

## Feedback Control System Numerical Example

### Numerically Solving Feedback Control Problems

Most of the emphasis in this course is in the analytic solutions for fairly simple processes. We can obviously do direct numerical solutions, too, though. There may be some difficulty in defining the integral & derivative terms for the controller however.

Let's consider a constant volume, well-mixed stirred tank with heating provided by a steam coil. The tank operates with 2,000 L (2.0 m<sup>3</sup>) of liquid. The inlet flow rate is 250 L/min (0.25 m<sup>3</sup>/min) at a temperature of 20°C. At steady state 350°C steam is sufficient to heat the effluent to 80°C. Let's further assume the liquid density is a constant 1 kg/L and the heat capacity is a constant 1 kcal/kg·°C.

The heat balance around the tank will be:

$$\rho \hat{C}_p V \frac{dT_1}{dt} = \rho F_0 \hat{C}_p (T_0 - T_1) + (UA)(T_s - T_1).$$

The traditional method to analyze the control scheme is to linearize the ODE:

$$\rho \hat{C}_p V \frac{dT'_1}{dt} = \rho F_0^* \hat{C}_p (T'_0 - T'_1) + \rho F_0' \hat{C}_p (T_0^* - T_1^*) + (UA)(T'_s - T'_1)$$

and put into a form that leads to transfer functions:

$$\frac{\rho \hat{C}_p V}{\rho F_0^* \hat{C}_p + UA} \cdot \frac{dT'_1}{dt} + T'_1 = \frac{\rho F_0^* \hat{C}_p}{\rho F_0^* \hat{C}_p + UA} T'_0 + \frac{(UA)}{\rho F_0^* \hat{C}_p + UA} T'_s + \frac{\rho \hat{C}_p (T_0^* - T_1^*)}{\rho F_0^* \hat{C}_p + UA} F'_0$$

$$\left( \frac{\rho \hat{C}_p V}{\rho F_0^* \hat{C}_p + UA} s + 1 \right) \bar{T}'_1 = \frac{\rho F_0^* \hat{C}_p}{\rho F_0^* \hat{C}_p + UA} \bar{T}'_0 + \frac{(UA)}{\rho F_0^* \hat{C}_p + UA} \bar{T}'_s + \frac{\rho \hat{C}_p (T_0^* - T_1^*)}{\rho F_0^* \hat{C}_p + UA} \bar{F}'_0$$

$$\bar{T}'_1 = \frac{K_0}{\tau s + 1} \bar{T}'_0 + \frac{K_s}{\tau s + 1} \bar{T}'_s + \frac{K_F}{\tau s + 1} \bar{F}'_0$$

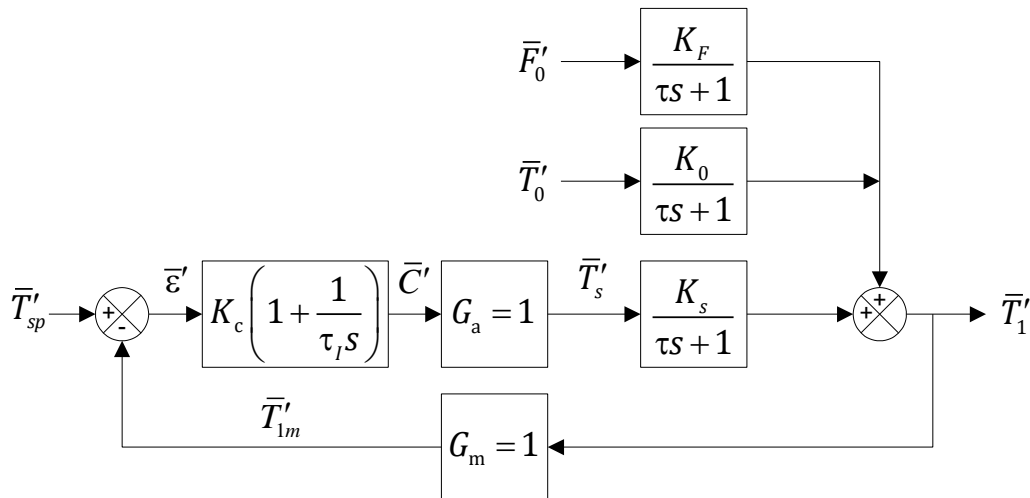
where:  $\tau \equiv \frac{\rho \hat{C}_p V}{\rho F_0^* \hat{C}_p + UA}$

$$K_0 \equiv \frac{\rho F_0^* \hat{C}_p}{\rho F_0^* \hat{C}_p + UA}$$

$$K_s \equiv \frac{(UA)}{\rho F_0^* \hat{C}_p + UA}$$

$$K_F \equiv \frac{\rho \hat{C}_p (T_0^* - T_1^*)}{\rho F_0^* \hat{C}_p + UA}$$

Typically we would manipulate the steam temperature to deal with any disturbances from the inlet temperature and/or flowrate. The following is an information diagram for a PI controller scheme.



