $C(S^1)$, with inner product (H.7), does not. Appendix A describes a process of constructing the completion of a metric space. When appied to an incomplete inner product space, it produces a Hilbert space. When this process is applied to $C(S^1)$, the completion is the space $L^2(S^1)$, briefly discussed in §13. This result is essentially the content of Propositions A and B, stated in §13.

There is a great deal more to be said about Hilbert space theory, but further material is not needed here. One can consult a book on functional analysis, or the appendix on functional analysis in Vol. 1 of [T2].

N. The matrix exponential

Let A be an $n \times n$ matrix with complex entries; we write $A \in M(n, \mathbb{C})$. Parallel to (0.49), we define

$$e^{tA} = \sum_{j=0}^{\infty} \frac{t^j}{j!} A^j. \tag{N.1}$$

To establish convergence of this series, we use the matrix norm

$$\|A\| = \sup\{\|Au\| : u \in \mathbb{C}^n,\ \|u\| \le 1\}, \tag{N.2}$$

where, if $u = (u_1, \dots, u_n)^t \in \mathbb{C}^n$, we set

$$\|u\| = (|u_1|^2 + \cdots + |u_n|^2)^{1/2}, \tag{N.3}$$

or equivalently, $\|u\|^2 = (u, u)$, where, if also $v = (v_1, \dots, v_n)^t \in \mathbb{C}^n$,

$$(u, v) = \sum_{k=1}^{n} u_k \overline{v}_k. \tag{N.4}$$

This makes $\mathbb{C}^n$ an inner product space, as treated in Appendix H. An equivalent characterization of $\|A\|$ is that it is the smallest constant K such that the estimate

$$\|Au\| \le K\|u\|, \quad \forall\, u \in \mathbb{C}^n \tag{N.5}$$

is valid. Given this, it is readily verified that, if also $B \in M(n, \mathbb{C})$, then

$$\|A + B\| \le \|A\| + \|B\|, \quad \text{and} \quad \|AB\| \le \|A\| \cdot \|B\|.$$

Consequently

$$\Bigl\| \sum_{j=\ell}^{\ell+m} \frac{t^j}{j!} A^j \Bigr\| \le \sum_{j=\ell}^{\ell+m} \frac{|t|^j}{j!} \|A\|^j. \tag{N.6}$$

Hence absolute convergence of (N.1) (uniformly for t in any bounded set) follows, via the ratio test, as it does in (0.49). We have

$$\|e^{tA}\| \le e^{|t| \cdot \|A\|}. \tag{N.7}$$

Just as in Proposition 0.4, we can differentiate (N.1) term by term, obtaining

$$\begin{aligned} \frac{d}{dt} e^{tA} &= \sum_{j=1}^{\infty} \frac{t^{j-1}}{(j-1)!} A^j \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} A^{k+1} \\ &= A e^{tA} = e^{tA} A. \end{aligned} \tag{N.8}$$

Next, we can differentiate the product $e^{-tA} e^{(s+t)A}$ to prove the following.

Proposition N.1. *For all $s, t \in \mathbb{R},\ A \in M(n, \mathbb{C})$,*

$$e^{(s+t)A} = e^{tA} e^{sA}. \tag{N.9}$$

Proof. To start, we have

$$\frac{d}{dt} e^{-tA} e^{(s+t)A} = -e^{-tA} A e^{(s+t)A} + e^{-tA} A e^{(s+t)A} = 0, \tag{N.10}$$

so $e^{-tA} e^{(s+t)A}$ is independent of $t \in \mathbb{R}$. Evaluating at $t = 0$, we have

$$e^{-tA} e^{(s+t)A} = e^{sA}, \quad \forall\, s, t \in \mathbb{R}. \tag{N.11}$$

Taking $s = 0$, we have

$$e^{-tA} e^{tA} = I, \quad \forall\, t \in \mathbb{R}, \tag{N.12}$$

where I is the identity matrix. With this in hand, we can multiply (N.11) on the left by e^{tA} and obtain (N.9).

Here is a useful extension of (N.7).

Proposition N.2. *Let $A, B \in M(n, \mathbb{C})$, and assume these matrices commute, i.e.,*

$$AB = BA. \tag{N.13}$$

Then, for all $t \in \mathbb{R}$,

$$e^{t(A+B)} = e^{tA} e^{tB}. \tag{N.14}$$

Proof. We compute that

$$\begin{aligned} \frac{d}{dt} e^{t(A+B)} e^{-tB} e^{-tA} &= e^{t(A+B)} (A+B) e^{-tB} e^{-tA} \\ &\quad - e^{t(A+B)} B e^{-tB} e^{-tA} \\ &\quad - e^{t(A+B)} e^{-tB} A e^{-tA}. \end{aligned} \tag{N.15}$$

We claim that (N.13) implies

$$A e^{-tB} = e^{-tB} A, \quad \forall\, t \in \mathbb{R}. \tag{N.16}$$

Given this, we see that (N.15) is 0, and then evaluating the triple matrix product at $t = 0$ yields

$$e^{t(A+B)} e^{-tB} e^{-tA} = I, \quad \forall\, t \in \mathbb{R}. \tag{N.17}$$

Using (N.12), we obtain (N.14) from this.

It remains to prove (N.16), which we rephrase as $Ae^{tB} = e^{tB}A$, for all $t \in \mathbb{R}$. Using the power series, we have

$$Ae^{tB} = \sum_{j=0}^{\infty} \frac{t^j}{j!} AB^j. \tag{N.18}$$

Then (N.13) yields $AB^j = B^j A$, so (N.18) is equal to

$$\sum_{j=0}^{\infty} \frac{t^j}{j!} B^j A = e^{tB} A, \tag{N.19}$$

and we are done.

These results will suffice for our needs in §15. Further material on the matrix exponential can be found in Chapter 3 of [T4].

G. The Weierstrass and Runge approximation theorems

In this appendix we discuss several approximation results, to the effect that a certain class of functions can be approximated uniformly by a sequence of functions of a more special type.

We start with the following result of Weierstrass, on the approximation by polynomials of an arbitrary continuous function on a closed bounded interval $[a, b] \subset \mathbb{R}$.

Theorem G.1. *If $f : [a, b] \to \mathbb{C}$ is continuous, then there exist polynomials $p_k(x)$ such that $p_k(x) \to f(x)$ uniformly on $[a, b]$.*

For the proof, first extend f to be continuous on $[a-1, b+1]$ and vanish at the endpoints. We leave it to the reader to do this. Then extend f to be 0 on $(-\infty, a-1]$ and on $[b+1, \infty)$. Then we have a continuous function $f : \mathbb{R} \to \mathbb{C}$ with compact support. We write $f \in C_0(\mathbb{R})$.

As seen in §14, if we set

$$H_\varepsilon(x) = \frac{1}{\sqrt{4\pi\varepsilon}}\, e^{-x^2/4\varepsilon}, \tag{G.1}$$

for $\varepsilon > 0$ and form

$$f_\varepsilon(x) = \int_{-\infty}^{\infty} H_\varepsilon(x-y) f(y)\, dy, \tag{G.2}$$

then $f_\varepsilon(x) \to f(x)$ as $\varepsilon \to 0$, uniformly for $x \in \mathbb{R}$, whenever $f \in C_0(\mathbb{R})$. In particular, given $\delta > 0$, there exists $\varepsilon > 0$ such that

$$|f_\varepsilon(x) - f(x)| < \delta, \quad \forall\, x \in \mathbb{R}. \tag{G.3}$$

Now note that, for each $\varepsilon > 0$, $f_\varepsilon(x)$ extends from $x \in \mathbb{R}$ to the entire holomorphic function

$$F_\varepsilon(z) = \frac{1}{\sqrt{4\pi\varepsilon}} \int_{-\infty}^{\infty} e^{-(z-y)^2/4\varepsilon} f(y)\, dy. \tag{G.4}$$

That this integral is absolutely convergent for each $z \in \mathbb{C}$ is elementary, and that it is holomorphic in z can be deduced from Morera's theorem. It follows that $F_\varepsilon(z)$ has a power series expansion,

$$F_\varepsilon(z) = \sum_{n=0}^{\infty} a_n(\varepsilon) z^n, \tag{G.5}$$

converging locally uniformly on $\mathbb{C}$. In particular, there exists $N = N(\varepsilon) \in \mathbb{Z}^+$ such that

$$|F_\varepsilon(z) - p_{N,\varepsilon}(z)| < \delta, \quad |z| \le R = \max\{|a|, |b|\}, \tag{G.6}$$

where

$$P_{N,\varepsilon}(z) = \sum_{n=0}^{N} a_n(\varepsilon) z^n. \tag{G.7}$$

Consequently, by (G.3) and (G.6),

$$|f(x) - p_{N,\varepsilon}(x)| < 2\delta, \quad \forall\, x \in [a,b]. \tag{G.8}$$

This proves Theorem G.1.

We next produce a useful variant of Theorem G.1. It will play a role in the proof of Karamata's Tauberian theorem, in Appendix R.

Proposition G.2. *Let $f \in C([0,\infty])$. Then for each $\varepsilon > 0$, there exists a function g of the form*

$$g(x) = \sum_{k=0}^{N} a_k e^{-kx} \tag{G.9}$$

such that $\sup_x |f(x) - g(x)| < \varepsilon$.

Proof. We use the homeomorphism $\varphi : [0,\infty] \to [0,1]$, given by

$$\begin{aligned} \varphi(x) = e^{-x}&, \quad 0 \le x < \infty, \\ 0&, \quad x = \infty, \end{aligned} \tag{G.10}$$

with inverse $\psi : [0,1] \to [0,\infty]$. Given $f \in C([0,\infty])$, take $F = f \circ \psi \in C([0,1])$. Use Theorem G.1 to produce a polynomial

$$G(t) = \sum_{k=0}^{N} a_k t^k \tag{G.11}$$

such that $\sup_t |F(t) - G(t)| < \varepsilon$, and then take $g = G \circ \varphi$, which has the form (G.9).

While Proposition G.2 is easy to establish, it is valuable to view it as a special case of a far reaching extension of the Weierstrass approximation theorem, due to M. Stone. The following result is known as the Stone-Weierstrass theorem.

Theorem G.3. *Let X be a compact metric space, $\mathcal{A}$ a subalgebra of $C_{\mathbb{R}}(X)$, the algebra of real valued continuous functions on X. Suppose $1 \in \mathcal{A}$ and that $\mathcal{A}$ separates points of X, i.e., for distinct $p, q \in X$, there exists $h_{pq} \in \mathcal{A}$ with $h_{pq}(p) \ne h_{pq}(q)$. Then the closure $\overline{\mathcal{A}}$ is equal to $C_{\mathbb{R}}(X)$.*

We present the proof in eight steps.

Step 1. Let $f \in \overline{\mathcal{A}}$ and assume $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous. If $\sup |f| \le A$, we can apply the Weierstrass approximation theorem to get polynomials $p_k \to \varphi$ uniformly on $[-A, A]$. Then $p_k \circ f \to \varphi \circ f$ uniformly on X, so $\varphi \circ f \in \overline{\mathcal{A}}$.

STEP 2. Consequently, if $f_j \in \overline{\mathcal{A}}$, then

$$\max(f_1, f_2) = \frac{1}{2}|f_1 - f_2| + \frac{1}{2}(f_1 + f_2) \in \overline{\mathcal{A}}, \tag{G.12}$$

and similarly $\min(f_1, f_2) \in \overline{\mathcal{A}}$.

STEP 3. It follows from the hypotheses that if $p, q \in X$ and $p \neq q$, then there exists $f_{pq} \in \mathcal{A}$, equal to 1 at p and to 0 at q.

STEP 4. Apply an appropriate continuous $\varphi : \mathbb{R} \to \mathbb{R}$ to get $g_{pq} = \varphi \circ f_{pq} \in \overline{\mathcal{A}}$, equal to 1 on a neighborhood of p and to 0 on a neighborhood of q, and satisfying $0 \le g_{pq} \le 1$ on X.

STEP 5. Fix $p \in X$ and let U be an open neighborhood of p. By Step 4, given $q \in X \setminus U$, there exists $g_{pq} \in \overline{\mathcal{A}}$ such that $g_{pq} = 1$ on a neighborhood $\mathcal{O}_q$ of p, equal to 0 on a neighborhood Ω_q of q, satisfying $0 \le g_{pq} \le 1$ on X.

Now $\{\Omega_q\}$ is an open cover of $X \setminus U$, so there exists a finite subcover $\Omega_{q_1}, \dots, \Omega_{q_N}$. Let

$$g_{pU} = \min_{1 \le j \le N} g_{pq_j} \in \overline{\mathcal{A}}. \tag{G.13}$$

Then $g_{pU} = 1$ on $\mathcal{O} = \cap_1^N \mathcal{O}_{q_j}$, an open neighborhood of p, $g_{pU} = 0$ on $X \setminus U$, and $0 \le g_{pU} \le 1$ on X.

STEP 6. Take $K \subset U \subset X$, K closed, U open. By Step 5, for each $p \in K$, there exists $g_{pU} \in \overline{\mathcal{A}}$, equal to 1 on a neighborhood $\mathcal{O}_p$ of p, and equal to 0 on $X \setminus U$.

Now $\{\mathcal{O}_p\}$ covers K, so there exists a finite subcover $\mathcal{O}_{p_1}, \dots, \mathcal{O}_{p_m}$. Let

$$g_{KU} = \max_{1 \le j \le M} g_{p_j U} \in \overline{\mathcal{A}}. \tag{G.14}$$

We have

$$g_{KU} = 1 \ \text{ on } \ K, \quad 0 \ \text{ on } \ X \setminus U, \quad \text{and} \ \ 0 \le g_{KU} \le 1 \ \text{ on } \ X. \tag{G.15}$$

STEP 7. Take $f \in C_{\mathbb{R}}(X)$ such that $0 \le f \le 1$ on X. Fix $k \in \mathbb{N}$ and set

$$K_\ell = \Big\{ x \in X : f(x) \ge \frac{\ell}{k} \Big\}, \tag{G.16}$$

so $X = K_0 \supset \cdots \supset K_\ell \supset K_{\ell+1} \supset \cdots K_k \supset K_{k+1} = \emptyset$. Define open $U_\ell \supset K_\ell$ by

$$U_\ell = \Big\{ x \in X : f(x) > \frac{\ell - 1}{k} \Big\}, \quad \text{so} \ \ X \setminus U_\ell = \Big\{ x \in X : f(x) \le \frac{\ell - 1}{k} \Big\}. \tag{G.17}$$

By Step 6, there exist $\psi_\ell \in \overline{\mathcal{A}}$ such that

$$\psi_\ell = 1 \ \text{ on } \ K_\ell, \quad \psi_\ell = 0 \ \text{ on } \ X \setminus U_\ell, \quad \text{and} \ \ 0 \le \psi_\ell \le 1 \ \text{ on } \ X. \tag{G.18}$$

Let

$$f_k = \max_{0 \le \ell \le k} \frac{\ell}{k} \psi_\ell \in \overline{\mathcal{A}}. \tag{G.19}$$

It follows that $f_k \ge \ell/k$ on K_ℓ and $f_k \le (\ell-1)/k$ on $X \setminus U_\ell$, for all ℓ. Hence $f_k \ge (\ell-1)/k$ on $K_{\ell-1}$ and $f_k \le \ell/k$ on $U_{\ell+1}$. In other words,

$$\frac{\ell-1}{k} \le f(x) \le \frac{\ell}{k} \Longrightarrow \frac{\ell-1}{k} \le f_k(x) \le \frac{\ell}{k}, \tag{G.20}$$

so

$$|f(x) - f_k(x)| \le \frac{1}{k}, \quad \forall\, x \in X. \tag{G.21}$$

Step 8. It follows from Step 7 that if $f \in C_{\mathbb{R}}(X)$ and $0 \le f \le 1$ on X, then $f \in \overline{\mathcal{A}}$. It is an easy final step to see that $f \in C_{\mathbb{R}}(X) \Rightarrow f \in \overline{\mathcal{A}}$.

Theorem G.3 has a complex analogue.

Theorem G.4. *Let X be a compact metric space, $\mathcal{A}$ a subalgebra (over $\mathbb{C}$) of $C(X)$, the algebra of complex valued continuous functions on X. Suppose $1 \in \mathcal{A}$ and that $\mathcal{A}$ separates the points of X. Furthermore, assume*

$$f \in \mathcal{A} \Longrightarrow \overline{f} \in \mathcal{A}. \tag{G.22}$$

Then the closure $\overline{\mathcal{A}} = C(X)$.

Proof. Set $\mathcal{A}_{\mathbb{R}} = \{f + \overline{f} : f \in \mathcal{A}\}$. One sees that Theorem G.3 applies to $\mathcal{A}_{\mathbb{R}}$.

Here are a couple of applications of Theorems G.3–G.4.

Corollary G.5. *If X is a compact subset of $\mathbb{R}^n$, then every $f \in C(X)$ is a uniform limit of polynomials on $\mathbb{R}^n$.*

Corollary G.6. *The space of trigonometric polynomials, given by*

$$\sum_{k=-N}^{N} a_k \, e^{ik\theta}, \tag{G.23}$$

is dense in $C(\mathbb{T}^1)$.

Proof. It suffices to note that

$$e^{ik\theta}e^{i\ell\theta} = e^{i(k+\ell)\theta}, \quad \text{and} \quad \overline{e^{ik\theta}} = e^{-ik\theta}, \tag{G.24}$$

to see that the space of trigonometric polynomials is an algebra of functions on $\mathbb{T}^1$ that satisfies the hypotheses of Theorem G.4.

Corollary G.6 is closely related to results on Fourier series in §13. In fact, this corollary can be deduced from Proposition 13.1 (and vice versa). The proof given above is apparently quite different from that of Proposition 13.1 given in §13, though a closer look reveals points in common between the proof of Proposition 13.1 and that of Theorem G.1. Both proofs have an argument that involves convolution with a highly peaked function.

We move on to some results of Runge, concerning the approximation of holomorphic functions by rational functions. Here is the setting. Take

$$\text{open } \Omega \subset \mathbb{C}, \quad \text{compact } K \subset \Omega, \tag{G.25}$$

and

$$f : \Omega \longrightarrow \mathbb{C}, \quad \text{holomorphic.} \tag{G.26}$$

Here is a preliminary result.

Proposition G.7. *Given Ω, K, and f as in (G.25)–(G.26), and $\varepsilon > 0$, there is a rational function g, with poles in $\Omega \setminus K$, such that*

$$\sup_{z\in K} |f(z) - g(z)| < \varepsilon. \tag{G.27}$$

We will obtain this result vis the Cauchy integral formula. The following result will prove useful.

Lemma G.8. *Given (G.25), there exists a piecewise smoothly bounded open set $\mathcal{O}$ such that*

$$K \subset \mathcal{O} \subset \overline{\mathcal{O}} \subset \Omega. \tag{G.28}$$

Proof. If $\Omega = \mathbb{C}$, the result is trivial. Otherwise, set $A = \inf\{|z - w| : z \in K, w \in \mathbb{C} \setminus \Omega\}$, and tile the complex plane $\mathbb{C}$ with closed squares $\overline{Q}_\nu$ of edge length $A/8$. Let $\{\overline{Q}_\nu : \nu \in S\}$ denote the set of these squares that have nonempty intersection with K, so $\cup_{\nu\in S}\overline{Q}_\nu \supset K$. Say $\mu \in S_2$ if $\mu \in S$ or $\overline{Q}_\mu$ has nonempty intersection with $\cup_{\nu\in S}\overline{Q}_\nu$. Then (G.28) holds with

$$\overline{\mathcal{O}} = \bigcup_{\mu\in S_2} \overline{Q}_\mu. \tag{G.29}$$

Let $\mathcal{O}$ be the interior of $\overline{\mathcal{O}}$. Note that $\partial\mathcal{O}$ is a finite union of line segments.

Proof of Proposition G.7. Take $\mathcal{O}$ as in Lemma G.8. By the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i} \int\limits_{\partial\mathcal{O}} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \quad \forall\, z \in K. \tag{G.30}$$

Dividing this curve into small segments, via a choice of $\zeta_{k\nu} \in \partial\mathcal{O}$, $1 \le k \le \nu$, we see from the defining results on the integral that the right side of (G.30) is equal to

$$\lim_{\nu\to\infty} \frac{1}{2\pi i} \sum_{k=1}^{\nu} \frac{f(\zeta_{k\nu})}{\zeta_{k\nu} - z}\, (\zeta_{k\nu} - \zeta_{k-1,\nu}), \tag{G.31}$$

and this convergence holds uniformly for $z \in K$. Since each function in (G.31) is a rational function of Z, with poles at $\{\zeta_{k\nu} : 1 \le k \le \nu\}$, we have Proposition G.7.

The following is an important strengthening of Proposition G.7.

Proposition G.9. *Take Ω, K, and f as in (G.25)–(G.26). Pick one point p_k in each connected component of $\mathbb{C} \setminus K$. Then, given $\varepsilon > 0$, there exists a rational function g, whose poles are contained in $\{p_k\}$, such that*

$$\sup_{z\in K} |f(z) - g(z)| < \varepsilon. \tag{G.32}$$

Proof. In light of Proposition G.7, it suffices to establish the following.

Lemma G.10. *Assume $R(z)$ is a rational function, with one pole, at $q \in \mathbb{C} \setminus K$. Assume $p \in \mathbb{C} \setminus K$ belongs to the same connected component U of $\mathbb{C} \setminus K$ as does q. Then, given $\varepsilon > 0$, there exists a rational function $G(z)$, with pole only at p, such that*

$$\sup_{z\in K} |R(z) - G(z)| < \varepsilon. \tag{G.33}$$

Proof. Take a smooth curve $\gamma : [0, 1] \to U$ such that $\gamma(0) = q$ and $\gamma(1) = p$. Take N sufficiently large that, with

$$q_\nu = \gamma(\nu/N), \quad q_0 = q, \quad q_N = p, \tag{G.34}$$

we have

$$|q_{\nu+1} - q_\nu| < \operatorname{dist}(q_\nu, K). \tag{G.35}$$

In particular,

$$|q - q_1| < \operatorname{dist}(q, K). \tag{G.36}$$

Now we make a Laurent expansion of $R(z)$ about q_1, obtaining

$$R(z) = \sum_{n=-\infty}^{\infty} a_n (z-q_1)^n, \quad \text{for} \ \ |z-q_1| > |q-q_1|, \tag{G.37}$$

converging uniformly on K. Truncating this series, one can produce a rational function $R_1(z)$, with a pole only at q_1, such that $|R - R_1| < \varepsilon/N$ on K. One can continue this process, obtaining for each ν a rational function R_ν with pole only at q_ν, and in the end obtain $G(z) = R_N(z)$, with pole only at p, satisfying (G.33).

In the special case that $\mathbb{C} \setminus K$ is connected, we have the following interesting variant of Theorem G.9.

Theorem G.11. *Take Ω, K, and f as in (G.25)–(G.26), and assume $\mathbb{C} \setminus K$ is connected. Then, for each $\varepsilon > 0$, there exists a polynomial $P(z)$ such that*

$$\sup_{z \in K} |f(z) - P(z)| < \varepsilon. \tag{G.38}$$

Proof. Say $K \subset D_R(0)$, and pick $p \in \mathbb{C} \setminus K$ such that $|p| > R$. By Theorem G.9, there exists a rational function $g(z)$, with pole only at p, such that $|f(z) - g(z)| < \varepsilon/2$ for $z \in K$. But then g is holomorphic on a neighborhood of $\overline{D_R(0)}$, so its power series about 0 converges to g uniformly on $\overline{D_R(0)}$. Then an appropriate truncation of this power series yields a polynomial $P(z)$ satisfying (G.38).

Note that Theorem G.11 does not require K to be connected. For example, suppose $\overline{D}_0$ and $\overline{D}_1$ are two disjoint disks in $\mathbb{C}$ and set $K = \overline{D}_0 \cup \overline{D}_1$. Then we can take disjoint open neighborhoods Ω_j of $\overline{D}_j$ and define a holomorphic function f on $\Omega = \Omega_0 \cup \Omega_1$ to be 0 on Ω_0 and 1 on Ω_1. By Theorem G.11, there exist polynomials $P_\nu(z)$ such that

$$\begin{aligned} P_\nu(z) \longrightarrow 0 \quad & \text{uniformly on } \overline{D}_0, \\ 1 \quad & \text{uniformly on } \overline{D}_1. \end{aligned} \tag{G.39}$$

One can put Theorems G.1 and G.11 together to get further polynomial approximation results. For example, for $\nu \in \mathbb{N}$, consider the line segments

$$L_\nu = \{x + iy : y = \nu^{-1}, \ |x| \le y\}, \tag{G.40}$$

and form

$$K_N = \bigcup_{\nu=1}^{N} L_\nu, \tag{G.41}$$

a compact set with connected complement. Take $f \in C(K_N)$, and take $\varepsilon > 0$. By Theorem G.1, there exist polynomials P_ν such that $\sup_{L_\nu} |f - P_\nu| < \varepsilon$. Take disjoint neighborhoods Ω_ν of L_ν, of the form

$$\Omega_\nu = \{x + iy : |y - \nu^{-1}| < 4^{-\nu}, \ |x| < y + 4^{-\nu}\}. \tag{G.42}$$

Then $\Omega = \cup_{\nu=1}^{N} \Omega_\nu$ is a neighborhood of K_N and we have a holomorphic function g_ε on Ω given by $g_\varepsilon|_{\Omega_\nu} = P_\nu|_{\Omega_\nu}$. Then, by Theorem G.11, there exists a polynomial P such that

$$\sup_{K_N} |g_\varepsilon - P| < \varepsilon. \tag{G.43}$$

Consequently, $\sup_{K_N} |f - P| < 2\varepsilon$. With a little more effort, which we leave to the reader, one can establish the following.

Proposition G.12. *With L_ν as in (G.40), set*

$$K = \{0\} \cup \bigcup_{\nu=1}^{\infty} L_\nu, \tag{G.44}$$

a compact subset of $\mathbb{C}$ with connected complement. Given $f \in C(K)$, there exist polynomials P_k such that

$$P_k \longrightarrow f, \quad \textit{uniformly on } K. \tag{G.45}$$

Results like (G.39) and Proposition G.12 are special cases of the following, known as Mergelyan's theorem.

Theorem G.13. *Assume $K \subset \mathbb{C}$ is compact and $\mathbb{C} \setminus K$ is connected. Take $f \in C(K)$ such that f is holomorphic on the interior of K. Then there exist polynomials P_k such that $P_k \to f$ uniformly on K. In particuler, this holds for all $f \in C(K)$ if K has empty interior.*

When $\mathbb{C} \setminus K$ is not compact, there is the following.

Theorem G.14. *Assume $K \subset \mathbb{C}$ is compact and $\mathbb{C} \setminus K$ has a finite number of connected components, $\Omega_1, \dots, \Omega_N$. Take $p_j \in \Omega_j$. Let $f \in C(K)$, and assume f is holomorphic on the interior of K. Then there exists a sequence of rational functions R_k, with poles contained in $\{p_j : 1 \le j \le N\}$, such that $R_k \to f$ uniformly on K.*

Proofs of these last two results are more elaborate than those given above of Theorems G.9 and G.11. Proofs of Theorem G.13 can be found in [Ru] and [GK]. A proof of Theorem G.14 can be found in [Gam2].

Chapter 4. Residue calculus, the argument principle, and two very special functions

The first two sections of this chapter give key applications of the Cauchy integral formula. The first, called residue calculus, leads to the evaluation of certain types of definite integrals. It involves identities of the form

$$\int_{\gamma} f(z)\,dz = 2\pi i \sum_{p} \operatorname{Res}_p f, \tag{4.0.1}$$

The second involves counting zeros of a function holomorphic on a domain Ω (and nonvanishing on $\partial\Omega$), via the identity

$$\nu(f,\Omega) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'(z)}{f(z)}\,dz. \tag{4.0.2}$$

In (4.0.1) the sum is over a set of poles of f, and $\operatorname{Res}_p f$, called the residue of f at p, is the coefficient of $(z-p)^{-1}$ in the Laurent series expansion of f about the pole p. In (4.0.2), $\nu(f,\Omega)$ is the number of zeros of f in Ω, counting multiplicity. The quantity (4.0.2) is also equal to the winding number of the image curve $f(\partial\Omega)$ about 0, giving rise to the interpretation of this result as the *argument principle.*

Fourier analysis is a rich source of integrals to which one can apply residue calculus. Examples include

$$\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{1+x^2}\,dx = \pi e^{-|\xi|}, \quad \xi \in \mathbb{R}, \tag{4.0.3}$$

$$\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{2\cosh x/2}\,dx = \frac{\pi}{\cosh \pi\xi}, \quad \xi \in \mathbb{R}, \tag{4.0.4}$$

and

$$\int_{0}^{\infty} \frac{x^{\alpha-1}}{1+x}\,dx = \frac{\pi}{\sin \pi\alpha}, \quad 0 < \operatorname{Re}\alpha < 1. \tag{4.0.5}$$

The integrals (4.0.3)–(4.0.4) are Fourier transforms, and (4.0.5) is a Mellin transform. None of these integrals has an elementary treatment. One cannot write down antiderivatives of the integrands that appear.

The integral

$$\int_{0}^{\infty} e^{-st} t^{z-1}\,dt \tag{4.0.6}$$

is simultaneously a Laplace transform and a Mellin transform. A change of variable yields, for $\operatorname{Re} s > 0$, the result $s^{-z}\Gamma(z)$, where

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt, \quad \operatorname{Re} z > 0, \tag{4.0.7}$$

is the Gamma function. This is the first "higher transcendental function." It plays a key role in the study of many other special functions, and it is the object of §18. The identity

$$\Gamma(z+1) = z\Gamma(z) \tag{4.0.8}$$

leads to an analytic continuation of $\Gamma(z)$ to all of $\mathbb{C}$, except for simple poles at $\{0, -1, -2, \dots\}$. Another important identity is

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{4.0.9}$$

We give two proofs of this, one using (4.0.4) and the other using (4.0.5). One consequence of (4.0.9) is that $\Gamma(z)$ is nowhere zero, so $1/\Gamma(z)$ is an entire function, holomorphic on all of $\mathbb{C}$. It is seen to have an interesting infinite product expansion.

The other special function studied in this chapter is the Riemann zeta function,

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}, \quad \operatorname{Re} s > 1, \tag{4.0.10}$$

the object of §19. This function also has an analytic continuation to all of $\mathbb{C}$, except for one simple pole at $s = 1$. This fact is related to an identity known the Riemann functional equation, which can be formulated as follows. Set

$$\xi(s) = \frac{s(s-1)}{2} \Gamma\Big(\frac{s}{2}\Big) \pi^{-s/2} \zeta(s). \tag{4.0.11}$$

Then $\xi(s)$ is an entire function, satisfying

$$\xi(1-s) = \xi(s). \tag{4.0.12}$$

The zeta function is intimately connected to the study of prime numbers, first through the Euler product

$$\zeta(s) = \prod_{p \in \mathcal{P}} \big(1 - p^{-s}\big)^{-1}, \tag{4.0.13}$$

where $\mathcal{P} = \{2, 3, 5, \dots\}$ is the set of primes. From the behavior of $\zeta(s)$ as $s \to 1$ and basic results on infinite products, one has

$$\sum_{p \in \mathcal{P}} \frac{1}{p^s} \sim \log \frac{1}{s-1}, \quad \text{as} \quad s \searrow 1. \tag{4.0.14}$$

This result is consistent with the prime number theorem, which states that

$$\pi(x) \sim \frac{\log x}{x}, \quad \text{as} \quad x \to \infty, \tag{4.0.15}$$

where, for $x > 0$, $\pi(x)$ denotes the number of primes that are $\leq x$. This result was independently conjectured by Gauss and Legendre, in the 1790s. Riemann's work on the zeta function in the 1850s provided a path to its proof, which was completed independently by Hadamard and de la Vallée Poussin in the 1890s. We give a proof in §19. Connecting $\pi(x)$ to the zeta function is done here in the language of the Stieltjes integral. Also, as is typical, the endgame of the proof involves a Tauberian theorem. These topics are treated in Appendices M and R, at the end of this text.

Meanwhile, this chapter has two appendices. One studies a special number known as Euler's constant,

$$\gamma = \lim_{n \to \infty} \Bigl(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\Bigr), \tag{4.0.16}$$

which arises in the analysis of $\Gamma(z)$. The other treats Hadamard's factorization theorem, with application to the infinite product expansion of the entire function $\xi(s)$ given in (4.0.11).

16. Residue calculus

Let f be holomorphic on an open set Ω except for isolated singularities, at points $p_j \in \Omega$. Each p_j is contained in a disk $D_j \subset\subset \Omega$ on a neighborhood of which f has a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(p_j)(z - p_j)^n. \tag{16.1}$$

The coefficient $a_{-1}(p_j)$ of $(z - p_j)^{-1}$ is called the *residue* of f at p_j, and denoted $\mathrm{Res}_{p_j}(f)$. We have

$$\mathrm{Res}_{p_j}(f) = \frac{1}{2\pi i} \int\limits_{\partial D_j} f(z)\,dz. \tag{16.2}$$

If in addition Ω is bounded, with piecewise smooth boundary, and $f \in C(\overline{\Omega} \setminus \{p_j\})$, assuming $\{p_j\}$ is a finite set, we have, by the Cauchy integral theorem,

$$\int\limits_{\partial\Omega} f(z)\,dz = \sum_j \int\limits_{\partial D_j} f(z)\,dz = 2\pi i \sum_j \mathrm{Res}_{p_j}(f). \tag{16.3}$$

This identity provides a useful tool for computing a number of interesting integrals of functions whose anti-derivatives we cannot write down. Examples almost always combine the identity (16.3) with further limiting arguments. We illustrate this method of residue calculus to compute integrals with a variety of examples.

To start with a simple case, let us compute

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}. \tag{16.4}$$

In this case we can actually write down an anti-derivative, but never mind. The function $f(z) = (1 + z^2)^{-1}$ has simple poles at $z = \pm i$. Whenever a meromorphic function $f(z)$ has a simple pole at $z = p$, we have

$$\mathrm{Res}_p(f) = \lim_{z \to p} (z - p) f(z). \tag{16.5}$$

In particular,

$$\mathrm{Res}_i\,(1 + z^2)^{-1} = \lim_{z \to i} (z - i) \frac{1}{(z + i)(z - i)} = \frac{1}{2i}. \tag{16.6}$$

Let γ_R be the path formed by the path α_R from $-R$ to R on the real line, followed by the semicircle β_R of radius R in the upper half-plane, running from $z = R$ to $z = -R$. Then γ_R encloses the pole of $(1+z^2)^{-1}$ at $z = i$, and we have

$$\int_{\gamma_R} \frac{dz}{1+z^2} = 2\pi i \ \mathrm{Res}_i\,(1+z^2)^{-1} = \pi, \tag{16.7}$$

provided $R > 1$. On the other hand, considering the length of β_R and the size of $(1+z^2)^{-1}$ on β_R, it is clear that

$$\lim_{R\to\infty} \int_{\beta_R} \frac{dz}{1+z^2} = 0. \tag{16.8}$$

Hence

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{R\to\infty} \int_{\gamma_R} \frac{dz}{1+z^2} = \pi. \tag{16.9}$$

This is consistent with the result one gets upon recalling that $d\ \tan^{-1} x/dx = (1+x^2)^{-1}$.

For a more elaborate example, consider

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}. \tag{16.10}$$

Now $(1+z^4)^{-1}$ has poles at the four 4th roots of -1:

$$p_1 = e^{\pi i/4}, \quad p_2 = e^{3\pi i/4}, \quad p_3 = e^{-3\pi i/4}, \quad p_4 = e^{-\pi i/4}. \tag{16.11}$$

A computation using (16.5) gives

$$\begin{aligned} \mathrm{Res}_{p_1}\,(1+z^4)^{-1} &= \frac{1}{4} e^{-3\pi i/4}, \\ \mathrm{Res}_{p_2}\,(1+z^4)^{-1} &= \frac{1}{4} e^{-\pi i/4}. \end{aligned} \tag{16.12}$$

Using the family of paths γ_R as in (16.7), we have

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \lim_{R\to\infty} \int_{\gamma_R} \frac{dz}{1+z^4} &= 2\pi i \sum_{j=1}^{2} \mathrm{Res}_{p_j}\,(1+z^4)^{-1} \\ &= \frac{\pi i}{2}(e^{-3\pi i/4} + e^{-\pi i/4}) = \frac{\pi}{\sqrt{2}}, \end{aligned} \tag{16.13}$$

where we use the identity

$$e^{\pi i/4} = \frac{1+i}{\sqrt{2}}. \tag{16.14}$$

The evaluation of Fourier transforms provides a rich source of examples to which to apply residue calculus. Consider the problem of computing

$$\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{1+x^2}\,dx, \tag{16.15}$$

for $\xi \in \mathbb{R}$. Note that $f(z) = e^{i\xi z}/(1+z^2)$ has simple poles at $z = \pm i$, and

$$\operatorname{Res}_i \frac{e^{i\xi z}}{1+z^2} = \frac{e^{-\xi}}{2i}. \tag{16.16}$$

Hence, making use of γ_R as in (16.7)–(16.9), we have

$$\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{1+x^2}\,dx = \pi\, e^{-\xi}, \quad \xi \ge 0. \tag{16.17}$$

For $\xi < 0$ one can make a similar argument, replacing γ_R by its image $\overline{\gamma}_R$ reflected across the real axis. Alternatively, we can see directly that (16.15) defines an even function of ξ, so

$$\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{1+x^2}\,dx = \pi\, e^{-|\xi|}, \quad \xi \in \mathbb{R}. \tag{16.18}$$

The reader can verify that this result is consistent with the computation of

$$\int_{-\infty}^{\infty} e^{-|x|}\, e^{-ix\xi}\,dx$$

and the Fourier inversion formula; cf. Exercise 4 in §14.

As another example of a Fourier transform, we look at

$$A = \int_{-\infty}^{\infty} \frac{e^{ix\xi}}{2\cosh\frac{x}{2}}\,dx. \tag{16.19}$$

To evaluate this, we compare it with the integral over the path $\gamma(x) = x - 2\pi i$. See Fig. 16.1. We have

$$\int_{\gamma} \frac{e^{iz\xi}}{2\cosh\frac{z}{2}}\,dz = -\int_{-\infty}^{\infty} \frac{e^{2\pi\xi+ix\xi}}{2\cosh\frac{x}{2}}\,dx = -e^{2\pi\xi}A, \tag{16.20}$$

since $\cosh(y-\pi i) = -\cosh y$. Now the integrand has a pole at $z = -\pi i$, and a computation gives

$$\operatorname{Res}_{-\pi i} \frac{e^{iz\xi}}{2\cosh\frac{z}{2}} = i\, e^{\pi\xi}. \tag{16.21}$$

We see that

$$-A - e^{2\pi\xi}A = 2\pi i\ \operatorname{Res}_{-\pi i} \frac{e^{iz\xi}}{2\cosh\frac{z}{2}}, \tag{16.22}$$

and hence

$$\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{2\cosh\frac{x}{2}}\, dx = \frac{\pi}{\cosh \pi\xi}. \tag{16.23}$$

The evaluation of (16.19) involved an extra wrinkle compared to the other examples given above, involving how the integrand on one path is related to the integrand on another path. Here is another example of this sort. Given $\alpha \in (0,1)$, consider the evaluation of

$$B = \int_0^{\infty} \frac{x^\alpha}{1+x^2}\, dx. \tag{16.24}$$

Let us define $z^\alpha = r^\alpha e^{i\alpha\theta}$ upon writing $z = re^{i\theta},\ 0 \le \theta \le 2\pi$. Thus in contrast with our treatment in §4, we here define z^α to be holomorphic on $\mathbb{C} \setminus \mathbb{R}^+$. It has distinct boundary values as $z = x + iy \to x > 0$ as $y \searrow 0$ and as $y \nearrow 0$. Let γ_R be the curve starting with r going from 0 to R, while $\theta = 0$, then keeping $r = R$ and taking θ from 0 to 2π, and finally keeping $\theta = 2\pi$ and taking r from R to 0. See Fig. 16.2. We have

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{z^\alpha}{1+z^2}\, dz = B - e^{2\pi i\alpha}B. \tag{16.25}$$

On the other hand, for $R > 1$ we have

$$\begin{aligned} \int_{\gamma_R} \frac{z^\alpha}{1+z^2}\, dz &= 2\pi i \sum_{p=\pm i} \operatorname{Res}_p \frac{z^\alpha}{1+z^2} \\ &= \pi\big(e^{\pi i\alpha/2} - e^{3\pi i\alpha/2}\big), \end{aligned} \tag{16.26}$$

so

$$\begin{aligned} B &= \pi\, \frac{e^{\pi i\alpha/2} - e^{3\pi i\alpha/2}}{1 - e^{2\pi i\alpha}} \\ &= \pi\, \frac{\sin \pi\alpha/2}{\sin \pi\alpha}. \end{aligned} \tag{16.27}$$

While all of the examples above involved integrals over infinite intervals, one can also use residue calculus to evaluate integrals of the form

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta, \tag{16.28}$$

when $R(u, v)$ is a rational function of its arguments. Indeed, if we consider the curve $\gamma(\theta) = e^{i\theta}$, $0 \le \theta \le 2\pi$, we see that (16.28) is equal to

$$\int_\gamma R\Bigl(\frac{z}{2} + \frac{1}{2z}, \frac{z}{2i} - \frac{1}{2iz}\Bigr)\, \frac{dz}{iz}, \tag{16.29}$$

which can be evaluated by residue calculus.

To illustrate this, we consider

$$I(r) = \int_0^{2\pi} \frac{d\theta}{1 - 2r\cos\theta + r^2}, \quad 0 < r < 1, \tag{16.30}$$

which arose in §13, in connection with the Poisson integral. This integral was evaluated, by other means, in (13.8)–(13.9). Here, we rewrite it in the form (16.29). We get

$$\begin{aligned} I(r) &= \int_\gamma \frac{1}{1 - r(z + 1/z) + r^2}\, \frac{dz}{iz} \\ &= i\int_\gamma \frac{dz}{rz^2 - (1 + r^2)z + r}. \end{aligned} \tag{16.31}$$

The polynomial $p(z) = rz^2 - (1+r^2)z + r$ appearing in the denominator in the last integral has two roots, r and $1/r$. We have the factorization

$$rz^2 - (1 + r^2)z + r = r(z - r)\Bigl(z - \frac{1}{r}\Bigr), \tag{16.32}$$

so, for $0 < r < 1$,

$$\begin{aligned} I(r) &= \frac{i}{r}\int_\gamma \frac{1}{z - 1/r}\, \frac{dz}{z - r} \\ &= \frac{i}{r} \cdot 2\pi i g(r), \end{aligned} \tag{16.33}$$

where $g(z) = (z - 1/r)^{-1}$, so

$$\begin{aligned} I(r) &= -\frac{2\pi}{r}\, \frac{1}{r - 1/r} \\ &= \frac{2\pi}{1 - r^2}, \end{aligned} \tag{16.34}$$

for $0 < r < 1$. This agrees with (13.8)–(13.9).

So far, the examples to which we have applied residue calculus have all involved integrands $f(z)$ all of whose singularities were simple poles. A pole of order 2 arises in the following integral:

$$u(\xi) = \int_{-\infty}^{\infty} \frac{e^{ix\xi}}{(1+x^2)^2}\, dx, \quad \xi \in \mathbb{R}. \tag{16.35}$$

This is similar to (16.15), except that in this case $f(z) = e^{i\xi z}/(1+z^2)^2$ has poles of order 2 at $z = \pm i$. We have

$$\begin{aligned} \operatorname{Res}_i \frac{e^{i\xi z}}{(1+z^2)^2} &= \operatorname{Res}_i \frac{e^{i\xi z}}{(z+i)^2} \frac{1}{(z-i)^2} \\ &= g'(i), \end{aligned} \tag{16.36}$$

where

$$g(z) = \frac{e^{i\xi z}}{(z+i)^2} \tag{16.37}$$

(see Exercise 6 below). A computation gives

$$g'(i) = -\frac{i}{4}(1+\xi)e^{-\xi}. \tag{16.38}$$

Arguing as in (16.15)–(16.18), we see that, if $\xi > 0$, then, with γ_R as in (16.7),

$$\begin{aligned} u(\xi) &= \lim_{R\to\infty} \int\limits_{\gamma_R} \frac{e^{i\xi z}}{(1+z^2)^2}\, dz \\ &= 2\pi i \operatorname{Res}_i \frac{e^{i\xi z}}{(1+z^2)^2} \\ &= \frac{\pi}{2}(1+\xi)e^{-\xi}. \end{aligned} \tag{16.39}$$

Noting that (16.35) is an even function of ξ, we deduce that

$$\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{(1+x^2)^2}\, dx = \frac{\pi}{2}(1+|\xi|)e^{-|\xi|}, \quad \forall\, \xi \in \mathbb{R}. \tag{16.40}$$

We next show how residue calculus can be used to compute an interesting class of infinite series.

Proposition 16.1. *Let $P(z)$ and $Q(z)$ be polynomials, of degree $d_Q \geq d_P + 2$, and*

$$f(z) = \frac{P(z)}{Q(z)}. \tag{16.41}$$

Let $\mathcal{Z}_Q = \{z \in \mathbb{C} : Q(z) = 0\}$, and assume none of these roots belong to $\mathbb{Z}$. Then

$$\sum_{k=-\infty}^{\infty} f(k) = -\pi \sum_{p \in \mathcal{Z}_Q} \operatorname{Res}_p f(z) \cot \pi z. \tag{16.42}$$

Proof. For $N \in \mathbb{N}$, take

$$\Omega_N = \Big\{ z \in \mathbb{C} : |\operatorname{Re} z| < N + \frac{1}{2}, \, |\operatorname{Im} z| \leq N + \frac{1}{2} \Big\}. \tag{16.43}$$

As long as

$$N > \max_{p \in \mathcal{Z}_Q} |p|, \tag{16.44}$$

$\partial \Omega_N$ is disjoint from $\mathcal{Z}_Q$. One can show that there exists $K < \infty$ such that

$$|\cot \pi z| \leq K, \quad \forall \, z \in \partial \Omega_N, \; N \in \mathbb{N}. \tag{16.45}$$

It readily follows that

$$\lim_{N \to \infty} \int\limits_{\partial \Omega_N} f(z) \, \cot \pi z \, dz = 0. \tag{16.46}$$

Meanwhile, by (16.3), for N satisfying (16.44),

$$\int\limits_{\partial \Omega_N} f(z) \, \cot \pi z \, dx = 2\pi i \sum_{p \in (\Omega_N \cap \mathbb{Z}) \cup \mathcal{Z}_Q} \operatorname{Res}_p \, f(z) \cot \pi z. \tag{16.47}$$

One now checks that

$$\operatorname{Res}_k f(z) \cot \pi z = \frac{1}{\pi} f(k), \quad \forall \, k \in \mathbb{Z}. \tag{16.48}$$

Hence taking $N \to \infty$ in (16.47) yields (16.42).

In connection with the formula (16.42), let us note that if p is a simple root of $Q(z)$ (and $p \notin \mathbb{Z}$), then

$$\operatorname{Res}_p f(z) \cot \pi z = \frac{P(p)}{Q'(p)} \, \cot \pi p. \tag{16.49}$$

(See Exercise 6 below.)

To take an example, set

$$f(z) = \frac{1}{z^2 + a^2}, \quad a \in \mathbb{C} \setminus i\mathbb{Z}. \tag{16.50}$$

We get

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \frac{1}{k^2 + a^2} &= -\pi\Bigl(\mathrm{Res}_{ia} \frac{\cot \pi z}{z^2 + a^2} + \mathrm{Res}_{-ia} \frac{\cot \pi z}{z^2 + a^2}\Bigr) \\ &= -\pi\Bigl(\frac{\cot \pi ia}{2ia} + \frac{\cot(-\pi ia)}{-2ia}\Bigr) \\ &= \frac{\pi}{a} \coth \pi a.\end{aligned} \tag{16.51}$$

See the exercises below for further applications of Proposition 16.1.

Exercises

1. Use residue calculus to evaluate the following definite integrals.

(a)
$$\int_{-\infty}^{\infty} \frac{x^2}{1 + x^6}\, dx,$$

(b)
$$\int_0^{\infty} \frac{\cos x}{1 + x^2}\, dx,$$

(c)
$$\int_0^{\infty} \frac{x^{1/3}}{1 + x^3}\, dx,$$

(d)
$$\int_0^{2\pi} \frac{\sin^2 x}{2 + \cos x}\, dx.$$

2. Let γ_R go from 0 to R on $\mathbb{R}^+$, from R to $Re^{2\pi i/n}$ on $\{z : |z| = R\}$, and from $Re^{2\pi i/n}$ to 0 on a ray. Assume $n > 1$. Take $R \to \infty$ and evaluate

$$\int_0^{\infty} \frac{dx}{1 + x^n}.$$

More generally, evaluate

$$\int_0^{\infty} \frac{x^a}{1 + x^n}\, dx,$$

for $0 < a < n - 1$.

3. If $\binom{n}{k}$ denotes the binomial coefficient, show that

$$\binom{n}{k} = \frac{1}{2\pi i} \int_{\gamma} \frac{(1+z)^n}{z^{k+1}}\, dz,$$

where γ is any simple closed curve about the origin.
Hint. $\binom{n}{k}$ is the coefficient of z^k in $(1+z)^n$.

4. Use residue calculus to compute the inverse Laplace transforms in Exercise 3 (parts (a)–(d)) of §15.

5. Extend the calculations in (16.30)–(16.34) to compute

$$\int_0^{2\pi} \frac{d\theta}{1 - 2r\,\cos\theta + r^2},$$

for all $r \in \mathbb{C}$ such that $|r| \neq 1$.

6. Let f and g be holomorphic in a neighborhood of p.
(a) If g has a simple zero at p, show that

$$\operatorname{Res}_p \frac{f}{g} = \frac{f(p)}{g'(p)}. \tag{16.52}$$

(b) Show that

$$\operatorname{Res}_p \frac{f(z)}{(z-p)^2} = f'(p). \tag{16.53}$$

7. In the setting of Exercise 6, show that, for $k \geq 1$,

$$\operatorname{Res}_p \frac{f(z)}{(z-p)^k} = \frac{1}{(k-1)!}\, f^{(k-1)}(p).$$

Hint. Use (5.11), in concert with (16.2). See how this generalizes (16.5) and (16.41).

8. Compute

$$\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{1+x^4}\, dx,$$

for $\xi \in \mathbb{R}$.

9. Deduce from (16.23) that

$$f(x) = \frac{1}{\cosh\sqrt{\pi/2}x} \Rightarrow \hat{f}(\xi) = f(\xi).$$

10. Show that the computation (16.24)–(16.27) extends from $\alpha \in (0,1)$ to $\operatorname{Re}\alpha \in (0,1)$.

11. Using an analysis parallel to (16.24)–(16.27), show that

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx = \frac{\pi}{\sin\pi\alpha}, \quad 0 < \operatorname{Re}\alpha < 1. \tag{16.54}$$

12. Compute

$$\int_{-\infty}^\infty \frac{\log|x|}{x^2+a^2}\,dx,$$

for $a \in \mathbb{R}\setminus 0$.
Hint. Bring in a semicircular contour.

13. Apply Proposition 16.1 to show that

$$\sum_{k=-\infty}^\infty \frac{1}{(k+a)^2} = \frac{\pi^2}{\sin^2\pi a}, \quad a \in \mathbb{C}\setminus\mathbb{Z}. \tag{16.55}$$

For another approach to this identity, see (30.2).

14. Establish the following variant of Proposition 16.1. Take $f(z) = P(z)/Q(z)$ as in that proposition. Show that

$$\sum_{k=-\infty}^\infty (-1)^k f(k) = -\pi \sum_{p\in\mathcal{Z}_Q} \operatorname{Res}_p \frac{f(z)}{\sin\pi z}.$$

Apply this to compute

$$\sum_{k=-\infty}^\infty \frac{(-1)^k}{k^2+a^2}, \quad a \in \mathbb{C}\setminus i\mathbb{Z}.$$

15. Extend Proposition 16.1 to allow $\mathcal{Z}_Q \cap \mathbb{Z} \neq \emptyset$. Show that the result still holds, with the conclusion modified to

$$\sum_{k\in\mathbb{Z}\setminus\mathcal{Z}_Q} f(k) = -\pi \sum_{p\in\mathcal{Z}_Q} \operatorname{Res}_p f(z)\cot\pi z. \tag{16.56}$$

16. Deduce from Exercise 15 (with $f(z) = z^{-2n}$) that, for $n \in \mathbb{N}$,

$$2 \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = -\pi \operatorname{Res}_0 \frac{\cot \pi z}{z^{2n}}. \tag{16.57}$$

Using this, show that, for $|z| < 1$,

$$\pi \cot \pi z = \frac{1}{z} - 2 \sum_{n=1}^{\infty} \zeta(2n) z^{2n-1}, \tag{16.58}$$

where, for $\operatorname{Re} s > 1$, we define the Riemann zeta function

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s},$$

a function we will study further in §19. For another derivation of (16.58), see the exercises in §30, which also connect this identity to formulas for $\zeta(2n)$ in terms of Bernoulli numbers.

17. The argument principle

Suppose $\Omega \subset \mathbb{C}$ is a bounded domain with piecewise smooth boundary and $f \in C^2(\overline{\Omega})$ is holomorphic on Ω, and nowhere zero on $\partial\Omega$. We desire to express the number of zeros of f in Ω in terms of the behavior of f on $\partial\Omega$. We count zeros with *multiplicity*, where we say $p_j \in \Omega$ is a zero of multiplicity k provided $f^{(\ell)}(p_j) = 0$ for $\ell \le k-1$ while $f^{(k)}(p_j) \neq 0$. The following consequence of Cauchy's integral theorem gets us started.

Proposition 17.1. *Under the hypotheses stated above, the number $\nu(f,\Omega)$ of zeros of f in Ω, counted with multiplicity, is given by*

$$\nu(f,\Omega) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'(z)}{f(z)}\, dz. \tag{17.1}$$

Proof. Suppose the zeros of f in Ω occur at p_j, with multiplicity m_j, $1 \le j \le K$. The function f'/f is then C^1 near $\partial\Omega$ and holomorphic on $\Omega \setminus \{p_j\}$. Hence, by Cauchy's integral theorem, the right side of (17.1) is equal to

$$\frac{1}{2\pi i} \sum_{j=1}^{K} \int_{\partial D_j} \frac{f'(z)}{f(z)}\, dz, \tag{17.2}$$

for sufficiently small disks D_j centered at p_j. It remains to check that

$$m_j = \frac{1}{2\pi i} \int_{\partial D_j} \frac{f'(z)}{f(z)}\, dz. \tag{17.3}$$

Indeed we have, on a neighborhood of $\overline{D}_j$,

$$f(z) = (z-p_j)^{m_j} g_j(z), \tag{17.4}$$

with $g_j(z)$ nonvanishing on $\overline{D}_j$. Hence on $\overline{D}_j$,

$$\frac{f'(z)}{f(z)} = \frac{m_j}{z-p_j} + \frac{g_j'(z)}{g_j(z)}. \tag{17.5}$$

The second term on the right is holomorphic on $\overline{D}_j$, so it integrates to 0 over ∂D_j. Hence the identity (17.3) is a consequence of the known result

$$m_j = \frac{m_j}{2\pi i} \int_{\partial D_j} \frac{dz}{z-p_j}. \tag{17.6}$$

Proposition 17.1 has an interesting interpretation in terms of *winding numbers*, which we now discuss. Denote the connected components of $\partial\Omega$ by C_j (with proper orientations). Say C_j is parametrized by $\varphi_j : S^1 \to \mathbb{C}$. Then

$$f \circ \varphi_j : S^1 \longrightarrow \mathbb{C} \setminus 0 \tag{17.7}$$

parametrizes the image curve $\gamma_j = f(C_j)$. The following is an important complement to (17.1).

Proposition 17.2. *With C_j and γ_j as above, we have*

$$\frac{1}{2\pi i}\int\limits_{C_j}\frac{f'(z)}{f(z)}\,dz=\frac{1}{2\pi i}\int\limits_{\gamma_j}\frac{dz}{z}. \tag{17.8}$$

Proof. Parametrize S^1 by $t\in\mathbb{R}/2\pi\mathbb{Z}$. In general,

$$\int\limits_{C_j}u(z)\,dz=\int_0^{2\pi}u(\varphi_j(t))\varphi_j'(t)\,dt,$$

and

$$\begin{aligned}\int\limits_{\gamma_j}v(z)\,dz&=\int_0^{2\pi}v(f(\varphi_j(t))\frac{d}{dt}f\circ\varphi_j(t)\,dt\\&=\int_0^{2\pi}v(f(\varphi_j(t))f'(\varphi_j(t))\varphi_j'(t)\,dt.\end{aligned}$$

In particular,

$$\int_{C_j}\frac{f'(z)}{f(z)}\,dz=\int_0^{2\pi}\frac{f'(\varphi_j(t))}{f(\varphi_j(t))}\varphi_j'(t)\,dt, \tag{17.9}$$

and

$$\int\limits_{\gamma_j}\frac{dz}{z}=\int_0^{2\pi}\frac{1}{f(\varphi_j(t))}f'(\varphi_j(t))\varphi_j'(t)\,dt, \tag{17.10}$$

an agreement that yields the asserted identity (17.8).

To analyze the right side of (17.8), let γ be an arbitrary continuous, piecewise C^1 curve in $\mathbb{C}\setminus 0$. Say it is given by

$$\gamma:[0,2\pi]\longrightarrow\mathbb{C}\setminus 0,\quad \gamma(t)=r(t)e^{i\theta(t)}, \tag{17.11}$$

with $r(t)$ and $\theta(t)$ both continuous, piecewise C^1, real-valued functions of t, $r(t)>0$. We have

$$\gamma'(t)=[r'(t)+ir(t)\theta'(t)]e^{i\theta(t)},$$

and hence

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{\gamma}\frac{dz}{z}&=\frac{1}{2\pi i}\int_0^{2\pi}\frac{\gamma'(t)}{\gamma(t)}\,dt\\&=\frac{1}{2\pi i}\int_0^{2\pi}\Big[\frac{r'(t)}{r(t)}+i\theta'(t)\Big]\,dt.\end{aligned} \tag{17.12}$$

Now

$$\begin{aligned} \int_0^{2\pi} \frac{r'(t)}{r(t)}\,dt &= \int_0^{2\pi} \frac{d}{dt}\log r(t)\,dt \\ &= \log r(2\pi) - \log r(0) \\ &= 0, \end{aligned} \tag{17.13}$$

so

$$\begin{aligned} \frac{1}{2\pi i}\int\limits_{\gamma} \frac{dz}{z} &= \frac{1}{2\pi}\int_0^{2\pi} \theta'(t)\,dt \\ &= \frac{1}{2\pi}[\theta(2\pi) - \theta(0)] \\ &= n(\gamma, 0). \end{aligned} \tag{17.14}$$

Since $r(0)e^{i\theta(0)} = r(2\pi)e^{i\theta(2\pi)}$, this is an integer, called the *winding number* of γ about 0.

The following is an important stability property of the winding number.

Proposition 17.3. *If γ_0 and γ_1 are smoothly homotopic in $\mathbb{C} \setminus 0$, then*

$$n(\gamma_0, 0) = n(\gamma_1, 0). \tag{17.15}$$

Proof. If γ_s is a smooth family of curves in $\mathbb{C} \setminus 0$, for $0 \le s \le 1$, then

$$n(\gamma_s, 0) = \frac{1}{2\pi}\int\limits_{\gamma_s} d\theta \tag{17.16}$$

is a continuous function of $s \in [0, 1]$, taking values in $\mathbb{Z}$. Hence it is constant.

Comparing (17.8) and (17.14), we have

Proposition 17.4. *With the winding number given by (17.14),*

$$\frac{1}{2\pi i}\int\limits_{C_j} \frac{f'(z)}{f(z)}\,dz = n(\gamma_j, 0), \quad \gamma_j = f(C_j). \tag{17.17}$$

In concert with Proposition 17.1, this yields:

Proposition 17.5. *In the setting of Proposition 17.1, with C_j denoting the connected components of $\partial\Omega$,*

$$\nu(f, \Omega) = \sum_j n(\gamma_j, 0), \quad \gamma_j = f(C_j). \tag{17.18}$$

That is, the total number of zeros of f in Ω, counting multiplicity, is equal to the sum of the winding numbers of $f(C_j)$ about 0.

This result is called the *argument principle.* It is of frequent use, since the right side of (17.18) is often more readily calculable directly than the left side. In evaluating this sum, take care as to the orientation of each component C_j, as that affects the sign of the winding number. We mention without further ado that the smoothness hypothesis on $\partial\Omega$ can be relaxed via limiting arguments.

The following useful result, known as Rouché's theorem or the "dog-walking theorem," can be derived as a corollary of Proposition 17.5.

Proposition 17.6. *Let $f, g \in C^1(\overline{\Omega})$ be holomorphic in Ω, and nowhere zero on $\partial\Omega$. Assume*

$$|f(z) - g(z)| < |f(z)|, \quad \forall\, z \in \partial\Omega. \tag{17.19}$$

Then

$$\nu(f, \Omega) = \nu(g, \Omega). \tag{17.20}$$

Proof. The hypothesis (17.19) implies that f and g are smoothly homotopic as maps from $\partial\Omega$ to $\mathbb{C} \setminus 0$, e.g., via the homotopy

$$f_\tau(z) = f(z) - \tau[f(z) - g(z)], \quad 0 \le \tau \le 1. \tag{17.21}$$

Hence, by Proposition 17.3, $f|_{C_j}$ and $g|_{C_j}$ have the same winding numbers about 0, for each boundary component C_j.

Second proof. With f_τ as in (17.21), we see that

$$\psi(\tau) = \nu(f_\tau, \Omega) = \frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{f_\tau'(z)}{f_\tau(z)}\, dz$$

is continuous in $\tau \in [0, 1]$. Since $\psi : [0, 1] \to \mathbb{Z}$, it must be constant.

As an example of how Rouché's theorem applies, we can give another proof of the fundamental theorem of algebra. Consider

$$f(z) = z^n, \quad g(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_0. \tag{17.22}$$

Clearly there exists $R < \infty$ such that

$$|f(z) - g(z)| < R^n \quad \text{for} \quad |z| = R. \tag{17.23}$$

Hence Proposition 17.6 applies, with $\Omega = D_R(0)$. It follows that

$$\nu(g, D_R(0)) = \nu(f, D_R(0)) = n, \tag{17.24}$$

so the polynomial $g(z)$ has complex roots.

Here is another useful stability result.

Proposition 17.7. *Take Ω as in Proposition 17.1, and let $f \in C^1(\overline{\Omega})$ be holomorphic on Ω. Assume $S \subset \mathbb{C}$ is connected and $S \cap f(\partial\Omega) = \emptyset$. Then*

$$\nu(f - q, \Omega) \text{ is independent of } q \in S. \tag{17.25}$$

Proof. The hypotheses imply that

$$\varphi(q) = \nu(f - q, \Omega) = \frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{f'(z)}{f(z) - q} \, dz \tag{17.26}$$

is a continuous function of $q \in S$. Since $\varphi : S \to \mathbb{Z}$, and S is connected, this forces φ to be constant on S.

The next result is known as the *open mapping theorem* for holomorphic functions.

Proposition 17.8. *If $\Omega \subset \mathbb{C}$ is open and connected and $f : \Omega \to \mathbb{C}$ is holomorphic and non-constant, then f maps open sets to open sets.*

Proof. Suppose $p \in \Omega$ and $q = f(p)$. We have a power series expansion

$$f(z) = f(p) + \sum_{n=k}^{\infty} a_n (z - p)^n, \tag{17.27}$$

where we pick a_k to be the first nonzero coefficient. It follows that there is a disk $D_\rho(p)$ such that $f\big|_{\partial D_\rho(p)}$ is bounded away from q. Thus Proposition 17.6 applies to $S = D_\varepsilon(q)$ for some $\varepsilon > 0$. Hence, for all $q' \in D_\varepsilon(q)$,

$$\nu(f - q', D_\rho(p)) = \nu(f - q, D_\rho(p)) = k. \tag{17.28}$$

Hence such points q' are contained in the range of f, and the proposition is proved.

The argument principle also holds for meromorphic functions. We have the following result.

Proposition 17.9. *Assume f is meromorphic on a bounded domain Ω, and C^1 on a neighborhood of $\partial\Omega$. Then the number of zeros of f minus the number of poles of f (counting multiplicity) in Ω is equal to the sum of the winding numbers of $f(C_j)$ about 0, where C_j are the connected components of $\partial\Omega$.*

Proof. The identity (17.1), with $\nu(f, \Omega)$ equal to zeros minus poles, follows by the same reasoning as used in the proof of Proposition 17.1. Now, in (17.3)–(17.5), m_j is a positive integer if f has a zero at p_j and it is a negative integer if f has a pole at p_j. The interpretation of the right side of (17.1) in terms of winding numbers follows as before.

Another application of Proposition 17.1 yields the following result, known as Hurwitz' theorem.

Proposition 17.10. *Assume f_n are holomorphic on a connected region Ω and $f_n \to f$ locally uniformly on Ω. Assume each f_n is nowhere vanishing in Ω. Then f is either nowhere vanishing or identically zero in Ω.*

Proof. We know f is holomorphic in Ω and $f_n' \to f'$ locally uniformly on Ω; see Proposition 5.10. Assume f is not identically zero. If it has zeros in Ω, they are isolated. Say D is a disk in Ω such that f has zeros in D but not in ∂D. It follows that $1/f_n \to 1/f$ uniformly on ∂D. By (17.1),

$$\frac{1}{2\pi i}\int\limits_{\partial D}\frac{f_n'(z)}{f_n(z)}\,dz = 0, \quad \forall\, n. \tag{17.29}$$

Then passing to the limit gives

$$\nu(f,D) = \frac{1}{2\pi i}\int\limits_{\partial D}\frac{f'(z)}{f(z)}\,dz = 0, \tag{17.30}$$

contradicting the possibility that f has zeros in D.

Exercises

1. Let $f(z) = z^3 + iz^2 - 2iz + 2$. Compute the change in the argument of $f(z)$ as z varies along:
   a) the real axis from 0 to ∞,
   b) the imaginary axis from 0 to ∞,
   c) the quarter circle $z = Re^{i\theta}$, where R is large and $0 \le \theta \le \pi/2$.

Use this information to determine the number of zeros of f in the first quadrant.

2. Prove that for any $\varepsilon > 0$ the function

$$\frac{1}{z+i} + \sin z$$

has infinitely many zeros in the strip $|\operatorname{Im} z| < \varepsilon$.
Hint. Rouché's theorem.

3. Suppose $f : \Omega \to \mathbb{C}$ is holomorphic and one-to-one. Show that $f'(p) \neq 0$ for all $p \in \Omega$, using the argument principle.
Hint. Compare the proof of Proposition 17.8, the open mapping theorem.

4. Make use of Exercise 7 in §5 to produce another proof of the open mapping theorem.

5. Let $\Omega \subset \mathbb{C}$ be open and connected and assume $g_n : \Omega \to \mathbb{C}$ are holomorphic and each is one-to one (we say univalent). Assume $g_n \to g$ locally uniformly on Ω. Show that g is either univalent or constant.
Hint. Pick arbitrary $b \in \Omega$ and consider $f_n(z) = g_n(z) - g_n(b)$.

6. Let D be a disk in $\mathbb{C}$. Assume $f \in C^1(\overline{D})$ is holomorphic in D. Show that $f(\partial D)$ cannot be a figure 8.

7. Let $\lambda > 1$. Show that $ze^{\lambda - z} = 1$ for exactly one $z \in D = \{z \in \mathbb{C} : |z| < 1\}$.
Hint. With $f(z) = ze^{\lambda - z}$, show that

$$|z| = 1 \Longrightarrow |f(z)| > 1.$$

Compare the number of solutions to $f(z) = 0$. Use either Proposition 17.7, with $S = \overline{D}$, or Rouché's theorem. with $f(z) = ze^{\lambda - z}$ and $g(z) = ze^{\lambda - z} - 1$.

In Exercises 8–11, we consider

$$\varphi(z) = ze^{-z}, \quad \varphi : \overline{D} \to \mathbb{C}, \quad \gamma = \varphi\Big|_{\partial D} : \partial D \to \mathbb{C}.$$

8. Show that $\gamma : \partial D \to \mathbb{C}$ is one-to-one.
Hint. $\gamma(z) = \gamma(w) \Rightarrow z/w = e^{z-w} \Rightarrow z - w \in i\mathbb{R} \Rightarrow z = \overline{w}$
$\Rightarrow \varphi(z) = \overline{\varphi(z)} \Rightarrow e^{i(\theta - \sin\theta)} = \pm 1$ if $z = e^{i\theta} \Rightarrow \cdots$

Given Exercise 8, it is a consequence of the *Jordan curve theorem* (which we assume here) that $\mathbb{C} \setminus \gamma(\partial D)$ has exactly two connected components. Say Ω_+ is the component that contains 0 and Ω_- is the other one.

9. Show that

$$p \in \Omega_+ \Rightarrow \nu(\varphi - p, D) = 1, \quad p \in \Omega_- \Rightarrow \nu(\varphi - p, D) = 0.$$

Hint. For the first case, take $p = 0$. For the second, let $p \to \infty$.

10. Deduce that $\varphi : D \to \Omega_+$ is one-to-one and onto.

11. Recalling Exercise 7, show that $\{z \in \mathbb{C} : |z| < 1/e\} \subset \Omega_+$.

12. Imagine walking a dog, on a 6-foot leash. You walk around a tree, keeping a distance of at least 8 feet. Why might Proposition 17.6 be called the "dog-walking theorem"?

18. The Gamma function

The Gamma function has been previewed in (15.17)–(15.18), arising in the computation of a natural Laplace transform:

$$f(t) = t^{z-1} \Longrightarrow \mathcal{L}f(s) = \Gamma(z)\, s^{-z}, \tag{18.1}$$

for Re $z > 0$, with

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \quad \text{Re } z > 0. \tag{18.2}$$

Here we develop further properties of this special function, beginning with the following crucial identity:

$$\begin{aligned} \Gamma(z+1) &= \int_0^\infty e^{-t} t^z\, dt \\ &= -\int_0^\infty \frac{d}{dt}(e^{-t})\, t^z\, dt \\ &= z\,\Gamma(z), \end{aligned} \tag{18.3}$$

for Re $z > 0$, where we use integration by parts. The definition (18.2) clearly gives

$$\Gamma(1) = 1, \tag{18.4}$$

so we deduce that for any integer $k \ge 1$,

$$\Gamma(k) = (k-1)\Gamma(k-1) = \cdots = (k-1)!. \tag{18.5}$$

While $\Gamma(z)$ is defined in (18.2) for Re $z > 0$, note that the left side of (18.3) is well defined for Re $z > -1$, so this identity extends $\Gamma(z)$ to be meromorphic on $\{z : \text{Re } z > -1\}$, with a simple pole at $z = 0$. Iterating this argument, we extend $\Gamma(z)$ to be meromorphic on $\mathbb{C}$, with simple poles at $z = 0, -1, -2, \dots$. Having such a meromorphic continuation of $\Gamma(z)$, we establish the following identity.

Proposition 18.1. *For $z \in \mathbb{C} \setminus \mathbb{Z}$ we have*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{18.6}$$

Proof. It suffices to establish this identity for $0 < \text{Re } z < 1$. In that case we have

$$\begin{aligned} \Gamma(z)\Gamma(1-z) &= \int_0^\infty \int_0^\infty e^{-(s+t)} s^{-z} t^{z-1}\, ds\, dt \\ &= \int_0^\infty \int_0^\infty e^{-u} v^{z-1} (1+v)^{-1}\, du\, dv \\ &= \int_0^\infty (1+v)^{-1} v^{z-1}\, dv, \end{aligned} \tag{18.7}$$

where we have used the change of variables $u = s + t,\ v = t/s$.

One way of showing the last integral in (18.7) is equal to $\pi/\sin \pi z$ is indicated in (16.54). Another approach goes as follows. With $v = e^x$, the last integral in (18.7) is equal to

$$\int_{-\infty}^{\infty} (1 + e^x)^{-1} e^{xz}\, dx, \tag{18.8}$$

which is holomorphic on $0 < \text{Re}\, z < 1$. We want to show that this is equal to the right side of (18.6) on this strip. It suffices to prove identity on the line $z = 1/2 + i\xi,\ \xi \in \mathbb{R}$. Then (18.8) is equal to the Fourier integral

$$\int_{-\infty}^{\infty} \Bigl(2 \cosh \frac{x}{2}\Bigr)^{-1} e^{ix\xi}\, dx. \tag{18.9}$$

This was evaluated in §16; by (16.23) it is equal to

$$\frac{\pi}{\cosh \pi\xi}, \tag{18.10}$$

and since

$$\frac{\pi}{\sin \pi(\frac{1}{2} + i\xi)} = \frac{\pi}{\cosh \pi\xi}, \tag{18.11}$$

we again have (18.6).

Corollary 18.2. *The function $\Gamma(z)$ has no zeros, so $1/\Gamma(z)$ is an entire function.*

For our next result, we begin with the following estimate:

Lemma 18.3. *We have*

$$0 \le e^{-t} - \Bigl(1 - \frac{t}{n}\Bigr)^n \le \frac{t^2}{n} e^{-t}, \quad 0 \le t \le n, \tag{18.12}$$

the latter inequality holding provided $n \ge 4$.

Proof. The first inequality in (18.12) is equivalent to the simple estimate $e^{-y} - (1 - y) \ge 0$ for $0 \le y \le 1$. To see this, denote the function by $f(y)$ and note that $f(0) = 0$ while $f'(y) = 1 - e^{-y} \ge 0$ for $y \ge 0$.

As for the second inequality in (18.12), write

$$\begin{aligned} \log\Bigl(1 - \frac{t}{n}\Bigr)^n &= n \log\Bigl(1 - \frac{t}{n}\Bigr) = -t - X, \\ X &= \frac{t^2}{n}\Bigl(\frac{1}{2} + \frac{1}{3}\frac{t}{n} + \frac{1}{4}\Bigl(\frac{t}{n}\Bigr)^2 + \cdots\Bigr). \end{aligned} \tag{18.13}$$

We have $(1 - t/n)^n = e^{-t-X}$ and hence, for $0 \le t \le n$,

$$e^{-t} - \Bigl(1 - \frac{t}{n}\Bigr)^n = (1 - e^{-X})e^{-t} \le X e^{-t},$$

using the estimate $x - (1 - e^{-x}) \geq 0$ for $x \geq 0$ (as above). It is clear from (18.13) that $X \leq t^2/n$ if $t \leq n/2$. On the other hand, if $t \geq n/2$ and $n \geq 4$ we have $t^2/n \geq 1$ and hence $e^{-t} \leq (t^2/n)e^{-t}$.

We use (18.12) to obtain, for Re $z > 0$,

$$\begin{aligned} \Gamma(z) &= \lim_{n\to\infty} \int_0^n \Bigl(1 - \frac{t}{n}\Bigr)^n t^{z-1}\, dt \\ &= \lim_{n\to\infty} n^z \int_0^1 (1-s)^n s^{z-1}\, ds. \end{aligned}$$

Repeatedly integrating by parts gives

$$\Gamma(z) = \lim_{n\to\infty} n^z \, \frac{n(n-1)\cdots 1}{z(z+1)\cdots(z+n-1)} \int_0^1 s^{z+n-1}\, ds, \tag{18.14}$$

which yields the following result of Euler:

Proposition 18.4. *For* Re $z > 0$*, we have*

$$\Gamma(z) = \lim_{n\to\infty} n^z \, \frac{1\cdot 2\cdots n}{z(z+1)\cdots(z+n)}, \tag{18.15}$$

Using the identity (18.3), analytically continuing $\Gamma(z)$, we have (18.15) for all $z \in \mathbb{C}$ other than $0, -1, -2, \dots$. In more detail, we have

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \lim_{n\to\infty} n^{z+1} \frac{1}{z} \, \frac{1\cdot 2\cdots n}{(z+1)(z+2)\cdots(z+1+n)},$$

for $\operatorname{Re} z > -1 (z \neq 0)$. We can rewrite the right side as

$$\begin{aligned} & n^z \, \frac{1\cdot 2\cdots n\cdot n}{z(z+1)\cdots(z+n+1)} \\ &= (n+1)^z \, \frac{1\cdot 2\cdots(n+1)}{z(z+1)\cdots(z+n+1)} \cdot \Bigl(\frac{n}{n+1}\Bigr)^{z+1}, \end{aligned}$$

and $(n/(n+1))^{z+1} \to 1$ as $n \to \infty$. This extends (18.15) to $\{z \neq 0 : \operatorname{Re} z > -1\}$, and iteratively we get further extensions.

We can rewrite (18.15) as

$$\Gamma(z) = \lim_{n\to\infty} n^z \, z^{-1}(1+z)^{-1}\Bigl(1+\frac{z}{2}\Bigr)^{-1} \cdots \Bigl(1+\frac{z}{n}\Bigr)^{-1}. \tag{18.16}$$

To work on this formula, we define Euler's constant:

$$\gamma = \lim_{n\to\infty} \Bigl(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\Bigr). \tag{18.17}$$

Then (18.16) is equivalent to

$$\Gamma(z) = \lim_{n\to\infty} e^{-\gamma z} \, e^{z(1+1/2+\cdots+1/n)} \, z^{-1}(1+z)^{-1}\Bigl(1+\frac{z}{2}\Bigr)^{-1} \cdots \Bigl(1+\frac{z}{n}\Bigr)^{-1}, \tag{18.18}$$

which leads to the following Euler product expansion.

Proposition 18.5. *For all $z \in \mathbb{C}$, we have*

$$\frac{1}{\Gamma(z)} = z\, e^{\gamma z} \prod_{n=1}^{\infty} \Bigl(1 + \frac{z}{n}\Bigr) e^{-z/n}. \tag{18.19}$$

Regarding convergence issues for (18.19), compare the treatment of (9.22).

We can combine (18.6) and (18.19) to produce a product expansion for $\sin \pi z$. In fact, it follows from (18.19) that the entire function $1/\Gamma(z)\Gamma(-z)$ has the product expansion

$$\frac{1}{\Gamma(z)\Gamma(-z)} = -z^2 \prod_{n=1}^{\infty} \Bigl(1 - \frac{z^2}{n^2}\Bigr). \tag{18.20}$$

Since $\Gamma(1-z) = -z\Gamma(-z)$, we have by (18.6) that

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \Bigl(1 - \frac{z^2}{n^2}\Bigr). \tag{18.21}$$

This result was established, by another method, in §9, Proposition 9.4. For another proof of (18.21), see §30, Exercise 2.

Here is another application of (18.6). If we take $z = 1/2$, we get $\Gamma(1/2)^2 = \pi$. Since (18.2) implies $\Gamma(1/2) > 0$, we have

$$\Gamma\Bigl(\frac{1}{2}\Bigr) = \sqrt{\pi}. \tag{18.22}$$

Another way to obtain (18.22) is the following. A change of variable gives

$$\int_0^{\infty} e^{-x^2}\, dx = \frac{1}{2} \int_0^{\infty} e^{-t} t^{-1/2}\, dt = \frac{1}{2} \Gamma\Bigl(\frac{1}{2}\Bigr). \tag{18.23}$$

It follows from (10.6) that the left side of (18.23) is equal to $\sqrt{\pi}/2$, so we again obtain (18.22). Note that application of (18.3) then gives, for each integer $k \ge 1$,

$$\Gamma\Bigl(k + \frac{1}{2}\Bigr) = \pi^{1/2} \Bigl(k - \frac{1}{2}\Bigr)\Bigl(k - \frac{3}{2}\Bigr) \cdots \Bigl(\frac{1}{2}\Bigr). \tag{18.24}$$

One can calculate the area A_{n-1} of the unit sphere $S^{n-1} \subset \mathbb{R}^n$ by relating Gaussian integrals to the Gamma function. To see this, note that the argument giving (10.6) yields

$$\int_{\mathbb{R}^n} e^{-|x|^2}\, dx = \Bigl(\int_{-\infty}^{\infty} e^{-x^2}\, dx\Bigr)^n = \pi^{n/2}. \tag{18.25}$$

On the other hand, using spherical polar coordinates to compute the left side of (18.24) gives

$$
\begin{aligned}
\int_{\mathbb{R}^n} e^{-|x|^2}\,dx &= A_{n-1}\int_0^\infty e^{-r^2}\,r^{n-1}\,dr \\
&= \frac{1}{2}A_{n-1}\int_0^\infty e^{-t}\,t^{n/2-1}\,dt,
\end{aligned} \tag{18.26}
$$

where we use $t = r^2$. Recognizing the last integral as $\Gamma(n/2)$, we have

$$
A_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{18.27}
$$

More details on this argument are given at the end of Appendix D.

The following is a useful integral representation of $1/\Gamma(z)$.

Proposition 18.6. *Take $\rho > 0$ and let σ be the path that goes from $-\infty - i0$ to $-\rho - i0$, then counterclockwise from $-\rho - i0$ to $-\rho + i0$, along the circle $|\zeta| = \rho$, and then from $-\rho + i0$ to $-\infty + i0$. Then*

$$
\frac{2\pi i}{\Gamma(z)} = \int_\sigma e^\zeta \zeta^{-z}\,d\zeta, \quad \forall\, z \in \mathbb{C}.
$$

Proof. Denote the right side by $I(z)$. Note that it is independent of the choice of $\rho \in (0,\infty)$. Both sides are holomorphic in $z \in \mathbb{C}$, so it suffices to establish identity for $z = x \in (0,1)$. In this case, as $\rho \to 0$, the contribution to the integral from the circle vanishes, so we can pass to the limit and write

$$
\begin{aligned}
I(x) &= \int_0^\infty e^{-t}t^{-x}e^{\pi i x}\,dt - \int_0^\infty e^{-t}t^{-x}e^{-\pi i x}\,dt \\
&= 2i(\sin \pi x)\Gamma(1-x) \\
&= \frac{2\pi i}{\Gamma(x)},
\end{aligned}
$$

the last identity by (18.6).

Exercises

1. Use the product expansion (18.19) to prove that

$$
\frac{d}{dz}\frac{\Gamma'(z)}{\Gamma(z)} = \sum_{n=0}^\infty \frac{1}{(z+n)^2}. \tag{18.28}
$$

Hint. Go from (18.19) to

$$\log \frac{1}{\Gamma(z)} = \log z + \gamma z + \sum_{n=1}^{\infty} \Bigl[\log\Bigl(1 + \frac{z}{n}\Bigr) - \frac{z}{n}\Bigr],$$

and note that

$$\frac{d}{dz}\frac{\Gamma'(z)}{\Gamma(z)} = \frac{d^2}{dz^2}\log\Gamma(z).$$

2. Let

$$\gamma_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log(n+1).$$

Show that $\gamma_n \nearrow$ and that $0 < \gamma_n < 1$. Deduce that $\gamma = \lim_{n\to\infty} \gamma_n$ exists, as asserted in (18.17).

3. Using $(\partial/\partial z)t^{z-1} = t^{z-1}\log t$, show that

$$f_z(t) = t^{z-1}\log t, \quad (\mathrm{Re}\ z > 0)$$

has Laplace transform

$$\mathcal{L}f_z(s) = \frac{\Gamma'(z) - \Gamma(z)\log s}{s^z}, \quad \mathrm{Re}\ s > 0.$$

4. Show that (18.19) yields

$$\Gamma(z+1) = z\Gamma(z) = e^{-\gamma z}\prod_{n=1}^{\infty}\Bigl(1 + \frac{z}{n}\Bigr)^{-1} e^{z/n}, \quad |z| < 1. \tag{18.29}$$

Use this to show that

$$\Gamma'(1) = \frac{d}{dz}\bigl(z\Gamma(z)\bigr)\big|_{z=0} = -\gamma. \tag{18.30}$$

5. Using Exercises 3–4, show that

$$f(t) = \log t \Longrightarrow \mathcal{L}f(s) = -\frac{\log s + \gamma}{s}, \tag{18.31}$$

and that

$$\gamma = -\int_0^{\infty} (\log t)e^{-t}\,dt. \tag{18.32}$$

6. Show that $\gamma = \gamma_a - \gamma_b$, with

$$\gamma_a = \int_0^1 \frac{1-e^{-t}}{t}\,dt, \quad \gamma_b = \int_1^\infty \frac{e^{-t}}{t}\,dt. \tag{18.33}$$

Consider how to obtain accurate numerical evaluations of these quantities.
Hint. Split the integral for γ in Exercise 5 into two pieces. Integrate each piece by parts, using $e^{-t} = -(d/dt)(e^{-t}-1)$ for one and $e^{-t} = -(d/dt)e^{-t}$ for the other. See Appendix J for more on this.

7. Use the Laplace transform identity (18.1) for $f_z(t) = t^{z-1}$ (on $t \ge 0$, given $\mathrm{Re}\, z > 0$) plus the results of Exercises 5–6 of §15 to show that

$$B(z,\zeta) = \frac{\Gamma(z)\Gamma(\zeta)}{\Gamma(z+\zeta)}, \quad \mathrm{Re}\, z, \mathrm{Re}\,\zeta > 0, \tag{18.34}$$

where the *beta function* $B(z,\zeta)$ is defined by

$$B(z,\zeta) = \int_0^1 s^{z-1}(1-s)^{\zeta-1}\,ds, \quad \mathrm{Re}\, z, \mathrm{Re}\,\zeta > 0. \tag{18.35}$$

The identity (18.34) is known as Euler's formula for the beta function.

8. Show that, for any $z \in \mathbb{C}$, when $n \ge 2|z|$, we have

$$\Bigl(1+\frac{z}{n}\Bigr)e^{-z/n} = 1 + w_n \tag{18.36}$$

with $\log(1+w_n) = \log(1+z/n) - z/n$ satisfying

$$\bigl|\log(1+w_n)\bigr| \le \frac{|z|^2}{n^2}.$$

Show that this estimate implies the convergence of the product on the right side of (18.19), locally uniformly on $\mathbb{C}$.

18A. The Legendre duplication formula

The Legendre duplication formula relates $\Gamma(2z)$ and $\Gamma(z)\Gamma(z+1/2)$. Note that each of these functions is meromorphic, with poles precisely at $\{0,-1/2,-1,-3/2,-2,\dots\}$, all simple, and both functions are nowhere vanishing. Hence their quotient is an entire holomorphic function, and it is nowhere vanishing, so

$$\Gamma(2z) = e^{A(z)}\Gamma(z)\Gamma\Bigl(z+\frac{1}{2}\Bigr), \tag{18.37}$$

with $A(z)$ holomorphic on $\mathbb{C}$. We seek a formula for $A(z)$. We will be guided by (18.19), which implies that

$$\frac{1}{\Gamma(2z)} = 2ze^{2\gamma z}\prod_{n=1}^{\infty}\Bigl(1+\frac{2z}{n}\Bigr)e^{-2z/n}, \tag{18.38}$$

and (via results given in §18B)

$$\begin{aligned}
&\frac{1}{\Gamma(z)\Gamma(z+1/2)} \\
&= z\Bigl(z+\frac{1}{2}\Bigr)e^{\gamma z}e^{\gamma(z+1/2)}\Bigl\{\prod_{n=1}^{\infty}\Bigl(1+\frac{z}{n}\Bigr)e^{-z/n}\Bigl(1+\frac{z+1/2}{n}\Bigr)e^{-(z+1/2)/n}\Bigr\}.
\end{aligned} \tag{18.39}$$

Setting

$$1+\frac{z+1/2}{n} = \frac{2z+2n+1}{2n} = \Bigl(1+\frac{2z}{2n+1}\Bigr)\Bigl(1+\frac{1}{2n}\Bigr), \tag{18.40}$$

and

$$e^{-(z+1/2)/n} = e^{-2z/(2n+1)}e^{-2z[(1/2n)-1/(2n+1)]}e^{-1/2n}, \tag{18.41}$$

we can write the infinite product on the right side of (18.39) as

$$\begin{aligned}
&\Bigl\{\prod_{n=1}^{\infty}\Bigl(1+\frac{2z}{2n}\Bigr)e^{-2z/2n}\Bigl(1+\frac{2z}{2n+1}\Bigr)e^{-2z/(2n+1)}\Bigr\} \\
&\qquad \times\Bigl\{\prod_{n=1}^{\infty}\Bigl(1+\frac{1}{2n}\Bigr)e^{-1/2n}\Bigr\}\times\prod_{n=1}^{\infty}e^{-2z[(1/2n)-1/(2n+1)]}.
\end{aligned} \tag{18.42}$$

Hence

$$\begin{aligned}
\frac{1}{\Gamma(z)\Gamma(z+1/2)} &= ze^{2\gamma z}e^{\gamma/2}\cdot\frac{e^{2z}}{2}(1+2z)e^{-2z}\times(18.42) \\
&= 2ze^{2\gamma z}e^{\gamma/2}\frac{e^{2z}}{4}\Bigl\{\prod_{k=1}^{\infty}\Bigl(1+\frac{2z}{k}\Bigr)e^{-2z/k}\Bigr\} \\
&\qquad\times\Bigl\{\prod_{n=1}^{\infty}\Bigl(1+\frac{1}{2n}\Bigr)e^{-1/2n}\Bigr\}\prod_{n=1}^{\infty}e^{-2z[(1/2n)-1/(2n+1)]}.
\end{aligned} \tag{18.43}$$

Now, setting $z=1/2$ in (18.19) gives

$$\frac{1}{\Gamma(1/2)} = \frac{1}{2}e^{\gamma/2}\prod_{n=1}^{\infty}\Bigl(1+\frac{1}{2n}\Bigr)e^{-1/2n}, \tag{18.44}$$

so taking (18.38) into account yields

$$\begin{aligned}\frac{1}{\Gamma(z)\Gamma(z+1/2)} &= \frac{1}{\Gamma(1/2)\Gamma(2z)}\,\frac{e^{2z}}{2}\prod_{n=1}^{\infty} e^{-2z[(1/2n)-1/(2n+1)]} \\ &= \frac{1}{\Gamma(1/2)\Gamma(2z)}\,\frac{e^{2\alpha z}}{2},\end{aligned} \tag{18.45}$$

where

$$\begin{aligned}\alpha &= 1-\sum_{n=1}^{\infty}\Bigl(\frac{1}{2n}-\frac{1}{2n+1}\Bigr) \\ &= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}\cdots \\ &= \log 2.\end{aligned} \tag{18.46}$$

Hence $e^{2\alpha z} = 2^{2z}$, and we get

$$\Gamma\Bigl(\frac{1}{2}\Bigr)\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\Bigl(z+\frac{1}{2}\Bigr). \tag{18.47}$$

This is the Legendre duplication formula. Recall that $\Gamma(1/2)=\sqrt{\pi}$.

An equivalent formulation of (18.47) is

$$(2\pi)^{1/2}\Gamma(z) = 2^{z-1/2}\Gamma\Bigl(\frac{z}{2}\Bigr)\Gamma\Bigl(\frac{z+1}{2}\Bigr). \tag{18.48}$$

This generalizes to the following formula of Gauss,

$$(2\pi)^{(n-1)/2}\Gamma(z) = n^{z-1/2}\Gamma\Bigl(\frac{z}{n}\Bigr)\Gamma\Bigl(\frac{z+1}{n}\Bigr)\cdots\Gamma\Bigl(\frac{z+n-1}{n}\Bigr), \tag{18.49}$$

valid for $n=3,4,\dots$.

19. The Riemann zeta function and the prime number theorem

The Riemann zeta function is defined by

$$\zeta(s)=\sum_{n=1}^{\infty} n^{-s}, \quad \operatorname{Re} s>1. \tag{19.1}$$

Some special cases of this arose in §13, namely

$$\zeta(2)=\frac{\pi^2}{6}, \quad \zeta(4)=\frac{\pi^4}{90}. \tag{19.2}$$

See also (16.57). This function is of great interest in number theory, due to the following result, known as Euler's product formula.

Proposition 19.1. *Let* $\{p_j : j\ge 1\}=\{2,3,5,7,11,\dots\}$ *denote the set of prime numbers in* $\mathbb{N}$. *Then, for* $\operatorname{Re} s>1$,

$$\zeta(s)=\prod_{j=1}^{\infty}(1-p_j^{-s})^{-1}. \tag{19.3}$$

Proof. The convergence of the infinite product in (19.3) for $\operatorname{Re} s>1$ follows from Proposition 9.1 and the fact that $p_j\ge j$. Now we can write the right side of (19.3) as

$$\begin{aligned}&\prod_{j=1}^{\infty}(1+p_j^{-s}+p_j^{-2s}+p_j^{-3s}+\cdots)\\ &=1+\sum_j p_j^{-s}+\sum_{j_1\le j_2}(p_{j_1}p_{j_2})^{-s}+\sum_{j_1\le j_2\le j_3}(p_{j_1}p_{j_2}p_{j_3})^{-s}+\cdots.\end{aligned} \tag{19.4}$$

That this is identical to the right side of (19.1) follows from the fundamental theorem of arithmetic, which says that each integer $n\ge 2$ has a unique factorization into a product of primes.

From (19.1) we see that

$$s\searrow 1 \Longrightarrow \zeta(s)\nearrow +\infty. \tag{19.5}$$

Hence

$$\prod_{j=1}^{\infty}(1-p_j^{-1})=0. \tag{19.6}$$

Applying (9.10), we deduce that

$$\sum_{j=1}^{\infty} \frac{1}{p_j} = \infty, \tag{19.7}$$

which is a quantitative strengthening of the result that there are infinitely many primes. In fact, applying Exercise 2 of §9, we have the more precise result

$$\frac{1}{2} \log \zeta(s) \le \sum_{j=1}^{\infty} \frac{1}{p_j^s} \le \log \zeta(s), \quad \forall\, s > 1. \tag{19.8}$$

These results suggest that $p_j \to \infty$ somewhat faster than j, but not as fast as $j(\log j)^{1+a}$, given $a > 0$. Just how fast p_j increases is the subject of the prime number theorem; see Theorem 19.10 below. See also Exercises 5–8 below for a variant of (19.8) and a hint of how that might lead one to conjecture the prime number theorem.

Another application of (9.10) gives

$$\zeta(s) \neq 0, \quad \text{for} \quad \operatorname{Re} s > 1. \tag{19.9}$$

Our next goal is to establish the following.

Proposition 19.2. *The function $\zeta(s)$ extends to a meromorphic function on $\mathbb{C}$, with one simple pole, at $s = 1$. In fact*

$$\zeta(s) - \frac{1}{s-1} \tag{19.10}$$

extends to an entire function of s.

To start the demonstration, we relate the Riemann zeta function to the function

$$g(t) = \sum_{n=1}^{\infty} e^{-n^2 \pi t}. \tag{19.11}$$

Indeed, we have

$$\begin{aligned} \int_0^\infty g(t) t^{s-1}\, dt &= \sum_{n=1}^{\infty} n^{-2s} \pi^{-s} \int_0^\infty e^{-t} t^{s-1}\, dt \\ &= \zeta(2s) \pi^{-s} \Gamma(s). \end{aligned} \tag{19.12}$$

This gives rise to further useful identities, via the Jacobi identity (14.43), i.e.,

$$\sum_{\ell=-\infty}^{\infty} e^{-\pi \ell^2 t} = \sqrt{\frac{1}{t}} \sum_{k=-\infty}^{\infty} e^{-\pi k^2/t}, \tag{19.13}$$

which implies

$$g(t) = -\frac{1}{2} + \frac{1}{2}t^{-1/2} + t^{-1/2}g\Big(\frac{1}{t}\Big). \tag{19.14}$$

To use this, we first note from (19.12) that, for Re $s > 1$,

$$\begin{aligned} \Gamma\Big(\frac{s}{2}\Big)\pi^{-s/2}\zeta(s) &= \int_0^\infty g(t)t^{s/2-1}\,dt \\ &= \int_0^1 g(t)t^{s/2-1}\,dt + \int_1^\infty g(t)t^{s/2-1}\,dt. \end{aligned} \tag{19.15}$$

Into the integral over $[0,1]$ we substitute the right side of (19.14) for $g(t)$, to obtain

$$\begin{aligned} \Gamma\Big(\frac{s}{2}\Big)\pi^{-s/2}\zeta(s) = &\int_0^1 \Big(-\frac{1}{2} + \frac{1}{2}t^{-1/2}\Big)t^{s/2-1}\,dt \\ &+ \int_0^1 g(t^{-1})t^{s/2-3/2}\,dt + \int_1^\infty g(t)t^{s/2-1}\,dt. \end{aligned} \tag{19.15A}$$

We evaluate the first integral on the right, and replace t by $1/t$ in the second integral, to obtain, for Re $s > 1$,

$$\Gamma\Big(\frac{s}{2}\Big)\pi^{-s/2}\zeta(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \big[t^{s/2} + t^{(1-s)/2}\big]g(t)t^{-1}dt. \tag{19.16}$$

Note that $g(t) \le Ce^{-\pi t}$ for $t \in [1,\infty)$, so the integral on the right side of (19.16) defines an entire function of s. Since $1/\Gamma(s/2)$ is entire, with simple zeros at $s = 0, -2, -4, \dots$, as seen in §18, this implies that $\zeta(s)$ is continued as a meromorphic function on $\mathbb{C}$, with one simple pole, at $s = 1$. The regularity of (19.10) at $s = 1$ follows from the identity $\Gamma(1/2) = \pi^{1/2}$. This finishes the proof of Proposition 19.2.

The formula (19.16) does more than establish the meromorphic continuation of the zeta function. Note that the right side of (19.16) is *invariant* under replacing s by $1 - s$. Thus we have an identity known as Riemann's functional equation:

Proposition 19.3. *For $s \neq 1$,*

$$\Gamma\Big(\frac{s}{2}\Big)\pi^{-s/2}\zeta(s) = \Gamma\Big(\frac{1-s}{2}\Big)\pi^{-(1-s)/2}\zeta(1-s). \tag{19.17}$$

The following is an important strengthening of (19.9).

Proposition 19.4. *For all real $t \neq 0$, $\zeta(1+it) \neq 0$. Hence*

$$\zeta(s) \neq 0 \quad for \quad \mathrm{Re}\, s \ge 1. \tag{19.18}$$

To prove this, we start with the following lemma.

Lemma 19.5. *For all* $\sigma > 1,\ t \in \mathbb{R}$,

$$\zeta(\sigma)^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| \geq 1. \tag{19.19}$$

Proof. Since $\log|z| = \operatorname{Re}\log z$, (19.19) is equivalent to

$$3\log\zeta(\sigma) + 4\operatorname{Re}\log\zeta(\sigma + it) + \operatorname{Re}\log\zeta(\sigma + 2it) \geq 0. \tag{19.20}$$

Now (19.3) is equivalent to

$$\log\zeta(s) = \sum_{p\in\mathcal{P}}\sum_{k=1}^{\infty} \frac{1}{k}\,\frac{1}{p^{ks}}, \tag{19.21}$$

where $\mathcal{P}$ denotes the set of prime numbers in $\mathbb{N}$, hence

$$\log\zeta(s) = \sum_{n\geq 1} \frac{a(n)}{n^s}, \quad \text{with each} \ \ a(n) \geq 0. \tag{19.22}$$

Thus the left side of (19.20) is equal to

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^\sigma}\ \operatorname{Re}(3 + 4n^{-it} + n^{-2it}). \tag{19.23}$$

But, with $\theta_n = t\log n$,

$$\begin{aligned} \operatorname{Re}(3 + 4n^{-it} + n^{-2it}) &= 3 + 4\cos\theta_n + \cos 2\theta_n \\ &= 2 + 4\cos\theta_n + 2\cos^2\theta_n \\ &= 2(1 + \cos\theta_n)^2, \end{aligned} \tag{19.24}$$

which is ≥ 0 for each n, so we have (19.20), hence the lemma.

We are now ready for the

Proof of Proposition 19.4. Recall from Proposition 19.2 that $\zeta(s) - 1/(s-1)$ is an entire function. Suppose that $t \in \mathbb{R}\setminus 0$ and $\zeta(1 + it) = 0$. Then

$$\lim_{\sigma\searrow 1} \frac{\zeta(\sigma + it)}{\sigma - 1} = \zeta'(1 + it). \tag{19.25}$$

If $\Phi(\sigma)$ denotes the left side of (19.19), then

$$\Phi(\sigma) = \big((\sigma - 1)\zeta(\sigma)\big)^3 \Big(\frac{|\zeta(\sigma + it)|}{\sigma - 1}\Big)^4 \big|(\sigma - 1)\zeta(\sigma + 2it)\big|. \tag{19.26}$$

But, as we know,

$$\lim_{\sigma \searrow 1} (\sigma - 1)\zeta(\sigma) = 1. \tag{19.27}$$

Thus if (19.25) holds we must have $\lim_{\sigma \searrow 0} \Phi(\sigma) = 0$, contradicting (19.19). This finishes the proof.

It is of great interest to determine where $\zeta(s)$ can vanish. By (19.16), it must vanish at all the poles of $\Gamma(s/2)$, other than $s = 0$, i.e.,

$$\zeta(s) = 0 \quad \text{on} \quad \{-2, -4, -6, \dots\}. \tag{19.28}$$

These are known as the "trivial zeros" of $\zeta(s)$. It follows from Proposition 19.4 and the functional equation (19.17) that all the other zeros of $\zeta(s)$ are contained in the "critical strip"

$$\Omega = \{s \in \mathbb{C} : 0 < \operatorname{Re} s < 1\}. \tag{19.29}$$

Concerning where in Ω these zeros can be, there is the following famous conjecture.

The Riemann hypothesis. All zeros in Ω of $\zeta(s)$ lie on the critical line

$$\Big\{\frac{1}{2} + it : t \in \mathbb{R}\Big\}. \tag{19.30}$$

Many zeros of $\zeta(s)$ have been computed and shown to lie on this line, but after over a century, a proof (or refutation) of the Riemann hypothesis eludes the mathematics community.

Returning to what has been established, we know that $1/\zeta(s)$ is meromorphic on $\mathbb{C}$, with a simple zero at $s = 1$, and all the poles satisfy $\operatorname{Re} s < 1$. We will want to establish a quantitative refinement of Proposition 19.4. Along the way to doing this, we will find it useful to obtain another representation of $\zeta(s)$ valid beyond $\operatorname{Re} s > 1$. We start with

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \int_0^{\infty} x^{-s} \, d\nu(x), \tag{19.31}$$

valid for $\operatorname{Re} s > 1$, where

$$\nu(x) = [x] = n \quad \text{for} \quad x \in [n, n+1). \tag{19.32}$$

The right side of (19.31) is an elementary case of a Stieltjes integral. We have, for each $N \in \mathbb{N}$,

$$\begin{aligned} \zeta(s) &= \sum_{n=1}^{N} \frac{1}{n^s} + \sum_{n=N+1}^{\infty} \frac{1}{n^s} \\ &= \sum_{n=1}^{N} \frac{1}{n^s} + \int_N^{\infty} x^{-s} \, d\nu(x) \\ &= \sum_{n=1}^{N} \frac{1}{n^s} + s \int_N^{\infty} x^{-s-1} [x] \, dx - N^{-s} \nu(N), \end{aligned} \tag{19.33}$$

the last identity by integration by parts. Note that $N^{-s}\nu(N) = N^{1-s}$. We can add and subtract

$$s \int_N^\infty x^{-s-1} x\, dx = \frac{s}{s-1} N^{1-s}, \tag{19.34}$$

to obtain, for $\operatorname{Re} s > 1$,

$$\zeta(s) = \sum_{n=1}^N \frac{1}{n^s} - s \int_N^\infty \frac{x-[x]}{x^{s+1}}\, dx + \frac{N^{1-s}}{s-1}. \tag{19.35}$$

Noting that the integral on the right side of (19.35) is holomorphic on $\operatorname{Re} s > 0$, we have:

Proposition 19.6. *For each $N \in \mathbb{N}$, the identity (19.35) holds for* $\operatorname{Re} s > 0$.

Proof. Both sides of (19.35) are holomorphic on $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$, except for a pole at $s = 1$, and we have shown that the identity holds for $\operatorname{Re} s > 1$.

REMARK. In the setting of Proposition 19.6, we can apply d/ds to (19.35), obtaining

$$\begin{aligned} \zeta'(s) = & -\sum_{n=1}^N \frac{\log n}{n^s} + s \int_N^\infty \frac{x-[x]}{x^{s+1}} (\log x)\, dx \\ & - \int_N^\infty \frac{x-[x]}{x^{s+1}}\, dx - \frac{N^{1-s}}{s-1}(\log N) - \frac{N^{1-s}}{(s-1)^2}, \end{aligned} \tag{19.36}$$

for $\operatorname{Re} s > 0$.

We next apply (19.35)–(19.36) to obtain upper bounds on $|\zeta(s)|$ and $|\zeta'(s)|$.

Proposition 19.7. *There exists $K < \infty$ such that*

$$|\zeta(s)| \le K \log t, \quad |\zeta'(s)| \le K (\log t)^2, \tag{19.37}$$

for $s = \sigma + it,\ \sigma \ge 1,\ t \ge e$.

Proof. It is elementary that

$$\operatorname{Re} s \ge 2 \Longrightarrow |\zeta(s)| \le \zeta(2) \ \text{ and } \ |\zeta'(s)| \le |\zeta'(2)|. \tag{19.38}$$

Hence it suffices to assume $1 \le \sigma \le 2$ and $t \ge e$. In such a case,

$$|s| \le \sigma + t \le 2t, \quad |s-1| \ge t, \quad \frac{1}{|s-1|} \le \frac{1}{t}. \tag{19.39}$$

Using (19.35), we have

$$
\begin{aligned}
|\zeta(s)| &\le \sum_{n=1}^{N} \frac{1}{n^\sigma} + 2t \int_N^\infty x^{-(\sigma+1)}\, dx + \frac{N^{1-\sigma}}{t} \\
&= \sum_{n=1}^{N} \frac{1}{n^\sigma} + \frac{2t}{\sigma N^\sigma} + \frac{N^{1-\sigma}}{t}.
\end{aligned}
\tag{19.40}
$$

Now take $N = [t]$, so $N \le t \le N+1$, and $\log n \le \log t$ if $n \le N$. We have

$$
\frac{2t}{\sigma N^\sigma} \le 2 \frac{N+1}{N}, \quad \text{and} \quad \frac{N^{1-\sigma}}{t} = \frac{N}{t} \frac{1}{N^\sigma},
\tag{19.41}
$$

so

$$
\begin{aligned}
|\zeta(s)| &= O\Bigl(\sum_{n=1}^{N} \frac{1}{n}\Bigr) + O(1) \\
&= O(\log N) + O(1) \\
&= O(\log t).
\end{aligned}
\tag{19.42}
$$

This gives the result (19.37) for $|\zeta(s)|$. To obtain (19.37) for $|\zeta'(s)|$, we apply the same sort of argument to (19.36). There appears an extra factor of $\log N = O(\log t)$.

We now refine Proposition 19.4.

Proposition 19.8. *There is a constant $C < \infty$ such that*

$$
\Bigl|\frac{1}{\zeta(s)}\Bigr| \le C(\log t)^7,
\tag{19.43}
$$

for

$$
s = \sigma + it, \quad \sigma \ge 1, \quad t \ge e.
\tag{19.44}
$$

Proof. To start, for $\operatorname{Re} s > 1$ we have

$$
\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad \mu(n) \in \{-1, 0, 1\},
\tag{19.45}
$$

cf. Exercise 4, part (d). Hence

$$
\operatorname{Re} s \ge 2 \Longrightarrow \Bigl|\frac{1}{\zeta(s)}\Bigr| \le \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2).
\tag{19.46}
$$

For the rest, it suffices to take $\sigma \in [1,2]$ and $t \geq e$.

To tackle this, we rewrite (19.19) as

$$\frac{1}{|\zeta(\sigma+it)|} \leq \zeta(\sigma)^{3/4}|\zeta(\sigma+2it)|^{1/4}, \tag{19.47}$$

for $\sigma > 1,\ t \in \mathbb{R}$. Clearly $(\sigma-1)\zeta(\sigma)$ is bounded for $\sigma \in [1,2]$, i.e., for some $M < \infty$,

$$\zeta(\sigma) \leq \frac{M}{\sigma-1}, \quad 1 < \sigma \leq 2. \tag{19.48}$$

Also $\zeta(\sigma+2it) = O(\log t)$, by Proposition 19.7, so, for $\sigma \in (1,2]$,

$$\frac{1}{|\zeta(\sigma+it)|} \leq C\frac{(\log t)^{1/4}}{(\sigma-1)^{3/4}}, \tag{19.49}$$

or equivalently

$$|\zeta(\sigma+it)| \geq B\frac{(\sigma-1)^{3/4}}{(\log t)^{1/4}}, \quad \text{if} \ \ 1 < \sigma \leq 2 \ \ \text{and} \ \ t \geq e. \tag{19.50}$$

Clearly this also holds at $\sigma = 1$.

Now take $\alpha \in (1,2)$. If $1 \leq \sigma \leq \alpha$ and $t \geq e$, we have via Proposition 19.7 that

$$\begin{aligned} |\zeta(\sigma+it)-\zeta(\alpha+it)| &\leq \int_\sigma^\alpha |\zeta'(u+it)|\,du \\ &\leq K(\alpha-\sigma)(\log t)^2. \end{aligned} \tag{19.51}$$

Hence, via the triangle inequality,

$$\begin{aligned} |\zeta(\sigma+it)| &\geq |\zeta(\alpha+it)| - |\zeta(\sigma+it)-\zeta(\alpha+it)| \\ &\geq B\frac{(\alpha-1)^{3/4}}{(\log t)^{1/4}} - K(\alpha-1)(\log t)^2. \end{aligned} \tag{19.52}$$

This holds not only for $\sigma \in [1,\alpha]$, but also for $\sigma \in [\alpha,2]$, since then $(\sigma-1)^{3/4} \geq (\alpha-1)^{3/4}$. Consequently, if $\sigma \in [1,2]$ and $t \geq e$, we have

$$|\zeta(\sigma+it)| \geq B\frac{(\alpha-1)^{3/4}}{(\log t)^{1/4}} - K(\alpha-1)(\log t)^2, \quad \forall\, \alpha \in (1,2). \tag{19.53}$$

Our choice of α will depend on t. We pick α so that the first term on the right side of (19.53) is equal to twice the second. This requires

$$\alpha = 1 + \Bigl(\frac{B}{2K}\Bigr)^4 \frac{1}{(\log t)^9}. \tag{19.54}$$

Clearly such $\alpha > 1$. Also,

$$\exists\, T_0 \in (e, \infty) \ \text{ such that } \ t \geq T_0 \Rightarrow \alpha < 2. \tag{19.55}$$

Thus, if $t \geq T_0$,

$$\begin{aligned} |\zeta(\sigma + it)| &\geq K(\alpha - 1)(\log t)^2 \\ &= \frac{C}{(\log t)^7}, \qquad \forall\, \sigma \in [1, 2]. \end{aligned} \tag{19.56}$$

On the other hand, since T_0 is a fixed finite quantity, the inequality (19.56) also holds for $t \in [e, T_0]$, perhaps with a modified value of C. This yields (19.43).

Combining Proposition 19.8 and the estimate on $|\zeta'(s)|$ in (19.37), we have:

Corollary 19.9. *There exists $C < \infty$ such that*

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leq C(\log t)^9, \tag{19.57}$$

for $s = \sigma + it,\ \sigma \geq 1,\ t \geq e$.

Remark. By (19.10) and Proposition 19.4, we also have

$$\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} \ \text{ holomorphic on a neighborhood of } \ \{s \in \mathbb{C} : \operatorname{Re} s \geq 1\}. \tag{19.58}$$

Counting primes

We turn to an examination of the prime counting function

$$\pi(x) = \#\{p \in \mathcal{P} : p \leq x\}, \tag{19.59}$$

where $\mathcal{P}$ denotes the set of positive integers that are primes. Results established above on $\zeta(s)$ will allow for an asymptotic analysis of $\pi(x)$ as $x \to \infty$ that is much more precise than (19.8). It is convenient to use the language of Stieltjes integrals, such as

$$\int_0^\infty f(x)\, d\pi(x) = \sum_{p \in \mathcal{P}} f(p), \tag{19.60}$$

given f continuous on $(0, \infty)$ and sufficiently rapidly decreasing at ∞. The result (19.8) suggests relating $\pi(x)$ to the formula you get by applying log to (19.3), which yields, for $\operatorname{Re} s > 1$,

$$\begin{aligned} \log \zeta(s) &= -\sum_{p \in \mathcal{P}} \log(1 - p^{-s}) \\ &= \sum_{p \in \mathcal{P}} \sum_{k=1}^\infty \frac{1}{k} p^{-ks} \\ &= \int_0^\infty x^{-s}\, dJ(x), \end{aligned} \tag{19.61}$$

where

$$J(x) = \sum_{k \ge 1} \frac{1}{k} \pi(x^{1/k}). \tag{19.62}$$

Note that π and J are supported on $[2, \infty)$. That is, they vanish outside this set. We write $\operatorname{supp} \pi \subset [2, \infty)$. Also note that, for each $x \in [2, \infty)$, the sum in (19.62) is finite, containing at most $\log_2 x$ terms. Applying d/ds to (19.62) yields, for $\operatorname{Re} s > 1$,

$$-\frac{\zeta'(s)}{\zeta(s)} = \int_0^\infty (\log x) x^{-s} \, dJ(x). \tag{19.63}$$

It is then suggestive to introduce $\psi(x)$, so that

$$d\psi(x) = (\log x) \, dJ(x). \tag{19.64}$$

Equivalently, if we want $\operatorname{supp} \psi \subset [2, \infty)$,

$$\begin{aligned} \psi(x) &= \int_0^x (\log y) \, dJ(y) \\ &= (\log x) J(x) - \int_0^x \frac{J(y)}{y} \, dy, \end{aligned} \tag{19.65}$$

where we have integrated by parts. Another equivalent identity is $dJ(x) = (1/\log x) \, d\psi(x)$, yielding

$$\begin{aligned} J(x) &= \int_0^x \frac{1}{\log x} \, d\psi(x) \\ &= \frac{\psi(x)}{\log x} + \int_0^x \frac{1}{(\log y)^2} \frac{\psi(y)}{y} \, dy. \end{aligned} \tag{19.66}$$

We can now write (19.63) as

$$\begin{aligned} -\frac{\zeta'(s)}{\zeta(s)} &= \int_1^\infty x^{-s} \, d\psi(x) \\ &= s \int_1^\infty \psi(x) x^{-s-1} \, dx. \end{aligned} \tag{19.67}$$

Analysis of the functions $J(x)$ and $\psi(x)$ will help us analyze $\pi(x)$, as we will see below.

The prime number theorem

The key result we aim to establish is the following.

Theorem 19.10. *The prime counting function $\pi(x)$ has the property that*

$$\pi(x) \sim \frac{x}{\log x}, \quad \text{as} \;\; x \to +\infty, \tag{19.68}$$

that is, $\pi(x)(\log x/x) \to 1$ as $x \to \infty$.

This result was conjectured independently by Gauss and Legendre in the 1790s, and, following Riemann's work in the 1850s, finally proved independently by Hadamard and de la Vallée Poussin in the 1890s.

We first show how the functions $J(x)$ and $\psi(x)$ play a role in the proof of this result.

Proposition 19.11. *If one shows that*

$$\psi(x) \sim x, \quad \text{as} \;\; x \to \infty, \tag{19.69}$$

i.e., $\psi(x)/x \to 1$ as $x \to \infty$, it follows that

$$J(x) \sim \frac{x}{\log x}, \quad \text{as} \;\; x \to \infty, \tag{19.70}$$

and this in turn implies (19.68).

Proof. Suppose

$$\psi(x) = x + E(x), \quad E(x) = o(x) \;\; \text{as} \;\; x \to \infty. \tag{19.71}$$

Then (19.66) yields

$$J(x) = \frac{x}{\log x} + \int_2^x \frac{dy}{(\log y)^2} + \frac{E(x)}{\log x} + \int_2^\infty \frac{1}{(\log y)^2} \frac{E(y)}{y}\, dy. \tag{19.72}$$

Note that

$$\int_2^x \frac{dy}{(\log y)^2} = \frac{x}{(\log x)^2} - \frac{2}{(\log 2)^2} + 2\int_2^x \frac{dy}{(\log y)^3}. \tag{19.73}$$

From here we have that (19.71) implies (19.70). In order to deduce (19.68), note that (19.62) implies

$$\begin{aligned} \pi(x) < J(x) &< \pi(x) + \pi(x^{1/2}) \sum_{1 \le k \le \log_2 x} \frac{1}{k} \\ &\le \pi(x) + \pi(x^{1/2}) \log(\log_2 x), \end{aligned} \tag{19.74}$$

and since clearly $\pi(x^{1/2}) \le x^{1/2}$, we have

$$J(x) - \pi(x) = O\Big(x^{1/2} \log(\log_2 x)\Big), \tag{19.75}$$

so indeed (19.70) implies (19.68).

Remark. To restate (19.72), we can bring in the logarithmic integral,

$$\begin{aligned}\mathrm{Li}(x) &= \int_2^\infty \frac{dx}{\log x} \\ &= \frac{x}{\log x} + \int_2^x \frac{dy}{(\log y)^2} - \frac{2}{\log 2}.\end{aligned} \tag{19.76}$$

Then (19.72) becomes

$$J(x) = \mathrm{Li}(x) + \frac{E(x)}{\log x} + \int_2^\infty \frac{1}{(\log y)^2}\,\frac{E(y)}{y}\,dy + \frac{2}{\log 2}. \tag{19.77}$$

Though we do not prove it here, it is known that $\mathrm{Li}(x)$ is a better approximation to $\pi(x)$ than $x/\log x$ is. In fact, the conjecture of Gauss used $\mathrm{Li}(x)$ in place of $x/\log x$ in (19.68).

Now, to prove Theorem 19.10, we have the task of establishing (19.69). Note that the right side of (19.67) involves an integral that is the Mellin transform of ψ. Ideally, one would like to analyze $\psi(x)$ by inverting this Mellin transform. Our ability to do this is circumscribed by our knowledge (or lack of knowledge) of the behavior of $\zeta'(s)/\zeta(s)$, appearing on the left side of (19.67). A number of approaches have been devised to pass from (19.67) to (19.69), all of them involving Proposition 19.4. Notable examples include [Wie] and [New]. The approach we take here is adapted from [Ap]. We bring in results on the Fourier transform, developed in §14.

To proceed, we bring in the function

$$\psi_1(x) = \int_0^x \psi(y)\,dy. \tag{19.78}$$

This function is also supported on $[2,\infty)$. Note that the crude estimate $J(x) \le 2x$ (from (19.74)) implies $\psi(x) \le 2x\log x$, and hence $\psi_1(x) \le 2x^2 \log x$. From (19.67), we have, for $\mathrm{Re}\, s > 1$,

$$\begin{aligned}-\frac{\zeta'(s)}{\zeta(s)} &= s\int_1^\infty \psi_1'(x) x^{-s-1}\,dx \\ &= s(s+1)\int_1^\infty \psi_1(x) x^{-s-2}\,dx,\end{aligned} \tag{19.79}$$

or equivalently

$$-\frac{1}{s(s+1)}\,\frac{\zeta'(s)}{\zeta(s)} = \int_1^\infty \frac{\psi_1(x)}{x^2}\,x^{-s}\,dx. \tag{19.80}$$

We want to produce the Mellin transform of an elementary function to cancel the singularity of the left side of (19.80) at $s = 1$, and arrange that the difference be nicely behaved on $\{s : \operatorname{Re} s = 1\}$. Using

$$\int_1^\infty x^{-s}\,dx = \frac{1}{s-1}, \tag{19.81}$$

valid for $\operatorname{Re} s > 1$, we have

$$\begin{aligned}\frac{1}{2}\int_1^\infty \Big(1-\frac{1}{x}\Big)^2 x^{-s}\,dx &= \frac{1}{2}\int_1^\infty (x^{-s} - 2x^{-s-1} + x^{-s-2})\,dx \\ &= \frac{1}{(s-1)s(s+1)},\end{aligned} \tag{19.82}$$

so we achieve such cancellation with

$$-\frac{1}{s(s+1)}\Big(\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}\Big) = \int_1^\infty \Big[\frac{\psi_1(x)}{x^2} - \frac{1}{2}\Big(1-\frac{1}{x}\Big)^2\Big] x^{-s}\,dx. \tag{19.83}$$

To simplify notation, let us set

$$\Phi(x) = \Big[\frac{\psi_1(x)}{x^2} - \frac{1}{2}\Big(1-\frac{1}{x}\Big)^2\Big]\chi_{[1,\infty)}(x), \tag{19.84}$$

where $\chi_S(x)$ is 1 for $x \in S$ and 0 for $x \notin S$. We make the change of variable $x = e^y$ and set $s = 1 + \sigma + it$, with $\sigma > 0$, $t \in \mathbb{R}$, and write the right side of (19.83) as

$$\int_{-\infty}^\infty \Phi(e^y) e^{-\sigma y} e^{-ity}\,dy, \quad \sigma > 0. \tag{19.85}$$

Note that $|\Phi(x)| \le C \log(2 + |x|)$, so

$$|\Phi(e^y)| \le B(1 + |y|)^C. \tag{19.86}$$

Also $\Phi(e^y) = 0$ for $y \le 0$, and hence $\Phi(e^y)e^{-\sigma y}$ is integrable for each $\sigma > 0$. Meanwhile, for $\sigma > 0$ we can write the left side of (19.83) as

$$\Psi_\sigma(t) = -\frac{1}{s(s+1)}\Big(\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}\Big)\Big|_{s=1+\sigma+it}. \tag{19.87}$$

Making use of (19.57)–(19.58), we have the following.

Lemma 19.12. *There exists $C < \infty$ such that, for $\sigma \ge 0$, $t \in \mathbb{R}$,*

$$|\Psi_\sigma(t)| \le \frac{C}{1+t^2}\Big(\log(2+|t|)\Big)^9. \tag{19.88}$$

We see that, for $\sigma \geq 0$, $|\Psi_\sigma|$ is uniformly bounded by a function that is integrable on $\mathbb{R}$. Now the identity (19.83) takes the form

$$\Psi_\sigma(t) = \int_{-\infty}^{\infty} \Phi(e^y) e^{-\sigma y} e^{-ity}\, dy, \quad \forall\, \sigma > 0, \tag{19.89}$$

the integrand being supported on $\mathbb{R}^+$. Thanks to the integrability, for $\sigma > 0$ we can apply the Fourier inversion formula, to write

$$\Phi(e^y) e^{-\sigma y} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi_\sigma(t) e^{ity}\, dt. \tag{19.90}$$

Then (19.88) allows us to pass to the limit $\sigma \searrow 0$, obtaining

$$\Phi(e^y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi_0(t) e^{ity}\, dt. \tag{19.91}$$

Since Ψ_0 is integrable, the Riemann-Lebesgue lemma, from §14, implies that $\Phi(e^y) \to 0$ as $y \to \infty$, or equivalently

$$\Phi(x) \longrightarrow 0, \quad \text{as } \ x \to +\infty. \tag{19.92}$$

In fact, since Ψ_0 is also C^∞ on $\mathbb{R}$, this implication is elementary. Take $f_N \in C^1(\mathbb{R})$ such that

$$0 \leq f_N \leq 1, \quad f_N(t) = 1 \text{ for } |t| \leq N, \ \ 0 \text{ for } |t| \geq N+1. \tag{19.93}$$

Then the Fourier integral on the right side of (19.91) is equal to

$$\int_{-\infty}^{\infty} f_N(t) \Psi_0(t) e^{ity}\, dt + \int_{-\infty}^{\infty} (1 - f_N(t)) \Psi_0(t) e^{ity}\, dt. \tag{19.94}$$

The first integral is $\leq C_N/(1+|y|)$, by (14.24), and the second is

$$\leq \int\limits_{|t| \geq N} |\Psi_0(t)|\, dt = \varepsilon_N \to 0, \quad \text{as } \ N \to \infty, \tag{19.95}$$

so

$$\limsup_{|y| \to \infty} \Phi(e^y) \leq \frac{\varepsilon_N}{2\pi}, \quad \forall\, N, \tag{19.96}$$

and (19.92) follows.

The result (19.92) establishes the following.

Proposition 19.13. *The function ψ_1, given by (19.78), satisfies*

$$\lim_{x\to\infty} \frac{\psi_1(x)}{x^2} = \frac{1}{2}. \tag{19.97}$$

The next result allows us to pass from (19.97) to (19.69). It is a Tauberian theorem, and its proof is given in Appendix R; see Proposition R.11.

Proposition 19.14. *Let $\psi : [0,\infty) \to [0,\infty)$ be an increasing function, and set $\psi_1(x) = \int_0^x \psi(y)\,dy$. Take $B \in (0,\infty)$, $\alpha \in [1,\infty)$. Then*

$$\begin{aligned} &\psi_1(x) \sim Bx^\alpha, \quad \text{as} \ \ x \to \infty \\ &\Longrightarrow \psi(x) \sim \alpha B x^{\alpha-1}, \quad \text{as} \ \ x \to \infty. \end{aligned} \tag{19.98}$$

Applying this result to (19.97), with $\alpha = 2,\ B = 1/2$, yields (19.69) and completes the proof of the prime number theorem.

Exercises

1. Show that, for $\operatorname{Re} s > 1$,

$$\sum_{k=1}^{\infty} k^{-s} = \sum_{k=1}^{\infty} \int_k^{k+1} (k^{-s} - x^{-s})\, dx + \int_1^{\infty} x^{-s}\, dx,$$

which one can write as

$$\zeta(s) = Z(s) + \frac{1}{s-1}.$$

Show that the infinite series for $Z(s)$ is absolutely convergent for $\operatorname{Re} s > 0$, and hence $Z(s)$ is holomorphic on $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$. Use this to produce another proof (independent of, and admittedly weaker than, Proposition 19.2) that

$$\zeta(s) - \frac{1}{s-1}$$

is holomorphic on $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$.
Hint. Use the identity $k^{-s} - x^{-s} = s \int_k^x y^{-s-1}\, dy$ to show that

$$|k^{-s} - x^{-s}| \le |s| k^{-\sigma-1},$$

if $k \le x \le k+1,\ s = \sigma + it,\ \sigma > 0,\ t \in \mathbb{R}$.
Note. $Z(1) = \gamma$, Euler's constant.
Remark. Also relate the result of this exercise to the case $N = 1$ of Proposition 19.6.

2. Use the functional equation (19.17) together with (18.6) and the Legendre duplication formula (18.47) to show that

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \Bigl(\cos \frac{\pi s}{2}\Bigr) \Gamma(s) \zeta(s).$$

3. Sum the identity

$$\Gamma(s) n^{-s} = \int_0^{\infty} e^{-nt} t^{s-1}\, dt$$

over $n \in \mathbb{Z}^+$ to show that

$$\Gamma(s)\zeta(s) = \int_0^{\infty} \frac{t^{s-1}}{e^t - 1}\, dt = \int_0^1 \frac{t^{s-1}}{e^t - 1}\, dt + \int_1^{\infty} \frac{t^{s-1}}{e^t - 1}\, dt = A(s) + B(s).$$

Show that $B(s)$ continues as an entire function. Use a Laurent series

$$\frac{1}{e^t - 1} = \frac{1}{t} + a_0 + a_1 t + a_2 t^2 + \cdots$$

to show that

$$\int_0^1 \frac{t^{s-1}}{e^t - 1}\, dt = \frac{1}{s-1} + \frac{a_0}{s} + \frac{a_1}{s+1} + \cdots$$

provides a meromorphic continuation of $A(s)$, with poles at $\{1, 0, -1, \dots\}$. Use this to give a second proof that $\zeta(s)$ has a meromorphic continuation with one simple pole, at $s = 1$.

4. Show that, for Re $s > 1$ (or, in cases (b) and (c), for Re $s > 2$), the following identities hold:

(a)
$$\zeta(s)^2 = \sum_{n=1}^\infty \frac{d(n)}{n^s},$$

(b)
$$\zeta(s)\zeta(s-1) = \sum_{n=1}^\infty \frac{\sigma(n)}{n^s},$$

(c)
$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^\infty \frac{\varphi(n)}{n^s},$$

(d)
$$\frac{1}{\zeta(s)} = \sum_{n=1}^\infty \frac{\mu(n)}{n^s},$$

(e)
$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^\infty \frac{\Lambda(n)}{n^s},$$

where

$$\begin{aligned} d(n) &= \#\text{ divisors of } \ n, \\ \sigma(n) &= \text{sum of divisors of } \ n, \\ \varphi(n) &= \#\text{ positive integers } \ \le n,\ \text{relatively prime to } n, \\ \mu(n) &= (-1)^{\#\text{ prime factors}},\ \text{if } n \text{ is square-free},\ 0 \text{ otherwise}, \\ \Lambda(n) &= \log p \text{ if } n = p^m \text{ for some prime } p,\ 0 \text{ otherwise}. \end{aligned}$$

Relate cse (e) to (19.63).

5. Supplement (19.8) with the result

(19.A)
$$\sum_{j=1}^\infty \frac{1}{p_j^s} \sim \log \frac{1}{s-1}, \quad \text{as } \ s \searrow 1.$$

Hint. Derive the following variant of the result of Exercise 2 in §9:

$$0 \le x \le \delta \Longrightarrow x \le \log \frac{1}{1-x} \le (1 + \varepsilon(\delta))x,$$

for $\delta \in (0, 1/2]$, where $\varepsilon(\delta) \to 0$ as $\delta \to 0$. Use this to show that, if $0 \le a_k \le \delta$ for $k \ge M$, then

$$\sum_{k \ge M} a_k \le \log \prod_{k \ge M} (1 - a_k)^{-1} \le (1 + \varepsilon(\delta)) \sum_{k \ge M} a_k.$$

Apply this to $a_k = p_k^{-s},\ s > 1$.

6. Show that the prime number theorem, in the form (19.68), is equivalent to

$$p_j \sim j \log j, \quad \text{as } j \to \infty. \tag{19.B}$$

7. Show that

$$\int_e^\infty \frac{dx}{x^s (\log x)^s} \sim \log \frac{1}{s-1}, \quad \text{as } s \searrow 0. \tag{19.C}$$

Hint. Using first $y = \log x,\ s = 1 + u$, and then $t = uy$, write the left side of (19.C) as

$$\begin{aligned}
&\int_1^\infty y^{-1-u} e^{-uy}\, dy = u^u \int_u^\infty t^{-1-u} e^{-t}\, dt \\
&\sim \int_u^\infty t^{-1-u} e^{-t}\, dt \sim \int_u^1 t^{-1-u}\, dt \\
&= \frac{e^{-u \log u} - 1}{u} \\
&\sim \log \frac{1}{u}, \quad \text{as } u \searrow 0.
\end{aligned}$$

8. See if you can modify the analysis behind (19.C) to show that (19.B) $\Rightarrow$ (19.A).

9. Set

$$\xi(s) = \frac{s(s-1)}{2} \Gamma\Big(\frac{s}{2}\Big) \pi^{-s/2} \zeta(s).$$

Using (19.16)–(19.17), show that $\xi(s)$ is an entire function, satisfying

$$\xi(1-s) = \xi(s),$$

and that the zeros of $\xi(z)$ coincide with the nontrivial zeros of $\zeta(s)$, i.e., by Proposition 19.4, those zeros in the critical strip (19.29).

J. Euler's constant

Here we say more about Euler's constant, introduced in (18.17), in the course of producing the Euler product expansion for $1/\Gamma(z)$. The definition

$$\gamma = \lim_{n\to\infty} \Bigl(\sum_{k=1}^{n} \frac{1}{k} - \log(n+1) \Bigr) \tag{J.1}$$

of Euler's constant involves a very slowly convergent sequence. In order to produce a numerical approximation of γ, it is convenient to use other formulas, involving the Gamma function $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$. Note that

$$\Gamma'(z) = \int_0^\infty (\log t) e^{-t} t^{z-1}\, dt. \tag{J.2}$$

Meanwhile the Euler product formula $1/\Gamma(z) = z e^{\gamma z} \prod_{n=1}^\infty (1 + z/n) e^{-z/n}$ implies

$$\Gamma'(1) = -\gamma. \tag{J.3}$$

Thus we have the integral formula

$$\gamma = -\int_0^\infty (\log t) e^{-t}\, dt. \tag{J.4}$$

To evaluate this integral numerically it is convenient to split it into two pieces:

$$\begin{aligned} \gamma &= -\int_0^1 (\log t) e^{-t}\, dt - \int_1^\infty (\log t) e^{-t}\, dt \\ &= \gamma_a - \gamma_b. \end{aligned} \tag{J.5}$$

We can apply integration by parts to both the integrals in (5), using $e^{-t} = -(d/dt)(e^{-t} - 1)$ on the first and $e^{-t} = -(d/dt)e^{-t}$ on the second, to obtain

$$\gamma_a = \int_0^1 \frac{1 - e^{-t}}{t}\, dt, \quad \gamma_b = \int_1^\infty \frac{e^{-t}}{t}\, dt. \tag{J.6}$$

Using the power series for e^{-t} and integrating term by term produces a rapidly convergent series for γ_a:

$$\gamma_a = \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k \cdot k!}. \tag{J.7}$$

Before producing infinite series representations for γ_b, we note that the change of variable $t = s^m$ gives

$$\gamma_b = m \int_1^\infty \frac{e^{-s^m}}{s}\, ds, \tag{J.8}$$

which is very well approximated by the integral over $s \in [1, 10)$ if $m = 2$, for example.

To produce infinite series for γ_b, we can break up $[1, \infty)$ into intervals $[k, k+1)$ and take $t = s + k$, to write

$$\gamma_b = \sum_{k=1}^\infty \frac{e^{-k}}{k}\, \beta_k, \quad \beta_k = \int_0^1 \frac{e^{-t}}{1 + t/k}\, dt. \tag{J.9}$$

Note that $0 < \beta_k < 1 - 1/e$ for all k. For $k \ge 2$ we can write

$$\beta_k = \sum_{j=0}^\infty \Bigl(-\frac{1}{k}\Bigr)^j \alpha_j, \quad \alpha_j = \int_0^1 t^j e^{-t}\, dt. \tag{J.10}$$

One convenient way to integrate $t^j e^{-t}$ is the following. Write

$$E_j(t) = \sum_{\ell=0}^j \frac{t^\ell}{\ell!}. \tag{J.11}$$

Then

$$E_j'(t) = E_{j-1}(t), \tag{J.12}$$

hence

$$\frac{d}{dt}\bigl(E_j(t)e^{-t}\bigr) = \bigl(E_{j-1}(t) - E_j(t)\bigr)e^{-t} = -\frac{t^j}{j!}\, e^{-t}, \tag{J.13}$$

so

$$\int t^j e^{-t}\, dt = -j! E_j(t) e^{-t} + C. \tag{J.14}$$

In particular,

$$\begin{aligned} \alpha_j = \int_0^1 t^j e^{-t}\, dt &= j!\Bigl(1 - \frac{1}{e} \sum_{\ell=0}^j \frac{1}{\ell!}\Bigr) \\ &= \frac{j!}{e} \sum_{\ell=j+1}^\infty \frac{1}{\ell!} \\ &= \frac{1}{e}\Bigl(\frac{1}{j+1} + \frac{1}{(j+1)(j+2)} + \cdots\Bigr). \end{aligned} \tag{J.15}$$

To evaluate β_1 as an infinite series, it is convenient to write

$$
\begin{aligned}
e^{-1}\beta_1 &= \int_1^2 \frac{e^{-t}}{t}\,dt \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \int_1^2 t^{j-1}\,dt \\
&= \log 2 + \sum_{j=1}^{\infty} \frac{(-1)^j}{j\cdot j!}(2^j-1).
\end{aligned} \tag{J.16}
$$

To summarize, we have $\gamma = \gamma_a - \gamma_b$, with γ_a given by the convenient series (J.7) and

$$
\gamma_b = \sum_{k=2}^{\infty}\sum_{j=0}^{\infty} \frac{e^{-k}}{k}\Big(-\frac{1}{k}\Big)^j \alpha_j + \log 2 + \sum_{j=1}^{\infty} \frac{(-1)^j}{j\cdot j!}(2^j-1), \tag{J.17}
$$

with α_j given by (J.15). We can reverse the order of summation of the double series and write

$$
\gamma_b = \sum_{j=0}^{\infty}(-1)^j \zeta_j \alpha_j + \log 2 + \sum_{j=1}^{\infty} \frac{(-1)^j}{j\cdot j!}(2^j-1). \tag{J.18}
$$

with

$$
\zeta_j = \sum_{k=2}^{\infty} \frac{e^{-k}}{k^{j+1}}. \tag{J.19}
$$

Note that

$$
0 < \zeta_j < 2^{-(j+1)} \sum_{k=2}^{\infty} e^{-k} < 2^{-(j+3)}, \tag{J.20}
$$

while (J.15) readily yields $0 < \alpha_j < 1/ej$. So one can expect 15 digits of accuracy by summing the first series in (J.18) over $0 \le j \le 50$ and the second series over $0 \le j \le 32$, assuming the ingredients α_j and ζ_j are evaluated sufficiently accurately. It suffices to sum (J.19) over $2 \le k \le 40 - 2j/3$ to evaluate ζ_j to sufficient accuracy.

Note that the quantities α_j do not have to be evaluated independently. Say you are summing the first series in (J.18) over $0 \le j \le 50$. First evaluate α_{50} using 20 terms in (J.15), and then evaluate inductively $\alpha_{49}, \dots, \alpha_0$ using the identity

$$
\alpha_{j-1} = \frac{1}{je} + \frac{\alpha_j}{j}, \tag{J.21}
$$

equivalent to $\alpha_j = j\alpha_{j-1} - 1/e$, which follows by integration by parts of $\int_0^1 t^j e^{-t}\,dt$.

If we sum the series (J.7) for γ_a over $1 \le k \le 20$ and either sum the series (J.18) as described above or have Mathematica numerically integrate (J.8), with $m = 2$, to high precision, we obtain

$$\gamma \approx 0.577215664901533, \tag{J.22}$$

which is accurate to 15 digits.

We give another series for γ. This one is more slowly convergent than the series in (J.7) and (J.18), but it makes clear why γ exceeds $1/2$ by a small amount, and it has other interesting aspects. We start with

$$\gamma = \sum_{n=1}^{\infty} \gamma_n, \quad \gamma_n = \frac{1}{n} - \int_n^{n+1} \frac{dx}{x} = \frac{1}{n} - \log\Bigl(1 + \frac{1}{n}\Bigr). \tag{J.23}$$

Thus γ_n is the area of the region

$$\Omega_n = \Bigl\{(x, y) : n \le x \le n+1,\ \frac{1}{x} \le y \le \frac{1}{n}\Bigr\}. \tag{J.24}$$

This region contains the triangle T_n with vertices $(n, 1/n)$, $(n{+}1, 1/n)$, and $(n{+}1, 1/(n{+}1))$. The region $\Omega_n \setminus T_n$ is a little sliver. Note that

$$\text{Area } T_n = \delta_n = \frac{1}{2}\Bigl(\frac{1}{n} - \frac{1}{n+1}\Bigr), \tag{J.25}$$

and hence

$$\sum_{n=1}^{\infty} \delta_n = \frac{1}{2}. \tag{J.26}$$

Thus

$$\gamma - \frac{1}{2} = (\gamma_1 - \delta_1) + (\gamma_2 - \delta_2) + (\gamma_3 - \delta_3) + \cdots . \tag{J.27}$$

Now

$$\gamma_1 - \delta_1 = \frac{3}{4} - \log 2, \tag{J.28}$$

while, for $n \ge 2$, we have power series expansions

$$\begin{aligned} \gamma_n &= \frac{1}{2n^2} - \frac{1}{3n^3} + \frac{1}{4n^4} - \cdots \\ \delta_n &= \frac{1}{2n^2} - \frac{1}{2n^3} + \frac{1}{2n^4} - \cdots , \end{aligned} \tag{J.29}$$

the first expansion by $\log(1+z) = z - z^2/2 + z^3/3 - \cdots$, and the second by

$$\delta_n = \frac{1}{2n(n+1)} = \frac{1}{2n^2}\frac{1}{1+\frac{1}{n}}, \tag{J.30}$$

and the expansion $(1+z)^{-1} = 1 - z + z^2 - \cdots$. Hence we have

$$\gamma - \frac{1}{2} = (\gamma_1 - \delta_1) + \Bigl(\frac{1}{2} - \frac{1}{3}\Bigr)\sum_{n\ge 2}\frac{1}{n^3} - \Bigl(\frac{1}{2} - \frac{1}{4}\Bigr)\sum_{n\ge 2}\frac{1}{n^4} + \cdots, \tag{J.31}$$

or, with

$$\zeta(k) = \sum_{n\ge 1}\frac{1}{n^k}, \tag{J.32}$$

we have

$$\gamma - \frac{1}{2} = \Bigl(\frac{3}{4} - \log 2\Bigr) + \Bigl(\frac{1}{2} - \frac{1}{3}\Bigr)[\zeta(3) - 1] - \Bigl(\frac{1}{2} - \frac{1}{4}\Bigr)[\zeta(4) - 1] + \cdots, \tag{J.33}$$

an alternating series from the third term on. We note that

$$\begin{aligned} \frac{3}{4} - \log 2 &\approx 0.0568528, \\ \frac{1}{6}[\zeta(3) - 1] &\approx 0.0336762, \\ \frac{1}{4}[\zeta(4) - 1] &\approx 0.0205808, \\ \frac{3}{10}[\zeta(5) - 1] &\approx 0.0110783. \end{aligned} \tag{J.34}$$

The estimate

$$\sum_{n\ge 2}\frac{1}{n^k} < 2^{-k} + \int_2^\infty x^{-k}\,dx \tag{J.35}$$

implies

$$0 < \Bigl(\frac{1}{2} - \frac{1}{k}\Bigr)[\zeta(k) - 1] < 2^{-k-1}, \tag{J.36}$$

so the series (J.33) is geometrically convergent. If k is even, $\zeta(k)$ is a known rational multiple of π^k. However, for odd k, the values of $\zeta(k)$ are more mysterious. Note that to get $\zeta(3)$ to 16 digits by summing (J.32) one needs to sum over $1 \le n \le 10^8$. To be sure, one can do this sum on a personal computer in a few seconds. Nevertheless, this is a vastly slower approach to evaluating γ than summing (J.7) and (J.18) over the ranges discussed above.

Returning to (J.3), we complement this with an evaluation of $\Gamma'(\ell)$ for general $\ell \in \mathbb{N}$. In fact, for use in §35, we derive a formula for $\beta'(\ell)$, for $\ell \in \mathbb{Z}$, where

$$\beta(z) = \frac{1}{\Gamma(z)}, \tag{J.37}$$

so

$$\beta'(z) = -\frac{\Gamma'(z)}{\Gamma(z)^2}, \tag{J.38}$$

for $\ell \notin \{0, -1, -2, \dots\}$, though β itself is holomorphic on all $\mathbb{C}$, and we can also evaluate β' at these points. By (J.3) and (J.38),

$$\beta'(1) = \gamma. \tag{J.39}$$

To evaluate $\beta'(\ell)$ for other $\ell \in \mathbb{Z}$, we can use

$$\beta(z) = z\beta(z+1), \tag{J.40}$$

which implies

$$\beta'(z) = \beta(z+1) + z\beta'(z+1), \tag{J.41}$$

hence

$$\beta'(0) = \beta(1) = 1. \tag{J.42}$$

For $\ell = -m$, a negative integer, we have $\beta(-m+1) = 0$, hence

$$\beta'(-m) = -m\beta'(-m+1), \tag{J.43}$$

hence $\beta'(-1) = -\beta'(0) = -1$, $\beta'(-2) = -2\beta'(-1) = 2$, and, inductively,

$$\beta'(-m) = (-1)^m m!, \tag{J.44}$$

when m is a positive integer. To evaluate $\beta'(\ell)$ for an integer $\ell \geq 2$, we can turn (J.41) around:

$$\beta'(z+1) = \frac{\beta'(z) - \beta(z+1)}{z}. \tag{J.45}$$

We have

$$\beta'(2) = \beta'(1) - \beta(2) = \gamma - 1, \tag{J.46}$$

and generally

$$\beta'(\ell+1) = \frac{\beta'(\ell)}{\ell} - \frac{1}{\ell \cdot \ell!}. \tag{J.47}$$

S. Hadamard's factorization theorem

As stated in §9, the Hadamard factorization theorem provides an incisive analysis of entire holomorphic functions f that satisfy an estimate of the form

$$|f(z)| \le C e^{B|z|^\rho}, \quad \forall\, z \in \mathbb{C}, \tag{S.1}$$

with $B, C, \rho < \infty$. Such a function is said to be of finite order. The infimum of the numbers ρ such that (S.1) holds is called the order of f. Here is Hadamard's theorem:

Theorem S.1. *Assume $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and satisfies (S.1). Take $p \in \mathbb{Z}^+$ such that $p \le \rho < p + 1$. Let (z_k) be the zeros in $\mathbb{C} \setminus 0$ of f, repeated according to multiplicity, and assume f vanishes to order m at 0. Then*

$$\sum_k |z_k|^{-\sigma} < \infty, \quad \forall\, \sigma > \rho, \tag{S.2}$$

and there exists a polynomial $q(z)$ of degree $\le p$ such that

$$f(z) = z^m e^{q(z)} \prod_k E\Bigl(\frac{z}{z_k}, p\Bigr). \tag{S.3}$$

We recall from §9 that

$$E(z, p) = (1 - z) \exp\Bigl(\sum_{k=1}^p \frac{z^k}{k}\Bigr). \tag{S.4}$$

According to Proposition 9.11, if (S.2) holds, with $p \le \sigma < p + 1$, then

$$g(z) = \prod_k E\Bigl(\frac{z}{z_k}, p\Bigr) \tag{S.5}$$

is holomorphic on $\mathbb{C}$ and satisfies (S.1) (with ρ replaced by σ). Also, $z^m e^{q(z)}$ clearly satisfies (S.1) if q is a polynomial of degree $\le p \le \rho$.

In case f has no zeros, Theorem S.1 boils down to the following.

Proposition S.2. *Suppose the entire function f satisfies (S.1) (with $p \le \rho < p + 1$) and has no zeros. Then there is a polynomial $q(z)$ of degree $\le p$ such that*

$$f(z) = e^{q(z)}. \tag{S.6}$$

Proof. Given that f has no zeros, Proposition 10.2 implies the existence of a holomorphic function q on $\mathbb{C}$ such that (S.6) holds. Given (S.1),

$$|f(z)| = e^{\operatorname{Re} q(z)} \le C e^{B|z|^{\rho}}, \tag{S.7}$$

so

$$\operatorname{Re} q(z) \le B_0 + B|z|^{\rho}, \quad \forall\, z \in \mathbb{C}. \tag{S.8}$$

Then, by Proposition 29.5, the harmonic function $u(z) = \operatorname{Re} q(z)$ must be a polynomial in x and y, of degree $\le p$. But then its harmonic conjugate is also a polynomial in x and y of degree $\le p$, so $|q(z)| \le A_0 + A_1|z|^p$. This forces $q(z)$ to be a polynomial in z of degree $\le p$ (cf. Exercise 9 of §6).

We next tackle (S.2). For this we use the following result, known as Jensen's formula.

Proposition S.3. *Take $r \in (0,\infty)$, $R > r$, and assume f is holomorphic on $D_R(0)$. Assume f has no zeros on $\partial D_r(0)$, that $f(0) \ne 0$, and that z_k, $1 \le k \le n$, are the zeros of f in $D_r(0)$, repeated according to multiplicity. Then*

$$\log\Big(r^n |f(0)| \prod_{k=1}^{n} |z_k|^{-1}\Big) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta. \tag{S.9}$$

Proof. First we treat the case $r = 1$. We bring in the linear fractional transformations

$$\varphi_k(z) = \frac{z - z_k}{1 - \overline{z}_k z}, \tag{S.10}$$

which preserve $D_1(0)$ and $\partial D_1(0)$, and map z_k to 0 and 0 to $-z_k$, and form

$$g(z) = f(z)\varphi_1(z)^{-1} \cdots \varphi_k(z)^{-1}, \tag{S.11}$$

which is holomorphic on $D_R(0)$ $(R > 1)$ and nowhere vanishing on $\overline{D_1(0)}$. (Note also that $|g| = |f|$ on $\partial D_1(0)$.) We see that $\log g(z)$ is holomorphic on a neighborhood of $\overline{D_1(0)}$, so $\log|g(z)|$ is harmonic on this set. Thus the mean value theorem for harmonic functions yields

$$\log\Big(|f(0)| \prod_{k=1}^{n} |z_k|^{-1}\Big) = \log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(ze^{i\theta})|\, d\theta. \tag{S.12}$$

This takes care of the case $r = 1$.

For general $r \in (0,\infty)$, we can apply the argument given above to $f_r(z) = f(rz)$, whose zeros in $D_1(0)$ are z_k/r, and obtain (S.9).

Using Jensen's formula, we obtain the following estimate on the number of zeros of an entire function.

Proposition S.4. *Assume f is an entire function and $f(0) = 1$. For $r \in (0, \infty)$, let*

$$\begin{aligned} n(r) &= \#\ \textit{zeros of } f \textit{ in } \ D_r(0), \\ M(r) &= \sup_{|z|=r} |f(z)|. \end{aligned} \tag{S.13}$$

Then

$$n(r) \le \frac{\log M(2r)}{\log 2}. \tag{S.14}$$

Proof. Set $n = n(r)$, $m = n(2r)$. Let z_k, $1 \le k \le m$ denote the zeros of f in $D_{2r}(0)$, ordered so $|z_k| \le |z_{k+1}|$. Then (S.9), with r replaced by $2r$, yields

$$\log\Bigl| f(0) \frac{2r}{z_1} \cdots \frac{2r}{z_m} \Bigr| \le \log M(2r). \tag{S.15}$$

Since $2r/|z_k| > 2$ for $k \le n$ and $2r/|z_k| > 1$ for $n < k \le m$, we get

$$\log |f(0) 2^n| \le \log M(2r). \tag{S.16}$$

Since $f(0) = 1$, this says $n \log 2 \le \log M(2r)$, giving (S.14).

We are now prepared to establish the following result, which implies (S.2).

Proposition S.5. *Let f be an entire function, and assume f satisfies (S.1). Let z_k, $k \ge 1$, denote its zeros in $\mathbb{C} \setminus 0$, ordered so that $|z_k| \le |z_{k+1}|$. Then there exists $C > 0$ such that*

$$|z_k|^\rho \ge Ck, \quad \forall\, k \ge 1. \tag{S.17}$$

Proof. It suffices to prove the result in the case $f(0) = 1$. Set $r_k = |z_k|$. By Proposition S.4,

$$k \le \frac{\log M(2r_k)}{\log 2}, \tag{S.18}$$

where $M(2r_k) = \sup_{|z|=2r_k} |f(z)|$. Now (S.1) implies $M(2r_k) \le Ce^{B(2r_k)^\rho}$, so $\log M(2r_k) \le c + B(2r_k)^\rho$, and (S.18) yields

$$k \le c' + B'(2r_k)^\rho, \tag{S.19}$$

which readily implies (S.17).

We have already noted that the estimate (S.2) implies that the product $g(z)$ in (S.5) is an entire function satisfying an estimate similar to (S.1) (with ρ replaced by σ). It follows that $f(z)/g(z)$ is an entire function with no zeros in $\mathbb{C} \setminus 0$. To proceed with the proof of Theorem 8.1, we need an estimate on this quotient, and for this the following result is useful.

Lemma S.6. *Let $p \in \mathbb{Z}^+$ and $\rho \in [p, p+1]$. Then*

$$|E(z,p)^{-1}| \le e^{p|z|^\rho}, \quad for \quad |z| \ge 2, \tag{S.20}$$

and

$$|E(z,p)^{-1}| \le e^{2|z|^\rho}, \quad for \quad |z| \le \frac{1}{2}. \tag{S.21}$$

Proof. For $|z| \ge 2$, we have $|1-z| \ge 1$ and $|z^k/k| \le |z|^\rho$ for $k \le p$. hence

$$|E(z,p)^{-1}| = |1-z|^{-1} \exp\Bigl(-z - \frac{z^2}{2} - \cdots - \frac{z^p}{p}\Bigr) \le e^{p|z|^\rho}, \tag{S.22}$$

giving (S.20).

For $|z| \le 1/2$, we use

$$E(z,p) = \exp\Bigl(-\sum_{k \ge p+1} \frac{z^k}{k}\Bigr) \tag{S.23}$$

to obtain

$$\begin{aligned} |E(z,p)^{-1}| &\le \exp\Bigl(|z|^{p+1} \sum_{k\ge 0} |z|^k\Bigr) \\ &\le \exp(2|z|^{p+1}) \\ &\le e^{2|z|^\rho}, \end{aligned} \tag{S.24}$$

and we have (S.21).

To complete the proof of Theorem S.1, we proceed to estimate

$$H(z) = \frac{f(z)}{z^m g(z)}, \tag{S.25}$$

when f satisfies the hypotheses of Theorem S.1 and $g(z)$ is given by (S.5). We have already seen that $H(z)$ is an entire function with no zeros. Recalling that f satisfies (S.1) with $\rho \in [p, p+1)$, pick $\sigma \in (\rho, p+1)$. Pick $r \in (1, \infty)$, and write

$$H(z) = H_1(z) H_2(z), \tag{S.26}$$

where

$$\begin{aligned} H_1(z) &= f(z) z^{-m} \prod_{|z_k| \le 2r} E\Bigl(\frac{z}{z_k}, p\Bigr)^{-1}, \\ H_2(z) &= \prod_{|z_k| > 2r} E\Bigl(\frac{z}{z_k}, p\Bigr)^{-1}. \end{aligned} \tag{S.27}$$

By (S.21),

$$\text{(S.28)}\qquad \begin{aligned} |z| = r \Rightarrow |H_2(z)| &\le \exp\Bigl(2 \sum_{|z_k|>2r} \Bigl|\frac{r}{z_k}\Bigr|^\sigma\Bigr) \\ &\le e^{B_1 r^\sigma}, \end{aligned}$$

thanks to (S.2). Similarly, by (S.20),

$$\text{(S.29)}\qquad \begin{aligned} |z| = 4r \Rightarrow \Bigl|\prod_{|z_k|\le 2r} E\Bigl(\frac{z}{z_k}, p\Bigr)^{-1}\Bigr| &\le \exp\Bigl(p \sum_{|z_k|\le 2r} \Bigl|\frac{4r}{z_k}\Bigr|^\sigma\Bigr) \\ &\le e^{B_2 r^\sigma}. \end{aligned}$$

Since H_1 is an entire function,

$$\text{(S.30)}\qquad \begin{aligned} \max_{|z|=r} |H_1(z)| &\le \max_{|z|=4r} |H_1(z)| \\ &\le (4r)^{-m} \max_{|z|=4r} |f(z)| \cdot e^{B_2 r^\sigma} \\ &\le C e^{B_3 r^\sigma}. \end{aligned}$$

Consequently $H(z)$ in (S.25) satisfies

$$\text{(S.31)}\qquad |H(z)| \le C e^{B|z|^\sigma}.$$

Since $H(z)$ is entire and nowhere vanishing, and $p < \sigma < p+1$, Proposition S.2 implies

$$\text{(S.32)}\qquad H(z) = e^{q(z)},$$

where $q(z)$ is a polynomial of degree $\le p$. Now (S.25) gives

$$\text{(S.33)}\qquad f(z) = z^m e^{q(z)} g(z),$$

and completes the proof of Theorem S.1.

Let us apply Hadamard's theorem to the function $f(z) = \sin \pi z$. In this case, (S.1) holds with $\rho = 1$, and the zeros of f are the integers, all simple. Hence (S.3) gives

$$\text{(S.34)}\qquad \begin{aligned} \sin \pi z &= z e^{q(z)} \prod_{k=1}^\infty \Bigl(1 - \frac{z}{k}\Bigr) e^{z/k} \cdot \prod_{k=1}^\infty \Bigl(1 + \frac{z}{k}\Bigr) e^{-z/k} \\ &= z e^{q(z)} \prod_{k=1}^\infty \Bigl(1 - \frac{z^2}{k^2}\Bigr), \end{aligned}$$

with $q(z) = az + b$. Since $\sin \pi z$ is an odd function, $e^{q(z)}$ must be even, so $a = 0$ and $e^{q(z)} = e^b$. Now

$$\text{(S.35)}\qquad \lim_{z\to 0} \frac{\sin \pi z}{z} = \pi \Longrightarrow e^b = \pi,$$

so we obtain

$$\sin \pi z = \pi z \prod_{k=1}^{\infty} \Bigl(1 - \frac{z^2}{k^2}\Bigr), \tag{S.36}$$

an identity obtained by other means in (9.46) and (18.21).

Hadamard applied his factorization theorem to the study of the Riemann zeta function. Recall from §19 that $\zeta(z)$, defined initially by

$$\zeta(z) = \sum_{k=1}^{\infty} k^{-z}, \quad \text{for} \quad \operatorname{Re} z > 1, \tag{S.37}$$

extends to a meromorphic function on $\mathbb{C}$, with one simple pole, at $z = 1$. It has "trivial" zeros at $z \in \{-2, -4, -6, \dots\}$, and (cf Proposition 19 4) all of its other zeros are contained in the "critical strip"

$$\Omega = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}. \tag{S.38}$$

Furthermore, $\Gamma(z/2)\pi^{-z/2}\zeta(z)$ satisfies the functional equation (19.17). Another way to state these results is the following:

Proposition S.7. *The function ξ, defined by*

$$\xi(z) = \frac{z(z-1)}{2}\,\Gamma\Bigl(\frac{z}{2}\Bigr)\pi^{-z/2}\zeta(z), \tag{S.39}$$

is an entire holomorphic function on $\mathbb{C}$. Its zeros coincide precisely with the seros of ζ that lie in the critical strip (S.38). Furthermore,

$$\xi(1-z) = \xi(z). \tag{S.40}$$

The following estimate on $\xi(z)$ allows us to apply Theorem S.1.

Proposition S.8. *There exists $R < \infty$ such that for all $|z| \ge R$,*

$$\Bigl|\xi\Bigl(z + \frac{1}{2}\Bigr)\Bigr| \le e^{|z| \log |z|}. \tag{S.41}$$

A proof of this estimate can be found in §2.3 of [Ed]. We mention that an ingredient in the proof is that

$$\xi(z) = \sum_{k=0}^{\infty} a_k \Bigl(z - \frac{1}{2}\Bigr)^{2k}, \quad \text{and each} \quad a_k > 0. \tag{S.42}$$

Thus (S.1) holds for all $\rho > 1$, so Theorem S.1 applies with $p = 1$. Since $\xi(0) \neq 0$, we obtain from (S.3) that

$$\xi(z) = e^{az+b} \prod_{\rho \in \mathcal{Z}} \Bigl(1 - \frac{z}{\rho}\Bigr) e^{z/\rho}, \tag{S.43}$$

where $\mathcal{Z}$ denotes the set of zeros of ξ (i.e., the zeros of ζ in Ω), repeated according to multiplicity (if any are not simple). Now (S.42) implies $\xi(x) > 0$ for $x \in \mathbb{R}$, and then (S.40) implies

$$\mathcal{Z} = \{\rho, 1 - \rho : \rho \in \mathcal{Z}^+\}, \quad \mathcal{Z}^+ = \{\rho \in \mathcal{Z} : \operatorname{Im} \rho > 0\}, \tag{S.44}$$

and we can write (S.43) as

$$\xi(z) = e^{\alpha z+b} \prod_{\rho \in \mathcal{Z}^+} \Bigl(1 - \frac{z}{\rho}\Bigr)\Bigl(1 - \frac{z}{1-\rho}\Bigr), \tag{S.45}$$

where

$$\alpha = a + \sum_{\rho \in \mathcal{Z}^+} \frac{1}{\rho(1-\rho)}. \tag{S.46}$$

Note that (S.2) implies

$$\sum_{\rho \in \mathcal{Z}} |\rho|^{-\sigma} < \infty, \quad \forall\, \sigma > 1, \tag{S.47}$$

so the sum over $\rho \in \mathcal{Z}^+$ in (S.46) is absolutely convergent. So is the infinite product in (S.45), since

$$\Bigl(1 - \frac{z}{\rho}\Bigr)\Bigl(1 - \frac{z}{1-\rho}\Bigr) = 1 - \frac{z(1-z)}{\rho(1-\rho)}. \tag{S.48}$$

Note that (S.48) is invariant under $z \mapsto 1 - z$, so

$$\xi(1-z) = e^{\alpha(1-z)+b} \prod_{\rho \in \mathcal{Z}^+} \Bigl(1 - \frac{z}{\rho}\Bigr)\Bigl(1 - \frac{z}{1-\rho}\Bigr), \tag{S.49}$$

which, by (S.40), is equal to $\xi(z)$. This implies $\alpha = 0$, so we have the following conclusion.

Proposition S.9. *The function $\xi(z)$ has the product expansion*

$$\xi(z) = \xi(0) \prod_{\rho \in \mathcal{Z}^+} \Bigl(1 - \frac{z}{\rho}\Bigr)\Bigl(1 - \frac{z}{1-\rho}\Bigr). \tag{S.50}$$

Hence

$$\zeta(z) = \frac{2\xi(0)}{z(z-1)} \frac{\pi^{z/2}}{\Gamma(z/2)} \prod_{\rho \in \mathcal{Z}^+} \Bigl(1 - \frac{z}{\rho}\Bigr)\Bigl(1 - \frac{z}{1-\rho}\Bigr). \tag{S.51}$$

Chapter 5. Conformal maps and geometrical aspects of complex function theory

In this chapter we turn to geometrical aspects of holomorphic function theory, starting with the characterization of a holomorphic function $f : \Omega \to \mathbb{C}$ as a conformal map, when $f'(z) \neq 0$. To say f is conformal at $z_0 \in \Omega$ is to say that if γ_1 and γ_2 are two smooth paths that intersect at z_0, then the images $\sigma_j = f(\gamma_j)$ intersect at $f(z_0)$ at the same angle as γ_0 and γ_1. We show that a smooth function on Ω is holomorphic if and only if it is conformal and preserves orientation, i.e., the 2×2 matrix $Df(z_0)$ has positive determinant.

The notion of conformal map is not confined to maps between planar domains. One can take 2D surfaces S and Σ in $\mathbb{R}^3$ and say a smooth map $f : S \to \Sigma$ is conformal at $z_0 \in S$ provided that when smooth curves γ_0 and γ_1 in S intersect at z_0, then their images $\sigma = f(\gamma_j)$ in Σ intersect at the same angle. A key example is the stereographic map

$$\mathcal{S} : S^2 \setminus \{e_3\} \longrightarrow \mathbb{R}^2 \approx \mathbb{C}, \tag{5.0.1}$$

where S^2 is the unit sphere in $\mathbb{R}^3$ and $e_3 = (0, 0, 1)$, given by

$$\mathcal{S}(x_1, x_2, x_3) = (1 - x_3)^{-1}(x_1, x_2), \tag{5.0.2}$$

which is seen to have this property. This map is complemented by

$$\mathcal{S}_- : S^2 \setminus \{-e_3\} \longrightarrow \mathbb{R}^2, \quad \mathcal{S}_-(x_1, x_2, x_3) = (1 + x_3)^{-1}(x_1, -x_2). \tag{5.0.3}$$

One has

$$\mathcal{S}_- \circ \mathcal{S}(z) = \frac{1}{z},$$

and, as explained in §22, this gives S^2 the structure of a Riemann surface. A related object is the Riemann sphere,

$$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}, \tag{5.0.4}$$

which is also given the structure of a Riemann surface. Given two Riemann surfaces, S and Σ, there is a natural notion of when a smooth map $f : S \to \Sigma$ is holomorphic. It is seen that the map $\mathcal{S}$ in (5.0.1) has a natural extension to

$$\widetilde{\mathcal{S}} : S^2 \longrightarrow \widehat{\mathbb{C}}, \tag{5.0.5}$$

equal to on $S^2 \setminus \{e_3\}$ and satisfying $\widetilde{\mathcal{S}}(e_3) = \infty$, and that $\widetilde{\mathcal{S}}$ is a holomorphic diffeomorphism.

An important class of conformal maps is the class of linear fractional transformations,

$$L_A(z) = \frac{az + b}{cz + d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{5.0.6}$$

given $A \in M(2,\mathbb{C})$, $\det A \neq 0$. This map has a pole at $z = -d/c$ if $c \neq 0$, but we can extend the map to

$$L_A : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}, \tag{5.0.7}$$

via

$$\begin{aligned} L_A\Big(-\frac{d}{c}\Big) &= \infty, \quad \text{if } \ c \neq 0, \\ L_A(\infty) &= \frac{a}{c}, \quad \text{if } \ c \neq 0, \\ &\quad\ \infty, \quad \text{if } \ c = 0. \end{aligned} \tag{5.0.8}$$

Then L_A is a holomorphic diffeomorphism of $\widehat{\mathbb{C}}$ to itself (i.e., a holomorphic automorphism), and all holomorphic automorphisms of $\widehat{\mathbb{C}}$ have this form. An important subset acts as a set of holomorphic automorphisms of the unit disk,

$$\begin{gathered} L_B : D_1(0) \longrightarrow D_1(0), \quad B \in SU(1,1), \ \text{ i.e.,} \\ B = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}, \quad |\alpha|^2 - |\beta|^2 = 1. \end{gathered} \tag{5.0.9}$$

This set of transformations acts transitively on $D_1(0)$, i.e., given $z_0, z_1 \in D_1(0)$, there exists $B \in SU(1,1)$ such that $L_B(z_0) = z_1$. It is very useful to understand the automorphisms of objects like $\widehat{\mathbb{C}}$ and $D_1(0)$.

The unit disk $D = D_1(0)$ is a special domain in a number of respects. One of its most special properties is that it is not so special, in the sense that one has the following result, known as the Riemann mapping theorem.

Theorem. If $\Omega \subset \mathbb{C}$ is a simply connected domain and $\Omega \neq \mathbb{C}$, then there exists a holomorphic diffeomorphism

$$F : \Omega \longrightarrow D. \tag{5.0.10}$$

This result is established in §23. The map F is constructed to maximize $g'(p)$ among all one-to-one holomorphic maps $g : \Omega \to D$ such that $g(p) = 0$ and $g'(p) > 0$. Here $p \in \Omega$ is chosen arbitrarily. An important ingredient in the proof that such a maximizer exists is the theory of *normal families*, introduced in §21. Results about transformations of the form (5.0.9) play a role in showing that this maximizer is a holomorphic diffeomorphism in (5.0.10).

For some domains Ω, one can take the Riemann mapping function (5.0.10) and apply Schwarz reflection to analytically continue F to a larger domain. For example, if Ω is a rectangle or an equilateral triangle, one can use this to extend F to a meromorphic function on all of $\mathbb{C}$, giving examples of elliptic functions, which will be studied in Chapter 6. In another very important example, studied in §26, Ω is a region in the unit disk $D = D_1(0)$, bounded by three arcs of circles, intersecting ∂D at right angles. Repeated application of Schwarz reflection extends F to the entire disk D, yielding

$$\Phi : D \longrightarrow \mathbb{C} \setminus \{0, 1\}, \tag{5.0.11}$$

which is seen to be a holomorphic covering map, a concept introduced in §25. As seen in §§27–28, this has some very important implications for complex function theory. One is the following, known as Montel's theorem.

Theorem. Given a connected domain $\Omega \subset \mathbb{C}$, the family of holomorphic maps $f : \Omega \to \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ is a normal family of maps from Ω to $\widehat{\mathbb{C}}$.

This in turn leads to Picard's theorem, which is a significant inmrovement on the Casorati-Weierstrass theorem, established in Chapter 2.

Theorem. If p and q are distinct and

$$f : D \setminus \{0\} \longrightarrow \mathbb{C} \setminus \{p, q\}, \tag{5.0.12}$$

is holomorphic, then the singularity at 0 is either a pole or a removable singularity.

Section 27 also has a sequence of exercises on iterates of a holomorphic map $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, regarding the beautiful theory of sets on which these iterates behave tamely or wildly, known respectively as Fatou sets and Julia sets.

In §29 we return to the subject of harmonic functions. We draw connections between conformal maps and the Dirichlet problem for harmonic functions. We indicate how one can use results on the Dirichlet problem to construct a holomorphic diffeomorphism from a domain that is homeomorphic to an annulus to an actual annulus.

Appendices D and E at the end of this chapter make contact with some basic concepts of differential geometry. The connection with conformal maps arises from the fact that if γ_0 and γ_1 are smooth curves on a surface S, with $\gamma_0(0) = \gamma_1(0) = p \in S$, then the angle θ at which they meet satisfies

$$\langle \gamma_0'(0), \gamma_1'(0) \rangle = \|\gamma_0'(0)\| \cdot \|\gamma_1'(0)\| \cos \theta. \tag{5.0.13}$$

Here $\langle\ ,\ \rangle$ is the inner product of vectors tangent to S at p, given by the standard dot product on $\mathbb{R}^3$ is S is a surface in $\mathbb{R}^3$. The assignment of an inner product to tangent vectors to a surface at each point defines a *metric tensor* on S. Appendix D studies how a smooth map $F : S \to \Sigma$ between surfaces pulls back a metric tensor on Σ to one on S, and how to use this to say when F is a conformal map.

Not all metric tensors on a surface S arise from considering how S sits in $\mathbb{R}^3$. An important example, studied in Appendix E, is the Poincaré metric on the unit disk $D = D_1(0)$. It is seen that this metric is invariant under all the maps (5.0.9). This together with a geometrical interpretation of the Schwarz lemma yields a powerful tool in complex function theory. In Appendix E we show how use of the Poincaré disk leads to alternative arguments in proofs of both the Riemann mapping theorem and Picard's theorem.

20. Conformal maps

In this section we explore geometrical properties of holomorphic diffeomorphisms $f : \Omega \to \mathcal{O}$ between various domains in $\mathbb{C}$. These maps are also called biholomorphic maps, and they are also called conformal maps. Let us explain the latter terminology.

A diffeomorphism $f : \Omega \to \mathcal{O}$ between two planar domains is said to be conformal provided it preserves angles. That is, if two curves γ_1 and γ_2 in Ω meet at an angle α at p, then $\sigma_1 = f(\gamma_1)$ and $\sigma_2 = f(\gamma_2)$ meet at the same angle α at $q = f(p)$. The condition for this is that, for each $p \in \Omega$, $Df(p) \in M(2,\mathbb{R})$ is a positive multiple $\lambda(p)$ of an orthogonal matrix:

$$Df(p) = \lambda(p)R(p). \tag{20.1}$$

Now $\det Df(p) > 0 \Leftrightarrow \det R(p) = +1$ and $\det Df(p) < 0 \Leftrightarrow \det R(p) = -1$. In the former case, $R(p)$ has the form

$$R(p) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{20.2}$$

of a rotation matrix, and we see that $Df(p)$ commutes with J, given by (1.39). In the latter case, $R(p)$ has the form

$$R(p) = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \tag{20.3}$$

and $C\,Df(p)$ commutes with J, where

$$C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{20.4}$$

i.e., $Cz = \overline{z}$. In the former case, f is said to be orientation preserving, and in the latter case it is said to be orientation reversing. We have the following result.

Proposition 20.1. *Given planar regions Ω, $\mathcal{O}$, the class of orientation-preserving conformal diffeomorphisms $f : \Omega \to \mathcal{O}$ coincides with the class of holomorphic diffeomorphisms. The class of orientation-reversing conformal diffeomorphisms $f : \Omega \to \mathcal{O}$ coincides with the class of conjugate-holomorphic diffeomorphisms.*

There are some particular varieties of conformal maps that have striking properties. Among them we first single out the linear fractional transformations. Given an invertible 2×2 complex matrix A (we say $A \in Gl(2,\mathbb{C})$), set

$$L_A(z) = \frac{az+b}{cz+d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{20.5}$$

We do not want the denominator in this fraction to be $\equiv 0$, and we do not want the numerator to be a constant multiple of the denominator. This leads to the requirement that $\det A \neq 0$. If $c \neq 0$, L_A is holomorphic on $\mathbb{C} \setminus \{-d/c\}$. We extend L_A to

$$L_A : \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C} \cup \{\infty\} \tag{20.6}$$

by setting

$$L_A(-d/c) = \infty, \quad \text{if} \;\; c \neq 0, \tag{20.7}$$

and

$$\begin{aligned} L_A(\infty) = \frac{a}{c} \quad &\text{if} \;\; c \neq 0, \\ \infty \quad &\text{if} \;\; c = 0. \end{aligned} \tag{20.8}$$

If also $B \in Gl(2, \mathbb{C})$, a calculation gives

$$L_A \circ L_B = L_{AB}. \tag{20.9}$$

In particular L_A is bijective in (20.6), with inverse $L_{A^{-1}}$. If we give $\mathbb{C} \cup \{\infty\}$ its natural topology as the one-point compactification of $\mathbb{C}$, we have L_A a homeomorphism in (20.6). Later we will give $\mathbb{C} \cup \{\infty\}$ the structure of a Riemann surface and see that L_A is biholomorphic on this surface.

Note that $L_{sA} = L_A$ for any nonzero $s \in \mathbb{C}$. In particular, $L_A = L_{A_1}$ for some A_1 of determinant 1; we say $A_1 \in Sl(2, \mathbb{C})$. Given $A_j \in Sl(2, \mathbb{C})$, $L_{A_1} = L_{A_2}$ if and only if $A_2 = \pm A_1$. In other words, the group of linear fractional transformations (20.5) is isomorphic to

$$PSl(2, \mathbb{C}) = Sl(2, \mathbb{C})/(\pm I). \tag{20.10}$$

Note that if a, b, c, d are all real, then L_A in (20.5) preserves $\mathbb{R} \cup \{\infty\}$. In this case we have $A \in Gl(2, \mathbb{R})$. We still have $L_{sA} = L_A$ for all nonzero s, but we need $s \in \mathbb{R}$ to get $sA \in Gl(2, \mathbb{R})$. We can write $L_A = L_{A_1}$ for $A_1 \in Sl(2, \mathbb{R})$ if $A \in Gl(2, \mathbb{R})$ and $\det A > 0$. We can also verify that

$$A \in Sl(2, \mathbb{R}) \Longrightarrow L_A : \mathcal{U} \to \mathcal{U}, \tag{20.11}$$

where

$$\mathcal{U} = \{z : \operatorname{Im} z > 0\}$$

is the upper half-plane. In more detail, for $a, b, c, d \in \mathbb{R}$, $z = x + iy$,

$$\frac{az + b}{cz + d} = \frac{(az + b)(c\overline{z} + d)}{(cz + d)(c\overline{z} + d)} = \frac{R}{P} + iy\frac{ad - bc}{P},$$

with

$$R = ac|z|^2 + bd + (ad+bc)x \in \mathbb{R}, \quad P = |cz+d|^2 > 0, \text{ if } y \neq 0,$$

which gives (20.11). Again $L_A = L_{-A}$, so the group

$$PSl(2,\mathbb{R}) = Sl(2,\mathbb{R})/(\pm I) \tag{20.12}$$

acts on $\mathcal{U}$.

We now single out for attention the following linear fractional transformation:

$$\varphi(z) = \frac{z-i}{z+i}, \quad \varphi(z) = L_{A_0}(z),\ A_0 = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}. \tag{20.13}$$

Note that

$$\varphi(x+iy) = \frac{x+i(y-1)}{x+i(y+1)} \Longrightarrow |\varphi(x+iy)|^2 = \frac{x^2+(y-1)^2}{x^2+(y+1)^2}.$$

In particular, $|\varphi(x+iy)| < 1$ if and only if $y > 0$. We have

$$\varphi : \mathcal{U} \to D, \quad \varphi : \mathbb{R} \cup \{\infty\} \to S^1 = \partial D, \tag{20.14}$$

where

$$D = \{z : |z| < 1\}$$

is the unit disk. The bijectivity of φ on $\mathbb{C} \cup \{\infty\}$ implies that φ is bijective in (20.14).

Conjugating the $Sl(2,\mathbb{R})$ action on $\mathcal{U}$ by φ yields the mappings

$$M_A = L_{A_0 A A_0^{-1}} : D \longrightarrow D. \tag{20.15}$$

In detail, if A is as in (20.5), with a, b, c, d real, and if A_0 is as in (20.13),

$$\begin{aligned} A_0 A A_0^{-1} &= \frac{1}{2i} \begin{pmatrix} (a+d)i - b + c & (a-d)i + b + c \\ (a-d)i - b - c & (a+d)i + b - c \end{pmatrix} \\ &= \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}. \end{aligned} \tag{20.16}$$

Note that

$$|\alpha|^2 - |\beta|^2 = \det A = ad - bc.$$

It follows that

$$\begin{aligned} A_0\, Sl(2,\mathbb{R})\, A_0^{-1} &= \left\{ \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in Gl(2,\mathbb{C}) : |\alpha|^2 - |\beta|^2 = 1 \right\} \\ &= SU(1,1), \end{aligned} \tag{20.17}$$

the latter identity defining the group $SU(1,1)$. Hence we have linear fractional transformations

$$(20.18)\qquad \begin{aligned} L_B : D \to D, \quad L_B(z) &= \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}}, \\ B = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in SU(1,1), &\qquad |\alpha|^2 - |\beta|^2 = 1. \end{aligned}$$

Note that for such B as in (20.18),

$$L_B(e^{i\theta}) = \frac{\alpha e^{i\theta} + \beta}{\overline{\beta} e^{i\theta} + \overline{\alpha}} = e^{i\theta} \frac{\alpha + \beta e^{-i\theta}}{\overline{\alpha} + \overline{\beta} e^{i\theta}},$$

and in the last fraction the numerator is the complex conjugate of the denominator. This directly implies the result $L_B : D \to D$ for such B.

We have the following important transitivity properties.