What if we want to look at the response to a change in the inlet flow rate. If we use analytic techniques we get an answer to the linearized ODE. We could use numerical techniques to determine how closely this predicted response is to what we would expect from the non-linearized ODE.

First, let's use the initial steady state condition to determine the  $UA$  value (which we will hold constant). The heat balance at the steady state is:

$$0 = \rho F_0^* \hat{C}_p (T_0^* - T_1^*) + (UA)(T_s^* - T_1^*) \Rightarrow UA = \frac{\rho F_0^* \hat{C}_p (T_1^* - T_0^*)}{T_s^* - T_1^*}$$

$$= \frac{(1)(0.25)(1)(80 - 20)}{350 - 80}$$

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$$= 0.05556 \text{ kcal/}^\circ\text{C}\cdot\text{min}$$

**POLYMATH 5.0 beta test #193 - [Ordinary Differential Equations Solver]**

File Edit Program Window Examples Help

Open LEQ NLE DEQ REG Calculate Units Const Setup

Indep Var:  Initial Value:

Solve with:  Final Value:

☒ Comments

|    | Differential equations / explicit equations  | Initial value | Comments   |
|----|--|---------------|--|
| 1  | $d(T1)/dt = (\rho \cdot F0 \cdot Cp \cdot (T0 - T1) + UA \cdot (Ts - T1)) / (\rho \cdot Cp \cdot V)$ | 80            | Heat Balance Equation                                      |
| 2  | $d(ErrorSum)/dt = Error$   | 0             | Integrate the Error  |
| 3  | $V = 2.0$  | n.a.          | Volume   |
| 4  | $\rho = 1.0$   | n.a.          | Density  |
| 5  | $Cp = 1.0$   | n.a.          | Heat capacity  |
| 6  | $F0 = 0.50$  | n.a.          | Inlet flow rate  |
| 7  | $T0 = T0Star$  | n.a.          | Inlet temperature  |
| 8  | $UA = \rho \cdot Cp \cdot F0Star \cdot (T1Star - T0Star) / (TsStar - T1Star)$                        | n.a.          | Back-calculate product of heat transfer coefficient & area |
| 9  | $Tm = T1$  | n.a.          | Measured temperature                                       |
| 10 | $Error = 80 - Tm$  | n.a.          | Controller error - value is set point                      |
| 11 | $Ts = TsStar + Kc \cdot (Error + ErrorSum \cdot InvIntTime)$   | n.a.          | Manipulated steam temperature                              |
| 12 | $Kc = 0.$  | n.a.          | Controller Gain  |
| 13 | $InvIntTime = 0.$  | n.a.          | Inverse of Integral Time                                   |
| 14 | $F0Star = 0.25$  | n.a.          | Initial Steady State                                       |
| 15 | $T0Star = 20$  | n.a.          | Initial Steady State                                       |
| 16 | $TsStar = 350$   | n.a.          | Initial Steady State                                       |
| 17 | $T1Star = 80.$   | n.a.          | Initial Steady State                                       |

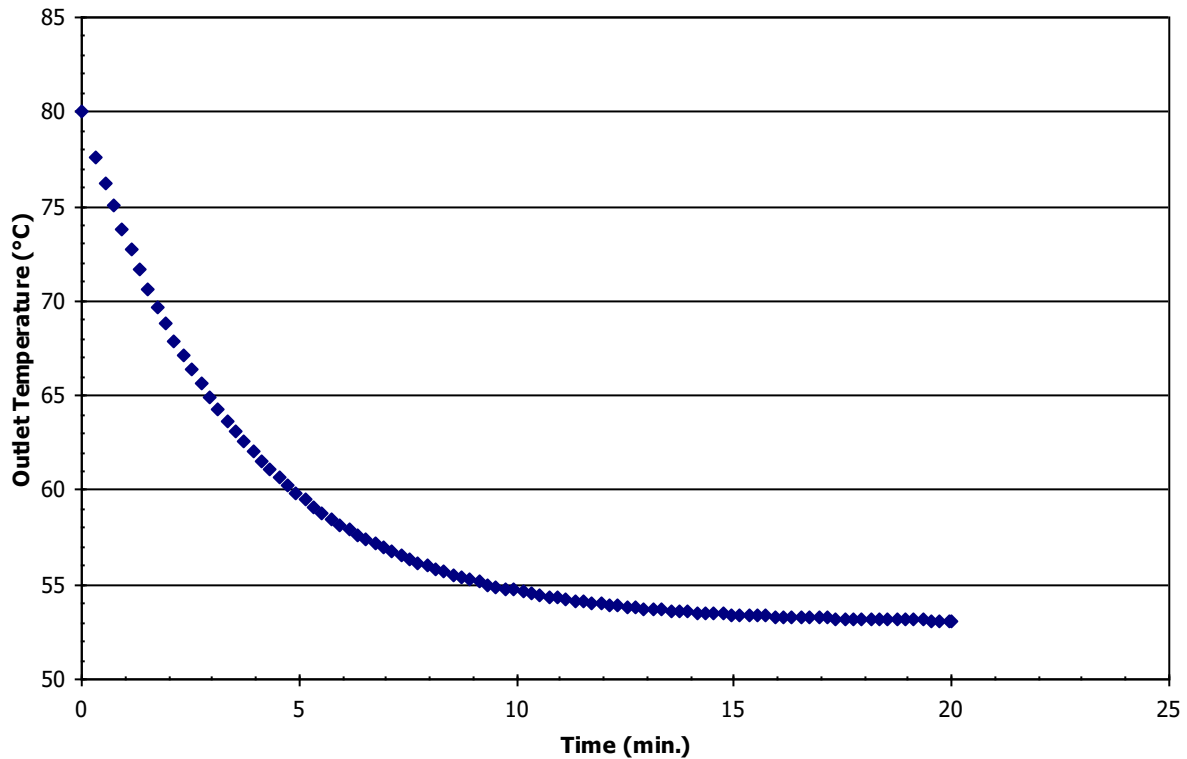
Differential Equations: 2 Auxiliary Equations: 15