Proposition 20.2. *Given the groups $Sl(2,\mathbb{R})$ and $SU(1,1)$ defined above*

(a) *$Sl(2,\mathbb{R})$ acts transitively in $\mathcal{U}$, via (20.5), and*

(b) *$SU(1,1)$ acts transitively on D, via (20.18).*

Proof. To demonstrate (a), take $p = a + ib \in \mathcal{U}$ $(a \in \mathbb{R},\ b > 0)$. Then $L_p(z) = bz + a = (b^{1/2} z + b^{-1/2} a)/b^{-1/2}$ maps i to p. Given another $q \in \mathcal{U}$, we see that $L_p L_q^{-1}$ maps q to p, so (a) holds. The conjugation (20.17) implies that (a) and (b) are equivalent.

We can also demonstrate (b) directly. Given $p \in D$, we can pick $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 - |\beta|^2 = 1$ and $\beta/\overline{\alpha} = p$. Then $L_B(0) = p$, so (b) holds.

The following converse to Proposition 20.2 is useful.

Proposition 20.3. *If $f : D \to D$ is a holomorphic diffeomorphism, then $f = L_B$ for some $B \in SU(1,1)$. Hence if $F : \mathcal{U} \to \mathcal{U}$ is a holomorphic diffeomorphism, then $F = L_A$ for some $A \in Sl(2,\mathbb{R})$.*

Proof. Say $f(0) = p \in D$. By Proposition 20.2 there exists $B_1 \in SU(1,1)$ such that $L_{B_1}(p) = 0$, so $g = L_{B_1} \circ f : D \to D$ is a holomorphic diffeomorphism satisfying $g(0) = 0$. Now we claim that $g(z) = cz$ for a constant c with $|c| = 1$.

To see this, note that, $h(z) = g(z)/z$ has a removable singularity at 0 and yields a holomorphic map $h : D \to \mathbb{C}$. A similar argument applies to $z/g(z)$. Furthermore, with $\gamma_\rho = \{z \in D : |z| = \rho\}$ one has, because $g : D \to D$ is a homeomorphism,

$$\lim_{\rho \nearrow 1} \sup_{z \in \gamma_\rho} |g(z)| = \lim_{\rho \nearrow 1} \inf_{z \in \gamma_\rho} |g(z)| = 1.$$

Hence the same can be said for $h|_{\gamma_\rho}$, and then a maximum principle argument yields $|g(z)/z| \le 1$ on D and also $|z/g(z)| \le 1$ on D; hence $|g(z)/z| \equiv 1$ on D. This implies

$g(z)/z = c$, a constant, and that $|c| = 1$, as asserted. (Compare the proof of the Schwartz lemma, Proposition 6.2.)

To proceed, we have $g = L_{B_2}$ with $B_2 = a\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}$, and we can take $a = \pm c^{-1/2}$, to obtain $B_2 \in SU(1,1)$. We have $f = L_{B_1^{-1}B_2}$, and the proof is complete.

We single out some building blocks for the group of linear fractional transformations, namely (with $a \neq 0$)

$$\delta_a(z) = az, \quad \tau_b(z) = z + b, \quad \iota(z) = \frac{1}{z}. \tag{20.19}$$

We call these respectively (complex) dilations, translations, and inversion about the unit circle $\{z : |z| = 1\}$. These have the form (20.5), with A given respectively by

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{20.20}$$

We can produce the inversion ι_D about the boundary of a disk $D = D_r(p)$ as

$$\iota_D = \tau_p \circ \delta_r \circ \iota \circ \delta_{1/r} \circ \tau_{-p}. \tag{20.21}$$

The reader can work out the explicit form of this linear fractional transformation. Note that ι_D leaves ∂D invariant and interchanges p and ∞.

Recall that the linear fractional transformation φ in (20.13) was seen to map $\mathbb{R} \cup \{\infty\}$ to S^1. Similarly its inverse, given by $-i\psi(z)$ with

$$\psi(z) = \frac{z+1}{z-1}, \tag{20.22}$$

maps S^1 to $\mathbb{R} \cup \{\infty\}$; equivalently ψ maps S^1 to $i\mathbb{R} \cup \{\infty\}$. To see this directly, write

$$\psi(e^{i\theta}) = \frac{e^{i\theta}+1}{e^{i\theta}-1} = \frac{-i}{\tan\theta/2}. \tag{20.23}$$

These are special cases of an important general property of linear fractional transformations. To state it, let us say that an extended line is a set $\ell \cup \{\infty\}$, where ℓ is a line in $\mathbb{C}$.

Proposition 20.4. *If L is a linear fractional transformation, then L maps each circle to a circle or an extended line, and L maps each extended line to a circle or an extended line.*

To begin the proof, suppose $D \subset \mathbb{C}$ is a disk. We investigate where L maps ∂D.

Claim 1. *If L has a pole at $p \in \partial D$, then L maps ∂D to an extended line.*

Proof. Making use of the transformations (20.19), we have $L(\partial D) = L'(S^1)$ for some linear fractional transformation L', so we need check only the case $D = \{z : |z| < 1\}$, with L having a pole on S^1, and indeed we can take the pole to be at $z = 1$. Thus we look at

$$
\begin{aligned}
L(e^{i\theta}) &= \frac{ae^{i\theta} + b}{e^{i\theta} - 1} \\
&= -\frac{a+b}{2} \frac{i}{\tan \theta/2} + \frac{a-b}{2},
\end{aligned}
\tag{20.24}
$$

whose image is clearly an extended line.

Claim 2. *If L has no pole on ∂D, then L maps ∂D to a circle.*

Proof. One possibility is that L has no pole in $\mathbb{C}$. Then $c = 0$ in (20.5). This case is elementary.

Next, suppose L has a pole at $p \in D$. Composing (on the right) with various linear fractional transformations, we can reduce to the case $D = \{z : |z| < 1\}$, and making further compositions (via Proposition 20.2), we need only deal with the case $p = 0$. So we are looking at

$$
L(z) = \frac{az + b}{z}, \quad L(e^{i\theta}) = a + be^{-i\theta}. \tag{20.25}
$$

Clearly the image $L(S^1)$ is a circle.

If L has a pole at $p \in \mathbb{C} \setminus \overline{D}$, we can use an inversion about ∂D to reduce the study to that done in the previous paragraph. This finishes Claim 2.

To finish the proof of Proposition 20.4, there are two more claims to establish:

Claim 3. *If $\ell \subset \mathbb{C}$ is a line and L has a pole on ℓ, or if L has no pole in $\mathbb{C}$, then L maps $\ell \cup \{\infty\}$ to an extended line.*

Claim 4. *If $\ell \subset \mathbb{C}$ is a line and L has a pole in $\mathbb{C} \setminus \ell$, then L maps $\ell \cup \{\infty\}$ to a circle.*

We leave Claims 3–4 as exercises for the reader.

We present a variety of examples of conformal maps in Figs. 20.1–20.3. The domains pictured there are all simply connected domains, and one can see that they are all conformally equivalent to the unit disk. The Riemann mapping theorem, which we will prove in §23, says that any simply connected domain $\Omega \subset \mathbb{C}$ such that $\Omega \neq \mathbb{C}$ is conformally equivalent to the disk. Here we make note of the following.

Proposition 20.5. *If $\Omega \subset \mathbb{C}$ is simply connected and $\Omega \neq \mathbb{C}$, then there is a holomorphic diffeomorphism $f : \Omega \to \mathcal{O}$, where $\mathcal{O} \subset \mathbb{C}$ is a bounded, simply connected domain.*

Proof. Pick $p \in \mathbb{C} \setminus \Omega$ and define a holomorphic branch on Ω of

$$
g(z) = (z - p)^{1/2}. \tag{20.26}
$$

The simple connectivity of Ω guarantees the existence of such g, as shown in §10; see (10.17). Now g is one-to-one on Ω and it maps Ω diffeomorphically onto a simply connected region $\widetilde{\Omega}$ having the property that

$$z \in \widetilde{\Omega} \Longrightarrow -z \notin \widetilde{\Omega}. \tag{20.27}$$

It follows that there is a disk $D \subset \mathbb{C} \setminus \widetilde{\Omega}$, and if we compose g with inversion across ∂D we obtain such a desired holomorphic diffeomorphism.

Exercises

1. Find conformal mappings of each of the following regions onto the unit disk. In each case, you can express the map as a composition of various conformal maps.

(a) $\Omega = \{z = x + iy : y > 0, \ |z| > 1\}$.

(b) $\Omega = \mathbb{C} \setminus \big((-\infty, -1] \cup [1, \infty)\big)$.

(c) $\Omega = \{z : |z + 1/2| < 1\} \cap \{z : |z - 1/2| < 1\}$.

2. Consider the quarter-plane

$$\Omega = \{z = x + iy : x > 0, \ y > 0\}.$$

Find a conformal map Φ of Ω onto itself such that

$$\Phi(1 + i) = 2 + i.$$

3. Let $f : \Omega \to \mathcal{O}$ be a conformal diffeomorphism. Show that if $u : \mathcal{O} \to \mathbb{R}$ is harmonic, so is $u \circ f : \Omega \to \mathbb{R}$.

4. Write out the details to establish Claims 3–4 in the proof of Proposition 20.4.

5. Look ahead at Exercise 1b) of §21, and map the region U defined there conformally onto the unit disk.

6. Given $q \in D = \{z : |z| < 1\}$, define

$$\varphi_q(z) = \frac{z - q}{1 - \overline{q} z}. \tag{20.28}$$

Show that $\varphi_q : D \to D$ and $\varphi_q(q) = 0$. Write φ_q in the form (20.18). Relate this to the proof of Proposition 20.2.

21. Normal families

Here we discuss a certain class of sets of mappings, the class of *normal families*. In a general context, suppose Ω is a locally compact metric space, i.e., Ω is a metric space and each $p \in \Omega$ has a closed neighborhood that is compact. Supppose S is a complete metric space. Let $C(\Omega, S)$ denote the set of continuous maps $f : \Omega \to S$. We say a subset $\mathcal{F} \subset C(\Omega, S)$ is a normal family (with respect to (Ω, S)) if and only if the following property holds:

$$\text{Every sequence } \ f_\nu \in \mathcal{F} \ \text{ has a locally uniformly} \\ \text{convergent subsequence } \ f_{\nu_k} \to f \in C(\Omega, S). \tag{21.1}$$

If the identity of the pair (Ω, S) is understood, we omit the phrase "with respect to (Ω, S)." The main case of interest to us here is where $\Omega \subset \mathbb{C}$ is an open set and $S = \mathbb{C}$. However, in later sections the case $S = \mathbb{C} \cup \{\infty\} \approx S^2$ will also be of interest.

A major technique to identify normal families is the following result, known as the Arzela-Ascoli theorem.

Proposition 21.1. *Let X and Y be compact metric spaces and fix a modulus of continuity $\omega(\delta)$. Then*

$$\mathcal{C}_\omega = \{f \in C(X, Y) : d(f(x), f(y)) \le \omega(d(x, y)), \forall\ x, y \in X\} \tag{21.2}$$

is a compact subset of $C(X, Y)$, hence a normal family.

This result is given as Proposition A.18 in Appendix A and proven there. See also this appendix for a discussion of $C(X, Y)$ as a metric space. The defining condition

$$d\big(f(x), f(y)\big) \le \omega\big(d(x, y)\big), \quad \forall\ x, y \in X,\ f \in \mathcal{F}, \tag{21.3}$$

for some modulus of continuity ω is called *equicontinuity* of $\mathcal{F}$. The following result is a simple extension of Proposition 21.1.

Proposition 21.2. *Assume there exists a countable family $\{K_j\}$ of compact subsets of Ω such that any compact $K \subset \Omega$ is contained in some finite union of these K_j. Consider a family $\mathcal{F} \subset C(\Omega, S)$. Assume that, for each j, there exist compact $L_j \subset S$ such that $f : K_j \to L_j$ for all $f \in \mathcal{F}$, and that $\{f|_{K_j} : f \in \mathcal{F}\}$ is equicontinuous. Then $\mathcal{F}$ is a normal family.*

Proof. Let f_ν be a sequence in $\mathcal{F}$. By Proposition 21.1 there is a uniformly convergent subsequence $f_{\nu_k} : K_1 \to L_1$. This has a further subsequence converging uniformly on K_2, etc. A diagonal argument finishes the proof.

The following result is sometimes called Montel's theorem, though there is a deeper result, discussed in §27, which is more properly called Montel's theorem. In light of this, one might call the next result "Montel's little theorem."

Proposition 21.3. *Let $\Omega \subset \mathbb{C}$ be open. A family $\mathcal{F}$ of holomorphic functions $f_\alpha : \Omega \to \mathbb{C}$ is normal (with respect to $(\Omega, \mathbb{C})$) if and only if this family is uniformly bounded on each compact subset of Ω.*

Proof. We can write $\Omega = \cup_j \overline{D}_j$ for a countable family of closed disks $\overline{D}_j \subset \Omega$, and satisfy the hypothesis on Ω in Proposition 21.2. Say $\operatorname{dist}(z, \partial\Omega) \geq 2\varepsilon_j > 0$ for all $z \in \overline{D}_j$. Let the disk $\widetilde{D}_j$ be concentric with $\overline{D}_j$ and have radius ε_j greater. The local uniform bounds hypothesis implies

$$|f_\alpha| \leq A_j \quad \text{on} \quad \widetilde{D}_j, \quad \forall\, f_\alpha \in \mathcal{F}. \tag{21.4}$$

This plus Cauchy's estimate (5.32) gives

$$|f'_\alpha| \leq \frac{A_j}{\varepsilon_j} \quad \text{on} \quad \overline{D}_j, \quad \forall\, f_\alpha \in \mathcal{F}, \tag{21.5}$$

hence

$$|f_\alpha(z) - f_\alpha(w)| \leq \frac{A_j}{\varepsilon_j}|z - w|, \quad \forall z, w \in \overline{D}_j,\ f_\alpha \in \mathcal{F}. \tag{21.6}$$

This equicontinuity on each $\overline{D}_j$ makes Proposition 21.2 applicable. This establishes one implication in Proposition 21.3, and the reverse implication is easy.

Exercises

1. Show whether each of the following families is or is not normal (with respect to $(\Omega, \mathbb{C})$).

(a) $\{n^{-1} \cos nz : n = 1, 2, 3, \dots\}$, $\quad \Omega = \{z = x + iy : x > 0,\ y > 0\}$.

(b) The set of holomorphic maps $g : D \to U$ such that $g(0) = 0$, with

$$\Omega = D = \{z : |z| < 1\}, \quad U = \{z : -2 < \operatorname{Re} z < 2\}.$$

2. Suppose that $\mathcal{F}$ is a normal family (with respect to $(\Omega, \mathbb{C})$). Show that $\{f' : f \in \mathcal{F}\}$ is also a normal family. (Compare Exercise 1 in §22.)

3. Let $\mathcal{F}$ be the set of entire functions f such that $f'(z)$ has a zero of order one at $z = 3$ and satisfies

$$|f'(z)| \leq 5|z - 3|, \quad \forall\, z \in \mathbb{C}.$$

Find all functions in $\mathcal{F}$. Determine whether $\mathcal{F}$ is normal (with respect to $(\mathbb{C}, \mathbb{C})$).

4. Let $\mathcal{F} = \{z^n : n \in \mathbb{Z}^+\}$. For which regions Ω is $\mathcal{F}$ normal with respect to $(\Omega, \mathbb{C})$? (Compare Exercise 3 in §22.)

22. The Riemann sphere (and other Riemann surfaces)

Our main goal here is to describe how the unit sphere $S^2 \subset \mathbb{R}^3$ has a role as a "conformal compactification" of the complex plane $\mathbb{C}$. To begin, we consider a map

$$\mathcal{S} : S^2 \setminus \{e_3\} \longrightarrow \mathbb{R}^2, \tag{22.1}$$

known as stereographic projection; here $e_3 = (0,0,1)$. We define $\mathcal{S}$ as follows:

$$\mathcal{S}(x_1,x_2,x_3) = (1-x_3)^{-1}(x_1,x_2). \tag{22.2}$$

See Fig. 22.1. A computation shows that $\mathcal{S}^{-1} : \mathbb{R}^2 \to S^2 \setminus \{e_3\}$ is given by

$$\mathcal{S}^{-1}(x,y) = \frac{1}{1+r^2}\,(2x, 2y, r^2-1), \quad r^2 = x^2+y^2. \tag{22.3}$$

The following is a key geometrical property.

Proposition 22.1. *The map $\mathcal{S}$ is a conformal diffeomorphism of $S^2 \setminus \{e_3\}$ onto $\mathbb{R}^2$.*

In other words, we claim that if two curves γ_1 and γ_2 in S^2 meet at an angle α at $p \neq e_3$, then their images under $\mathcal{S}$ meet at the same angle at $q = \mathcal{S}(p)$. It is equivalent, and slightly more convenient, to show that $F = \mathcal{S}^{-1}$ is conformal. We have

$$DF(q) : \mathbb{R}^2 \longrightarrow T_pS^2 \subset \mathbb{R}^3. \tag{22.4}$$

See Appendix D for more on this. Conformality is equivalent to the statement that there is a positive function $\lambda(p)$ such that, for all $v, w \in \mathbb{R}^2$,

$$DF(q)v \cdot DF(q)w = \lambda(q)\, v \cdot w, \tag{22.5}$$

or in other words,

$$DF(q)^t\, DF(q) = \lambda(q) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{22.6}$$

To check (22.6), we compute DF via (22.3). A calculation gives

$$DF(x,y) = \frac{2}{(1+r^2)^2} \begin{pmatrix} 1-x^2+y^2 & -2xy \\ -2xy & 1+x^2-y^2 \\ -2x & -2y \end{pmatrix}, \tag{22.7}$$

and hence

$$DF(x,y)^t\, DF(x,y) = \frac{4}{(1+r^2)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{22.8}$$

This gives Proposition 22.1.

Similarly we can define a conformal diffeomorphism

$$\mathcal{S}_- : S^2 \setminus \{-e_3\} \longrightarrow \mathbb{R}^2. \tag{22.9}$$

To do this we take $x_3 \mapsto -x_3$. This reverses orientation, so we also take $x_2 \mapsto -x_2$. Thus we set

$$\mathcal{S}_-(x_1, x_2, x_3) = (1 + x_3)^{-1}(x_1, -x_2). \tag{22.10}$$

Comparing this with (22.3), we see that $\mathcal{S}_- \circ \mathcal{S}^{-1} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ is given by

$$\mathcal{S}_- \circ \mathcal{S}^{-1}(x, y) = \frac{1}{r^2}\,(x, -y). \tag{22.11}$$

Identifying $\mathbb{R}^2$ with $\mathbb{C}$ via $z = x + iy$, we have

$$\mathcal{S}_- \circ \mathcal{S}^{-1}(z) = \frac{\overline{z}}{|z|^2} = \frac{1}{z}. \tag{22.12}$$

Clearly the composition of conformal transformations is conformal, so we could predict in advance that $\mathcal{S}_1 \circ \mathcal{S}^{-1}$ would be conformal and orientation-preserving, hence holomorphic, and (22.12) bears this out.

There is a natural one-to-one map from S^2 onto the space $\mathbb{C} \cup \{\infty\}$, introduced in (20.6),

$$\widetilde{\mathcal{S}} : S^2 \longrightarrow \mathbb{C} \cup \{\infty\}, \tag{22.13}$$

given by

$$\begin{aligned} \widetilde{\mathcal{S}}(p) &= \mathcal{S}(p), \quad p \in S^2 \setminus \{e_3\} \ \ \text{(given by (22.2))}, \\ \widetilde{\mathcal{S}}(e_3) &= \infty, \end{aligned} \tag{22.14}$$

with inverse $\widetilde{\mathcal{S}}^{-1} : \mathbb{C} \cup \{\infty\} \to S^2$, given by

$$\begin{aligned} \widetilde{\mathcal{S}}^{-1}(z) &= \mathcal{S}^{-1}(z), \quad z \in \mathbb{C} \ \ \text{(given by (22.3))}, \\ \widetilde{\mathcal{S}}^{-1}(\infty) &= e_3, \end{aligned} \tag{22.15}$$

Going forward, we use the notation

$$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}, \tag{22.16}$$

and we call $\widehat{\mathbb{C}}$ the *Riemann sphere.*

The concept of a normal family of maps $\Omega \to S$, introduced in §21, is of great interest when $S = \widehat{\mathbb{C}}$. The following result produces a key link with results established in §21.

Proposition 22.2. *Assume $\Omega \subset \mathbb{C}$ is a connected open set. A family $\mathcal{F}$ of holomorphic functions $\Omega \to \mathbb{C}$ is normal with respect to $(\Omega, \widehat{\mathbb{C}})$ if and only if for each sequence f_ν from $\mathcal{F}$ one of the following happens:*
(a) A subsequence f_{ν_k} converges uniformly on each compact $K \subset \Omega$, as a sequence $f_{\nu_k} : K \to \mathbb{C}$, or
(b) A subsequence f_{ν_k} tends to ∞ uniformly on each compact $K \subset \Omega$.

Proof. Assume $\mathcal{F}$ is a normal family with respect to $(\Omega, \widehat{\mathbb{C}})$, and f_ν is a sequence of elements of $\mathcal{F}$. Take a subsequence f_{ν_k}, uniformly convergent on each compact K, as a sequence of maps $f_{\nu_k} : K \to \widehat{\mathbb{C}}$. Say $f_{\nu_k} \to f : \Omega \to \widehat{\mathbb{C}}$. Pick $p \in \Omega$. We consider two cases.

Case I. First suppose $f(p) = \infty$. Then there exists $N \in \mathbb{Z}^+$ and a neighborhood U of p in Ω such that $|f_{\nu_k}(z)| \geq 1$ for $z \in U, k \geq N$. Set $g_{\nu_k}(z) = 1/f_{\nu_k}(z)$, for $z \in U, k \geq N$. We have $|g_{\nu_k}| \leq 1$ on U, $g_{\nu_k}(z) \neq 0$, and $g_{\nu_k}(z) \to 1/f(z)$, locally uniformly on U (with $1/\infty = 0$), and in particular $g_{\nu_k}(p) \to 0$. By Hurwitz' theorem (Proposition 17.8), this implies $1/f(z) = 0$ on all of U, i.e., $f = \infty$ on U, hence $f = \infty$ on Ω. Hence Case I $\Rightarrow$ Case (b).

Case II. Suppose $f(p) \in \mathbb{C}$, i.e., $f(p) \in \widehat{\mathbb{C}} \setminus \{\infty\}$. By the analysis in Case I it follows that $f(z) \in \mathbb{C}$ for all $z \in \Omega$. It is now straightforward to verify Case (a) here.

This gives one implication in Proposition 22.2. The reverse implication is easily established.

The surface S^2 is an example of a Riemann surface, which we define as follows. A Riemann surface is a two-dimensional manifold M covered by open sets $\mathcal{O}_j$ with coordinate charts $\varphi_j : \Omega_j \to \mathcal{O}_j$ having the property that, if $\mathcal{O}_j \cap \mathcal{O}_k \neq \emptyset$, and if $\Omega_{jk} = \varphi_j^{-1}(\mathcal{O}_j \cap \mathcal{O}_k)$, then the diffeomorphism

$$\varphi_k^{-1} \circ \varphi_j : \Omega_{jk} \longrightarrow \Omega_{kj} \tag{22.17}$$

is holomorphic. See Appendix D for general background on manifolds and coordinate charts.

This concept applies to S^2 in the following way. We can take

$$\mathcal{O}_1 = S^2 \setminus \{e_3\}, \quad \mathcal{O}_2 = S^2 \setminus \{-e_3\}, \quad \Omega_1 = \Omega_2 = \mathbb{C}, \tag{22.18}$$

and set

$$\varphi_1 = \mathcal{S}^{-1}, \quad \varphi_2 = \mathcal{S}_-^{-1}, \quad \varphi_j : \mathbb{C} \to \mathcal{O}_j. \tag{22.19}$$

Then $\mathcal{O}_1 \cap \mathcal{O}_2 = S^2 \setminus \{e_3, -e_3\}$, $\Omega_{12} = \Omega_{21} = \mathbb{C} \setminus 0$, and, by (22.10),

$$\varphi_2^{-1} \circ \varphi_1(z) = \frac{1}{z}, \quad z \in \mathbb{C} \setminus 0, \tag{22.20}$$

and similarly for $\varphi_1^{-1} \circ \varphi_2$.

Similarly, $\widehat{\mathbb{C}}$ has the structure of a Riemann surface. This time, we take

$$\mathcal{O}_1 = \mathbb{C} = \widehat{\mathbb{C}} \setminus \{\infty\}, \quad \mathcal{O}_2 = \widehat{\mathbb{C}} \setminus \{0\}, \quad \Omega_1 = \Omega_2 = \mathbb{C}, \tag{22.21}$$

and set $\varphi_1(z) = z$ and

$$\begin{aligned} \varphi_2(z) &= \frac{1}{z}, \quad \text{if } \ z \neq 0, \\ \varphi_2(0) &= \infty. \end{aligned} \tag{22.22}$$

Then $\mathcal{O}_1 \cap \mathcal{O}_2 = \mathbb{C} \setminus 0 = \Omega_{12} = \Omega_{21}$, and again we have (22.20).

We next discuss a natural extension of the notion of a holomorphic map in the setting of Riemann surfaces. Let M be a Riemann surface, as defined above, let $U \subset \mathbb{C}$ be open, and let $f : U \to M$ be continuous. Assume $f(p) = q$, with $p \in U, q \in M$. We say f is holomorphic in a neighborhood of p provided that, if $\varphi_j : \Omega_j \to \mathcal{O}_j$ is a coordinate chart on M with $q \in \mathcal{O}_j$, there is a neighborhood U_p of p such that $f : U_p \to \mathcal{O}_j$ and

$$\varphi_j^{-1} \circ f : U_p \longrightarrow \Omega_j \subset \mathbb{C} \ \text{ is holomorphic.} \tag{22.23}$$

We say $f : \Omega \to M$ is holomorphic provided it is holomorphic on a neighborhood of each $p \in \Omega$.

Suppose more generally that X is also a Riemann surface, covered by open sets V_k, with coordinate charts $\psi_k : U_k \to V_k$, satisfying a compatibility condition parallel to (22.17). Let $f : X \to M$ be continuous, and assume $f(p) = q$, with $p \in X, q \in M$. We say f is holomorphic on a neighborhood of p provided that $p \in V_k$, for a coordinate chart $\psi_k : U_k \to V_k$, and

$$f \circ \psi_k : U_k \longrightarrow M \ \text{ is holomorphic near } \ p. \tag{22.24}$$

We say $f : X \to M$ is holomorphic provided it is holomorphic on a neighborhood of each $p \in X$.

Here is a basic case of holomorphic maps. We leave the verification as an exercise for the reader.

Proposition 22.3. *The maps*

$$\widetilde{\mathcal{S}} : S^2 \longrightarrow \widehat{\mathbb{C}}, \quad \widetilde{\mathcal{S}}^{-1} : \widehat{\mathbb{C}} \longrightarrow S^2, \tag{22.25}$$

defined by (22.14)–(22.15), are holomorphic.

Here is another basic result.

Proposition 22.4. *Let $\Omega \subset \mathbb{C}$ be open, $p \in \Omega$, and let $f : \Omega \setminus p \to \mathbb{C}$ be holomorphic. Assume*

$$|f(z)| \longrightarrow \infty, \quad \textit{as } \ z \to p, \tag{22.26}$$

i.e., f has a pole at p. Define $F:\Omega\to\widehat{\mathbb{C}}$ by

$$F(z)=\begin{cases} f(z), & z\in\Omega\setminus p,\\ \infty, & z=p.\end{cases}\tag{22.27}$$

Then F is holomorphic on Ω.

Proof. Clearly $F:\Omega\to\widehat{\mathbb{C}}$ is continuous. It suffices to check that F is holomorphic on a neighborhood of p. We can pick a neighborhood U_p of p such that $z\in U_p\setminus p\Rightarrow|f(z)|\geq 1$. Then F maps U_p to $\widehat{\mathbb{C}}\setminus\{0\}$. Our task is to verify that

$$\varphi_2^{-1}\circ F:U_p\longrightarrow\Omega_2=\mathbb{C}\ \text{ is holomorphic,}\tag{22.28}$$

with Ω_2 and φ_2 as in (22.21)–(22.22). In fact,

$$\varphi_2^{-1}\circ F(x)=\begin{cases}\dfrac{1}{f(z)}, & \text{for } z\in U_p\setminus p,\\ 0, & \text{for } z=p,\end{cases}\tag{22.29}$$

and the proof is finished by observing that the right side of (22.28) is holomorphic on U_p.

The following is a useful corollary to Proposition 22.4.

Proposition 22.5. *Given $A\in Gl(2,\mathbb{C})$, define*

$$L_A=\frac{az+b}{cz+d},\quad A=\begin{pmatrix}a&b\\c&d\end{pmatrix},\tag{22.30}$$

and extend this to

$$L_A:\widehat{\mathbb{C}}\longrightarrow\widehat{\mathbb{C}},\tag{22.31}$$

as a continuous map, as in (20.6). Then L_A is holomorphic in (22.31).

Corollary 22.6. *For $A\in Gl(2,\mathbb{C})$ as above, define*

$$\Lambda_A:S^2\longrightarrow S^2,\quad \Lambda_A=\widetilde{\mathcal{S}}^{-1}\circ L_A\circ\widetilde{\mathcal{S}},\tag{22.32}$$

with $\widetilde{\mathcal{S}}$ as in (22.13)–(22.14). Then Λ_A is holomorphic.

Another important class of Riemann surfaces is given as follows. Let $\Lambda\subset\mathbb{R}^2\approx\mathbb{C}$ be the image of $\mathbb{Z}^2$ under any matrix in $Gl(2,\mathbb{R})$. Then the torus

$$\mathbb{T}_\Lambda=\mathbb{C}/\Lambda\tag{22.33}$$

is a Riemann surface in a natural fashion.

We now look at conformal coordinates on general surfaces of revolution in $\mathbb{R}^3$. Let γ be a smooth curve in the (x_1, x_3)-plane,

$$\gamma(v) = (r(v), z(v)), \quad v \in I, \tag{22.34}$$

where $I \subset \mathbb{R}$ is an interval. For now, we assume $r(v) > 0$ for $v \in I$. The associated surface of revolution Σ can be described as

$$\begin{aligned} x_1(u,v) &= r(v)\cos u, \\ x_2(u,v) &= r(v)\sin u, \\ x_3(u,v) &= z(v). \end{aligned} \tag{22.35}$$

As we will see, the (u, v) coordinates are not typically conformal. Instead, we will take

$$u = x, \quad v = v(y), \quad y \in J,$$

with $J \subset \mathbb{R}$ an interval. We will obtain an equation for $v(y)$ that yields conformality for

$$G : \mathbb{R} \times J \longrightarrow \Sigma, \tag{22.36}$$

given by

$$G(x,y) = \bigl(r(v(y))\cos x, r(v(y))\sin x, z(v(y))\bigr). \tag{22.37}$$

We calculate

$$DG(x,y) = \begin{pmatrix} -r(v)\sin x & r'(v)v'(y)\cos x \\ r(v)\cos x & r'(v)v'(y)\sin x \\ 0 & z'(v)v'(y) \end{pmatrix}, \tag{22.38}$$

and hence

$$DG(x,y)^t DG(x,y) = \begin{pmatrix} r(v)^2 & 0 \\ 0 & (r'(v)^2 + z'(v)^2)v'(y)^2 \end{pmatrix}. \tag{22.39}$$

Hence the condition that G be conformal is

$$\frac{dv}{dy} = \frac{r(v)}{\sqrt{r'(v)^2 + z'(v)^2}} = \frac{r(v)}{|\gamma'(v)|}. \tag{22.40}$$

This (typically) nonlinear differential equation for v is separable, yielding

$$\int \frac{|\gamma'(v)|}{r(v)}\, dv = y + C. \tag{22.41}$$

As an example, suppose γ is a circle, of radius b, centered at $(a,0)$ in the (x_1,x_3)-plane, so

$$\gamma(v) = (a + b\cos v, b\sin v). \tag{22.42}$$

We assume for now that $0 < b < a$, and take $v \in I = \mathbb{R}$. The differential equation (22.40) becomes

$$\frac{dv}{dy} = A + \cos v, \quad A = \frac{a}{b}, \tag{22.43}$$

with $A \in (1,\infty)$. This has a solution for all $y \in \mathbb{R}$, periodic in y, with period

$$T = \int_0^{2\pi} \frac{dv}{A + \cos v}. \tag{22.44}$$

One can convert this integral into one over the unit circle σ, using $z = e^{iv}$, obtaining

$$\begin{aligned} T &= \int\limits_\sigma \frac{dz}{iz(A + z/2 + 1/2z)} \\ &= \frac{2}{i} \int\limits_\sigma \frac{dz}{z^2 + 2Az + 1} \\ &= \frac{2\pi}{\sqrt{A^2-1}}, \end{aligned} \tag{22.45}$$

assuming $A \in (1,\infty)$, the last integral via residue calculus. We see that if γ is given by (22.42), with $0 < b < a$, the conformal map G from $\mathbb{R}^2 \approx \mathbb{C}$ onto the resulting surface of revolution Σ (an "inner tube") is periodic of period 2π in x and of period $T = 2\pi(A^2-1)^{-1/2}$ in y. Consequently, we have a holomorphic diffeomorphism

$$G : \mathbb{T}_\Lambda \longrightarrow \Sigma, \tag{22.46}$$

wth lattice $\Lambda \subset \mathbb{C}$ given by

$$\Lambda = \{2\pi k + i\ell T : k, \ell \in \mathbb{Z}\}, \tag{22.47}$$

and T given by (22.45).

Let us now take γ in (22.42) to be a more general smooth, simple closed curve, periodic of period 2π in v, with velocity $\gamma'(v)$ bounded away from 0 and $r(v) \ge \delta > 0$. Then the differential equation (22.40) is solvable for all $y \in \mathbb{R}$, and the solution $v(y)$ is periodic, of period

$$T = \int_0^{2\pi} \frac{|\gamma'(v)|}{r(v)}\, dv. \tag{22.48}$$

In this setting, one continues to have a holomorphic diffeomorphism $F : \mathbb{T}_\Lambda \to \Sigma$, as in (22.46), with Λ given by (22.47).

Going further, we can drop the requirement that $r(v)$ be bounded away from 0, and the differential equation (22.40) is still solvable for all y. In this more general setting, we generally will not obtain a holomorphic diffeomorphism of the form (22.46). As an example, take $a = 0, b = 1$ in (22.42), so

$$\gamma(v) = (\cos v, \sin v). \tag{22.49}$$

This is the unit circle centered at 0, and the surface of revolution produced is of course the unit sphere S^2. In this case, the differential equation (22.40) becomes

$$\frac{dv}{dy} = \cos v, \tag{22.50}$$

so, if $v = 0$ at $y = 0$, the solution is given implicitly by

$$y(v) = \int_0^v \sec t \, dt, \quad |v| < \frac{\pi}{2}. \tag{22.51}$$

We see that the solution v to (22.50) with $v(0) = 0$ defines a diffeomorphism $v : (-\infty, \infty) \to (-\pi/2, \pi/2)$. Another way to write (22.51) is

$$\sinh y = \tan v, \quad \text{hence} \quad \cosh y = \sec v. \tag{22.52}$$

See Exercise 11 of §4. Consequently, in this case (22.37) takes the form

$$G(x, y) = \Bigl(\frac{\cos x}{\cosh y}, \frac{\sin x}{\cosh y}, \frac{\sinh y}{\cosh y}\Bigr), \tag{22.53}$$

yielding a conformal diffeomorphism

$$G : (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R} \longrightarrow S^2 \setminus \{e_3, -e_3\}. \tag{22.54}$$

Bringing in $\mathcal{S}$ from (22.1)–(22.2), we have

$$\begin{aligned} \mathcal{S} \circ G(x, y) &= \frac{\cosh y}{\cosh y - \sinh y} \Bigl(\frac{\cos x}{\cosh y}, \frac{\sin x}{\cosh y}\Bigr) \\ &= e^y (\cos x, \sin x), \end{aligned} \tag{22.55}$$

or, in complex notation,

$$\mathcal{S} \circ G(z) = e^{i\bar{z}}. \tag{22.56}$$

This is actually a conjugate holomorphic map, signifying that $\mathcal{S}$ and G induce opposite orientations on $S^2 \setminus \{e_3, -e_3\}$.

There are many other Riemann surfaces. For example, any oriented two-dimensional Riemannian manifold has a natural structure of a Riemann surface. A proof of this can be found in Chapter 5 of [T2]. An important family of Riemann surfaces holomorphically diffeomorphic to surfaces of the form (22.33) will arise in §34, with implications for the theory of elliptic functions.

Exercises

1. Give an example of a family $\mathcal{F}$ of holomorphic functions $\Omega \to \mathbb{C}$ with the following two properties:

(a) $\mathcal{F}$ is normal with respect to $(\Omega, \widehat{\mathbb{C}})$.
(b) $\{f' : f \in \mathcal{F}\}$ is *not* normal, with respect to $(\Omega, \mathbb{C})$.

Compare Exercise 2 of §21. See also Exercise 11 below.

2. Given $\Omega \subset \mathbb{C}$ open, let

$$\mathcal{F} = \{f : \Omega \to \mathbb{C} : \operatorname{Re} f > 0 \text{ on } \Omega, f \text{ holomorphic}\}.$$

Show that $\mathcal{F}$ is normal with respect to $(\Omega, \widehat{\mathbb{C}})$. Is $\mathcal{F}$ normal with respect to $(\Omega, \mathbb{C})$?

3. Let $\mathcal{F} = \{z^n : n \in \mathbb{Z}^+\}$. For which regions Ω is $\mathcal{F}$ normal with respect to $(\Omega, \widehat{\mathbb{C}})$? Compare Exercise 4 in §21.

4. Show that the set of orientation-preserving conformal diffeomorphisms $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is precisely the set of linear fractional transformations of the form (22.30), with $A \in Gl(2, \mathbb{C})$. *Hint.* Given such $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, take L_A such that $L_A \circ \varphi$ takes ∞ to ∞, so $\psi = L_A \circ \varphi|_{\mathbb{C}}$ is a holomorphic diffeomorphism of $\mathbb{C}$ onto itself. What form must ψ have? (Cf. Proposition 11.4.)

5. Let M_j be Riemann surfaces. Show that if $\varphi : M_1 \to M_2$ and $\psi : M_2 \to M_3$ are holomorphic, then so is $\psi \circ \varphi : M_1 \to M_3$.

6. Let $p(z)$ and $q(z)$ be polynomials on $\mathbb{C}$. Assume the roots of $p(z)$ are disjoint from the roots of $q(z)$. Form the meromorphic function

$$R(z) = \frac{p(z)}{q(z)}.$$

Show that $R(z)$ has a unique continuous extension $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, and this is holomorphic.

Exercises 7–9 deal with holomorphic maps $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Assume F is not constant.

7. Show that there are only finitely many $p_j \in \widehat{\mathbb{C}}$ such that $DF(p_j) : T_{p_j}\widehat{\mathbb{C}} \to T_{q_j}\widehat{\mathbb{C}}$ is singular (hence zero), where $q_j = F(p_j)$. The points q_j are called critical values of F.

8. Suppose ∞ is not a critical value of F and that $F^{-1}(\infty) = \{\infty, p_1, \dots, p_k\}$. Show that

$$f(z) = F(z)(z - p_1) \cdots (z - p_k) : \mathbb{C} \longrightarrow \mathbb{C},$$

and $|f(z)| \to \infty$ as $|z| \to \infty$. Deduce that $f(z)$ is a polynomial in z. (Cf. Proposition 11.4.)

9. Show that every holomorphic map $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is of the form treated in Exercise 6 (except for the constant map $F \equiv \infty$).
Hint. Compose with linear fractional transformations and transform F to a map satisfying the conditions of Exercise 8.

10. Given a holomorphic map $f : \Omega \to \mathbb{C}$, set

$$g = \mathcal{S}^{-1} \circ f : \Omega \longrightarrow S^2.$$

For $z \in \Omega$, set $q = f(z),\ p = g(z)$, and consider

$$Dg(z) : \mathbb{R}^2 \longrightarrow T_p S^2.$$

Using (22.8) (where $F = \mathcal{S}^{-1}$), show that

$$Dg(z)^t\ Dg(z) = 4\Big(\frac{|f'(z)|}{1 + |f(z)|^2}\Big)^2\ I,$$

where I is the identity matrix. The quantity

$$f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2}$$

is sometimes called the "spherical derivative" of f.

11. Using Exercise 10, show that a family $\mathcal{F}$ of holomorphic functions on Ω is normal with respect to $(\Omega, \widehat{\mathbb{C}})$ if and only if for each compact $K \subset \Omega$,

$$\{f^{\#}(z) : f \in \mathcal{F},\, z \in K\} \ \text{ is bounded.}$$

Hint. Check Proposition 21.1.

12. Show that if $\widetilde{G}(x, y) = G(x, -y)$, with G as in (22.53), then $\widetilde{G}$ yields a variant of (22.54) and, in place of (22.56), we have $\mathcal{S} \circ \widetilde{G}(z) = e^{iz}$.

13. For $\theta \in \mathbb{R}$, define $\rho_\theta : \mathbb{C} \to \mathbb{C}$ by $\rho_\theta(z) = e^{i\theta} z$. Also set $\rho_\theta(\infty) = \infty$, so $\rho_\theta : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Take $\widetilde{\mathcal{S}}$ as in (22.13). Show that

$$R_\theta = \widetilde{\mathcal{S}}^{-1} \circ \rho_\theta \circ \widetilde{\mathcal{S}} \Longrightarrow R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 1 \end{pmatrix}.$$

14. Let $R : S^2 \to S^2$ be a conformal diffeomorphism with the properties

$$R(e_3) = e_3, \quad R : E \to E,$$

where $E = \{(x_1, x_2, x_3) \in S^2 : x_3 = 0\}$. Show that $R = R_\theta$ (as in Exercise 13) for some $\theta \in \mathbb{R}$.
Hint. Consider $\rho = \widetilde{\mathcal{S}} \circ R \circ \widetilde{\mathcal{S}}^{-1}$. Show that $\rho : \mathbb{C} \to \mathbb{C}$ is bijective and ρ preserves $\{z \in \mathbb{C} : |z| = 1\}$. Deduce that $\rho = \rho_\theta$ for some $\theta \in \mathbb{R}$.

15. For $\theta \in \mathbb{R}$, consider the linear fractional transformation

$$f_\theta(z) = \frac{(\cos\theta)z - \sin\theta}{(\sin\theta)z + \cos\theta}, \quad f_\theta : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}.$$

Set

$$\varphi_\theta = \widetilde{\mathcal{S}}^{-1} \circ f_\theta \circ \widetilde{\mathcal{S}}, \quad \varphi_\theta : S^2 \to S^2,$$

with $\widetilde{\mathcal{S}}$ as in (22.13). Show that φ_θ is a conformal diffeomorphism.

16. In the setting of Exercise 15, show that, for all $\theta \in \mathbb{R}$,

$$\varphi_\theta(e_2) = e_2, \quad \varphi_\theta : \widetilde{E} \to \widetilde{E},$$

where $\widetilde{E} = \{(x_1, x_2, x_3) \in S^2 : x_2 = 0\}$. Show also that

$$\varphi_\theta(e_3) = (\sin 2\theta, 0, \cos 2\theta).$$

Hint. To get started, show that

$$f_\theta(i) = i, \quad f_\theta : \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\}, \quad f_\theta(\infty) = \frac{\cos\theta}{\sin\theta}.$$

17. In the setting of Exercises 15–16, show that

$$\varphi_\theta = \begin{pmatrix} \cos 2\theta & & \sin 2\theta \\ & 1 & \\ -\sin 2\theta & & \cos 2\theta \end{pmatrix}.$$

Hint. Translate the result of Exercise 14 to this setting.

18. Show that if $A \in G\ell(2,\mathbb{C})$, $\Lambda_A : S^2 \to S^2$ (defined by (22.32)) is a holomorphic diffeomorphism. If also $B \in G\ell(2,\mathbb{C})$, $\Lambda_{AB} = \Lambda_A \circ \Lambda_B$.

Exercise 4 implies that the class of holomorphic diffeomorphisms $\varphi : S^2 \to S^2$ is equal to the class of maps Λ_A as A runs over $G\ell(2,\mathbb{C})$.

19. Given 3 distinct points $a, b, c \in \mathbb{C}$, show that there exists a linear fractional transformation

$$L(z) = \alpha \, \frac{z-a}{z-c} \quad \text{such that} \quad L(a) = 0, \; L(b) = 1, \; L(c) = \infty.$$

Deduce that if $\{p, q, r\}$ and $\{p', q', r'\}$ are two sets of 3 distinct points in S^2, then there exists a holomorphic diffeomorphism $\varphi : S^2 \to S^2$ such that

$$\varphi(p) = p', \quad \varphi(q) = q', \quad \varphi(r) = r'.$$

Show that such φ is unique.

20. Let γ be a circle in S^2. Show that

$$\begin{aligned} &\mathcal{S}\gamma \text{ is a circle in } \mathbb{C} \text{ if } e_3 \notin \gamma, \text{ and} \\ &\widetilde{\mathcal{S}}\gamma \text{ is an exended line in } \widehat{\mathbb{C}} \text{ if } e_3 \in \gamma. \end{aligned}$$

Hint. For the first part, take a rotation R such that $\gamma_0 = R\gamma$ is a circle centered at e_3. Show directly that $\mathcal{S}\gamma_0 = \sigma$ is a circle in $\mathbb{C}$. Deduce that $\mathcal{S}\gamma = L_A^{-1}\sigma$ where $L_A = \mathcal{S} \circ R \circ \mathcal{S}^{-1}$ is a linear fractional transformation. Then apply Proposition 20.4 to $L_A^{-1}\sigma$.

21. Let $\sigma \subset \mathbb{C}$ be a circle. Show that $\mathcal{S}^{-1}(\sigma)$ is a circle in S^2.
Hint. Pick $a \in \sigma$ and let $p = \mathcal{S}^{-1}(a) \in S^2$. Pick a rotation R such that $R(p) = e_3$, so $R \circ \mathcal{S}^{-1}(a) = e_3$. Now $\gamma = R \circ \mathcal{S}^{-1}(\sigma)$ is a curve in S^2, and we want to show that it is a circle.

Indeed, $\mathcal{S}(\gamma) = \mathcal{S} \circ R \circ \mathcal{S}^{-1}(\sigma) = L(\sigma)$, with L a linear fractional transformation. $L(\sigma)$ contains $\mathcal{S}(e_3) = \infty$, so $\mathcal{S}(\gamma) = L(\sigma) = \ell$ is an extended line (by Proposition 20.4). Then $\gamma = \mathcal{S}^{-1}(\ell)$, which is seen to be a circle in S^2. In fact, $\mathcal{S}^{-1}(\ell)$ is the intersection of S^2 and the plane through ℓ and e_3.

22. Show that if γ is a circle in S^2 and $\varphi : S^2 \to S^2$ is a holomorphic diffeomorphism, then $\varphi(\gamma)$ is a circle in S^2.
Hint. Use Exercises 20–21 and Proposition 20.4.

23. The Riemann mapping theorem

The following result is known as the Riemann mapping theorem.

Theorem 23.1. *Assume $\Omega \subset \mathbb{C}$ is a simply connected domain, and $\Omega \neq \mathbb{C}$. Then there exists a holomorphic diffeomorphism*

$$f : \Omega \longrightarrow D \tag{23.1}$$

of Ω onto the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$.

The proof given here is due to P. Koebe. To begin the proof, we have shown in Proposition 20.5 that such a domain Ω is conformally equivalent to a bounded domain in $\mathbb{C}$, so it suffices to treat the bounded case. Thus from here on we assume Ω is bounded. Fix $p \in \Omega$.

We define $\mathcal{F}$ to be the set of holomorphic maps $g : \Omega \to D$ that have the following three properties:

(i) g is one-to-one (we say *univalent*),
(ii) $g(p) = 0$,
(iii) $g'(p) > 0$.

(Note that (i) implies g' is nowhere zero on Ω; cf. Exercise 3 in §17.) For $\Omega \subset \mathbb{C}$ bounded it is clear that $b(z - p)$ belongs to $\mathcal{F}$ for small $b > 0$, so $\mathcal{F}$ is nonempty.

Note that if $R = \operatorname{dist}(p, \partial\Omega)$, then, by (5.32),

$$|g'(p)| \le \frac{1}{R}, \quad \forall\, g \in \mathcal{F}, \tag{23.2}$$

so we can set

$$A = \sup \{g'(p) : g \in \mathcal{F}\}, \tag{23.3}$$

and we have $A < \infty$. Pick $g_\nu \in \mathcal{F}$ such that $g_\nu'(p) \to A$ as $\nu \to \infty$. A normal family argument from §21 shows that there exists a subsequence $g_\nu \to f$ locally uniformly on Ω, and

$$f : \Omega \longrightarrow D \tag{23.4}$$

is holomorphic and satisfies

$$f(p) = 0, \quad f'(p) = A. \tag{23.5}$$

We claim this function provides the holomorphic diffeomorphism (23.1). There are two parts to showing this, treated in the next two lemmas.

Lemma 23.2. *In (23.4), f is one-to-one.*

Proof. Suppose there exist distinct $z_1, z_2 \in \Omega$ such that $f(z_1) = f(z_2) = w \in D$. Let $\overline{U} \subset \Omega$ be a smoothly bounded region such that $z_1, z_2 \in U$ and such that $f(\gamma)$ is disjoint from w, where $\gamma = \partial U$.

By the argument principle (Proposition 17.5), $f(\gamma)$ winds (at least) twice about w. But each $g_\nu(\gamma)$ winds only once about w. Since $g_\nu \to f$ uniformly on γ, this is a contradiction.

Lemma 23.3. *In (23.4), f is onto.*

To prove Lemma 23.3, let $\mathcal{O} = f(\Omega)$ and assume $\mathcal{O}$ is not all of D. In other words, we have the following situation, for some $q \in D$:

$$\begin{aligned} &\mathcal{O} \subset D \ \text{ is open and simply connected, } \ 0 \in \mathcal{O}, \\ &q \in D \ \text{ but } \ q \notin \mathcal{O}. \end{aligned} \tag{23.6}$$

We will bring in the *Koebe transformation*

$$K = K_{\mathcal{O},q} : \mathcal{O} \longrightarrow D, \tag{23.7}$$

defined as follows. First, define

$$F : \mathcal{O} \longrightarrow D \tag{23.8}$$

by

$$F(z) = \sqrt{\frac{z - q}{1 - \overline{q}z}} = \sqrt{\varphi_q(z)}, \tag{23.9}$$

where, for a given $b \in D$, we take

$$\varphi_b(z) = \frac{z - b}{1 - \overline{b}z}, \quad \varphi_b : D \to D, \ \text{ one-to-one and onto.} \tag{23.10}$$

Cf. Exercise 6 of §20. Note that $\varphi_q(q) = 0$. The hypotheses of (23.6) imply $\varphi_q(z) \neq 0$ for $z \in \mathcal{O}$, and hence, since $\mathcal{O}$ is simply connected, there is a holomorphic function F on $\mathcal{O}$ satisfying (23.9), by Proposition 10.2 and its corollary (10.17), obtained once one specifies $F(0) = (-q)^{1/2}$. Having F, we define K in (23.7) by

$$K(z) = \frac{|F'(0)|}{F'(0)} \, \varphi_{F(0)}(F(z)). \tag{23.11}$$

We will verify below that $F'(0) \neq 0$. From (23.10) we see that

$$\varphi_b(b) = 0, \quad \text{hence } \ K(0) = 0. \tag{23.16}$$

The following result will enable us to prove Lemma 23.3.

Proposition 23.4. *In the setting of (23.6), the Koebe transformation (23.7) is one-to-one and satisfies* $K(0) = 0$, *and*

$$K'(0) > 1. \tag{23.13}$$

Given this proposition, we have the following proof of Lemma 23.3. If f is not onto in (23.4), then $\mathcal{O} = f(\Omega) \subset D$ satisfies (23.6) for some q. Then

$$g = K \circ f : \Omega \longrightarrow D, \tag{23.14}$$

is one-to-one, $g(p) = 0$, and

$$g'(p) = K'(0)f'(p) = K'(0)A > A. \tag{23.15}$$

This contradiction of (23.3) yields Lemma 23.3, and hence we have the Riemann mapping theorem, once we prove Proposition 23.4.

Proof of Proposition 23.4. That K is one-to-one in (23.7) is straightforward. Our task is to calculate $K'(0)$ and verify (23.13). To start, we have from (23.11) that

$$K'(z) = \frac{|F'(0)|}{F'(0)}\, \varphi'_{F(0)}(F(z))F'(z), \tag{23.16}$$

hence

$$K'(0) = |F'(0)|\varphi'_{F(0)}(F(0)). \tag{23.17}$$

Now (23.10) yields, for $b \in D$,

$$\varphi'_b(z) = \frac{1 - |b|^2}{(1 - \bar{b}z)^2}, \quad \text{hence} \;\; \varphi'_b(b) = \frac{1}{1 - |b|^2}, \quad \varphi'_b(0) = 1 - |b|^2, \tag{23.18}$$

so

$$K'(0) = \frac{|F'(0)|}{1 - |F(0)|^2}. \tag{23.19}$$

Next, (23.9) yields

$$F'(z) = \frac{1}{2}\varphi_q(z)^{-1/2}\varphi'_q(z) = \frac{\varphi'_q(z)}{2F(z)}, \tag{23.20}$$

hence

$$F'(0) = \frac{1 - |q|^2}{2F(0)}, \quad \text{so} \;\; |F'(0)| = \frac{1 - |q|^2}{2|q|^{1/2}}, \tag{23.21}$$

since $|F(0)| = |q|^{1/2}$. Consequently,

$$K'(0) = \frac{1-|q|^2}{2|q|^{1/2}} \cdot \frac{1}{1-|q|} = \frac{1+|q|}{2|q|^{1/2}}. \tag{23.22}$$

This yields (23.12), together with the more precise result

$$K'(0) = 1 + \frac{(1-|q|^{1/2})^2}{2|q|^{1/2}}. \tag{23.23}$$

At this point, the proof of the Riemann mapping theorem is complete.

Remark. While the computaton of $K'(0)$ in (23.16)–(23.23) is elementary, we point out that there is an interesting conceptual approach to this result, using the *Poincaré metric* on the unit disk, and its invariance under the linear fractional transformations φ_b. For this, see Appendix E, particularly Proposition E.6.

Riemann's original idea for proving Theorem 23.1 involved solving a Dirichlet problem to find a harmonic function on Ω satisfying a certain boundary condition. In §29 we show how a related construction works when $\Omega \subset \mathbb{C}$ is a smoothly bounded domain whose boundary consists of two smooth, simple closed curves. In such a case, there is a holomorphic diffeomorphism

$$f : \Omega \longrightarrow \mathfrak{A}_\rho \tag{23.23}$$

of Ω onto an annulus

$$\mathfrak{A}_\rho = \{z \in \mathbb{C} : \rho < |z| < 1\}, \tag{23.24}$$

for some $\rho \in (0,1)$. See Proposition 29.10.

Exercises

1. Suppose $h_\nu : D \to D$ are holomorphic, univalent maps satisfying

$$h_\nu(0) = 0, \quad h'_\nu(0) > 0, \quad h_\nu(D) \supset D_{\rho_\nu}, \quad \rho_\nu \to 1.$$

Show that, for $z \in D$,

$$\rho_\nu |z| \le |h_\nu(z)| \le |z|.$$

Then show that

$$h_\nu(z) \to z \quad \text{locally uniformly on} \quad D.$$

Hint. Use a normal families argument, and show that any limit $h_{\nu_k} \to g$ must have the property that $g(D) = D$, and conclude that $g(z) = z$. (The argument principle may be useful.)

2. Suppose Ω is a bounded, simply connected domain, $p \in \Omega$, and $f_\nu : \Omega \to D$ are univalent holomorphic maps satisfying

$$f_\nu(p) = 0, \quad f_\nu'(p) > 0, \quad f_\nu(\Omega) \supset D_{\rho_\nu}, \quad \rho_\nu \to 1.$$

Show that $f_\nu \to f$ locally uniformly on Ω, where $f : \Omega \to D$ is the Riemann mapping function given by Theorem 23.1.
Hint. Consider $h_\nu = f_\nu \circ f^{-1} : D \to D$.

3. Let $f_1 : \Omega \to D$ be a univalent holomorphic mapping satisfying $f_1(p) = 0$, $f_1'(p) = A_1 > 0$ (i.e., an element of $\mathcal{F}$). Asuming f_1 is not onto, choose $q_1 \in D \setminus f_1(\Omega)$ with minimal possible absolute value. Construct $f_2 \in \mathcal{F}$ as

$$f_2(z) = \frac{|F_1'(p)|}{F_1'(p)}\, \varphi_{F_1(p)}\big(F_1(z)\big), \quad F_1(z) = \sqrt{\varphi_{q_1}\big(f_1(z)\big)}.$$

Show that

$$A_2 = f_2'(p) = \frac{1 + |q_1|}{2|q_1|^{1/2}}\, A_1.$$

Take $q_2 \in D \setminus f_2(\Omega)$ with minimal absolute value and use this to construct f_3. Continue, obtaining $f_4, f_5, \dots$. Show that at least one of the following holds:

$$f_\nu'(p) \to A, \quad \text{or} \;\; f_\nu(\Omega) \supset D_{\rho_\nu}, \quad \rho_\nu \to 1,$$

with A as in (23.3). Deduce that $f_\nu \to f$ locally uniformly on Ω, where $f : \Omega \to D$ is the Riemann mapping function.
Hint. If $|q_1|$ is not very close to 1, then A_2 is somewhat larger than A_1. Similarly for $A_{\nu+1}$ compared with A_ν.

4. Give an example of a bounded, simply connected domain $\Omega \subset \mathbb{C}$ whose boundary is not path connected.

5. Give an example of a bounded, simply connected domain $\Omega \subset \mathbb{C}$ whose boundary is path connected, but such that $\partial\Omega$ is not a simple closed curve.

24. Boundary behavior of conformal maps

Throughout this section we assume that $\Omega \subset \mathbb{C}$ is a simply connected domain and $f : \Omega \to D$ is a holomorphic diffeomorphism, where $D = \{z : |z| < 1\}$ is the unit disk. We look at some cases where we can say what happens to $f(z)$ as z approaches the boundary $\partial\Omega$. The following is a simple but useful result.

Lemma 24.1. *We have*

$$z \to \partial\Omega \Longrightarrow |f(z)| \to 1. \tag{24.1}$$

Proof. For each $\varepsilon > 0$, $\overline{D}_{1-\varepsilon} = \{z : |z| \le 1-\varepsilon\}$ is a compact subset of D, and $K_\varepsilon = f^{-1}(\overline{D}_{1-\varepsilon})$ is a compact subset of Ω. As soon as $z \notin K_\varepsilon$, $|f(z)| > 1-\varepsilon$.

We now obtain a local regularity result.

Proposition 24.2. *Assume $\gamma : (a,b) \to \mathbb{C}$ is a simple real analytic curve, satisfying $\gamma'(t) \ne 0$ for all t. Assume γ is part of $\partial\Omega$, with all points near to and on the left side of γ (with its given orientation) belonging to Ω, and all points on the right side of and sufficiently near to γ belong to $\mathbb{C} \setminus \overline{\Omega}$. Then there is a neighborhood $\mathcal{V}$ of γ in $\mathbb{C}$ and a holomorphic extension F of f to $F : \Omega \cup \mathcal{V} \to \mathbb{C}$. We have $F(\gamma) \subset \partial D$ and $F'(\zeta) \ne 0$ for all $\zeta \in \gamma$.*

Proof. The hypothesis on γ implies (via Proposition 10.5) that there exists a neighborhood $\mathcal{O}$ of (a,b) in $\mathbb{C}$ and a univalent holomorphic map $\Gamma : \mathcal{O} \to \mathbb{C}$ extending γ. Say $\mathcal{V} = \Gamma(\mathcal{O})$. See Fig. 24.1. We can assume $\mathcal{O}$ is symmetric with respect to reflection across $\mathbb{R}$. Say $\mathcal{O}^\pm = \{\zeta \in \mathcal{O} : \pm \operatorname{Im} \zeta > 0\}$.

We have $f \circ \Gamma : \mathcal{O}^+ \to D$ and

$$z_\nu \in \mathcal{O}^+,\ z_\nu \to L = \mathcal{O} \cap \mathbb{R} \Longrightarrow |f \circ \Gamma(z_\nu)| \to 1. \tag{24.2}$$

It follows from the form of the Schwarz reflection principle given in §13 that $g = f \circ \Gamma|_{\mathcal{O}^+}$ has a holomorphic extension $G : \mathcal{O} \to \mathbb{C}$, and $G : L \to \partial D$. Say $\mathcal{U} = G(\mathcal{O})$, as in Fig. 24.1. Note that $\mathcal{U}$ is invariant under $z \mapsto \overline{z}^{-1}$.

Then we have a holomorphic map

$$F = G \circ \Gamma^{-1} : \mathcal{V} \longrightarrow \mathcal{U}. \tag{24.3}$$

It is clear that $F = f$ on $\mathcal{V} \cap \Omega$. It remains to show that $F'(\zeta) \ne 0$ for $\zeta \in \gamma$. It is equivalent to show that $G'(t) \ne 0$ for $t \in L$. To see this, note that G is univalent on $\mathcal{O}^+$; $G|_{\mathcal{O}^+} = g|_{\mathcal{O}^+} : \mathcal{O}^+ \to D$. Hence G is univalent on $\mathcal{O}^-$; $G|_{\mathcal{O}^-} : \mathcal{O}^- \to \mathbb{C} \setminus \overline{D}$. The argument principle then gives $G'(t) \ne 0$ for $t \in L$. This finishes the proof.

Using Proposition 24.2 we can show that if $\partial\Omega$ is real analytic then f extends to a homeomorphism from $\overline{\Omega}$ to $\overline{D}$. We want to look at a class of domains Ω with non-smooth boundaries for which such a result holds. Clearly a necessary condition is that $\partial\Omega$ be homeomorphic to S^1, i.e., that $\partial\Omega$ be a Jordan curve. C. Caratheodory proved that this condition is also sufficient. A proof can be found in [Ts]. Here we establish a simpler result, which nevertheless will be seen to have interesting consequences.

Proposition 24.3. *In addition to the standing assumptions on Ω, assume it is bounded and that $\partial\Omega$ is a simple closed curve that is a finite union of real analytic curves. Then the Riemann mapping function f extends to a homeomorphism $f : \overline{\Omega} \to \overline{D}$.*

Proof. By Proposition 24.2, f extends to the real analytic part of $\partial\Omega$, and the extended f maps these curves diffeomorphically onto open intervals in ∂D. Let J_1 and J_2 be real analytic curves in $\partial\Omega$, meeting at p, as illustrated in Fig. 24.2, and denote by I_ν the images in ∂D. We claim that I_1 and I_2 meet, i.e., the endpoints q_1 and q_2 pictured in Fig. 24.2 coincide.

Let γ_r be the intersection $\Omega \cap \{z : |z-p| = r\}$, and let $\ell(r)$ be the length of $f(\gamma_r) = \sigma_r$. Clearly $|q_1 - q_2| \le \ell(r)$ for all (small) $r > 0$, so we would like to show that $\ell(r)$ is small for (some) small r.

We have $\ell(r) = \int_{\gamma_r} |f'(z)|\, ds$, and Cauchy's inequality implies

$$\frac{\ell(r)^2}{r} \le 2\pi \int_{\gamma_r} |f'(z)|^2\, ds. \tag{24.4}$$

If $\ell(r) \ge \delta$ for $\varepsilon \le r \le R$, then integrating over $r \in [\varepsilon, R]$ yields

$$\delta^2 \log \frac{R}{\varepsilon} \le 2\pi \iint\limits_{\Omega(\varepsilon,R)} |f'(z)|^2\, dx\, dy = 2\pi \cdot \operatorname{Area} f\big(\Omega(\varepsilon, R)\big) \le 2\pi^2, \tag{24.5}$$

where $\Omega(\varepsilon, R) = \Omega \cap \{z : \varepsilon \le |z-p| \le R\}$. Since $\log(1/\varepsilon) \to \infty$ as $\varepsilon \searrow 0$, there exists arbitrarily small $r > 0$ such that $\ell(r) < \delta$. Hence $|q_1 - q_2| < \delta$, so $q_1 = q_2$, as asserted.

It readily follows that taking $f(p) = q_1 = q_2$ extends f continuously at p. Such an extension holds at other points of $\partial\Omega$ where two real analytic curves meet, so we have a continuous extension $f : \overline{\Omega} \to \overline{D}$. This map is also seen to be one-to-one and onto. Since $\overline{\Omega}$ and $\overline{D}$ are compact, this implies it is a homeomorphism, i.e., $f^{-1} : \overline{D} \to \overline{\Omega}$ is continuous (cf. Exercise 9 below).

Exercises

1. Suppose $f : \Omega \to D$ is a holomorphic diffeomorphism, extending to a homeomorphism $f : \overline{\Omega} \to \overline{D}$. Let $g \in C(\partial\Omega)$ be given. Show that the Dirichlet problem

$$\Delta u = 0 \ \text{ in } \ \Omega, \quad u\big|_{\partial\Omega} = g \tag{24.6}$$

has a unique solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$, given by $u = v \circ f$, where

$$\Delta v = 0 \ \text{ in } \ D, \quad v\big|_{\partial D} = g \circ f^{-1}\big|_{\partial D}, \tag{24.7}$$

the solution to (24.7) having been given in §13.

2. Verify that for $f : \Omega \to \mathbb{C}$ holomorphic, if we consider $Df(z) \in M(2,\mathbb{R})$, then $\det Df(z) = |f'(z)|^2$, and explain how this yields the identity (24.5).

3. Let $\Omega \subset \mathbb{C}$ satisfy the hypotheses of Proposition 24.3. Pick distinct $p_1, p_2, p_3 \in \partial\Omega$ such that, with its natural orientation, $\partial\Omega$ runs from p_1 to p_2 to p_3, and back to p_1. Pick $q_1, q_2, q_3 \in \partial D$ with the analogous properties. Show that there exists a unique holomorphic diffeomorphism $f : \Omega \to D$ whose continuous extension to $\overline{\Omega}$ takes p_j to q_j, $1 \le j \le 3$.
Hint. First tackle the case $\Omega = D$.

In Exercises 4–6, pick $p > 0$ and let $\mathcal{R} \subset \mathbb{C}$ be the rectangle with vertices at $-1, 1, 1 + ip$, and $-1 + ip$. Let $\varphi : \overline{\mathcal{R}} \to \overline{D}$ be the Riemann mapping function such that

$$\varphi(-1) = -i, \quad \varphi(0) = 1, \quad \varphi(1) = i. \tag{24.8}$$

Define $\Phi : \mathcal{R} \to \mathcal{U}$ (the upper half plane) by

$$\Phi(z) = -i\,\frac{\varphi(z) - 1}{\varphi(z) + 1}. \tag{24.9}$$

4. Show that $\varphi(ip) = -1$ (so $\Phi(z) \to \infty$ as $z \to ip$). Show that Φ extends continuously to $\overline{\mathcal{R}} \setminus \{ip\} \to \mathbb{C}$ and

$$\Phi(-1) = -1, \quad \Phi(0) = 0, \quad \Phi(1) = 1. \tag{24.10}$$

5. Show that you can apply the Schwarz reflection principle repeatedly and extend Φ to a meromorphic function on $\mathbb{C}$, with simple poles at $ip + 4k + 2i\ell p$, $k, \ell \in \mathbb{Z}$. Show that

$$\Phi(z + 4) = \Phi(z + 2ip) = \Phi(z). \tag{24.11}$$

Hint. To treat reflection across the top boundary of $\mathcal{R}$, apply Schwarz reflection to $1/\Phi$.
Remark. We say Φ is doubly periodic, with periods 4 and $2ip$.

6. Say $\Phi(1 + ip) = r$. Show that $r > 0$, that $\Phi(-1 + ip) = -r$, and that

$$\Phi\Big(\frac{ip}{2} - z\Big) = \frac{r}{\Phi(z)}. \tag{24.12}$$

7. Let $\mathcal{T} \subset \mathbb{C}$ be the equilateral triangle with vertices at $-1, 1$, and $\sqrt{3}i$. Let $\Psi : \mathcal{T} \to \mathcal{U}$ be the holomorphic diffeomorphism with boundary values

$$\Psi(-1) = -1, \quad \Psi(0) = 0, \quad \Psi(1) = 1. \tag{24.13}$$

Use Schwarz reflection to produce a meromorphic extension of Ψ to $\mathbb{C}$, which is doubly periodic. Show that

$$\Psi(z + 2\sqrt{3}i) = \Psi(z + 3 + \sqrt{3}i) = \Psi(z).$$

What are the poles of Ψ? Cf. Fig. 24.3.

8. In the context of Exercise 7, show that

$$\Psi(i\sqrt{3}) = \infty.$$

Let

$$\Psi^{\#}\Big(\frac{i}{\sqrt{3}} + z\Big) = \Psi\Big(\frac{i}{\sqrt{3}} + e^{2\pi i/3} z\Big).$$

Show that $\Psi^{\#} : \mathcal{T} \to \mathcal{U}$ is a holomorphic diffeomorphism satisfying

$$\Psi^{\#}(-1) = 1, \quad \Psi^{\#}(1) = \infty, \quad \Psi^{\#}(i\sqrt{3}) = -1.$$

Conclude that

$$\Psi^{\#}(z) = \varphi \circ \Psi(z),$$

where $\varphi : \mathcal{U} \to \mathcal{U}$ is the holomorphic diffeomorphism satisfying

$$\varphi(-1) = 1, \quad \varphi(1) = \infty, \quad \varphi(\infty) = -1,$$

so

$$\varphi(z) = -\frac{z+3}{z-1}.$$

9. Let X and Y be compact metric spaces, and assume $f : X \to Y$ is continuous, one-to-one, and onto. Show that f is a homeomorphism, i.e., $f^{-1} : Y \to X$ is continuous.
Hint. You are to show that if $f(x_j) = y_j \to y$, then $x_j \to f^{-1}(y)$. Now since X is compact, every subsequence of (x_j) has a further subsequence that converges to some point in X...

25. Covering maps

The concept of covering map comes from topology. Generally, if E and X are topological spaces, a continuous map $\pi : E \to X$ is said to be a covering map provided every $p \in X$ has a neighborhood U such that $\pi^{-1}(U)$ is a disjoint union of open sets $S_j \subset E$, each of which is mapped homeomorphically by π onto U. In topology one studies conditions under which a continuous map $f : Y \to X$ *lifts* to a continuous map $\tilde{f} : Y \to E$, so that $f = \pi \circ \tilde{f}$. Here is one result, which holds when Y is simply connected. By definition, a connected, locally path connected space Y is said to be simply connected provided that whenever γ is a closed path in Y, there exists a continuous family γ_s $(0 \le s \le 1)$ of closed paths in Y such that $\gamma_1 = \gamma$ and the image of γ_0 consists of a single point. (Cf. §5.) To say Y is locally path connected is to say that if $q \in Y$ and V is an open neighborhood of q, then there exists an open set $V_0 \subset V$ such that $p \in V_0$ and V_0 is path connected. Clearly each nonempty open subset of $\mathbb{C}$ is locally path connected.

Proposition 25.1. *Assume E, X, and Y are all connected and locally path-connected, and $\pi : E \to X$ is a covering map. If Y is simply connected, any continuous map $f : Y \to X$ lifts to $\tilde{f} : Y \to E$.*

A proof can be found in Chapter 6 of [Gr]. See also Chapter 8 of [Mun]. Here our interest is in holomorphic covering maps $\pi : \Omega \to \mathcal{O}$, where Ω and $\mathcal{O}$ are domains in $\mathbb{C}$. The following is a simple but useful result.

Proposition 25.2. *Let U, Ω and $\mathcal{O}$ be connected domains in $\mathbb{C}$. Assume $\pi : \Omega \to \mathcal{O}$ is a holomorphic covering map and $f : U \to \mathcal{O}$ is holomorphic. Then any continuous lift $\tilde{f} : U \to \Omega$ of f is also holomorphic.*

Proof. Take $q \in U$, $p = f(q) \in \mathcal{O}$. Let $\mathcal{V}$ be a neighborhood of p such that $\pi^{-1}(\mathcal{V})$ is a disjoint union of open sets $S_j \subset \Omega$ with $\pi : S_j \to \mathcal{V}$ a (holomorphic) homeomorphism. As we have seen (in several previous exercises) this implies $\pi : S_j \to \mathcal{V}$ is actually a holomorphic diffeomorphism.

Now $\tilde{f}(q) \in S_k$ for some k, and $\tilde{f}^{-1}(S_k) = f^{-1}(\mathcal{V}) = U_q$ is a neighborhood of q in U. We have

$$\tilde{f}\big|_{U_q} = \pi^{-1} \circ f\big|_{U_q}, \tag{25.1}$$

which implies $\tilde{f}$ is holomorphic on U_q, for each $q \in U$. This suffices.

Proposition 25.3. *The following are holomorphic covering maps:*

$$\exp : \mathbb{C} \longrightarrow \mathbb{C} \setminus \{0\}, \quad \mathrm{Sq} : \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C} \setminus \{0\}, \tag{25.2}$$

where $\exp(z) = e^z$ *and* $\mathrm{Sq}(z) = z^2$.

Proof. Exercise.

Combined with Propositions 25.1–25.2, this leads to another demonstration of Proposition 10.2.

Corollary 25.4. *If $U \subset \mathbb{C}$ is simply connected and $f : U \to \mathbb{C} \setminus \{0\}$ is holomorphic, then there exist holomorphic functions g and h on U such that*

$$f(z) = e^{g(z)}, \quad f(z) = h(z)^2. \tag{25.3}$$

We say $g(z)$ is a branch of $\log f(z)$ and $h(z)$ is a branch of $\sqrt{f(z)}$, over U.

Exercises

1. Here we recall the aproach to Corollary 25.4 taken in §10. As in Corollary 25.4, suppose $U \subset \mathbb{C}$ is simply connected and $f : U \to \mathbb{C} \setminus \{0\}$ is holomorphic.

(a) Show that $f'/f : U \to \mathbb{C}$ is holomorphic.

(b) Pick $p \in U$ and define

$$\varphi(z) = \int_p^z \frac{f'(\zeta)}{f(\zeta)}\, d\zeta.$$

Show this is independent of the choice of path in U from p to z, and it gives a holomorphic function $\varphi : U \to \mathbb{C}$.

(c) Use φ to give another proof of Corollary 25.4 (particularly the existence of $g(z)$ in (25.3)).

2. Let $\widetilde{\Sigma}_+ = \{z \in \mathbb{C} : |\operatorname{Re} z| < \pi/2 \text{ and } \operatorname{Im} z > 0\}$. Use the analysis of (4.20)–(4.24) to show that

$$\sin : \widetilde{\Sigma}_+ \longrightarrow U \text{ is a holomorphic diffeomorphism,}$$

where $U = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. With this in hand, use the Schwarz reflection principle to show that

$$\sin : U \longrightarrow \mathbb{C} \setminus [-1, 1] \text{ is a covering map.}$$

3. Produce a holomorphic covering map of

$$\{z \in \mathbb{C} : |\operatorname{Re} z| < 1\} \text{ onto } \{z \in \mathbb{C} : 1 < |z| < 2\}.$$

4. Produce a holomorphic covering of

$$\{z \in \mathbb{C} : |z| > 1\}$$

by some simply connected domain $\Omega \subset \mathbb{C}$.

5. Produce a holomorphic covering map of

$$\{z \in \mathbb{C} : 0 < |z| < 1\}$$

by the upper half plane $U = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.

26. The disk covers $\mathbb{C} \setminus \{0,1\}$

Our main goal in this section is to prove that the unit disk D covers the complex plane with two points removed, holomorphically. Formally:

Proposition 26.1. *There exists a holomorphic covering map*

$$\Phi : D \longrightarrow \mathbb{C} \setminus \{0,1\}. \tag{26.1}$$

The proof starts with an examination of the following domain Ω. It is the subdomain of the unit disk D whose boundary consists of three circular arcs, intersecting ∂D at right angles, at the points $\{1, e^{2\pi i/3}, e^{-2\pi i/3}\}$. See Fig. 26.1. If we denote by

$$\varphi : D \longrightarrow \mathcal{U} = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\} \tag{26.2}$$

the linear fractional transformation of D onto $\mathcal{U}$ with the property that $\varphi(1) = 0$, $\varphi(e^{2\pi i/3}) = 1$, and $\varphi(e^{-2\pi i/3}) = \infty$, the image $\widetilde{\Omega} = \varphi(\Omega)$ is pictured in Fig. 26.2.

The Riemann mapping theorem guarantees that there is a holomorphic diffeomorphism

$$\psi : \Omega \longrightarrow D, \tag{26.3}$$

and by Proposition 24.3 this extends to a homeomorphism $\psi : \overline{\Omega} \to \overline{D}$. We can take ψ to leave the points 1 and $e^{\pm 2\pi i/3}$ fixed. Conjugation with the linear fractional transformation φ gives a holomorphic diffeomorphism

$$\Psi = \varphi \circ \psi \circ \varphi^{-1} : \widetilde{\Omega} \longrightarrow \mathcal{U}, \tag{26.4}$$

and Ψ extends to map $\partial\widetilde{\Omega}$ onto the real axis, with $\Psi(0) = 0$ and $\Psi(1) = 1$.

Now the Schwarz reflection principle can be applied to Ψ, reflecting across the vertical lines in $\partial\widetilde{\Omega}$, to extend Ψ to the regions $\widetilde{\mathcal{O}}_2$ and $\widetilde{\mathcal{O}}_3$ in Fig. 26.2. A variant extends Ψ to $\widetilde{\mathcal{O}}_1$. (Cf. Exercise 1 in §8.) Note that this extension maps the closure in $\mathcal{U}$ of $\widetilde{\Omega} \cup \widetilde{\mathcal{O}}_1 \cup \widetilde{\mathcal{O}}_2 \cup \widetilde{\mathcal{O}}_3$ onto $\mathbb{C} \setminus \{0,1\}$. Now we can iterate this reflection process indefinitely, obtaining

$$\Psi : \mathcal{U} \longrightarrow \mathbb{C} \setminus \{0,1\}. \tag{26.5}$$

Furthermore, this is a holomorphic covering map. Then $\Phi = \Psi \circ \varphi$ gives the desired holomorphic covering map (26.1).

Exercises

1. Show that the map $\varphi : D \to \mathcal{U}$ in (26.2) is given by

$$\varphi(z) = -\omega \frac{z-1}{z-\omega^2}, \quad \omega = e^{2\pi i/3}.$$

Show that

$$\varphi(-\omega) = -1, \quad \varphi(-\omega^2) = \frac{1}{2}.$$

For use below, in addition to $z \mapsto \overline{z}$, we consider the following anti-holomorphic involutions of $\mathbb{C}$: $z \mapsto z^*$, $z \mapsto z^\circ$, $z \mapsto z^\dagger$, and $z \mapsto z^c$, given by

$$\begin{gathered} z^* = \frac{1}{\overline{z}}, \quad \Bigl(\frac{1}{2} + z\Bigr)^\circ = \frac{1}{2} + \frac{z^*}{4}, \\ (x+iy)^\dagger = -x + iy, \quad \Bigl(\frac{1}{2} + z\Bigr)^c = \frac{1}{2} + z^\dagger. \end{gathered} \tag{26.6}$$

2. With $\Psi : \mathcal{U} \to \mathbb{C} \setminus \{0,1\}$ as in (26.5), show that

$$\Psi(z^\dagger) = \overline{\Psi(z)}, \quad \Psi(z^\circ) = \overline{\Psi(z)}, \quad \Psi(z+2) = \Psi(z). \tag{26.7}$$

3. Show that

$$\Psi(z^c) = \Psi(z)^c. \tag{26.8}$$

4. Show that $z \mapsto z^*$ leaves $\widetilde{\Omega}$ invariant, and that

$$\Psi(z^*) = \Psi(z)^*. \tag{26.9}$$

Hint. First establish this identity for $z \in \widetilde{\Omega}$. Use Exercise 1 to show that (26.9) is equivalent to the statement that ψ in (26.3) commutes with reflection across the line through 0 and $e^{2\pi i/3}$, while (26.8) is equivalent to the statement that ψ commutes with reflection across the line through 0 and $e^{-2\pi i/3}$.

5. Show that $\overline{z^*} = 1/z$ and $(z^*)^\dagger = -1/z$. Deduce from (26.7) and (26.9) that

$$\Psi\Bigl(-\frac{1}{z}\Bigr) = \frac{1}{\Psi(z)}. \tag{26.10}$$

6. Show that $(z^\dagger)^c = z+1$ and $(\overline{z})^c = 1-z$. Deduce from (26.7) and (26.8) that

$$\Psi(z+1) = 1 - \Psi(z). \tag{26.11}$$

As preparation for Exercises 7–9, the reader should recall §22.

7. Show that F_{01}, $F_{0\infty} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ ($\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$), given by

$$F_{01}(w) = 1 - w, \quad F_{0\infty}(w) = \frac{1}{w}, \tag{26.12}$$

are holomorphic automorphisms of $\widehat{\mathbb{C}}$ that leave $\mathbb{C} \setminus \{0, 1\}$ invariant, F_{01} fixes ∞ and switches 0 and 1, while $F_{0\infty}$ fixes 1 and switches 0 and ∞. Show that these maps generate a group $\mathcal{G}$ of order 6, of automorphisms of $\widehat{\mathbb{C}}$ and of $\mathbb{C} \setminus \{0, 1\}$, that permutes $\{0, 1, \infty\}$ and is isomorphic to the permutation group S_3 on three objects. Show that

$$F_{1\infty}(w) = F_{0\infty} \circ F_{01} \circ F_{0\infty}(w) = \frac{w}{w-1} \tag{26.13}$$

is the element of $\mathcal{G}$ that fixes 0 and switches 1 and ∞. Show that the rest of the elements of $\mathcal{G}$ consist of the identity map, $w \mapsto w$, and the following two maps:

$$\begin{aligned} F_{01\infty}(w) &= F_{0\infty} \circ F_{01}(w) = \frac{1}{1-w}, \\ F_{\infty 10}(w) &= F_{01} \circ F_{0\infty}(w) = \frac{w-1}{w}. \end{aligned} \tag{26.14}$$

8. Show that the transformations in $\mathcal{G}$ listed in (26.12)–(26.14) have the following fixed points.

Element	Fixed points
$F_{0\infty}$	$1,\ -1 = A_1$
F_{01}	$\infty,\ \frac{1}{2} = A_2$
$F_{1\infty}$	$0,\ 2 = A_3$
$F_{01\infty}$	$e^{\pm \pi i/3} = B_{\pm}$
$F_{\infty 10}$	$e^{\pm \pi i/3} = B_{\pm}$

See Figure 26.3. Show that the elements of $\mathcal{G}$ permute $\{A_1, A_2, A_3\}$ and also permute $\{B_+, B_-\}$.

9. We claim there is a holomorphic map

$$H : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}, \tag{26.15}$$

satisfying

$$H(F(w)) = H(w), \quad \forall\, F \in \mathcal{G}, \tag{26.16}$$

such that

$$H(0) = H(1) = H(\infty) = \infty, \tag{26.17}$$

with poles of order 2 at each of these points,

$$H(e^{\pm \pi i/3}) = 0, \tag{26.18}$$

with zeros of order 3 at each of these points, and

$$H(-1) = H(\tfrac{1}{2}) = H(2) = 1, \tag{26.19}$$

$H(w) - 1$ having zeros of order 2 at each of these points.

To obtain the properties (26.17)–(26.18), we try

$$H(w) = C\frac{(w - e^{\pi i/3})^3 (w - e^{-\pi i/3})^3}{w^2 (w-1)^2} = C\frac{(w^2 - w + 1)^3}{w^2(w-1)^2}, \tag{26.20}$$

and to achieve (26.19), we set

$$C = \frac{4}{27}, \tag{26.21}$$

so

$$H(w) = \frac{4}{27} \, \frac{(w^2 - w + 1)^3}{w^2 (w-1)^2}. \tag{26.22}$$

Verify that

$$H\Bigl(\frac{1}{w}\Bigr) = H(w), \quad \text{and} \ \ H(1-w) = H(w), \tag{26.23}$$

and show that (26.16) follows.

Remark. The map Ψ in (26.5) is a variant of the "elliptic modular function," and the composition $H \circ \Psi$ is a variant of the "j-invariant." Note from (26.10)–(26.11) that

$$H \circ \Psi\Bigl(-\frac{1}{z}\Bigr) = H \circ \Psi(z+1) = H \circ \Psi(z), \quad \forall \, z \in \mathcal{U}. \tag{26.24}$$

For more on these maps, see the exercises at the end of §34. For a different approach, see the last two sections in Chapter 7 of [Ahl].

27. Montel's theorem

Here we establish a theorem of Montel, giving a highly nontrivial and very useful sufficient condition for a family of maps from a domain Ω to the Riemann sphere $\widehat{\mathbb{C}}$ to be a normal family.

Theorem 27.1. *Fix a connected domain $U \subset \mathbb{C}$ and let $\mathcal{F}$ be the family of all holomorphic maps $f : U \to \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with range in $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$. Then $\mathcal{F}$ is a normal family.*

Proof. There is no loss of generality in treating the case $U = D$, the unit disk; in particular we can assume henceforth that U is simply connected.

Take $f_\nu \in \mathcal{F}$, and let K be any compact connected subset of U. We aim to show that some subsequence f_{ν_k} converges uniformly on K. Two cases arise:

Case A. There exist $p_{\nu_k} \in K$ such that $f_{\nu_k}(p_{\nu_k})$ is bounded away from $\{0, 1, \infty\}$ in $\widehat{\mathbb{C}}$.

Case B. Such a subsequence does not exist.

We first take care of Case B. In fact, if Case A fails, then, given any open neighborhood U of $\{0, 1, \infty\}$ in $\widehat{\mathbb{C}}$, there exists N such that

$$\nu \geq N \Longrightarrow f_\nu(K) \subset U.$$

If U is small enough, it must have (at least) three connected components, and K connected implies one of them must contain $f_\nu(K)$ for all $\nu \geq N$. This readily implies that f_ν converges to either 0, 1, or ∞, uniformly on K.

It remains to deal with Case A. First, to simplify the notation, relabel ν_k as ν. We make use of the holomorphic covering map $\Phi : D \to \mathbb{C} \setminus \{0, 1\}$ given in §26, and refer to Fig. 26.1 in that section. Pick

$$\zeta_\nu \in (\overline{\Omega} \cup \overline{\mathcal{O}}_1 \cup \overline{\mathcal{O}}_2 \cup \overline{\mathcal{O}}_3) \cap D$$

such that $\Phi(\zeta_\nu) = f(p_\nu)$. At this point it is crucial to observe that $|\zeta_\nu| \leq 1 - \varepsilon$ for some $\varepsilon > 0$, independent of ν. Now we take the unique lifting $g_\nu : U \to D$ of f_ν such that $g_\nu(p_\nu) = \zeta_\nu$. That it is a lifting means $f_\nu = \Phi \circ g_\nu$. The existence of such a lifting follows from the hypothesized simple connectivity of U.

The uniform boundedness of $\{g_\nu\}$ implies that a subsequence (which again we relabel g_ν) converges locally uniformly on U; we have $g_\nu \to g : U \to \overline{D}$. Furthermore, again passing to a subsequence, we can assume $p_\nu \to p \in K$ and

$$g_\nu(p_\nu) = \zeta_\nu \to \zeta \in \overline{D}_{1-\varepsilon}. \tag{27.1}$$

Hence $g(p)=\zeta \in D$, so we actually have

$$g_\nu \to g : U \to D. \tag{27.2}$$

It follows that, for some $\delta > 0$,

$$g_\nu(K) \subset \overline{D}_{1-\delta}, \quad \forall\, \nu. \tag{27.3}$$

This in turn gives the desired convergence $f_\nu \to \Phi \circ g$, uniformly on K.

The last argument shows that in Case A the limit function $f = \Phi \circ g$ maps U to $\widehat{\mathbb{C}} \setminus \{0,1,\infty\}$, so we have the following. (Compare Proposition 22.2.)

Corollary 27.2. *In the setting of Theorem 27.1, if $f_\nu \in \mathcal{F}$ and $f_\nu \to f$ locally uniformly, then either*

$$f \equiv 0, \quad f \equiv 1, \quad f \equiv \infty, \quad \textit{or} \ \ f : U \to \widehat{\mathbb{C}} \setminus \{0,1,\infty\}. \tag{27.4}$$

Exercises on Fatou sets and Julia sets

Let $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be holomorphic, having the form $R(z) = p(z)/q(z)$ with $p(z)$ and $q(z)$ polynomials with no common zeros. We set $d = \deg R = \max\{\deg p, \deg q\}$, called the degree of the map R.