13\_FeedbackControl.pol Serial Cascade of 3 Identical CSTRs with Feedback PID Contr 3:29 PM 12/28/2004 CAPS NUM

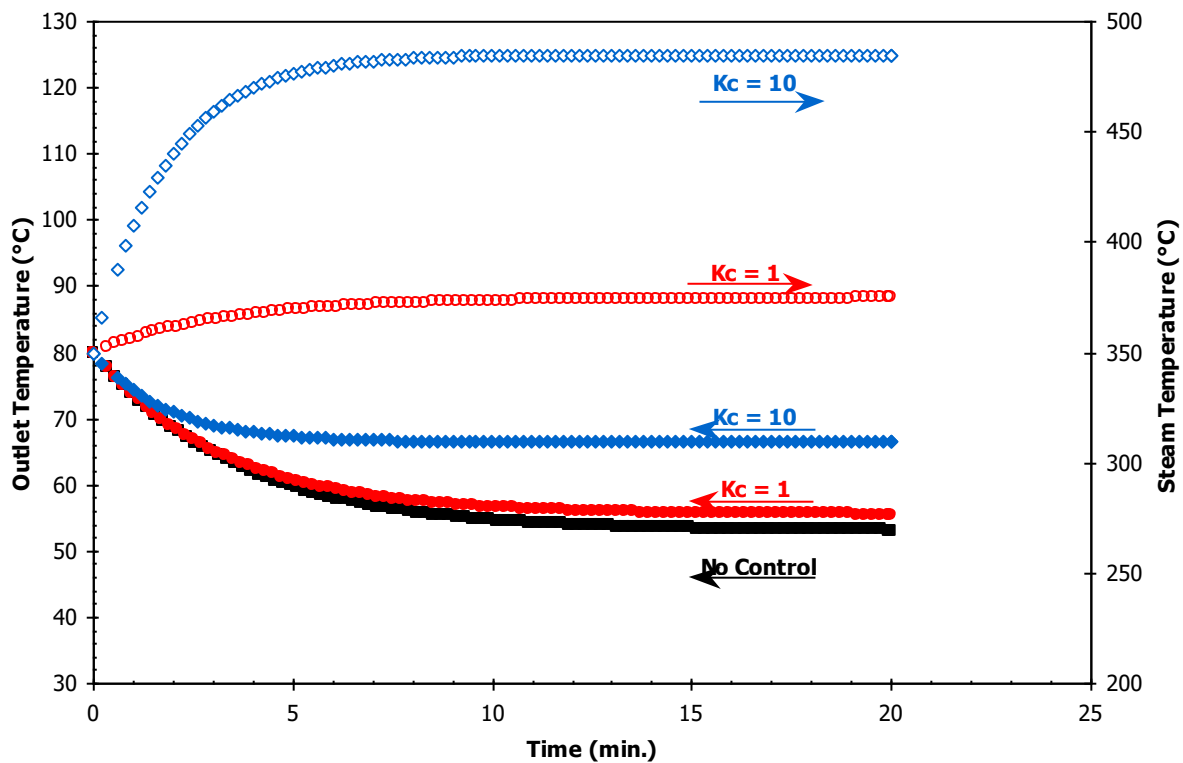
This screen shot shows the input to *POLYMATH* to numerically solve this problem of when the controller is turned off ( $K_c = 0$  and  $1/\tau_i = 0$ ). Note that there is a term **ErrorSum** that integrates the difference between the measured outlet temperature and the set point and is needed for the integral control.