1. Show that if $p_1 \in \widehat{\mathbb{C}}$ is not a critical value of R, then $R^{-1}(p_1)$ consists of d points.

2. Define $R^2 = R \circ R,\ R^3 = R \circ R^2, \dots, R^n = R \circ R^{n-1}$. Show that $\deg R^n = d^n$.

3. Show that if $d \geq 2$ then $\{R^n : n \geq 1\}$ is not a normal family of maps $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.
Hint. If R^{n_k} is uniformly close to $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, the maps must have the same degree, as shown in basic topology courses.

We say a point $\zeta \in \widehat{\mathbb{C}}$ belongs to the *Fatou set* of R provided there exists a neighborhood Ω of ζ such that $\{R^n\big|_\Omega : n \geq 1\}$ is a normal family, with respect to $(\Omega, \widehat{\mathbb{C}})$. The Fatou set of R is denoted $\mathcal{F}_R$.

4. Show that $\mathcal{F}_R$ is open, $R : \mathcal{F}_R \to \mathcal{F}_R$, and $\{R^n\big|_{\mathcal{F}_R} : n \geq 1\}$ is normal with respect to $(\mathcal{F}_R, \widehat{\mathbb{C}})$.

The complement of the Fatou set is called the Julia set, $\mathcal{J}_R = \widehat{\mathbb{C}} \setminus \mathcal{F}_R$. By Exercise 3, $\mathcal{J}_R \neq \emptyset$, whenever $\deg R \geq 2$, which we assume from here on.

5. Given $\zeta \in \mathcal{J}_R$, and any neighborhood $\mathcal{O}$ of ζ in $\widehat{\mathbb{C}}$, consider

$$E_{\mathcal{O}} = \widehat{\mathbb{C}} \setminus \bigcup_{n \geq 0} R^n(\mathcal{O}). \tag{27.5}$$

Use Theorem 27.1 to show that $E_{\mathcal{O}}$ contains at most 2 points.

6. Set

$$E_\zeta = \bigcup \{E_{\mathcal{O}} : \mathcal{O} \text{ neighborhood of } \zeta\}. \tag{27.6}$$

Show that $E_\zeta = E_{\mathcal{O}}$ for some neighborhood $\mathcal{O}$ of ζ. Show that $R : E_\zeta \to E_\zeta$.

7. Consider the function $\mathrm{Sq} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, given by $\mathrm{Sq}(z) = z^2$, $\mathrm{Sq}(\infty) = \infty$. Show that

$$\mathcal{J}_{\mathrm{Sq}} = \{z : |z| = 1\}, \quad E_\zeta = \{0, \infty\}, \quad \forall\, \zeta \in \mathcal{J}_{\mathrm{Sq}}. \tag{27.7}$$

8. Suppose E_ζ consists of one point. Show that R is conjugate to a polynomial, i.e., there exists a linear fractional transformation L such that $L^{-1}RL$ is a polynomial.
Hint. Consider the case $E_\zeta = \{\infty\}$.

9. Suppose E_ζ consists of two points. Show that R is conjugate to P_m for some $m \in \mathbb{Z}$, where $P_m(z) = z^m$, defined appropriately at 0 and ∞.
Hint. Suppose $E_\zeta = \{0, \infty\}$. Then R either fixes 0 and ∞ or interchanges them.

Conclusion. Typically, $E_\zeta = \emptyset$, for $\zeta \in \mathcal{J}_R$. Furthermore, if $E_\zeta \neq \emptyset$, then $E_\zeta = E$ is independent of $\zeta \in \mathcal{J}_R$, and $E \subset \mathcal{F}_R$.

10. Show that

$$R : \mathcal{J}_R \longrightarrow \mathcal{J}_R \tag{27.8}$$

and

$$R^{-1}(\mathcal{J}_R) \subset \mathcal{J}_R. \tag{27.9}$$

Hint. Use Exercise 4. For (27.8), if $R(\zeta) \in \mathcal{F}_R$, then ζ has a neighborhood $\mathcal{O}$ such that $R(\mathcal{O}) \subset \mathcal{F}_R$. For (27.9), use $R : \mathcal{F}_R \to \mathcal{F}_R$.

11. Show that either $\mathcal{J}_R = \widehat{\mathbb{C}}$ or $\mathcal{J}_R$ has empty interior.
Hint. If $\zeta \in \mathcal{J}_R$ has a neighborhood $\mathcal{O} \subset \mathcal{J}_R$, then, by (27.8), $R^n(\mathcal{O}) \subset \mathcal{J}_R$. Now use Exercise 5.

12. Show that, if $p \in \mathcal{J}_R$, then

$$\bigcup_{k\ge 0} R^{-k}(p) \ \text{ is dense in } \ \mathcal{J}_R. \tag{27.10}$$

Hint. Use Exercises 5–6, the “conclusion” following Exercise 9, and Exercise 10.

13. Show that

$$R(z) = 1 - \frac{2}{z^2} \Longrightarrow \mathcal{J}_R = \widehat{\mathbb{C}}. \tag{27.11}$$

REMARK. This could be tough. See [CG], p. 82, for a proof of (27.11), using results not developed in these exercises.

14. Show that

$$R(z) = z^2 - 2 \Longrightarrow \mathcal{J}_R = [-2, 2] \subset \mathbb{C}. \tag{27.12}$$

Hint. Show that $R : [-2, 2] \to [-2, 2]$, and that if $z_0 \in \mathbb{C} \setminus [-2, 2]$, then $R^k(z_0) \to \infty$ as $k \to \infty$.

The next exercise will exploit the following general result.

Proposition 27.3. *Let X be a compact metric space, $F : X \to X$ a continuous map. Assume that for each nonempty open $U \subset X$, there exists $N(U) \in \mathbb{N}$ such that*

$$\bigcup_{0\le j\le N(U)} F^j(U) = X.$$

Then there exists $p \in X$ such that

$$\bigcup_{j\ge 1} F^j(p) \ \textit{ is dense in } \ X.$$

15. Show that Proposition 27.3 applies to $R : \mathcal{J}_R \to \mathcal{J}_R$.
Hint. Use Exercise 5, and the conclusion after Exercise 9.

16. Show that, for $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ as above,

$$\exists\, p \in \mathcal{J}_R \ \text{ such that } \ \bigcup_{j\ge 1} R^j(p) \ \text{ is dense in } \ \mathcal{J}_R.$$

17. Prove Proposition 27.3.

Hint. Take a countable dense subset $\{q_j : j \ge 1\}$ of X. Try to produce a shrinking family $K_j \supset K_{j+1} \supset \cdots$ of nonempty, compact subsets of X, and $N_j \in \mathbb{N}$, such that, for all $j \in \mathbb{N}$,

$$F^{N_j}(K_j) \subset B_{2^{-j}}(q_j).$$

Then take $p \in \cap_{j\ge 1} K_j$, so

$$F^{N_j}(p) \in B_{2^{-j}}(q_j), \quad \forall\, j \ge 1.$$

18. Show that $R : \mathcal{J}_R \to \mathcal{J}_R$ is surjective.
Hint. Consider Exercise 15.

19. Show that, for each $k \in \mathbb{N}$, $\mathcal{J}_{R^k} = \mathcal{J}_R$.
Hint. Clearly $\mathcal{F}_{R^k} \supset \mathcal{F}_R$. To get the converse, use $R^j = R^\ell R^{mk},\ 0 \le \ell \le k-1$.

20. Show that $\mathcal{J}_R$ must be infinite. (Recall that we assume $\deg R \ge 2$.)
Hint. If $\mathcal{J}_R$ is finite, we can replace R by $R^k = \widetilde{R}$ and find $p \in \mathcal{J}_R$ such that $\widetilde{R}(p) = p$. Then take a small neighborhood $\mathcal{O}$ of p (disjoint from the rest of $\mathcal{J}_R$) and apply Exercise 5 (and Exercise 15) to $\widetilde{R}$, to get a contradiction.

21. Show that $\mathcal{J}_R$ has no isolated points.
Hint. If $p \in \mathcal{J}_R$ is isolated, let $\mathcal{O}$ be a small neighborhood of p in $\widehat{\mathbb{C}}$, disjoint from the rest of $\mathcal{J}_R$, and (again) apply Exercise 5 (and Exercise 15) to R, to get a contradiction, taking into account Exercise 20.

These exercises provide a glimpse at an area known as Complex Dynamics. More material on this area can be found in Chapter 5 of [Sch] (a brief treatment), and in [CG] and [Mil]. As opposed to (27.7) and (27.12), typically $\mathcal{J}_R$ has an extremely complicated, "fractal" structure, as explained in these references.

28. Picard's theorems

Here we establish two theorems of E. Picard. The first, known as "Picard's little theorem," is an immediate consequence of the fact that the disk holomorphically covers $\mathbb{C} \setminus \{0, 1\}$.

Proposition 28.1. *If p and q are distinct points in $\mathbb{C}$ and if $f : \mathbb{C} \to \mathbb{C} \setminus \{p, q\}$ is holomorphic, then it is constant.*

Proof. Without loss of generality, we can take $p = 0, q = 1$. Via the covering map $\Phi : D \to \mathbb{C} \setminus \{0, 1\}$ produced in §26, f lifts to a holomorphic map

$$g : \mathbb{C} \longrightarrow D, \quad f = \Phi \circ g. \tag{28.1}$$

Liouville's theorem then implies g is constant, so also f is constant.

The following sharper result is called "Picard's big theorem." It is proved using Montel's theorem.

Proposition 28.2. *If p and q are distinct and*

$$f : D \setminus \{0\} \longrightarrow \mathbb{C} \setminus \{p, q\} \tag{28.2}$$

is holomorphic, then the singularity at 0 is either a pole or a removable singularity. Equivalently, if $f : D \setminus \{0\} \longrightarrow \widehat{\mathbb{C}}$ is holomorphic and has range in $\widehat{\mathbb{C}} \setminus \{p, q, r\}$ with $p, q, r \in \widehat{\mathbb{C}}$ distinct, then f extends to $\tilde{f} : D \to \widehat{\mathbb{C}}$, holomorphic.

Proof. Again there is no loss of generality in taking $p = 0, q = 1$, and $r = \infty$. Take $\Omega = D \setminus \{0\}$ and define $f_\nu : \Omega \to \mathbb{C} \setminus \{0, 1\}$ by $f_\nu(z) = f(2^{-\nu} z)$. By Montel's theorem (Theorem 27.1), $\{f_\nu\}$ is a normal family of maps from Ω to $\widehat{\mathbb{C}}$. In particular, there exists a subsequence f_{ν_k} converging locally uniformly on Ω:

$$f_{\nu_k} \to g : \Omega \to \widehat{\mathbb{C}}, \tag{28.3}$$

and g is a holomorphic map. Pick $r \in (0, 1)$ and set $\Gamma = \{z : |z| = r\} \subset \Omega$. The convergence in (28.3) is uniform on Γ.

We consider two cases.

CASE A. $\infty \notin g(\Omega)$.
Then there exists $A < \infty$ such that

$$|f_{\nu_k}(z)| \le A, \quad |z| = r, \tag{28.4}$$

for large k, or equivalently

$$|f(z)| \le A, \quad |z| = 2^{-\nu_k} r. \tag{28.5}$$

The maximum principle then implies $|f(z)| \leq A$ for all z close to 0, so 0 is a removable singularity of f.

Case B. $\infty \in g(\Omega)$.
By Corollary 27.2, $g \equiv \infty$ on Ω. Consequently, $1/f_{\nu_k} \to 0$ uniformly on Γ, and then analogues of (28.4)–(28.5) hold with f replaced by $1/f$ (recall that f is nowhere vanishing on $D \setminus \{0\}$), so f has a pole at 0.

Exercises

1. Fix distinct points $a, b \in \mathbb{C}$, let γ be the line segment joining a and b, and assume $f : \mathbb{C} \to \mathbb{C} \setminus \gamma$ is holomorphic. Show that f is constant. Show this in an elementary fashion, not using Picard theorems.

2. Suppose $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function that is not a polynomial. (We say f is transcendental.) Show that for every $w \in \mathbb{C}$, with at most one exception, there are infinitely many $z_\nu \in \mathbb{C}$ such that $f(z_\nu) = w$.

3. Test the conclusion in Exercise 2 for the function $f(z) = e^z$.

4. Show that if

$$f(z) = ze^z,$$

then $f : \mathbb{C} \to \mathbb{C}$ is onto. Relate this to Exercise 2.
Hint. If $f(z)$ is never equal to w, then

$$ze^z - w = e^{g(z)}.$$

What can $g(z)$ be?

5. Suppose that f is *meromorphic* on $\mathbb{C}$, and not constant.. Show that f can omit at most *two* complex values. Give an example where such an omission occurs.

29. Harmonic functions II

Here we study further properties of harmonic functions. A key tool will be the explicit formula for the solution to the Dirichlet problem

$$\Delta u = 0 \;\text{ on }\; D_1(0), \quad u\big|_{S^1} = f, \tag{29.1}$$

given in §13, namely

$$u(z) = \int_0^{2\pi} p(z,\theta) f(\theta)\, d\theta, \tag{29.2}$$

where

$$p(z,\theta) = \frac{1}{2\pi} \frac{1-|z|^2}{|w-z|^2}, \quad w = e^{i\theta}, \tag{29.3}$$

as in (13.76), or equivalently (cf. (13.85))

$$p(z,\theta) = \frac{1}{2\pi} \operatorname{Re} \frac{e^{i\theta}+z}{e^{i\theta}-z} = \frac{1}{\pi} \operatorname{Re} \frac{e^{i\theta}}{e^{i\theta}-z} - \frac{1}{2\pi}. \tag{29.4}$$

We restrict attention to harmonic functions on domains in $\mathbb{C} = \mathbb{R}^2$. However we point out that results presented here can be extended to treat harmonic functions on domains in $\mathbb{R}^n$. In such a case we replace (29.2)–(29.3) by

$$u(x) = \int\limits_{S^{n-1}} p(x,\omega) f(\omega)\, dS(\omega), \tag{29.5}$$

for $x \in B_1(0) = \{x \in \mathbb{R}^n : |x| < 1\}$, with

$$p(x,\omega) = \frac{1}{A_{n-1}} \frac{1-|x|^2}{|x-\omega|^n}, \tag{29.6}$$

where A_{n-1} is the $(n-1)$-dimensional area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. A proof of (29.5)–(29.6) can be found in Chapter 5 of [T2]. But here we will say no more about the higher-dimensional case.

We use (29.2)–(29.3) to establish a Harnack estimate, concerning the set of harmonic functions

$$\mathcal{A} = \{u \in C^2(D) \cap C(\overline{D}) : \Delta u = 0,\ u \ge 0 \text{ on } D,\ u(0) = 1\}, \tag{29.7}$$

where $D = D_1(0) = \{z \in \mathbb{C} : |z| < 1\}$.

Proposition 29.1. *For $z \in D$,*

$$u \in \mathcal{A} \Longrightarrow u(z) \geq \frac{1-|z|}{1+|z|}. \tag{29.8}$$

Proof. The mean value property of harmonic functions implies that $f = u|_{\partial D}$ satisfies

$$\frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta = 1, \tag{29.9}$$

if $u \in \mathcal{A}$. Given this, (29.8) follows from (29.2)–(29.3) together with the estimate

$$\frac{1-|z|^2}{|w-z|^2} \geq \frac{1-|z|}{1+|z|}, \quad \text{for } \ |z| < 1,\ |w| = 1. \tag{29.10}$$

To get (29.10), note that if $w = e^{i\varphi},\ z = re^{i\theta}$, and $\psi = \varphi - \theta$, then

$$\begin{aligned} |w-z|^2 &= (e^{i\psi} - r)(e^{-i\psi} - r) \\ &= 1 - 2r\cos\psi + r^2 \\ &\leq 1 + 2r + r^2 = (1+r)^2, \end{aligned} \tag{29.11}$$

from which (29.10) follows.

By translating and scaling, we deduce that if u is ≥ 0 and harmonic on $D_R(p)$, then

$$|z-p| = a \in [0, R) \Longrightarrow u(z) \geq \frac{R-a}{R+a} u(p). \tag{29.12}$$

This leads to the following extension of the Liouville theorem, Proposition 7.5 (with a different approach from that suggested in Exercise 1 of §7).

Proposition 29.2. *If u is harmonic on all of $\mathbb{C}$ (and real valued) and bounded from below, then u is constant.*

Proof. Adding a constant if necessary, we can assume $u \geq 0$ on $\mathbb{C}$. Pick $z_0,\ z_1 \in \mathbb{C}$ and set $a = |z_0 - z_1|$. By (29.12),

$$u(z_0) \geq \frac{R-a}{R+a} u(z_1), \quad \forall\, R \in (a, \infty), \tag{29.13}$$

hence, taking $R \to \infty$,

$$u(z_0) \geq u(z_1), \tag{29.14}$$

for all $z_0,\ z_1 \in \mathbb{C}$. Reversing roles gives

$$u(z_0) = u(z_1), \tag{29.15}$$

and proves the proposition.

The following result will lead to further extensions of Liouville's theorem.

Proposition 29.3. *There exists a number $A \in (0,\infty)$ with the following property. Let u be harmonic in $D_R(0)$. Assume*

$$u(0) = 0, \quad u(z) \le M \quad \text{on} \ \ D_R(0). \tag{29.16}$$

Then

$$u(z) \ge -AM \quad \text{on} \ \ D_{R/2}(0). \tag{29.17}$$

Proof. Set $v(z) = M - u(z)$, so $v \ge 0$ on $D_R(0)$ and $v(0) = M$. Take

$$p \in \overline{D_{R/2}(0)}, \quad u(p) = \inf_{D_{R/2}(0)} u. \tag{29.18}$$

From (29.12), with R replaced by $R/2$, a by $R/4$, and u by v, we get

$$v(z) \ge \frac{1}{3}\big(M - u(p)\big) \ \ \text{on} \ \ D_{R/2}(p). \tag{29.19}$$

Hence

$$\frac{1}{\text{Area } D_R(0)} \iint\limits_{D_R(0)} v(z)\, dx\, dy \ge \frac{1}{16} \cdot \frac{1}{3}\big(M - u(p)\big). \tag{29.20}$$

On the other hand, the mean value property for harmonic functions implies that the left side of (29.20) equals $v(0) = M$, so we get

$$M - u(p) \le 48M, \tag{29.21}$$

which implies (29.17).

Note that Proposition 29.3 gives a second proof of Proposition 29.2. Namely, under the hypotheses of Proposition 29.2, if we set $v(z) = u(0) - u(z)$, we have $v(0) = 0$ and $v(z) \le u(0)$ on $\mathbb{C}$ (if $u \ge 0$ on $\mathbb{C}$), hence, by Proposition 29.3, $v(z) \ge -Au(0)$ on $\mathbb{C}$, so v is bounded on $\mathbb{C}$, and Proposition 7.5 implies v is constant. Note however, that the first proof of Proposition 29.2 did not depend upon Proposition 7.5.

Here is another corollary of Proposition 29.3.

Proposition 29.4. *Assume u is harmonic on $\mathbb{C}$ and there exist $C_0,\ C_1 \in (0,\infty)$, and $k \in \mathbb{Z}^+$ such that*

$$u(z) \le C_0 + C_1|z|^k, \quad \forall\, z \in \mathbb{C}. \tag{29.22}$$

Then there exist $C_2, C_3 \in (0,\infty)$ such that

$$u(z) \ge -C_2 - C_3|z|^k, \quad \forall\, z \in \mathbb{C}. \tag{29.23}$$

Proof. Apply Proposition 29.3 to $u(z) - u(0)$, $M = C_0 + |u(0)| + C_1 R^k$.

Note that as long as $C_2 \geq C_0$ and $C_3 \geq C_1$, the two one-sided bounds (29.22) and (29.23) imply

$$|u(z)| \leq C_2 + C_3|z|^k, \quad \forall\, z \in \mathbb{C}. \tag{29.24}$$

We aim to show that if $u(z)$ is harmonic on $\mathbb{C}$ and satisfies the bound (29.24), then u must be a polynomial in x and y. For this, it is useful to have estimates on derivatives $\partial_x^i \partial_y^j u$, which we turn to.

If $u \in C^2(D) \cap C(\overline{D})$ is harmonic on $D = D_1(0)$, the formula (25.2) yields

$$\partial_x^i \partial_y^j u(z) = \int_0^{2\pi} p_{ij}(z,\theta) f(\theta)\, d\theta, \tag{29.25}$$

where

$$p_{ij}(z,\theta) = \partial_x^i \partial_y^j p(z,\theta). \tag{29.26}$$

From (29.3) it is apparent that $p(z,\theta)$ is smooth in $z \in D$. We have bounds of the form

$$|p_{ij}(z,\theta)| \leq K_{ij}, \quad |z| \leq \frac{1}{2}. \tag{29.27}$$

For example, from (29.4) we get

$$\begin{aligned} \frac{\partial}{\partial x} p(z,\theta) &= \frac{1}{\pi} \operatorname{Re} \frac{e^{i\theta}}{(e^{i\theta} - z)^2}, \\ \frac{\partial}{\partial y} p(z,\theta) &= \frac{1}{\pi} \operatorname{Re} \frac{ie^{i\theta}}{(e^{i\theta} - z)^2}. \end{aligned} \tag{29.28}$$

Hence

$$\begin{aligned} |\nabla_{x,y} p(z,\theta)| &\leq \frac{1}{\pi} \frac{1}{|e^{i\theta} - z|^2} \\ &\leq \frac{1}{\pi(1 - |z|)^2}, \end{aligned} \tag{29.29}$$

the last estimate by a variant of (29.11).

Applied to (29.25), the bounds (29.27) imply

$$\sup_{|z| \leq 1/2} |\partial_x^i \partial_y^j u(z)| \leq 2\pi K_{ij} \sup_D |u(z)|, \tag{29.30}$$

whenever $u \in C^2(D) \cap C(\overline{D})$ is harmonic on D.

We are now are ready for the following.

Proposition 29.5. *If* $u : \mathbb{C} \to \mathbb{R}$ *is harmonic and, for some* $k \in \mathbb{Z}^+$, $C_j \in (0, \infty)$,

$$u(z) \le C_0 + C_1 |z|^k, \quad \forall\, z \in \mathbb{C}, \tag{29.31}$$

then u *is a polynomial in* x *and* y *of degree* k.

Proof. By Proposition 29.4, we have the two-sided bound

$$|u(z)| \le C_2 + C_3 |z|^k, \quad \forall\, z \in \mathbb{C}. \tag{29.32}$$

Now set $v_R(z) = R^{-k} u(Rz)$. We have $v_R|_D$ bounded, independent of $R \in [1, \infty)$, where $D = D_1(0)$. Hence, by (29.30), $\partial_x^i \partial_y^j v_R$ is bounded on $D_{1/2}(0)$, independent of R, for each $i, j \in \mathbb{Z}^+$, so

$$R^{i+j-k} |\partial_x^i \partial_y^j u(Rz)| \le C_{ij}, \quad |z| \le \frac{1}{2}, \ R \in [1, \infty). \tag{29.33}$$

Taking $R \to \infty$ yields $\partial_x^i \partial_y^j u = 0$ on $\mathbb{C}$ for $i + j > k$, which proves the proposition.

Remark. In case $k = 0$, the argument above gives another proof of Proposition 7.5.

Recall from §7 the mean value property satisfied by a harmonic function $u \in C^2(\Omega)$, namely, if $\overline{D_r(z_0)} \subset \Omega$, then

$$u(z_0) = \frac{1}{\pi r^2} \iint\limits_{D_r(z_0)} u(x, y)\, dx\, dy, \tag{29.34}$$

a result mentioned above in connection with (29.20). We now establish the following converse.

Proposition 29.6. *Let* $\Omega \subset \mathbb{C}$ *be open and let* $u : \Omega \to \mathbb{R}$ *be continuous and satisfy the mean value property (29.34) whenever* $\overline{D_r(z_0)} \subset \Omega$. *Then* u *is harmonic on* Ω.

Proof. It suffices to prove that u is harmonic on $D_\rho(z_1)$ whenever $\overline{D_\rho(z_1)} \subset \Omega$. To see this, let $f = u|_{\partial D_\rho(z_1)}$, and let $v \in C^2(D_\rho(z_1)) \cap C(\overline{D_\rho(z_1)})$ solve the Dirichlet problem

$$\Delta v = 0 \ \text{ on } \ D_\rho(z_1), \quad v\big|_{\partial D_\rho(z_1)} = f. \tag{29.35}$$

The function v is given by a construction parallel to that for (29.1). It then suffices to show that

$$v = u \ \text{ on } \ D_\rho(z_1). \tag{29.36}$$

To see this, consider $w = v - u \in C(\overline{D_\rho(z_1)})$, which satisfies

$$w\big|_{\partial D_\rho(z_1)} = 0, \tag{29.37}$$

and

$$w(z_2)=\frac{1}{\pi r^2}\iint\limits_{D_r(z_2)} w(x,y)\,dx\,dy, \quad \text{whenever } \overline{D_r(z_2)}\subset D_\rho(z_1). \tag{29.38}$$

Now, argunig as in the proof of Proposition 7.4 (cf. also Proposition 6.1), we see that

$$\sup_{D_\rho(z_1)} |w| = \sup_{\partial D_\rho(z_1)} |w|, \tag{29.39}$$

and hence, by (29.37), $w\equiv 0$ on $D_\rho(z_1)$. This gives (29.36) and finishes the proof of Proposition 29.6.

We next use the solvability of the Dirichlet problem (29.1) to establish the following removable singularity theorem, which extends the scope of Theorem 11.1.

Proposition 29.7. *Let $\Omega\subset\mathbb{C}$ be open, $p\in\Omega$, and $u\in C^2(\Omega\setminus p)$ be harmonic. If u is bounded, then there is a harmonic $\tilde u\in C^2(\Omega)$ such that $\tilde u=u$ on $\Omega\setminus p$.*

Proof. It suffices to assume that $p=0$, $\overline{D_1(0)}\subset\Omega$, and that u is real valued. Say $|u|\le M$ on $D_1^*=D_1(0)\setminus 0$. Take $v\in C^2(D_1(0))\cap C(\overline{D_1(0)})$ to satisfy

$$\Delta v=0 \ \text{ on } \ D_1(0), \quad v\big|_{S^1}=u\big|_{S^1}, \tag{29.40}$$

so $w=u-v$ is C^2 on $D_1(0)\setminus 0$, continuous on $\overline{D_1(0)}\setminus 0$, and satisfies

$$\Delta w=0 \ \text{ on } D_1(0)\setminus 0, \quad w\big|_{S^1}=0. \tag{29.41}$$

Also, the maximum principle applied to (29.40) yields $|v|\le M$, hence $|w|\le 2M$, on D_1^*. If we restrict w to $\{z\in\mathbb{C}: a\le|z|\le 1\}$ and compare with

$$\psi(z)=\Bigl(\log\frac{1}{a}\Bigr)^{-1}\log\frac{1}{|z|}, \tag{29.42}$$

which is harmonic on $z\neq 0$, and vanishes at $|z|=1$, the maximum principle implies

$$|w(z)|\le 2M\Bigl(\log\frac{1}{a}\Bigr)^{-1}\log\frac{1}{|z|}, \quad \text{for } \ a\le|z|\le 1, \tag{29.43}$$

whenever $a\in(0,1)$. Letting $a\searrow 0$ yields $w\equiv 0$ on $0<|z|\le 1$, hence

$$u(z)=v(z) \ \text{ for } \ 0<|z|\le 1. \tag{29.44}$$

Thus we can set $\tilde u=u$ on $\Omega\setminus 0$ and $\tilde u=v$ on $D_1(0)$, to complete the proof of Proposition 29.7.

We turn now to the Dirichlet problem on more general domains, that is, to the problem of finding $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying

$$\Delta u = 0 \ \text{ on } \ \Omega, \quad u\big|_{\partial\Omega} = \varphi, \tag{29.45}$$

given $\varphi \in C(\partial\Omega)$, given $\Omega \subset \mathbb{C}$ a bounded open set. It is a consequence of the maximum principle, which implies that

$$\sup_{\Omega} |u| = \sup_{\partial\Omega} |u|, \tag{29.46}$$

that such a solution, if it exists, is unique. We focus on those bounded, simply connected domains Ω such that the Riemann mapping function

$$f : D \longrightarrow \Omega \tag{29.47}$$

extends to a homeomorphism $f : \overline{D} \to \overline{\Omega}$, with inverse

$$g : \overline{\Omega} \longrightarrow \overline{D}, \tag{29.48}$$

holomorphic on Ω. Here, $D = D_1(0)$ is the unit disk. A class of domains satisfying this condition is given in Proposition 24.3. In such a case, by Proposition 7.9, we have

$$u(z) = v(g(z)), \tag{29.49}$$

where $v \in C^2(D) \cap C(\overline{D})$ satisfies

$$\Delta v = 0 \ \text{ on } \ D, \quad v\big|_{\partial D} = \psi = \varphi \circ f. \tag{29.50}$$

From (29.2), we have

$$v(z) = \int_0^{2\pi} p(z,\theta)\psi(e^{i\theta})\, d\theta, \tag{29.51}$$

with $p(z,\theta)$ given by (29.3)–(29.4). We have the following conclusion.

Proposition 29.8. *Assume $\Omega \subset \mathbb{C}$ is a bounded, simply connected domain with the property that the Riemann mapping function $f : D \to \Omega$ extends to a homeomorphism from $\overline{D}$ to $\overline{\Omega}$, with inverse g, as in (29.48). Then given $\varphi \in C(\partial\Omega)$, the Dirichlet problem (29.45) has a unique solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$, given by*

$$\begin{aligned} u(z) &= \int_0^{2\pi} p(g(z),\theta)\varphi(f(e^{i\theta}))\, d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-|g(z)|^2}{|e^{i\theta}-g(z)|^2}\, \varphi(f(e^{i\theta}))\, d\theta. \end{aligned} \tag{29.52}$$

We examine how this works out when $\Omega = \mathcal{H}^+$ is the upper half plane

$$\mathcal{H}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}, \tag{29.53}$$

for which (29.47)–(29.48) hold with

$$g(z) = \frac{i - z}{i + z}, \quad f(z) = i\,\frac{1 - z}{1 + z}. \tag{29.54}$$

This is slightly outside the results of (29.47)–(29.48) since $\mathcal{H}^+$ is not bounded, but it works if we regard $\mathcal{H}^+$ as a relatively compact subset of the Riemann sphere $\mathbb{C} \cup \{\infty\}$, with boundary $\partial\mathcal{H}^+ = \mathbb{R} \cup \{\infty\}$. In this case, we have

$$t = f(e^{i\theta}) = i\,\frac{1 - e^{i\theta}}{1 + e^{i\theta}} = \tan\frac{\theta}{2}, \tag{29.55}$$

for which $d\theta = 2\,dt/(1 + t^2)$, and (29.52) yields

$$u(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1 - |g(z)|^2}{|g(t) - g(z)|^2}\,\varphi(t)\,\frac{dt}{1 + t^2}. \tag{29.56}$$

For $g(z)$ as in (29.54), we calculate that

$$\frac{1 - |g(z)|^2}{|g(t) - g(z)|^2} = \frac{y(1 + t^2)}{(x - t)^2 + y^2}, \quad z = x + iy \in \mathcal{H}^+, \tag{29.57}$$

so we obtain the formula

$$u(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x - t)^2 + y^2}\,\varphi(t)\,dt, \tag{29.58}$$

for the solution to (29.45) with $\Omega = \mathcal{H}^+$. Note that we are assuming that $\varphi : \mathbb{R} \cup \{\infty\} \to \mathbb{C}$ is continuous.

It is of interest to consider an alternative derivation of (29.58), using the Fourier transform, introduced in §14. Namely, consider

$$u(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y|\xi|}\hat{\varphi}(\xi)e^{ix\xi}\,d\xi, \tag{29.59}$$

with $\varphi \in \mathcal{A}(\mathbb{R})$, defined in (14.21). We have, for $y > 0$,

$$\begin{aligned} \frac{\partial^2}{\partial y^2} u(x, y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y|\xi|}\xi^2\hat{\varphi}(\xi)e^{ix\xi}\,d\xi \\ &= -\frac{\partial^2}{\partial x^2} u(x, y). \end{aligned} \tag{29.60}$$

Hence $u(x,y)$ solves

$$\Delta u = 0 \quad \text{on} \quad \mathcal{H}^+, \quad u(x,0) = \varphi(x). \tag{29.61}$$

Furthermore, substituting the definition of $\hat{\varphi}(\xi)$ into (29.59) yields

$$u(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-y|\xi| + i(x-t)\xi} \varphi(t)\, d\xi\, dt, \tag{29.62}$$

and a computation of the integral

$$\int_{-\infty}^{\infty} e^{-y|\xi| + i(x-t)\xi}\, d\xi \tag{29.63}$$

(cf. Exercise 4 of §14), yields (29.58).

Let us note that direct applications of (29.58) and (29.59) yield, respectively,

$$|u(x,y)| \le \sup_t |\varphi(t)|, \quad |u(x,y)| \le (2\pi)^{-1/2} \|\hat{\varphi}\|_{L^1}, \tag{29.64}$$

for all $x \in \mathbb{R},\ y > 0$. Also, from both (29.58) and (29.59), it follows that

$$\sup_{x \in \mathbb{R}} |u(x,y) - \varphi(x)| \longrightarrow 0, \quad \text{as} \quad y \searrow 0, \tag{29.65}$$

via (29.58) for $\varphi \in C(\mathbb{R})$ vanishing at ∞, and, via (29.59), for $\varphi \in \mathcal{A}(\mathbb{R})$. In light of these results, the following uniqueness result is of interest.

Proposition 29.9. *Let $u \in C^2(\mathcal{H}^+) \cap C(\overline{\mathcal{H}^+})$ satisfy $\Delta u = 0$ on $\mathcal{H}^+$ and $u(x,0) = 0$. Assume $|u(x,y)| \le M < \infty$ on $\mathcal{H}^+$. Then $u \equiv 0$.*

Proof. For $x \in \mathbb{R}$, set

$$\begin{aligned} v(x,y) = {} & u(x,y), && y \ge 0, \\ & -u(x,-y), && y < 0. \end{aligned} \tag{29.66}$$

Then, by Proposition 13.8, v is harmonic on $\mathbb{C}$. Also $|v(x,y)| \le M$ for all $(x,y) \in \mathbb{R}^2$. It follows from Liouville's theorem, Proposition 7.5, that u is constant. Of course, the constant must be 0.

We next look at the Dirichlet problem (29.45) when

$$\Omega = \mathcal{D}^+ = \{z \in \mathbb{C} : |z| < 1 \text{ and } \operatorname{Im} z > 0\}. \tag{29.67}$$

In this case, a construction of a conformal map $g : \overline{\Omega} \to \overline{D}$ is indicated in Fig. 22.1. Using this, we could implement (29.49). Rather than recording further details of that approach, we now describe another way to solve

$$\Delta u = 0 \quad \text{on} \quad \mathcal{D}^+, \quad u = \varphi \quad \text{on} \quad \partial\mathcal{D}^+ = [-1,1] \cup S^+, \tag{29.68}$$

where $S^+ = \{e^{is} : 0 \le s \le \pi\}$, which will involve breaking φ into pieces. First, subtracting a harmonic function of the form $a + bx$, we can reduce our problem to the case where $\varphi(-1) = \varphi(1) = 0$. Then write $\varphi = \varphi_0 + \varphi_1$, where $\varphi_j \in C(\partial\mathcal{D}^+)$, with φ_0 supported on $[-1, 1]$ and φ_1 supported on S^+. We define φ_0 on all of $\mathbb{R}$ to be 0 on $\mathbb{R} \setminus [-1, 1]$, and let u_0 solve

$$\Delta u_0 = 0 \ \text{ on } \ \mathcal{H}^+, \quad u_0(x, 0) = \varphi_0(x), \tag{29.69}$$

as in (29.61). The solution is given by (29.58), with $\varphi(t)$ replaced by $\varphi_0(t)$. We then see that the solution to (29.68) is given by $u = u_0 + u_1$, where

$$\Delta u_1 = 0 \ \text{ on } \ \mathcal{D}^+, \quad u_1\big|_{\partial\mathcal{D}^+} = \psi_0 = \varphi - u_0\big|_{\partial\mathcal{D}^+}. \tag{29.70}$$

Note that $\psi_0 \in C(\partial\mathcal{D}^+)$ is supported on S^+. We can now define $\psi \in C(S^1)$ by

$$\begin{aligned} \psi(x, y) = \psi_0(x, y), \qquad & (x, y) \in S^+, \\ -\psi_0(x, -y), \quad & (x, -y) \in S^+, \end{aligned} \tag{29.71}$$

and obtain u_1 as the restriction to $\mathcal{D}^+$ of v, solving

$$\Delta v = 0 \ \text{ on } \ D_1(0), \quad v\big|_{S^1} = \psi. \tag{29.72}$$

Note that $v(x, -y) = -v(x, y)$.

We move on to a consideration of the Dirichlet problem on an "exterior domain." Let $U \subset \mathbb{C}$ be a bounded open set, and let $\Omega = \mathbb{C} \setminus \overline{U}$. We seek a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ to

$$\Delta u = 0 \ \text{ on } \ \Omega, \quad u\big|_{\partial\Omega} = f, \tag{29.73}$$

given $f \in C(\partial\Omega)$. We also impose the condition that

$$u \text{ is bounded on } \Omega. \tag{29.74}$$

To get this, we take $z_0 \in U$ and set

$$u(z) = v\Big(\frac{1}{z - z_0}\Big), \quad \text{i.e.,} \ \ v(\zeta) = u\Big(\frac{1}{\zeta} + z_0\Big), \tag{29.75}$$

where

$$\Delta v(\zeta) = 0 \ \text{ for } \ \zeta^{-1} + z_0 \in \Omega. \tag{29.76}$$

Note that

$$\{\zeta \in \mathbb{C} \setminus 0 : \zeta^{-1} + z_0 \in \Omega\} = \Omega^* \setminus 0, \tag{29.77}$$

where Ω^* is the image of $\Omega \cup \{\infty\} \subset \mathbb{C} \cup \{\infty\}$ under the map $z \mapsto 1/(z - z_0)$. Thus a bounded harmonic function u on Ω corresponds to a bounded harmonic function v on $\Omega^* \setminus 0$. Proposition 29.7 implies that such v must extend to be harmonic on Ω^*. Thus (29.73)–(29.74) translates to the Dirichlet problem

$$\Delta v = 0 \ \text{ on } \ \Omega^*, \quad v\big|_{\partial\Omega^*} = g, \tag{29.78}$$

where

$$g(\zeta) = f(\zeta^{-1} + z_0), \quad \zeta \in \partial\Omega^*. \tag{29.79}$$

One simple case of (29.73)–(29.79) arises from

$$\Omega = \mathbb{C} \setminus \overline{D_1(0)}, \quad \text{with } \ \Omega^* = D_1(0). \tag{29.80}$$

A more elaborate case is

$$\Omega = \mathbb{C} \setminus \bigl\{\overline{D_1(-a)} \cup \overline{D_1(a)}\bigr\}, \tag{29.81}$$

given $a > 1$, mentioned in the exercises below.

To close, we mention that the Dirichlet problem (29.45), for a general bounded open donain $\Omega \subset \mathbb{C}$, is not always solvable for $u \in C(\overline{\Omega})$, given arbitrary $\varphi \in C(\partial\Omega)$. For example, let

$$\Omega = \{z \in \mathbb{C} : 0 < |z| < 1\}, \quad \partial\Omega = S^1 \cup \{0\}, \tag{29.82}$$

and take $\varphi \in C(S^1 \cup \{0\})$. By Proposition 29.7, the solution u is uniquely determined by $u|_{S^1}$ alone. If this solution does not agree with $\varphi(0)$ at $z = 0$, then, strictly speaking, we do not have (29.45). We say 0 is an *irregular boundary point* for Ω in (29.82). It is of interest to characterize domains without irregular boundary points. Furthermore, there is a notion of a generalized solution to (29.45) in the presence of irregular boundary points. Material on this, both for planar domains and higher dimensional domains, can be found in Chapter 5 of [T2].

In addition, results in §§4–6 of that Chapter 5 allow for a proof of the Riemann mapping theorem, as a consequence of solving the Dirichlet problem, thus going in the opposite direction from Proposition 29.8 of this section. As an alternative to presenting that argument, we will show how solving a Dirichlet problem on another class of domains gives rise to a variant of the Riemann mapping theorem.

Namely, suppose $\Omega \subset \mathbb{C}$ is a smoothly bounded domain whose boundary consists of *two* smooth, simple closed curves, say γ_0 and γ_1. Let u be the solution to the Dirichlet problem

$$\Delta u = 0 \ \text{ on } \ \Omega, \quad u\big|_{\gamma_0} = 0, \quad u\big|_{\gamma_1} = -1. \tag{29.83}$$

If $\partial\Omega$ is C^∞, then $u \in C^\infty(\overline{\Omega})$. We know from Proposition 7.4 that $-1 < u < 0$ on Ω. In addition, a "strong maximum principle" (cf. Chapter 5, §2 of [T2]) implies

$$\frac{\partial u}{\partial \nu} > 0 \ \text{ on } \ \gamma_0, \quad \frac{\partial u}{\partial \nu} < 0 \ \text{ on } \ \gamma_1, \tag{29.84}$$

where ν is the unit outward normal to $\partial\Omega$. We set

$$G = cu, \tag{29.85}$$

where c is a positive constant to be determined, and, following (7.22), try to produce a harmonic conjugate to G, of the form

$$H(z) = \int_{\gamma_{\alpha z}} \Big(-\frac{\partial G}{\partial y}\,dx + \frac{\partial G}{\partial x}\,dy\Big), \tag{29.86}$$

where we pick $\alpha \in \Omega$ and then $\gamma_{\alpha z}$ is a smooth path from α to z. By Lemma 7.7, if $\gamma_{\alpha z}$ and $\tilde{\gamma}_{\alpha z}$ are homotopic paths from α to z, the integrals coincide. However, in this case Ω is not simply connected, and (27.86) does not yield a single-valued harmonic conjugate of G. Here is what we get instead. We have

$$\begin{aligned} \int\limits_{\gamma_0} \Big(-\frac{\partial G}{\partial y}\,dx + \frac{\partial G}{\partial x}\,dy\Big) &= \int\limits_{\gamma_0} \frac{\partial G}{\partial \nu}\,ds \\ &= c\int\limits_{\gamma_0} \frac{\partial u}{\partial \nu}\,ds \\ &> 0. \end{aligned} \tag{29.87}$$

Hence there is a unique value of $c \in (0,\infty)$ such that (29.87) is equal to 2π. For such c, the function H in (29.86) is well defined, up to an additive integral multiple of 2π. Hence

$$\Psi = e^{G+iH} \tag{29.88}$$

is a single-valued, holomorphic function on Ω, extending smoothly to $\overline{\Omega}$. It maps γ_0 to $C_0 = \{z \in \mathbb{C} : |z| = 1\}$, and it maps γ_1 to $C_1 = \{z \in \mathbb{C} : |z| = e^{-c}\}$. Consequently,

$$\Psi : \Omega \longrightarrow \mathfrak{A}_\rho = \{z \in \mathbb{C} : \rho < |z| < 1\}, \quad \rho = e^{-c}, \tag{29.89}$$

extending smoothly to $\Psi : \overline{\Omega} \to \overline{\mathfrak{A}}_\rho$. It follows from (29.84) and the Cauchy-Riemann equations that the tangential derivative of H is never vanishing on $\partial\Omega$, hence

$$\begin{aligned} &\psi_j = \Psi\big|_{\gamma_j}, \quad \psi_j : \gamma_j \to C_j \\ &\Rightarrow \psi_j' \ \text{ is nowhere vanishing on } \ \gamma_j. \end{aligned} \tag{29.90}$$

In fact $|\psi_j'(z)| = |\partial_\nu G(z)|$ for $z \in \gamma_j$. Thus the choice of c yields

$$\int\limits_{\gamma_0} |\psi_0'(z)| \cdot |dz| = 2\pi, \tag{29.91}$$

so $\psi_0 : \gamma_0 \to C_0$ is a diffeomorphism.

Now if we pick $w \in \mathfrak{A}_\rho$, we see that $\psi_0 : \gamma_0 \to C_0$ has winding number 1 about w. We can apply the argument principle (Proposition 17.5), to deduce that there is a unique $z \in \Omega$ such that $\Psi(z) = w$. We have the following variant of the Riemann mapping theorem for annular regions.

Proposition 29.10. *If $\Omega \subset \mathbb{C}$ is a smoothly bounded domain with two boundary components, and Ψ is constructed as in (29.83)–(29.89), then Ψ is a holomorphic diffeomorphism from Ω onto $\mathfrak{A}_\rho$, extending to a smooth diffeomorphism from $\overline{\Omega}$ onto $\overline{\mathfrak{A}}_\rho$*

We note that if $0 < \rho < \sigma < 1$, then $\mathfrak{A}_\rho$ and $\mathfrak{A}_\sigma$ are not holomorphically equivalent. Indeed, if there were a holomorphic diffeomorphism $F : \mathfrak{A}_\rho \to \mathfrak{A}_\sigma$, then, using an inversion if necessary, we could assume F maps $\{|z| = 1\}$ to itself and maps $\{|z| = \rho\}$ to $\{|z| = \sigma\}$. Then, applying the Schwarz reflection principle an infinite number of times, we can extend F to a holomorphic diffeomorphism of $\overline{D}_1(0) = \{|z| \le 1\}$ onto itself, preserving the origin. Then we must have $F(z) = az,\ |a| = 1$, which would imply $\rho = \sigma$.

Exercises

1. Suppose $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and $|f(z)| \le Ae^{B|z|}$. Show that if f has only finitely many zeros then f has the form

$$f(z) = p(z)\, e^{az+b},$$

for some polynomial $p(z)$.
Hint. If $p(z)$ has the same zeros as $f(z)$, write $f(z)/p(z) = e^{g(z)}$ and apply Proposition 29.5 to Re $g(z)$.

2. Show that the function $e^z - z$ (considered in Exercise 10 of §6) has infinitely many zeros. More generally, show that, for each $a \in \mathbb{C}$, $e^z - z - a$ has infinitely many zeros.

3. The proof given above of Proposition 29.3 shows that (29.17) holds with $A = 47$. Can you show it holds with some smaller A?

4. Given $\rho \in (0, 1)$, consider the annulus $\mathfrak{A}_\rho = \{z \in \mathbb{C} : \rho < |z| < 1\}$, and the Dirichlet problem

$$\Delta u = 0 \ \text{ on } \ \mathfrak{A}_\rho, \quad u(e^{i\theta}) = f(\theta),\ u(\rho e^{i\theta}) = g(\theta). \tag{29.92}$$

Produce the solution in the form

$$u(re^{i\theta}) = \sum_{k \in \mathbb{Z} \setminus 0} (a_k r^k + b_k r^{-k}) e^{ik\theta} + a_0 + b_0 \log r.$$

Solve for a_k and b_k in terms of $\hat{f}(k)$ and $\hat{g}(k)$. Show that, if $f, g \in \mathcal{A}(S^1)$, defined in (13.28), then this series converges absolutely and uniformly on $\overline{\mathfrak{A}}_\rho$ to the solution to (29.92).

5. Consider $\Omega = \mathbb{C} \setminus \{\overline{D_1(-a)} \cup \overline{D_1(a)}\}$, with $a > 1$, as in (29.81). Find a linear fractional transformation mapping $\Omega \cup \{\infty\}$ conformally onto an annulus $\mathfrak{A}_\rho$, given as in Exercise 4. Use this to analyze the Dirichlet problem (29.73)–(29.74), in this setting.

D. Surfaces and metric tensors

A smooth m-dimensional surface $M \subset \mathbb{R}^n$ is characterized by the following property. Given $p \in M$, there is a neighborhood U of p in M and a smooth map $\varphi : \mathcal{O} \to U$, from an open set $\mathcal{O} \subset \mathbb{R}^m$ bijectively to U, with injective derivative at each point, and continuous inverse $\varphi^{-1} : U \to \mathcal{O}$. Such a map φ is called a *coordinate chart* on M. We call $U \subset M$ a coordinate patch. If all such maps φ are smooth of class C^k, we say M is a surface of class C^k.

There is an abstraction of the notion of a surface, namely the notion of a "manifold," which we discuss later in this appendix.

If $\varphi : \mathcal{O} \to U$ is a C^k coordinate chart, such as described above, and $\varphi(x_0) = p$, we set

$$T_pM = \operatorname{Range} D\varphi(x_0), \tag{D.1}$$

a linear subspace of $\mathbb{R}^n$ of dimension m, and we denote by N_pM its orthogonal complement. It is useful to consider the following map. Pick a linear isomorphism $A : \mathbb{R}^{n-m} \to N_pM$, and define

$$\Phi : \mathcal{O} \times \mathbb{R}^{n-m} \longrightarrow \mathbb{R}^n, \quad \Phi(x, z) = \varphi(x) + Az. \tag{D.2}$$

Thus Φ is a C^k map defined on an open subset of $\mathbb{R}^n$. Note that

$$D\Phi(x_0, 0) \begin{pmatrix} v \\ w \end{pmatrix} = D\varphi(x_0)v + Aw, \tag{D.3}$$

so $D\Phi(x_0, 0) : \mathbb{R}^n \to \mathbb{R}^n$ is surjective, hence bijective, so the Inverse Function Theorem applies; Φ maps some neighborhood of $(x_0, 0)$ diffeomorphically onto a neighborhood of $p \in \mathbb{R}^n$.

Suppose there is another C^k coordinate chart, $\psi : \Omega \to U$. Since φ and ψ are by hypothesis one-to-one and onto, it follows that $F = \psi^{-1} \circ \varphi : \mathcal{O} \to \Omega$ is a well defined map, which is one-to-one and onto. See Fig. D.1. Also F and F^{-1} are continuous. In fact, we can say more.

Lemma D.1. *Under the hypotheses above, F is a C^k diffeomorphism.*

Proof. It suffices to show that F and F^{-1} are C^k on a neighborhood of x_0 and y_0, respectively, where $\varphi(x_0) = \psi(y_0) = p$. Let us define a map Ψ in a fashion similar to (5.2). To be precise, we set $\widetilde{T}_pM = \operatorname{Range} D\psi(y_0)$, and let $\widetilde{N}_pM$ be its orthogonal complement. (Shortly we will show that $\widetilde{T}_pM = T_pM$, but we are not quite ready for that.) Then pick a linear isomorphism $B : \mathbb{R}^{n-m} \to \widetilde{N}_pM$ and set $\Psi(y, z) = \psi(y) + Bz$, for $(y, z) \in \Omega \times \mathbb{R}^{n-m}$. Again, Ψ is a C^k diffeomorphism from a neighborhood of $(y_0, 0)$ onto a neighborhood of p.

To be precise, there exist neighborhoods $\widetilde{\mathcal{O}}$ of $(x_0, 0)$ in $\mathcal{O} \times \mathbb{R}^{n-m}$, $\widetilde{\Omega}$ of $(y_0, 0)$ in $\Omega \times \mathbb{R}^{n-m}$, and $\widetilde{U}$ of p in $\mathbb{R}^n$ such that

$$\Phi : \widetilde{\mathcal{O}} \longrightarrow \widetilde{U}, \quad \text{and} \quad \Psi : \widetilde{\Omega} \longrightarrow \widetilde{U}$$

are C^k diffeomorphisms. It follows that $\Psi^{-1} \circ \Phi : \widetilde{\mathcal{O}} \to \widetilde{\Omega}$ is a C^k diffeomorphism. Now note that, for $(x_0, 0) \in \widetilde{\mathcal{O}}$ and $(y, 0) \in \widetilde{\Omega}$,

$$\Psi^{-1} \circ \Phi(x, 0) = \big(F(x), 0\big), \quad \Phi^{-1} \circ \Psi(y, 0) = \big(F^{-1}(y), 0\big). \tag{D.4}$$

In fact, to verify the first identity in (D.4), we check that

$$\begin{aligned} \Psi(F(x), 0) &= \psi(F(x)) + B0 \\ &= \psi(\psi^{-1} \circ \varphi(x)) \\ &= \varphi(x) \\ &= \Phi(x, 0). \end{aligned}$$

The identities in (D.4) imply that F and F^{-1} have the desired regularity.

Thus, when there are two such coordinate charts, $\varphi : \mathcal{O} \to U$, $\psi : \Omega \to U$, we have a C^k diffeomorphism $F : \mathcal{O} \to \Omega$ such that

$$\varphi = \psi \circ F. \tag{D.5}$$

By the chain rule,

$$D\varphi(x) = D\psi(y)\, DF(x), \quad y = F(x). \tag{D.6}$$

In particular this implies that $\text{Range}\, D\varphi(x_0) = \text{Range}\, D\psi(y_0)$, so T_pM in (D.1) is independent of the choice of coordinate chart. It is called the *tangent space* to M at p.

We next define an object called the *metric tensor* on M. Given a coordinate chart $\varphi : \mathcal{O} \to U$, there is associated an $m \times m$ matrix $G(x) = \big(g_{jk}(x)\big)$ of functions on $\mathcal{O}$, defined in terms of the inner product of vectors tangent to M :

$$g_{jk}(x) = D\varphi(x)e_j \cdot D\varphi(x)e_k = \frac{\partial \varphi}{\partial x_j} \cdot \frac{\partial \varphi}{\partial x_k} = \sum_{\ell=1}^{n} \frac{\partial \varphi_\ell}{\partial x_j} \frac{\partial \varphi_\ell}{\partial x_k}, \tag{D.7}$$

where $\{e_j : 1 \le j \le m\}$ is the standard orthonormal basis of $\mathbb{R}^m$. Equivalently,

$$G(x) = D\varphi(x)^t\, D\varphi(x). \tag{D.8}$$

We call (g_{jk}) the metric tensor of M, on U, with respect to the coordinate chart $\varphi : \mathcal{O} \to U$. Note that this matrix is positive-definite. From a coordinate-independent point of view, the metric tensor on M specifies inner products of vectors tangent to M, using the inner product of $\mathbb{R}^n$.

If we take another coordinate chart $\psi : \Omega \to U$, we want to compare (g_{jk}) with $H = (h_{jk})$, given by

$$h_{jk}(y) = D\psi(y)e_j \cdot D\psi(y)e_k, \quad \text{i.e.,} \;\; H(y) = D\psi(y)^t \; D\psi(y). \tag{D.9}$$

As seen above we have a diffeomorphism $F : \mathcal{O} \to \Omega$ such that (5.5)–(5.6) hold. Consequently,

$$G(x) = DF(x)^t \, H(y) \, DF(x), \tag{D.10}$$

or equivalently,

$$g_{jk}(x) = \sum_{i,\ell} \frac{\partial F_i}{\partial x_j} \frac{\partial F_\ell}{\partial x_k} \, h_{i\ell}(y). \tag{D.11}$$

We now define the notion of surface integral on M. If $f : M \to \mathbb{R}$ is a continuous function supported on U, we set

$$\int\limits_M f \, dS = \int\limits_{\mathcal{O}} f \circ \varphi(x) \, \sqrt{g(x)} \; dx, \tag{D.12}$$

where

$$g(x) = \det G(x). \tag{D.13}$$

We need to know that this is independent of the choice of coordinate chart $\varphi : \mathcal{O} \to U$. Thus, if we use $\psi : \Omega \to U$ instead, we want to show that (D.12) is equal to $\int_\Omega f \circ \psi(y) \, \sqrt{h(y)} \, dy$, where $h(y) = \det H(y)$. Indeed, since $f \circ \psi \circ F = f \circ \varphi$, we can apply the change of variable formula for multidimensional integrals, to get

$$\int\limits_\Omega f \circ \psi(y) \, \sqrt{h(y)} \; dy = \int\limits_{\mathcal{O}} f \circ \varphi(x) \, \sqrt{h(F(x))} \; |\det \, DF(x)| \; dx. \tag{D.14}$$

Now, (D.10) implies that

$$\sqrt{g(x)} = |\det \, DF(x)| \; \sqrt{h(y)}, \tag{D.15}$$

so the right side of (D.14) is seen to be equal to (D.12), and our surface integral is well defined, at least for f supported in a coordinate patch. More generally, if $f : M \to \mathbb{R}$ has compact support, write it as a finite sum of terms, each supported on a coordinate patch, and use (D.12) on each patch.

Let us pause to consider the special cases $m = 1$ and $m = 2$. For $m = 1$, we are considering a curve in $\mathbb{R}^n$, say $\varphi : [a,b] \to \mathbb{R}^n$. Then $G(x)$ is a 1×1 matrix, namely $G(x) = |\varphi'(x)|^2$. If we denote the curve in $\mathbb{R}^n$ by γ, rather than M, the formula (D.12) becomes

$$\int_\gamma f \, ds = \int_a^b f \circ \varphi(x) \, |\varphi'(x)| \, dx. \tag{D.16}$$

In case $m = 2$, let us consider a surface $M \subset \mathbb{R}^3$, with a coordinate chart $\varphi : \mathcal{O} \to U \subset M$. For f supported in U, an alternative way to write the surface integral is

$$\int_M f \, dS = \int_{\mathcal{O}} f \circ \varphi(x) \, |\partial_1 \varphi \times \partial_2 \varphi| \, dx_1 dx_2, \tag{D.17}$$

where $u \times v$ is the cross product of vectors u and v in $\mathbb{R}^3$. To see this, we compare this integrand with the one in (D.12). In this case,

$$g = \det \begin{pmatrix} \partial_1\varphi \cdot \partial_1\varphi & \partial_1\varphi \cdot \partial_2\varphi \\ \partial_2\varphi \cdot \partial_1\varphi & \partial_2\varphi \cdot \partial_2\varphi \end{pmatrix} = |\partial_1\varphi|^2 |\partial_2\varphi|^2 - (\partial_1\varphi \cdot \partial_2\varphi)^2. \tag{D.18}$$

Recall that $|u \times v| = |u| \; |v| \; |\sin\theta|$, where θ is the angle between u and v. Equivalently, since $u \cdot v = |u| \; |v| \; \cos\theta$,

$$|u \times v|^2 = |u|^2 |v|^2 \big(1 - \cos^2\theta\big) = |u|^2|v|^2 - (u \cdot v)^2. \tag{D.19}$$

Thus we see that $|\partial_1\varphi \times \partial_2\varphi| = \sqrt{g}$, in this case, and (D.17) is equivalent to (D.12).

An important class of surfaces is the class of graphs of smooth functions. Let $u \in C^1(\Omega)$, for an open $\Omega \subset \mathbb{R}^{n-1}$, and let M be the graph of $z = u(x)$. The map $\varphi(x) = \big(x, u(u)\big)$ provides a natural coordinate system, in which the metric tensor is given by

$$g_{jk}(x) = \delta_{jk} + \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_k}. \tag{D.20}$$

If u is C^1, we see that g_{jk} is continuous. To calculate $g = \det(g_{jk})$, at a given point $p \in \Omega$, if $\nabla u(p) \neq 0$, rotate coordinates so that $\nabla u(p)$ is parallel to the x_1 axis. We see that

$$\sqrt{g} = \big(1 + |\nabla u|^2\big)^{1/2}. \tag{D.21}$$

In particular, the $(n-1)$-dimensional volume of the surface M is given by

$$V_{n-1}(M) = \int_M dS = \int_\Omega \big(1 + |\nabla u(x)|^2\big)^{1/2} \, dx. \tag{D.22}$$

Particularly important examples of surfaces are the unit spheres S^{n-1} in $\mathbb{R}^n$,

$$S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}. \tag{D.23}$$

Spherical polar coordinates on $\mathbb{R}^n$ are defined in terms of a smooth diffeomorphism

$$R : (0,\infty) \times S^{n-1} \longrightarrow \mathbb{R}^n \setminus 0, \quad R(r,\omega) = r\omega. \tag{D.24}$$

Let $(h_{\ell m})$ denote the metric tensor on S^{n-1} (induced from its inclusion in $\mathbb{R}^n$) with respect to some coordinate chart $\varphi : \mathcal{O} \to U \subset S^{n-1}$. Then, with respect to the coordinate chart $\Phi : (0,\infty) \times \mathcal{O} \to \mathcal{U} \subset \mathbb{R}^n$ given by $\Phi(r,y) = r\varphi(y)$, the Euclidean metric tensor can be written

$$(e_{jk}) = \begin{pmatrix} 1 & \\ & r^2 h_{\ell m} \end{pmatrix}. \tag{D.25}$$

To see that the blank terms vanish, i.e., $\partial_r \Phi \cdot \partial_{x_j} \Phi = 0$, note that $\varphi(x) \cdot \varphi(x) = 1 \Rightarrow \partial_{x_j} \varphi(x) \cdot \varphi(x) = 0$. Now (D.25) yields

$$\sqrt{e} = r^{n-1} \sqrt{h}. \tag{D.26}$$

We therefore have the following result for integrating a function in spherical polar coordinates.

$$\int\limits_{\mathbb{R}^n} f(x)\, dx = \int\limits_{S^{n-1}} \Big[\int_0^\infty f(r\omega) r^{n-1}\, dr \Big]\, dS(\omega). \tag{D.27}$$

We next compute the $(n-1)$-dimensional area A_{n-1} of the unit sphere $S^{n-1} \subset \mathbb{R}^n$, using (D.27) together with the computation

$$\int\limits_{\mathbb{R}^n} e^{-|x|^2}\, dx = \pi^{n/2}, \tag{D.28}$$

which can be reduced to the case $n = 2$ and done there in polar coordinates (see (10.6)). First note that, whenever $f(x) = \varphi(|x|)$, (D.27) yields

$$\int\limits_{\mathbb{R}^n} \varphi(|x|)\, dx = A_{n-1} \int_0^\infty \varphi(r) r^{n-1}\, dr. \tag{D.29}$$

In particular, taking $\varphi(r) = e^{-r^2}$ and using (D.28), we have

$$\pi^{n/2} = A_{n-1} \int_0^\infty e^{-r^2} r^{n-1}\, dr = \frac{1}{2} A_{n-1} \int_0^\infty e^{-s} s^{n/2-1}\, ds, \tag{D.30}$$

where we used the substitution $s = r^2$ to get the last identity. We hence have

$$A_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \tag{D.31}$$

where $\Gamma(z)$ is Euler's Gamma function, defined for $z > 0$ by

$$\Gamma(z) = \int_0^\infty e^{-s} s^{z-1} \, ds. \tag{D.32}$$

The gamma function is developed in §18.

Having discussed surfaces in $\mathbb{R}^n$, we turn to the more general concept of a *manifold*, useful for the construction in §34. A manifold is a metric space with an "atlas," i.e., a covering by open sets U_j together with homeomorphisms $\varphi_j : U_j \to V_j$, V_j open in $\mathbb{R}^n$. The number n is called the dimension of M. We say that M is a smooth manifold of class C^k provided the atlas has the following property. If $U_{jk} = U_j \cap U_k \neq \emptyset$, then the map

$$\psi_{jk} : \varphi_j(U_{jk}) \to \varphi_k(U_{jk})$$

given by $\varphi_k \circ \varphi_j^{-1}$, is a smooth diffeomorphism of class C^k from the open set $\varphi_j(U_{jk})$ to the open set $\varphi_k(U_{jk})$ in $\mathbb{R}^n$. By this, we mean that ψ_{jk} is C^k, with a C^k inverse. The pairs (U_j, φ_j) are called local coordinate charts.

A continuous map from M to another smooth manifold N is said to be smooth of class C^k if it is smooth of class C^k in local coordinates. Two different atlasses on M, giving a priori two structures of M as a smooth manifold, are said to be equivalent if the identity map on M is smooth (of class C^k) from each one of these two manifolds to the other. Really a smooth manifold is considered to be defined by equivalence classes of such atlasses, under this equivalence relation.

It follows from Lemma D.1 that a C^k surface in $\mathbb{R}^n$ is a smooth manifold of class C^k. Other examples of smooth manifolds include tori $\mathbb{T}/\Lambda$, introduced in §22 and used in §34, as well as other Riemann surfaces discussed in §34. One can find more material on manifolds in [Sp] and in [T2].

The notion of a metric tensor generalizes readily from surfaces in $\mathbb{R}^n$ to smooth manifolds; this leads to the notion of an integral on a manifold with a metric tensor (i.e., a Riemannian manifold). For use in Appendix E, we give details about metric tensors, in case M is covered by one coordinate chart,

$$\varphi_1 : U_1 \longrightarrow M, \tag{D.33}$$

with $U_1 \subset \mathbb{R}^n$ open. In such a case, a metric tensor on M is defined by an $n \times n$ matrix

$$G(x) = \big(g_{jk}(x)\big), \quad g_{jk} \in C^k(U_1), \tag{D.34}$$

which is taken to be symmetric and positive definite, generalizing the set-up in (D.7). If there is another covering of M by a coordinate chart $\varphi_2 : U_2 \to M$, a positive definite matrix H on U_2 defines the same metric tensor on M provided

$$G(x) = DF(x)^t H(y) DF(x), \quad \text{for} \ \ y = F(x), \tag{D.35}$$

as in (D.10), where F is the diffeomorphism

$$F = \varphi_2^{-1} \circ \varphi_1 : U_1 \longrightarrow U_2. \tag{D.36}$$

We also say that G defines a metric tensor on U_1 and H defines a metric tensor on U_2, and the diffeomorphism $F : U_1 \to U_2$ pulls H back to G.

Let $\gamma : [a, b] \to U_1$ be a C^1 curve. The following is a natural generalization of (D.16), defining the integral of a function $f \in C(U_1)$ over γ, with respect to arc length:

$$\int\limits_\gamma f\, ds = \int_a^b f(\gamma(t)) \Big[\gamma'(t) \cdot G(\gamma(t))\gamma'(t)\Big]^{1/2}\, dt. \tag{D.37}$$

If $\tilde{\gamma} = F \circ \gamma$ is the associated curve on U_2 and if $\tilde{f} = f \circ F \in C(U_2)$, we have

$$\begin{aligned} \int\limits_{\tilde{\gamma}} \tilde{f}\, ds &= \int_a^b \tilde{f}(\tilde{\gamma}(t)) \Big[\tilde{\gamma}'(t) \cdot H(\tilde{\gamma}(t))\tilde{\gamma}'(t)\Big]^{1/2}\, dt \\ &= \int_a^b f(\gamma(t)) \Big[DF(\gamma(t))\gamma'(t) \cdot H(\tilde{\gamma}(t))DF(\gamma(t))\gamma'(t)\Big]^{1/2}\, dt \\ &= \int_a^b f(\gamma(t)) \Big[\gamma'(t) \cdot G(x)\gamma'(t)\Big]^{1/2}\, dt \\ &= \int\limits_\gamma f\, ds, \end{aligned} \tag{D.38}$$

the second identity by the chain rule $\tilde{\gamma}'(t) = DF(\gamma(t))\gamma'(t)$ and the third identity by (D.35). Another property of this integral is parametrization invariance. Say $\psi : [\alpha, \beta] \to [a, b]$ is an order preserving C^1 diffeomorphism and $\sigma = \gamma \circ \psi : [\alpha, \beta] \to U_1$. Then

$$\begin{aligned} \int\limits_\sigma f\, ds &= \int_\alpha^\beta f(\sigma(t)) \Big[\sigma'(t) \cdot G(\sigma(t))\sigma'(t)\Big]^{1/2}\, dt \\ &= \int_\alpha^\beta f(\gamma \circ \psi(t)) \Big[\psi'(t)^2 \gamma'(\psi(t)) \cdot G(\gamma \circ \psi(t))\gamma'(\psi(t))\Big]^{1/2}\, dt \\ &= \int_a^b f(\gamma(\tau)) \Big[\gamma'(\tau) \cdot G(\gamma(\tau))\gamma'(\tau)\Big]^{1/2}\, d\tau \\ &= \int\limits_\gamma f\, ds, \end{aligned} \tag{D.39}$$

the second identity by the chain rule $\sigma'(t) = \psi'(t)\gamma'(\sigma(t))$ and the third identity via the change of variable $\tau = \psi(t), d\tau = \psi'(t)\, dt$.

The arc length of these curves is defined by integrating 1. We have

$$\ell_G(\gamma) = \int_a^b \Big[\gamma'(t) \cdot G(\gamma(t))\gamma'(t)\Big]^{1/2}\, dt, \tag{D.40}$$

and a parallel definition of $\ell_H(\tilde{\gamma})$. With $\gamma, \tilde{\gamma}$, and σ related as in (D.38)–(D.39), we have

$$\ell_G(\gamma) = \ell_G(\sigma) = \ell_H(\tilde{\gamma}). \tag{D.41}$$

Another useful piece of notation associated with the metric tensor G is

$$ds^2 = \sum_{j,k} g_{jk}(x)\, dx_j\, dx_k. \tag{D.42}$$

In case $U_1 \subset \mathbb{R}^2 \approx \mathbb{C}$, this becomes

$$ds^2 = g_{11}(x,y)\, dx^2 + 2g_{12}(x,y)\, dx\, dy + g_{22}(x,y)\, dy^2. \tag{D.43}$$

In case

$$g_{jk}(x) = A(x)^2\, \delta_{jk}, \tag{D.44}$$

we have

$$ds^2 = A(x)^2 (dx_1^2 + \cdots + dx_n^2). \tag{D.45}$$

For $n = 2$, this becomes

$$ds^2 = A(x,y)^2 (dx^2 + dy^2) = A(z)^2\, |dz|^2, \tag{D.46}$$

or

$$ds = A(z)\, |dz|. \tag{D.47}$$

In such a case, (D.40) becomes

$$\ell_G(\gamma) = \int_a^b A(\gamma(t))\, |\gamma'(t)|\, dt. \tag{D.48}$$

Under the change of variable $y = F(x)$, the formula (D.35) for the metric tensor $H = (h_{jk})$ on U_2 that pulls back to G under $F : U_1 \to U_2$ is equivalent to

$$\sum_{j,k} h_{jk}(y)\, dy_j\, dy_k = \sum_{j,k} g_{jk}(x)\, dx_j\, dx_k, \quad dy_j = \sum_\ell \frac{\partial F_j}{\partial x_\ell}\, dx_\ell. \tag{D.49}$$

E. Poincaré metrics

Recall from §20 that the upper half plane

$$\mathcal{U} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\} \tag{E.1}$$

is invariant under the group of linear fractional transformations

$$L_A(z) = \frac{az+b}{cz+d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{E.2}$$

of the form

$$A \in Sl(2,\mathbb{R}), \quad \text{i.e.,} \quad a,b,c,d \in \mathbb{R}, \ ad - bc = 1, \tag{E.3}$$

while the disk

$$D = \{z \in \mathbb{C} : |z| < 1\} \tag{E.4}$$

is invariant under the group of transformations L_B with

$$B \in SU(1,1), \quad \text{i.e.,} \quad B = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}, \quad |a|^2 - |b|^2 = 1. \tag{E.5}$$

These groups are related by the holomorphic diffeomorphism

$$L_{A_0} = \varphi : \mathcal{U} \longrightarrow D, \quad A_0 = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}, \quad \text{i.e.,} \quad \varphi(z) = \frac{z-i}{z+i}. \tag{E.6}$$

Here we produce metric tensors on $\mathcal{U}$ and D that are invariant under these respective group actions. These are called Poincaré metrics.

We start with $\mathcal{U}$, and the metric tensor

$$g_{jk}(z) = \frac{1}{y^2}\,\delta_{jk}, \tag{E.7}$$

i.e., $ds_{\mathcal{U}}^2 = (dx^2 + dy^2)/y^2$, or equivalently

$$ds_{\mathcal{U}} = \frac{1}{y}\,|dz|. \tag{E.8}$$

This is easily seen to be invariant under horizontal translations and dilations,

$$\tau_\xi(z) = z + \xi, \quad \xi \in \mathbb{R}, \quad \delta_r(z) = rz, \quad r \in (0,\infty). \tag{E.9}$$

Note that

$$\tau_\xi = L_{T_\xi}, \quad \delta_r = L_{D_r}, \quad T_\xi = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}, \quad D_r = \begin{pmatrix} r^{1/2} & 0 \\ 0 & r^{-1/2} \end{pmatrix}. \tag{E.10}$$

The group generated by these operations is not all of $Sl(2,\mathbb{R})$. We proceed as follows.

We pull the metric tensor (E.7) back to the disk D, via $\psi = \varphi^{-1}$, given by

$$\psi(z) = \frac{1}{i}\frac{z+1}{z-1}, \quad \psi : D \longrightarrow \mathcal{U}. \tag{E.11}$$

We have

$$\psi^* dz = \psi'(z)\, dz = \frac{2i}{(z-1)^2}\, dz, \tag{E.12}$$

and hence

$$\begin{aligned} \psi^* ds_{\mathcal{U}} &= \frac{1}{\operatorname{Im}\psi(z)}\, |\psi^* dz| \\ &= \frac{2i}{\psi(z) - \overline{\psi(z)}} |\psi'(z)| \cdot |dz| \\ &= -\frac{4}{\frac{z+1}{z-1} + \frac{\overline{z}+1}{\overline{z}-1}} \cdot \frac{|dz|}{(z-1)(\overline{z}-1)} \\ &= \frac{2}{1-|z|^2}\, |dz|. \end{aligned} \tag{E.13}$$

Thus we arrive at the metric tensor on the disk, given by

$$ds_D = \frac{2}{1-|z|^2}\, |dz|, \tag{E.14}$$

or

$$h_{jk}(z) = \frac{4}{(1-|z|^2)^2}\, \delta_{jk}. \tag{E.15}$$

This metric tensor is invariant under

$$L_{\widetilde{T}_\xi}, \ L_{\widetilde{D}_r}, \quad \widetilde{T}_\xi = A_0 T_\xi A_0^{-1}, \ \widetilde{D}_r = A_0 D_r A_0^{-1}. \tag{E.16}$$

In addition, the metric tensor (E.15) is clearly invariant under rotations

$$\rho_\theta(z) = e^{i\theta} z, \quad \theta \in \mathbb{R}. \tag{E.17}$$

Here

$$\rho_\theta = L_{R_\theta}, \quad R_\theta = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}. \tag{E.18}$$

The transformations $\widetilde{T}_\xi$, $\widetilde{D}_r$ and R_θ can be seen to generate all of $SU(1,1)$, which implies the metric tensor (E.11) on D is invariant under all the linear fractional transformations (E.5), and hence the metric tensor (E.7) on $\mathcal{U}$ is invariant under all the linear fractional transformations (E.2)–(E.3). Alternatively, one can check directly that

$$\varphi_{a,b}^* \, ds_D = ds_D, \tag{E.19}$$

when

$$\varphi_{a,b}(z) = \frac{az+b}{\bar{b}z+\overline{a}}, \quad |a|^2 - |b|^2 = 1. \tag{E.20}$$

In fact,

$$\varphi_{a,b}^* \, dz = \varphi_{a,b}'(z)\, dz = \frac{dz}{(\bar{b}z+\overline{a})^2}, \tag{E.21}$$

and hence

$$\begin{aligned} \varphi_{a,b}^* \, ds_D &= \frac{2}{1-|\varphi_{a,b}(z)|^2} |\varphi_{a,b}'(z)| \cdot |dz| \\ &= \frac{2\,|dz|}{|\bar{b}z+\overline{a}|^2 - |az+b|^2} \\ &= \frac{2\,|dz|}{1-|z|^2} \\ &= ds_D. \end{aligned} \tag{E.22}$$

Let us record the result formally.

Proposition E.1. *The metric tensor (E.15) on D is invariant under all linear fractional transformations of the form (E.20). Hence the metric tensor (E.7) on $\mathcal{U}$ is invariant under all linear fractional transformations of the form (E.2)–(E.3).*

These metric tensors are called Poincaré metrics, and $\mathcal{U}$ and D, equipped with these metric tensors, are called the Poincaré upper half plane and the Poincaré disk.

Let $\Omega \subset \mathbb{C}$ be a simply connected domain, $\neq \mathbb{C}$. The Riemann mapping theorem produces a holomorphic diffeomorphism

$$\Phi : \Omega \longrightarrow D. \tag{E.23}$$

The pull back of the Poincaré metric on D via Φ is called the Poincaré metric on Ω. Note that if $\widetilde{\Phi} : \Omega \to D$ is another holomorphic diffeomorphism, then $\Phi \circ \widetilde{\Phi}^{-1} : D \to D$ is a holomorphic diffeomorphism, hence, as seen in §22, a linear fractional transformation of the form (E.20), hence it preserves the Poincaré metric on D, so Φ and $\widetilde{\Phi}$ pull back the Poincaré metric to the same metric tensor on Ω.

More generally, a domain $\Omega \subset \mathbb{C}$ inherits a Poincaré metric whenever there is a holomorphic covering map

$$\psi : D \longrightarrow \Omega. \tag{E.24}$$

In fact, for each $q \in \Omega$, if you choose $p \in \psi^{-1}(q)$, ψ is a holomorphic diffeomorphism from a neighborhood $\mathcal{O}_p$ of p onto a neighborhood $\mathcal{O}_q$ of q, and the Poincaré metric on D pulls back via $\psi^{-1} : \mathcal{O}_q \to \mathcal{O}_p$. This is independent of the choice of $p \in \psi^{-1}(q)$, since two such inverses ψ^{-1} and $\tilde{\psi}^{-1} : \mathcal{O}_q \to \mathcal{O}_{p'}$ are related by a covering map on D, which must be of the form (E.20). For the same reason, any other covering map $D \to \Omega$ produces the same metric on Ω, so one has a well defined Poincaré metric on Ω, whenever there is a holomorphic covering map (E.24). Such a metric tensor is always a multiple of δ_{jk} in standard (x, y)-coordinates,

$$g_{jk}(x, y) = A_\Omega(x, y)^2 \, \delta_{jk}, \tag{E.25}$$

or

$$ds_\Omega = A_\Omega(z) \, |dz|, \tag{E.26}$$

where $A_\Omega : \Omega \to (0, \infty)$. In fact, on a neighborhood $\mathcal{O}_q$ of $q \in \Omega$ where there is a local inverse φ_q to ψ,

$$A_\Omega(z) = A_D(\varphi_q(z)) \, |\varphi_q'(z)|, \quad \text{i.e.,} \ \ A_\Omega(\psi(z)) = \frac{A_D(z)}{|\psi'(z)|}, \tag{E.27}$$

with A_D given by (E.14), i.e.,

$$A_D(z) = \frac{2}{1 - |z|^2}. \tag{E.28}$$

The following is a definitive result on the class of domains to which such a construction applies.

Theorem E.2. *If $\Omega \subset \mathbb{C}$ is a connected open set and $\mathbb{C} \setminus \Omega$ contains at least two points, then there is a holomorphic covering map (E.24).*

This is part of the celebrated Uniformization Theorem, of which one can read a careful account in [For]. We will not give a proof of Theorem E.2 here. A proof using basic results about partial differential equations is given in [MaT]. We recall that §26 establishes this result for $\Omega = \mathbb{C} \setminus \{0, 1\}$.

To see how Theorem E.2 works for $D^* = D \setminus \{0\}$, note that

$$\Psi : \mathcal{U} \longrightarrow D^*, \quad \Psi(z) = e^{iz}, \tag{E.29}$$

is a holomorphic covering map, and composing with the inverse of φ in (E.6) yields a holomorphic covering map $D \to D^*$. We can calculate $ds_{D^*}(z)$ as follows. Parallel to (E.27), we can write

$$ds_{D^*}(z) = A_{D^*}(z)\,|dz|, \quad A_{D^*}(\Psi(z)) = \frac{A_{\mathcal{U}}(z)}{|\Psi'(z)|}, \tag{E.29A}$$

where $ds_{\mathcal{U}}(z) = A_{\mathcal{U}}(z)\,|dz|$, so $A_{\mathcal{U}}(z) = 1/\operatorname{Im} z$. Then from $\Psi(z) = e^{iz}$, we get

$$A_{D^*}(z) = \frac{1}{|z| \log 1/|z|}, \quad \text{i.e., } ds_{D^*}(z) = \frac{|dz|}{|z| \log 1/|z|}. \tag{E.29B}$$

The following interesting result is a geometric version of the Schwarz lemma.

Proposition E.3. *Assume $\mathcal{O}$ and Ω are domains in $\mathbb{C}$ with Poincaré metrics, inherited from holomorphic coverings by D, and $F : \mathcal{O} \to \Omega$ is holomorphic. Then F is distance-decreasing.*

What is meant by "distance decreasing" is the following. Let $\gamma : [a, b] \to \mathcal{O}$ be a smooth path. Its length, with respect to the Poincaré metric on $\mathcal{O}$, is

$$\ell_{\mathcal{O}}(\gamma) = \int_a^b A_{\mathcal{O}}(\gamma(s))|\gamma'(s)|\,ds. \tag{E.30}$$

The assertion that F is distance decreasing is that, for all such paths γ,

$$\ell_\Omega(F \circ \gamma) \le \ell_{\mathcal{O}}(\gamma). \tag{E.31}$$

Note that $\ell_\Omega(F \circ \gamma) = \int_a^b A_\Omega(F \circ \gamma(s))|(F \circ \gamma)'(s)|\,ds$ and $(F \circ \gamma)(s) = F'(\gamma(s))\gamma'(s)$, so the validity of (E.31) for all paths is equivalent to

$$A_\Omega(F(z))|F'(z)| \le A_{\mathcal{O}}(z), \quad \forall\, z \in \mathcal{O}. \tag{E.32}$$

To prove Proposition E.3, note that under the hypotheses given there, $F : \mathcal{O} \to \Omega$ lifts to a holomorphic map $G : D \to D$, and it suffices to show that any such G is distance decreasing, for the Poincaré metric on D, i.e.,

$$G : D \to D \ \text{ holomorphic } \Longrightarrow A_D(G(z_0))|G'(z_0)| \le A_D(z_0), \quad \forall\, z_0 \in D. \tag{E.33}$$

Now, given $z_0 \in D$, we can compose G on the right with a linear fractional transformation of the form (E.20), taking 0 to z_0, and on the left by such a linear fractional transformation, taking $G(z_0)$ to 0, obtaining

$$H : D \longrightarrow D \ \text{ holomorphic}, \ H(0) = 0, \tag{E.34}$$

and the desired conclusion is that

$$|H'(0)| \le 1, \tag{E.35}$$

which follows immediately from the inequality

$$|H(z)| \le |z|. \tag{E.36}$$

This in turn is the conclusion of the Schwarz lemma, Proposition 6.2.

Using these results, we will give another proof of Picard's big theorem, Proposition 26.2.

Proposition E.4. *If $f : D^* \to \mathbb{C} \setminus \{0,1\}$ is holomorphic, then the singularity at 0 is either a pole or a removable singularity.*

To start, we assume 0 is not a pole or removable singularity, and apply the Casorati-Weierstrass theorem, Proposition 11.3, to deduce the existence of $a_j, b_j \in D^*$ such that $a_j, b_j \to 0$ and

$$p_j = f(a_j) \to 0, \quad q_j = f(b_j) \to 1, \tag{E.37}$$

as $j \to \infty$. Let γ_j be the circle centered at 0 of radius $|a_j|$ and σ_j the circle centered at 0 of radius $|b_j|$. An examination of (E.29B) reveals that

$$\ell_{D^*}(\gamma_j) \to 0, \quad \ell_{D^*}(\sigma_j) \to 0. \tag{E.38}$$

Applying Proposition E.3 (with $\mathcal{O} = D^*,\ \Omega = \mathbb{C} \setminus \{0,1\}$), we obtain for

$$\tilde{\gamma}_j = f \circ \gamma_j, \quad \tilde{\sigma}_j = f \circ \sigma_j \tag{E.39}$$

that (with $\mathbb{C}_{**} = \mathbb{C} \setminus \{0,1\}$)

$$\ell_{\mathbb{C}_{**}}(\tilde{\gamma}_j) \to 0, \quad \ell_{\mathbb{C}_{**}}(\tilde{\sigma}_j) \to 0. \tag{E.40}$$

We now bring in the following:

Lemma E.5. *Fix $z_0 \in \mathbb{C}_{**} = \mathbb{C} \setminus \{0,1\}$. Let τ_j be paths from z_0 to $p_j \to 0$. Then $\ell_{\mathbb{C}_{**}}(\tau_j) \to \infty$. A parallel result holds for paths from z_0 to $q_j \to 1$.*

Proof. Let $\psi : D \to \mathbb{C}_{**}$ be a holomorphic covering, $\tilde{z}_0 \in \psi^{-1}(z_0)$, and $\tilde{\tau}_j$ a lift of τ_j to a path in D starting at $\tilde{z}_0$. Then

$$\ell_{\mathbb{C}_{**}}(\tau_j) = \ell_D(\tilde{\tau}_j). \tag{E.41}$$

Now $\tilde{\tau}_j$ contains points that tend to ∂D (in the Euclidean metric) as $j \to \infty$, so the fact that $\ell_D(\tilde{\tau}_j) \to \infty$ follows easily from the formula (E.14).

Returning to the setting of (E.37)–(E.40), we deduce that

$$\begin{aligned} &\text{Given } \ \varepsilon > 0,\ \exists\, N < \infty \ \text{ such that whenever } j \ge N \\ &f \circ \gamma_j = \tilde{\gamma}_j \subset \{z \in \mathbb{C} : |z| < \varepsilon\}, \ \text{ and} \\ &f \circ \sigma_j = \tilde{\sigma}_j \subset \{z \in \mathbb{C} : |z-1| < \varepsilon\}. \end{aligned} \tag{E.42}$$

If, e.g., $\varepsilon < 1/4$, this cannot hold for a function f that is holomorphic on a neighborhood of the annular region bounded by γ_j and σ_j. (Cf. §6, Exercise 2.) Thus our assumption on f contradicts the Casorati-Weierstrass theorem, and Proposition E.4 is proven.

Returning to the setting of Proposition E.3, let us explicitly check the conclusion for

$$\mathrm{Sq} : D \longrightarrow D, \quad \mathrm{Sq}(z) = z^2. \tag{E.43}$$

In this case, we have $\mathrm{Sq}^*\, dz = 2z\, dz$, hence

$$\begin{aligned}\mathrm{Sq}^*\, ds_D(z) &= \frac{2}{1-|z|^4}\,|\,\mathrm{Sq}^*\, dz| = \frac{2}{1-|z|^4}\cdot|2z|\cdot|dz| \\ &= \frac{|2z|}{1+|z|^2}\cdot\frac{2}{1-|z|^2}\,|dz| = \frac{|2z|}{1+|z|^2}\,ds_D(z) \\ &= \Big[1-\frac{(1-|z|)^2}{1+|z|^2}\Big]ds_D(z).\end{aligned} \tag{E.44}$$

We relate how this distance-decreasing result for $\mathrm{Sq}: D \to D$ leads to a "non-computational" proof of Proposition 23.4, an important ingredient in the proof of the Riemann mapping theorem. We recall the result.

Proposition E.6. *Let $\mathcal{O} \subset D$ be an open, simply connected subset. Assume $0 \in \mathcal{O}$ but $q \notin \mathcal{O}$ for some $q \in D$. Then there exists a one-to-one holomorphic map*

$$K : \mathcal{O} \to D, \quad \textit{satisfying} \;\; K(0) = 0,\;\; K'(0) > 1. \tag{E.45}$$

Proof. As in §23, we define $F : \mathcal{O} \to D$ by

$$F(z) = \sqrt{\varphi_q(z)}, \tag{E.46}$$

where, for each $b \in D$, $\varphi_b : D \to D$ is the automorphism

$$\varphi_b(z) = \frac{z-b}{1-\bar{b}z}. \tag{E.47}$$

Then we take

$$K(z) = \alpha\varphi_{F(0)}(F(z)), \quad \alpha = \frac{|F'(0)|}{F'(0)}. \tag{E.48}$$

We have $F(0) = \sqrt{-q}$ and $K(0) = \varphi_{F(0)}(F(0)) = 0$. The maps φ_q and $\varphi_{F(0)}$ are both automorphisms of D, preserving the Poincaré metric. The image set $U = \varphi_q(\mathcal{O})$ is a simply connected set that does not contain 0, and the square root function

$$\mathrm{Sqrt} : U \to D \tag{E.49}$$

is well defined (up to a choice of sign). We have

$$K(z) = \alpha\varphi_{F(0)} \circ \mathrm{Sqrt} \circ \varphi_q. \tag{E.50}$$

As observed, φ_q and $\varphi_{F(0)}$ preserve the Poincaré metric. On the other hand, in light of the fact that $\mathrm{Sq} : D \to D$ strictly decreases the Poincaré metric, we see that $\mathrm{Sqrt} : U \to D$ strictly increases the Poincaré metric. The result $K'(0) > 1$ is simply the observation that K strictly increases the Poincaré metric at 0.

Chapter 6. Elliptic functions and elliptic integrals

Elliptic functions are meromorphic functions on $\mathbb{C}$ satisfying a periodicity condition of the form

$$f(z+\omega) = f(z), \quad \forall\, \omega \in \Lambda, \tag{6.0.1}$$

where $\Lambda \subset \mathbb{C}$ is a lattice, of the form

$$\Lambda = \{j\omega_1 + k\omega_2 : j, k \in \mathbb{Z}\}, \tag{6.0.2}$$

with $\omega_1, \omega_2 \in \mathbb{C}$, linearly independent over $\mathbb{R}$. If such f had no poles it would be bounded, hence constant, so the interesting elliptic functions have poles. Such functions can also be regarded as meromorphic functions on the compact Riemann surface

$$\mathbb{T}_\Lambda = \mathbb{C}/\Lambda. \tag{6.0.3}$$

As a first attempt to construct such a function, one might try

$$f(z) = \sum_{\omega\in\Lambda} \frac{1}{(z-\omega)^2}, \tag{6.0.4}$$

but this series does not quite converge. The following modification works,

$$\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{0\neq\omega\in\Lambda} \Bigl(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\Bigr), \tag{6.0.5}$$

and defines the Weierstrass $\wp$ function.

The Weierstrass $\wp$ function plays a central role in elliptic function theory. This arises from the fact that every meromorphic function satisfying (6.0.1) has the form

$$f(z) = Q(\wp(z)) + R(\wp(z))\wp'(z), \tag{6.0.6}$$

for some rational functions Q and R. (We drop the Λ from the notation.) A key example is the identity

$$\wp'(z)^2 = 4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3), \tag{6.0.7}$$

where

$$e_j = \wp\Bigl(\frac{\omega_j}{2}\Bigr), \quad j = 1, 2, 3, \tag{6.0.8}$$

with ω_1, ω_2 as in (6.0.2) and $\omega_3 = \omega_1 + \omega_2$. The equation (6.0.7) can be regarded as a differential equation for $\wp$. This is amenable to separation of variables, yielding

$$\frac{1}{2} \int \frac{d\wp}{\sqrt{(\wp - e_1)(\wp - e_2)(\wp - e_3)}} = z + c. \tag{6.0.9}$$

The left side of (6.0.9) is an elliptic integral.

Generally, an elliptic integral is an integral of the form

$$\int_{\zeta_0}^{\zeta_1} R(\zeta, \sqrt{q(\zeta)})\, d\zeta, \tag{6.0.10}$$

where R is a rational function of its arguments, and

$$q(\zeta) = (\zeta - e_1)(\zeta - e_2)(\zeta - e_3), \tag{6.0.11}$$

with $e_j \in \mathbb{C}$ distinct. By a coordinate translation, we can arrange that

$$e_1 + e_2 + e_3 = 0. \tag{6.0.12}$$

One has the following result.

Theorem. Given distinct e_j satisfying (6.0.12), there exists a lattice, of the form (6.0.2), such that if $\wp(z) = \wp(z; \Lambda)$, then

$$\wp\Big(\frac{\omega_j}{2}\Big) = e_j, \quad 1 \le j \le 3, \tag{6.0.13}$$

where $\omega_3 = \omega_1 + \omega_2$.

Given this result, we have (6.0.7), and hence

$$\frac{1}{2} \int_{\wp(z_0)}^{\wp(z)} \frac{d\zeta}{\sqrt{q(\zeta)}} = z - z_0, \quad \text{mod} \quad \Lambda.$$

The problem of proving this theorem is known as the Abel inversion problem. The proof given in this chapter involves constructing a compact Riemann surface associated to the "double valued" function $\sqrt{q(\zeta)}$ and showing that it is holomorphically diffeomorphic to $\mathbb{T}_\Lambda$ for a lattice Λ.

Elliptic integrals arise in many situations, from computing lengths of ellipses to integrating equations of basic physics, such as the pendulum. A discussion of such applications is given in §33. In that section there is also a discussion of reducing (6.0.10) to more basic forms, using identities of the form

$$R(\wp(z), \tfrac{1}{2}\wp'(z)) = R_1(\wp(z)) + R_2(\wp(z))\wp'(z). \tag{6.0.15}$$

One can also treat integrals of the form

$$\int R(\zeta, \sqrt{Q(\zeta)})\, d\zeta, \tag{6.0.16}$$

where $Q(\zeta)$ is a quartic polynomial with distinct roots, and convert it to an integral of the form (6.0.10).

Another topic, covered in §32, is the production of formulas for $\wp(z)$ and $\wp'(z)$ in terms of "theta functions." Among other things, such functions are useful for the rapid evaluation of $\wp$, which is convenient, since the infinite series (6.0.5) converges very slowly. This is discussed in Appendix K, at the end of this chapter.

30. Periodic and doubly periodic functions - infinite series representations

We can obtain periodic meromorphic functions by summing translates of z^{-k}. For example,

$$f_1(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} \tag{30.1}$$

is meromorphic on $\mathbb{C}$, with poles in $\mathbb{Z}$, and satisfies $f_1(z+1) = f_1(z)$. In fact, we have

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = \frac{\pi^2}{\sin^2 \pi z}. \tag{30.2}$$

To see this, note that both sides have the same poles, and their difference $g_1(z)$ is seen to be an entire function, satisfying $g_1(z+1) = g_1(z)$. Also it is seen that, for $z = x + iy$, both sides of (30.2) tend to 0 as $|y| \to \infty$. This forces $g_1 \equiv 0$. For another derivation, see (16.55).

A second example is

$$\begin{aligned} f_2(z) = \lim_{m\to\infty} \sum_{n=-m}^{m} \frac{1}{z-n} &= \frac{1}{z} + \sum_{n\neq 0} \Bigl(\frac{1}{z-n} + \frac{1}{n} \Bigr) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}. \end{aligned} \tag{30.3}$$

This is also meromorphic on $\mathbb{C}$, with poles in $\mathbb{Z}$, and it is seen to satisfy $f_2(z+1) = f_2(z)$. We claim that

$$\frac{1}{z} + \sum_{n\neq 0} \Bigl(\frac{1}{z-n} + \frac{1}{n} \Bigr) = \pi \cot \pi z. \tag{30.4}$$

In this case again we see that the difference $g_2(z)$ is entire. Furthermore, applying $-d/dz$ to both sides of (30.4), we get the two sides of (30.2), so g_2 is constant. Looking at the last term in (30.3), we see that the left side of (30.4) is odd in z; so is the right side; hence $g_2 = 0$.

As a third example, we consider

$$\begin{aligned} \lim_{m\to\infty} \sum_{n=-m}^{m} \frac{(-1)^n}{z-n} &= \frac{1}{z} + \sum_{n\neq 0} (-1)^n \Bigl(\frac{1}{z-n} + \frac{1}{n} \Bigr) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \frac{2z}{z^2 - n^2} \\ &= \frac{1}{z} - 4 \sum_{k=1}^{\infty} \frac{z(1-2k)}{[z^2 - (2k-1)^2][z^2 - (2k)^2]}. \end{aligned} \tag{30.5}$$

We claim that

$$\frac{1}{z}+\sum_{n\neq 0}(-1)^n\Bigl(\frac{1}{z-n}+\frac{1}{n}\Bigr)=\frac{\pi}{\sin \pi z}. \tag{30.6}$$

In this case we see that their difference $g_3(z)$ is entire and satisfies $g_3(z+2)=g_3(z)$. Also, for $z=x+iy$, both sides of (30.6) tend to 0 as $|y|\to\infty$, so $g_3\equiv 0$.

We now use a similar device to construct doubly periodic meromorphic functions, following K. Weierstrass. These functions are also called elliptic functions. Further introductory material on this topic can be found in [Ahl] and [Hil]. Pick $\omega_1,\omega_2\in\mathbb{C}$, linearly independent over $\mathbb{R}$, and form the lattice

$$\Lambda=\{j\omega_1+k\omega_2 : j,k\in\mathbb{Z}\}. \tag{30.7}$$

In partial analogy with (30.4), we form the "Weierstrass $\wp$-function,"

$$\wp(z;\Lambda)=\frac{1}{z^2}+\sum_{0\neq\omega\in\Lambda}\Bigl(\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\Bigr). \tag{30.8}$$

Convergence on $\mathbb{C}\setminus\Lambda$ is a consequence of the estimate

$$\Bigl|\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\Bigr|\le C\frac{|z|}{|\omega|^3},\quad \text{for}\ \ |\omega|\ge 2|z|. \tag{30.9}$$

To verify that

$$\wp(z+\omega;\Lambda)=\wp(z;\Lambda),\quad \forall\ \omega\in\Lambda, \tag{30.10}$$

it is convenient to differentiate both sides of (30.8), obtaining

$$\wp'(z;\Lambda)=-2\sum_{\omega\in\Lambda}\frac{1}{(z-\omega)^3}, \tag{30.11}$$

which clearly satisfies

$$\wp'(z+\omega;\Lambda)=\wp'(z;\Lambda),\quad \forall\ \omega\in\Lambda. \tag{30.12}$$

Hence

$$\wp(z+\omega;\Lambda)-\wp(z;\Lambda)=c(\omega),\quad \omega\in\Lambda. \tag{30.13}$$

Now (30.8) implies $\wp(z;\Lambda)=\wp(-z;\Lambda)$. Hence, taking $z=-\omega/2$ in (30.13) gives $c(\omega)=0$ for all $\omega\in\Lambda$, and we have (30.10).

Another analogy with (30.4) leads us to look at the function (not to be confused with the Riemann zeta function)

$$\zeta(z;\Lambda) = \frac{1}{z} + \sum_{0\neq\omega\in\Lambda} \Bigl(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\Bigr). \tag{30.14}$$

We note that the sum here is obtained from the sum in (30.8) (up to sign) by integrating from 0 to z along any path that avoids the poles. This is enough to establish convergence of (30.14) in $\mathbb{C}\setminus\Lambda$, and we have

$$\zeta'(z;\Lambda) = -\wp(z;\Lambda). \tag{30.15}$$

In view of (30.10), we hence have

$$\zeta(z+\omega;\Lambda) - \zeta(z;\Lambda) = \alpha_\Lambda(\omega), \quad \forall\omega\in\Lambda. \tag{30.16}$$

In this case $\alpha_\Lambda(\omega)\neq 0$, but we can take $a,b\in\mathbb{C}$ and form

$$\zeta_{a,b}(z;\Lambda) = \zeta(z-a;\Lambda) - \zeta(z-b;\Lambda), \tag{30.17}$$

obtaining a meromorphic function with poles at $(a+\Lambda)\cup(b+\Lambda)$, all simple (if $a-b\notin\Lambda$).

Let us compare the doubly periodic function Φ constructed in (24.8)–(24.11), which maps the rectangle with vertices at $-1, 1, 1+ip, -1+ip$ conformally onto the upper half plane $\mathcal{U}$, with $\Phi(-1)=-1, \Phi(0)=0, \Phi(1)=1$. (Here p is a given positive number.) As seen there,

$$\Phi(z+\omega) = \Phi(z), \quad \omega\in\Lambda = \{4k+2i\ell p : k,\ell\in\mathbb{Z}\}. \tag{30.18}$$

Furthermore, this function has simple poles at $(ip+\Lambda)\cup(ip+2+\Lambda)$, and the residues at ip and at $ip+2$ cancel. Thus there exist constants A and B such that

$$\Phi(z) = A\zeta_{ip,ip+2}(z;\Lambda) + B. \tag{30.19}$$

The constants A and B can be evaluated by taking $z=0,1$, though the resulting formulas give A and B in terms of special values of $\zeta(z;\Lambda)$ rather than in elementary terms.

Exercises

1. Setting $z=1/2$ in (30.2), show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Compare (13.79). Differentiate (30.2) repeatedly and obtain formulas for $\sum_{n\ge1} n^{-k}$ for even integers k.
Hint. Denoting the right side of (30.2) by $f(z)$, show that

$$f^{(\ell)}(z) = (-1)^\ell(\ell+1)! \sum_{n=-\infty}^{\infty} (z-n)^{-(\ell+2)}.$$

Deduce that, for $k \ge 1$,

$$f^{(2k-2)}\Big(\frac{1}{2}\Big) = (2k-1)!2^{2k+1} \sum_{n\ge1,\mathrm{odd}} n^{-2k}.$$

Meanwhile, use

$$\sum_{n=1}^{\infty} n^{-2k} = \sum_{n\ge1,\mathrm{odd}} n^{-2k} + 2^{-2k}\sum_{n=1}^{\infty} n^{-2k}$$

to get a formula for $\sum_{n=1}^{\infty} n^{-2k}$, in terms of $f^{(2k-2)}(1/2)$.

1A. Set $F(z) = (\pi\cot\pi z) - 1/z$, and use (30.4) to compute $F^{(\ell)}(0)$. Show that, for $|z| < 1$,

$$\pi\cot\pi z = \frac{1}{z} - 2\sum_{k=1}^{\infty} \zeta(2k) z^{2k-1}, \quad \zeta(2k) = \sum_{n=1}^{\infty} n^{-2k}.$$

1B. Recall from Exercise 6 in §12 that, for $|z|$ sufficiently small,

$$\frac{1}{2}\,\frac{e^z+1}{e^z-1} = \frac{1}{z} + \sum_{k=1}^{\infty} (-1)^{k-1}\frac{B_k}{(2k)!} z^{2k-1},$$

with B_k (called the Bernoulli numbers) *rational* numbers for each k. Note that

$$\frac{e^{2\pi i z}+1}{e^{2\pi i z}-1} = \frac{1}{i}\cot\pi z.$$

Deduce from this and Exercise 1A that, for $k \ge 1$,

$$2\zeta(2k) = (2\pi)^{2k}\frac{B_k}{(2k)!}.$$

Relate this to results of Exercise 1.

1C. For an alternative aproach to the results of Exercise 1B, show that

$$G(z) = \pi\cot\pi z \Longrightarrow G'(z) = -\pi^2 - G(z)^2.$$

Using

$$G(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^{2n-1},$$

compute the Laurent series expansions of $G'(z)$ and $G(z)^2$ and deduce that $a_1 = -\pi^2/3$, while, for $n \geq 2$,

$$a_n = -\frac{1}{2n+1} \sum_{\ell=1}^{n-1} a_{n-\ell} a_\ell.$$

In concert with Exercise 1A, show that $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, and also compute $\zeta(6)$ and $\zeta(8)$.

2. Set

$$F(z) = \pi z \prod_{n=1}^{\infty} \Bigl(1 - \frac{z^2}{n^2}\Bigr).$$

Show that

$$\frac{F'(z)}{F(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

Using this and (30.3)–(30.4), deduce that

$$F(z) = \sin \pi z,$$

obtaining another proof of (18.21).
Hint. Show that if F and G are meromorphic and $F'/F \equiv G'/G$, then $F = cG$ for some constant c. To find c in this case, note that $F'(0) = \pi$.

3. Show that if Λ is a lattice of the form (30.7) then a meromorphic function satisfying

$$f(z + \omega) = f(z), \quad \forall\, \omega \in \Lambda \tag{30.20}$$

yields a meromorphic function on the torus $\mathbb{T}_\Lambda$, defined by (22.14). Show that if such f has no poles then it must be constant.

We say a parallelogram $\mathcal{P} \subset \mathbb{C}$ is a period parallelogram for a lattice Λ (of the form (30.7)) provided it has vertices of the form $p, p + \omega_1, p + \omega_2, p + \omega_1 + \omega_2$. Given a meromorphic function f satisfying (30.20), pick a period parallelogram $\mathcal{P}$ whose boundary is disjoint from the set of poles of f.

4. Show that

$$\int_{\partial \mathcal{P}} f(z)\, dz = 0.$$

Deduce that

$$\sum_{p_j \in \mathcal{P}} \operatorname{Res}_{p_j}(f) = 0.$$

Deduce that if f has just one pole in $\mathcal{P}$ then that pole cannot be simple.

5. For ζ defined by (30.14), show that, if $\operatorname{Im}(\omega_2/\omega_1) > 0$,

$$\int_{\partial \mathcal{P}} \zeta(z;\Lambda)\, dz = \alpha_\Lambda(\omega_1)\omega_2 - \alpha_\Lambda(\omega_2)\omega_1 = 2\pi i. \tag{30.21}$$

6. Show that α_Λ in (30.16) satisfies

$$\alpha_\Lambda(\omega + \omega') = \alpha_\Lambda(\omega) + \alpha_\Lambda(\omega'), \quad \omega, \omega' \in \Lambda. \tag{30.22}$$

Show that if $\omega \in \Lambda,\ \omega/2 \notin \Lambda$, then

$$\alpha_\Lambda(\omega) = 2\zeta(\omega/2;\Lambda).$$

7. Apply Green's theorem

$$\iint_{\Omega} \Bigl(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\Bigr)\, dx\, dy = \int_{\partial \Omega} (f\, dx + g\, dy)$$

in concert with $\zeta'(z;\Lambda) = -\wp(z;\Lambda)$, written as

$$\frac{1}{2}\Bigl(\frac{\partial}{\partial x} + \frac{1}{i}\,\frac{\partial}{\partial y}\Bigr)\zeta(z;\Lambda) = -\wp(z;\Lambda),$$

and with $\Omega = \mathcal{P}$, as in Exercise 5, to establish that

$$\alpha_\Lambda(\omega_1)\overline{\omega}_2 - \alpha_\Lambda(\omega_2)\overline{\omega}_1 = 2i\mathcal{I}(\Lambda), \tag{30.23}$$

with

$$\mathcal{I}(\Lambda) = \lim_{\varepsilon \to 0} \iint_{\mathcal{P}\setminus D_\varepsilon(0)} \wp(z;\Lambda)\, dx\, dy, \tag{30.24}$$

assuming $\mathcal{P}$ is centered at 0.

8. Solve the pair of equations (30.21) and (30.23) for $\alpha_\Lambda(\omega_1)$ and $\alpha_\Lambda(\omega_2)$. Use this in concert with (30.22) to show that

$$\alpha_\Lambda(\omega) = \frac{1}{A(\mathcal{P})}\Bigl(-\mathcal{I}(\Lambda)\omega + \pi\overline{\omega}\Bigr), \quad \omega \in \Lambda, \tag{30.25}$$

where $\mathcal{I}(\Lambda)$ is as in (30.24) and $A(\mathcal{P})$ is the area of $\mathcal{P}$.

9. Show that the constant A in (30.19) satisfies

$$A = \mathrm{Res}_{ip}(\Phi).$$

10. Show that the constants A and B in (30.19) satisfy

$$[\zeta(1 - ip; \Lambda) - \zeta(-1 - ip; \Lambda)]A + B = 1,$$

and

$$\alpha_\Lambda(4)A + 2B = 0,$$

with Λ given by (30.18).
Hint. Both $\Phi(z)$ and $\zeta(z; \Lambda)$ are odd in z.

In Exercises 11–12, given $p_j \in \mathbb{T}_\Lambda,\ n_j \in \mathbb{Z}^+$, set $\vartheta = \sum n_j p_j$ and define

$$\mathcal{M}_\vartheta(\mathbb{T}_\Lambda) = \{f \ \text{ meromorphic on } \ \mathbb{T}_\Lambda : \text{poles of } f \text{ are at } p_j \text{ and of order} \le n_j\}. \tag{30.26}$$

Set $|\vartheta| = \sum n_j$.

11. Show that $|\vartheta| = 2 \Rightarrow \dim \mathcal{M}_\vartheta(\mathbb{T}_\Lambda) = 2$, and that this space is spanned by 1 and ζ_{p_1,p_2} if $n_1 = n_2 = 1$, and by 1 and $\wp(z - p_1)$ if $n_1 = 2$.
Hint. Use Exercise 4.

12. Show that

$$|\vartheta| = k \ge 2 \Longrightarrow \dim \mathcal{M}_\vartheta(\mathbb{T}_\Lambda) = k. \tag{30.27}$$

Hint. Argue by induction on k, noting that you can augment $|\vartheta|$ by 1 either by adding another p_j or by increasing some positive n_j by 1.

31. The Weierstrass $\wp$ in elliptic function theory

Recall that an elliptic function with period lattice Λ is a meromorphic function f on $\mathbb{C}$ satisfying $f(z+\omega)=f(z)$, for each $\omega\in\Lambda$. It turns out that such a function can be expressed in terms of $\wp(z;\Lambda)$ and its first derivative. Before discussing a general result, we illustrate this in the case of the functions $\zeta_{a,b}(z;\Lambda)$, given by (30.17). Henceforth we simply denote these functions by $\wp(z)$ and $\zeta_{a,b}(z)$, respectively.

We claim that, if $2\beta\notin\Lambda$, then

$$\frac{\wp'(\beta)}{\wp(z)-\wp(\beta)}=\zeta_{\beta,-\beta}(z)+2\zeta(\beta). \tag{31.1}$$

To see this, note that both sides have simple poles at $z=\pm\beta$. (As will be shown below, the zeros α of $\wp'(z)$ satisfy $2\alpha\in\Lambda$.) The factor $\wp'(\beta)$ makes the poles cancel, so the difference is entire, hence constant. Both sides vanish at $z=0$, so this constant is zero. We also note that

$$\zeta_{a,b}(z)=\zeta_{\beta,-\beta}(z-\alpha),\quad \alpha=\frac{a+b}{2},\ \beta=\frac{a-b}{2}. \tag{31.2}$$

As long as $a-b\notin\Lambda$, (31.1) applies, giving

$$\zeta_{a,b}(z)=\frac{\wp'(\beta)}{\wp(z-\alpha)-\wp(\beta)}-2\zeta(\beta),\quad \alpha=\frac{a+b}{2},\ \beta=\frac{a-b}{2}. \tag{31.3}$$

We now prove the result on the zeros of $\wp'(z)$ stated above. Assume Λ has the form (30.7).

Proposition 31.1. *The three points $\omega_1/2$, $\omega_2/2$ and $(\omega_1+\omega_2)/2$ are (mod Λ) all the zeros of $\wp'(z)$.*

Proof. Symmetry considerations (oddness of $\wp'(z)$) imply $\wp'(z)=0$ at each of these three points. Since $\wp'(z)$ has a single pole of order 3 in a period parallelogram, these must be all the zeros. (Cf. Exercise 1 below to justify this last point.)

The general result hinted at above is the following.

Proposition 31.2. *Let f be an elliptic function with period lattice Λ. There exist rational functions Q and R such that*

$$f(z)=Q(\wp(z))+R(\wp(z))\wp'(z). \tag{31.4}$$

Proof. First assume f is even, i.e., $f(z)=f(-z)$. The product of $f(z)$ with factors of the form $\wp(z)-\wp(a)$ lowers the degree of a pole of f at any point $a\notin\Lambda$, so there exists a polynomial P such that $g(z) = P(\wp(z))f(z)$ has poles only in Λ. Note that $g(z)$ is also even. Then there exists a polynomial P_2 such that $g(z) - P_2(\wp(z))$ has its poles annihilated. This function must hence be constant. Hence any even elliptic f must be a rational function of $\wp(z)$.

On the other hand, if $f(z)$ is odd, then $f(z)/\wp'(z)$ is even, and the previous argument applies, so a general elliptic function must have the form (31.4).

The right side of (31.3) does not have the form (31.4), but we can come closer to this form via the identity

$$\wp(z-\alpha) = -\wp(z) - \wp(\alpha) + \frac{1}{4}\Big(\frac{\wp'(z)+\wp'(\alpha)}{\wp(z)-\wp(\alpha)}\Big)^2. \tag{31.5}$$

This identity can be verified by showing that the difference of the two sides is pole free and vanishes at $z = 0$. The right side of (31.5) has the form (31.4) except for the occurrence of $\wp'(z)^2$, which we will dispose of shortly.

Note that $\wp'(z)^2$ is even, with poles (of order 6) on Λ. We can explicitly write this as $P(\wp(z))$, as follows. Set

$$e_j = \wp\Big(\frac{\omega_j}{2}\Big), \quad j = 1, 2, 3, \tag{31.6}$$

where we set $\omega_3 = \omega_1 + \omega_2$. We claim that

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3). \tag{31.7}$$

In fact, both sides of (31.7) have poles of order 6, precisely at points of Λ. Furthermore, by Proposition 31.1, the zeros of $\wp'(z)^2$ occur precisely at $z = \omega_j \pmod{\Lambda}$, each zero having multiplicity 2. We also see that the right side of (31.7) has a double zero at $z = \omega_j$, $j = 1, 2, 3$. So the quotient is entire, hence constant. The factor 4 arises by examining the behavior as $z \to 0$.

The identity (31.7) is a differential equation for $\wp(z)$. Separation of variables yields

$$\frac{1}{2}\int \frac{d\wp}{\sqrt{(\wp - e_1)(\wp - e_2)(\wp - e_3)}} = z + c. \tag{31.8}$$

The left side of (31.8) is known as an elliptic integral.

Any cubic polynomial in u is a constant multiple of $(u - e_1)(u - e_2)(u - e_3)$ for some $e_j \in \mathbb{C}$. However, it is not quite the case that every cubic polynomial fits into the current setting. Here is one constraint; another will be produced in (31.15) below.

Proposition 31.2. *Given a lattice $\Lambda \subset \mathbb{C}$, the quantities e_j in (31.6) are all distinct.*

Proof. Note that $\wp(z) - e_j$ has a double pole at each $z \in \Lambda$, and a double zero at $z = \omega_j/2$. Hence, in an appropriate period parallelogram, it has no other zeros (again cf. Exercise 1 below). Hence $\wp(\omega_k/2) - e_j = e_k - e_j \neq 0$ for $j \neq k$.

We can get more insight into the differential equation (31.7) by comparing Laurent series expansions of the two sides about $z = 0$. First, we can deduce from (30.8) that

$$\wp(z) = \frac{1}{z^2} + az^2 + bz^4 + \cdots . \tag{31.9}$$

Of course, only even powers of z arise. Regarding the absence of the constant term and the values of a and b, see Exercise 3 below. We have

$$a = 3 \sum_{0 \neq \omega \in \Lambda} \frac{1}{\omega^4}, \quad b = 5 \sum_{0 \neq \omega \in \Lambda} \frac{1}{\omega^6}. \tag{31.10}$$

Hence

$$\wp'(z) = -\frac{2}{z^3} + 2az + 4bz^3 + \cdots . \tag{31.11}$$

It follows, after some computation, that

$$\frac{1}{4}\wp'(z)^2 = \frac{1}{z^6} - \frac{2a}{z^2} - 4b + \cdots , \tag{31.12}$$

while

$$\begin{aligned} &(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3) \\ &\quad = \wp(z)^3 - \tau_1 \wp(z)^2 + \tau_2 \wp(z) - \tau_3 \\ &\quad = \frac{1}{z^6} - \frac{\tau_1}{z^4} + \frac{3a + \tau_2}{z^2} + (3b - 2a\tau_1 - \tau_3) + \cdots , \end{aligned} \tag{31.13}$$

where

$$\begin{aligned} \tau_1 &= e_1 + e_2 + e_3, \\ \tau_2 &= e_1 e_2 + e_2 e_3 + e_3 e_1, \\ \tau_3 &= e_1 e_2 e_3. \end{aligned} \tag{31.14}$$

Comparing coefficients in (31.12)–(31.13) gives the following relation:

$$e_1 + e_2 + e_3 = 0. \tag{31.15}$$

It also gives

$$\begin{aligned} e_1 e_2 + e_2 e_3 + e_1 e_3 &= -5a, \\ e_1 e_2 e_3 &= 7b, \end{aligned} \tag{31.16}$$

where a and b are given by (31.10). Hence we can rewrite the differential equation (31.7) as

$$\wp'(z)^2 = 4\wp(z)^3 - g_2 \wp(z) - g_3, \tag{31.17}$$

where g_2 and g_3, known as the Weierstrass invariants of the lattice Λ, are given by

$$g_2 = 60 \sum_{0\neq\omega\in\Lambda} \frac{1}{\omega^4}, \quad g_3 = 140 \sum_{0\neq\omega\in\Lambda} \frac{1}{\omega^6}. \tag{31.18}$$

Exercises

1. Assume f is meromorphic (and not identically zero) on $\mathbb{T}_\Lambda = \mathbb{C}/\Lambda$. Show that the number of poles of f is equal to the number of zeros of f, counting multiplicity.
Hint. Let γ bound a period parallelogram, avoiding the zeros and poles of f, and examine

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz.$$

Recall the argument principle, discussed in §17.

2. Show that, given a lattice $\Lambda \subset \mathbb{C}$, and given $\omega \in \mathbb{C}$,

$$\omega \in \Lambda \Longleftrightarrow \wp\Bigl(\frac{\omega}{2} + z; \Lambda\Bigr) = \wp\Bigl(\frac{\omega}{2} - z; \Lambda\Bigr), \quad \forall\, z. \tag{31.19}$$

Relate this to the proof of Proposition 31.1.

3. Consider

$$\Phi(z) = \wp(z) - \frac{1}{z^2} = \sum_{\omega\in\Lambda\setminus 0} \Bigl(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\Bigr), \tag{31.20}$$

which is holomorphic near $z = 0$. Show that $\Phi(0) = 0$ and that, for $k \geq 1$,

$$\frac{1}{k!}\Phi^{(k)}(0) = (k+1) \sum_{\omega\in\Lambda\setminus 0} \omega^{-(k+2)}. \tag{31.21}$$

(These quantities vanish for k odd.) Relate these results to (31.9)–(31.10).

4. Complete the sketch of the proof of (31.5).
Hint. Use the fact that $\wp(z) - z^{-2}$ is holomorphic near $z = 0$ and vanishes at $z = 0$.

5. Deduce from (31.17) that

$$\wp''(z) = 6\wp(z)^2 - \frac{1}{2} g_2. \tag{31.22}$$

6. Say that, near $z = 0$,

$$\wp(z) = \frac{1}{z^2} + \sum_{n\geq 1} b_n z^{2n}, \tag{31.23}$$

where b_n are given by (31.21), with $k = 2n$. Deduce from Exercise 5 that for $n \geq 3$,

$$b_n = \frac{3}{(2n+3)(n-2)} \sum_{k=1}^{n-2} b_k b_{n-k-1}. \tag{31.24}$$

In particular, we have

$$b_3 = \frac{1}{3} b_1^2, \quad b_4 = \frac{3}{11} b_1 b_2,$$

and

$$b_5 = \frac{1}{13}(b_2^2 + 2b_1 b_3) = \frac{1}{13}\Big(b_2^2 + \frac{2}{3} b_1^3\Big).$$

7. Deduce from Exercise 6 that if

$$\sigma_n = \sum_{\omega\in\Lambda\setminus 0} \frac{1}{\omega^{2n}}, \tag{31.25}$$

then for $n \geq 3$,

$$\sigma_n = P_n(\sigma_2, \sigma_3), \tag{31.26}$$

where $P_n(\sigma_2, \sigma_3)$ is a polynomial in σ_2 and σ_3 with coefficients that are positive, rational numbers. Use (31.16) to show that

$$\sigma_2 = -\frac{1}{15}(e_1e_2 + e_2e_3 + e_1e_3), \quad \sigma_3 = \frac{1}{35}\, e_1e_2e_3. \tag{31.27}$$

Note that $b_n = (2n+1)\sigma_{n+1}$. Note also that g_k in (31.17)–(31.18) satisfy $g_2 = 60\sigma_2$ and $g_3 = 140\sigma_3$.

8. Given f as in Exercise 1, show that

$$\frac{1}{2\pi i} \int_\sigma \frac{f'(z)}{f(z)}\, dz \in \mathbb{Z}, \tag{31.28}$$

whenever σ is a closed curve in $\mathbb{T}_\Lambda$ that avoids the zeros and poles of f.

9. Again take f as in Exercise 1. Assume f has zeros at $p_j \in \mathbb{T}_\Lambda$, of order m_j, and poles at $q_j \in \mathbb{T}_\Lambda$, of order n_j, and no other zeros or poles. Show that

$$\sum m_j p_j - \sum n_j q_j = 0 \pmod{\Lambda}. \tag{31.29}$$

Hint. Take γ as in Exercise 1, and consider

$$\frac{1}{2\pi i} \int\limits_\gamma \frac{f'(z)}{f(z)}\, z\, dz. \tag{31.30}$$

On the one hand, Cauchy's integral theorem (compare (5.19)) implies (31.30) is equal to the left side of (31.29), provided p_j and q_j are all in the period domain. On the other hand, if γ consists of four consecutive line segments, $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, periodicity of $f'(z)/f(z)$ implies that (31.30) equals

$$-\frac{\omega_2}{2\pi i} \int\limits_{\sigma_1} \frac{f'(z)}{f(z)}\, dz + \frac{\omega_1}{2\pi i} \int\limits_{\sigma_4} \frac{f'(z)}{f(z)}\, dz. \tag{31.31}$$

Use Exercise 8 to deduce that the coefficients of ω_1 and ω_2 in (31.31) are integers.

10. Deduce from (31.5) that

$$u + v + w = 0 \Rightarrow \det \begin{pmatrix} \wp(u) & \wp'(u) & 1 \\ \wp(v) & \wp'(v) & 1 \\ \wp(w) & \wp'(w) & 1 \end{pmatrix} = 0. \tag{31.32}$$

11. Deduce from (31.5) that

$$\wp(2z) = \frac{1}{4} \Big(\frac{\wp''(z)}{\wp'(z)} \Big)^2 - 2\wp(z). \tag{31.33}$$

Hint. Set $\alpha = -z + h$ in (31.5) and let $h \to 0$.

12. Deduce from (31.33), in concert with (31.17) and (31.22), that

$$\wp(2z) = R(\wp(z)),$$

with

$$R(\zeta) = \frac{\zeta^4 + (g_2/2)\zeta^2 + 2g_3\zeta + (g_2/4)^2}{4\zeta^3 - g_2\zeta - g_3}.$$

13. Use (31.3) and (31.5), plus (31.7), to write $\zeta_{a,b}(z)$ (as in (31.3)) in the form (31.4), i.e.,

$$\zeta_{a,b}(z) = Q(\wp(z)) + R(\wp(z))\wp'(z).$$

32. Theta functions and $\wp$

We begin with the function

$$\theta(x,t)=\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} e^{2\pi i n x}, \tag{32.1}$$

defined for $x\in\mathbb{R},\ t>0$, which solves the "heat equation"

$$\frac{\partial\theta}{\partial t}=\frac{1}{4\pi}\frac{\partial^2\theta}{\partial x^2}. \tag{32.2}$$

Note that θ is actually holomorphic on $\{(x,t)\in\mathbb{C}\times\mathbb{C}:\operatorname{Re}t>0\}$. It is periodic of period 1 in x; $\theta(x+1,t)=\theta(x,t)$. Also one has

$$\theta(x+it,t)=e^{\pi t-2\pi i x}\theta(x,t). \tag{32.3}$$

This identity will ultimately lead us to a connection with $\wp(z)$. In addition, we have

$$\theta\Bigl(x+\frac{1}{2},t\Bigr)=\sum_{n\in\mathbb{Z}}(-1)^n e^{-\pi n^2 t}e^{2\pi i n x}, \tag{32.4}$$

and

$$\theta\Bigl(x+\frac{i}{2}t,t\Bigr)=e^{\pi t/4}\sum_{n\in\mathbb{Z}} e^{-\pi(n+1/2)^2 t}e^{2\pi i n x}, \tag{32.5}$$

which introduces series related to but slightly different from that in (32.1).

Following standard terminology, we set $-t=i\tau$, with $\operatorname{Im}\tau>0$, and denote $\theta(z,-i\tau)$ by

$$\vartheta_3(z,\tau)=\sum_{n\in\mathbb{Z}} e^{n^2\pi i\tau}e^{2n\pi i z}=\sum_{n\in\mathbb{Z}} p^{2n}q^{n^2}, \tag{32.6}$$

where

$$p=e^{\pi i z},\quad q=e^{\pi i\tau}. \tag{32.7}$$

This theta function has three partners, namely

$$\vartheta_4(z,\tau)=\sum_{n\in\mathbb{Z}}(-1)^n e^{n^2\pi i\tau}e^{2\pi i n z}=\sum_{n\in\mathbb{Z}}(-1)^n p^{2n}q^{n^2}, \tag{32.8}$$

and

$$\vartheta_1(z,\tau) = i\sum_{n\in\mathbb{Z}} (-1)^n e^{(n-1/2)^2\pi i\tau} e^{(2n-1)\pi iz} = i\sum_{n\in\mathbb{Z}} (-1)^n p^{2n-1} q^{(n-1/2)^2}, \tag{32.9}$$

and finally

$$\vartheta_2(z,\tau) = \sum_{n\in\mathbb{Z}} e^{(n-1/2)^2\pi i\tau} e^{(2n-1)\pi iz} = \sum_{n\in\mathbb{Z}} p^{2n-1} q^{(n-1/2)^2}. \tag{32.10}$$

We will explore these functions, with the goal of relating them to $\wp(z)$. These results are due to Jacobi; we follow the exposition of [MM].

To begin, we record how $\vartheta_j(z+\alpha)$ is related to $\vartheta_k(z)$ for various values of α. Here and (usually) below we will suppress the τ and denote $\vartheta_j(z,\tau)$ by $\vartheta_j(z)$, for short. In the table below we use

$$a = p^{-1}q^{-1/4} = e^{-\pi iz-\pi i\tau/4}, \quad b = p^{-2}q^{-1} = e^{-2\pi iz-\pi i\tau}.$$

Proofs of the tabulated relations are straightforward analogues of (32.3)–(32.5).

Table of Relations among Various Translations of ϑ_j

	$z+1/2$	$z+\tau/2$	$z+1/2+\tau/2$	$z+1$	$z+\tau$	$z+1+\tau$
ϑ_1	ϑ_2	$ia\vartheta_4$	$a\vartheta_3$	$-\vartheta_1$	$-b\vartheta_1$	$b\vartheta_1$
ϑ_2	$-\vartheta_1$	$a\vartheta_3$	$-ia\vartheta_4$	$-\vartheta_2$	$b\vartheta_2$	$-b\vartheta_2$
ϑ_3	ϑ_4	$a\vartheta_2$	$ia\vartheta_1$	ϑ_3	$b\vartheta_3$	$b\vartheta_3$
ϑ_4	ϑ_3	$ia\vartheta_1$	$a\vartheta_2$	ϑ_4	$-b\vartheta_4$	$-b\vartheta_4$

An inspection shows that the following functions

$$F_{jk}(z) = \Bigl(\frac{\vartheta_j(z)}{\vartheta_k(z)}\Bigr)^2 \tag{32.11}$$

satisfy

$$F_{jk}(z+\omega) = F_{jk}(z), \quad \forall\, \omega \in \Lambda, \tag{32.12}$$

where

$$\Lambda = \{k + \ell\tau : k,\ell \in \mathbb{Z}\}. \tag{32.13}$$

Note also that

$$G_j(z) = \frac{\vartheta_j'(z)}{\vartheta_j(z)} \tag{32.14}$$

satisfies

$$G_j(z+1) = G_j(z), \quad G_j(z+\tau) = G_j(z) - 2\pi i. \tag{32.15}$$

To relate the functions F_{jk} to previously studied elliptic functions, we need to know the zeros of $\vartheta_k(z)$. Here is the statement:

Proposition 32.1. *We have*

$$\begin{aligned}\vartheta_1(z)=0\Leftrightarrow z\in\Lambda,\quad \vartheta_2(z)=0\Leftrightarrow z\in\Lambda+\frac{1}{2},\\ \vartheta_3(z)=0\Leftrightarrow z\in\Lambda+\frac{1}{2}+\frac{\tau}{2},\quad \vartheta_4(z)=0\Leftrightarrow z\in\Lambda+\frac{\tau}{2}.\end{aligned} \tag{32.16}$$

Proof. In view of the relations tabulated above, it suffices to treat $\vartheta_1(z)$. We first note that

$$\vartheta_1(-z)=-\vartheta_1(z). \tag{32.17}$$

To see this, replace z by $-z$ in (32.8) and simultaneously replace n by $-m$. Then replace m by $n-1$ and observe that (32.17) pops out. Hence ϑ_1 has a zero at $z=0$. We claim it is simple and that ϑ_1 has no others, mod Λ. To see this, let γ be the boundary of a period parallelogram containing 0 in its interior. Then use of (32.15) with $j=1$ easily gives

$$\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\vartheta_1'(z)}{\vartheta_1(z)}\,dz=1,$$

completing the proof.

Let us record the following complement to (32.17):

$$2\le j\le 4\Longrightarrow \vartheta_j(-z)=\vartheta_j(z). \tag{32.18}$$

The proof is straightforward from the defining formulas (32.6)–(32.9).

We are now ready for the following important result. For consistency with [MM], we slightly reorder the quantities e_1,e_2,e_3. Instead of using (31.6), we set

$$e_1=\wp\Bigl(\frac{\omega_1}{2}\Bigr),\quad e_2=\wp\Bigl(\frac{\omega_1+\omega_2}{2}\Bigr),\quad e_3=\wp\Bigl(\frac{\omega_2}{2}\Bigr), \tag{32.19}$$

where, in the current setting, with Λ given by (32.13), we take $\omega_1=1$ and $\omega_2=\tau$.

Proposition 32.2. *For $\wp(z)=\wp(z;\Lambda)$, with Λ of the form (32.13) and $\vartheta_j(z)=\vartheta_j(z,\tau)$,*

$$\begin{aligned}\wp(z)&=e_1+\Bigl(\frac{\vartheta_1'(0)}{\vartheta_1(z)}\cdot\frac{\vartheta_2(z)}{\vartheta_2(0)}\Bigr)^2\\ &=e_2+\Bigl(\frac{\vartheta_1'(0)}{\vartheta_1(z)}\cdot\frac{\vartheta_3(z)}{\vartheta_3(0)}\Bigr)^2\\ &=e_3+\Bigl(\frac{\vartheta_1'(0)}{\vartheta_1(z)}\cdot\frac{\vartheta_4(z)}{\vartheta_4(0)}\Bigr)^2.\end{aligned} \tag{32.20}$$

Proof. We have from (32.11)–(32.13) that each function $P_j(z)$ on the right side of (32.20) is Λ-periodic. Also Proposition 32.1 implies each P_j has poles of order 2, precisely on

Λ. Furthermore, we have arranged that each such function has leading singularity z^{-2} as $z \to 0$, and each P_j is even, by (32.17) and (32.18), so the difference $\wp(z) - P_j(z)$ is constant for each j. Evaluating at $z = 1/2$, $(1+\tau)/2$, and $\tau/2$, respectively, shows that these constants are zero, and completes the proof.

Part of the interest in (32.20) is that the series (32.6)–(32.10) for the theta functions are extremely rapidly convergent. To complete this result, we want to express the quantities e_j in terms of theta functions. The following is a key step.

Proposition 32.3. *In the setting of Proposition 32.2,*

$$
\begin{aligned}
e_1 - e_2 &= \Bigl(\frac{\vartheta_1'(0)\vartheta_4(0)}{\vartheta_2(0)\vartheta_3(0)}\Bigr)^2 = \pi^2\vartheta_4(0)^4, \\
e_1 - e_3 &= \Bigl(\frac{\vartheta_1'(0)\vartheta_3(0)}{\vartheta_2(0)\vartheta_4(0)}\Bigr)^2 = \pi^2\vartheta_3(0)^4, \\
e_2 - e_3 &= \Bigl(\frac{\vartheta_1'(0)\vartheta_2(0)}{\vartheta_3(0)\vartheta_4(0)}\Bigr)^2 = \pi^2\vartheta_2(0)^4.
\end{aligned}
\tag{32.21}
$$

Proof. To get the first part of the first line, evaluate the second identity in (32.20) at $z = 1/2$, to obtain

$$
e_1 - e_2 = \Bigl(\frac{\vartheta_1'(0)}{\vartheta_1(1/2)} \cdot \frac{\vartheta_3(1/2)}{\vartheta_3(0)}\Bigr)^2,
$$

and then consult the table to rewrite $\vartheta_3(1/2)/\vartheta_1(1/2)$. Similar arguments give the first identity in the second and third lines of (32.21). The proof of the rest of the identities then follows from the next result.

Proposition 32.4. *We have*

$$
\vartheta_1'(0) = \pi\vartheta_2(0)\vartheta_3(0)\vartheta_4(0). \tag{32.22}
$$

Proof. To begin, consider

$$
\varphi(z) = \vartheta_1(2z)^{-1}\vartheta_1(z)\vartheta_2(z)\vartheta_3(z)\vartheta_4(z).
$$

Consultation of the table shows that $\varphi(z+\omega) = \varphi(z)$ for each $\omega \in \Lambda$. Also φ is free of poles, so it is constant. The behavior as $z \to 0$ reveals the constant, and yields the identity

$$
\vartheta_1(2z) = 2\,\frac{\vartheta_1(z)\vartheta_2(z)\vartheta_3(z)\vartheta_4(z)}{\vartheta_2(0)\vartheta_3(0)\vartheta_4(0)}. \tag{32.23}
$$

Now applying log, taking $(d/dz)^2$, evaluating at $z = 0$, and using

$$
\vartheta_1''(0) = \vartheta_2'(0) = \vartheta_3'(0) = \vartheta_4'(0) = 0, \tag{32.24}
$$

(a consequence of (32.17)–(32.18)), yields

$$\frac{\vartheta_1'''(0)}{\vartheta_1'(0)} = \frac{\vartheta_2''(0)}{\vartheta_2(0)} + \frac{\vartheta_3''(0)}{\vartheta_3(0)} + \frac{\vartheta_4''(0)}{\vartheta_4(0)}. \tag{32.25}$$

Now, from (32.2) we have

$$\frac{\partial \vartheta_j}{\partial \tau} = \frac{1}{4\pi i} \frac{\partial^2 \vartheta_j}{\partial z^2}, \tag{32.26}$$

and computing

$$\frac{\partial}{\partial \tau} \Bigl[\log \vartheta_2(0,\tau) + \log \vartheta_3(0,\tau) + \log \vartheta_4(0,\tau) - \log \vartheta_1'(0,\tau) \Bigr] \tag{32.27}$$

and comparing with (32.25) shows that

$$\vartheta_2(0,\tau)\vartheta_3(0,\tau)\vartheta_4(0,\tau)/\vartheta_1'(0,\tau) \ \text{ is independent of } \ \tau. \tag{32.28}$$

Thus

$$\vartheta_1'(0) = A\vartheta_2(0)\vartheta_3(0)\vartheta_4(0), \tag{32.29}$$

with A independent of τ, hence independent of $q = e^{\pi i \tau}$. As $q \to 0$, we have

$$\vartheta_1'(0) \sim 2\pi q^{1/4}, \quad \vartheta_2(0) \sim 2q^{1/4}, \quad \vartheta_3(0) \sim 1, \quad \vartheta_4(0) \sim 1, \tag{32.30}$$

which implies $A = \pi$, proving (32.22).

Now that we have Proposition 32.3, we can use $(e_1 - e_3) - (e_1 - e_2) = e_2 - e_3$ to deduce that

$$\vartheta_3(0)^4 = \vartheta_2(0)^4 + \vartheta_4(0)^4. \tag{32.31}$$

Next, we can combine (32.21) with

$$e_1 + e_2 + e_3 = 0 \tag{32.32}$$

to deduce the following.

Proposition 32.5. *In the setting of Proposition 32.2, we have*

$$\begin{aligned} e_1 &= \frac{\pi^2}{3} \bigl[\vartheta_3(0)^4 + \vartheta_4(0)^4 \bigr], \\ e_2 &= \frac{\pi^2}{3} \bigl[\vartheta_2(0)^4 - \vartheta_4(0)^4 \bigr], \\ e_3 &= -\frac{\pi^2}{3} \bigl[\vartheta_2(0)^4 + \vartheta_3(0)^4 \bigr]. \end{aligned} \tag{32.33}$$

Thus we have an efficient method to compute $\wp(z;\Lambda)$ when Λ has the form (32.13). To pass to the general case, we can use the identity

$$\wp(z;a\Lambda)=\frac{1}{a^2}\,\wp\Bigl(\frac{z}{a};\Lambda\Bigr). \tag{32.34}$$

See Appendix K for more on the rapid evaluation of $\wp(z;\Lambda)$.

Exercises

1. Show that

$$\frac{d}{dz}\frac{\vartheta_1'(z)}{\vartheta_1(z)}=a\wp(z)+b, \tag{32.35}$$

with $\wp(z)=\wp(z;\Lambda)$, Λ as in (32.13). Show that

$$a=-1,\quad b=e_1+\frac{\vartheta_1''(\omega_1/2)\vartheta_1(\omega_1/2)-\vartheta_1'(\omega_1/2)^2}{\vartheta_1(\omega_1/2)^2}, \tag{32.36}$$

where $\omega_1=1,\ \omega_2=\tau$.

2. In the setting of Exercise 1, deduce that $\zeta_{a,b}(z;\Lambda)$, given by (30.17), satisfies

$$\begin{aligned}\zeta_{a,b}(z;\Lambda)&=\frac{\vartheta_1'(z-a)}{\vartheta_1(z-a)}-\frac{\vartheta_1'(z-b)}{\vartheta_1(z-b)}\\&=\frac{d}{dz}\log\frac{\vartheta_1(z-a)}{\vartheta_1(z-b)}.\end{aligned} \tag{32.37}$$

3. Show that, if $a\neq e_j$,

$$\frac{1}{\wp(z)-a}=A\zeta_{\alpha,-\alpha}(z)+B, \tag{32.38}$$

where $\wp(\pm\alpha)=a$. Show that

$$A=\frac{1}{\wp'(\alpha)}. \tag{32.39}$$

Identify B.

4. Give a similar treatment of $1/(\wp(z)-a)$ for $a=e_j$. Relate these functions to $\wp(z-\tilde{\omega}_j)$, with $\tilde{\omega}_j$ found from (32.19).

5. Express g_2 and g_3, given in (31.17)–(31.18), in terms of theta functions.
Hint. Use Exercise 7 of §31, plus Proposition 32.5

33. Elliptic integrals

The integral (31.8) is a special case of a class of integrals known as elliptic integrals, which we explore in this section. Let us set

$$q(\zeta) = (\zeta - e_1)(\zeta - e_2)(\zeta - e_3). \tag{33.1}$$

We assume $e_j \in \mathbb{C}$ are distinct and that (as in (31.15))

$$e_1 + e_2 + e_3 = 0, \tag{33.2}$$

which could be arranged by a coordinate translation. Generally, an elliptic integral is an integral of the form

$$\int_{\zeta_0}^{\zeta_1} R(\zeta, \sqrt{q(\zeta)})\, d\zeta, \tag{33.3}$$

where $R(\zeta, \eta)$ is a rational function of its arguments. The relevance of (31.8) is reinforced by the following result.

Proposition 33.1. *Given distinct e_j satisfying (33.2), there exists a lattice Λ, generated by $\omega_1, \omega_2 \in \mathbb{C}$, linearly independent over $\mathbb{R}$, such that if $\wp(z) = \wp(z; \Lambda)$, then*

$$\wp\Bigl(\frac{\omega_j}{2}\Bigr) = e_j, \quad 1 \le j \le 3, \tag{33.4}$$

where $\omega_3 = \omega_1 + \omega_2$.

Given this result, we have from (31.7) that

$$\wp'(z)^2 = 4q(\wp(z)), \tag{33.5}$$

and hence, as in (31.8),

$$\frac{1}{2}\int_{\wp(z_0)}^{\wp(z)} \frac{d\zeta}{\sqrt{q(\zeta)}} = z - z_0, \quad \operatorname{mod} \Lambda. \tag{33.6}$$

The problem of proving Proposition 33.1 is known as the Abel inversion problem. The proof requires new tools, which will be provided in §34. We point out here that there is no difficulty in identifying what the lattice Λ must be. We have

$$\frac{1}{2}\int_{\infty}^{e_j} \frac{d\zeta}{\sqrt{q(\zeta)}} = \frac{\omega_j}{2}, \quad \operatorname{mod} \Lambda, \tag{33.7}$$

by (33.6). One can also verify directly from (33.7) that if the branches are chosen appropriately then $\omega_3 = \omega_1 + \omega_2$. It is not so clear that if Λ is constructed directly from (33.7) then the values of $\wp(z;\Lambda)$ at $z = \omega_j/2$ are given by (33.4), unless one already knows that Proposition 33.1 is true.

Given Proposition 33.1, we can rewrite the elliptic integral (33.3) as follows. The result depends on the particular path γ_{01} from ζ_0 to ζ_1 and on the particular choice of path σ_{01} in $\mathbb{C}/\Lambda$ such that $\wp$ maps σ_{01} homeomorphically onto γ_{01}. With these choices, (33.3) becomes

$$\int\limits_{\sigma_{01}} R\Big(\wp(z), \frac{1}{2}\wp'(z)\Big)\wp'(z)\,dz, \tag{33.8}$$

or, as we write more loosely,

$$\int_{z_0}^{z_1} R\Big(\wp(z), \frac{1}{2}\wp'(z)\Big)\wp'(z)\,dz, \tag{33.9}$$

where z_0 and z_1 are the endpoints of σ_{01}, satisfying $\wp(z_j) = \zeta_j$. It follows from Proposition 31.2 that

$$R\Big(\wp(z), \frac{1}{2}\wp'(z)\Big)\wp'(z) = R_1(\wp(z)) + R_2(\wp(z))\wp'(z), \tag{33.10}$$

for some rational functions $R_j(\zeta)$. In fact, one can describe computational rules for producing such R_j, by using (33.5). Write $R(\zeta,\eta)$ as a quotient of polynomials in (ζ,η) and use (33.5) to obtain that the left side of (33.10) is equal to

$$\frac{P_1(\wp(z)) + P_2(\wp(z))\wp'(z)}{Q_1(\wp(z)) + Q_2(\wp(z))\wp'(z)}, \tag{33.11}$$

for some polynomials $P_j(\zeta), Q_j(\zeta)$. Then multiply the numerator and denominator of (33.11) by $Q_1(\wp(z)) - Q_2(\wp(z))\wp'(z)$ and use (33.5) again on the new denominator to obtain the right side of (33.10).

The integral of (33.3) is now transformed to the integral of the right side of (33.10). Note that

$$\int_{z_0}^{z_1} R_2(\wp(z))\wp'(z)\,dz = \int_{\zeta_0}^{\zeta_1} R_2(\zeta)\,d\zeta, \quad \zeta_j = \wp(z_j). \tag{33.12}$$

This leaves us with the task of analyzing

$$\int_{z_0}^{z_1} R_1(\wp(z))\,dz, \tag{33.13}$$

when $R_1(\zeta)$ is a rational function.

We first analyze (33.13) when $R_1(\zeta)$ is a polynomial. To begin, we have

$$\int_{z_0}^{z_1} \wp(z)\,dz = \zeta(z_0) - \zeta(z_1), \tag{33.14}$$

by (30.15), where $\zeta(z) = \zeta(z;\Lambda)$ is given by (30.14). See (32.35)–(32.36) for a formula in terms of theta functions. Next, differentiating (33.5) gives (as mentioned in Exercise 5 of §31)

$$\wp''(z) = 2q'(\wp(z)) = 6\wp(z)^2 - \frac{1}{2}g_2, \tag{33.15}$$

so

$$6\int_{z_0}^{z_1} \wp(z)^2\,dz = \wp'(z_1) - \wp'(z_0) + \frac{g_2}{2}(z_1 - z_0). \tag{33.16}$$

We can integrate $\wp(z)^{k+2}$ for $k \in \mathbb{N}$ via the following inductive procedure. We have

$$\frac{d}{dz}\big(\wp'(z)\wp(z)^k\big) = \wp''(z)\wp(z)^k + k\wp'(z)^2\wp(z)^{k-1}. \tag{33.17}$$

Apply (33.15) to $\wp''(z)$ and (33.5) (or equivalently (31.17)) to $\wp'(z)^2$ to obtain

$$\frac{d}{dz}\big(\wp'(z)\wp(z)^k\big) = (6+4k)\wp(z)^{k+2} - (3+k)g_2\wp(z)^k - kg_3\wp(z)^{k-1}. \tag{33.18}$$

From here the inductive evaluation of $\int_{z_0}^{z_1} \wp(z)^{k+2}\,dz$, for $k = 1, 2, 3, \dots$, is straightforward.

To analyze (33.13) for a general rational function $R_1(\zeta)$, we see upon making a partial fraction decomposition that it remains to analyze

$$\int_{z_0}^{z_1} (\wp(z) - a)^{-\ell}\,dz, \tag{33.19}$$

for $\ell = 1, 2, 3, \dots$. One can also obtain inductive formulas here, by replacing $\wp(z)^k$ by $(\wp(z) - a)^k$ in (33.18) and realizing that k need not be positive. We get

$$\frac{d}{dz}\big(\wp'(z)(\wp(z)-a)^k\big) = \wp''(z)(\wp(z)-a)^k + k\wp'(z)^2(\wp(z)-a)^{k-1}. \tag{33.20}$$

Now write

$$\begin{aligned} \wp'(z)^2 &= 4\alpha_3(\wp(z)-a)^3 + 4\alpha_2(\wp(z)-a)^2 + 4\alpha_1(\wp(z)-a) + 4\alpha_0, \\ \wp''(z) &= 2A_2(\wp(z)-a)^2 + 2A_1(\wp(z)-a) + 2A_0, \end{aligned} \tag{33.21}$$

where

$$\alpha_j = \frac{q^{(j)}(a)}{j!}, \quad A_j = \frac{q^{(j+1)}(a)}{j!}, \tag{33.22}$$

to obtain

$$\tag{33.23} \begin{aligned} &\frac{d}{dz}\big(\wp'(z)(\wp(z)-a)^k\big) \\ &\quad = (2A_2+4k\alpha_3)(\wp(z)-a)^{k+2} + (2A_1+4k\alpha_2)(\wp(z)-a)^{k+1} \\ &\qquad + (2A_0+4k\alpha_1)(\wp(z)-a)^k + 4k\alpha_0(\wp(z)-a)^{k-1}. \end{aligned}$$

Note that

$$\tag{33.24} \alpha_0 = q(a), \quad 2A_0 + 4k\alpha_1 = (2+4k)q'(a).$$

Thus, if a is not equal to e_j for any j and if we know the integral (33.19) for integral $\ell \le -k$ (for some negative integer k), we can compute the integral for $\ell = 1-k$, as long as $k \neq 0$. If $a = e_j$ for some j, and if we know (33.19) for integral $\ell \le -k-1$, we can compute it for $\ell = -k$, since then $q'(a) \neq 0$.

At this point, the remaining case of (33.19) to consider is the case $\ell = 1$, i.e.,

$$\tag{33.25} \int_{z_0}^{z_1} \frac{dz}{\wp(z)-a}.$$

See Exercises 2–4 of §32 for expressions of $(\wp(z)-a)^{-1}$ in terms of logarithmic derivatives of quotients of theta functions.

Note that the cases $\ell = 0,\ -1$, and 1 of (33.19) are, under the correspondence of (33.3) with (33.8), respectively equal to

$$\tag{33.26} \int_{\zeta_0}^{\zeta_1} \frac{d\zeta}{\sqrt{q(\zeta)}}, \quad \int_{\zeta_0}^{\zeta_1} (\zeta-a)\,\frac{d\zeta}{\sqrt{q(\zeta)}}, \quad \int_{\zeta_0}^{\zeta_1} \frac{1}{\zeta-a}\,\frac{d\zeta}{\sqrt{q(\zeta)}}.$$

These are called, respectively, elliptic integrals of the first, second, and third kind. The material given above expresses the general elliptic integral (33.3) in terms of these cases.

There is another family of elliptic integrals, namely those of the form

$$\tag{33.27} I = \int R(\zeta, \sqrt{Q(\zeta)})\, d\zeta,$$

where $R(\zeta,\eta)$ is a rational function of its arguments and $Q(\zeta)$ is a fourth degree polynomial:

$$\tag{33.28} Q(\zeta) = (\zeta-a_0)(\zeta-a_1)(\zeta-a_2)(\zeta-a_3),$$

with $a_j \in \mathbb{C}$ distinct. Such integrals can be transformed to integrals of the form (33.3), via the change of variable

$$\tag{33.29} \tau = \frac{1}{\zeta-a_0}, \quad d\zeta = -\frac{1}{\tau^2}\, d\tau.$$

One has

$$
\begin{aligned}
Q\Big(\frac{1}{\tau}+a_0\Big) &= \frac{1}{\tau}\Big(\frac{1}{\tau}+a_0-a_1\Big)\Big(\frac{1}{\tau}+a_0-a_2\Big)\Big(\frac{1}{\tau}+a_0-a_3\Big) \\
&= -\frac{A}{\tau^4}(\tau-e_1)(\tau-e_2)(\tau-e_3),
\end{aligned} \tag{33.30}
$$

where

$$
A=(a_1-a_0)(a_2-a_0)(a_3-a_0), \quad e_j=\frac{1}{a_j-a_0}. \tag{33.31}
$$

Then we have

$$
I=-\int R\Big(\frac{1}{\tau}+a_0, \frac{\sqrt{-A}}{\tau^2}\sqrt{q(\tau)}\Big)\frac{1}{\tau^2}\,d\tau, \tag{33.32}
$$

with $q(\tau)$ as in (33.1). After a further coordinate translation, one can alter e_j to arrange (33.2).

Elliptic integrals are frequently encountered in many areas of mathematics. Here we give two examples, one from differential equations and one from geometry.

Our first example involves the differential equation for the motion of a simple pendulum, which takes the form

$$
\ell\frac{d^2\theta}{dt^2}+g\,\sin\theta=0, \tag{33.33}
$$

where ℓ is the length of the pendulum g the acceleration of gravity (32 ft./sec.^{2} on the surface of the earth), and θ is the angle the pendulum makes with the downward-pointing vertical axis. The total energy of the pendulum is proportional to

$$
E=\frac{1}{2}\Big(\frac{d\theta}{dt}\Big)^2-\frac{g}{\ell}\,\cos\theta. \tag{33.34}
$$

Applying d/dt to (33.34) and comparing with (33.33) shows that E is constant for each solution to (33.33), so one has

$$
\frac{1}{\sqrt{2}}\,\frac{d\theta}{dt}=\pm\sqrt{E+\frac{g}{\ell}\cos\theta}, \tag{33.35}
$$

or

$$
\pm\int\frac{d\theta}{\sqrt{E+a\cos\theta}}=\sqrt{2}t+c, \tag{33.36}
$$

with $a=g/\ell$. With $\varphi=\theta/2,\ \cos 2\varphi=1-2\sin^2\varphi$, we have

$$
\pm\int\frac{d\varphi}{\sqrt{\alpha-\beta\sin^2\varphi}}=\frac{t}{\sqrt{2}}+c', \tag{33.37}
$$

with $\alpha = E + a,\ \beta = 2a$. Then setting $\zeta = \sin\varphi,\ d\zeta = \cos\varphi\, d\varphi$, we have

$$\pm \int \frac{d\zeta}{\sqrt{(\alpha - \beta\zeta^2)(1-\zeta^2)}} = \frac{t}{\sqrt{2}} + c', \tag{33.38}$$

which is an integral of the form (33.27). If instead in (33.36) we set $\zeta = \cos\theta$, so $d\zeta = -\sin\theta\, d\theta$, we obtain

$$\mp \int \frac{d\zeta}{\sqrt{(E + a\zeta)(1-\zeta^2)}} = \sqrt{2}\, t + c, \tag{33.39}$$

which is an integral of the form (33.3).

In our next example we produce a formula for the arc length $L(\theta)$ of the portion of the ellipse

$$z(t) = (a\,\cos t, b\,\sin t), \tag{33.40}$$

from $t = 0$ to $t = \theta$. We assume $a > b > 0$. Note that

$$\begin{aligned} |z'(t)|^2 &= a^2 \sin^2 t + b^2 \cos^2 t \\ &= b^2 + c^2 \sin^2 t, \end{aligned} \tag{33.41}$$

with $c^2 = a^2 - b^2$, so

$$L(\theta) = \int_0^\theta \sqrt{b^2 + c^2 \sin^2 t}\, dt. \tag{33.42}$$

With $\zeta = \sin t,\ u = \sin\theta$, this becomes

$$\begin{aligned} &\int_0^u \sqrt{b^2 + c^2\zeta^2}\, \frac{d\zeta}{\sqrt{1-\zeta^2}} \\ &= \int_0^u \frac{b^2 + c^2\zeta^2}{\sqrt{(1-\zeta^2)(b^2+c^2\zeta^2)}}\, d\zeta, \end{aligned} \tag{33.43}$$

which is an integral of the form (33.27).

Exercises

1. Using (33.7) and the comments that follow it, show that, for $j = 1, 2$,

$$\frac{1}{2} \int_{e_j}^{e_3} \frac{d\zeta}{\sqrt{q(\zeta)}} = \frac{\omega_{j'}}{2}, \quad \mathrm{mod}\ \Lambda, \tag{33.44}$$

where $j' = 2$ if $j = 1$, $j' = 1$ if $j = 2$.

2. Setting $e_{kj} = e_k - e_j$, show that

$$\frac{1}{2}\int_{e_1}^{e_1+\eta} \frac{d\zeta}{\sqrt{q(\zeta)}} = \frac{1}{2\sqrt{e_{12}e_{13}}} \sum_{k,\ell=0}^{\infty} \binom{-1/2}{k}\binom{-1/2}{\ell} \frac{1}{e_{12}^k e_{13}^\ell} \frac{\eta^{k+\ell+1/2}}{k+\ell+1/2} \tag{33.45}$$

is a convergent power series provided $|\eta| < \min(|e_1 - e_2|, |e_1 - e_3|)$. Using this and variants to integrate from e_j to $e_j + \eta$ for $j = 2$ and 3, find convergent power series for $\omega_j/2$ (mod Λ).

3. Given $k \neq \pm 1$, show that

$$\int \frac{d\zeta}{\sqrt{(1-\zeta^2)(k^2-\zeta^2)}} = -\frac{1}{\sqrt{2(1-k^2)}} \int \frac{d\tau}{\sqrt{q(\tau)}}, \tag{33.46}$$

with

$$\tau = \frac{1}{\zeta+1}, \quad q(\tau) = \Big(\tau - \frac{1}{2}\Big)\Big(\tau - \frac{1}{1-k}\Big)\Big(\tau - \frac{1}{1+k}\Big).$$

In Exercises 4–9, we assume e_j are real and $e_1 < e_2 < e_3$. We consider

$$\omega_3 = \int_{e_1}^{e_2} \frac{d\zeta}{\sqrt{(\zeta-e_1)(\zeta-e_2)(\zeta-e_3)}}.$$

4. Show that

$$\begin{aligned} \omega_3 &= 2\int_0^{\pi/2} \frac{d\theta}{\sqrt{(e_3-e_2)\sin^2\theta + (e_3-e_1)\cos^2\theta}} \\ &= 2I\big(\sqrt{e_3-e_2}, \sqrt{e_3-e_1}\big), \end{aligned} \tag{33.48}$$

where

$$I(r,s) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{r^2\sin^2\theta + s^2\cos^2\theta}}. \tag{33.49}$$

Note that in (33.48), $0 < r < s$.

Exercises 5–7 will be devoted to showing that

$$I(r,s) = \frac{\pi}{2M(s,r)}, \tag{33.50}$$

if $0 < r \le s$, where $M(s,r)$ is the Gauss arithmetic-geometric mean, defined below.

5. Given $0 < b \le a$, define inductively

$$(a_0, b_0) = (a, b), \quad (a_{k+1}, b_{k+1}) = \Bigl(\frac{a_k + b_k}{2}, \sqrt{a_k b_k}\Bigr). \tag{33.51}$$

Show that

$$a_0 \ge a_1 \ge a_2 \ge \cdots \ge b_2 \ge b_1 \ge b_0.$$

Show that

$$a_{k+1}^2 - b_{k+1}^2 = (a_{k+1} - a_k)^2.$$

Monotonicity implies $a_k - a_{k+1} \to 0$. Deduce that $a_{k+1} - b_{k+1} \to 0$, and hence

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = M(a, b), \tag{33.52}$$

the latter identity being the definition of $M(a, b)$. Show also that

$$a_{k+1}^2 - b_{k+1}^2 = \frac{1}{4}(a_k - b_k)^2,$$

hence

$$a_{k+1} - b_{k+1} = \frac{(a_k - b_k)^2}{8a_{k+2}}. \tag{33.53}$$

Deduce from (33.53) that convergence in (33.52) is quite rapid.

6. Show that the asserted identity (33.50) holds if it can be demonstrated that, for $0 < r \le s$,

$$I(r, s) = I\Bigl(\sqrt{rs}, \frac{r + s}{2}\Bigr). \tag{33.54}$$

Hint. Show that (33.54) $\Rightarrow I(r, s) = I(m, m)$, with $m = M(s, r)$, and evaluate $I(m, m)$.

7. Take the following steps to prove (33.54). Show that you can make the change of variable from θ to φ, with

$$\sin\theta = \frac{2s\sin\varphi}{(s + r) + (s - r)\sin^2\varphi}, \quad 0 \le \varphi \le \frac{\pi}{2}, \tag{33.55}$$

and obtain

$$I(r, s) = \int_0^{\pi/2} \frac{2\,d\varphi}{\sqrt{4rs\sin^2\varphi + (s + r)^2\cos^2\varphi}}. \tag{33.56}$$

Show that this yields (33.54).

8. In the setting of Exercise 4, deduce that

$$\omega_3 = \frac{\pi}{M(\sqrt{e_3 - e_1}, \sqrt{e_3 - e_2})}. \tag{33.57}$$

9. Similarly, show that

$$\begin{aligned} \omega_1 &= \int_{e_2}^{e_3} \frac{d\zeta}{\sqrt{(\zeta - e_1)(\zeta - e_2)(\zeta - e_3)}} \\ &= 2i \int_0^{\pi/2} \frac{d\theta}{\sqrt{(e_2 - e_1)\sin^2\theta + (e_3 - e_1)\cos^2\theta}} \\ &= \frac{\pi i}{M(\sqrt{e_3 - e_1}, \sqrt{e_2 - e_1})}. \end{aligned} \tag{33.58}$$

10. Set $x = \sin\theta$ to get

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \beta^2 \sin^2\theta}} = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - \beta^2 x^2)}}.$$

Write $1 - \beta^2 \sin^2\theta = (1 - \beta^2)\sin^2\theta + \cos^2\theta$ to deduce that

$$\int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - \beta^2 x^2)}} = \frac{\pi}{2M(1, \sqrt{1 - \beta^2})}, \tag{33.59}$$

if $\beta \in (-1, 1)$.

11. Parallel to Exercise 10, show that

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 + \beta^2 \sin^2\theta}} = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 + \beta^2 x^2)}},$$

and deduce that

$$\int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 + \beta^2 x^2)}} = \frac{\pi}{2M(\sqrt{1 + \beta^2}, 1)}, \tag{33.60}$$

if $\beta \in \mathbb{R}$. A special case is

$$\int_0^1 \frac{dx}{\sqrt{1 - x^4}} = \frac{\pi}{2M(\sqrt{2}, 1)}. \tag{33.61}$$

For more on the arithmetic-geometric mean (AGM), see [BB].

34. The Riemann surface of $\sqrt{q(\zeta)}$

Recall from §33 the cubic polynomial

$$q(\zeta) = (\zeta - e_1)(\zeta - e_2)(\zeta - e_3), \tag{34.1}$$

where $e_1, e_2, e_3 \in \mathbb{C}$ are distinct. Here we will construct a compact Riemann surface M associated with the "double valued" function $\sqrt{q(\zeta)}$, together with a holomorphic map

$$\varphi : M \longrightarrow \widehat{\mathbb{C}}, \tag{34.2}$$

and discuss some important properties of M and φ. Here $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere, introduced in §22 and identified there with S^2. We will use the construction of φ to prove Proposition 33.1. Material developed below will use some basic results on manifolds, particularly on surfaces. Background material on this is given in Appendix D. Further material can be found in basic topology and geometry texts, such as [Mun] and [Sp].

To begin, we set $e_4 = \infty$ in $\widehat{\mathbb{C}}$. Reordering if necessary, we arrange that the geodesic γ_{12} from e_1 to e_2 is disjoint from the geodesic γ_{34} from e_3 to e_4. We slit $\widehat{\mathbb{C}}$ along γ_{12} and along γ_{34}, obtaining X, a manifold with boundary, as illustrated in the top right portion of Fig. 34.1. Now

$$M = X_1 \cup X_2 / \sim, \tag{34.3}$$

where X_1 and X_2 are two copies of X, and the equivalence relation $\sim$ identifies the upper boundary of X_1 along the slit γ_{12} with the lower boundary of X_2 along this slit and vice-versa, and similarly for γ_{34}. This is illustrated in the middle and bottom parts of Fig. 34.1. The manifold M is seen to be topologically equivalent to a torus.

The map $\varphi : M \to \widehat{\mathbb{C}}$ in (34.2) is tautological. It is two-to-one except for the four points $p_j = \varphi^{-1}(e_j)$. Recall the definition of a Riemann surface given in §22, in terms of coordinate covers. The space M has a unique Riemann surface structure for which φ is holomorphic. A coordinate taking a neighborhood of p_j in M bijectively onto a neighborhood of the origin in $\mathbb{C}$ is given by $\varphi_j(x) = (\varphi(x) - e_j)^{1/2}$, for $1 \le j \le 3$, with $\varphi(x) \in \widehat{\mathbb{C}}$, and a coordinate mapping a neighborhood of p_4 in M bijectively onto a neighborhood of the origin in $\mathbb{C}$ is given by $\varphi_4(x) = \varphi(x)^{-1/2}$.

Now consider the double-valued form $d\zeta/\sqrt{q(\zeta)}$ on $\widehat{\mathbb{C}}$, having singularities at $\{e_j\}$. This pulls back to a single-valued 1-form α on M. Noting that if $w^2 = \zeta$ then

$$\frac{d\zeta}{\sqrt{\zeta}} = 2\, dw, \tag{34.4}$$

and that if $w^2 = 1/\zeta$ then

$$\frac{d\zeta}{\sqrt{\zeta^3}} = -2\, dw, \tag{34.5}$$

we see that α is a smooth, holomorphic 1-form on M, with no singularities, and also that α has no zeros on M. Using this, we can prove the following.

Proposition 34.1. *There is a lattice $\Lambda_0 \subset \mathbb{C}$ and a holomorphic diffeomorphism*

$$\psi : M \longrightarrow \mathbb{C}/\Lambda_0. \tag{34.6}$$

Proof. Given M homeomorphic to $S^1 \times S^1$, we have closed curves c_1 and c_2 through p_1 in M such that each closed curve γ in M is homotopic to a curve starting at p_1, winding n_1 times along c_1, then n_2 times along c_2, with $n_j \in \mathbb{Z}$. Say $\omega_j = \int_{c_j} \alpha$. We claim ω_1 and ω_2 are linearly independent over $\mathbb{R}$. First we show that they are not both 0. Indeed, if $\omega_1 = \omega_2 = 0$, then

$$\Psi(z) = \int_{p_0}^{z} \alpha \tag{34.7}$$

would define a non-constant holomorphic map $\Psi : M \to \mathbb{C}$, which would contradict the maximum principle. Let us say $\omega_2 \neq 0$, and set $\beta = \omega_2^{-1}\alpha$. Then $\Psi_1(z) = \int_{p_0}^{z} \beta$ is well defined modulo an additive term of the form $j + k(\omega_1/\omega_2)$, with $j, k \in \mathbb{Z}$. If ω_1/ω_2 were real, then $\operatorname{Im}\Psi_1 : M \to \mathbb{R}$ would be a well defined harmonic function, hence (by the maximum principle) constant, forcing Ψ constant, and contradicting the fact that $\alpha \neq 0$.

Thus we have that $\Lambda_1 = \{n_1\omega_1 + n_2\omega_2 : n_j \in \mathbb{Z}\}$ is a lattice, and that (34.7) yields a well defined holomorphic map

$$\Psi : M \longrightarrow \mathbb{C}/\Lambda_1. \tag{34.8}$$

Since α is nowhere vanishing, Ψ is a local diffeomorphism. Hence it must be a covering map. This gives (34.6), where Λ_0 is perhaps a sublattice of Λ_1.

We now prove Proposition 33.1, which we restate here.

Proposition 34.2. *Let e_1, e_2, e_3 be distinct points in $\mathbb{C}$, satisfying*

$$e_1 + e_2 + e_3 = 0. \tag{34.9}$$

There exists a lattice $\Lambda \subset \mathbb{C}$, generated by ω_1, ω_2, linearly independent over $\mathbb{R}$, such that if $\wp(z) = \wp(z; \Lambda)$, then

$$\wp\Big(\frac{\omega_j}{2}\Big) = e_j, \quad 1 \le j \le 3, \tag{34.10}$$

where $\omega_3 = \omega_1 + \omega_2$.

Proof. We have from (34.2) and (34.6) a holomorphic map

$$\Phi : \mathbb{C}/\Lambda_0 \longrightarrow \widehat{\mathbb{C}}, \tag{34.11}$$

which is a branched double cover, branching over e_1, e_2, e_3, and ∞. We can regard Φ as a meromorphic function on $\mathbb{C}$, satisfying

$$\Phi(z + \omega) = \Phi(z), \quad \forall\, \omega \in \Lambda_0. \tag{34.12}$$

Furthermore, translating coordinates, we can assume Φ has a double pole, precisely at points in Λ_0. It follows that there are constants a and b such that

$$\Phi(z) = a\wp_0(z) + b, \quad a \in \mathbb{C}^*, \ b \in \mathbb{C}, \tag{34.13}$$

where $\wp_0(z) = \wp(z;\Lambda_0)$. Hence $\Phi'(z) = a\wp_0'(z)$, so by Proposition 31.1 we have

$$\Phi'(z) = 0 \Longleftrightarrow z = \frac{\omega_{0j}}{2}, \quad \text{mod } \Lambda_0, \tag{34.14}$$

where ω_{01}, ω_{02} generate Λ_0 and $\omega_{03} = \omega_{01} + \omega_{02}$. Hence (perhaps after some reordering)

$$e_j = a\wp_0\Big(\frac{\omega_{0j}}{2}\Big) + b. \tag{34.15}$$

Now if $e_j' = \wp_0(\omega_{0j}/2)$, we have by (31.15) that $e_1' + e_2' + e_3' = 0$, so (34.9) yields

$$b = 0. \tag{34.16}$$

Finally, we set $\Lambda = a^{-1/2}\Lambda_0$ and use (32.34) to get

$$\wp(z;\Lambda) = a\,\wp(a^{1/2}z;\Lambda_0). \tag{34.17}$$

Then (34.10) is achieved.

We mention that a similar construction works to yield a compact Riemann surface $M \to \widehat{\mathbb{C}}$ on which there is a single valued version of $\sqrt{q(\zeta)}$ when

$$q(\zeta) = (\zeta - e_1)\cdots(\zeta - e_m), \tag{34.18}$$

where $e_j \in \mathbb{C}$ are distinct, and $m \geq 2$. If $m = 2g+1$, one has slits from e_{2j-1} to e_{2j}, for $j = 1, \dots, g$, and a slit from e_{2g+1} to ∞, which we denote e_{2g+2}. If $m = 2g+2$, one has slits from e_{2j-1} to e_{2j}, for $j = 1, \dots, g+1$. Then X is constructed by opening the slits, and M is constructed as in (34.3). The picture looks like that in Fig. 34.1, but instead of two sets of pipes getting attached, one has $g+1$ sets. One gets a Riemann surface M with g holes, called a surface of genus g. Again the double-valued form $d\zeta/\sqrt{q(\zeta)}$ on $\widehat{\mathbb{C}}$ pulls back to a single-valued 1-form α on M, with no singularities, except when $m = 2$ (see the exercises). If $m = 4$ (so again $g = 1$), α has no zeros. If $m \geq 5$ (so $g \geq 2$), α has a zero at $\varphi^{-1}(\infty)$. Proposition 34.1 extends to the case $m = 4$. If $m \geq 5$ the situation changes. It is a classical result that M is covered by the disk D rather than by $\mathbb{C}$. The pull-back of α to D is called an automorphic form. For much more on such matters, and on more general constructions of Riemann surfaces, we recommend [FK] and [MM].

We end this section with a brief description of a Riemann surface, conformally equivalent to M in (34.3), appearing as a submanifold of complex projective space $\mathbb{CP}^2$. More details on such a construction can be found in [Cl] and [MM].

To begin, we define complex projective space $\mathbb{CP}^n$ as $(\mathbb{C}^{n+1} \setminus 0)/\sim$, where we say z and $z' \in \mathbb{C}^{n+1} \setminus 0$ satisfy $z \sim z'$ provided $z' = az$ for some $a \in \mathbb{C}^*$. Then $\mathbb{CP}^n$ has the structure of a complex manifold. Denote by $[z]$ the equivalence class in $\mathbb{CP}^n$ of $z \in \mathbb{C}^{n+1} \setminus 0$. We note that the map

$$\kappa : \mathbb{CP}^1 \longrightarrow \mathbb{C} \cup \{\infty\} = \widehat{\mathbb{C}}, \tag{34.19}$$

given by

$$\begin{aligned} \kappa([(z_1, z_2)]) &= z_1/z_2, \quad z_2 \neq 0, \\ \kappa([(1,0)]) &= \infty, \end{aligned} \tag{34.20}$$

is a holomorphic diffeomorphism, so $\mathbb{CP}^1 \approx \widehat{\mathbb{C}} \approx S^2$.

Now given distinct $e_1, e_2, e_3 \in \mathbb{C}$, we can define $M_e \subset \mathbb{CP}^2$ to consist of elements $[(w, \zeta, t)]$ such that $(w, \zeta, t) \in \mathbb{C}^3 \setminus 0$ satisfies

$$w^2 t = (\zeta - e_1 t)(\zeta - e_2 t)(\zeta - e_3 t). \tag{34.21}$$

One can show that M_e is a smooth complex submanifold of $\mathbb{CP}^2$, possessing then the structure of a compact Riemann surface. An analogue of the map (34.2) is given as follows.

Set $p = [(1, 0, 0)] \in \mathbb{CP}^2$. Then there is a holomorphic map

$$\psi : \mathbb{CP}^2 \setminus p \longrightarrow \mathbb{CP}^1, \tag{34.22}$$

given by

$$\psi([(w, \zeta, t)]) = [(\zeta, t)]. \tag{34.23}$$

This restricts to $M_e \setminus p \to \mathbb{CP}^1$. Note that $p \in M_e$. While ψ in (34.22) is actually singular at p, for the restriction to $M_e \setminus p$ this is a removable singularity, and one has a holomorphic map

$$\varphi_e : M_e \longrightarrow \mathbb{CP}^1 \approx \widehat{\mathbb{C}} \approx S^2, \tag{34.24}$$

given by (34.22) on $M_e \setminus p$ and taking p to $[(1, 0)] \in \mathbb{CP}^1$, hence to $\infty \in \mathbb{C} \cup \{\infty\}$. This map can be seen to be a 2-to-1 branched covering, branching over $\mathcal{B} = \{e_1, e_2, e_3, \infty\}$. Given $q \in \mathbb{C}$, $q \notin \mathcal{B}$, and a choice $r \in \varphi^{-1}(q) \subset M$ and $r_e \in \varphi_e^{-1}(q) \subset M_e$, there is a unique holomorphic diffeomorphism

$$\Gamma : M \longrightarrow M_e, \tag{34.25}$$

such that $\Gamma(r) = r_e$ and $\varphi = \varphi_e \circ \Gamma$.

Exercises

1. Show that the covering map Ψ in (34.8) is actually a diffeomorphism, and hence $\Lambda_0 = \Lambda_1$.

2. Suppose Λ_0 and Λ_1 are two lattices in $\mathbb{C}$ such that $\mathbb{T}_{\Lambda_0}$ and $\mathbb{T}_{\Lambda_1}$ are conformally equivalent, via a holomorphic diffeomorphism

$$f : \mathbb{C}/\Lambda_0 \longrightarrow \mathbb{C}/\Lambda_1. \tag{34.26}$$

Show that f lifts to a holomorphic diffeomorphism F of $\mathbb{C}$ onto itself, such that $F(0) = 0$, and hence that $F(z) = az$ for some $a \in \mathbb{C}^*$. Deduce that $\Lambda_1 = a\Lambda_0$.

3. Consider the upper half-plane $\mathcal{U} = \{\tau \in \mathbb{C} : \operatorname{Im} \tau > 0\}$. Given $\tau \in \mathcal{U}$, define

$$\Lambda(\tau) = \{m + n\tau : m, n \in \mathbb{Z}\}. \tag{34.27}$$

Show that each lattice $\Lambda \subset \mathbb{C}$ has the form $\Lambda = a\Lambda(\tau)$ for some $a \in \mathbb{C}^*$, $\tau \in \mathcal{U}$.

4. Define the maps $\alpha, \beta : \mathcal{U} \to \mathcal{U}$ by

$$\alpha(\tau) = -\frac{1}{\tau}, \quad \beta(\tau) = \tau + 1. \tag{34.28}$$

Show that, for each $\tau \in \mathcal{U}$,

$$\Lambda(\alpha(\tau)) = \tau^{-1}\,\Lambda(\tau), \quad \Lambda(\beta(\tau)) = \Lambda(\tau). \tag{34.29}$$

5. Let $\mathcal{G}$ be the group of automorphisms of $\mathcal{U}$ generated by α and β, given in (34.28). Show that if $\tau, \tau' \in \mathcal{U}$,

$$\mathbb{C}/\Lambda(\tau) \approx \mathbb{C}/\Lambda(\tau'), \tag{34.30}$$

in the sense of being holomorphically diffeomorphic, if and only if

$$\tau' = \gamma(\tau), \quad \text{for some } \ \gamma \in \mathcal{G}. \tag{34.31}$$

6. Show that the group $\mathcal{G}$ consists of linear fractional transformations of the form

$$L_A(\tau) = \frac{a\tau + b}{c\tau + d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{34.32}$$

where $a, b, c, d \in \mathbb{Z}$ and $\det A = 1$, i.e., $A \in Sl(2, \mathbb{Z})$. Show that

$$\mathcal{G} \approx Sl(2, \mathbb{Z})/\{\pm I\}.$$

In Exercises 7–8, we make use of the covering map $\Psi : \mathcal{U} \to \mathbb{C} \setminus \{0,1\}$, given by (26.5), and results of Exercises 1–8 of §26, including (26.10)–(26.11), i.e.,

$$\Psi(\alpha(\tau)) = \frac{1}{\Psi(\tau)}, \quad \Psi(\beta(\tau)) = 1 - \Psi(\tau). \tag{34.33}$$

7. Given $\tau, \tau' \in \mathcal{U}$, we say $\tau \sim \tau'$ if and only if (34.30) holds. Show that, given

$$\tau,\ \tau' \in \mathcal{U}, \quad w = \Psi(\tau),\ w' = \Psi(\tau') \in \mathbb{C} \setminus \{0,1\}, \tag{34.34}$$

we have

$$\tau \sim \tau' \Longleftrightarrow w' = F(w) \ \text{ for some } \ F \in \mathcal{G}, \tag{34.35}$$

where $\mathcal{G}$ is the group (of order 6) of automorphisms of $\mathbb{C} \setminus \{0,1\}$ arising in Exercise 7 of §26.

8. Bringing in the map $H : S^2 \to S^2$ arising in Exercise 9 of §26, i.e.,

$$H(w) = \frac{4}{27} \frac{(w^2 - w + 1)^3}{w^2 (w-1)^2}, \tag{34.36}$$

satisfying (26.23), i.e.,

$$H\Big(\frac{1}{w}\Big) = H(w), \quad H(1-w) = H(w), \tag{34.37}$$

show that

$$w' = F(w) \ \text{ for some } \ F \in \mathcal{G} \Longleftrightarrow H(w') = H(w). \tag{34.38}$$

Deduce that, for $\tau, \tau' \in \mathcal{U}$,

$$\tau \sim \tau' \Longleftrightarrow H \circ \Psi(\tau') = H \circ \Psi(\tau). \tag{34.39}$$

Exercises 9–14 deal with the Riemann surface M of $\sqrt{q(\zeta)}$ when

$$q(\zeta) = (\zeta - e_1)(\zeta - e_2), \tag{34.40}$$

and $e_1, e_2 \in \mathbb{C}$ are distinct.

9. Show that the process analogous to that pictured in Fig. 34.1 involves the attachment of one pair of pipes, and M is topologically equivalent to a sphere. One gets a branched covering $\varphi : M \to \widehat{\mathbb{C}}$, as in (34.2).

10. Show that the double-valued form $d\zeta/\sqrt{q(\zeta)}$ on $\widehat{\mathbb{C}}$ pulls back to a single-valued form α on M. Using (34.4), show that α is a smooth nonvanishing form except at $\{p_1, p_2\} = \varphi^{-1}(\infty)$. In a local coordinate system about p_j of the form $\varphi_j(x) = \varphi(x)^{-1}$, use a variant of (34.4)–(34.5) to show that α has the form

$$\alpha = (-1)^j \frac{g(z)}{z}\, dz, \tag{34.41}$$

where $g(z)$ is holomorphic and $g(0) \neq 0$.

11. Let c be a curve in $M \setminus \{p_1, p_2\}$ with winding number 1 about p_1. Set

$$\omega = \int_c \alpha, \quad L = \{k\omega : k \in \mathbb{Z}\} \subset \mathbb{C}. \tag{34.42}$$

Note that Exercise 10 implies $\omega \neq 0$. Pick $q \in M \setminus \{p_1, p_2\}$. Show that

$$\Psi(z) = \int_q^z \alpha \tag{34.43}$$

yields a well defined holomorphic map

$$\Psi : M \setminus \{p_1, p_2\} \longrightarrow \mathbb{C}/L. \tag{34.44}$$

12. Show that Ψ in (34.44) is a holomorphic diffeomorphism of $M \setminus \{p_1, p_2\}$ onto $\mathbb{C}/L$. *Hint.* To show Ψ is onto, use (34.41) to examine the behavior of Ψ near p_1 and p_2.

13. Produce a holomorphic diffeomorphism $\mathbb{C}/L \approx \mathbb{C} \setminus \{0\}$, and then use (34.44) to obtain a holomorphic diffeomorphism

$$\Psi_1 : M \setminus \{p_1, p_2\} \longrightarrow \widehat{\mathbb{C}} \setminus \{0, \infty\}. \tag{34.45}$$

Show that this extends uniquely to a holomorphic diffeomorphism

$$\Psi_1 : M \longrightarrow \widehat{\mathbb{C}}. \tag{34.46}$$

14. Note that with a linear change of variable we can arrange $e_j = (-1)^j$ in (34.40). Relate the results of Exercises 9–13 to the identity

$$\int_0^z (1 - \zeta^2)^{-1/2}\, d\zeta = \sin^{-1} z \pmod{2\pi\mathbb{Z}}. \tag{34.47}$$

K. Rapid evaluation of the Weierstrass $\wp$-function

Given a lattice $\Lambda \subset \mathbb{C}$, the associated Weierstrass $\wp$-function is defined by

$$\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{0\neq\beta\in\Lambda} \Bigl(\frac{1}{(z-\beta)^2} - \frac{1}{\beta^2}\Bigr). \tag{K.1}$$

This converges rather slowly, so another method must be used to evaluate $\wp(z;\Lambda)$ rapidly. The classical method, which we describe below, involves a representation of $\wp$ in terms of theta functions. It is most conveniently described in case

$$\Lambda \text{ generated by } 1 \text{ and } \tau, \quad \operatorname{Im}\tau > 0. \tag{K.2}$$

To pass from this to the general case, we can use the identity

$$\wp(z;a\Lambda) = \frac{1}{a^2}\wp\Bigl(\frac{z}{a};\Lambda\Bigr). \tag{K.3}$$

The material below is basically a summary of material from §32, assembled here to clarify the important application to the task of the rapid evaluation of (K.1).

To evaluate $\wp(z;\Lambda)$, which we henceforth denote $\wp(z)$, we use the following identity:

$$\wp(z) = e_1 + \Bigl(\frac{\vartheta_1'(0)}{\vartheta_2(0)}\,\frac{\vartheta_2(z)}{\vartheta_1(z)}\Bigr)^2. \tag{K.4}$$

See (32.20). Here $e_1 = \wp(\omega_1/2) = \wp(1/2)$, and the theta functions $\vartheta_j(z)$ (which also depend on ω) are defined as follows (cf. (32.6)–(32.10)):

$$\begin{aligned}
\vartheta_1(z) &= i\sum_{n=-\infty}^{\infty} (-1)^n p^{2n-1} q^{(n-1/2)^2},\\
\vartheta_2(z) &= \sum_{n=-\infty}^{\infty} p^{2n-1} q^{(n-1/2)^2},\\
\vartheta_3(z) &= \sum_{n=-\infty}^{\infty} p^{2n} q^{n^2},\\
\vartheta_4(z) &= \sum_{n=-\infty}^{\infty} (-1)^n p^{2n} q^{n^2}.
\end{aligned} \tag{K.5}$$

Here

$$p = e^{\pi i z}, \quad q = e^{\pi i \tau}, \tag{K.6}$$

with τ as in (K.2).

The functions ϑ_1 and ϑ_2 appear in (K.4). Also ϑ_3 and ϑ_4 arise to yield a rapid evaluation of e_1 (cf. (32.33)):

$$e_1 = \frac{\pi^2}{3}\big[\vartheta_3(0)^4 + \vartheta_4(0)^4\big]. \tag{K.7}$$

Note that $(d/dz)p^{2n-1} = \pi i(2n-1)p^{2n-1}$ and hence

$$\vartheta_1'(0) = -\pi \sum_{n=-\infty}^{\infty} (-1)^n (2n-1) q^{(n-1/2)^2}. \tag{K.8}$$

It is convenient to rewrite the formulas for $\vartheta_1(z)$ and $\vartheta_2(z)$ as

$$\begin{aligned} \vartheta_1(z) &= i\sum_{n=1}^{\infty}(-1)^n q^{(n-1/2)^2}(p^{2n-1} - p^{1-2n}), \\ \vartheta_2(z) &= \sum_{n=1}^{\infty} q^{(n-1/2)^2}(p^{2n-1} + p^{1-2n}). \end{aligned} \tag{K.9}$$

also formulas for $\vartheta_1'(0)$ and $\vartheta_j(0)$, which appear in (K.4) and (K.7), can be rewritten:

$$\begin{aligned} \vartheta_1'(0) &= -2\pi\sum_{n=1}^{\infty}(-1)^n(2n-1)q^{(n-1/2)^2}, \\ \vartheta_2(0) &= 2\sum_{n=1}^{\infty} q^{(n-1/2)^2}, \\ \vartheta_3(0) &= 1 + 2\sum_{n=1}^{\infty} q^{n^2}, \\ \vartheta_4(0) &= 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}. \end{aligned} \tag{K.10}$$

Rectangular lattices

We specialize to the case where Λ is a rectangular lattice, of sides 1 and L, more precisely:

$$\Lambda \ \text{ generated by } \ 1 \ \text{ and } \ iL, \quad L > 0. \tag{K.11}$$

Now the formulas established above hold, with $\tau = iL$, hence

$$q = e^{-\pi L}. \tag{K.12}$$

Since q is real, we see that the quantities $\vartheta_1'(0)$ and $\vartheta_j(0)$ in (K.10) are real. It is also convenient to calculate the real and imaginary parts of $\vartheta_j(z)$ in this case. Say

$$z = u + iv, \tag{K.13}$$

with u and v real. Then

$$p^{2n-1} = e^{-(2n-1)\pi v}\big[\cos(2n-1)\pi u + i\sin(2n-1)\pi u\big]. \tag{K.14}$$

We then have

$$\begin{aligned} \operatorname{Re}(-i\vartheta_1(z)) &= -\sum_{n=1}^{\infty}(-1)^n q^{(n-1/2)^2}\left[e^{(2n-1)\pi v} - e^{-(2n-1)\pi v}\right]\cos(2n-1)\pi u, \\ \operatorname{Im}(-i\vartheta_1(z)) &= \sum_{n=1}^{\infty}(-1)^n q^{(n-1/2)^2}\left[e^{(2n-1)\pi v} + e^{-(2n-1)\pi v}\right]\sin(2n-1)\pi u, \end{aligned} \tag{K.15}$$

and

$$\begin{aligned} \operatorname{Re}\vartheta_2(z) &= \sum_{n=1}^{\infty} q^{(n-1/2)^2}\left[e^{(2n-1)\pi v} + e^{-(2n-1)\pi v}\right]\cos(2n-1)\pi u, \\ \operatorname{Im}\vartheta_2(z) &= -\sum_{n=1}^{\infty} q^{(n-1/2)^2}\left[e^{(2n-1)\pi v} - e^{-(2n-1)\pi v}\right]\sin(2n-1)\pi u. \end{aligned} \tag{K.16}$$

We can calculate these quantities accurately by summing over a small range. Let us insist that

$$-\frac{1}{2} \le u < \frac{1}{2}, \quad -\frac{L}{2} \le v < \frac{L}{2}, \tag{K.17}$$

and assume

$$L \ge 1. \tag{K.18}$$

Then

$$\left|q^{(n-1/2)^2} e^{(2n-1)\pi v}\right| \le e^{-(n^2-3n+5/4)\pi L}, \tag{K.19}$$

and since

$$e^{-\pi} < \frac{1}{20}, \tag{K.20}$$

we see that the quantity in (K.19) is

$$\begin{aligned} &< 0.5 \times 10^{-14} \quad \text{for} \quad n = 5, \\ &< 2 \times 10^{-25} \quad \text{for} \quad n = 6, \end{aligned} \tag{K.21}$$

with rapid decrease for $n > 6$. Thus, summing over $1 \le n \le 5$ will give adequate approximations.

For $z = u + iv$ very near 0, where ϑ_1 vanishes and $\wp$ has a pole, the identity

$$\frac{1}{\wp(z) - e_1} = \Bigl(\frac{\vartheta_2(0)}{\vartheta_1'(0)} \frac{\vartheta_1(z)}{\vartheta_2(z)}\Bigr)^2, \tag{K.22}$$

in concert with (K.10) and (K.15)–(K.16), gives an accurate approximation to $(\wp(z) - e_1)^{-1}$, which in this case is also very small. Note, however, that some care should be taken in evaluating $\mathrm{Re}(-i\vartheta_1(z))$, via the first part of (K.15), when $|z|$ is very small. More precisely, care is needed in evaluating

$$e^{k\pi v} - e^{-k\pi v}, \quad k = 2n - 1 \in \{1, 3, 5, 7, 9\}, \tag{K.23}$$

when v is very small, since then (K.23) is the difference between two quantities close to 1, so evaluating $e^{k\pi v}$ and $e^{-k\pi v}$ separately and subtracting can lead to an undesirable loss of accuracy. In case $k = 1$, one can effect this cancellation at the power series level and write

$$e^{\pi v} - e^{-\pi v} = 2 \sum_{j \ge 1, \mathrm{odd}} \frac{(\pi v)^j}{j!}. \tag{K.24}$$

If $|\pi v| \le 10^{-2}$, summing over $j \le 7$ yields substantial accuracy. (If $|\pi v| > 10^{-2}$, separate evaluation of $e^{k\pi v}$ and $e^{-k\pi v}$ should not be a problem.) For other values of k in (K.23), one can derive from

$$(x^k - 1) = (x - 1)(x^{k-1} + \cdots + 1) \tag{K.25}$$

the identity

$$e^{k\pi v} - e^{-k\pi v} = (e^{\pi v} - e^{-\pi v}) \sum_{\ell=0}^{k-1} e^{(2\ell - (k-1))\pi v}, \tag{K.26}$$

which in concert with (K.24) yields an accurate evaluation of each term in (K.23).

Remark. If (K.11) holds with $0 < L < 1$, one can use (K.3), with $a = iL$, to transfer to the case of a lattice generated by 1 and i/L.

This chapter has two sections. The first discusses a special differential equation, called Bessel's equation, and the special functions (Bessel functions) that arise in its solution. The second gives a general treatment of linear differential equations with holomorphic coefficients on a complex domain, with results on Bessel's equation serving as a guide.

Bessel's equation is the following:

$$\frac{d^2x}{dt^2} + \frac{1}{t}\frac{dx}{dt} + \Bigl(1 - \frac{\nu^2}{t^2}\Bigr)x = 0. \tag{7.0.1}$$

Its central importance comes from the fact that it arises in the analysis of wave equations on domains in Euclideam space $\mathbb{R}^n$ on which it is convenient to use polar coordinates. A discussion of the relevance of (7.0.1) to such wave equations is given in Appendix O, at the end of this chapter.

We apply power series techniques to obtain a solution to (7.0.1) of the form

$$J_\nu(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)}\Bigl(\frac{t}{2}\Bigr)^{2k+\nu}, \tag{7.0.2}$$

valid for $t \in \mathbb{C} \setminus (-\infty, 0]$. Clearly $J_{-\nu}(t)$ is also a solution to (7.0.1). If $\nu \notin \mathbb{Z}$, $J_\nu(t)$ and $J_{-\nu}(t)$ form a complete set of solutions, but if $n \in \mathbb{Z}$, $J_n(t) = (-1)^n J_{-n}(t)$, We bring in the Wronskian as a tool to produce another family of solutions $Y_\nu(t)$, which works with $J_\nu(t)$ to produce a complete set of solutions for all $\nu \in \mathbb{C}$.

At this point, let us take an opportunity to list (7.0.2) as an example of a point raised in §18 of Chapter 4, regarding the central role that the Gamma function plays in the study of many higher transcendental functions.

We complement series representations of Bessel functions with integral representations, such as

$$J_\nu(t) = \frac{(t/2)^\nu}{\Gamma(1/2)\Gamma(\nu+1/2)} \int_{-1}^{1} (1-s^2)^{\nu-1/2} e^{ist}\, ds, \tag{7.0.3}$$

for $\operatorname{Re}\nu > -1/2$ and $t \in \mathbb{C} \setminus (-\infty, 0]$. To celebrate the unity of various facets of analysis developed in this text, we point out that the integral in (7.0.3) is a Fourier transform. Section 35 also analyzes the behavior of $J_\nu(t)$ as $t \to \infty$, using results developed in Appendix P.

In addition to the functions $J_\nu(t)$ and $Y_\nu(t)$, related to wave equations, we have the functions

$$I_\nu(t) = e^{-\pi i\nu/2} J_\nu(it), \tag{7.0.4}$$

which solve

$$\frac{d^2u}{dt^2} + \frac{1}{t}\frac{du}{dt} - \Bigl(1 + \frac{\nu^2}{t^2}\Bigr)u = 0, \tag{7.0.5}$$

and arise in the study of diffusion equations. The simultaneous treatment of wave equations and diffusion equations gives evidence of the advantage of treating these equations in the complex domain.

Section 36 treats a general class of linear differential equations with holomorphic coefficients. Specifically, we look at first order $n \times n$ systems,

$$\frac{dv}{dz} = A(z)v + f(z), \quad v(z_0) = v_0, \tag{7.0.6}$$

given $z_0 \in \Omega,\ v_0 \in \mathbb{C}^n$, and

$$A : \Omega \to M(n, \mathbb{C}), \quad f : \Omega \to \mathbb{C}^n, \quad \text{holomorphic.} \tag{7.0.7}$$

Higher order equations, such as (7.0.1), can be converted to first order systems. Our first attack on (7.0.6) is via power series,

$$v(z) = \sum_{k=0}^{\infty} v_k (z - z_0)^k, \quad v_k \in \mathbb{C}^n. \tag{7.0.8}$$

Plugging this and the power series for $A(z)$ and $f(z)$ into (7.0.6) yields recursive formulas for the coefficients v_k, and we establish the following.

Theorem. If the power series for $A(z)$ and $f(z)$ converge for $|z - z_0| < R$, then the power series (7.0.6) obtained by the process described above also converges for $|z - z_0| < R$, to a solution to (7.0.6).

Unless Ω is a disk and z_0 its center, there is more work to do to investigate the solutions to (7.0.6). One approach to pursuing this brings in one of the most important applications of analytic continuation along a curve. A key result called the Monodromy Theorem states that analytic continuation along two curves in Ω from z_0 to z_1 yields identical holomorphic functions on a neighborhood of z_1 provided these curves are homotopic (as curves in Ω with fixed ends). It follows that (7.0.6) has a unique global holomorphic solution on Ω, provided Ω is simply connected.

In connection with this, we note that Bessel's equation (7.0.1) has coefficients that are holomorphic on $\mathbb{C} \setminus \{0\}$, which is not simply connected. We get single valued solutions, like $J_\nu(z)$, by working on the simply connected domain $\mathbb{C} \setminus (-\infty, 0]$.

To dwell on (7.0.1), we note that the coefficients are singular at $z = 0$ in a special way. This equation can be converted to a first order system of the form

$$z \frac{dv}{dz} = A(z)v(z), \tag{7.0.9}$$

with $A(z)$ a holomorphic 2×2 matrix on $\mathbb{C}$. Generally, an $n \times n$ system of the form (7.0.9). with

$$A : D_a(0) \longrightarrow M(n, \mathbb{C}), \quad \text{holomorphic,} \tag{7.0.10}$$

is said to have a regular singular point at $z = 0$. In case $A(z) \equiv A_0$, (7.0.9) is a matrix Euler equation,

$$z \frac{dv}{dz} = A_0 v, \tag{7.0.11}$$

with solution

$$v(z) = e^{(\log z) A_0} v(1), \quad z \in \mathbb{C} \setminus (-\infty, 0]. \tag{7.0.12}$$

This provides a starting point for the treatment of (7.0.9) given in §36.

Many other differential equations have regular singular points, including the Legendre equation, given in (36.106), and the confluent hypergeometric and hypergeometric equations, given in (36.112)–(36.113). We also have a brief discussion of generalized hypergeometric equations, one of which provides a formula for the Bring radical, arising in Appendix Q in connection with quintic equations.

35. Bessel functions

Bessel functions are an important class of special functions that arise in the analysis of wave equations on domains in Euclidean space $\mathbb{R}^n$ on which it is convenient to use polar coordinates. Such wave equations give rise to Bessel's equation,

$$\frac{d^2x}{dt^2} + \frac{1}{t}\frac{dx}{dt} + \Bigl(1 - \frac{\nu^2}{t^2}\Bigr)x = 0. \tag{35.1}$$

Details on how this equation arises are given in Appendix O. The analysis of the solutions to (35.1) carried out in this section provide an illustration of the power of various techniques of complex analysis developed in this text. We first treat (35.1) for $t > 0$, and then extend the solutions to complex t.

Note that if the factor $(1 - \nu^2/t^2)$ in front of x had the term 1 dropped, one would have the Euler equation

$$t^2x'' + tx' - \nu^2 x = 0, \tag{35.2}$$

with solutions

$$x(t) = t^{\pm\nu}, \tag{35.3}$$

In light of this, we are motivated to set

$$x(t) = t^{\nu}y(t), \tag{35.4}$$

and study the resulting differential equation for y:

$$\frac{d^2y}{dt^2} + \frac{2\nu+1}{t}\frac{dy}{dt} + y = 0. \tag{35.5}$$

This might seem only moderately less singular than (35.1) at $t = 0$, but in fact it has a smooth solution. To obtain it, let us note that if $y(t)$ solves (35.5), so does $y(-t)$, hence so does $y(t) + y(-t)$, which is even in t. Thus, we look for a solution to (35.5) in the form

$$y(t) = \sum_{k=0}^{\infty} a_k t^{2k}. \tag{35.6}$$

Substitution into (35.5) yields for the left side of (35.5) the power series

$$\sum_{k=0}^{\infty}\Bigl\{(2k+2)(2k+2\nu+2)a_{k+1} + a_k\Bigr\}t^{2k}, \tag{35.7}$$

assuming convergence, which we will examine shortly. From this we see that, as long as

$$\nu \notin \{-1,-2,-3,\dots\}, \tag{35.8}$$

we can fix $a_0 = a_0(\nu)$ and solve recursively for a_{k+1}, for each $k \ge 0$, obtaining

$$a_{k+1} = -\frac{1}{4}\,\frac{a_k}{(k+1)(k+\nu+1)}. \tag{35.9}$$

Given (35.8), this recursion works, and one can readily apply the ratio test to show that the power series (35.6) converges for all $t \in \mathbb{C}$.

We will find it useful to produce an explicit solution to the recursive formula (35.9). For this, it is convenient to write

$$a_k = \alpha_k \beta_k \gamma_k, \tag{35.10}$$

with

$$\alpha_{k+1} = -\frac{1}{4}\alpha_k, \quad \beta_{k+1} = \frac{\beta_k}{k+1}, \quad \gamma_{k+1} = \frac{\gamma_k}{k+\nu+1}. \tag{35.11}$$

Clearly the first two equations have the explicit solutions

$$\alpha_k = \Bigl(-\frac{1}{4}\Bigr)^k \alpha_0, \quad \beta_k = \frac{\beta_0}{k!}. \tag{35.12}$$

We can solve the third if we have in hand a function $\Gamma(z)$ satisfying

$$\Gamma(z+1) = z\Gamma(z). \tag{35.13}$$

Indeed, the Euler gamma function $\Gamma(z)$, discussed §18, is a holomorphic function on $\mathbb{C} \setminus \{0,-1,-2,\dots\}$ that satisfies (35.13). With this function in hand, we can write

$$\gamma_k = \frac{\tilde{\gamma}_0}{\Gamma(k+\nu+1)}, \tag{35.14}$$

and putting together (35.10)–(35.14) yields

$$a_k = \Bigl(-\frac{1}{4}\Bigr)^k \frac{\tilde{a}_0}{k!\,\Gamma(k+\nu+1)}. \tag{35.15}$$

We initialize this with $\tilde{a}_0 = 2^{-\nu}$. At this point it is useful to recall from §18 that

$$\frac{1}{\Gamma(z)} \text{ is well defined and holomorphic in } z \in \mathbb{C}$$
$$\text{vanishing for } z \in \{0,-1,-2,\dots\}.$$

Consequently we obtain

$$\mathcal{J}_\nu(t) = \sum_{k=0}^{\infty} 2^{-(2k+\nu)} \frac{(-1)^k}{k!\Gamma(k+\nu+1)} t^{2k}, \tag{35.16}$$
$$\text{holomorphic in } t \in \mathbb{C}, \quad \forall\, \nu \in \mathbb{C},$$

as a solution $y(t) = \mathcal{J}_\nu(t)$ to (35.5), and hence the solution $x(t) = J_\nu(t) = t^\nu \mathcal{J}_\nu(t)$ to (35.1),

$$J_\nu(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \Big(\frac{t}{2}\Big)^{2k+\nu}. \tag{35.17}$$

is defined for each $\nu \in \mathbb{C}$, and is holomorphic in $\mathbb{C} \setminus \gamma$, the complex plane slit along a ray γ through 0, typically $\gamma = (-\infty, 0]$. Furthermore,

$$J_\nu \text{ and } J_{-\nu} \text{ solve (16.1), for } \nu \in \mathbb{C}. \tag{35.18}$$

Let us examine the behavior of $J_\nu(t)$ as $t \to 0$. We have

$$J_\nu(t) = \frac{1}{\Gamma(\nu+1)} \Big(\frac{t}{2}\Big)^\nu + O(t^{\nu+1}), \quad \text{as } \ t \to 0. \tag{35.19}$$

As long as ν satisfies (35.8), the coefficient $1/\Gamma(\nu+1)$ is nonzero. Furthermore,

$$J_{-\nu}(t) = \frac{1}{\Gamma(1-\nu)} \Big(\frac{t}{2}\Big)^{-\nu} + O(t^{-\nu+1}), \quad \text{as } \ t \to 0, \tag{35.20}$$

and as long as $\nu \notin \{1, 2, 3, \dots\}$, the coefficient $1/\Gamma(1-\nu)$ is nonzero. In particular, we see that

$$\begin{gathered} \text{If } \nu \notin \mathbb{Z},\ J_\nu \text{ and } J_{-\nu} \text{ are linearly independent solutions} \\ \text{to (35.1) on } (0, \infty), \end{gathered} \tag{35.21}$$

and in fact on $t \in \mathbb{C} \setminus \gamma$. In contrast to this, we have the following:

$$\text{If } n \in \mathbb{Z},\ J_n(t) = (-1)^n J_{-n}(t). \tag{35.22}$$

To see this, we assume $n \in \{1, 2, 3, \dots\}$, and note that

$$\frac{1}{\Gamma(k-n+1)} = 0, \quad \text{for } \ 0 \le k \le n-1. \tag{35.23}$$

We use this, together with the restatement of (35.17) that

$$J_\nu(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)} \Big(\frac{t}{2}\Big)^{2k+\nu}, \tag{35.24}$$

which follows from the identity $\Gamma(k+1) = k!$, to deduce that, for $n \in \mathbb{N}$,

$$\begin{aligned} J_{-n}(t) &= \sum_{k=n}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k-n+1)} \Big(\frac{t}{2}\Big)^{2k-n} \\ &= \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+n}}{\Gamma(\ell+1)\Gamma(\ell+n+1)} \Big(\frac{t}{2}\Big)^{2\ell+n} \\ &= (-1)^n J_n(t). \end{aligned} \tag{35.25}$$

Consequently $J_\nu(t)$ and $J_{-\nu}(t)$ are linearly independent solutions to (35.1) as long as $\nu \notin \mathbb{Z}$, but this fails for $\nu \in \mathbb{Z}$. We now seek a family of solutions $Y_\nu(t)$ to (35.1) with the property that J_ν and Y_ν are linearly independent solutions, for all $\nu \in \mathbb{R}$. The key to this construction lies in an analysis of the Wronskian

$$W_\nu(t) = W(J_\nu, J_{-\nu})(t) = J_\nu(t) J'_{-\nu}(t) - J'_\nu(t) J_{-\nu}(t). \tag{35.26}$$

Applying d/dt to (35.26) and then applying (35.1) to replace J''_ν and $J''_{-\nu}$, one gets

$$\frac{dW_\nu}{dt} = -\frac{1}{t} W_\nu, \tag{35.27}$$

hence

$$W_\nu(t) = \frac{K(\nu)}{t}. \tag{35.28}$$

To evaluate $K(\nu)$, we calculate

$$\begin{aligned} W(J_\nu, J_{-\nu}) &= W(t^\nu \mathcal{J}_\nu, t^{-\nu} \mathcal{J}_{-\nu}) \\ &= W(\mathcal{J}_\nu, \mathcal{J}_{-\nu}) - \frac{2\nu}{t} \mathcal{J}_\nu(t) \mathcal{J}_{-\nu}(t). \end{aligned} \tag{35.29}$$

Since $\mathcal{J}_\nu(t)$ and $\mathcal{J}_{-\nu}(t)$ are smooth in t, so is $W(\mathcal{J}_\nu, \mathcal{J}_{-\nu})$, and we deduce from (35.28)–(35.29) that

$$W_\nu(t) = -\frac{2\nu}{t} \mathcal{J}_\nu(0) \mathcal{J}_{-\nu}(0). \tag{35.30}$$

Now, since $\mathcal{J}_\nu(0) = 1/2^\nu \Gamma(\nu+1)$, we have

$$\begin{aligned} \nu \mathcal{J}_\nu(0) \mathcal{J}_{-\nu}(0) &= \frac{\nu}{\Gamma(\nu+1)\Gamma(1-\nu)} \\ &= \frac{1}{\Gamma(\nu)\Gamma(1-\nu)}. \end{aligned} \tag{35.31}$$

An importent gamma function identity, established in (18.6), is

$$\Gamma(\nu)\Gamma(1-\nu) = \frac{\pi}{\sin \pi\nu}. \tag{35.32}$$

Hence (35.30)–(35.31) yields

$$W(J_\nu, J_{-\nu})(t) = -\frac{2}{\pi}\,\frac{\sin \pi\nu}{t}. \tag{35.33}$$

This motivates the following. For $\nu \notin \mathbb{Z}$, set

$$Y_\nu(t) = \frac{J_\nu(t)\cos \pi\nu - J_{-\nu}(t)}{\sin \pi\nu}. \tag{35.34}$$

Note that, by (35.25), numerator and denominator both vanish for $\nu \in \mathbb{Z}$. Now, for $\nu \notin \mathbb{Z}$, we have

$$\begin{aligned} W(J_\nu, Y_\nu)(t) &= -\frac{1}{\sin \pi\nu} W(J_\nu, J_{-\nu})(t) \\ &= \frac{2}{\pi t}. \end{aligned} \tag{35.35}$$

Consequently, for $n \in \mathbb{Z}$, we set

$$Y_n(t) = \lim_{\nu \to n} Y_\nu(t) = \frac{1}{\pi}\Big[\frac{\partial J_\nu(t)}{\partial \nu} - (-1)^n \frac{\partial J_{-\nu}(t)}{\partial \nu}\Big]\Big|_{\nu=n}, \tag{35.36}$$

and we also have (35.35) for $\nu \in \mathbb{Z}$. Note that

$$Y_{-n}(t) = (-1)^n Y_n(t). \tag{35.37}$$

We next desire to parallel (35.17) with a series representation for $Y_n(t)$, when $n \in \mathbb{Z}^+$. We can rewrite (35.17) as

$$J_\nu(t) = \sum_{k=0}^\infty \alpha_k(\nu)\Big(\frac{t}{2}\Big)^{2k+\nu}, \quad \alpha_k(\nu) = \frac{(-1)^k}{k!\Gamma(k+\nu+1)}. \tag{35.38}$$

Hence

$$\frac{\partial}{\partial \nu} J_\nu(t) = \sum_{k=0}^\infty \alpha_k'(\nu)\Big(\frac{t}{2}\Big)^{2k+\nu} + \Big(\log\frac{t}{2}\Big)\sum_{k=0}^\infty \alpha_k(\nu)\Big(\frac{t}{2}\Big)^{2k+\nu}. \tag{35.39}$$

The last sum is equal to $J_\nu(t)$. Hence

$$\frac{\partial}{\partial \nu} J_\nu(t)\Big|_{\nu-n} = \sum_{k=0}^\infty \alpha_k'(n)\Big(\frac{t}{2}\Big)^{2k+n} + \Big(\log\frac{t}{2}\Big) J_n(t). \tag{35.40}$$

Similarly,

$$\frac{\partial}{\partial \nu} J_{-\nu}(t)\Big|_{\nu=n} = -\sum_{k=0}^{\infty} \alpha_k'(-n)\Big(\frac{t}{2}\Big)^{2k-n} - \Big(\log \frac{t}{2}\Big) J_{-n}(t). \tag{35.41}$$

Plugging (35.40) and (35.41) into (35.36) then yields a series for $Y_n(t)$, convergent for $t \in \mathbb{C} \setminus \gamma$.

In detail, if n is a positive integer,

$$\begin{aligned} Y_n(t) = \frac{2}{\pi}\Big(\log \frac{t}{2}\Big) J_n(t) + \frac{(-1)^n}{\pi} \sum_{k=0}^{\infty} \alpha_k'(-n)\Big(\frac{t}{2}\Big)^{2k-n} \\ + \frac{1}{\pi} \sum_{k=0}^{\infty} \alpha_k'(n)\Big(\frac{t}{2}\Big)^{2k+n}, \end{aligned} \tag{35.42}$$

and

$$Y_0(t) = \frac{2}{\pi}\Big(\log \frac{t}{2}\Big) J_0(t) + \frac{2}{\pi} \sum_{k=0}^{\infty} \alpha_k'(0)\Big(\frac{t}{2}\Big)^{2k}. \tag{35.43}$$

The coefficients $\alpha_k'(\pm n)$ can be evaluated using

$$\alpha_k'(\nu) = \frac{(-1)^k}{k!} \beta'(k+\nu+1),$$

where

$$\beta(z) = \frac{1}{\Gamma(z)}.$$

The evaluation of $\beta'(\ell)$ for $\ell \in \mathbb{Z}$ is discussed in Appendix J.

We see that

$$Y_0(t) \sim \frac{2}{\pi} \log t, \quad \text{as} \quad t \to 0, \tag{35.44}$$

while, if n is a positive integer,

$$Y_n(t) \sim \frac{(-1)^n}{\pi} \alpha_0'(-n)\Big(\frac{t}{2}\Big)^{-n}, \quad \text{as} \quad t \to 0. \tag{35.45}$$

As seen in Appendix J, when n is a positive integer,

$$\alpha_0'(-n) = \beta'(-n+1) = (-1)^{n-1}(n-1)!. \tag{35.46}$$

We next obtain an integral formula for $J_\nu(t)$, which plays an important role in further investigations, such as the behavior of $J_\nu(t)$ for large t.

Proposition 35.1. *If* $\operatorname{Re}\nu > -1/2$ *and* $t \in \mathbb{C} \setminus \gamma$, *then*

$$J_\nu(t) = \frac{(t/2)^\nu}{\Gamma(1/2)\Gamma(\nu+1/2)} \int_{-1}^{1} (1-s^2)^{\nu-1/2} e^{ist}\, ds. \tag{35.47}$$

Proof. To verify (35.47), we replace e^{ist} by its power series, integrate term by term, and use some identites from §18. To begin, the integral on the right side of (35.47) is equal to

$$\sum_{k=0}^\infty \frac{1}{(2k)!} \int_{-1}^{1} (ist)^{2k} (1-s^2)^{\nu-1/2}\, ds. \tag{35.48}$$

The identity (18.31) implies

$$\int_{-1}^{1} s^{2k} (1-s^2)^{\nu-1/2}\, ds = \frac{\Gamma(k+1/2)\Gamma(\nu+1/2)}{\Gamma(k+\nu+1)}, \tag{35.49}$$

for $\nu > -1/2$, so the right side of (35.47) equals

$$\frac{(t/2)^\nu}{\Gamma(1/2)\Gamma(\nu+1/2)} \sum_{k=0}^\infty \frac{1}{(2k)!} (it)^{2k} \frac{\Gamma(k+1/2)\Gamma(\nu+1/2)}{\Gamma(k+\nu+1)}. \tag{35.50}$$

From (18.34) (see also (18.47)), we have

$$\Gamma\Bigl(\frac{1}{2}\Bigr)(2k)! = 2^{2k} k!\, \Gamma\Bigl(k+\frac{1}{2}\Bigr), \tag{35.51}$$

so (35.50) is equal to

$$\Bigl(\frac{t}{2}\Bigr)^\nu \sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \Bigl(\frac{t}{2}\Bigr)^{2k}, \tag{35.52}$$

which agrees with our formula (35.17) for $J_\nu(t)$.

We will provide a second proof of Proposition 35.1, which incidentally yields further interesting identities for Bessel functions.

Second proof. Denote by $x_\nu(t)$ the right side of (35.47), and set

$$c(\nu) = \Bigl[\Gamma\Bigl(\frac{1}{2}\Bigr)\Gamma\Bigl(\nu+\frac{1}{2}\Bigr)\Bigr]^{-1} \cdot 2^{-\nu}. \tag{35.53}$$

Applying d/dt to the right side of (35.47) yields

$$\begin{aligned} \frac{dx_\nu}{dt} &= \Bigl(\frac{\nu c(\nu)}{t}\Bigr) t^\nu \int_{-1}^{1} e^{ist} (1-s^2)^{\nu-1/2}\, ds \\ &\quad + ic(\nu) t^\nu \int_{-1}^{1} e^{ist} s (1-s^2)^{\nu-1/2}\, ds. \end{aligned} \tag{35.54}$$

The first term on the right side of (35.54) is equal to $(\nu/t)x_\nu(t)$. If we assume $\operatorname{Re}\nu > 1/2$, the second is equal to

$$\begin{aligned}
&\frac{c(\nu)}{t}\, t^\nu \int_{-1}^{1} \frac{d}{ds}\Big(e^{ist}\Big) s(1-s^2)^{\nu-1/2}\, ds \\
&= -\frac{c(\nu)}{t}\, t^\nu \int_{-1}^{1} \Big[(1-s^2)^{\nu-1/2} - (2\nu-1)s^2(1-s^2)^{\nu-1-1/2}\Big]\, ds \\
&= -\frac{c(\nu)}{t}\, t^\nu \int_{-1}^{1} \Big[2\nu(1-s^2)^{\nu-1/2} - (2\nu-1)(1-s^2)^{\nu-1-1/2}\Big]\, ds \\
&= -\frac{2\nu}{t} x_\nu(t) + \frac{(2\nu-1)c(\nu)}{c(\nu-1)} x_{\nu-1}(t).
\end{aligned} \tag{35.55}$$

Since $c(\nu)/c(\nu-1) = 1/(2\nu-1)$, we have

$$\frac{dx_\nu}{dt} = -\frac{\nu}{t} x_\nu(t) + x_{\nu-1}(t), \tag{35.56}$$

or

$$\Big(\frac{d}{dt} + \frac{\nu}{t}\Big) x_\nu(t) = x_{\nu-1}(t), \tag{35.57}$$

at least for $\operatorname{Re}\nu > 1/2$. Next, we have

$$\begin{aligned}
x_{\nu+1}(t) &= c(\nu+1)t^{\nu+1} \int_{-1}^{1} e^{ist}(1-s^2)^{\nu+1-1/2}\, ds \\
&= -ic(\nu+1)t^\nu \int_{-1}^{1} \Big(\frac{d}{ds} e^{ist}\Big)(1-s^2)^{\nu+1-1/2}\, ds \\
&= -ic(\nu+1)(2\nu+1)t^\nu \int_{-1}^{1} e^{ist} s(1-s^2)^{\nu-1/2}\, ds,
\end{aligned} \tag{35.58}$$

and since $c(\nu+1) = c(\nu)/(2\nu+1)$, this is equal to the negative of the second term on the right side of (35.54). Hence

$$\Big(\frac{d}{dt} - \frac{\nu}{t}\Big) x_\nu(t) = -x_{\nu+1}(t), \tag{35.59}$$

complementing (35.57). Putting these two formulas together, we get

$$\Big(\frac{d}{dt} - \frac{\nu-1}{t}\Big)\Big(\frac{d}{dt} + \frac{\nu}{t}\Big) x_\nu(t) = -x_\nu(t), \tag{35.60}$$

which is equivalent to Bessel's equation (35.1). It follows that

$$x_\nu(t) = a_\nu J_\nu(t) + b_\nu J_{-\nu}(t), \tag{35.61}$$

for $\operatorname{Re}\nu > 1/2,\ \nu \notin \mathbb{N}$. We can evaluate the constants a_ν and b_ν from the behavior of $x_\nu(t)$ as $t \to 1$. Since taking $k = 0$ in (35.49) yields

$$\int_{-1}^{1} (1 - s^2)^{\nu - 1/2}\, ds = \frac{\Gamma(1/2)\Gamma(\nu + 1/2)}{\Gamma(\nu + 1)}, \tag{35.62}$$

we see from inspecting the right side of (35.47) that

$$x_\nu(t) \sim \frac{1}{\Gamma(\nu + 1)} \Big(\frac{t}{2}\Big)^\nu, \quad \text{as } \ t \to 0. \tag{35.63}$$

Comparison with (35.18)–(35.19) then gives $a_\nu = 1,\ b_\nu = 0$, so we have

$$x_\nu(t) = J_\nu(t), \tag{35.64}$$

for $\operatorname{Re}\nu > 1/2,\ \nu \notin \mathbb{N}$. From here, we have (35.64) for all $\nu \in \mathbb{C}$ such that $\operatorname{Re}\nu > -1/2$, by analytic continuation (in ν). This completes the second proof of Proposition 35.1.

Having (35.64), we can rewrite the identities (35.57) and (35.59) as

$$\begin{aligned} \Big(\frac{d}{dt} + \frac{\nu}{t}\Big) J_\nu(t) &= J_{\nu-1}(t), \\ \Big(\frac{d}{dt} - \frac{\nu}{t}\Big) J_\nu(t) &= -J_{\nu+1}(t), \end{aligned} \tag{35.65}$$

first for $\operatorname{Re}\nu > 1/2$, and then, by analytic continuation in ν, for all $\nu \in \mathbb{C}$.

Remark. The identity (35.47) implies

$$J_\nu(t) = \frac{\sqrt{2}}{\Gamma(\nu + 1/2)} \Big(\frac{t}{2}\Big)^\nu \hat{\psi}_\nu(t), \quad \text{for } \operatorname{Re}\nu > -\frac{1}{2}, \tag{35.66}$$

for $t \in \mathbb{R}$, where $\hat{\psi}_\nu$ is the Fourier transform of ψ_ν, given by

$$\begin{aligned} \psi_\nu(s) = (1 - s^2)^{\nu - 1/2} \quad & \text{for } \ |s| \le 1, \\ 0 \qquad\qquad & \text{for } \ |s| > 1. \end{aligned} \tag{35.67}$$

We next seek an integral formula for $J_\nu(t)$ that works for all $\nu \in \mathbb{C}$. To get this, we replace the identity (35.49) by

$$\frac{1}{\Gamma(k + \nu + 1)} = \frac{1}{2\pi i} \int\limits_\sigma e^\zeta \zeta^{-(k+\nu+1)}\, d\zeta, \quad \forall\, \nu \in \mathbb{C}, \tag{35.68}$$

where the path σ runs from $-\infty - i0$ to $-\rho - i0$, then counterclockwise from $-\rho - i0$ to $-\rho + i0$ on the circle $|\zeta| = \rho$, and then from $-\rho + i0$ to $-\infty + i0$ (cf. Figure 35.1). This is a rewrite of the identity (cf. Proposition 18.6)

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_\sigma e^\zeta \zeta^{-z}\, d\zeta, \quad \forall\, z \in \mathbb{C}. \tag{35.69}$$

We pick $\rho \in (0, \infty)$. By Cauchy's integral theorem, this integral is independent of the choice of such ρ. Plugging (35.68) into the series (35.17), we have

$$\begin{aligned} J_\nu(z) &= \Bigl(\frac{t}{2}\Bigr)^\nu \sum_{k=0}^\infty \frac{(-1)^k}{k!} \Bigl(\frac{1}{2\pi i} \int_\sigma e^\zeta \zeta^{-(k+\nu+1)}\, d\zeta\Bigr) \Bigl(\frac{t}{2}\Bigr)^{2k} \\ &= \Bigl(\frac{t}{2}\Bigr)^\nu \frac{1}{2\pi i} \int_\sigma e^\zeta \zeta^{-\nu-1} \sum_{k=0}^\infty \frac{(-1)^k}{k!} \Bigl(\frac{t^2}{4\zeta}\Bigr)^k\, d\zeta, \end{aligned} \tag{35.70}$$

hence

$$J_\nu(t) = \Bigl(\frac{t}{2}\Bigr)^\nu \frac{1}{2\pi i} \int_\sigma e^{\zeta - t^2/4\zeta}\, \zeta^{-\nu-1}\, d\zeta, \tag{35.71}$$

for each $\nu \in \mathbb{C},\ t \in \mathbb{C} \setminus \gamma$.

To proceed, we can scale, setting $\zeta = st/2$, if $t > 0$, and obtain

$$J_\nu(t) = \frac{1}{2\pi i} \int_{\sigma'} e^{(t/2)(s-1/s)}\, s^{-\nu-1}\, ds, \tag{35.72}$$

where σ' is a scaled version of σ. We first have (35.72) for $t > 0$, and then, by analytic continuation, for $\operatorname{Re} t > 0$. Choosing σ' so that the circular part has radius 1, we have

$$\begin{aligned} J_\nu(t) = &\frac{1}{\pi} \int_0^\pi \cos(t \sin\theta - \nu\theta)\, d\theta \\ &- \frac{\sin \pi\nu}{\pi} \int_1^\infty e^{-(t/2)(x-1/x)}\, x^{-\nu-1}\, dx. \end{aligned} \tag{35.73}$$

Taking $x = e^u$, we obtain the following.

Proposition 35.2. *For $\nu \in \mathbb{C},\ \operatorname{Re} t > 0$,*

$$\begin{aligned} J_\nu(t) = &\frac{1}{\pi} \int_0^\pi \cos(t \sin\theta - \nu\theta)\, d\theta \\ &- \frac{\sin \pi\nu}{\pi} \int_0^\infty e^{-t \sinh u - \nu u}\, du. \end{aligned} \tag{35.74}$$

Bringing in the formula (35.34) for $Y_\nu(t)$ (complemented by (35.36)), we obtain from (35.74) the integral

$$
\begin{aligned}
Y_\nu(t) = \frac{1}{\pi}\int_0^\pi &\sin(t\sin\theta - \nu\theta)\,d\theta \\
- \frac{1}{\pi}\int_0^\infty &e^{-t\sinh u}(e^{\nu u} + \cos\pi\nu\, e^{-\nu u})\,du,
\end{aligned}
\tag{35.75}
$$

for $\mathrm{Re}\,t > 0,\ \nu \in \mathbb{C}$ (first for $\nu \notin \mathbb{Z}$, then, by analytic continuation, for all $\nu \in \mathbb{C}$).

Another useful basis of solutions to (35.1) is the pair of Hankel functions, defined by

$$
\begin{aligned}
H_\nu^{(1)}(t) &= J_\nu(t) + iY_\nu(t), \\
H_\nu^{(2)}(t) &= J_\nu(t) - iY_\nu(t),
\end{aligned}
\tag{35.76}
$$

for $\nu \in \mathbb{C},\ t \in \mathbb{C} \setminus \gamma$. Parallel to (35.35), we have

$$
W(H_\nu^{(1)}, H_\nu^{(2)})(t) = \frac{4}{\pi i t}. \tag{35.77}
$$

From (35.74)–(35.75) we obtain the integral formula

$$
\begin{aligned}
H_\nu^{(1)}(t) = \frac{1}{\pi i}\int_0^\pi &e^{i(t\sin\theta - \nu\theta)}\,d\theta \\
+ \frac{1}{\pi i}\int_0^\infty &e^{-t\sinh u}\big[e^{\nu u} + e^{-\nu(u+\pi i)}\big]\,du,
\end{aligned}
\tag{35.78}
$$

for $\nu \in \mathbb{C},\ \mathrm{Re}\,t > 0$, which we can rewrite as

$$
H_\nu^{(1)}(t) = \frac{1}{\pi i}\int_{C_1} e^{t\sinh u - \nu u}\,du, \tag{35.79}
$$

for $\nu \in \mathbb{C},\ \mathrm{Re}\,t > 0$, where the path C_1 goes from $-\infty$ to 0 along the negative real axis, then from 0 to πi, then from πi to $\pi i + \infty$ along a horizontal ray (cf. Figure 35.2). Similarly

$$
H_\nu^{(2)}(t) = -\frac{1}{\pi i}\int_{\overline{C}_1} e^{t\sinh u - \nu u}\,du, \tag{35.80}
$$

for $\nu \in \mathbb{C},\ \mathrm{Re}\,t > 0$, where the path $\overline{C}_1$ is the reflection of C_1 across the real axis. Shifting the contour down $\pi/2$ units and making the associated change of variable, we can write

$$
H_\nu^{(1)}(t) = \frac{1}{\pi i}e^{-\nu\pi i/2}\int_{C_2} e^{it\cosh u - \nu u}\,du, \quad \mathrm{Re}\,t > 0, \tag{35.81}
$$

where $C_2 = C_1 - \pi i/2$ (cf. Figure 35.3).

It is convenient to deform the path C_2 to a new path, C_3, asymptotic to C_2 at $\mp(\infty + \pi i/2)$, on which the real part of $\cosh u$ is constant. Writing

$$\cosh(x+iy) = (\cosh x)(\cos y) + i(\sinh x)(\sin y), \tag{35.82}$$

we see that we want

$$(\cosh x)(\cos y) \equiv 1 \tag{35.83}$$

if C_2 is to pass through the origin. Wanting $y \to \mp\pi i/2$ as $x \to \mp\infty$, we obtain

$$y(x) = \tan^{-1}(\sinh x), \quad x \in \mathbb{R}, \tag{35.84}$$

defining the path C_3. Then, changing the path in (35.81) from C_2 to C_3, we get

$$H_\nu^{(1)}(t) = \frac{1}{\pi i} e^{i(t-\nu\pi/2)} \int_{-\infty}^{\infty} e^{-t\psi(x)-\nu u(x)} u'(x)\, dx, \tag{35.85}$$

where

$$\begin{aligned} u(x) &= x + i\tan^{-1}(\sinh x), \\ u'(x) &= 1 + \frac{i}{\cosh x}, \end{aligned} \tag{35.86}$$

and

$$\begin{aligned} \psi(x) &= (\sinh x)(\sin y(x)) \\ &= (\sinh x)(\tanh x) \\ &= \frac{\sinh^2 x}{\cosh x}. \end{aligned} \tag{35.87}$$

The second identity in (35.86) uses

$$\frac{dy}{dx} = \frac{\cosh x}{1+\sinh^2 x} = \frac{1}{\cosh x}, \tag{35.88}$$

and the second identity in (35.87) uses

$$\sin(\tan^{-1} v) = \frac{v}{\sqrt{1+v^2}}. \tag{35.89}$$

Let us note that $\psi(x)$ is positive for all real $x \neq 0$, tending exponentially fast to ∞ as $|x| \to \infty$, and that

$$\psi''(0) = 2, \quad u(0) = 0, \quad u'(0) = 1 + i. \tag{35.90}$$

Hence the Laplace asymptotic method, treated in Appendix P, gives the following result on the asymptotic behavior of $H_\nu^{(1)}(t)$. To state it, set

$$A_\beta = \{re^{i\theta} : r > 0,\ -\beta \le \theta \le \beta\}, \tag{35.91}$$

for $\beta \in (0, \pi)$.

Proposition 35.3. *Given $\delta \in (0, \pi/2)$, we have*

$$H_\nu^{(1)}(t) = e^{i(t-\nu\pi/2-\pi/4)}\Big[\Big(\frac{2}{\pi t}\Big)^{1/2} + O(|t|^{-3/2})\Big], \tag{35.92}$$

as $t \to \infty$ in $A_{\pi/2-\delta}$.

The following is a useful complement to (35.92):

$$\frac{d}{dt}H_\nu^{(1)}(t) = ie^{i(t-\nu\pi/2-\pi/4)}\Big[\Big(\frac{2}{\pi t}\Big)^{1/2} + O(|t|^{-3/2})\Big], \tag{35.93}$$

as $t \to \infty$ in $A_{\pi/2-\delta}$. One way to deduce this from (35.92) is to apply the identity

$$f'(t) = \frac{1}{2\pi i}\int\limits_{\partial D_1(t)} \frac{f(\zeta)}{(\zeta - t)^2}\,d\zeta \tag{35.94}$$

to $f(\zeta) = H_\nu^{(1)}(\zeta)$. Another is to bring in the identity

$$\frac{d}{dt}H_\nu^{(1)}(t) = H_{\nu-1}^{(1)}(t) - \frac{\nu}{t}H_\nu^{(1)}(t), \tag{35.95}$$

which is parallel to (35.65), and then apply (35.92) with ν replaced by $\nu - 1$.

We now restrict attention to $\nu \in \mathbb{R}$ and $t \in (0, \infty)$. Then $J_\nu(t)$ and $Y_\nu(t)$ are the real and imaginary parts of $H_\nu^{(1)}(t)$, and we have the following.

Proposition 35.4. *For $\nu \in \mathbb{R}$ and $t \in (0, \infty)$,*

$$\begin{aligned}
J_\nu(t) &= \Big(\frac{2}{\pi t}\Big)^{1/2}\cos\Big(t - \frac{\nu\pi}{2} - \frac{\pi}{4}\Big) + O(t^{-3/2}),\\
J_\nu'(t) &= -\Big(\frac{2}{\pi t}\Big)^{1/2}\sin\Big(t - \frac{\nu\pi}{2} - \frac{\pi}{4}\Big) + O(t^{-3/2}),\\
Y_\nu(t) &= \Big(\frac{2}{\pi t}\Big)^{1/2}\sin\Big(t - \frac{\nu\pi}{2} - \frac{\pi}{4}\Big) + O(t^{-3/2}),\\
Y_\nu'(t) &= \Big(\frac{2}{\pi t}\Big)^{1/2}\cos\Big(t - \frac{\nu\pi}{2} - \frac{\pi}{4}\Big) + O(t^{-3/2}),
\end{aligned} \tag{35.96}$$

as $t \to +\infty$.

More generally, if $\nu \in \mathbb{R}$, any real valued solutiuon u_ν to Bessel's equation (35.1) is a linear combination $u_\nu = aJ_\nu + bY_\nu$, with $a, b \in \mathbb{R}$, and hence there exist $\theta \in [0, 2\pi]$ and $A \in \mathbb{R}$ such that

$$\begin{aligned}
u_\nu(t) &= At^{-1/2}\cos(t - \theta) + O(t^{-3/2})\\
u_\nu'(t) &= -At^{-1/2}\sin(t - \theta) + O(t^{-3/2}),
\end{aligned} \tag{35.97}$$

as $t \to +\infty$. We assume u_ν is not identically 0, so $A \neq 0$. Now set

$$\alpha_k = k\pi + \theta, \quad \beta_k = \alpha_k + \frac{\pi}{2}. \tag{35.98}$$

We have, with $t = \alpha_k$,

$$\begin{aligned} u_\nu(\alpha_k) &= (-1)^k A t^{-1/2} + O(t^{-3/2}), \\ u_\nu(\beta_k) &= O(t^{-3/2}), \end{aligned} \tag{35.99}$$

and

$$\begin{aligned} u'_\nu(\alpha_k) &= O(t^{-3/2}), \\ u'_\nu(\beta_k) &= (-1)^{k+1} A t^{-1/2} + O(t^{-3/2}). \end{aligned} \tag{35.100}$$

The first part of (35.99) guarantees that there exists $K < \infty$ such that whenever $k \geq K$, the numbers $u_\nu(\alpha_k)$ and $u_\nu(\alpha_{k+1}) = u_\nu(\alpha_k + \pi)$ have opposite signs. This leads to the following.

Proposition 35.5. *Let $\nu \in \mathbb{R}$ and take u_ν to be a real valued solution of (35.1), so (35.97)–(35.100) hold. Then there exists $K < \infty$ such that whenever $k \geq K$, there is exactly one*

$$\vartheta_k \in (\alpha_k, \alpha_k + \pi) \ \ \textit{such that} \ \ u_\nu(\vartheta_k) = 0. \tag{35.101}$$

Furthermore,

$$\vartheta_k = \beta_k + O(k^{-1}). \tag{35.102}$$

Proof. We can assume $A = 1$. The estimates in (35.97) imply that there exists $B \in (0, \infty)$ such that, for all sufficiently large k,

$$\begin{aligned} (-1)^k u_\nu(t) > 0 \ \ &\text{for} \ \ \alpha_k \leq t \leq \beta_k - \frac{B}{k}, \\ (-1)^k u_\nu(t) < 0 \ \ &\text{for} \ \ \beta_k + \frac{B}{k} \leq t \leq \alpha_k + \pi. \end{aligned} \tag{35.103}$$

Thus there is a zero of u_ν in the subinterval $[\beta_k - B/k, \beta_k + B/k]$, and all zeros of u_ν in $[\alpha_k, \alpha_k + \pi]$ must lie in this subinterval. If u_ν had more than one zero in this subinterval, then $u'_\nu(t)$ must vanish for some $t \in [\beta_k - B/k, \beta_k + B/k]$, but this contradicts the second part of (35.97).

We turn to a variant of Bessel's equation. As shown in Appendix O, while wave equations on certain domains in $\mathbb{R}^n$ give rise to (35.1), diffusion equations on such domains lead to

$$\frac{d^2u}{dt^2} + \frac{1}{t}\frac{du}{dt} - \Big(1 + \frac{\nu^2}{t^2}\Big)u = 0. \tag{35.104}$$

This differs from (35.1) only in the change $1 \mapsto -1$. One readily verifies that

$$x(t) \text{ solves } (35.1) \ \Rightarrow u(t) = x(it) \text{ solves } (35.104).$$

In this correspondence, if $x(t)$ is holomorphic in $t \in \mathbb{C} \setminus \gamma$, then $u(t)$ is holomorphic in $t \in \mathbb{C} \setminus \gamma/i$. Standard terminology takes

$$I_\nu(t) = e^{-\pi i\nu/2} J_\nu(it), \tag{35.105}$$

and

$$K_\nu(t) = \frac{\pi i}{2} e^{\pi i\nu/2} H_\nu^{(1)}(it). \tag{35.106}$$

Parallel to (35.35) and (35.77), we have

$$W(I_\nu, K_\nu)(t) = -\frac{1}{t}. \tag{35.107}$$

From Proposition 35.1 we obtain the integral formula

$$I_\nu(t) = \frac{(t/2)^\nu}{\Gamma(1/2)\Gamma(\nu+1/2)} \int_{-1}^{1} (1-s^2)^{\nu-1/2} e^{st}\, ds, \tag{35.108}$$

for $\operatorname{Re}\nu > -1/2,\ \operatorname{Re} t > 0$. We also have

$$K_\nu(t) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-t\cosh u - \nu u}\, du, \tag{35.109}$$

for $\operatorname{Re} t > 0,\ \nu \in \mathbb{C}$, which is obtained from the following variant of (35.81):

$$H_\nu^{(1)}(t) = \frac{1}{\pi i} e^{-\nu\pi i/2} \int_{-\infty}^{\infty} e^{it\cosh u - \nu u}\, du, \quad \operatorname{Im} t > 0. \tag{35.110}$$

Parallel to Proposition 35.3, one can show that

$$\begin{aligned} I_\nu(t) &= e^t \Big[\Big(\frac{1}{2\pi t}\Big)^{1/2} + O(|t|^{-3/2}) \Big], \\ K_\nu(t) &= e^{-t} \Big[\Big(\frac{\pi}{2t}\Big)^{1/2} + O(|t|^{-2/3}) \Big], \end{aligned} \tag{35.111}$$

as $t \to \infty$ in $A_{\pi/2-\delta}$. Details on these last mentioned results can be found in Chapter 5 of [Leb].

Applications of Bessel functions to partial differential equations, particularly to wave equations, can be found in Chapters 8 and 9 of [T2]. Also, one can find a wealth of further material on Bessel functions in the classic treatise [W].

Exercises

1. Show directly from the power series (35.17), and its analogues with ν replaced by $\nu - 1$ and $\nu + 1$, that the Bessel functions J_ν satisfy the following recursion relations:

$$\frac{d}{dt}\Big(t^\nu J_\nu(t)\Big) = t^\nu J_{\nu-1}(t), \quad \frac{d}{dt}\Big(t^{-\nu} J_\nu(t)\Big) = -t^{-\nu} J_{\nu+1}(t),$$

or equivalently

$$\begin{aligned} J_{\nu+1}(t) &= -J_\nu'(t) + \frac{\nu}{t} J_\nu(t), \\ J_{\nu-1}(t) &= J_\nu'(t) + \frac{\nu}{t} J_\nu(t). \end{aligned}$$

This provides another derivation of (35.69).

2. Show that $\mathcal{J}_{-1/2}(t) = \sqrt{2/\pi}\cos t$, and deduce that

$$J_{-1/2}(t) = \sqrt{\frac{2}{\pi t}}\cos t, \quad J_{1/2}(t) = \sqrt{\frac{2}{\pi t}}\sin t.$$

Deduce from Exercise 1 that, for $n \in \mathbb{Z}^+$,

$$\begin{aligned} J_{n+1/2}(t) &= (-1)^n \Big\{\prod_{j=1}^n \Big(\frac{d}{dt} - \frac{j-1/2}{t}\Big)\Big\} \frac{\sin t}{\sqrt{2\pi t}}, \\ J_{-n-1/2}(t) &= \Big\{\prod_{j=1}^n \Big(\frac{d}{dt} - \frac{j-1/2}{t}\Big)\Big\} \frac{\cos t}{\sqrt{2\pi t}}. \end{aligned}$$

Hint. The differential equation (16.5) for $\mathcal{J}_{-1/2}$ is $y'' + y = 0$. Since $\mathcal{J}_{-1/2}(t)$ is even in t, $\mathcal{J}_{-1/2}(t) = C\cos t$, and the evaluation of C comes from $\mathcal{J}_{-1/2}(0) = \sqrt{2}/\Gamma(1/2) = \sqrt{2/\pi}$, thanks to (18.22).

3. Show that the functions Y_ν satisfy the same recursion relations as J_ν, i.e.,

$$\frac{d}{dt}\Big(t^\nu Y_\nu(t)\Big) = t^\nu Y_{\nu-1}(t), \quad \frac{d}{dt}\Big(t^{-\nu} Y_\nu(t)\Big) = -t^{-\nu} Y_{\nu+1}(t).$$

4. The Hankel functions $H_\nu^{(1)}(t)$ and $H_\nu^{(2)}(t)$ are defined to be

$$H_\nu^{(1)}(t) = J_\nu(t) + iY_\nu(t), \quad H_\nu^{(2)}(t) = J_\nu(t) - iY_\nu(t).$$

Show that they satisfy the same recursion relations as J_ν, i.e.,

$$\frac{d}{dt}\Big(t^\nu H_\nu^{(j)}(t)\Big) = t^\nu H_{\nu-1}^{(j)}(t), \quad \frac{d}{dt}\Big(t^{-\nu} H_\nu^{(j)}(t)\Big) = -t^{-\nu} H_{\nu+1}^{(j)}(t),$$

for $j = 1, 2$.

5. Show that
$$H_{-\nu}^{(1)}(t) = e^{\pi i\nu} H_\nu^{(1)}(t), \quad H_{-\nu}^{(2)}(t) = e^{-\pi i\nu} H_\nu^{(2)}(t).$$

6. Show that $Y_{1/2}(t) = -J_{-1/2}(t)$, and deduce that

$$H_{1/2}^{(1)}(t) = -i\sqrt{\frac{2}{\pi t}}e^{it}, \quad H_{1/2}^{(2)}(t) = i\sqrt{\frac{2}{\pi t}}e^{-it}.$$

7. Show that if $\nu \in \mathbb{R}$, then $H_\nu^{(1)}(t)$ dos not vanish at any $t \in \mathbb{R} \setminus 0$.

36. Differential equations on a complex domain

Here we study differential equations, such as

$$a_n(z)\frac{d^n u}{dz^n} + a_{n-1}(z)\frac{d^{n-1}u}{dz^{n-1}} + \cdots + a_1(z)\frac{du}{dz} + a_0(z)u = g(z), \tag{36.1}$$

given a_j and g holomorphic on a connected, open domain $\Omega \subset \mathbb{C}$, and $a_n(z) \neq 0$ for $z \in \Omega$. An example, studied in §35, is Bessel's equation, i.e.,

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \Big(1 - \frac{\nu^2}{z^2}\Big)u = 0, \tag{36.2}$$

for which $\Omega = \mathbb{C} \setminus 0$. We confine our attention to linear equations. Some basic material on nonlinear holomorphic differential equations can be found in Chapter 1 (particularly Sections 6 and 9) of [T2].

We will work in the setting of a first-order $n \times n$ system,

$$\frac{dv}{dz} = A(z)v + f(z), \quad v(z_0) = v_0, \tag{36.3}$$

given $z_0 \in \Omega,\ v_0 \in \mathbb{C}^n$, and

$$A : \Omega \longrightarrow M(n, \mathbb{C}), \quad f : \Omega \longrightarrow \mathbb{C}^n, \ \text{ holomorphic.} \tag{36.4}$$

A standard conversion of (36.1) to (36.3) takes $v = (u_0, \dots, u_{n-1})^t$, with

$$u_j = \frac{d^j u}{dz^j}, \quad 0 \le j \le n-1. \tag{36.5}$$

Then v satisfies the $n \times n$ system

$$\frac{du_j}{dz} = u_{j+1}, \quad 0 \le j \le n-2, \tag{36.6}$$

$$\frac{du_{n-1}}{dz} = -a_n(z)^{-1}\Big[g(z) - \sum_{j=0}^{n-1} a_j(z)u_j\Big]. \tag{36.7}$$

We pick $z_0 \in \Omega$ and impose on this system the initial condition

$$v(z_0) = (u(z_0), u'(z_0), \dots, u^{(n-1)}(z_0))^t = v_0 \in \mathbb{C}^n. \tag{36.8}$$

In the general framework of (36.3)–(36.4), we seek a solution $v(z)$ that is holomorphic on a neighborhood of z_0 in Ω. Such $v(z)$ would have a power series expansion

$$v(z) = \sum_{k=0}^{\infty} v_k (z - z_0)^k, \quad v_k \in \mathbb{C}^n. \tag{36.9}$$

Assume $A(z)$ and $f(z)$ ar egiven by convergent power series,

$$A(z) = \sum_{k=0}^{\infty} A_k (z - z_0)^k, \quad f(z) = \sum_{k=0}^{\infty} f_k (z - z_0)^k, \tag{36.10}$$

with

$$A_k \in M(n, \mathbb{C}), \quad f_k \in \mathbb{C}^n. \tag{36.11}$$

If (36.9) is a convergent power series, then the coefficients v_k are obtained, recursively, as follows. We have

$$\frac{dv}{dz} = \sum_{k=1}^{\infty} k v_k (z - z_0)^{k-1} = \sum_{k=0}^{\infty} (k+1) v_{k+1} z^k, \tag{36.12}$$

and

$$\begin{aligned} A(z)v(z) &= \sum_{j=0}^{\infty} A_j (z - z_0)^j \sum_{\ell=0}^{\infty} v_\ell (z - z_0)^\ell \\ &= \sum_{k=0}^{\infty} \Bigl(\sum_{j=0}^{k} A_{k-j} v_j \Bigr) (z - z_0)^k, \end{aligned} \tag{36.13}$$

so the power series for the left side and the right side of (36.3) agree if and only if, for each $k \geq 0$,

$$(k+1) v_{k+1} = \sum_{j=0}^{k} A_{k-j} v_j + f_k. \tag{36.14}$$

In particular, the first three recursions are

$$\begin{aligned} v_1 &= A_0 v_0 + f_0, \\ 2v_2 &= A_1 v_0 + A_0 v_1 + f_1, \\ 3v_3 &= A_2 v_0 + A_1 v_1 + A_0 v_2 + f_2. \end{aligned} \tag{36.15}$$

To start the recursion, the initial condition in (36.3) specifies v_0.

The following key result addresses the issue of convergence of the power series thus produced for $v(z)$.

Proposition 36.1. *Assume the power series for $A(z)$ and $f(z)$ in (36.10) converge for*

$$|z - z_0| < R_0, \tag{36.16}$$

and let v_k be defined recursively by (36.14). Then the power series (36.9) for $v(z)$ also converges in the region (36.16).

Proof. To estimate the terms in (36.9), we use matrix and vector norms, as treated in Appendix N. The hypotheses on (36.10) imply that for each $R < R_0$, there exist $a, b \in (0, \infty)$ such that

$$\|A_k\| \le aR^{-k}, \quad \|f_k\| \le bR^{-k}, \quad \forall\, k \in \mathbb{Z}^+. \tag{36.17}$$

We will show that, given $r \in (0, R)$, there exists $C \in (0, \infty)$ such that

$$\|v_j\| \le Cr^{-j}, \quad \forall\, j \in \mathbb{Z}^+. \tag{36.18}$$

Such estimates imply that the power series (36.9) converges for $|z - z_0| < r$, for each $r < R_0$, hence for $|z - z_0| < R_0$.

We will prove (36.18) by induction. The inductive step is to assume it holds for all $j \le k$ and to deduce it holds for $j = k + 1$. This deduction proceeds as follows. We have, by (36.14), (36.17), and (36.18) for $j \le k$,

$$\begin{aligned} (k+1)\|v_{k+1}\| &\le \sum_{j=0}^{k} \|A_{k-j}\| \cdot \|v_j\| + \|f_k\| \\ &\le aC \sum_{j=0}^{k} R^{j-k} r^{-j} + bR^{-k} \\ &= aCr^{-k} \sum_{j=0}^{k} \Bigl(\frac{r}{R}\Bigr)^{k-j} + bR^{-k}. \end{aligned} \tag{36.19}$$

Now, given $0 < r < R$,

$$\sum_{j=0}^{k} \Bigl(\frac{r}{R}\Bigr)^{k-j} < \sum_{j=0}^{\infty} \Bigl(\frac{r}{R}\Bigr)^{j} = \frac{1}{1 - r/R} = M(R, r) < \infty. \tag{36.20}$$

Hence

$$(k+1)\|v_{k+1}\| \le aCM(R, r)r^{-k} + br^{-k}. \tag{36.21}$$

We place on C the constraint that

$$C \ge b, \tag{36.22}$$

and obtain

$$\|v_{k+1}\| \le \frac{aM(R, r) + 1}{k+1} r \cdot Cr^{-k-1}. \tag{36.23}$$

This gives the desired result

$$\|v_{k+1}\| \le C r^{-k-1}, \tag{36.24}$$

as long as

$$\frac{aM(R,r)+1}{k+1} r \le 1. \tag{36.25}$$

Thus, to finish the argument, we pick $K \in \mathbb{N}$ such that

$$K+1 \ge \bigl[aM(R,r)+1\bigr] r. \tag{36.26}$$

Recall that we have a, R, r and $M(R,r)$. Then we pick $C \in (0,\infty)$ large enough that (36.18) holds for all $j \in \{0,1,\dots,K\}$, i.e., we take (in addition to (36.22))

$$C \ge \max_{0\le j\le K} r^j \|v_j\|. \tag{36.27}$$

Then for all $k \ge K$, the inductive step yielding (36.24) from the validity of (36.18) for all $j \le k$ holds, and the inductive proof of (36.18) is complete.

Under the hypothesis (36.4), Proposition 36.1 applies whenever

$$D_{R_0}(z_0) = \{z \in \mathbb{C} : |z - z_0| < R_0\} \subset \Omega. \tag{36.28}$$

Having the solution to (36.3) on $D_{R_0}(z_0)$, we next want to analytically continue this solution to a larger domain in Ω. To start, we discuss analytic continuation of v along a continuous curve

$$\gamma : [a,b] \longrightarrow \Omega, \quad \gamma(a) = z_0, \quad \gamma(b) = z_1. \tag{36.29}$$

To get this, using compactness we pick $R_1 > 0$ and a partition

$$a = a_0 \le a_1 \le \cdots \le a_{n+1} = b \tag{36.30}$$

of $[a,b]$ such that, for $0 \le j \le n$,

$$\begin{aligned} D_j = D_{R_1}(\gamma(a_j)) \text{ contains } \gamma([a_j, a_{j+1}]), \text{ and} \\ \text{is contained in } \Omega. \end{aligned} \tag{36.31}$$

Proposition 36.1 implies the power series (36.9), given via (36.14), is convergent and holomorphic on D_0. Call this solution v_{D_0}. Now $\gamma(a_1) \in D_0$, and we can apply Proposition 36.1, with z_0 replaced by $\gamma(a_1)$ and R_0 replaced by R_1, to produce a holomorphic function v_{D_1} on D_1, satisfying (36.9), and agreeing with v_{D_0} on $D_0 \cap D_1$. We can continue this construction for each j, obtaining at the end v_{D_n}, holomorphic on D_n, which contains $\gamma(a_{n+1}) = z_1$. Also, v_{D_n} solves (36.9) on D_n.

It is useful to put this notion of analytic continuation along γ in a more general context, as follows. Given a continuous path γ as in (36.29), we say a chain along γ is a partition of $[a, b]$ as in (36.30) together with a collection of open, convex sets D_j, satisfying

$$\gamma([a_j, a_{j+1}]) \subset D_j \subset \Omega, \quad \text{for } \ 0 \le j \le n. \tag{36.32}$$

Given $v = v_{D_0}$, holomorphic on D_0, we say an analytic continuation of v along this chain is a collection

$$\begin{aligned} &v_{D_j} : D_j \to \mathbb{C}, \text{ holomorphic, such that} \\ &v_{D_j} = v_{D_{j+1}} \text{ on } D_j \cap D_{j+1}, \text{ for } 0 \le j \le n-1. \end{aligned} \tag{36.33}$$

Note that (36.32) implies $D_j \cap D_{j+1} \neq \emptyset$.

The following is a key uniqueness result.

Proposition 36.2. *Let $\{\widetilde{D}_0, \dots, \widetilde{D}_m\}$ be another chain along γ, associated to a partition*

$$a = \tilde{a}_0 \le \tilde{a}_1 \le \cdots \le \tilde{a}_{m+1} = b. \tag{36.34}$$

Assume we also have an analytic continuation of v along this chain, given by holomorphic functions $\tilde{v}_{\widetilde{D}_j}$ on $\widetilde{D}_j$. Then

$$\tilde{v}_{\widetilde{D}_m} = v_{D_n} \quad \textit{on the neighborhood } \ \widetilde{D}_m \cap D_n \ \textit{ of } \ \gamma(b). \tag{36.35}$$

Proof. We first show the conclusion holds when the two partitions are equal, i.e., $m = n$ and $\tilde{a}_j = a_j$, but the sets D_j and $\widetilde{D}_j$ may differ. In such a case, $D_j^{\#} = D_j \cap \widetilde{D}_j$ is also an open, convex set satisfying $D_j^{\#} \supset \gamma([a_j, a_{j+1}])$. Application of Proposition 10.1 and induction on j shows that if $v|_{D_0} = \tilde{v}|_{D_0^{\#}}$ on $D_0^{\#} \subset D_0$, then $v|_{D_j} = \tilde{v}|_{D_j^{\#}}$ for each j, which yields (36.35).

It remains to show that two analytic continuations of v on D_0, along chains associated to two different partitions of $[a, b]$, agree on a neighborhood of $\gamma(b)$. To see this, note that two partitions of $[a, b]$ have a common refinement, so it suffices to show such agreement when the partition (36.30) is augmented by adding one element, say $\tilde{a}_\ell \in (a_\ell, a_{\ell+1})$. We can obtain a chain associated to this partition by taking the chain $\{D_0, \dots, D_n\}$ associated to the partition (36.30), as in (36.32), and "adding" $\widetilde{D}_\ell = D_\ell \supset \gamma([a_\ell, a_{\ell+1}]) = \gamma([a_\ell, \tilde{a}_\ell]) \cup \gamma([\tilde{a}_\ell, a_{\ell+1}])$. That the conclusion of Proposition 36.2 holds in this case is clear.

Proposition 36.2 motivates us to introduce the following notation. Let $\mathcal{O}_0$ and $\mathcal{O}_1$ be open convex neighborhoods of z_0 and z_1 in Ω, and let γ be a continuous path from z_0 to z_1, as in (36.29). Let v be holomorphic on $\mathcal{O}_0$. Assume there exists a chain $\{D_0, \dots, D_n\}$ along γ such that $D_0 = \mathcal{O}_0$ and $D_n = \mathcal{O}_1$, and an analytic continuation of v along this chain, as in (36.33). We set

$$v_\gamma : \mathcal{O}_1 \longrightarrow \mathbb{C} \tag{36.36}$$

equal to v_{D_n} on $D_n = \mathcal{O}_1$. By Proposition 36.2, this is independent of the choice of chain along γ for which such an analytic continuation exists.

We next supplement Proposition 36.2 with another key uniqueness result, called the *monodromy theorem.* To formulate it, suppose we have a homotopic family of curves

$$\gamma_s : [a, b] \longrightarrow \Omega, \quad \gamma_s(a) \equiv z_0, \ \gamma_s(b) \equiv z_1, \quad 0 \le s \le 1, \tag{36.37}$$

so $\gamma(s,t) = \gamma_s(t)$ defines a continuous map $\gamma : [0,1] \times [a,b] \to \Omega$. As above, assume $\mathcal{O}_1$ and $\mathcal{O}_1$ are convex open neighbrhoods of z_0 and z_1 in Ω, and that v is holomorphic on $\mathcal{O}_0$.

Proposition 36.3. *Assume that for each $s \in [0,1]$ there is a chain along γ_s from $\mathcal{O}_0$ to $\mathcal{O}_1$ and an analytic continuation of v along this chain, producing*

$$v_{\gamma_s} : \mathcal{O}_1 \longrightarrow \mathbb{C}. \tag{36.38}$$

Then v_{γ_s} is independent of $s \in [0,1]$.

Proof. Take $s_0 \in [0,1]$, and let $\{D_0, \dots, D_n\}$ be a chain along γ_{s_0}, associated to a partition of $[a,b]$ of the form (36.30), satisfying $D_0 = \mathcal{O}_0$, $D_n = \mathcal{O}_1$, along which v has an analytic continuation. Then there exists $\varepsilon > 0$ such that, for each $s \in [0,1]$ such that $|s - s_0| \le \varepsilon$, $\{D_0, \dots, D_n\}$ is also a chain along γ_s. The fact that $v_{\gamma_s} = v_{\gamma_{s_0}}$ for all such s follows from Proposition 36.2. This observation readily leads to the conclusion of Proposition 36.3.

Let us return to the setting of Proposition 36.1, and the discussion of analytic continuation along a curve in Ω, involving (36.28)–(36.31). Combining these results with Proposition 36.3, we have the following.

Proposition 36.4. *Consider the differential equation (36.3), where A and f satisfy (36.4) and $z_0 \in \Omega$. If Ω is simply connected, then the solution v on $D_{R_0}(z_0)$ produced in Proposition 36.1 has a unique extension to a holomorphic function $v : \Omega \to \mathbb{C}$, satisfying (36.3) on Ω.*

Returning to Bessel's equation (36.2) and the first-order 2×2 system arising from it, we see that Proposition 36.4 applies, not to $\Omega = \mathbb{C} \setminus 0$, but to a simply connected subdomain of $\mathbb{C} \setminus 0$, such as $\Omega = \mathbb{C} \setminus (-\infty, 0]$. Indeed, as we saw in (35.16)–(35.17), the solution $J_\nu(z)$ to (36.2) has the form

$$J_\nu(z) = z^\nu \mathcal{J}_\nu(z), \quad \text{for} \ \ z \in \mathbb{C} \setminus (-\infty, 0], \tag{36.39}$$

With $\mathcal{J}_\nu(z)$ holomorphic in $z \in \mathbb{C}$, for each $\nu \in \mathbb{C}$. Subsequent calculations, involving (35.34)–(35.43), show that the solution $Y_\nu(z)$ to (36.2) has the form

$$Y_\nu(z) = z^\nu \mathcal{A}_\nu(z) + z^{-\nu} \mathcal{B}_\nu(z), \quad \text{for} \ \ z \in \mathbb{C} \setminus (-\infty, 0], \tag{36.40}$$

with $\mathcal{A}_\nu(z)$ and $\mathcal{B}_\nu(z)$ holomorphic in $z \in \mathbb{C}$, for each $\nu \in \mathbb{C} \setminus \mathbb{Z}$, and, for $n \in \mathbb{Z}$,

$$Y_n(z) = \frac{2}{\pi}\Bigl(\log \frac{z}{2}\Bigr) J_n(z) + \mathcal{A}_n(z), \quad \text{for} \ \ z \in \mathbb{C} \setminus (-\infty, 0], \tag{36.41}$$

where $\mathcal{A}_n(z)$ is holomorphic on $\mathbb{C}$ if $n = 0$, and is meromorphic with a pole of order n at $z = 0$ if $n \in \mathbb{N}$.

Bessel's equation (36.2) belongs to the more general class of second order linear differential equations of the form

$$z^2u''(z) + zb(z)u'(z) + c(z)u(z) = 0, \tag{36.42}$$

where $c(z)$ and $b(z)$ are holomorphic on some disk $D_a(0)$. For (36.2),

$$b(z) = 1, \quad c(z) = z^2 - \nu^2. \tag{36.43}$$

To convert (36.42) to a first-order system, instead of (36.5), it is convenient to set

$$v(z) = \begin{pmatrix} u(z) \\ zu'(z) \end{pmatrix}. \tag{36.44}$$

Then the differential equation for v becomes

$$z\frac{dv}{dz} = A(z)v(z), \tag{36.45}$$

with

$$A(z) = \begin{pmatrix} 0 & 1 \\ -c(z) & 1 - b(z) \end{pmatrix}. \tag{36.46}$$

Generally, if

$$A : D_a(0) \longrightarrow M(n, \mathbb{C}) \ \text{ is holomorphic,} \tag{36.47}$$

the $n \times n$ system (36.45) is said to have a regular singular point at $z = 0$. By Proposition 36.4, if (36.47) holds and we set

$$\Omega = D_a(0) \setminus (-a, 0], \quad z_0 \in \Omega, \tag{36.48}$$

and take $v_0 \in \mathbb{C}^n$, then the system (36.45) has a unique solution, holomorphic on Ω, satisfying $v(z_0) = v_0$. We now desire to understand more precisely how such a solution $v(z)$ behaves as $z \to 0$, in light of the results (36.39)–(36.41).

As a simple example, take $A(z) \equiv A_0 \in M(n, \mathbb{C})$, so (36.45) becomes

$$z\frac{dv}{dz} = A_0 v. \tag{36.49}$$

If we set $z_0 = 1$ and $v(1) = v_0$, the solution is

$$v(z) = e^{(\log z)A_0}v_0, \quad z \in \mathbb{C} \setminus (-\infty, 0], \tag{36.50}$$

where $e^{\zeta A_0}$ is the matrix exponential, discussed in Appendix N. If v_0 is an eigenvector of A_0,

$$A_0 v_0 = \lambda v_0 \Longrightarrow v(z) = z^\lambda v_0. \tag{36.51}$$

From (36.42)–(36.46), we see that, for the system arising from Bessel's equation,

$$A(z) = A_0 + A_2 z^2, \quad A_0 = \begin{pmatrix} 0 & 1 \\ \nu^2 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \tag{36.52}$$

and the eigenvalues of A_0 are $\pm\nu$. This is a preliminary indication of a connection between (36.51) and (36.39)–(36.41). The following result will lead to a strengthening of this connection.

Proposition 36.5. *Assume $B : D_a(0) \to M(n, \mathbb{C})$ is holomorphic, and set $B_0 = B(0)$. Assume that 0 is an eigenvalue of B_0 but that B_0 has no eigenvalues that are positive integers. If $B_0 v_0 = 0$, then the system*

$$z \frac{dv}{dz} = B(z)v \tag{36.53}$$

has a solution v, holomorphic on $D_a(0)$, satisfying $v(0) = v_0$.

Proof. Parallel to the proof of Proposition 36.1, we set

$$v(z) = \sum_{k=0}^{\infty} z^k v_k, \tag{36.54}$$

and produce a recursive formula for the coefficients v_k, given $v_0 \in \mathbb{C}^n$ such that $B_0 v_0 = 0$. Granted convergence of (36.54), we have

$$z \frac{dv}{dz} = \sum_{k \geq 1} k v_k z^k, \tag{36.55}$$

and

$$\begin{aligned} B(z)v &= \sum_{j \geq 0} B_j z^j \sum_{\ell \geq 0} v_\ell z^\ell \\ &= B_0 v_0 + \sum_{k \geq 1} \sum_{\ell=0}^{k} B_{k-\ell} v_\ell z^k. \end{aligned} \tag{36.56}$$

Having $B_0 v_0 = 0$, we obtain the following recursive formula, for $k \geq 1$,

$$k v_k = B_0 v_k + \sum_{\ell=0}^{k-1} B_{k-\ell} v_\ell, \tag{36.57}$$

or equivalently

$$(kI - B_0) v_k = \sum_{\ell=0}^{k-1} B_{k-\ell} v_\ell. \tag{36.58}$$

We can solve uniquely for v_k provided k is not an eigenvalue of B_0. If no positive integer is an eigenvalue of B_0, we can solve for the coefficients v_k for all $k \in \mathbb{N}$, obtaining the series (36.54). Estimates on these coefficients, verifying that (36.54) converges, are similar to those made in the proof of Proposition 36.1.

To apply Proposition 36.5 to the system (36.45), under the hypothesis (36.47), assume λ is an eigenvalue of $A_0 = A(0)$, so

$$(A_0 - \lambda) v_0 = 0, \tag{36.59}$$

for some nonzero $v_0 \in \mathbb{C}^n$. If we set

$$v(z) = z^\lambda w(z), \quad z \in \Omega = D_a(0) \setminus (-a, 0], \tag{36.60}$$

the equation (36.45) is converted to

$$z \frac{dw}{dz} = (A(z) - \lambda) w. \tag{36.61}$$

Hence Proposition 36.5 applies, with $B(z) = A(z) - \lambda$, and we have:

Proposition 36.6. *Assume $A : D_a(0) \to M(n, \mathbb{C})$ is holomorphic and λ is an eigenvalue of $A_0 = A(0)$, with eigenvector v_0. Assume that $\lambda + k$ is not an eigenvalue of A_0 for any $k \in \mathbb{N}$. Then the system (36.45) has a solution of the form (36.60), where*

$$w : D_a(0) \to \mathbb{C}^n \ \text{ is holomorphic and } \ w(0) = v_0. \tag{36.62}$$

For Bessel's equation, whose associated first-order system (36.45) takes $A(z)$ as in (36.52), we have seen that

$$A_0 = \begin{pmatrix} 0 & 1 \\ \nu^2 & 0 \end{pmatrix} \ \text{ has eigenvalues } \pm \nu, \tag{36.63}$$

and

$$(A_0 - \nu I) \begin{pmatrix} 1 \\ \nu \end{pmatrix} = 0. \tag{36.64}$$

Thus Proposition 36.6 provides a solution to (36.45) of the form

$$v_\nu(z) = z^\nu w_\nu(z), \quad w_\nu \text{ holomorphic on } \mathbb{C}, \quad w_\nu(0) = \begin{pmatrix} 1 \\ \nu \end{pmatrix}, \tag{36.65}$$

as long as

$$-2\nu \notin \mathbb{N}. \tag{36.66}$$

Comparison with (36.39)–(36.41), and the supporting calculations from §35, shows that the condition (36.66) is stronger than necessary for the conclusion given in (36.65). The condition ought to be

$$-\nu \notin \mathbb{N}. \tag{36.67}$$

To get this stronger conclusion, we can take into account the special structure of (36.52) and bring in the following variant of Proposition 36.5.

Proposition 36.7. *In the setting of Proposition 36.5, assume that* $B : D_a(0) \to M(n, \mathbb{C})$ *is holomorphic and even in* z*, and set* $B_0 = B(0)$*. Assume that* 0 *is an eigenvalue of* B_0 *(with associated eigenvector* v_0*) and that* B_0 *has no eigenvalues that are positive even integers. Then the equation (36.53) has a solution* v*, holomorphic and even on* $D_a(0)$*, satisfying* $v(0) = v_0$*.*

Proof. This is a simple variant of the proof of Proposition 36.5. Replace (36.54) by

$$v(z) = \sum_{k=0}^{\infty} z^{2k} v_{2k}, \tag{36.68}$$

and replace (36.56) by

$$\begin{aligned} B(z)v &= \sum_{j \geq 0} B_{2j} z^{2j} \sum_{\ell \geq 0} v_{2\ell} z^{2\ell} \\ &= B_0 v_0 + \sum_{k \geq 1} \sum_{\ell=0}^{k} B_{2k-2\ell} v_{2\ell} z^{2k}. \end{aligned} \tag{36.69}$$

Then the recursion (36.58) is replaced by

$$(2kI - B_0) v_{2k} = \sum_{\ell=0}^{k-1} B_{2k-2\ell} v_{2\ell}, \tag{36.70}$$

and the result follows.

Proposition 36.8. *In the setting of Proposition 36.6, assume* $A : D_a(0) \to M(n, \mathbb{C})$ *is holomorphic and even in* z*, and* λ *is an eigenvalue of* $A_0 = A(0)$*, with eigenvector* v_0*. Assume* $\lambda + 2k$ *is not an eigenvalue of* A_0 *for any* $k \in \mathbb{N}$*. Then the system (36.45) has a solution of the form (36.60), with (36.62) holding.*

We see that Proposition 36.8 applies to the Bessel system

$$z \frac{dv}{dz} = \begin{pmatrix} 0 & 1 \\ \nu^2 - z^2 & 0 \end{pmatrix} v, \tag{36.71}$$

to produce a 2D space of solutions, holomorphic on $\Omega = \mathbb{C} \setminus (-\infty, 0]$, if $\nu \in \mathbb{C} \setminus \mathbb{Z}$, and a 1D space of solutions to this system if $\nu = n \in \mathbb{Z}$ (which one verifies to be the same for $\nu = n$ and $\nu = -n$). Results of §35 show that the 2D space of solutions to (36.71) is spanned by

$$\begin{pmatrix} J_\nu(z) \\ z J_\nu'(z) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} J_{-\nu}(z) \\ z J_{-\nu}'(z) \end{pmatrix}, \quad \text{for} \quad \nu \in \mathbb{C} \setminus \mathbb{Z}, \tag{36.72}$$

and the 1D space of solutions given by Corollary 36.8 is spanned by

$$\begin{pmatrix} J_n(z) \\ z J_n'(z) \end{pmatrix}, \quad \text{for} \quad \nu = n \in \mathbb{Z}. \tag{36.73}$$

Also results of §35 imply that the 2D space of solutions to (36.71) is spanned by (36.73) and

$$\begin{pmatrix} Y_n(z) \\ zY_n'(z) \end{pmatrix}, \quad \text{for} \ \ \nu = n \in \mathbb{Z}. \tag{36.74}$$

As seen in (36.41), these last solutions do not behave as in (36.65), since factors of $\log z$ appear.

To obtain results more general than those of Propositions 36.6 and 36.8, we take the following approach. We seek a holomorphic map

$$U : D_a(0) \longrightarrow M(n,\mathbb{C}), \quad U(0) = I, \tag{36.75}$$

such that, under the change of variable

$$v(z) = U(z)w(z), \tag{36.76}$$

(36.45) becomes

$$z\frac{dw}{dz} = A_0 w, \tag{36.77}$$

a case already treated in (36.49)–(36.50). To reiterate, the general solution to (36.77) on $\mathbb{C} \setminus (-\infty, 0]$ is

$$w(z) = e^{(\log z)A_0} v_0 = z^{A_0} v_0, \quad v_0 \in \mathbb{C}^n, \tag{36.78}$$

the latter identity defining z^{A_0} for $z \in \mathbb{C} \setminus (-\infty, 0]$. To construct such $U(z) = \sum_{k\ge 0} U_k z^k$, we start with

$$A(z)U(z)w = z\frac{dv}{dz} = zU(z)\frac{dw}{dz} + zU'(z)w, \tag{36.79}$$

which leads to (36.77) provided $U(z)$ satisfies

$$z\frac{dU}{dz} = A(z)U(z) - U(z)A_0. \tag{36.80}$$

This equation has the same form as (36.45), i.e.,

$$z\frac{dU}{dz} = \mathcal{A}(z)U(z), \tag{36.81}$$

where $U(z)$ takes values in $M(n,\mathbb{C})$ and $\mathcal{A}(z)$ is a linear transformation on the vector space $M(n,\mathbb{C})$:

$$\begin{aligned} \mathcal{A}(z)U &= A(z)U - UA_0 \\ &= \sum_{k\ge 0} \mathcal{A}_k z^k U. \end{aligned} \tag{36.82}$$

In particular,

$$\mathcal{A}_0 U = A_0 U - U A_0 = [A_0, U] = C_{A_0} U, \tag{36.83}$$

the latter identity defining

$$C_{A_0} : M(n, \mathbb{C}) \longrightarrow M(n, \mathbb{C}). \tag{36.84}$$

Note that $C_{A_0} I = [A_0, I] = 0$, so 0 is an eigenvalue of C_{A_0}, with eigenvector $I = U(0)$. Hence Proposition 36.5 applies, with $\mathbb{C}^n$ replaced by $M(n, \mathbb{C})$ and $B_0 = C_{A_0}$. The hypothesis that B_0 has no eigenvalue that is a positive integer is hence that C_{A_0} has no eigenvalue that is a positive integer. One readily verifies that the set $\operatorname{Spec} C_{A_0}$ of eigenvalues of C_{A_0} is described as follows:

$$\operatorname{Spec} A_0 = \{\lambda_j\} \Longrightarrow \operatorname{Spec} C_{A_0} = \{\lambda_j - \lambda_k\}. \tag{36.85}$$

Thus the condition that $\operatorname{Spec} C_{A_0}$ contains no positive integer is equivalent to the condition that A_0 have no pair of eigenvalues that differ by a nonzero integer. This establishes the following.

Proposition 36.9. *Assume $A : D_a(0) \to M(n, \mathbb{C})$ is holomorphic, and $A(0) = A_0$. Assume A_0 has no two eigenvalues that differ by a nonzero integer. Then there is a holomorphic map U, as in (36.75), such that the general solution to (36.45) on $D_a(0) \setminus (-a, 0]$ is given by*

$$v(z) = U(z) z^{A_0} v_0, \quad v_0 \in \mathbb{C}^n. \tag{36.86}$$

Let us comment that, in this setting, the recursion (36.57)–(36.58) becomes

$$k U_k = [A_0, U_k] + \sum_{\ell=0}^{k-1} A_{k-\ell} U_\ell, \tag{36.87}$$

i.e.,

$$(kI - C_{A_0}) U_k = \sum_{\ell=0}^{k-1} A_{k-\ell} U_\ell, \tag{36.88}$$

for $k \geq 1$.

In cases where $A(z)$ is an even function of z, we can replace Proposition 36.5 by Proposition 36.7, and obtain the following.

Proposition 36.10. *In the setting of Proposition 36.9, assume $A(z)$ is even. Then the conclusion holds as long as A_0 has no two eigenvalues that differ by a nonzero even integer.*

Comparing Propositions 36.6 and 36.8 with Propositions 36.9 and 36.10, we see that the latter pair do not involve more general hypotheses on A_0 but they do have a stronger conclusion, when A_0 is not diagonalizable. To illustrate this, we take the case $\nu = 0$ of the Bessel system (36.71), for which

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{36.89}$$

In this case, $A_0^2 = 0$, and the power series representation of $e^{\zeta A_0}$ gives

$$z^{A_0} = e^{(\log z)A_0} = I + (\log z)A_0 = \begin{pmatrix} 1 & \log z \\ 0 & 1 \end{pmatrix}. \tag{36.90}$$

Thus we see that (36.86) captures the full 2D space of solutions to this Bessel equation, and the $\log z$ factor in the behavior of $Y_0(z)$ is manifest.

On the other hand, Propositions 36.9–36.10 are not effective for the Bessel system (36.71) when $\nu = n$ is a nonzero integer, in which case

$$A_0 = \begin{pmatrix} 0 & 1 \\ n^2 & 0 \end{pmatrix}, \quad \operatorname{Spec} A_0 = \{-n, n\}, \quad \operatorname{Spec} C_{A_0} = \{-2n, 0, 2n\}. \tag{36.91}$$

We next state a result that does apply to such situations.

Proposition 36.11. *Let $A : D_a(0) \to M(n, \mathbb{C})$ be holomorphic, with $A(0) = A_0$. Assume*

$$A_0 \text{ is diagonalizable,} \tag{36.92}$$

and

$$\operatorname{Spec} C_{A_0} \text{ contains exactly one positive integer } \ell. \tag{36.93}$$

Then there exists $B_\ell \in M(n, \mathbb{C})$ such that

$$[A_0, B_\ell] = \ell B_\ell, \tag{36.94}$$

and a holomorphic function $U(z)$, as in (36.75), such that the general solution to (36.45) on $D_a(0) \setminus (-a, 0]$ is given by

$$v(z) = U(z) z^{A_0} z^{B_\ell} v_0, \quad v_0 \in \mathbb{C}^n. \tag{36.95}$$

Furthermore,

$$B_\ell \text{ is nilpotent.} \tag{36.96}$$

We refer to Proposition 11.5 in Chapter 3 of [T4] for a proof, remarking that the crux of the matter involves modifying (36.77) to

$$z\frac{dw}{dz} = (A_0 + B_\ell z^\ell)w, \tag{36.97}$$

hence replacing (36.80) by

$$z\frac{dU}{dz} = A(z)U(z) - U(z)(A_0 + B_\ell z^\ell), \tag{36.98}$$

and consequently modifying (36.88) to something that works for $k = \ell$, as well as other values of $k \in \mathbb{N}$.

To say that B_ℓ is nilpotent is to say that $B_\ell^{m+1} = 0$ for some $m \in \mathbb{N}$. In such a case, we have

$$z^{B_\ell} = \sum_{k=0}^{m} \frac{1}{k!}(\log z)^k B_\ell^k. \tag{36.99}$$

If (36.45) is a 2×2 system, nilpotence implies $B_\ell^2 = 0$, and hence

$$z^{B_\ell} = I + (\log z)B_\ell, \tag{36.100}$$

as in (36.90). For larger systems, higher powers of $\log z$ can appear.

Note that if (36.45) is a 2×2 system and if (36.92) fails, then A_0 has just one eigenvalue, so Proposition 36.9 applies. On the other hand, if (36.92) holds, then

$$\operatorname{Spec} A_0 = \{\lambda_1, \lambda_2\} \Longrightarrow \operatorname{Spec} C_{A_0} = \{\lambda_1 - \lambda_2, 0, \lambda_2 - \lambda_1\}, \tag{36.101}$$

so either (36.93) holds or Proposition 36.9 applies.

We refer to Chapter 3 of [T4] for further results, more general than Proposition 36.11, applicable to $n \times n$ systems of the form (36.45) when $n \geq 3$.

We have discussed differential equations with a regular singular point at 0. Similarly, a forst-order system has a regular singular point at $z_0 \in \Omega$ if it is of the form

$$(z - z_0)\frac{dv}{dz} = A(z)v, \tag{36.102}$$

with $A : \Omega \to M(n, \mathbb{C})$ holomorphic. More generally, if $\Omega \subset \mathbb{C}$ is a connected open set, and $\mathcal{F} \subset \Omega$ is a finite subset, and if $B : \Omega \setminus \mathcal{F} \to M(n, \mathbb{C})$ is holomorphic, the system

$$\frac{dv}{dz} = B(z)v \tag{36.103}$$

is said to have a regular singular point at $z_0 \in \mathcal{F}$ provided $(z - z_0)B(z)$ extends to be holomorphic on a neighborhood of z_0. Related to this, if $A, B, C : \Omega \setminus \mathcal{F} \to \mathbb{C}$ are holomorphic, the second-order differential equation

$$A(z)u''(z) + B(z)u'(z) + C(z)u(z) = 0 \tag{36.104}$$

has a regular singular point at $z_0 \in \mathcal{F}$ provided

$$A(z), \quad (z-z_0)B(z), \quad \text{and} \quad (z-z_0)^2C(z) \\ \text{extend to be holomorphic on a neighborhood of } z_0, \\ \text{and} \quad A(z_0) \neq 0. \tag{36.105}$$

An example is the class of Legendre equations

$$(1-z^2)u''(z) - 2zu'(z) + \Big[\nu^2 - \frac{\mu^2}{1-z^2}\Big]u(z) = 0, \tag{36.106}$$

produced in (O.28), which has the form (36.104)–(36.105), upon division by $1-z^2$, with regular singular points at $z = \pm 1$, for all $\nu, \mu \in \mathbb{C}$, including $\mu = 0$, when (36.106) reduces to

$$(1-z^2)u''(z) + 2zu'(z) + \nu^2 u(z) = 0. \tag{36.107}$$

Proposition 36.4 implies that if Ω is a simply connected subdomain of $\mathbb{C} \setminus \{-1, 1\}$, then (36.106) has a two-dimensional space of solutions that are holomorphic on Ω. Such Ω might, for example, be

$$\Omega = \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}. \tag{36.108}$$

Propositions 36.9 and 36.11 apply to first-order systems, of the form (36.102), with $z_0 = \pm 1$, derivable from (36.106).

We next consider special functions defined by what are called hypergeometric series, and the differential equations they satisfy. The basic cases are

$${}_1F_1(a; b; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}, \tag{36.109}$$

nad

$${}_2F_1(a_1, a_2; b; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k}{(b)_k} \frac{z^k}{k!}. \tag{36.110}$$

Here $(a)_0 = 1$ and, for $k \in \mathbb{N}$,

$$(a)_k = a(a+1)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}, \tag{36.111}$$

the latter identity holding provided $a \notin \{0, -1, -2, \dots\}$. In (36.109)–(36.110), we require $b \notin \{0, -1, -2, \dots\}$. Then the series (36.109) converges for all $z \in \mathbb{C}$. The series (36.110) converges for $|z| < 1$. The function ${}_1F_1$ is called the confluent hypergeometric function, and ${}_2F_1$ is called the hypergeometric function. Differentiation of these power series implies that ${}_1F_1(a;b;z) = u(z)$ satisfies the differential equation

$$zu''(z) + (b - z)u'(z) - au(z) = 0, \tag{36.112}$$

and that ${}_2F_1(a_1, a_2; b; z) = u(z)$ satisfies the differential equation

$$z(1 - z)u''(z) + [b - (a_1 + a_2 + 1)z]u'(z) - a_1 a_2 u(z) = 0. \tag{36.113}$$

The equations (36.112)–(36.113) are called, respectively, the confluent hypergeometric equation and the hypergeometric equation. For (36.112), $z = 0$ is a regular singular point, and for (36.113), $z = 0$ and $z = 1$ are regular singular points. By Proposition 36.4, a solution to (36.113) has an analytic continuation to any simply connected domain $\Omega \subset \mathbb{C} \setminus \{0, 1\}$, such as

$$\Omega = \mathbb{C} \setminus \{(-\infty, 0] \cup [1, \infty)\}. \tag{36.114}$$

Such a continuation has jumps across $(-\infty, 0)$ and across $(1, \infty)$. However, we know that ${}_2F_1(a_1, a_2; b; z)$ is holomorphic in $|z| < 1$, so the jump across the segment $(-1, 0)$ in $(-\infty, 0)$ vanishes. Hence this jump vanishes on all of $(-\infty, 0)$. We deduce that, as long as $b \notin \{0, -1, -2, \dots\}$,

$$\begin{gathered} {}_2F_1(a_1, a_2; b; z) \ \text{ continues analytically to} \\ z \in \mathbb{C} \setminus [1, \infty). \end{gathered} \tag{36.115}$$

As we know, whenever $b \notin \{0, -1, -2, \dots\}$, both (36.112) and (36.113) have a 2D space of solutions, on their domains, respectively $\mathbb{C} \setminus (-\infty, 0]$ and (36.114). Concentrating on the regular singularity at $z = 0$, we see that each of these differential equations, when converted to a first-order system as in (36.42)–(36.46), yields a system of the form (36.45) with

$$A(0) = A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 1 - b \end{pmatrix}. \tag{36.116}$$

Clearly the eigenvalues of A_0 are 0 and $1 - b$. Given that we already require $b \notin \{0, -1, -2, \dots\}$, we see that Proposition 36.6 applies, with $\lambda = 1 - b$, as long as

$$b \notin \{2, 3, 4, \dots\}. \tag{36.117}$$

Then, as long as (36.117) holds, we expect to find another solution to each of (36.112) and (36.113), of the form (36.60), with $w(z)$ holomorphic on a neighborhood of $z = 0$ and $\lambda = 1 - b$. In fact, one can verify that, if (36.117) holds,

$$z^{1-b}\, {}_1F_1(1 + a - b; 2 - b; z) \tag{36.118}$$

is also a solution to (36.112). In addition,

$$z^{1-b}\,{}_2F_1(a_1-b+1, a_2-b+1; 2-b; z) \tag{36.119}$$

is also a solution to (36.113).

One reason the hypergeometric functions are so important is that many other functions can be expressed using them. These range from elementary functions, such as

$$\begin{aligned} e^z &= {}_1F_1(1;1;z), \\ \log(1-z) &= -z\,{}_2F_1(1,1;2;z), \\ \sin^{-1} z &= z\,{}_2F_1\Bigl(\frac{1}{2},\frac{1}{2};\frac{3}{2};z^2\Bigr), \end{aligned} \tag{36.120}$$

to special functions, such as Bessel functions,

$$J_\nu(z) = \frac{e^{-iz}}{\Gamma(\nu+1)}\Bigl(\frac{z}{2}\Bigr)^\nu\,{}_1F_1\Bigl(\nu+\frac{1}{2};2\nu+1;2iz\Bigr), \tag{36.121}$$

elliptic integrals,

$$\begin{aligned} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-z^2\sin^2\theta}} &= \frac{\pi}{2}\,{}_2F_1\Bigl(\frac{1}{2},\frac{1}{2};1;z^2\Bigr), \\ \int_0^{\pi/2} \sqrt{1-z^2\sin^2\theta}\,d\theta &= \frac{\pi}{2}\,{}_2F_1\Bigl(-\frac{1}{2},\frac{1}{2};1;z^2\Bigr), \end{aligned} \tag{36.122}$$

and many other examples, which can be found in Chapter 9 of [Leb].

The hypergeometric functions (36.109)–(36.110) are part of a heirarchy, defined as follows. Take $p, q \in \mathbb{N},\ p \le q+1$, and

$$a = (a_1,\dots,a_p), \quad b = (b_1,\dots,b_q), \quad b_j \notin \{0,-1,-2,\dots\}. \tag{36.123}$$

Then we set

$${}_pF_q(a;b;z) = \sum_{k=0}^\infty \frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\,\frac{z^k}{k!}, \tag{36.124}$$

with $(a_j)_k$ defined as in (36.111). Equivalently, if also $a_j \notin \{0,-1,-2,\dots\}$,

$${}_pF_q(a;b;z) = \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)} \sum_{k=0}^\infty \frac{\Gamma(a_1+k)\cdots\Gamma(a_p+k)}{\Gamma(b_1+k)\cdots\Gamma(b_q+k)}\,\frac{z^k}{k!}. \tag{36.125}$$

This power series converges for all $z \in \mathbb{C}$ if $p < q+1$, and for $|z| < 1$ if $p = q+1$. The function ${}_pF_q(a;b;z) = u(z)$ satisfies the differential equation

$$\Bigl[\frac{d}{dz}\prod_{j=1}^q\Bigl(z\frac{d}{dz}+b_j-1\Bigr) - \prod_{\ell=1}^p\Bigl(z\frac{d}{dz}+a_\ell\Bigr)\Bigr]u = 0, \tag{36.125}$$

which reduces to (36.112) if $p = q = 1$ and to (36.113) if $p = 2, q = 1$. In case $p = q + 1$, we see that this differential equation has the form

$$z^q(1-z)\frac{d^p u}{dz^p} + \cdots = 0, \tag{36.126}$$

with singular points at $z = 0$ and $z = 1$, leading to a holomorphic continuation of ${}_pF_q(a; b; z)$, first to the domain (36.114), and then, since the jump across $(-1, 0)$ vanishes, to $\mathbb{C} \setminus [1, \infty)$.

As an example of a special function expressible via ${}_4F_3$, we consider the Bring radical, Φ^{-1}, defined in Appendix Q as the holomorphic map

$$\Phi^{-1} : D_{(4/5)5^{-1/4}}(0) \longrightarrow \mathbb{C}, \quad \Phi^{-1}(0) = 0, \tag{36.127}$$

that inverts

$$\Phi(z) = z - z^5. \tag{36.128}$$

As shown in (Q.93), we have

$$\Phi^{-1}(z) = \sum_{k=0}^{\infty} \binom{5k}{k} \frac{z^{4k+1}}{4k+1}, \tag{36.129}$$

for $|z| < (4/5)5^{-1/4}$. This leads to the following identity.

Proposition 36.12. *The Bring radical Φ^{-1} is given by*

$$\Phi^{-1}(z) = z\,{}_4F_3\Bigl(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}; \frac{1}{2}, \frac{3}{4}, \frac{5}{4}; 5\Bigl(\frac{5z}{4}\Bigr)^4\Bigr). \tag{36.130}$$

Proof. To start, we write

$$\binom{5k}{k} = \frac{(5k)!}{(4k)!k!} = \frac{\Gamma(5k+1)}{\Gamma(4k+1)k!} = \frac{5}{4}\,\frac{\Gamma(5k)}{\Gamma(4k)}\,\frac{1}{k!}. \tag{36.131}$$

Next, the Gauss formula (18.40) yields

$$\frac{\Gamma(5k)}{\Gamma(4k)} = (2\pi)^{-1/2}\frac{5^{5k-1/2}}{4^{4k-1/2}}\,\frac{\Gamma(k+\frac{1}{5})\Gamma(k+\frac{2}{5})\Gamma(k+\frac{3}{5})\Gamma(k+\frac{4}{5})}{\Gamma(k+\frac{1}{4})\Gamma(k+\frac{2}{4})\Gamma(k+\frac{3}{4})}. \tag{36.132}$$

Hence, for $|z| < (4/5)5^{-1/4}$, $\Phi^{-1}(z)$ is equal to

$$\begin{aligned}(2\pi)^{-1/2}\frac{5^{1/2}}{4^{1/2}}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{1}{5}+k)\Gamma(\frac{2}{5}+k)\Gamma(\frac{3}{5}+k)\Gamma(\frac{4}{5}+k)}{\Gamma(\frac{1}{4}+k)\Gamma(\frac{2}{4}+k)\Gamma(\frac{3}{4}+k)}\cdot\frac{1}{4k+1}& \\ \times 5^k\Bigl(\frac{5}{4}\Bigr)^k z^{4k+1}&.\end{aligned} \tag{36.133}$$

Using

$$(4k+1)\Gamma(k+\tfrac{1}{4}) = 4\Gamma(k+\tfrac{5}{4}), \tag{36.134}$$

we obtain

$$\begin{aligned}\Phi^{-1}(z) = (2\pi)^{-1/2}\frac{5^{1/2}}{2}&\frac{\Gamma(\frac{1}{5})\Gamma(\frac{2}{5})\Gamma(\frac{3}{5})\Gamma(\frac{4}{5})}{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})\Gamma(\frac{3}{4})}\cdot z \\ &\times {}_4F_3\Bigl(\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5};\frac{1}{2},\frac{3}{4},\frac{5}{4};5\Bigl(\frac{5z}{4}\Bigr)^4\Bigr).\end{aligned} \tag{36.135}$$

It remains to evaluate the constant factor that precedes z. Using the identity

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}, \tag{36.136}$$

and $\Gamma(1/2) = \sqrt{\pi}$, we see that this factor is equal to

$$\frac{\sqrt{5}}{4}\cdot\frac{1}{\sin\frac{\pi}{5}\cdot\sin\frac{2\pi}{5}}, \tag{36.137}$$

since $\sin\pi/4 = 1/\sqrt{2}$. Now, by (Q.32),

$$\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}, \quad \cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}, \tag{36.138}$$

and then we can readily compute $(1-\cos^2(\pi/5))(1-\cos^2(2\pi/5))$ and show that

$$\sin\frac{\pi}{5}\cdot\sin\frac{2\pi}{5} = \frac{\sqrt{5}}{4}. \tag{36.139}$$

Hence (36.137) is equal to 1, and we have (36.130).

The results on analytic continuation of ${}_pF_q$, derived above from (36.125)–(36.126), give another proof of Proposition Q.4, on the analytic continuation of Φ^{-1}, established by a geometrical argument in Appendix Q.

We leave special functions and return to generalities. We have seen many cases of holomorphic differential equations defined on a domain $\Omega \subset \mathbb{C}$ that is connected but not simply connected, and have seen that, given $z_0 \in \Omega$, we can analytically continue a solution from a neighborhood $D_a(z_0)$ to any other point $z \in \Omega$ along a curve $\gamma_{z_0 z}$, in a way that depends only on the homotopy class of the curve from z_0 to z. If Ω were simply connected, there would be only one such homotopy class, leading to a uniquely defined analytic continuation to a holomorphic function on Ω. If Ω is not simply connected, there can be many such homotopy classes, and one would get a "multivalued" analytic continuation. Examples of this are readily seen in the representation (36.60) of a solution to (36.45) near a regular singular point, when λ is not an integer.

One way to get a single valued analytic continuation of a solution v to a system of the form (36.3) is to work on the univesal covering space $\widetilde{\Omega}$ of Ω,

$$\pi : \widetilde{\Omega} \longrightarrow \Omega. \tag{36.140}$$

Given a point $z_0 \in \Omega$, the space $\widetilde{\Omega}$ is defined so that, for $z \in \Omega$, $\pi^{-1}(z)$ consists of homotopy classes of continuous curves $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = z_0$ and $\gamma(1) = z$. The space $\widetilde{\Omega}$ has a natural structure of a Riemann surface, for which π in (36.140) yields local holomorphic coordinates. Then analytic continuation along a continuous curve $\gamma_{z_0 z}$, as in (36.36), defines a holomorphic function on $\widetilde{\Omega}$.

The obstruction to an analytic continuation of a solution to a first-order linear system on Ω being single valued on Ω, or equivalently the way in which its analytic continuation to $\widetilde{\Omega}$ differs at different points of $\pi^{-1}(z)$, for $z \in \Omega$, can be expressed in terms of monodromy, defined as follows.

Fix $z_0 \in \Omega$, and let $v : D_a(z_0) \to \mathbb{C}^n$ solve

$$\frac{dv}{dz} = A(z)v, \quad v(z_0) = v_0, \tag{36.141}$$

with $A : \Omega \to M(n,\mathbb{C})$ holomorphic. Consider a continuous curve

$$\gamma : [0,1] \longrightarrow \Omega, \quad \gamma(0) = \gamma(1) = z_0. \tag{36.142}$$

Then set

$$\varkappa(\gamma)v_0 = v_\gamma(z_0), \tag{36.143}$$

with v_γ defined as in (36.36). Clearly the dependence on v_0 is linear, so we have

$$\varkappa(\gamma) \in M(n,\mathbb{C}), \tag{36.144}$$

for each continuous path γ of the form (36.142). By Proposition 36.3, $\varkappa(\gamma_0) = \varkappa(\gamma_1)$ if γ_0 and γ_1 are homotopic curves satisfying (36.142). Hence $\varkappa$ induces a well defined map

$$\varkappa : \pi_1(\Omega, z_0) \longrightarrow M(n,\mathbb{C}), \tag{36.145}$$

where $\pi_1(\Omega, z_0)$ consists of homotopy classes of continuous curves satisfying (36.142). This is the monodromy map associated with the holomorphic system (36.141).

We can define a product on paths of the form (36.142), as follows. If γ_0 and γ_1 satisfy (36.142), we set

$$\begin{aligned} \gamma_1 \circ \gamma_0(t) = \gamma_0(2t), &\qquad 0 \le t \le \frac{1}{2}, \\ \gamma_1(2t-1), &\quad \frac{1}{2} \le t \le 1. \end{aligned} \tag{36.146}$$

We have

$$\varkappa(\gamma_1 \circ \gamma_0) = \varkappa(\gamma_1)\varkappa(\gamma_0). \tag{36.147}$$

If γ_j are homotopic to σ_j, satisfying (36.142), then $\gamma_1 \circ \gamma_0$ is homotopic to $\sigma_1 \circ \sigma_0$, so we have a product on $\pi_1(\Omega, z_0)$, and $\varkappa$ in (36.145) preserves products.

Note that the constant map

$$\varepsilon : [0,1] \longrightarrow \Omega, \quad \varepsilon(t) \equiv z_0, \tag{36.148}$$

acts as a multiplicative identity on $\pi_1(\Omega, z_0)$, in that for each γ satisfying (36.142), γ, $\varepsilon \circ \gamma$, and $\gamma \circ \varepsilon$ are homotopic. Clearly

$$\varkappa(\varepsilon) = I. \tag{36.149}$$

Furthermore, given γ as in (36.142), if we set

$$\gamma^{-1} : [0,1] \longrightarrow \Omega, \quad \gamma^{-1}(t) = \gamma(1-t), \tag{36.150}$$

then $\gamma^{-1} \circ \gamma$ and $\gamma \circ \gamma^{-1}$ are homotopic to ε. Hence

$$\varkappa(\gamma^{-1} \circ \gamma) = \varkappa(\gamma^{-1})\varkappa(\gamma) = I, \tag{36.151}$$

so

$$\varkappa(\gamma^{-1}) = \varkappa(\gamma)^{-1}, \tag{36.152}$$

and in particular $\varkappa(\gamma) \in M(n, \mathbb{C})$ is invertible. We have

$$\varkappa : \pi_1(\Omega, z_0) \longrightarrow G\ell(n, \mathbb{C}), \tag{36.153}$$

where $G\ell(n, \mathbb{C})$ denotes the set of invertible matrices in $M(n, \mathbb{C})$.

One additional piece of structure on $\pi_1(\Omega, z_0)$ worth observing is that, if γ_1, γ_2, and γ_3 satisfy (36.142), then

$$\gamma_3 \circ (\gamma_2 \circ \gamma_1) \text{ and } (\gamma_3 \circ \gamma_2) \circ \gamma_1 \text{ are homotopic}, \tag{36.154}$$

hence define the same element of $\pi_1(\Omega, z_0)$. We say that the product in $\pi_1(\Omega, z_0)$ is associative. A set with an associative product, and a multiplicative identity element, with the property that each element has a multiplicative inverse is called a *group*. The group $\pi_1(\Omega, z_0)$ is called the fundamental group of Ω. Matrix multiplication also makes $G\ell(n, \mathbb{C})$ a group. The fact that $\varkappa$ in (36.153) satisfies (36.147) and (36.149) makes it a *group homomorphism.*

To illustrate the monodromy map, let us take

$$\Omega = D_a(0) \setminus 0, \tag{36.155}$$

and consider the system (36.45), with A holomorphic on $D_a(0)$. Take

$$z_0 = r \in (0,a), \tag{36.156}$$

and set

$$\beta(t) = re^{2\pi it}, \quad 0 \le t \le 1, \tag{36.157}$$

so $\beta(0) = \beta(1) = z_0$. Note that

$$\begin{aligned} A(z) \equiv A_0 &\Longrightarrow v(z) = z^{A_0}(r^{-A_0}v_0) \\ &\Longrightarrow v(re^{2\pi it}) = e^{2\pi itA_0}v_0 \\ &\Longrightarrow \varkappa(\beta) = e^{2\pi iA_0}. \end{aligned} \tag{36.158}$$

When $A(z)$ is not constant, matters are more complicated. Here is a basic case.

Proposition 36.13. *Assume that Proposition 36.9 applies to the system (36.45), so (36.86) holds. Then*

$$\varkappa(\beta) = U(r)e^{2\pi iA_0}U(r)^{-1}. \tag{36.159}$$

Proof. From (36.86) we have

$$v(r) = U(r)r^{A_0}v_1, \quad v_1 \in \mathbb{C}^n, \tag{36.160}$$

and

$$\begin{aligned} v(e^{2\pi it}r) &= U(e^{2\pi it}r)r^{A_0}e^{2\pi itA_0}v_1 \\ &= U(e^{2\pi it}r)r^{A_0}e^{2\pi itA_0}r^{-A_0}U(r)^{-1}v(r) \\ &= U(e^{2\pi it}r)e^{2\pi itA_0}U(r)^{-1}v(r), \end{aligned} \tag{36.161}$$

and taking $t \to 1$ gives (36.159).

Suppose you have a second-order differential equation for which $z = 0$ is a regular singular point, as in (36.42). Then the conversion to the system (36.45) is accomplished by the substitution (36.44), and Proposition 36.13 bears on this system, in most cases. However, it might be preferable to use instead the substitution

$$v(z) = \begin{pmatrix} u(z) \\ u'(z) \end{pmatrix}, \tag{36.162}$$

producing a different first-order system, with a different monodromy map, though the two monodromy maps are related in an elementary fashion.

We briefly discuss the monodromy map associated to the hypergeometric equation (36.113), which is holomorphic on

$$\Omega = \mathbb{C} \setminus \{0, 1\}, \tag{36.163}$$

with regular singular points at $z = 0$ and $z = 1$. In this case, we take

$$z_0 = \frac{1}{2}, \quad \beta_0(t) = \frac{1}{2} e^{2\pi i t}, \quad \beta_1(t) = 1 - \frac{1}{2} e^{2\pi i t}. \tag{36.164}$$

We convert (36.113) to a first-order 2×2 system, using (36.162). In this case, every continuous path γ satisfying (36.142) is homotopic to a product

$$\beta_{j_1} \circ \beta_{j_2} \circ \cdots \circ \beta_{j_M}, \quad j_\nu \in \{0, 1\}. \tag{36.165}$$

(This product is not commutative!) Thus the monodromy map (36.153) is determined by

$$\varkappa(\beta_0), \ \varkappa(\beta_1) \in G\ell(2, \mathbb{C}). \tag{36.166}$$

Exercises

1. The following equation is called the Airy equation:

$$u''(z) - z u(z) = 0, \quad u(0) = u_0, \ u'(0) = u_1.$$

Show that taking $v = (u, u')^t$ yields the first-order system

$$\frac{dv}{dz} = (A_0 + A_1 z) v, \quad v(0) = v_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix},$$

with

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Using the recursion (36.14), show that the coefficients v_k in the power series

$$v(z) = \sum_{k=0}^{\infty} v_k z^k$$

satisfy

$$v_{k+3} = \frac{1}{k+3} \begin{pmatrix} \frac{1}{k+2} & 0 \\ 0 & \frac{1}{k+1} \end{pmatrix} v_k.$$

Use this to establish directly that this series converges for all $z \in \mathbb{C}$.

2. Let $v(z)$ solve the first-order system

$$z\frac{dv}{dz} = A(z)v(z), \tag{36.167}$$

on $|z| > R$, with $A(z)$ holomorphic in z in this region, and set

$$\tilde{v}(z) = v(z^{-1}), \quad 0 < |z| < R^{-1}. \tag{36.168}$$

Show that $\tilde{v}$ satisfies

$$z\frac{d\tilde{v}}{dz} = -A(z^{-1})\tilde{v}(z), \tag{36.169}$$

for $0 < |z| < R^{-1}$. We say (36.167) has a regular singular point at ∞ provided (36.169) has a regular singular point at 0. Note that this holds if and only if $\widetilde{A}(z) = A(z^{-1})$ extends to be holomorphic on a neighborhood of $z = 0$, hence if and only if $A(z)$ is bounded as $z \to \infty$.

3. Let $u(z)$ solve a second-order linear differential equation, e.g., (36.32), with coefficients that are holomorphic for $|z| > R$. We say this equation has a regular singular point at ∞ provided the first-order system, of the form (36.167), produced via (36.44)–(36.46), has a regular singular point at ∞. Show that the hypergeometric equation (36.113) has a regular singular point at ∞. Show that the confluent hypergeometric equation (36.112) does not have a regular singular point at ∞. Neither does the Bessel equation, (36.2).

4. Show that the function (36.119) solves the hypergeometric equation (36.113).

5. Demonstrate the identity (36.121) relating J_ν and ${}_1F_1$.

6. Demonstrate the identities (36.122) relating elliptic integrals and ${}_2F_1$.

7. Show that, if $b \notin \{0, -1, -2, \dots\}$,

$${}_1F_1(a; b; z) = \lim_{c \nearrow \infty} {}_2F_1(a, c; b; c^{-1}z).$$

O. From wave equations to Bessel and Legendre equations

Bessel functions, the subject of §35, arise from the natural generalization of the equation

$$\frac{d^2u}{dx^2} + k^2u = 0, \tag{O.1}$$

with solutions $\sin kx$ and $\cos kx$, to partial differential equations

$$\Delta u + k^2 u = 0, \tag{O.2}$$

where Δ is the Laplace operator, acting on a function u on a domain $\Omega \subset \mathbb{R}^n$ by

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}. \tag{O.3}$$

We can eliminate k^2 from (O.2) by scaling. Set $u(x) = v(kx)$. Then equation (O.2) becomes

$$(\Delta + 1)v = 0. \tag{O.4}$$

We specialize to the case $n = 2$ and write

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \tag{O.5}$$

For a number of special domains $\Omega \subset \mathbb{R}^2$, such as circular domains, annular domains, angular sectors, and pie-shaped domains, it is convenient to switch to polar coordinates (r, θ), related to (x, y)-coordinates by

$$x = r\cos\theta, \quad y = r\sin\theta. \tag{O.6}$$

In such coordinates,

$$\Delta v = \Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Big)v. \tag{O.7}$$

A special class of solutions to (O.4) has the form

$$v = w(r)e^{i\nu\theta}. \tag{O.8}$$

By (O.7), for such v,

$$(\Delta + 1)v = \Big[\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} + \Big(1 - \frac{\nu^2}{r^2}\Big)w\Big]e^{i\nu\theta}, \tag{O.9}$$

so (O.4) holds if and only if

$$\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} + \Bigl(1 - \frac{\nu^2}{r^2}\Bigr)w = 0. \tag{O.10}$$

This is Bessel's equation (35.1) (with different variables).

Note that if v solves (O.4) on $\Omega \subset \mathbb{R}^2$ and if Ω is a circular domain or an annular domain, centered at the origin, then ν must be an integer. However, if Ω is an angular sector or a pie-shaped domain, with vertex at the origin, ν need not be an integer.

In n dimensions, the Laplace operator (O.3) can be written

$$\Delta v = \Bigl(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_S\Bigr)v, \tag{O.11}$$

where Δ_S is a second-order differential operator acting on functions on the unit sphere $S^{n-1} \subset \mathbb{R}^n$, called the Laplace-Beltrami operator. Generalizing (O.8), one looks for solutions to (O.4) of the form

$$v(x) = w(r)\psi(\omega), \tag{O.12}$$

where $x = r\omega,\ r \in (0,\infty),\ \omega \in S^{n-1}$. Parallel to (O.9), for such v,

$$(\Delta + 1)v = \Bigl[\frac{d^2w}{dr^2} + \frac{n-1}{r}\frac{dw}{dr} + \Bigl(1 - \frac{\nu^2}{r^2}\Bigr)w\Bigr]\psi(\omega), \tag{O.13}$$

provided

$$\Delta_S\psi = -\nu^2\psi. \tag{O.14}$$

The equation

$$\frac{d^2w}{dr^2} + \frac{n-1}{r}\frac{dw}{dr} + \Bigl(1 - \frac{\nu^2}{r^2}\Bigr)w = 0 \tag{O.15}$$

is a variant of Bessel's equation. If we set

$$\varphi(r) = r^{n/2-1}w(r), \tag{O.16}$$

then (O.15) is converted into the Bessel equation

$$\frac{d^2\varphi}{dr^2} + \frac{1}{r}\frac{d\varphi}{dr} + \Bigl(1 - \frac{\mu^2}{r^2}\Bigr)\varphi = 0, \quad \mu^2 = \nu^2 + \Bigl(\frac{n-2}{2}\Bigr)^2. \tag{O.17}$$

The study of solutions to (O.14) gives rise to the study of spherical harmonics, and from there to other special functions, such as Legendre functions, more on which below.

The equation (O.2) arises from looking for solutions to the wave equation

$$\Big(\frac{\partial^2}{\partial t^2} - \Delta\Big)\varphi = 0, \tag{O.18}$$

in the form

$$\varphi(t,x) = e^{ikt}u(x). \tag{O.19}$$

Another classical partial differential equation is the heat equation, or diffusion equation,

$$\frac{\partial \varphi}{\partial t} = \Delta\varphi, \tag{O.20}$$

and if we seek purely decaying solutions, of the form

$$\varphi(t,x) = e^{-k^2 t}u(x), \tag{O.21}$$

we get, in place of (O.2), the equation

$$(\Delta - k^2)u = 0. \tag{O.22}$$

Again we can scale to take $k = 1$. This in turn gives rise to the following variant of Bessel's equation (O.10),

$$\frac{d^2w}{dr^2} + \frac{1}{r}\,\frac{dw}{dr} - \Big(1 - \frac{\nu^2}{r^2}\Big)w = 0, \tag{O.23}$$

in which 1 is changed to -1. The variants of Bessel functions that satisfy (O.23) are also considered in §35.

We return to (O.14) and look at it more closely in the case $n = 3$, so $S^{n-1} = S^2$ is the unit 2-sphere in $\mathbb{R}^3$. Standard spherical coordinates are $(\theta,\varphi) \in [0,\pi] \times (\mathbb{R}/2\pi\mathbb{Z})$, θ denoting geodesic distance to the "north pole" $(0,0,1) \in S^2 \subset \mathbb{R}^3$, and φ the angular coordinate in the (x,y)-plane. In these coordinates,

$$\Delta_S \psi = \frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\Big(\sin\theta\,\frac{\partial\psi}{\partial\theta}\Big) + \frac{1}{\sin^2\theta}\,\frac{\partial^2\psi}{\partial\varphi^2}. \tag{O.24}$$

Parallel to (O.8), we seek solutions of the form

$$\psi(\theta,\varphi) = f(\theta)e^{i\mu\varphi}. \tag{O.25}$$

Then (O.14) becomes

$$\frac{1}{\sin\theta}\,\frac{d}{d\theta}\Big(\sin\theta\,\frac{df}{d\theta}\Big) - \frac{\mu^2}{\sin^2\theta}\,f(\theta) = -\nu^2 f(\theta). \tag{O.26}$$

Since $\cos\theta$ gives the vertical coordinate of a point in $S^2 \subset \mathbb{R}^3$, it is natural to set

$$f(\theta) = g(\cos\theta). \tag{O.27}$$

Then (O.26) becomes

$$(1-z^2)g''(z) - 2zg'(z) + \Big[\nu^2 - \frac{\mu^2}{1-z^2}\Big]g(z) = 0, \tag{O.28}$$

known as Legendre's differential equation.

Appendices

In addition to various appendices scattered through Chapters 1–7, we have six "global" appendices, collected here.

In Appendix A we cover material on metric spaces and compactness, such as what one might find in a good advanced calculus course (cf. [T0] and [T]). This material applies both to subsets of the complex plane and to various sets of functions. In the latter category, we have the Arzela-Ascoli theorem, which is an important ingredient in the theory of normal families. We also have the contraction mapping theorem, of use in Appendix B.

In Appendix B we discuss the derivative of a function of several real variables and prove the inverse function theorem, in the real context, which is of use in §4 of Chapter 1 to get the inverse function theorem for holomorphic functions on domains in $\mathbb{C}$. It is also useful for the treatment of surfaces in Chapter 5.

Appendix P treats a method of analyzing an integral of the form

$$\int_{-\infty}^{\infty} e^{-t\varphi(x)} g(x)\, dx \tag{A.0.1}$$

for large t, known as the Laplace asymptotic method. This is applied here to analyze the behavior of $\Gamma(z)$ for large z (Stirling's formula). Also, in §35 of Chapter 7, this method is applied to analyze the behavior of Bessel functions for large argument.

Appendix M provides some basic results on the Stieltjes integral

$$\int_a^b f(x)\, du(x). \tag{A.0.2}$$

We assume that $f \in C([a,b])$ and $u : [a,b] \to \mathbb{R}$ is increasing. Possibly $b = \infty$, and then there are restrictions on the behavior of f and u at infinity. The Stieltjes integral provides a convenient language to use to relate functions that count primes to the Riemann zeta function, and we make use of it in §19 of Chapter 4. It also provides a convenient setting for the material in Appendix R.

Appendix R deals with Abelian theorems and Tauberian theorems. These are results to the effect that one sort of convergence implies another. In a certain sense, Tauberian theorems are partial converses to Abelian theorems. One source for such results is the following: in many proofs of the prime number theorem, including the one given in §19 of Chapter 4, the last step involves using a Tauberian theorem. The particular Tauberian theorem needed to end the analysis in §19 is given a short proof in Appendix R, as a consequence of a result of broad general use known as Karamata's Tauberian theorem.

In Appendix Q we show how the formula

$$\sin 3z = -4\sin^3 z + 3\sin z \tag{A.0.3}$$

enables one to solve cubic equations, and move on to seek formulas for solutions to quartic equations and quintic equations. In the latter case this cannot necessarily be done in terms of radicals, and this appendix introduces a special function, called the Bring radical, to treat quintic equations.

A. Metric spaces, convergence, and compactness

A metric space is a set X, together with a distance function $d : X \times X \to [0, \infty)$, having the properties that

$$
\begin{aligned}
&d(x, y) = 0 \Longleftrightarrow x = y, \\
&d(x, y) = d(y, x), \\
&d(x, y) \le d(x, z) + d(y, z).
\end{aligned}
\tag{A.1}
$$

The third of these properties is called the triangle inequality. An example of a metric space is the set of rational numbers $\mathbb{Q}$, with $d(x, y) = |x - y|$. Another example is $X = \mathbb{R}^n$, with

$$
d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.
$$

If (x_ν) is a sequence in X, indexed by $\nu = 1, 2, 3, \dots$, i.e., by $\nu \in \mathbb{Z}^+$, one says $x_\nu \to y$ if $d(x_\nu, y) \to 0$, as $\nu \to \infty$. One says (x_ν) is a Cauchy sequence if $d(x_\nu, x_\mu) \to 0$ as $\mu, \nu \to \infty$. One says X is a *complete* metric space if every Cauchy sequence converges to a limit in X. Some metric spaces are not complete; for example, $\mathbb{Q}$ is not complete. You can take a sequence (x_ν) of rational numbers such that $x_\nu \to \sqrt{2}$, which is not rational. Then (x_ν) is Cauchy in $\mathbb{Q}$, but it has no limit in $\mathbb{Q}$.

If a metric space X is not complete, one can construct its completion $\widehat{X}$ as follows. Let an element ξ of $\widehat{X}$ consist of an *equivalence class* of Cauchy sequences in X, where we say $(x_\nu) \sim (y_\nu)$ provided $d(x_\nu, y_\nu) \to 0$. We write the equivalence class containing (x_ν) as $[x_\nu]$. If $\xi = [x_\nu]$ and $\eta = [y_\nu]$, we can set $d(\xi, \eta) = \lim_{\nu\to\infty} d(x_\nu, y_\nu)$, and verify that this is well defined, and makes $\widehat{X}$ a complete metric space.

If the completion of $\mathbb{Q}$ is constructed by this process, you get $\mathbb{R}$, the set of real numbers. This construction provides a good way to develop the basic theory of the real numbers. A detailed construction of $\mathbb{R}$ using this method is given in Chapter 1 of [T0].

There are a number of useful concepts related to the notion of closeness. We define some of them here. First, if p is a point in a metric space X and $r \in (0, \infty)$, the set

$$
B_r(p) = \{x \in X : d(x, p) < r\}
\tag{A.2}
$$

is called the open ball about p of radius r. Generally, a *neighborhood* of $p \in X$ is a set containing such a ball, for some $r > 0$.

A set $U \subset X$ is called *open* if it contains a neighborhood of each of its points. The complement of an open set is said to be *closed.* The following result characterizes closed sets.

Proposition A.1. *A subset $K \subset X$ of a metric space X is closed if and only if*

$$
x_j \in K,\ x_j \to p \in X \Longrightarrow p \in K.
\tag{A.3}
$$

Proof. Assume K is closed, $x_j \in K$, $x_j \to p$. If $p \notin K$, then $p \in X \setminus K$, which is open, so some $B_\varepsilon(p) \subset X \setminus K$, and $d(x_j, p) \geq \varepsilon$ for all j. This contradiction implies $p \in K$.

Conversely, assume (A.3) holds, and let $q \in U = X \setminus K$. If $B_{1/n}(q)$ is not contained in U for any n, then there exists $x_n \in K \cap B_{1/n}(q)$, hence $x_n \to q$, contradicting (A.3). This completes the proof.

The following is straightforward.

Proposition A.2. *If U_α is a family of open sets in X, then $\cup_\alpha U_\alpha$ is open. If K_α is a family of closed subsets of X, then $\cap_\alpha K_\alpha$ is closed.*

Given $S \subset X$, we denote by $\overline{S}$ (the *closure* of S) the smallest closed subset of X containing S, i.e., the intersection of all the closed sets $K_\alpha \subset X$ containing S. The following result is straightforward.

Proposition A.3. *Given $S \subset X$, $p \in \overline{S}$ if and only if there exist $x_j \in S$ such that $x_j \to p$.*

Given $S \subset X$, $p \in X$, we say p is an *accumulation point* of S if and only if, for each $\varepsilon > 0$, there exists $q \in S \cap B_\varepsilon(p)$, $q \neq p$. It follows that p is an accumulation point of S if and only if each $B_\varepsilon(p)$, $\varepsilon > 0$, contains infinitely many points of S. One straightforward observation is that all points of $\overline{S} \setminus S$ are accumulation points of S.

The *interior* of a set $S \subset X$ is the largest open set contained in S, i.e., the union of all the open sets contained in S. Note that the complement of the interior of S is equal to the closure of $X \setminus S$.

We now turn to the notion of compactness. We say a metric space X is *compact* provided the following property holds:

(A) Each sequence (x_k) in X has a convergent subsequence.

We will establish various properties of compact metric spaces, and provide various equivalent characterizations. For example, it is easily seen that (A) is equivalent to:

(B) Each infinite subset $S \subset X$ has an accumulation point.

The following property is known as total boundedness:

Proposition A.4. *If X is a compact metric space, then*

(C) *Given* $\varepsilon > 0$, $\exists$ *finite set* $\{x_1, \dots, x_N\}$ *such that* $B_\varepsilon(x_1), \dots, B_\varepsilon(x_N)$ *covers* X.

Proof. Take $\varepsilon > 0$ and pick $x_1 \in X$. If $B_\varepsilon(x_1) = X$, we are done. If not, pick $x_2 \in X \setminus B_\varepsilon(x_1)$. If $B_\varepsilon(x_1) \cup B_\varepsilon(x_2) = X$, we are done. If not, pick $x_3 \in X \setminus [B_\varepsilon(x_1) \cup B_\varepsilon(x_2)]$. Continue, taking $x_{k+1} \in X \setminus [B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_k)]$, if $B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_k) \neq X$. Note that, for $1 \leq i, j \leq k$,

$$i \neq j \Longrightarrow d(x_i, x_j) \geq \varepsilon.$$

If one never covers X this way, consider $S = \{x_j : j \in \mathbb{N}\}$. This is an infinite set with no accumulation point, so property (B) is contradicted.

Corollary A.5. *If X is a compact metric space, it has a countable dense subset.*

Proof. Given $\varepsilon = 2^{-n}$, let S_n be a finite set of points x_j such that $\{B_\varepsilon(x_j)\}$ covers X. Then $\mathcal{C} = \cup_n S_n$ is a countable dense subset of X.

Here is another useful property of compact metric spaces, which will eventually be generalized even further, in (E) below.

Proposition A.6. *Let X be a compact metric space. Assume $K_1 \supset K_2 \supset K_3 \supset \cdots$ form a decreasing sequence of closed subsets of X. If each $K_n \neq \emptyset$, then $\cap_n K_n \neq \emptyset$.*

Proof. Pick $x_n \in K_n$. If (A) holds, (x_n) has a convergent subsequence, $x_{n_k} \to y$. Since $\{x_{n_k} : k \geq \ell\} \subset K_{n_\ell}$, which is closed, we have $y \in \cap_n K_n$.

Corollary A.7. *Let X be a compact metric space. Assume $U_1 \subset U_2 \subset U_3 \subset \cdots$ form an increasing sequence of open subsets of X. If $\cup_n U_n = X$, then $U_N = X$ for some N.*

Proof. Consider $K_n = X \setminus U_n$.

The following is an important extension of Corollary A.7.

Proposition A.8. *If X is a compact metric space, then it has the property:*

(D) *Every open cover $\{U_\alpha : \alpha \in \mathcal{A}\}$ of X has a finite subcover.*

Proof. Each U_α is a union of open balls, so it suffices to show that (A) implies the following:

(D') Every cover $\{B_\alpha : \alpha \in \mathcal{A}\}$ of X by open balls has a finite subcover.

Let $\mathcal{C} = \{z_j : j \in \mathbb{N}\} \subset X$ be a countable dense subset of X, as in Corollary A.2. Each B_α is a union of balls $B_{r_j}(z_j)$, with $z_j \in \mathcal{C} \cap B_\alpha$, r_j rational. Thus it suffices to show that

(D") Every countable cover $\{B_j : j \in \mathbb{N}\}$ of X by open balls has a finite subcover.

For this, we set

$$U_n = B_1 \cup \cdots \cup B_n$$

and apply Corollary A.7.

The following is a convenient alternative to property (D):

(E) If $K_\alpha \subset X$ are closed and $\bigcap_\alpha K_\alpha = \emptyset$, then some finite intersection is empty.

Considering $U_\alpha = X \setminus K_\alpha$, we see that

$$(D) \Longleftrightarrow (E).$$

The following result completes Proposition A.8.

Theorem A.9. *For a metric space X,*

$$(A) \Longleftrightarrow (D).$$

Proof. By Proposition A.8, $(A) \Rightarrow (D)$. To prove the converse, it will suffice to show that $(E) \Rightarrow (B)$. So let $S \subset X$ and assume S has no accumulation point. We claim:

$$\text{Such } \ S \ \text{ must be closed.}$$

Indeed, if $z \in \overline{S}$ and $z \notin S$, then z would have to be an accumulation point. Say $S = \{x_\alpha : \alpha \in \mathcal{A}\}$. Set $K_\alpha = S \setminus \{x_\alpha\}$. Then each K_α has no accumulation point, hence $K_\alpha \subset X$ is closed. Also $\cap_\alpha K_\alpha = \emptyset$. Hence there exists a finite set $\mathcal{F} \subset \mathcal{A}$ such that $\cap_{\alpha \in \mathcal{F}} K_\alpha = \emptyset$, if (E) holds. Hence $S = \cup_{\alpha \in \mathcal{F}} \{x_\alpha\}$ is finite, so indeed $(E) \Rightarrow (B)$.

Remark. So far we have that for every metric space X,

$$(A) \Longleftrightarrow (B) \Longleftrightarrow (D) \Longleftrightarrow (E) \Longrightarrow (C).$$

We claim that (C) implies the other conditions if X is *complete*. Of course, compactness implies completeness, but (C) may hold for incomplete X, e.g., $X = (0, 1) \subset \mathbb{R}$.

Proposition A.10. *If X is a complete metric space with property (C), then X is compact.*

Proof. It suffices to show that $(C) \Rightarrow (B)$ if X is a complete metric space. So let $S \subset X$ be an infinite set. Cover X by balls $B_{1/2}(x_1), \dots, B_{1/2}(x_N)$. One of these balls contains infinitely many points of S, and so does its closure, say $X_1 = \overline{B_{1/2}(y_1)}$. Now cover X by finitely many balls of radius $1/4$; their intersection with X_1 provides a cover of X_1. One such set contains infinitely many points of S, and so does its closure $X_2 = \overline{B_{1/4}(y_2)} \cap X_1$. Continue in this fashion, obtaining

$$X_1 \supset X_2 \supset X_3 \supset \cdots \supset X_k \supset X_{k+1} \supset \cdots, \quad X_j \subset \overline{B_{2^{-j}}(y_j)},$$

each containing infinitely many points of S. One sees that (y_j) forms a Cauchy sequence. If X is complete, it has a limit, $y_j \to z$, and z is seen to be an accumulation point of S.

If X_j, $1 \le j \le m$, is a finite collection of metric spaces, with metrics d_j, we can define a Cartesian product metric space

$$X = \prod_{j=1}^{m} X_j, \quad d(x, y) = d_1(x_1, y_1) + \cdots + d_m(x_m, y_m). \tag{A.4}$$

Another choice of metric is $\delta(x, y) = \sqrt{d_1(x_1, y_1)^2 + \cdots + d_m(x_m, y_m)^2}$. The metrics d and δ are *equivalent*, i.e., there exist constants $C_0, C_1 \in (0, \infty)$ such that

$$C_0 \delta(x, y) \le d(x, y) \le C_1 \delta(x, y), \quad \forall\, x, y \in X. \tag{A.5}$$

A key example is $\mathbb{R}^m$, the Cartesian product of m copies of the real line $\mathbb{R}$.

We describe some important classes of compact spaces.

Proposition A.11. *If X_j are compact metric spaces, $1 \le j \le m$, so is $X = \prod_{j=1}^m X_j$.*

Proof. If (x_ν) is an infinite sequence of points in X, say $x_\nu = (x_{1\nu}, \dots, x_{m\nu})$, pick a convergent subsequence of $(x_{1\nu})$ in X_1, and consider the corresponding subsequence of (x_ν), which we relabel (x_ν). Using this, pick a convergent subsequence of $(x_{2\nu})$ in X_2. Continue. Having a subsequence such that $x_{j\nu} \to y_j$ in X_j for each $j = 1, \dots, m$, we then have a convergent subsequence in X.

The following result is useful for calculus on $\mathbb{R}^n$.

Proposition A.12. *If K is a closed bounded subset of $\mathbb{R}^n$, then K is compact.*

Proof. The discussion above reduces the problem to showing that any closed interval $I = [a, b]$ in $\mathbb{R}$ is compact. This compactness is a corollary of Proposition A.10. For pedagogical purposes, we redo the argument here, since in this concrete case it can be streamlined.

Suppose S is a subset of I with infinitely many elements. Divide I into 2 equal subintervals, $I_1 = [a, b_1]$, $I_2 = [b_1, b]$, $b_1 = (a+b)/2$. Then either I_1 or I_2 must contain infinitely many elements of S. Say I_j does. Let x_1 be any element of S lying in I_j. Now divide I_j in two equal pieces, $I_j = I_{j1} \cup I_{j2}$. One of these intervals (say I_{jk}) contains infinitely many points of S. Pick $x_2 \in I_{jk}$ to be one such point (different from x_1). Then subdivide I_{jk} into two equal subintervals, and continue. We get an infinite sequence of distinct points $x_\nu \in S$, and $|x_\nu - x_{\nu+k}| \le 2^{-\nu}(b-a)$, for $k \ge 1$. Since $\mathbb{R}$ is complete, (x_ν) converges, say to $y \in I$. Any neighborhood of y contains infinitely many points in S, so we are done.

If X and Y are metric spaces, a function $f : X \to Y$ is said to be continuous provided $x_\nu \to x$ in X implies $f(x_\nu) \to f(x)$ in Y. An equivalent condition, which the reader is invited to verify, is

$$U \text{ open in } Y \Longrightarrow f^{-1}(U) \text{ open in } X. \tag{A.6}$$

Proposition A.13. *If X and Y are metric spaces, $f : X \to Y$ continuous, and $K \subset X$ compact, then $f(K)$ is a compact subset of Y.*

Proof. If (y_ν) is an infinite sequence of points in $f(K)$, pick $x_\nu \in K$ such that $f(x_\nu) = y_\nu$. If K is compact, we have a subsequence $x_{\nu_j} \to p$ in X, and then $y_{\nu_j} \to f(p)$ in Y.

If $F : X \to \mathbb{R}$ is continuous, we say $f \in C(X)$. A useful corollary of Proposition A.13 is:

Proposition A.14. *If X is a compact metric space and $f \in C(X)$, then f assumes a maximum and a minimum value on X.*

Proof. We know from Proposition A.13 that $f(X)$ is a compact subset of $\mathbb{R}$. Hence $f(X)$ is bounded, say $f(X) \subset I = [a, b]$. Repeatedly subdividing I into equal halves, as in the proof of Proposition A.12, at each stage throwing out intervals that do not intersect $f(X)$, and keeping only the leftmost and rightmost interval amongst those remaining, we obtain points $\alpha \in f(X)$ and $\beta \in f(X)$ such that $f(X) \subset [\alpha, \beta]$. Then $\alpha = f(x_0)$ for some $x_0 \in X$ is the minimum and $\beta = f(x_1)$ for some $x_1 \in X$ is the maximum.

If $S \subset \mathbb{R}$ is a nonempty, bounded set, Proposition A.12 implies $\overline{S}$ is compact. The function $\eta : \overline{S} \to \mathbb{R}$, $\eta(x) = x$ is continuous, so by Proposition A.14 it assumes a maximum and a minimum on $\overline{S}$. We set

$$\text{(A.7)} \qquad \sup S = \max_{s \in \overline{S}} x, \quad \inf S = \min_{x \in \overline{S}} x,$$

when S is bounded. More generally, if $S \subset \mathbb{R}$ is nonempty and bounded from above, say $S \subset (-\infty, B]$, we can pick $A < B$ such that $S \cap [A, B]$ is nonempty, and set

$$\text{(A.8)} \qquad \sup S = \sup S \cap [A, B].$$

Similarly, if $S \subset \mathbb{R}$ is nonempty and bounded from below, say $S \subset [A, \infty)$, we can pick $B > A$ such that $S \cap [A, B]$ is nonempty, and set

$$\text{(A.9)} \qquad \inf S = \inf S \cap [A, B].$$

If X is a nonempty set and $f : X \to \mathbb{R}$ is bounded from above, we set

$$\text{(A.10)} \qquad \sup_{x \in X} f(x) = \sup f(X),$$

and if $f : X \to \mathbb{R}$ is bounded from below, we set

$$\text{(A.11)} \qquad \inf_{x \in X} f(x) = \inf f(X).$$

If f is not bounded from above, we set $\sup f = +\infty$, and if f is not bounded from below, we set $\inf f = -\infty$.

Given a set X, $f : X \to \mathbb{R}$, and $x_n \to x$, we set

$$\text{(A.11A)} \qquad \limsup_{n \to \infty} f(x_n) = \lim_{n \to \infty} \Bigl(\sup_{k \ge n} f(x_k) \Bigr),$$

and

$$\text{(A.11B)} \qquad \liminf_{n \to \infty} f(x_n) = \lim_{n \to \infty} \Bigl(\inf_{k \ge n} f(x_k) \Bigr).$$

We return to the notion of continuity. A function $f \in C(X)$ is said to be *uniformly continuous* provided that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{(A.12)} \qquad x, y \in X, \ d(x, y) \le \delta \Longrightarrow |f(x) - f(y)| \le \varepsilon.$$

An equivalent condition is that f have a *modulus of continuity*, i.e., a monotonic function $\omega : [0, 1) \to [0, \infty)$ such that $\delta \searrow 0 \Rightarrow \omega(\delta) \searrow 0$, and such that

$$\text{(A.13)} \qquad x, y \in X, \ d(x, y) \le \delta \le 1 \Longrightarrow |f(x) - f(y)| \le \omega(\delta).$$

Not all continuous functions are uniformly continuous. For example, if $X = (0, 1) \subset \mathbb{R}$, then $f(x) = \sin 1/x$ is continuous, but not uniformly continuous, on X. The following result is useful, for example, in the development of the Riemann integral.

Proposition A.15. *If X is a compact metric space and $f \in C(X)$, then f is uniformly continuous.*

Proof. If not, there exist $x_\nu, y_\nu \in X$ and $\varepsilon > 0$ such that $d(x_\nu, y_\nu) \le 2^{-\nu}$ but

$$|f(x_\nu) - f(y_\nu)| \ge \varepsilon. \tag{A.14}$$

Taking a convergent subsequence $x_{\nu_j} \to p$, we also have $y_{\nu_j} \to p$. Now continuity of f at p implies $f(x_{\nu_j}) \to f(p)$ and $f(y_{\nu_j}) \to f(p)$, contradicting (A.14).

If X and Y are metric spaces, the space $C(X, Y)$ of continuous maps $f : X \to Y$ has a natural metric structure, under some additional hypotheses. We use

$$D(f, g) = \sup_{x \in X} d\big(f(x), g(x)\big). \tag{A.15}$$

This sup exists provided $f(X)$ and $g(X)$ are *bounded* subsets of Y, where to say $B \subset Y$ is bounded is to say $d : B \times B \to [0, \infty)$ has bounded image. In particular, this supremum exists if X is compact. The following is a natural completeness result.

Proposition A.16. *If X is a compact metric space and Y is a complete metric space, then $C(X, Y)$, with the metric (A.9), is complete.*

Proof. That $D(f, g)$ satisfies the conditions to define a metric on $C(X, Y)$ is straightforward. We check completeness. Suppose (f_ν) is a Cauchy sequence in $C(X, Y)$, so, as $\nu \to \infty$,

$$\sup_{k \ge 0} \sup_{x \in X} d\big(f_{\nu+k}(x), f_\nu(x)\big) \le \varepsilon_\nu \to 0. \tag{A.16}$$

Then in particular $(f_\nu(x))$ is a Cauchy sequence in Y for each $x \in X$, so it converges, say to $g(x) \in Y$. It remains to show that $g \in C(X, Y)$ and that $f_\nu \to g$ in the metric (A.9).

In fact, taking $k \to \infty$ in the estimate above, we have

$$\sup_{x \in X} d\big(g(x), f_\nu(x)\big) \le \varepsilon_\nu \to 0, \tag{A.17}$$

i.e., $f_\nu \to g$ uniformly. It remains only to show that g is continuous. For this, let $x_j \to x$ in X and fix $\varepsilon > 0$. Pick N so that $\varepsilon_N < \varepsilon$. Since f_N is continuous, there exists J such that $j \ge J \Rightarrow d(f_N(x_j), f_N(x)) < \varepsilon$. Hence

$$j \ge J \Rightarrow d\big(g(x_j), g(x)\big) \le d\big(g(x_j), f_N(x_j)\big) + d\big(f_N(x_j), f_N(x)\big) + d\big(f_N(x), g(x)\big) < 3\varepsilon.$$

This completes the proof.

In case $Y = \mathbb{R}$, $C(X, \mathbb{R}) = C(X)$, introduced earlier in this appendix. The distance function (A.15) can be written

$$D(f, g) = \|f - g\|_{\sup}, \quad \|f\|_{\sup} = \sup_{x \in X} |f(x)|.$$

$\|f\|_{\sup}$ is a *norm* on $C(X)$.

Generally, a norm on a vector space V is an assignment $f \mapsto \|f\| \in [0,\infty)$, satisfying

$$\|f\| = 0 \Leftrightarrow f = 0, \quad \|af\| = |a|\,\|f\|, \quad \|f+g\| \le \|f\| + \|g\|,$$

given $f, g \in V$ and a a scalar (in $\mathbb{R}$ or $\mathbb{C}$). A vector space equipped with a norm is called a normed vector space. It is then a metric space, with distance function $D(f,g) = \|f-g\|$. If the space is complete, one calls V a *Banach space.*

In particular, by Proposition A.16, $C(X)$ is a Banach space, when X is a compact metric space.

We next give a couple of slightly more sophisticated results on compactness. The following extension of Proposition A.11 is a special case of Tychonov's Theorem.

Proposition A.17. *If $\{X_j : j \in \mathbb{Z}^+\}$ are compact metric spaces, so is $X = \prod_{j=1}^{\infty} X_j$.*

Here, we can make X a metric space by setting

$$d(x,y) = \sum_{j=1}^{\infty} 2^{-j} \frac{d_j(p_j(x), p_j(y))}{1 + d_j(p_j(x), p_j(y))}, \tag{A.18}$$

where $p_j : X \to X_j$ is the projection onto the jth factor. It is easy to verify that, if $x_\nu \in X$, then $x_\nu \to y$ in X, as $\nu \to \infty$, if and only if, for each j, $p_j(x_\nu) \to p_j(y)$ in X_j.

Proof. Following the argument in Proposition A.11, if (x_ν) is an infinite sequence of points in X, we obtain a nested family of subsequences

$$(x_\nu) \supset (x^1{}_\nu) \supset (x^2{}_\nu) \supset \cdots \supset (x^j{}_\nu) \supset \cdots \tag{A.19}$$

such that $p_\ell(x^j{}_\nu)$ converges in X_ℓ, for $1 \le \ell \le j$. The next step is a *diagonal construction.* We set

$$\xi_\nu = x^\nu{}_\nu \in X. \tag{A.20}$$

Then, for each j, after throwing away a finite number $N(j)$ of elements, one obtains from (ξ_ν) a subsequence of the sequence $(x^j{}_\nu)$ in (A.19), so $p_\ell(\xi_\nu)$ converges in X_ℓ for all ℓ. Hence (ξ_ν) is a convergent subsequence of (x_ν).

The next result is known as the Arzela-Ascoli Theorem. It is useful in the theory of normal families, developed in §21.

Proposition A.18. *Let X and Y be compact metric spaces, and fix a modulus of continuity $\omega(\delta)$. Then*

$$\mathcal{C}_\omega = \bigl\{ f \in C(X,Y) : d\bigl(f(x), f(x')\bigr) \le \omega\bigl(d(x,x')\bigr)\ \forall\, x, x' \in X \bigr\} \tag{A.21}$$

is a compact subset of $C(X,Y)$.

Proof. Let (f_ν) be a sequence in $\mathcal{C}_\omega$. Let Σ be a countable dense subset of X, as in Corollary A.5. For each $x \in \Sigma$, $(f_\nu(x))$ is a sequence in Y, which hence has a convergent subsequence. Using a diagonal construction similar to that in the proof of Proposition A.17, we obtain a subsequence (φ_ν) of (f_ν) with the property that $\varphi_\nu(x)$ converges in Y, for *each* $x \in \Sigma$, say

$$\varphi_\nu(x) \to \psi(x), \tag{A.22}$$

for all $x \in \Sigma$, where $\psi : \Sigma \to Y$.

So far, we have not used (A.21). This hypothesis will now be used to show that φ_ν converges uniformly on X. Pick $\varepsilon > 0$. Then pick $\delta > 0$ such that $\omega(\delta) < \varepsilon/3$. Since X is compact, we can cover X by finitely many balls $B_\delta(x_j)$, $1 \le j \le N$, $x_j \in \Sigma$. Pick M so large that $\varphi_\nu(x_j)$ is within $\varepsilon/3$ of its limit for all $\nu \ge M$ (when $1 \le j \le N$). Now, for any $x \in X$, picking $\ell \in \{1, \dots, N\}$ such that $d(x, x_\ell) \le \delta$, we have, for $k \ge 0$, $\nu \ge M$,

$$\begin{aligned} d\big(\varphi_{\nu+k}(x), \varphi_\nu(x)\big) &\le d\big(\varphi_{\nu+k}(x), \varphi_{\nu+k}(x_\ell)\big) + d\big(\varphi_{\nu+k}(x_\ell), \varphi_\nu(x_\ell)\big) \\ &\qquad + d\big(\varphi_\nu(x_\ell), \varphi_\nu(x)\big) \\ &\le \varepsilon/3 + \varepsilon/3 + \varepsilon/3. \end{aligned} \tag{A.23}$$

Thus $(\varphi_\nu(x))$ is Cauchy in Y for all $x \in X$, hence convergent. Call the limit $\psi(x)$, so we now have (A.22) for all $x \in X$. Letting $k \to \infty$ in (A.23) we have uniform convergence of φ_ν to ψ. Finally, passing to the limit $\nu \to \infty$ in

$$d(\varphi_\nu(x), \varphi_\nu(x')) \le \omega(d(x, x')) \tag{A.24}$$

gives $\psi \in \mathcal{C}_\omega$.

We want to re-state Proposition A.18, bringing in the notion of *equicontinuity*. Given metric spaces X and Y, and a set of maps $\mathcal{F} \subset C(X, Y)$, we say $\mathcal{F}$ is equicontinuous at a point $x_0 \in X$ provided

$$\begin{aligned} &\forall\, \varepsilon > 0,\ \exists\, \delta > 0 \text{ such that } \forall\, x \in X,\ f \in \mathcal{F}, \\ &d_X(x, x_0) < \delta \Longrightarrow d_Y(f(x), f(x_0)) < \varepsilon. \end{aligned} \tag{A.25}$$

We say $\mathcal{F}$ is equicontinuous on X if it is equicontinuous at each point of X. We say $\mathcal{F}$ is *uniformly equicontinuous* on X provided

$$\begin{aligned} &\forall\, \varepsilon > 0,\ \exists\, \delta > 0 \text{ such that } \forall\, x, x' \in X,\ f \in \mathcal{F}, \\ &d_X(x, x') < \delta \Longrightarrow d_Y(f(x), f(x')) < \varepsilon. \end{aligned} \tag{A.26}$$

Note that (A.26) is equivalent to the existence of a modulus of continuity ω such that $\mathcal{F} \subset \mathcal{C}_\omega$, given by (A.21). It is useful to record the following result.

Proposition A.19. *Let X and Y be metric spaces, $\mathcal{F} \subset C(X,Y)$. Assume X is compact. then*

$$\mathcal{F} \text{ equicontinuous} \Longrightarrow \mathcal{F} \text{ is uniformly equicontinuous.} \tag{A.27}$$

Proof. The argument is a variant of the proof of Proposition A.15. In more detail, suppose there exist $x_\nu, x'_\nu \in X,\ \varepsilon > 0$, and $f_\nu \in \mathcal{F}$ such that $d(x_\nu, x'_\nu) \le 2^{-\nu}$ but

$$d(f_\nu(x_\nu), f_\nu(x'_\nu)) \ge \varepsilon. \tag{A.28}$$

Taking a convergent subsequence $x_{\nu_j} \to p \in X$, we also have $x'_{\nu_j} \to p$. Now equicontinuity of $\mathcal{F}$ at p implies that there esists $N < \infty$ such that

$$d(g(x_{\nu_j}), g(p)) < \frac{\varepsilon}{2}, \quad \forall\, j \ge N,\ g \in \mathcal{F}, \tag{A.29}$$

contradicting (A.28).

Putting together Propositions A.18 and A.19 then gives the following.

Proposition A.20. *Let X and Y be compact metric spaces. If $\mathcal{F} \subset C(X,Y)$ is equicontinuous on X, then it has compact closure in $C(X,Y)$.*

We next define the notion of a *connected* space. A metric space X is said to be connected provided that it cannot be written as the union of two disjoint nonempty open subsets. The following is a basic class of examples.

Proposition A.21. *Each interval I in $\mathbb{R}$ is connected.*

Proof. Suppose $A \subset I$ is nonempty, with nonempty complement $B \subset I$, and both sets are open. Take $a \in A,\ b \in B$; we can assume $a < b$. Let $\xi = \sup\{x \in [a,b] : x \in A\}$ This exists, as a consequence of the basic fact that $\mathbb{R}$ is complete.

Now we obtain a contradiction, as follows. Since A is closed $\xi \in A$. But then, since A is open, there must be a neighborhood $(\xi - \varepsilon, \xi + \varepsilon)$ contained in A; this is not possible.

We say X is path-connected if, given any $p, q \in X$, there is a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$. It is an easy consequence of Proposition A.21 that X is connected whenever it is path-connected.

The next result, known as the Intermediate Value Theorem, is frequently useful.

Proposition A.22. *Let X be a connected metric space and $f : X \to \mathbb{R}$ continuous. Assume $p, q \in X$, and $f(p) = a < f(q) = b$. Then, given any $c \in (a,b)$, there exists $z \in X$ such that $f(z) = c$.*

Proof. Under the hypotheses, $A = \{x \in X : f(x) < c\}$ is open and contains p, while $B = \{x \in X : f(x) > c\}$ is open and contains q. If X is connected, then $A \cup B$ cannot be all of X; so any point in its complement has the desired property.

The next result is known as the Contraction Mapping Principle, and it has many uses in analysis. In particular, we will use it in the proof of the Inverse Function Theorem, in Appendix B.

Theorem A.23. *Let X be a complete metric space, and let $T : X \to X$ satisfy*

$$d(Tx, Ty) \le r\, d(x, y), \tag{A.30}$$

for some $r < 1$. (We say T is a contraction.) Then T has a unique fixed point x. For any $y_0 \in X$, $T^k y_0 \to x$ as $k \to \infty$.

Proof. Pick $y_0 \in X$ and let $y_k = T^k y_0$. Then $d(y_k, y_{k+1}) \le r^k\, d(y_0, y_1)$, so

$$\begin{aligned} d(y_k, y_{k+m}) &\le d(y_k, y_{k+1}) + \cdots + d(y_{k+m-1}, y_{k+m}) \\ &\le (r^k + \cdots + r^{k+m-1})\, d(y_0, y_1) \\ &\le r^k (1-r)^{-1}\, d(y_0, y_1). \end{aligned} \tag{A.31}$$

It follows that (y_k) is a Cauchy sequence, so it converges; $y_k \to x$. Since $Ty_k = y_{k+1}$ and T is continuous, it follows that $Tx = x$, i.e., x is a fixed point. Uniqueness of the fixed point is clear from the estimate $d(Tx.Tx') \le r\, d(x, x')$, which implies $d(x, x') = 0$ if x and x' are fixed points. This proves Theorem A.23.

Exercises

1. If X is a metric space, with distance function d, show that

$$|d(x, y) - d(x', y')| \le d(x, x') + d(y, y'),$$

and hence

$$d : X \times X \longrightarrow [0, \infty) \ \text{ is continuous.}$$

2. Let $\varphi : [0, \infty) \to [0, \infty)$ be a C^2 function. Assume

$$\varphi(0) = 0, \quad \varphi' > 0, \quad \varphi'' < 0.$$

Prove that if $d(x, y)$ is symmetric and satisfies the triangle inequality, so does

$$\delta(x, y) = \varphi(d(x, y)).$$

Hint. Show that such φ satisfies $\varphi(s + t) \le \varphi(s) + \varphi(t)$, for $s, t \in \mathbb{R}^+$.

3. Show that the function $d(x, y)$ defined by (A.18) satisfies (A.1).
Hint. Consider $\varphi(r) = r/(1 + r)$.

4. Let X be a compact metric space. Assume $f_j, f \in C(X)$ and

$$f_j(x) \nearrow f(x), \quad \forall\, x \in X.$$

Prove that $f_j \to f$ uniformly on X. (This result is called Dini's theorem.)
Hint. For $\varepsilon > 0$, let $K_j(\varepsilon) = \{x \in X : f(x) - f_j(x) \geq \varepsilon\}$. Note that $K_j(\varepsilon) \supset K_{j+1}(\varepsilon) \supset \cdots$.

5. In the setting of (A.4), let

$$\delta(x, y) = \Big\{ d_1(x_1, y_1)^2 + \cdots + d_m(x_m, y_m)^2 \Big\}^{1/2}.$$

Show that

$$\delta(x, y) \leq d(x, y) \leq \sqrt{m}\, \delta(x, y).$$

6. Let X and Y be compact metric spaces. Show that if $\mathcal{F} \subset C(X, Y)$ is compact, then $\mathcal{F}$ is equicontinuous. (This is a converse to Proposition A.20.)

7. Recall that a Banach space is a complete normed linear space. Consider $C^1(I)$, where $I = [0, 1]$, with norm

$$\|f\|_{C^1} = \sup_I |f| + \sup_I |f'|.$$

Show that $C^1(I)$ is a Banach space.

8. Let $\mathcal{F} = \{f \in C^1(I) : \|f\|_{C^1} \leq 1\}$. Show that $\mathcal{F}$ has compact closure in $C(I)$. Find a function in the closure of $\mathcal{F}$ that is not in $C^1(I)$.

B. Derivatives and diffeomorphisms

To start this section off, we define the derivative and discuss some of its basic properties. Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$, and $F : \mathcal{O} \to \mathbb{R}^m$ a continuous function. We say F is differentiable at a point $x \in \mathcal{O}$, with derivative L, if $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation such that, for $y \in \mathbb{R}^n$, small,

$$F(x+y) = F(x) + Ly + R(x,y) \tag{B.1}$$

with

$$\frac{\|R(x,y)\|}{\|y\|} \to 0 \quad \text{as} \quad y \to 0. \tag{B.2}$$

We denote the derivative at x by $DF(x) = L$. With respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, $DF(x)$ is simply the matrix of partial derivatives,

$$DF(x) = \left(\frac{\partial F_j}{\partial x_k}\right), \tag{B.3}$$

so that, if $v = (v_1, \dots, v_n)^t$, (regarded as a column vector) then

$$DF(x)v = \Bigl(\sum_k \frac{\partial F_1}{\partial x_k} v_k, \dots, \sum_k \frac{\partial F_m}{\partial x_k} v_k\Bigr)^t. \tag{B.4}$$

It will be shown below that F is differentiable whenever all the partial derivatives exist and are *continuous* on $\mathcal{O}$. In such a case we say F is a C^1 function on $\mathcal{O}$. More generally, F is said to be C^k if all its partial derivatives of order $\le k$ exist and are continuous. If F is C^k for all k, we say F is C^∞.

In (B.2), we can use the *Euclidean* norm on $\mathbb{R}^n$ and $\mathbb{R}^m$. This norm is defined by

$$\|x\| = \bigl(x_1^2 + \cdots + x_n^2\bigr)^{1/2} \tag{B.5}$$

for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Any other norm would do equally well.

We now derive the *chain rule* for the derivative. Let $F : \mathcal{O} \to \mathbb{R}^m$ be differentiable at $x \in \mathcal{O}$, as above, let U be a neighborhood of $z = F(x)$ in $\mathbb{R}^m$, and let $G : U \to \mathbb{R}^k$ be differentiable at z. Consider $H = G \circ F$. We have

$$\begin{aligned} H(x+y) &= G(F(x+y)) \\ &= G\bigl(F(x) + DF(x)y + R(x,y)\bigr) \\ &= G(z) + DG(z)\bigl(DF(x)y + R(x,y)\bigr) + R_1(x,y) \\ &= G(z) + DG(z)DF(x)y + R_2(x,y) \end{aligned} \tag{B.6}$$

with

$$\frac{\|R_2(x,y)\|}{\|y\|} \to 0 \quad \text{as} \quad y \to 0.$$

Thus $G \circ F$ is differentiable at x, and

$$D(G \circ F)(x) = DG(F(x)) \cdot DF(x). \tag{B.7}$$

Another useful remark is that, by the Fundamental Theorem of Calculus, applied to $\varphi(t) = F(x + ty)$,

$$F(x+y) = F(x) + \int_0^1 DF(x+ty)y\,dt, \tag{B.8}$$

provided F is C^1. A closely related application of the Fundamental Theorem of Calculus is that, if we assume $F : \mathcal{O} \to \mathbb{R}^m$ is differentiable in each variable separately, and that each $\partial F/\partial x_j$ is continuous on $\mathcal{O}$, then

$$\begin{aligned} F(x+y) &= F(x) + \sum_{j=1}^n \big[F(x+z_j) - F(x+z_{j-1})\big] = F(x) + \sum_{j=1}^n A_j(x,y)y_j, \\ A_j(x,y) &= \int_0^1 \frac{\partial F}{\partial x_j}\big(x + z_{j-1} + ty_je_j\big)\,dt, \end{aligned} \tag{B.9}$$

where $z_0 = 0,\ z_j = (y_1, \dots, y_j, 0, \dots, 0)$, and $\{e_j\}$ is the standard basis of $\mathbb{R}^n$. Now (B.9) implies F is differentiable on $\mathcal{O}$, as we stated below (B.4). Thus we have established the following.

Proposition B.1. *If $\mathcal{O}$ is an open subset of $\mathbb{R}^n$ and $F : \mathcal{O} \to \mathbb{R}^m$ is of class C^1, then F is differentiable at each point $x \in \mathcal{O}$.*

As is shown in many calculus texts, one can use the Mean Value Theorem instead of the Fundamental Theorem of Calculus, and obtain a slightly sharper result.

For the study of higher order derivatives of a function, the following result is fundamental.

Proposition B.2. *Assume $F : \mathcal{O} \to \mathbb{R}^m$ is of class C^2, with $\mathcal{O}$ open in $\mathbb{R}^n$. Then, for each $x \in \mathcal{O},\ 1 \le j, k \le n$,*

$$\frac{\partial}{\partial x_j}\frac{\partial F}{\partial x_k}(x) = \frac{\partial}{\partial x_k}\frac{\partial F}{\partial x_j}(x). \tag{B.10}$$

To prove Proposition B.2, it suffices to treat real valued functions, so consider $f : \mathcal{O} \to \mathbb{R}$. For $1 \le j \le n$, set

$$\Delta_{j,h}f(x) = \frac{1}{h}\big(f(x+he_j) - f(x)\big), \tag{B.11}$$

where $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{R}^n$. The mean value theorem (for functions of x_j alone) implies that if $\partial_j f = \partial f/\partial x_j$ exists on $\mathcal{O}$, then, for $x \in \mathcal{O}$, $h > 0$ sufficiently small,

$$\Delta_{j,h} f(x) = \partial_j f(x + \alpha_j h e_j), \tag{B.12}$$

for some $\alpha_j \in (0,1)$, depending on x and h. Iterating this, if $\partial_j(\partial_k f)$ exists on $\mathcal{O}$, then, for $x \in \mathcal{O}$ and $h > 0$ sufficiently small,

$$\begin{aligned} \Delta_{k,h}\Delta_{j,h} f(x) &= \partial_k(\Delta_{j,h} f)(x + \alpha_k h e_k) \\ &= \Delta_{j,h}(\partial_k f)(x + \alpha_k h e_k) \\ &= \partial_j \partial_k f(x + \alpha_k h e_k + \alpha_j h e_j), \end{aligned} \tag{B.13}$$

with $\alpha_j, \alpha_k \in (0,1)$. Here we have used the elementary result

$$\partial_k \Delta_{j,h} f = \Delta_{j,h}(\partial_k f). \tag{B.14}$$

We deduce the following.

Proposition B.3. *If $\partial_k f$ and $\partial_j \partial_k f$ exist on $\mathcal{O}$ and $\partial_j \partial_k f$ is continuous at $x_0 \in \mathcal{O}$, then*

$$\partial_j \partial_k f(x_0) = \lim_{h \to 0} \Delta_{k,h}\Delta_{j,h} f(x_0). \tag{B.15}$$

Clearly

$$\Delta_{k,h}\Delta_{j,h} f = \Delta_{j,h}\Delta_{k,h} f, \tag{B.16}$$

so we have the following, which easily implies Proposition B.2.

Corollary B.4. *In the setting of Proposition B.3, if also $\partial_j f$ and $\partial_k \partial_j f$ exist on $\mathcal{O}$ and $\partial_k \partial_j f$ is continuous at x_0, then*

$$\partial_j \partial_k f(x_0) = \partial_k \partial_j f(x_0). \tag{B.17}$$

If U and V be open subsets of $\mathbb{R}^n$ and $F : U \to V$ is a C^1 map, we say F is a diffeomorphism of U onto V provided F maps U one-to-one and onto V, and its inverse $G = F^{-1}$ is a C^1 map. If F is a diffeomorphism, it follows from the chain rule that $DF(x)$ is invertible for each $x \in U$. We now present a partial converse of this, the Inverse Function Theorem, which is a fundamental result in multivariable calculus.

Theorem B.5. *Let F be a C^k map from an open neighborhood Ω of $p_0 \in \mathbb{R}^n$ to $\mathbb{R}^n$, with $q_0 = F(p_0)$. Assume $k \geq 1$. Suppose the derivative $DF(p_0)$ is invertible. Then there is a neighborhood U of p_0 and a neighborhood V of q_0 such that $F : U \to V$ is one-to-one and onto, and $F^{-1} : V \to U$ is a C^k map. (So $F : U \to V$ is a diffeomorphism.)*

First we show that F is one-to-one on a neighborhood of p_0, under these hypotheses. In fact, we establish the following result, of interest in its own right.

Proposition B.6. *Assume $\Omega \subset \mathbb{R}^n$ is open and convex, and let $f : \Omega \to \mathbb{R}^n$ be C^1. Assume that the symmetric part of $Df(u)$ is positive-definite, for each $u \in \Omega$. Then f is one-to-one on Ω.*

Proof. Take distinct points $u_1,\ u_2 \in \Omega$, and set $u_2 - u_1 = w$. Consider $\varphi : [0,1] \to \mathbb{R}$, given by

$$\varphi(t) = w \cdot f(u_1 + tw).$$

Then $\varphi'(t) = w \cdot Df(u_1 + tw)w > 0$ for $t \in [0,1]$, so $\varphi(0) \neq \varphi(1)$. But $\varphi(0) = w \cdot f(u_1)$ and $\varphi(1) = w \cdot f(u_2)$, so $f(u_1) \neq f(u_2)$.

To continue the proof of Theorem B.5, let us set

$$f(u) = A\big(F(p_0 + u) - q_0\big), \quad A = DF(p_0)^{-1}. \tag{B.18}$$

Then $f(0) = 0$ and $Df(0) = I$, the identity matrix. We show that f maps a neighborhood of 0 one-to-one and onto some neighborhood of 0. Proposition B.4 applies, so we know f is one-to-one on some neighborhood $\mathcal{O}$ of 0. We next show that the image of $\mathcal{O}$ under f contains a neighborhood of 0.

Note that

$$f(u) = u + R(u), \quad R(0) = 0,\ DR(0) = 0. \tag{B.19}$$

For v small, we want to solve

$$f(u) = v. \tag{B.20}$$

This is equivalent to $u + R(u) = v$, so let

$$T_v(u) = v - R(u). \tag{B.21}$$

Thus solving (B.20) is equivalent to solving

$$T_v(u) = u. \tag{B.22}$$

We look for a *fixed point* $u = K(v) = f^{-1}(v)$. Also, we want to prove that $DK(0) = I$, i.e., that $K(v) = v + r(v)$ with $r(v) = o(\|v\|)$, i.e., $r(v)/\|v\| \to 0$ as $v \to 0$. If we succeed in doing this, it follows easily that, for general x close to q_0, $G(x) = F^{-1}(x)$ is defined, and

$$DG(x) = \Big(DF\big(G(x)\big)\Big)^{-1}. \tag{B.23}$$

Then a simple inductive argument shows that G is C^k if F is C^k.

A tool we will use to solve (B.22) is the Contraction Mapping Principle, established in Appendix A, which states that if X is a complete metric space, and if $T : X \to X$ satisfies

$$\operatorname{dist}(Tx, Ty) \le r\ \operatorname{dist}(x, y), \tag{B.24}$$

for some $r < 1$ (we say T is a contraction), then T has a unique fixed point x.

In order to implement this, we consider

$$T_v : X_v \longrightarrow X_v \tag{B.25}$$

with

$$X_v = \{u \in \Omega : \|u - v\| \leq A_v\} \tag{B.26}$$

where we set

$$A_v = \sup_{\|w\| \leq 2\|v\|} \|R(w)\|. \tag{B.27}$$

We claim that (B.25) holds if $\|v\|$ is sufficiently small. To prove this, note that $T_v(u) - v = -R(u)$, so we need to show that, provided $\|v\|$ is small, $u \in X_v$ implies $\|R(u)\| \leq A_v$. But indeed, if $u \in X_v$, then $\|u\| \leq \|v\| + A_v$, which is $\leq 2\|v\|$ if $\|v\|$ is small, so then

$$\|R(u)\| \leq \sup_{\|w\| \leq 2\|v\|} \|R(w)\| = A_v.$$

This establishes (B.25).

Note that $T_v(u_1) - T_v(u_2) = R(u_2) - R(u_1)$, and R is a C^k map, satisfying $DR(0) = 0$. It follows that, if $\|v\|$ is small enough, the map (B.18) is a contraction map. Hence there exists a unique fixed point $u = K(v) \in X_v$. Also, since $u \in X_v$,

$$\|K(v) - v\| \leq A_v = o(\|v\|), \tag{B.28}$$

so the Inverse Function Theorem is proved.

Thus if DF is invertible on the domain of F, F is a local diffeomorphism. Stronger hypotheses are needed to guarantee that F is a global diffeomorphism onto its range. Proposition B.6 provides one tool for doing this. Here is a slight strengthening.

Corollary B.7. *Assume $\Omega \subset \mathbb{R}^n$ is open and convex, and that $F : \Omega \to \mathbb{R}^n$ is C^1. Assume there exist $n \times n$ matrices A and B such that the symmetric part of $A\,DF(u)\,B$ is positive definite for each $u \in \Omega$. Then F maps Ω diffeomorphically onto its image, an open set in $\mathbb{R}^n$.*

Proof. Exercise.

P. The Laplace asymptotic method and Stirling's formula

Recall that the Gamma function is given by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \tag{P.1}$$

for $\operatorname{Re} z > 0$. We aim to analyze its behavior for large z, particularly in a sector

$$A_\beta = \{re^{i\theta} : r > 0,\ |\theta| \le \beta\}, \tag{P.2}$$

for $\beta < \pi/2$. Let us first take $z > 0$, and set $t = sz$, and then $s = e^y$, to write

$$\begin{aligned} \Gamma(z) &= z^z \int_0^\infty e^{-z(s-\log s)} s^{-1}\, ds \\ &= z^z e^{-z} \int_{-\infty}^\infty e^{-z(e^y - y - 1)}\, dy. \end{aligned} \tag{P.3}$$

Having done this, we see that each side of (P.3) is holomorphic in the half plane $\operatorname{Re} z > 0$, so the identity holds for all such z. The last integral has the form

$$I(z) = \int_{-\infty}^\infty e^{-z\varphi(y)} A(y)\, dy, \tag{P.4}$$

with $A(y) \equiv 1$ in this case, and $\varphi(y) = e^y - y - 1$. Note that $\varphi(y)$ is real valued and has a nondegenerate minimum at $y = 0$,

$$\varphi(0) = 0, \quad \varphi'(0) = 0, \quad \varphi''(0) > 0. \tag{P.5}$$

Furthermore,

$$\begin{aligned} \varphi(y) \ge ay^2 &\quad \text{for} \quad |y| \le 1, \\ a &\quad \text{for} \quad |y| \ge 1, \end{aligned} \tag{P.6}$$

for some $a > 0$.

The Laplace asymptotic method analyzes the asymptotic behavior of such an integral, as $z \to \infty$ in a sector $A_{\pi/2-\delta}$. In addition to the hypotheses (P.5)–(P.6), we assume that φ and A are smooth, and we assume that, given $\alpha > 0$, there exists $\beta > 0$ such that

$$\Bigl| \int\limits_{|y|>\alpha} e^{-z\varphi(y)} A(y)\, dy \Bigr| \le Ce^{-\beta \operatorname{Re} z}, \quad \text{for} \quad \operatorname{Re} z \ge 1. \tag{P.7}$$

These hypotheses are readily verified for the integral that arises in (P.3).

Given these hypotheses, our first step to tackle (P.4) is to pick $b \in C^\infty(\mathbb{R})$ such that $b(y) = 1$ for $|y| \le \alpha$ and $b(y) = 0$ for $|y| \ge 2\alpha$, and set

$$A_0(y) = b(y)A(y), \quad A_1(y) = (1 - b(y))A(y), \tag{P.8}$$

so

$$\Bigl|\int_{-\infty}^{\infty} e^{-z\varphi(y)} A_1(y)\, dy\Bigr| \le Ce^{-\beta \operatorname{Re} z}, \tag{P.9}$$

for $\operatorname{Re} z \ge 1$. It remains to analyze

$$I_0(z) = \int_{-\infty}^{\infty} e^{-z\varphi(y)} A_0(y)\, dy. \tag{P.10}$$

Pick α sufficiently small that you can write

$$\varphi(y) = \xi(y)^2, \quad \text{for} \quad |y| \le 2\alpha, \tag{P.11}$$

where ξ maps $[-2\alpha, 2\alpha]$ diffeomorphically onto an interval about 0 in $\mathbb{R}$. Then

$$I_0(z) = \int_{-\infty}^{\infty} e^{-z\xi^2} B_0(\xi)\, d\xi, \tag{P.12}$$

with $B_0(\xi) = A_0(y(\xi))y'(\xi)$, where $y(\xi)$ denotes the map inverse to $\xi(y)$. Hence $B_0 \in C_0^\infty(\mathbb{R})$.

To analyze (P.12), we use the Fourier transform:

$$\widehat{B}_0(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} B_0(\xi) e^{-ix\xi}\, d\xi, \tag{P.13}$$

studied in §14. Arguing as in the calculation (14.11)–(14.12), we have, for $\operatorname{Re} z > 0$,

$$E_z(\xi) = e^{-z\xi^2} \Longrightarrow \widehat{E}_z(x) = \Bigl(\frac{1}{2z}\Bigr)^{1/2} e^{-x^2/4z}. \tag{P.14}$$

Hence, by Plancherel's theorem, for $\operatorname{Re} z > 0$,

$$\begin{aligned} I_0(z) &= (2\zeta)^{1/2} \int_{-\infty}^{\infty} e^{-\zeta x^2} \widehat{B}_0(x)\, dx \\ &= (2\zeta)^{1/2} \mathcal{I}_0(\zeta), \qquad \zeta = \frac{1}{4z}. \end{aligned} \tag{P.15}$$

Now, given $B_0 \in C_0^\infty(\mathbb{R})$, one has $\widehat{B}_0 \in C^\infty(\mathbb{R})$, and

$$\bigl|x^j \widehat{B}_0^{(k)}(x)\bigr| \le C_{jk}, \quad \forall x \in \mathbb{R}. \tag{P.16}$$

We say $\widehat{B}_0 \in \mathcal{S}(\mathbb{R})$. Using this, it follows that

$$\mathcal{I}_0(\zeta) = \int_{-\infty}^{\infty} e^{-\zeta x^2} \widehat{B}_0(x)\, dx \tag{P.17}$$

is holomorphic on $\{\zeta \in \mathbb{C} : \operatorname{Re}\zeta > 0\}$ and C^∞ on $\{\zeta \in \mathbb{C} : \operatorname{Re}\zeta \geq 0\}$. We have

$$\mathcal{I}_0(0) = \int_{-\infty}^{\infty} \widehat{B}_0(x)\, dx = \sqrt{2\pi} B_0(0). \tag{P.18}$$

It follows that, for $\operatorname{Re}\zeta \geq 0$,

$$\mathcal{I}_0(\zeta) = \sqrt{2\pi} B_0(0) + O(|\zeta|), \quad \text{as} \ \ \zeta \to 0, \tag{P.19}$$

hence, for $\operatorname{Re} z \geq 0$, $z \neq 0$,

$$I_0(z) = \Big(\frac{\pi}{z}\Big)^{1/2} B_0(0) + O(|z|^{-3/2}), \quad \text{as} \ \ z \to \infty. \tag{P.20}$$

If we apply this to (P.3)–(P.9), we obtain Stirling's formula,

$$\Gamma(z) = z^z e^{-z} \Big(\frac{2\pi}{z}\Big)^{1/2} \Big[1 + O(|z|^{-1})\Big], \tag{P.21}$$

for $z \in A_{\pi/2-\delta}$, taking into account that in this case $B_0(0) = \sqrt{2}$.

Asymptotic analysis of the Hankel function, done in §35, leads to an integral of the form (P.4) with

$$\varphi(y) = \frac{\sinh^2 y}{\cosh y}, \tag{P.22}$$

and

$$A(y) = e^{-\nu u(y)} u'(y), \quad u(y) = y + i \tan^{-1}(\sinh y), \tag{P.23}$$

so $u'(y) = 1 + i/\cosh y$. The conditions for applicability of (P.5)–(P.9) are readily verified for this case, yielding the asymptotic expansion in Proposition 35.3.

Returning to Stirling's formula, we mention another approach, which gives more precise information. It involves the ingenious identity

$$\log \Gamma(z) = \Big(z - \frac{1}{2}\Big) \log z - z + \frac{1}{2} \log 2\pi + \omega(z), \tag{P.24}$$

with a convenient integral formula for $\omega(z)$, namely

$$\omega(z) = \int_0^\infty f(t) e^{-tz}\, dt, \tag{P.25}$$

for $\operatorname{Re} z > 0$, with

$$
\text{(P.26)} \qquad \begin{aligned} f(t) &= \Bigl(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\Bigr)\frac{1}{t} \\ &= \frac{1}{t}\Bigl(\frac{1}{2}\,\frac{\cosh t/2}{\sinh t/2} - \frac{1}{t}\Bigr), \end{aligned}
$$

which is a smooth, even function of t on $\mathbb{R}$, asymptotic to $1/2t$ as $t \nearrow +\infty$. A proof can be found in §1.4 of [Leb].

We show how to derive a complete asymptotic expansion of the Laplace transform (3.2), valid for $z \to \infty$, $\operatorname{Re} z \ge 0$, just given that $f \in C^\infty([0,\infty))$ and that $f^{(j)}$ is integrable on $[0,\infty)$ for each $j \ge 1$. To start, integration by parts yields

$$
\text{(P.27)} \qquad \begin{aligned} \int_0^\infty f(t) e^{-zt}\,dt &= -\frac{1}{z}\int_0^\infty f(t)\,\frac{d}{dt} e^{-zt}\,dt \\ &= \frac{1}{z}\int_0^\infty f'(t) e^{-zt}\,dt + \frac{f(0)}{z}, \end{aligned}
$$

valid for $\operatorname{Re} z > 0$. We can iterate this argument to obtain

$$
\text{(P.28)} \qquad \omega(z) = \sum_{k=1}^{N} \frac{f^{(k-1)}(0)}{z^k} + \frac{1}{z^N}\int_0^\infty f^{(N)}(t) e^{-zt}\,dt,
$$

and

$$
\text{(P.29)} \qquad \Bigl|\int_0^\infty f^{(N)}(t) e^{-zt}\,dt\Bigr| \le \int_0^\infty |f^{(N)}(t)|\,dt < \infty, \quad \text{for } \ N \ge 1,\ \operatorname{Re} z \ge 0.
$$

By (P.24), $\omega(z)$ is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$. Meanwhile, the right side of (P.28) is continuous on $\{z \in \mathbb{C} : \operatorname{Re} z \ge 0,\ z \ne 0\}$, so equality in (P.28) holds on this region.

To carry on, we note that, for $|t| < 2\pi$,

$$
\text{(P.30)} \qquad \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k)!} B_k t^{2k-1},
$$

where B_k are the *Bernoulli numbers*, introduced in §12, Exercises 6–8, and related to $\zeta(2k)$ in §30. Hence, for $|t| < 2\pi$,

$$
\text{(P.31)} \qquad f(t) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell+2)!} B_{\ell+1} t^{2\ell}.
$$

Thus

$$
\text{(P.32)} \qquad \begin{aligned} f^{(j)}(0) = &\qquad 0 & &j \text{ odd}, \\ &\frac{(-1)^\ell B_{\ell+1}}{(2\ell+1)(2\ell+2)} & &j = 2\ell, \end{aligned}
$$

so

$$\omega(z) \sim \sum_{\ell \geq 0} \frac{(-1)^\ell B_{\ell+1}}{(2\ell+1)(2\ell+2)} \frac{1}{z^{2\ell+1}}, \quad z \to \infty, \ \operatorname{Re} z \geq 0. \tag{P.33}$$

Thus there are $A_k \in \mathbb{R}$ such that

$$e^{\omega(z)} \sim 1 + \sum_{k \geq 1} \frac{A_k}{z^k}, \quad z \to \infty, \ \operatorname{Re} z \geq 0. \tag{P.34}$$

This yields the following refinement of (P.21):

$$\Gamma(z) \sim z^z e^{-z} \Bigl(\frac{2\pi}{z}\Bigr)^{1/2} \Bigl[1 + \sum_{k \geq 1} A_k z^{-k}\Bigr], \quad |z| \to \infty, \ \operatorname{Re} z \geq 0. \tag{P.35}$$

We can push the asymptotic analysis of $\Gamma(z)$ into the left half-plane, using the identity

$$\Gamma(-z) \sin \pi z = -\frac{\pi}{z\Gamma(z)} \tag{P.36}$$

to extend (P.24), i.e.,

$$\Gamma(z) = \Bigl(\frac{z}{e}\Bigr)^z \sqrt{\frac{2\pi}{z}}\, e^{\omega(z)}, \quad \text{for} \quad \operatorname{Re} z \geq 0, \ z \neq 0, \tag{P.37}$$

to the rest of $\mathbb{C} \setminus \mathbb{R}^-$. If we define z^z and $\sqrt{z}$ in the standard fashion for $z \in (0, \infty)$ and to be holomorphic on $\mathbb{C} \setminus \mathbb{R}^-$, we get

$$\Gamma(z) = \frac{1}{1 - e^{2\pi i z}} \Bigl(\frac{z}{e}\Bigr)^z \sqrt{\frac{2\pi}{z}}\, e^{-\omega(-z)}, \quad \text{for} \quad \operatorname{Re} z \leq 0, \ \operatorname{Im} z > 0, \tag{P.38}$$

and

$$\Gamma(z) = \frac{1}{1 - e^{-2\pi i z}} \Bigl(\frac{z}{e}\Bigr)^z \sqrt{\frac{2\pi}{z}}\, e^{-\omega(-z)}, \quad \text{for} \quad \operatorname{Re} z \leq 0, \ \operatorname{Im} z < 0. \tag{P.39}$$

Comparing (P.37) and (P.38) for $z = iy, \ y > 0$, we see that

$$e^{-\omega(-iy)} = (1 - e^{-2\pi y}) e^{\omega(iy)}, \quad y > 0. \tag{P.40}$$

That $e^{-\omega(-iy)}$ and $e^{\omega(iy)}$ have the same asymptotic behavior as $y \to +\infty$ also follows from the fact that only odd powers of z^{-1} appear in (P.33).

M. The Stieltjes integral

Here we develop basic results on integrals of the form

$$\int_a^b f(x)\,du(x), \tag{M.1}$$

known as Stieltjes integrals. We assume that $f \in C([a,b])$ and that

$$u : [a,b] \longrightarrow \mathbb{R} \ \text{ is increasing,} \tag{M.2}$$

i.e., $x_1 < x_2 \Rightarrow u(x_1) \le u(x_2)$. We also assume u is right continuous, i.e.,

$$u(x) = \lim_{y \searrow x} u(y), \quad \forall\, x \in [a,b). \tag{M.3}$$

Note that (M.2) implies the existence for all $x \in [a,b]$ of

$$u^+(x) = \liminf_{y \searrow x} u(y), \quad u^-(x) = \limsup_{y \nearrow x} u(y) \tag{M.4}$$

(with the convention that $u^-(a) = u(a)$ and $u^+(b) = u(b)$). We have

$$u^-(x) \le u^+(x), \quad \forall\, x \in [a,b], \tag{M.5}$$

and (M.3) says $u(x) = u^+(x)$ for all x. Note that u is continuous at x if and only if $u^-(x) = u^+(x)$. If u is not continuous at x, it has a jump discontinuity there, and it is easy to see that u can have at most countably many such discontinuities.

We prepare to define the integral (M.1), mirroring a standard development of the Riemann integral when $u(x) = x$. For now, we allow f to be any bounded, real-valued function on $[a,b]$, say $|f(x)| \le M$. To start, we partition $[a,b]$ into smaller intervals. A *partition* $\mathcal{P}$ of $[a,b]$ is a finite collection of subintervals $\{J_k : 0 \le k \le N-1\}$, disjoint except for their endpoints, whose union is $[a,b]$. We order the J_k so that $J_k = [x_k, x_{k+1}]$, where

$$a = x_0 < x_1 < \cdots < x_N = b. \tag{M.6}$$

We call the points x_k the endpoints of $\mathcal{P}$. We set

$$\begin{aligned} \overline{I}_{\mathcal{P}}(f\,du) &= \sum_{k=0}^{N-1} (\sup_{J_k} f)[u(x_{k+1}) - u(x_k)], \\ \underline{I}_{\mathcal{P}}(f\,du) &= \sum_{k=0}^{N-1} (\inf_{J_k} f)[u(x_{k+1}) - u(x_k)]. \end{aligned} \tag{M.7}$$

Note that $\underline{I}_{\mathcal{P}}(f\,du) \le \overline{I}_{\mathcal{P}}(f\,du)$. These quantities should be approximations to (M.1) if the partition $\mathcal{P}$ is sufficiently "fine."

To be more precise, if $\mathcal{P}$ and $\mathcal{Q}$ are two partitions of $[a,b]$, we say $\mathcal{P}$ *refines* $\mathcal{Q}$, and write $\mathcal{P} \succ \mathcal{Q}$, if $\mathcal{P}$ is formed by partitioning the intervals in $\mathcal{Q}$. Equivalently, $\mathcal{P} \succ \mathcal{Q}$ if and only if all the endpoints of $\mathcal{Q}$ are endpoints of $\mathcal{P}$. It is easy to see that any two partitions have a common refinement; just take the union of their endpoints, to form a new partition. Note that

$$\text{(M.8)} \qquad \begin{aligned} \mathcal{P} \succ \mathcal{Q} \Rightarrow \overline{I}_{\mathcal{P}}(f\,du) &\le \overline{I}_{\mathcal{Q}}(f\,du), \quad \text{and} \\ \underline{I}_{\mathcal{P}}(f\,du) &\ge \underline{I}_{\mathcal{Q}}(f\,du). \end{aligned}$$

Consequently, if $\mathcal{P}_j$ are two partitions of $[a,b]$ and $\mathcal{Q}$ is a common refinement, we have

$$\text{(M.9)} \qquad \underline{I}_{\mathcal{P}_1}(f\,du) \le \underline{I}_{\mathcal{Q}}(f\,du) \le \overline{I}_{\mathcal{Q}}(f\,du) \le \overline{I}_{\mathcal{P}_2}(f\,du).$$

Thus, whenever $f : [a,b] \to \mathbb{R}$ is bounded, the following quantities are well defined:

$$\text{(M.10)} \qquad \begin{aligned} \overline{I}_a^b(f\,du) &= \inf_{\mathcal{P}\in\Pi[a,b]} \overline{I}_{\mathcal{P}}(f\,du), \\ \underline{I}_a^b(f\,du) &= \sup_{\mathcal{P}\in\Pi[a,b]} \underline{I}_{\mathcal{P}}(f\,du), \end{aligned}$$

where $\Pi[a,b]$ denotes the set of all partitions of $[a,b]$. Clearly, by (M.9),

$$\text{(M.11)} \qquad \underline{I}_a^b(f\,du) \le \overline{I}_a^b(f\,du).$$

We say a bounded function $f : [a,b] \to \mathbb{R}$ is Riemann-Stieltjes integrable provided there is equality in (M.11). In such a case, we set

$$\text{(M.12)} \qquad \int_a^b f(x)\,du(x) = \overline{I}_a^b(f\,du) = \underline{I}_a^b(f\,du),$$

and we write $f \in \mathcal{R}([a,b],du)$. Though we will not emphasize it, another notation for (M.12) is

$$\text{(M.13)} \qquad \int_I f(x)\,du(x), \quad I = (a,b].$$

Our first basic result is that each continuous function on $[a,b]$ is Riemann-Stieltjes integrable.

Proposition M.1. *If $f : [a,b] \to \mathbb{R}$ is continuous, then $f \in \mathcal{R}([a,b],du)$.*

Proof. Any continuous function on $[a,b]$ is uniformly continuous (cf. Appendix A). Thus there is a function $\omega(\delta)$ such that

$$\text{(M.14)} \qquad |x-y| \le \delta \Rightarrow |f(x)-f(y)| \le \omega(\delta), \quad \omega(\delta) \to 0 \ \text{ as } \ \delta \to 0.$$

Given $J_k = [x_k, x_{k+1}]$, let us set $\ell(J_k) = x_{k+1} - x_k$, and, for the partition $\mathcal{P}$ with endpoints as in (M.6), set

$$\text{(M.15)} \qquad \operatorname{maxsize}(\mathcal{P}) = \max_{0 \le k \le N-1} \ell(J_k).$$

Then

$$\text{(M.16)} \qquad \operatorname{maxsize}(\mathcal{P}) \le \delta \Longrightarrow \overline{I}_{\mathcal{P}}(f\,du) - \underline{I}_{\mathcal{P}}(f\,du) \le \omega(\delta)[u(b) - u(a)],$$

which yields the proposition.

We will concentrate on (M.1) for continuous f, but there are a couple of results that are conveniently established for more general integrable f.

Proposition M.2. *If $f, g \in \mathcal{R}([a,b], du)$, then $f + g \in \mathcal{R}([a,b], du)$, and*

$$\text{(M.17)} \qquad \int_a^b (f(x) + g(x))\,du(x) = \int_a^b f(x)\,du(x) + \int_a^b g(x)\,du(x).$$

Proof. If J_k is any subinterval of $[a,b]$, then

$$\text{(M.18)} \qquad \begin{aligned} &\sup_{J_k}\,(f+g) \le \sup_{J_k}\, f + \sup_{J_k}\, g, \quad \text{and} \\ &\inf_{J_k}\,(f+g) \ge \inf_{J_k}\, f + \inf_{J_k}\, g, \end{aligned}$$

so, for any partition $\mathcal{P}$, we have $\overline{I}_{\mathcal{P}}(f+g)\,du) \le \overline{I}_{\mathcal{P}}(f\,du) + \overline{I}_{\mathcal{P}}(g\,du)$. Also,using a common refinement of partitions, we can *simultaneously* approximate $\overline{I}_a^b(f\,du)$ and $\overline{I}_a^b(g\,du)$ by $\overline{I}_{\mathcal{P}}(f\,du)$ and $\overline{I}_{\mathcal{P}}(g\,du)$, and likewise for $\overline{I}_a^b((f+g)\,du)$. Then the characterization (M.10) implies $\overline{I}_a^b((f+g)\,du) \le \overline{I}_a^b(f\,du) + \overline{I}_a^b(g\,du)$. A parallel argument implies $\underline{I}_a^b((f+g)\,du) \ge \underline{I}_a^b(f\,du) + \underline{I}_a^b(g\,du)$, and the proposition follows.

Here is another useful additivity result.

Proposition M.3. *Let $a < b < c$, $f : [a,c] \to \mathbb{R}$, $f_1 = f|_{[a,b]}$, $f_2 = f|_{[b,c]}$. Assume $u : [a,c] \to \mathbb{R}$ is increasing and right continuous. Then*

$$\text{(M.19)} \qquad f \in \mathcal{R}([a,c], du) \Leftrightarrow f_1 \in \mathcal{R}([a,b], du) \text{ and } f_2 \in \mathcal{R}([b,c], du),$$

and, if this holds,

$$\text{(M.20)} \qquad \int_a^c f(x)\,du(x) = \int_a^b f_1(x)\,du(x) + \int_b^c f_2(x)\,du(x).$$

Proof. Since any partition of $[a,c]$ has a refinement for which b is an endpoint, we may as well consider a partition $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, where $\mathcal{P}_1$ is a partition of $[a,b]$ and $\mathcal{P}_2$ is a partition of $[b,c]$. Then

$$\text{(M.21)} \qquad \overline{I}_{\mathcal{P}}(f\,du) = \overline{I}_{\mathcal{P}_1}(f_1\,du) + \overline{I}_{\mathcal{P}_2}(f_2\,du),$$

with a parallel identity for $\underline{I}_{\mathcal{P}}(f\,du)$, so

$$\overline{I}_{\mathcal{P}}(f\,du)-\underline{I}_{\mathcal{P}}(f\,du)=\{\overline{I}_{\mathcal{P}_1}(f_1\,du)-\underline{I}_{\mathcal{P}_1}(f_1\,du)\}+\{\overline{I}_{\mathcal{P}_2}(f_2\,du)-\underline{I}_{\mathcal{P}_2}(f_2\,du)\}. \tag{M.22}$$

Since both terms in braces in (M.22) are ≥ 0, we have the equivalence in (M.19). Then (M.20) follows from (M.21) upon taking sufficiently fine partitions.

In the classical case $u(x)=x$, we denote $\mathcal{R}([a,b],du)$ by $\mathcal{R}([a,b])$, the space of Riemann integrable functions on $[a,b]$. We record a few standard results about the Riemann integral, whose proofs can be found in many texts, including [T0], Chapter 4, and [T], §0.

Proposition M.4. *If $u:[a,b]\to\mathbb{R}$ is increasing and right continuous, then $u\in\mathcal{R}([a,b])$.*

Proposition M.5. *If $f,g\in\mathcal{R}([a,b])$, then $fg\in\mathcal{R}([a,b])$.*

The next result is known as the Darboux theorem for the Riemann integral.

Proposition M.6. *Let $\mathcal{P}_\nu$ be a sequence of partitions of $[a,b]$, into ν intervals $J_{\nu k}$, $1\leq k\leq\nu$, such that*

$$\operatorname{maxsize}(\mathcal{P}_\nu)\longrightarrow 0, \tag{M.23}$$

and assume $f\in\mathcal{R}([a,b])$. Then

$$\int_a^b f(x)\,dx=\lim_{\nu\to\infty}\sum_{k=1}^{\nu} f(\xi_{\nu k})\ell(J_{\nu k}), \tag{M.24}$$

for arbitrary $\xi_{\nu k}\in J_{\nu k}$, where $\ell(J_{\nu k})$ is the length of the interval $J_{\nu k}$.

We now present a very useful result, known as integration by parts for the Stieltjes integral.

Proposition M.7. *Let $u:[a,b]\to\mathbb{R}$ be increasing and right continuous, and let $f\in C^1([a,b])$, so $f'\in C([a,b])$. Then*

$$\int_a^b f(x)\,du(x)=fu\Big|_a^b-\int_a^b f'(x)u(x)\,dx, \tag{M.29}$$

where

$$fu\Big|_a^b=f(b)u(b)-f(a)u(a). \tag{M.30}$$

Proof. Pick a partition $\mathcal{P}$ of $[a,b]$ with endpoints x_k, $0\leq k\leq N$, as in (M.6). Then

$$\int_a^b f(x)\,du(x)=\sum_{k=0}^{N-1}\int_{x_k}^{x_{k+1}} f(x)\,du(x). \tag{M.31}$$

Now, given $\varepsilon > 0$, pick $\delta > 0$ such that

$$\text{maxsize}(\mathcal{P}) \le \delta \Longrightarrow \sup_{\xi \in [x_k, x_{k+1}]} |f(\xi) - f(x_k)| \le \varepsilon. \tag{M.32}$$

Then

$$\int_a^b f(x)\,du(x) = \sum_{k=0}^{N-1} f(x_k)[u(x_{k+1}) - u(x_k)] + O(\varepsilon). \tag{M.33}$$

We can write this last sum as

$$\begin{aligned} -f(x_0)u(x_0) + [f(x_0) - f(x_1)]u(x_1) + \cdots \\ + [f(x_{N-1}) - f(x_N)]u(x_N) + f(x_N)u(x_N), \end{aligned}$$

so

$$\int_a^b f(x)\,du(x) = fu\Big|_a^b + \sum_{k=0}^{N-1} [f(x_k) - f(x_{k+1})]u(x_k) + O(\varepsilon). \tag{M.34}$$

Now the Mean Value Theorem implies

$$f(x_k) - f(x_{k+1}) = -f'(\zeta_k)(x_{k+1} - x_k), \tag{M.35}$$

for some $\zeta_k \in (x_k, x_{k+1})$. Since $f' \in C([a,b])$, we have in addition to (M.32) that, after perhaps shrinking δ,

$$\text{maxsize}(\mathcal{P}) \le \delta \Rightarrow \sup_{\zeta \in [x_k, x_{k+1}]} |f'(\zeta) - f'(x_k)| \le \varepsilon. \tag{M.36}$$

Hence

$$\int_a^b f(x)\,du(x) = fu\Big|_a^b - \sum_{k=0}^{N-1} f'(x_k)u(x_k)(x_{k+1} - x_k) + O(\varepsilon). \tag{M.36}$$

Now Propositions M.4–M.5 imply $f'u \in \mathcal{R}([a,b])$, and then Proposition M.6, applied to $f'u$, implies that, in the limit as $\text{maxsize}(\mathcal{P}) \to 0$, the sum on the right side of (M.36) tends to

$$\int_a^b f'(x)u(x)\,dx. \tag{M.37}$$

This proves (M.29).

We discuss some natural extensions of the integral (M.1). For one, we can take $w = u - v$, where $v : [a, b] :\to \mathbb{R}$ is also increasing and right continuous, and set

$$\int_a^b f(x)\, dw(x) = \int_a^b f(x)\, du(x) - \int_a^b f(x)\, dv(x). \tag{M.38}$$

Let us take $f \in C([a, b])$. To see (M.38) is well defined, suppose that also $w = u_1 - v_1$, where u_1 and v_1 are also increasing and right continuous. The identity of the right side of (M.38) with

$$\int_a^b f(x)\, du_1(x) - \int_a^b f(x)\, dv_1(x) \tag{M.39}$$

is equivalent to the identity

$$\int_a^b f(x)\, du(x) + \int_a^b f(x)\, dv_1(x) = \int_a^b f(x)\, du_1(x) + \int_a^b f(x)\, dv(x), \tag{M.40}$$

hence to

$$\int_a^b f(x)\, du(x) + \int_a^b f(x)\, dv_1(x) = \int_a^b f(x)\, d(u + v_1)(x), \tag{M.41}$$

which is readily established, via

$$\overline{I}_{\mathcal{P}}(f\, du) + \overline{I}_{\mathcal{P}}(f\, dv_1) = \overline{I}_{\mathcal{P}}(f\, d(u + v_1)), \tag{M.42}$$

and similar identities.

Another extension is to take $u : [0, \infty) \to \mathbb{R}$, increasing and right continuous, and define

$$\int_0^\infty f(x)\, du(x), \tag{M.43}$$

for a class of functions $f : [0, \infty) \to \mathbb{R}$ satisfying appropriate bounds at infinity. For example, we might take

$$\begin{aligned} u(x) &\le C_\varepsilon e^{\varepsilon x}, \quad \forall\, \varepsilon > 0, \\ |f(x)| &\le C e^{-ax}, \quad \text{for some } a > 0. \end{aligned} \tag{M.44}$$

There are many variants. One then sets

$$\int_0^\infty f(x)\, du(x) = \lim_{R \to \infty} \int_0^R f(x)\, du(x). \tag{M.45}$$

Extending the integration by parts formula (M.29), we have

$$
\begin{aligned}
\int_0^\infty f(x)\,du(x) &= \lim_{R\to\infty} fu\Big|_0^R - \int_0^R f'(x)u(x)\,dx \\
&= -f(0)u(0) - \int_0^\infty f'(x)u(x)\,dx,
\end{aligned} \tag{M.46}
$$

for $f \in C^1([0,\infty))$, under an appropriate additional condition on $f'(x)$, such as

$$
|f'(x)| \le Ce^{-ax}, \tag{M.47}
$$

when (M.44) holds.

In addition, one can also have $v : [0,\infty) \to \mathbb{R}$, increasing and right continuous, set $w = u - v$, and define $\int_0^\infty f(x)\,dw(x)$, in a fashion parallel to (M.38). If, for example, (M.44) also holds with u replaced by v, we can extend (M.46) to

$$
\int_0^\infty f(x)\,dw(x) = -f(0)w(0) - \int_0^\infty f'(x)w(x)\,dx. \tag{M.48}
$$

The material developed above is adequate for use in §19 and Appendix R, but we mention that further extension can be made, to the Lebesgue-Stieltjes integral. In this set-up, one associates a "measure" μ on $[a,b]$ to the function u, and places the integral (M.1) within the framework of the Lebesgue integral with respect to a measure. Material on this can be found in many texts on measure theory, such as [T3], Chapters 5 and 13. In this setting, the content of Proposition M.7 is that the measure μ is the "weak derivative" of u, and one can extend the identity (M.29) to a class of functions f much more general than $f \in C^1([a,b])$.

R. Abelian theorems and Tauberian theorems

Abelian theorems and Tauberian theorems are results to the effect that one sort of convergence leads to another. We start with the original Abelian theorem, due to Abel, and give some applications of that result, before moving on to other Abelian theorems, and to Tauberian theorems.

Proposition R.1. *Assume we have a convergent series*

$$\sum_{k=0}^{\infty} a_k = A. \tag{R.1}$$

Then

$$f(r) = \sum_{k=0}^{\infty} a_k r^k \tag{R.2}$$

converges uniformly on $[0,1]$*, so* $f \in C([0,1])$*. In particular,* $f(r) \to A$ *as* $r \nearrow 1$.

As a warm up, we look at the following somewhat simpler result. Compare Propositions 13.2 and L.4.

Proposition R.2. *Assume we have an absolutely convergent series*

$$\sum_{k=0}^{\infty} |a_k| < \infty. \tag{R.3}$$

Then the series (R.2) converges uniformly on $[-1,1]$*, so* $f \in C([-1,1])$.

Proof. Clearly

$$\Bigl|\sum_{k=m}^{m+n} a_k r^k\Bigr| \le \sum_{k=m}^{m+n} |a_k|,$$

for $r \in [-1,1]$, so if (R.3) holds, then (R.2) converges u niformly for $r \in [-1,1]$. Of course, a uniform limit of a sequence of continuous functions on $[-1,1]$ is also continuous on this set.

Proposition R.1 is much more subtle than Proposition R.2. One ingredient in the proof is the following *summation by parts* formula.

Proposition R.3. *Let* (a_j) *and* (b_j) *be sequences, and let*

$$s_n = \sum_{j=0}^{n} a_j. \tag{R.4}$$

If $m > n$, then

$$\sum_{k=n+1}^{m} a_k b_k = (s_m b_m - s_n b_{n+1}) + \sum_{k=n+1}^{m-1} s_k (b_k - b_{k+1}). \tag{R.5}$$

Proof. Write the left side of (M.5) as

$$\sum_{k=n+1}^{m} (s_k - s_{k-1}) b_k. \tag{R.6}$$

It is then straightforward to obtain the right side.

Before applying Proposition R.3 to the proof of Proposition R.1, we note that, by Proposition 0.3 and its proof, especially (0.32), the power series (R.2) converges uniformly on compact subsets of $(-1, 1)$, and defines $f \in C((-1, 1))$. Our task here is to get uniform convergence up to $r = 1$.

To proceed, we apply (R.5) with $b_k = r^k$ and $n + 1 = 0, s_{-1} = 0$, to get

$$\sum_{k=0}^{m} a_k r^k = (1 - r) \sum_{k=0}^{m-1} s_k r^k + s_m r^m. \tag{R.7}$$

Now, we want to add and subtract a function $g_m(r)$, defined for $0 \le r < 1$ by

$$\begin{aligned} g_m(r) &= (1 - r) \sum_{k=m}^{\infty} s_k r^k \\ &= A r^m + (1 - r) \sum_{k=m}^{\infty} \sigma_k r^k, \end{aligned} \tag{R.8}$$

with A as in (R.1) and

$$\sigma_k = s_k - A \longrightarrow 0, \quad \text{as} \quad k \to \infty. \tag{R.9}$$

Note that, for $0 \le r < 1$, $\mu \in \mathbb{N}$,

$$\begin{aligned} (1 - r) \Bigl| \sum_{k=\mu}^{\infty} \sigma_k r^k \Bigr| &\le \Bigl(\sup_{k \ge \mu} |\sigma_k| \Bigr) (1 - r) \sum_{k=\mu}^{\infty} r^k \\ &= \Bigl(\sup_{k \ge \mu} |\sigma_k| \Bigr) r^\mu. \end{aligned} \tag{R.10}$$

It follows that

$$g_m(r) = A r^m + h_m(r) \tag{R.11}$$

extends to be continuous on $[0,1]$ and

$$|h_m(r)| \le \sup_{k\ge m} |\sigma_k|, \quad h_m(1) = 0. \tag{R.12}$$

Now adding and subtracting $g_m(r)$ in (R.7) gives

$$\sum_{k=0}^{m} a_k r^k = g_0(r) + (s_m - A)r^m - h_m(r), \tag{R.13}$$

and this converges uniformly for $r \in [0,1]$ to $g_0(r)$. We have Theorem R.1, with $f(r) = g_0(r)$.

Here is one illustration of Proposition R.1. Let $a_k = (-1)^{k-1}/k$, which produces a convergent series by the alternating series test (Section 0, Exercise 8). By (4.33),

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} r^k = \log(1+r), \tag{R.14}$$

for $|r| < 1$. It follows from Proposition R.1 that this infinite series converges uniformly on $[0,1]$, and hence

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \log 2. \tag{R.15}$$

See Exercise 2 in §4 for a more direct approach to (R.15), using the special behavior of alternating series. Here is a more subtle generalization, which we will establish below.

Claim. For all $\theta \in (0, 2\pi)$, the series

$$\sum_{k=1}^{\infty} \frac{e^{ik\theta}}{k} = S(\theta) \tag{R.16}$$

converges.

Given this claim, it follows from Proposition R.1 that

$$\lim_{r\nearrow 1} \sum_{k=1}^{\infty} \frac{e^{ik\theta}}{k} r^k = S(\theta), \quad \forall\, \theta \in (0, 2\pi). \tag{R.17}$$

Note that taking $\theta = \pi$ gives (R.15). We recall from §4 that the function $\log : (0,\infty) \to \mathbb{R}$ has a natural extension to

$$\log : \mathbb{C} \setminus (-\infty, 0] \longrightarrow \mathbb{C}, \tag{R.18}$$

and

$$\sum_{k=1}^{\infty} \frac{1}{k} z^k = -\log(1-z), \quad \text{for} \ \ |z| < 1, \tag{R.19}$$

from which we deduce, via Proposition R.1, that $S(\theta)$ in (R.16) satisfies

$$S(\theta) = -\log(1-e^{i\theta}), \quad 0 < \theta < 2\pi. \tag{R.20}$$

We want to establish the convergence of (R.16) for $\theta \in (0, 2\pi)$. In fact, we prove the following more general result.

Proposition R.4. *If $b_k \searrow 0$, then*

$$\sum_{k=1}^{\infty} b_k e^{ik\theta} = F(\theta) \tag{R.21}$$

converges for all $\theta \in (0, 2\pi)$.

Given Proposition R.4, it then follows from Proposition R.1 that

$$\lim_{r \nearrow 1} \sum_{k=1}^{\infty} b_k r^k e^{ik\theta} = F(\theta), \quad \forall\, \theta \in (0, 2\pi). \tag{R.22}$$

In turn, Proposition R.4 is a special case of the following more general result, known as the *Dirichlet test* for convergence of an infinite series.

Proposition R.5. *If $b_k \searrow 0$, $a_k \in \mathbb{C}$, and there exists $B < \infty$ such that*

$$s_k = \sum_{j=1}^{k} a_j \Longrightarrow |s_k| \le B, \quad \forall\, k \in \mathbb{N}, \tag{R.23}$$

then

$$\sum_{k=1}^{\infty} a_k b_k \quad \textit{converges}. \tag{R.24}$$

To apply Proposition R.5 to Proposition R.4, take $a_k = e^{ik\theta}$ and observe that

$$\sum_{j=1}^{k} e^{ij\theta} = \frac{1-e^{ik\theta}}{1-e^{i\theta}} e^{i\theta}, \tag{R.25}$$

which is uniformly bounded (in k) for each $\theta \in (0, 2\pi)$.

To prove Proposition R.5, we use summation by parts, Proposition R.3. We have, via (R.5) with $n = 0,\ s_0 = 0$,

$$\sum_{k=1}^{m} a_k b_k = s_m b_m + \sum_{k=1}^{m-1} s_k (b_k - b_{k+1}). \tag{R.26}$$

Now, if $|s_k| \le B$ for all k and $b_k \searrow 0$, then

$$\sum_{k=1}^{\infty} |s_k (b_k - b_{k+1})| \le B \sum_{k=1}^{\infty} (b_k - b_{k+1}) = B b_1 < \infty, \tag{R.27}$$

so the infinite series

$$\sum_{k=1}^{\infty} s_k (b_k - b_{k+1}) \tag{R.28}$$

is absolutely convergent, and the convergence of the left side of (R.26) readily follows.

For a first generalization of Proposition R.1, let us make a change of variable, $r \mapsto e^{-s}$, so $r \nearrow 1 \Leftrightarrow s \searrow 0$. Also think of $\{k \in \mathbb{Z}^+\}$ as a discretization of $\{t \in \mathbb{R}^+\}$. To proceed, assume we have

$$u, v : [0, \infty) \longrightarrow [0, \infty), \quad \text{monotone increasing,} \tag{R.29}$$

e.g., $t_1 < t_2 \Rightarrow u(t_1) \le u(t_2)$, and right continuous. Also assume that $u(0) = v(0) = 0$, and that

$$u(t),\ v(t) \le C_\varepsilon e^{\varepsilon t}, \quad \forall\, \varepsilon > 0. \tag{R.29A}$$

Now form

$$f(t) = u(t) - v(t). \tag{R.30}$$

An example would be a piecewise constant $f(t)$, with jumps a_k at $t = k$. The following result generalizes Proposition R.1. We use the Stieltjes integral, discussed in Appendix M.

Proposition R.6. *Take f as above, and assume*

$$f(t) \longrightarrow A, \quad \textit{as} \ \ t \to \infty. \tag{R.31}$$

Then

$$\int_0^\infty e^{-st}\, df(t) \longrightarrow A, \quad \textit{as} \ \ s \searrow 0. \tag{R.32}$$

Proof. The hypothesis (R.29A) implies the left side of (R.31) is an absolutely convergent integral for each $s > 0$. Replacing summation by parts by integration by parts in the Stieltjes integral, we have

$$\begin{aligned}\int_0^\infty e^{-st}\,df(t) &= s\int_0^\infty e^{-st}f(t)\,dt \\ &= A + s\int_0^\infty e^{-st}[f(t)-A]\,dt.\end{aligned} \tag{R.33}$$

Pick $\varepsilon > 0$, and then take $K < \infty$ such that

$$t \ge K \Longrightarrow |f(t)-A| \le \varepsilon. \tag{R.34}$$

Then

$$\begin{aligned}&s\int_0^\infty e^{-st}|f(t)-A|\,dt \\ &\le s\int_0^K e^{-st}|f(t)-A|\,dt + \varepsilon s\int_K^\infty e^{-st}\,dt \\ &\le \Bigl(\sup_{t\le K}|f(t)-A|\Bigr)Ks + \varepsilon.\end{aligned} \tag{R.35}$$

Hence

$$\limsup_{s\searrow 0}\Bigl|\int_0^\infty e^{-st}df(t) - A\Bigr| \le \varepsilon, \quad \forall\,\varepsilon > 0, \tag{R.36}$$

and we have (R.32).

We next replace the hypothesis (R.31) by

$$f(t) \sim At^\alpha, \quad \text{as} \ \ t \to \infty, \tag{R.37}$$

given $\alpha \ge 0$.

Proposition R.7. *In the setting of Proposition R.6, if hypothesis (R.31) is replaced by (R.37), with $\alpha \ge 0$, then*

$$\int_0^\infty e^{-st}\,df(t) \sim A\Gamma(\alpha+1)s^{-\alpha}, \quad \textit{as} \ \ s \searrow 0. \tag{R.38}$$

Proof. Noting that

$$\int_0^\infty e^{-st}t^\alpha\,dt = \Gamma(\alpha+1)s^{-\alpha-1}, \tag{R.39}$$

we have, in place of (R.33),

$$\begin{aligned}\int_0^\infty e^{-st}\,df(t) &= s\int_0^\infty e^{-st} f(t)\,dt \\ &= A\Gamma(\alpha+1)s^{-\alpha} + s\int_0^\infty e^{-st}[f(t)-At^\alpha]\,dt.\end{aligned} \tag{R.40}$$

Now, in place of (R.74), pick $\varepsilon > 0$ and take $K < \infty$ such that

$$t \ge K \Longrightarrow |f(t) - At^\alpha| \le \varepsilon t^\alpha. \tag{R.41}$$

We have

$$\begin{aligned} &s^{1+\alpha}\int_0^\infty e^{-st}|f(t)-At^\alpha|\,dt \\ &\le s^{1+\alpha}\int_0^K e^{-st}|f(t)-At^\alpha|\,dt + \varepsilon s^{1+\alpha}\int_K^\infty e^{-st}t^\alpha\,dt \\ &\le \Bigl(\sup_{t\le K} |f(t)-At^\alpha|\Bigr) K s^{1+\alpha} + \varepsilon\Gamma(\alpha+1).\end{aligned} \tag{R.42}$$

Hence

$$\limsup_{s\searrow 0}\Bigl|s^\alpha\int_0^\infty e^{-st}\,df(t) - A\Gamma(\alpha+1)\Bigr| \le \varepsilon\Gamma(\alpha+1), \quad \forall\,\varepsilon>0, \tag{R.43}$$

and we have (R.38).

In the next result, we weaken the hypothesis (R.37).

Proposition R.8. *Let f be as in Proposition R.7, except that we replace hypothesis (R.37) by the hypothesis*

$$f_1(t) \sim Bt^{\alpha+1}, \quad \textit{as} \ \ t\to\infty, \tag{R.44}$$

where

$$f_1(t) = \int_0^t f(\tau)\,d\tau. \tag{R.45}$$

Then the conclusion (R.38) holds, with

$$A = (\alpha+1)B. \tag{R.46}$$

Proof. In place of (R.40), write

$$\begin{aligned}\int_0^\infty e^{-st}\,df(t) &= s\int_0^\infty e^{-st}f(t)\,dt \\ &= s\int_0^\infty e^{-st}\,df_1(t).\end{aligned} \tag{R.47}$$

We apply Proposition R.7, with f replaced by f_1 (and At^α replaced by $Bt^{\alpha+1}$) to deduce that

$$\int_0^\infty e^{-st}\, df_1(t) \sim B\Gamma(\alpha+2)s^{-\alpha-1}. \tag{R.48}$$

Multiplying both sides of (R.48) by s and noting that $\Gamma(\alpha+2) = (\alpha+1)\Gamma(\alpha+1)$, we have (R.38).

The Abelian theorems given above have been stated for real-valued f, but we can readily treat complex-valued f, simply by taking the real and imaginary parts.

Tauberian theorems are to some degree converse results to Abelian theorems. However, Tauberian theorems require some auxiliary structure on f, or, in the setting of Proposition R.1, on $\{a_k\}$. To see this, we bring in the geometric series

$$\sum_{k=0}^\infty z^k = \frac{1}{1-z}, \quad \text{for } \ |z| < 1. \tag{R.49}$$

If we take $a_k = e^{ik\theta}$, then

$$\sum_{k=0}^\infty a_k r^k = \frac{1}{1-re^{i\theta}} \longrightarrow \frac{1}{1-e^{i\theta}}, \quad \text{as } \ r \nearrow 1, \tag{R.50}$$

for $0 < \theta < 2\pi$. However, since $|a_k| \equiv 1$, the series $\sum_k a_k$ is certainly not convergent. A classical theorem of Littlewood does obtain the convergence (R.1) from convergence $f(r) \to A$ in (R.2), under the hypothesis that $|a_k| \le C/k$. One can see [Don] for such a result.

The Tauberian theorems we concentrate on here require

$$f(t) = u(t) \nearrow, \tag{R.51}$$

or, in the setting of Proposition R.1, $a_k \ge 0$. In the latter case, it is clear that

$$\lim_{r\nearrow 1} \sum a_k r^k = A \Longrightarrow \sum a_k < \infty, \tag{R.52}$$

and then the "easy" result Proposition R.2 applies.

However, converses of results like Proposition R.7 when $\alpha > 0$ are not at all trivial. In particular, we have the following important result, known as Karamata's Tauberian theorem.

Proposition R.9. *Let $u : [0,\infty) \to [0,\infty)$ be an increasing, right-continuous function, as in (R.29). Take $\alpha \in (0,\infty)$, and assume*

$$\int_0^\infty e^{-st}\, du(t) \sim Bs^{-\alpha}, \quad \text{as } \ s \searrow 0. \tag{R.53}$$

Then

$$u(t) \sim \frac{B}{\Gamma(\alpha+1)} t^{\alpha}, \quad \text{as } \ t \nearrow \infty. \tag{R.54}$$

Proof. Let us phrase the hypothesis (R.53) as

$$\int_0^\infty e^{-st}\, du(t) \sim B\varphi(s), \tag{R.55}$$

where

$$\varphi(s) = s^{-\alpha} = \int_0^\infty e^{-st} v(t)\, dt, \quad v(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}. \tag{R.56}$$

Our goal in (R.54) is equivalent to showing that

$$\int_0^{1/s} du(t) = B\int_0^{1/s} v(t)\, dt + o(s^{-\alpha}), \quad s \searrow 0. \tag{R.57}$$

We tackle this problem in stages, examining when we can show that

$$\int_0^\infty F(st)\, du(t) = B\int_0^\infty F(st) v(t)\, dt + o(s^{-\alpha}), \tag{R.58}$$

for various functions $F(t)$, ultimately including

$$\begin{aligned} \chi_I(t) = 1, &\quad \text{for } \ 0 \le t < 1, \\ 0, &\quad \text{for } \ t \ge 1. \end{aligned} \tag{R.59}$$

We start with the function space

$$\mathcal{E} = \Big\{ \sum_{k=1}^M \gamma_k e^{-kt} : \gamma_k \in \mathbb{R},\, M \in \mathbb{N} \Big\}. \tag{R.60}$$

As seen in Appendix G, as a consequence of the Weierstrass approximation theorem, the space $\mathcal{E}$ is dense in

$$C_0([0,\infty)) = \big\{ f \in C([0,\infty)) : \lim_{t\to\infty} f(t) = 0 \big\}. \tag{R.61}$$

Now if $F \in \mathcal{E}$, say

$$F(t) = \sum_{k=1}^M \gamma_k e^{-kt}, \tag{R.62}$$

then (R.55) implies

$$\begin{aligned} \int_0^\infty F(st)\, du(t) &= \sum_{k=1}^M \gamma_k \int_0^\infty e^{-skt}\, du(t) \\ &= B\sum_{k=1}^M \gamma_k \varphi(ks) + o\Big(\sum_{k=1}^M (ks)^{-\alpha}\Big) \\ &= B\int_0^\infty F(st) v(t)\, dt + o(s^{-\alpha}). \end{aligned} \tag{R.63}$$

Hence (R.58) holds for all $F \in \mathcal{E}$. The following moves us along.

Lemma R.10. *In the setting of Proposition R.9, the result (R.58) holds for all*

$$F \in C_0([0,\infty)) \quad \textit{such that} \quad e^t F \in C_0([0,\infty)). \tag{R.64}$$

Proof. Given such F and given $\varepsilon > 0$, we take $H \in \mathcal{E}$ such that $\sup |H(t) - e^t F(t)| \le \varepsilon$, and set $G(t) = e^{-t} H(t)$, so

$$G \in \mathcal{E}, \quad |F(t) - G(t)| \le \varepsilon e^{-t}. \tag{R.65}$$

This implies

$$\int_0^\infty |F(st) - G(st)|\, du(t) \le \varepsilon \int_0^\infty e^{-st}\, du(t) \tag{R.66}$$

and

$$\int_0^\infty |F(st) - G(st)| v(t)\, dt \le \varepsilon \int_0^\infty e^{-st} v(t)\, dt. \tag{R.67}$$

The facts that the right sides of (R.66) and (R.67) are both $\le C\varepsilon\varphi(s)$ follow from (R.55) and (R.56), respectively. But we know that (R.58) holds with G in place of F. Hence

$$\Bigl| \int_0^\infty F(st)\, du(t) - B \int_0^\infty F(st) v(t)\, dt \Bigr| \le 2C\varepsilon\varphi(s) + o(\varphi(s)), \tag{R.68}$$

for each $\varepsilon > 0$. Taking $\varepsilon \searrow 0$ yields the lemma.

We now tackle (R.58) for $F = \chi_I$, given by (R.59). For each $\delta \in (0, 1/2]$, take $f_\delta, g_\delta \in C_0([0,\infty))$ such that

$$0 \le f_\delta \le \chi_I \le g_\delta \le 1, \tag{R.69}$$

with

$$\begin{aligned} f_\delta(t) = 1 &\quad \text{for} \quad 0 \le t \le 1 - \delta, \\ 0 &\quad \text{for} \quad t \ge 1, \end{aligned} \tag{R.70}$$

and

$$\begin{aligned} g_\delta(t) = 1 &\quad \text{for} \quad 0 \le t \le 1, \\ 0 &\quad \text{for} \quad t \ge 1 + \delta. \end{aligned} \tag{R.71}$$

Note that Lemma R.10 is applicable to each f_δ and g_δ. Hence

$$\begin{aligned} \int_0^\infty \chi_I(st)\, du(t) &\le \int_0^\infty g_\delta(st)\, du(t) \\ &= A \int_0^\infty g_\delta(st) v(t)\, dt + o(\varphi(s)), \end{aligned} \tag{R.72}$$

and

$$\tag{R.73} \begin{aligned} \int_0^\infty \chi_I(st)\,du(t) &\ge \int_0^\infty f_\delta(st)\,du(t) \\ &= A\int_0^\infty f_\delta(st)v(t)\,dt + o(\varphi(s)). \end{aligned}$$

Complementing the estimates (R.72)–(R.73), we have

$$\tag{R.74} \begin{aligned} \int_0^\infty &\bigl[g_\delta(st) - f_\delta(st)\bigr]v(t)\,dt \\ &\le \int_{(1-\delta)/s}^{(1+\delta)/s} v(t)\,dt \\ &\le \frac{2\delta}{s}\cdot\max\Bigl\{v(t) : \frac{1-\delta}{s} \le t \le \frac{1+\delta}{s}\Bigr\} \\ &\le C\delta s^{-\alpha}. \end{aligned}$$

It then follows from (R.72)–(R.73) that

$$\tag{R.75} \begin{aligned} &\limsup_{s\searrow 0}\, s^\alpha\Bigl|\int_0^\infty \chi_I(st)\,du(t) - B\int_0^\infty \chi_I(st)v(t)\,dt\Bigr| \\ &\le \inf_{\delta\le 1/2} C\delta = 0. \end{aligned}$$

This yields (R.57) and hence completes the proof of Proposition R.9.

Arguments proving Proposition R.9 can also be used to establish variants of the implication (R.53) ⇒ (R.54), such as

$$\tag{R.76} \begin{aligned} &\int_0^\infty e^{-st}\,du(t) \sim A\Bigl(\log\frac{1}{s}\Bigr)s^{-\alpha}, \quad s\searrow 0, \\ &\Longrightarrow u(t) \sim \frac{A}{\Gamma(\alpha+1)}t^\alpha(\log t), \quad t\to\infty, \end{aligned}$$

provided $u : [0,\infty) \to [0,\infty)$ is increasing and $\alpha > 0$. The reader might like to verify this. Hint: replace the calculation in (R.56) by the Laplace transform identity

$$\tag{R.77} \int_0^\infty e^{-st}t^{\alpha-1}(\log t)\,dt = \bigl(\Gamma'(\alpha) - \Gamma(\alpha)\log s\bigr)s^{-\alpha}.$$

See Exercise 3 in §18.

Putting together Propositions R.8 and R.9 yields the following result, of use in §19. In fact, Proposition R.11 below is equivalent to Proposition 19.14, which plays a role in the proof of the prime number theorem.

Proposition R.11. *Let $\psi : [0,\infty) \to [0,\infty)$ be an increasing function, and set $\psi_1(t) = \int_0^t \psi(\tau)\,d\tau$. Take $B \in (0,\infty),\ \alpha \in [0,\infty)$. Then*

$$\begin{aligned} &\psi_1(t) \sim Bt^{\alpha+1}, \quad \text{as } t \to \infty \\ &\Longrightarrow \psi(t) \sim (\alpha+1)Bt^{\alpha}, \quad \text{as } t \to \infty. \end{aligned} \tag{R.78}$$

Proof. First, by Proposition R.8, the hypothesis on $\psi_1(t)$ in (R.78) implies

$$\int_0^\infty e^{-st}\,d\psi(t) \sim B\Gamma(\alpha+2)s^{-\alpha}, \quad s \searrow 0. \tag{R.79}$$

Then Karamata's Tauberian theorem applied to (R.79) yields the conclusion in (R.78), at least for $\alpha > 0$. But such a conclusion for $\alpha = 0$ is elementary.

Karamata's Tauberian theorem is a very important tool. In addition to the application we have made in the proof of the prime number theorem, it has uses in partial differential equations, which can be found in [T2].

We mention another Tauberian theorem, known as Ikehara's Tauberian theorem.

Proposition R.12. *Let $u : [0,\infty) \to [0,\infty)$ be increasing, and consider*

$$F(s) = \int_0^\infty e^{-st}\,du(t). \tag{R.80}$$

Assume the integral is absolutely convergent on $\{s \in \mathbb{C} : \operatorname{Re} s > 1\}$ and that

$$F(s) - \frac{A}{s-1} \quad \text{is continuous on} \quad \{s \in \mathbb{C} : \operatorname{Re} s \ge 1\}. \tag{R.81}$$

Then

$$e^{-t}u(t) \longrightarrow A \quad \text{as} \quad t \to \infty. \tag{R.82}$$

We refer to [Don] for a proof of Proposition R.12. This result is applicable to (19.67),

$$-\frac{\zeta'(s)}{\zeta(s)} = \int_1^\infty x^{-s}\,d\psi(x), \tag{R.83}$$

with ψ given by (19.65). In fact, setting $u(t) = \psi(e^t)$ gives

$$-\frac{\zeta'(s)}{\zeta(s)} = \int_0^\infty e^{-ts}\,du(t). \tag{R.84}$$

Then Propositions 19.2 and 19.4 imply (R.81), with $A = 1$, so (R.82) yields $e^{-t}u(t) \to 1$, hence

$$\frac{\psi(x)}{x} \longrightarrow 1, \quad \text{as} \quad x \to \infty. \tag{R.85}$$

In this way, we get another proof of (19.69), which yields the prime number theorem. This proof requires less information on the Riemann zeta function than was used in the proof of Theorem 19.10. It requires Proposition 19.4, but not its refinement, Proposition 19.8 and Corollary 19.9. On the other hand, the proof of Ikehara's theorem is more subtle than that of Proposition R.11. This illustrates the advantage of obtaining more insight into the Riemann zeta function.

Q. Cubics, quartics, and quintics

We take up the problem of finding formulas for the roots of polynomials, i.e., elements $z \in \mathbb{C}$ such that

$$P(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-1} + \cdots + a_1 z + a_0 = 0, \tag{Q.1}$$

given $a_j \in \mathbb{C}$, with emphasis on the cases $n = 3, 4$, and 5. We start with generalities, involving two elementary transformations. First, if $z = w - a_{n-1}/n$, then

$$P(z) = Q(w) = w^n + b_{n-2}w^{n-2} + \cdots + b_0, \tag{Q.2}$$

with $b_j \in \mathbb{C}$. We have arranged that the coefficient of w^{n-1} be zero. In case $n = 2$, we get

$$Q(w) = w^2 + b_0, \tag{Q.3}$$

with roots $w = \pm\sqrt{-b_0}$, leading to the familiar quadratic formula.

For $n \geq 3$, the form of $Q(w)$ is more complicated. We next take $w = \gamma u$, so

$$Q(w) = \gamma^n R(u) = \gamma^n(u^n + c_{n-2}u^{n-2} + \cdots + c_0), \quad c_j = \gamma^{j-n} b_j. \tag{Q.4}$$

In particular, $c_{n-2} = \gamma^{-2} b_{n-2}$. This has the following significance. If $b_{n-2} \neq 0$, we can preselect $c \in \mathbb{C} \setminus 0$ and choose $\gamma \in \mathbb{C}$ such that $\gamma^{-2} b_{n-2} = c$, i.e.,

$$\gamma = \bigl(c b_{n-2}^{-1}\bigr)^{1/2}, \tag{Q.5}$$

and therefore achieve that c is the coefficient of u^{n-2} in $R(u)$.

In case $n = 3$, we get

$$R(u) = u^3 + cu + d, \quad d = \gamma^{-3} c_0. \tag{Q.6}$$

Our task is to find a formula for the roots of $R(u)$, along the way making a convenient choice of c to facilitate this task. One neat approach involves a trigonometric identity, expressing $\sin 3\zeta$ as a polynomial in $\sin\zeta$. Starting with

$$\sin(\zeta + 2\zeta) = \sin\zeta\, \cos 2\zeta + \cos\zeta\, \sin 2\zeta, \tag{Q.7}$$

it is an exercise to obtain

$$\sin 3\zeta = -4\sin^3\zeta + 3\sin\zeta, \quad \forall\, \zeta \in \mathbb{C}. \tag{Q.8}$$

Consequently, we see that the equation

$$4u^3 - 3u + 4d = 0 \tag{Q.9}$$

is solved by

$$u = \sin\zeta, \quad \text{if } \ 4d = \sin 3\zeta. \tag{Q.10}$$

Here we have taken $c = -3/4$ in (Q.6). In this case, the other solutions to (Q.9) are

$$u_2 = \sin\Bigl(\zeta + \frac{2\pi}{3}\Bigr), \quad u_3 = \sin\Bigl(\zeta - \frac{2\pi}{3}\Bigr). \tag{Q.11}$$

Now (Q.10)–(Q.11) provide formulas for the solutions to (Q.9), but they involve the transcendental functions sin and $\sin^{-1}$. We can obtain purely algebraic formulas as follows. If $4d = \sin 3\zeta$, as in (Q.10),

$$\begin{aligned} e^{3i\zeta} = \eta &\Longrightarrow \eta - \eta^{-1} = 8id \\ &\Longrightarrow \eta = 4id \pm \sqrt{-(4d)^2 + 1}. \end{aligned} \tag{Q.12}$$

Then

$$u = \sin\zeta = \frac{1}{2i}\bigl(\eta^{1/3} - \eta^{-1/3}\bigr). \tag{Q.13}$$

Note that the two roots $\eta_\pm$ in (Q.12) are related by $\eta_- = -1/\eta_+$, so they lead to the same quantity $\eta^{1/3} - \eta^{-1/3}$. In (Q.13), the cube root is regarded as a multivalued function; for $a \in \mathbb{C}$,

$$a^{1/3} = \{b \in \mathbb{C} : b^3 = a\}.$$

Similarly, if $a \neq 0$, then

$$a^{1/3} - a^{-1/3} = \{b - b^{-1} : b^3 = a\}.$$

Taking the three cube roots of η in (Q.13) gives the three roots of $R(u)$.

We have obtained an algebraic formula for the roots of (Q.6), with the help of the functions sin and $\sin^{-1}$. Now we will take an alternative route, avoiding explicit use of these functions. To get it, note that, with $v = e^{i\zeta}$, the identity (Q.8) is equivalent to

$$v^3 - v^{-3} = (v - v^{-1})^3 + 3(v - v^{-1}), \tag{Q.14}$$

which is also directly verifiable via the binomial formula. Thus, if we set

$$u = v - v^{-1}, \tag{Q.15}$$

and take $c = 3$ in (Q.6), we see that $R(u) = 0$ is equivalent to

$$v^3 - v^{-3} = -d. \tag{Q.16}$$

This time, in place of (Q.12), we have

$$
\begin{aligned}
v^3 = \eta &\Longrightarrow \eta - \eta^{-1} = -d \\
&\Longrightarrow \eta = -\frac{d}{2} \pm \frac{1}{2}\sqrt{d^2 + 4}.
\end{aligned}
\tag{Q.17}
$$

Then, in place of (Q.13), we get

$$
u = \eta^{1/3} - \eta^{-1/3}. \tag{Q.18}
$$

Again the two roots $\eta_\pm$ in (Q.17) are related by $\eta_- = -1/\eta_+$, so they lead to the same quantity $\eta^{1/3} - \eta^{-1/3}$. Furthermore, taking the three cube roots of η gives, via (Q.18), the three roots of $R(u)$. The two formulas (Q.13) and (Q.18) have a different appearance simply because of the different choices of c: $c = -3/4$ for (Q.13) and $c = 3$ for (Q.18).

We move on to quartic polynomials,

$$
P(z) = z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0. \tag{Q.19}
$$

As before, setting $z = w - a_3/4$ yields

$$
P(z) = Q(w) = w^4 + bw^2 + cw + d. \tag{Q.20}
$$

We seek a formula for the solutions to $Q(w) = 0$. We can rewrite this equation as

$$
\Bigl(w^2 + \frac{b}{2}\Bigr)^2 = -cw - d + \frac{b^2}{4}. \tag{Q.21}
$$

The left side is a perfect square, but the right side is not, unless $c = 0$. We desire to add a certain quadratic polynomial in w to both sides of (Q.21) so that the resulting polynomials are both perfect squares. We aim for the new left side to have the form

$$
\Bigl(w^2 + \frac{b}{2} + \alpha\Bigr)^2, \tag{Q.22}
$$

with $\alpha \in \mathbb{C}$ to be determined. This requires adding $2\alpha(w^2 + b/2) + \alpha^2$ to the left side of (Q.21), and adding this to the right side of (Q.21) yields

$$
2\alpha w^2 - cw + \Bigl(\alpha^2 + b\alpha + \frac{b^2}{4} - d\Bigr). \tag{Q.22A}
$$

We want this to be a perfect square. If it were, it would have to be

$$
\Bigl(\sqrt{2\alpha}\,w - \frac{c}{2\sqrt{2\alpha}}\Bigr)^2. \tag{Q.23}
$$

This is equal to (Q.22A) if and only if

$$8\alpha^3 + 4b\alpha^2 + (2b^2 - 8d)\alpha - c = 0. \tag{Q.24}$$

This is a *cubic* equation for α, solvable by means discussed above. For (Q.23) to work, we need $\alpha \neq 0$. If $\alpha = 0$ solves (Q.24), this forces $c = 0$, hence $Q(w) = w^4 + bw^2 + d$, which is a quadratic polynomial in w^2, solvable by elementary means. Even if $c = 0$, (Q.24) has a nonzero root unless also $b = d = 0$, i.e., unless $Q(w) = w^4$.

Now, assuming $Q(w) \neq w^4$, we pick α to be *one* nonzero solution to (Q.24). Then the solutions to $Q(w) = 0$ are given by

$$w^2 + \frac{b}{2} + \alpha = \pm\Bigl(\sqrt{2\alpha}w - \frac{c}{2\sqrt{2\alpha}}\Bigr). \tag{Q.25}$$

This is a pair of quadratic equations. Each has two roots, and together they yield the four roots of $Q(w)$.

It is interesting to consider a particular quartic equation for which a different approach, not going through (Q.20), is effective, namely

$$z^4 + z^3 + z^2 + z + 1 = 0, \tag{Q.26}$$

which arises from factoring $z - 1$ out of $z^5 - 1$, therefore seeking the other fifth roots of unity. Let us multiply (Q.26) by z^{-2}, obtaining

$$z^2 + z + 1 + z^{-1} + z^{-2} = 0. \tag{Q.27}$$

The symmetric form of this equation suggests making the substitution

$$w = z + z^{-1}, \tag{Q.28}$$

so

$$w^2 = z^2 + 2 + z^{-2}, \tag{Q.29}$$

and (Q.27) becomes

$$w^2 + w - 1 = 0, \tag{Q.30}$$

a quadratic equation with solutions

$$w = -\frac{1}{2} \pm \frac{1}{2}\sqrt{5}. \tag{Q.31}$$

Then (Q.28) becomes a quadratic equation for z. We see that (Q.26) is solvable in a fashion that requires no extraction of cube roots. Noting that the roots of (Q.26) have absolute value 1, we see that $w = 2\,\mathrm{Re}\,z$, and (Q.31) says

$$\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}, \quad \cos\frac{4\pi}{5} = -\frac{\sqrt{5}+1}{4}. \tag{Q.32}$$

Such a calculation allows one to construct a regular pentagon with compass and straightedge.

Let us extend the scope of this, and look at solutions to $z^7 - 1 = 0$, arising in the construction of a regular 7-gon. Factoring out $z - 1$ yields

$$z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0, \tag{Q.33}$$

or equivalently

$$z^3 + z^2 + z + 1 + z^{-1} + z^{-2} + z^{-3} = 0. \tag{Q.34}$$

Again we make the substitution (Q.28). Complementing (Q.29) with

$$w^3 = z^3 + 3z + 3z^{-1} + z^{-3}, \tag{Q.35}$$

we see that (Q.34) leads to the *cubic* equation

$$q(w) = w^3 + w^2 - 2w - 1 = 0. \tag{Q.36}$$

Since $q(-1) > 0$ and $q(0) < 0$, we see that (Q.36) has three real roots, satisfying

$$w_3 < w_2 < 0 < w_1, \tag{Q.37}$$

and, parallel to (Q.32), we have

$$\cos\frac{2\pi}{7} = \frac{w_1}{2}, \quad \cos\frac{4\pi}{7} = \frac{w_2}{2}, \quad \cos\frac{6\pi}{7} = \frac{w_3}{2}. \tag{Q.38}$$

One major difference between (Q.32) and (Q.38) is that the computation of w_j involves the extraction of cube roots. In the time of Euclid, the problems of whether one could construct cube roots or a regular 7-gon by compass and straightedge were regarded as major mysteries. Much later, a young Gauss proved that one could make such a construction of a regular n-gon if and only if n is of the form 2^k, perhaps times a product of distinct Fermat primes, i.e., primes of the form $p = 2^{2^j} + 1$. The smallest examples are $p = 2^1 + 1 = 3$, $p = 2^2 + 1 = 5$, and $p = 2^4 + 1 = 17$. Modern treatments of these problems cast them in the framework of Galois theory; see [L].

We now consider fifth degree polynomials,

$$P(z) = z^5 + a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0. \tag{Q.39}$$

The treatment of this differs markedly from that of equations of degree ≤ 4, in that one cannot find a formula for the roots in terms of radicals, i.e., involving a finite number of arithmetical operations and extraction of nth roots. That no such general formula exists was proved by Abel. Then Galois showed that specific equations, such as

$$z^5 - 16z + 2 = 0, \tag{Q.40}$$

had roots that could not be obtained from the set $\mathbb{Q}$ of rational numbers via radicals. We will not delve into Galois theory here; see [L]. Rather, we will discuss how there are formulas for the roots of (Q.39) that bring in other special functions.

In our analysis, we will find it convenient to assume the roots of $P(z)$ are distinct. Otherwise, any double root of $P(z)$ is also a root of $P'(z)$, which has degree 4. We also assume $z = 0$ is not a root, i.e., $a_0 \neq 0$.

A key tool in the analysis of (Q.39) is the reduction to *Bring-Jerrard normal form*:

$$Q(w) = w^5 - w + a. \tag{Q.41}$$

That is to say, given $a_j \in \mathbb{C}$, one can find $a \in \mathbb{C}$ such that the roots of $P(z)$ in (Q.39) are related to the roots of $Q(w)$ in (Q.41) by means of solving polynomial equations of degree ≤ 4. Going from (Q.39) to (Q.41) is done via a *Tschirnhaus transformation*. Generally, such a transformation takes $P(z)$ in (Q.39) to a polynomial of the form

$$Q(w) = w^5 + b_4 w^4 + b_3 w^3 + b_2 w^2 + b_1 w + b_0, \tag{Q.42}$$

in a way that the roots of $P(z)$ and of $Q(w)$ are realted as described above. The ultimate goal is to produce a Tschirnhaus transformation that yields (Q.42) with

$$b_4 = b_3 = b_2 = 0. \tag{Q.43}$$

As we have seen, the linear change of variable $z = w - a_4/5$ achieves $b_4 = 0$, but here we want to achieve much more. This will involve a nonlinear change of variable.

Following [A4], we give a matrix formulation of Tschirnhaus transformations. Relevant linear algebra background can be found in §§6–7 of [T5]. To start, given (Q.39), pick a matrix $A \in M(5, \mathbb{C})$ whose characteristic polynomial is $P(z)$,

$$P(z) = \det(zI - A). \tag{Q.44}$$

For example, A could be the companion matrix of P. Note that the set of eigenvalues of A,

$$\operatorname{Spec} A = \{z_j : 1 \leq j \leq 5\}, \tag{Q.45}$$

is the set of roots of (Q.39). The Cayley-Hamilton theorem implies

$$P(A) = A^5 + a_4 A^4 + a_3 A^3 + a_2 A^2 + a_1 A + a_0 I = 0. \tag{Q.46}$$

It follows that

$$\mathcal{A} = \operatorname{Span}\{I, A, A^2, A^3, A^4\} \tag{Q.47}$$

is a commutative matrix algebra. The hypothesis that the roots of A are disinct implies that P is the minimal polynomial of A, so the 5 matrices listed in (Q.47) form a basis of $\mathcal{A}$.

In this setting, a Tschirnhaus transformation is produced by taking

$$B = \beta(A) = \sum_{j=0}^{m} \beta_j A^j, \tag{Q.48}$$

where $\beta(z)$ is a non-constant polynomial of degree $m \le 4$. Then $B \in \mathcal{A}$, with characteristic polynomial

$$Q(w) = \det(wI - B), \tag{Q.49}$$

of the form (Q.42). The set of roots of $Q(w)$ forms

$$\operatorname{Spec} B = \{\beta(z_j) : z_j \in \operatorname{Spec} A\}. \tag{Q.50}$$

We can entertain two possibilities, depending on the behavior of

$$\{I, B, B^2, B^3, B^4\}. \tag{Q.51}$$

Case I. The set (Q.51) is linearly dependent.
Then $q(B) = 0$ for some polynomial $q(w)$ of degree ≤ 4, so

$$\operatorname{Spec} B = \{w_j : 1 \le j \le 5\} \tag{Q.52}$$

and each w_j is a root of q. Methods described earlier in this appendix apply to solving for the roots of q, and to find the roots of $P(z)$, i.e., the elements of $\operatorname{Spec} A$, we solve

$$\beta(z_j) = w_j \tag{Q.53}$$

for z_j. Since, for each j, (Q.53) has m solutions, this may produce solutions not in $\operatorname{Spec} A$, but one can test each solution z_j to see if it is a root of $P(z)$.

Case II. The set (Q.51) is linearly independent.
Then this set spans $\mathcal{A}$, so we can find $\gamma_j \in \mathbb{C}$ such that

$$A = \sum_{j=0}^{4} \gamma_j B^j = \Gamma(B). \tag{Q.54}$$

It follows that

$$\operatorname{Spec} A = \{\Gamma(w_j) : w_j \in \operatorname{Spec} B\}. \tag{Q.55}$$

It remains to find $\operatorname{Spec} B$, i.e., the set of roots of $Q(w)$ in (Q.42). It is here that we want to implement (Q.43). The following result is relevant to this endeavor.

Lemma Q.1. *Let $Q(w)$, given by (Q.49), have the form (Q.42), and pick $\ell \in \{1, \dots, 5\}$. Then*

$$b_{5-j} = 0 \text{ for } 1 \le j \le \ell \Longleftrightarrow \operatorname{Tr} B^j = 0 \text{ for } 1 \le j \le \ell. \tag{Q.56}$$

Proof. To start, we note that $b_4 = -\operatorname{Tr} B$. More generally, b_{5-j} is given (up to a sign) as an elementary symmetric polynomial in the eigenvalues $\{w_1, \dots, w_5\}$ of B. The equivalence (Q.51) follows from the classical Newton formula for these symmetric polynomials in terms of the polynomials $w_1^j + \cdots + w_5^j = \operatorname{Tr} B^j$.

We illustrate the use of (Q.48) to achieve (Q.56) in case $\ell = 2$. In this case, we take $m = 2$ in (Q.40), and set

$$B = \beta_0 I + \beta_1 A + \beta_2 A^2, \quad \beta_2 = 1. \tag{Q.57}$$

Then

$$\begin{aligned} B^2 = \beta_0^2 I + 2\beta_0\beta_1 A + (2\beta_2 + \beta_1^2) A^2 \\ + 2\beta_1\beta_2 A^3 + \beta_2^2 A^4. \end{aligned} \tag{Q.57A}$$

Then, if

$$\xi_j = \operatorname{Tr} A^j, \tag{Q.58}$$

we obtain

$$\begin{aligned} \operatorname{Tr} B &= 5\beta_0 + \xi_1\beta_1 + \xi_2, \\ \operatorname{Tr} B^2 &= 5\beta_0^2 + 2\xi_1\beta_0\beta_1 + \xi_2(\beta_1^2 + 2) + 2\xi_3\beta_1 + \xi_4. \end{aligned} \tag{Q.59}$$

Set $\operatorname{Tr} B = \operatorname{Tr} B^2 = 0$. Then the first identity in (Q.55) yields

$$\beta_0 = -\frac{1}{5}(\xi_1\beta_1 + \xi_2), \tag{Q.60}$$

and substituting this into the second identity of (Q.59) gives

$$\frac{1}{5}(\xi_1\beta_1 + \xi_2)^2 - \frac{2}{5}\xi_1\beta_1(\xi_1\beta_1 + \xi_2) + \xi_2\beta_1^2 + 2\xi_3\beta_1 = -2\xi_2 - \xi_4, \tag{Q.61}$$

a quadratic equation for β_1, with leading term $(\xi_2 - \xi_1^2/5)\beta_1^2$. We solve for β_0 and β_1, and hence obtain $B \in \mathcal{A}$ with characteristic polynomial $Q(w)$ satisfying

$$Q(w) = w^5 + b_2 w^2 + b_1 w + b_0. \tag{Q.62}$$

This goes halfway from (Q.2) (with $n = 5$) to (Q.43).

Before discussing closing this gap, we make another comment on achieving (Q.62). Namely, suppose A has been prepped so that

$$\operatorname{Tr} A = 0, \tag{Q.63}$$

i.e., A is replaced by $A - (1/5)(\operatorname{Tr} A)I$. Apply (Q.57) to this new A. Then $\xi_1 = 0$, so (Q.60)–(Q.61) simplify to

$$\beta_0 = -\frac{1}{5}\xi_2, \tag{Q.64}$$

and

$$\xi_2\beta_1^2 + 2\xi_3\beta_1 = -\frac{1}{5}\xi_2^2 - 2\xi_2 - \xi_4. \tag{Q.65}$$

This latter equation is a quadratic equation for β_1 if $\xi_2 = \operatorname{Tr} A^2 \neq 0$. Of course, if $\operatorname{Tr} A^2 = 0$, we have already achieved our goal (Q.62), with $B = A$.

Moving forward, let us now assume we have

$$A \in M(5, \mathbb{C}), \quad \operatorname{Tr} A = \operatorname{Tr} A^2 = 0, \tag{Q.66}$$

having minimal polynomial of the form (Q.39) with $a_4 = a_3 = 0$, and we desire to construct B as in (Q.48), satisfying

$$\operatorname{Tr} B = \operatorname{Tr} B^2 = \operatorname{Tr} B^3 = 0. \tag{Q.67}$$

At this point, a first try would take

$$B = \beta_0 I + \beta_1 A + \beta_2 A^2 + \beta_3 A^3, \quad \beta_3 = 1. \tag{Q.68}$$

Calculations parallel to (Q.57)–(Q.65) yield first

$$\beta_0 = -\frac{1}{5}\xi_3, \tag{Q.69}$$

and then a pair of polynomial equations for (β_1, β_2), one of degree 2 and one of degree 3. However, this system is more complicated than the 5th degree equation we are trying to solve. Another attack is needed.

E. Bring, and, independently, G. Jerrard, succeeded in achieving (Q.43) by using a quartic transformation. In the current setting, this involves replacing (Q.68) by

$$B = \beta_0 I + \beta_1 A + \beta_2 A^2 + \beta_3 A^3 + \beta_4 A^4, \quad \beta_4 = 1. \tag{Q.70}$$

The extra parameter permits one to achieve (Q.67) with coefficients $\beta_0, \dots, \beta_3$ determined by fourth order equations. The computations are lengthy, and we refer to [Ki] for more details.

Once one has $Q(w)$ satisfying (Q.42)–(Q.43), i.e.,

$$Q(w) = w^5 + b_1 w + b_0, \tag{Q.71}$$

then, if $b_1 \neq 0$, one can take $w = \gamma u$ as in (Q.4), and, by a parallel computation, write

$$Q(w) = \gamma^5 R(u), \quad R(u) = u^5 - u + a, \tag{Q.72}$$

with

$$\gamma^4 = -b_1, \quad a = \gamma^{-5} b_0. \tag{Q.73}$$

$R(u)$ thus has the Bring-Jerrard normal form (Q.41). Solving $R(u) = 0$ is equivalent to solving

$$\Phi(z) = a, \quad \Phi(z) = z - z^5. \tag{Q.74}$$

Consequently our current task is to study mapping properties of $\Phi : \mathbb{C} \to \mathbb{C}$ and its inverse Φ^{-1}, a multi-valued function known as the *Bring radical.*

To start, note that

$$\Phi'(z) = 1 - 5z^4, \tag{Q.75}$$

hence

$$\Phi'(\zeta) = 0 \Longleftrightarrow \zeta \in \mathcal{C} = \{\pm 5^{-1/4}, \pm i 5^{-1/4}\}. \tag{Q.76}$$

If $z_0 \in \mathbb{C} \setminus \mathcal{C}$, the inverse function theorem, Theorem 4.2, implies that there exist neighborhoods $\mathcal{O}$ of z_0 and U of $\Phi(z_0)$ such that $\Phi : \mathcal{O} \to U$ is one-to-one and onto, with holomorphic inverse. This observation applies in particular to $z_0 = 0$, since

$$\Phi(0) = 0, \quad \Phi'(0) = 1. \tag{Q.77}$$

Note that, by (Q.75),

$$\begin{aligned} |z| < 5^{-1/4} &\Longrightarrow |\Phi'(z) - 1| < 1 \\ &\Longrightarrow \operatorname{Re} \Phi'(z) > 0, \end{aligned} \tag{Q.78}$$

so, by Proposition 4.3,

$$\Phi : D_{5^{-1/4}}(0) \longrightarrow \mathbb{C} \ \text{ is one-to-one,} \tag{Q.79}$$

where, for $\rho \in (0, \infty),\ z_0 \in \mathbb{C}$,

$$D_\rho(z_0) = \{z \in \mathbb{C} : |z - z_0| < \rho\}. \tag{Q.80}$$

Also note that

$$
\text{(Q.81)}\qquad \begin{aligned} |z| = 5^{-1/4} \Longrightarrow |\Phi(z)| &= |z - z^5| \\ &\geq 5^{-1/4} - 5^{-5/4} = (4/5)5^{-1/4}. \end{aligned}
$$

Hence, via results of §17 on the argument principle,

$$
\text{(Q.82)}\qquad \Phi(D_{5^{-1/4}}(0)) \supset D_{(4/5)5^{-1/4}}(0).
$$

We deduce that the map (Q.79) has holomorphic inverse

$$
\text{(Q.83)}\qquad \Phi^{-1} : D_{(4/5)5^{-1/4}}(0) \longrightarrow D_{5^{-1/4}}(0) \subset \mathbb{C},
$$

satisfying $\Phi^{-1}(0) = 0$.

Note that

$$
\text{(Q.84)}\qquad \Phi(ia) = i\Phi(a).
$$

Hence we can write, for $|a| < (4/5)5^{-1/4}$,

$$
\text{(Q.85)}\qquad \Phi^{-1}(a) = a\Psi(a^4),
$$

with

$$
\text{(Q.86)}\qquad \Psi(b) \ \text{ holomorphic in } \ |b| < \frac{4^4}{5^5},
$$

satisfying

$$
\text{(Q.87)}\qquad a\Psi(a^4) - a^5\Psi(a^4) = a, \quad \Psi(0) = 1,
$$

hence

$$
\text{(Q.88)}\qquad \Psi(b) = 1 + b\Psi(b)^5, \quad \Psi(0) = 1.
$$

Using (Q.88), we can work out the power series

$$
\text{(Q.89)}\qquad \Psi(b) = \sum_{k=0}^{\infty} \psi_k b^k, \quad \psi_0 = 1,
$$

as follows. First, (Q.89) yields

$$
\text{(Q.90)}\qquad \begin{aligned} \Psi(b)^5 &= \prod_{\nu=1}^{5} \sum_{\ell_\nu=0}^{\infty} \psi_{\ell_\nu} b^{\ell_\nu} \\ &= \sum_{k=0}^{\infty} \sum_{\ell\geq 0, |\ell|=k} \psi_{\ell_1} \cdots \psi_{\ell_5} b^k, \end{aligned}
$$

where $\ell = (\ell_1, \dots, \ell_5)$, $|\ell| = \ell_1 + \cdots + \ell_5$. Then (Q.88) yields

$$\begin{aligned} b \sum_{k \ge 0} \psi_{k+1} b^k &= b \Psi(b)^5 \\ &= b \sum_{k \ge 0} \sum_{\ell \ge 0, |\ell| = k} \psi_{\ell_1} \cdots \psi_{\ell_5} b^k, \end{aligned} \tag{Q.91}$$

hence

$$\psi_{k+1} = \sum_{\ell \ge 0, |\ell| = k} \psi_{\ell_1} \cdots \psi_{\ell_5}, \quad \text{for} \quad k \ge 0. \tag{Q.92}$$

While this recursive formula is pretty, it is desirable to have an explicit power series formula. Indeed, one has the following.

Proposition Q.2. *In the setting of (Q.83),*

$$\Phi^{-1}(a) = \sum_{j=0}^{\infty} \binom{5j}{j} \frac{a^{4j+1}}{4j+1}, \quad \textit{for} \quad |a| < \frac{4}{5} 5^{-1/4}. \tag{Q.93}$$

Proof. By Proposition 5.6, we have from (Q.79)–(Q.83) that

$$\Phi^{-1}(a) = \frac{1}{2\pi i} \int_{\partial \mathcal{O}} \frac{z \Phi'(z)}{\Phi(z) - a} \, dz, \tag{Q.94}$$

with $\mathcal{O} = D_{5^{-1/4}}(0)$, $|a| < (4/5) 5^{-1/4}$. We then have the convergent power series

$$\begin{aligned} \frac{1}{\Phi(z) - a} &= \frac{1}{\Phi(z)} \frac{1}{1 - a/\Phi(z)} \\ &= \sum_{k \ge 0} \frac{a^k}{\Phi(z)^{k+1}}, \end{aligned} \tag{Q.95}$$

given $z \in \partial \mathcal{O}$, $|a| < (4/5) 5^{-1/4}$. Hence

$$\Phi^{-1}(a) = \frac{1}{2\pi i} \sum_{k \ge 0} \Big(\int_{\partial \mathcal{O}} \frac{z \Phi'(z)}{\Phi(z)^{k+1}} \, dz \Big) a^k. \tag{Q.96}$$

Since $\Phi'(z) = 1 - 5z^4$ and $\Phi(z) = z(1 - z^4)$, the coefficient of a^k is

$$\frac{1}{2\pi i} \int_{\partial \mathcal{O}} \frac{1 - 5z^4}{(1 - z^4)^{k+1}} \frac{dz}{z^k}, \tag{Q.97}$$

or equivalently, the coefficient of a^k is equal to the coefficient of z^{k-1} in the power series expansion of $(1-5z^4)/(1-z^4)^{k+1}$. This requires $k = 4j+1$ for some $j \in \mathbb{Z}^+$, and we then seek

$$\text{the coefficient of } \zeta^j \text{ in } \frac{1-5\zeta}{(1-\zeta)^{k+1}}, \quad k = 4j+1. \tag{Q.98}$$

We have

$$(1-\zeta)^{-(k+1)} = \sum_{j=0}^{\infty} \binom{k+j}{j} \zeta^j, \tag{Q.99}$$

hence

$$\begin{aligned} -5\zeta(1-\zeta)^{-(k+1)} &= -5\sum_{\ell=0}^{\infty} \binom{k+\ell}{\ell} \zeta^{\ell+1} \\ &= -5\sum_{j=1}^{\infty} \binom{k+j-1}{j-1} \zeta^j. \end{aligned} \tag{Q.100}$$

Thus the coefficient specified in (Q.98) is 1 for $j = 0$, and, for $j \geq 1$, it is

$$\begin{aligned} &\binom{k+j}{j} - 5\binom{k+j-1}{j-1}, \quad \text{with } k = 4j+1, \\ &= \binom{5j+1}{j} - 5\binom{5j}{j-1} \\ &= \frac{1}{4j+1}\binom{5j}{j}, \end{aligned} \tag{Q.101}$$

giving (Q.93).

We next discuss some global aspects of the map $\Phi : \mathbb{C} \to \mathbb{C}$, making use of results on covering maps from §25. To state the result, let us take $\mathcal{C} = \{\pm 5^{-1/4}, \pm i5^{-1/4}\}$, as in (Q.76), and set

$$\mathcal{V} = \Phi(\mathcal{C}) = \frac{4}{5}\mathcal{C}, \quad \widetilde{\mathcal{C}} = \Phi^{-1}(\mathcal{V}). \tag{Q.102}$$

Lemma Q.3. *The map*

$$\Phi : \mathbb{C} \setminus \widetilde{\mathcal{C}} \longrightarrow \mathbb{C} \setminus \mathcal{V} \tag{Q.103}$$

is a 5-fold covering map.

Note that (Q.81) implies

$$D_{(4/5)5^{-1/4}}(0) \subset \mathbb{C} \setminus \mathcal{V}. \tag{Q.104}$$

The following is a consequence of Propositions 25.1–25.2.

Proposition Q.4. *Assume Ω is an open, connected, simply connected set satisfying*

$$\Omega \subset \mathbb{C} \setminus \mathcal{V}, \quad \Omega \supset D_{(4/5)5^{-1/4}}(0). \tag{Q.105}$$

Then Φ^{-1} in (Q.83) has a unique extension to a holomorphic map

$$\Phi^{-1} : \Omega \longrightarrow \mathbb{C} \setminus \widetilde{\mathcal{C}}. \tag{Q.106}$$

A direct path from Proposition Q.2 to Proposition Q.4 is provided by recognizing the power series (Q.93) as representing $Q^{-1}(a)$ in terms of a generalized hypergeometric function, namely

$$\Phi^{-1}(a) = a\,{}_4F_3\Big(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}; \frac{1}{2}, \frac{3}{4}, \frac{5}{4}; 5\Big(\frac{5a}{4}\Big)^4\Big). \tag{Q.107}$$

See §36 for a discussion of hypergeometric functions, their differential equations and analytic continuations, and a proof of (Q.107).

The analysis of the Bring radical was carried further by C. Hermite, who produced a formula for Φ^{-1} in terms of elliptic functions and their associated theta functions. This was also pursued by L. Kronecker and F. Klein. For more on this, we refer to [Kl], and to Chapter 5 of [MM]. Work on the application of theta functions to higher degree polynomials is given in [Um].

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