Let's calculate the dynamic response when the inlet flow rate is doubled to 500 L/min ( $0.50 \text{ m}^3/\text{min}$ ). From the original steady state equation we would expect the new steady state outlet temperature to be:

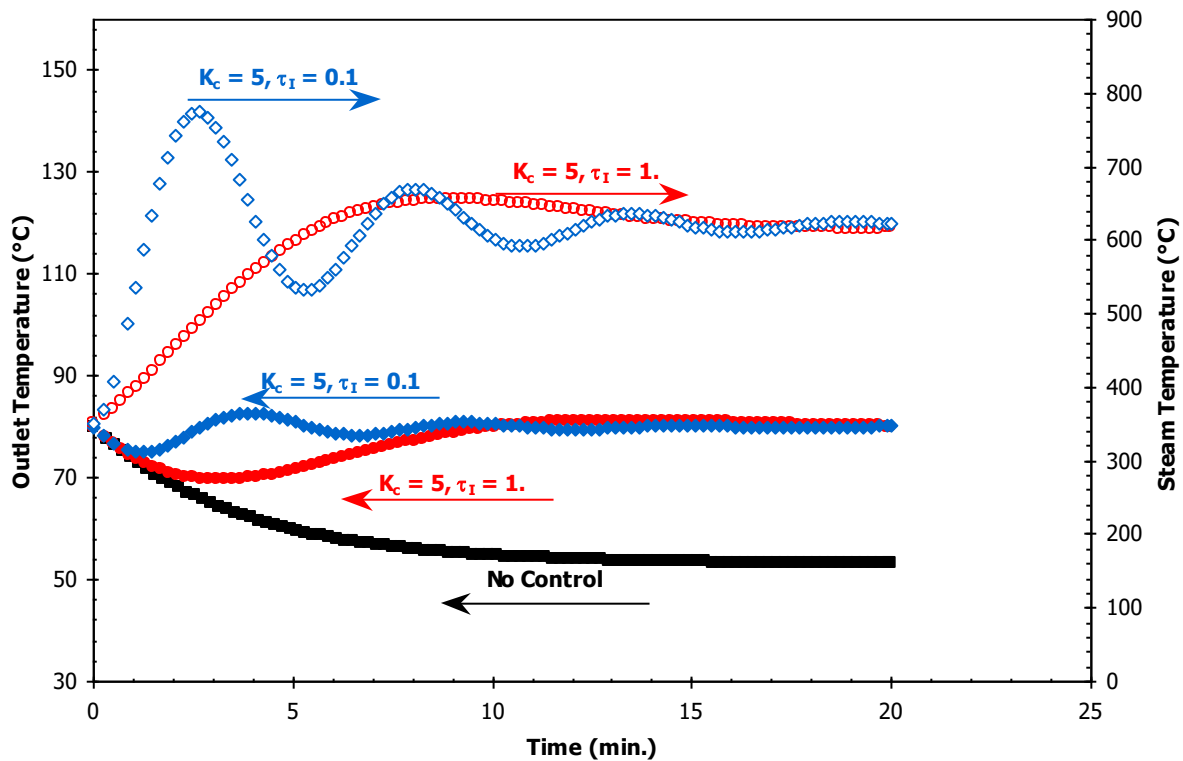
$$\begin{aligned}
 0 = \rho F_0^* \hat{C}_p (T_0^* - T_1^*) + (UA)(T_s^* - T_1^*) &\Rightarrow T_1^* = \frac{(UA)T_s^* + (\rho F_0^* \hat{C}_p)T_0^*}{(UA) + (\rho F_0^* \hat{C}_p)} \\
 &= \frac{(0.05556)(350) + (1)(0.50)(1)(20)}{(0.05556) + (1)(0.50)(1)} \\
 &= 53^\circ\text{C}
 \end{aligned}$$



This chart shows the response – the new outlet temperature is about 53°C and it is reached in about 15 minutes. This new temperature matches what we would expect from the steady state equation.



Let's look at the effects of various amounts of proportional control. This chart shows both the outlet temperature and the steam temperature for various values of  $K_c$ . As expected there is an offset that gets smaller the larger the controller gain. However, notice that the larger the gain, the larger the steam temperature.



Let's look at the effects of various amounts of proportional control integral control. This chart shows both the outlet temperature and the steam temperature for various values of  $\tau_I$  when  $K_c = 5$ . As expected there is zero offset. We have also picked controller settings that turn this 1<sup>st</sup> order system into an underdamped 2<sup>nd</sup> order system.

POLYMATH 5.0 beta test #193 - [Ordinary Differential Equations Solver]

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Solve with:  Final Value:

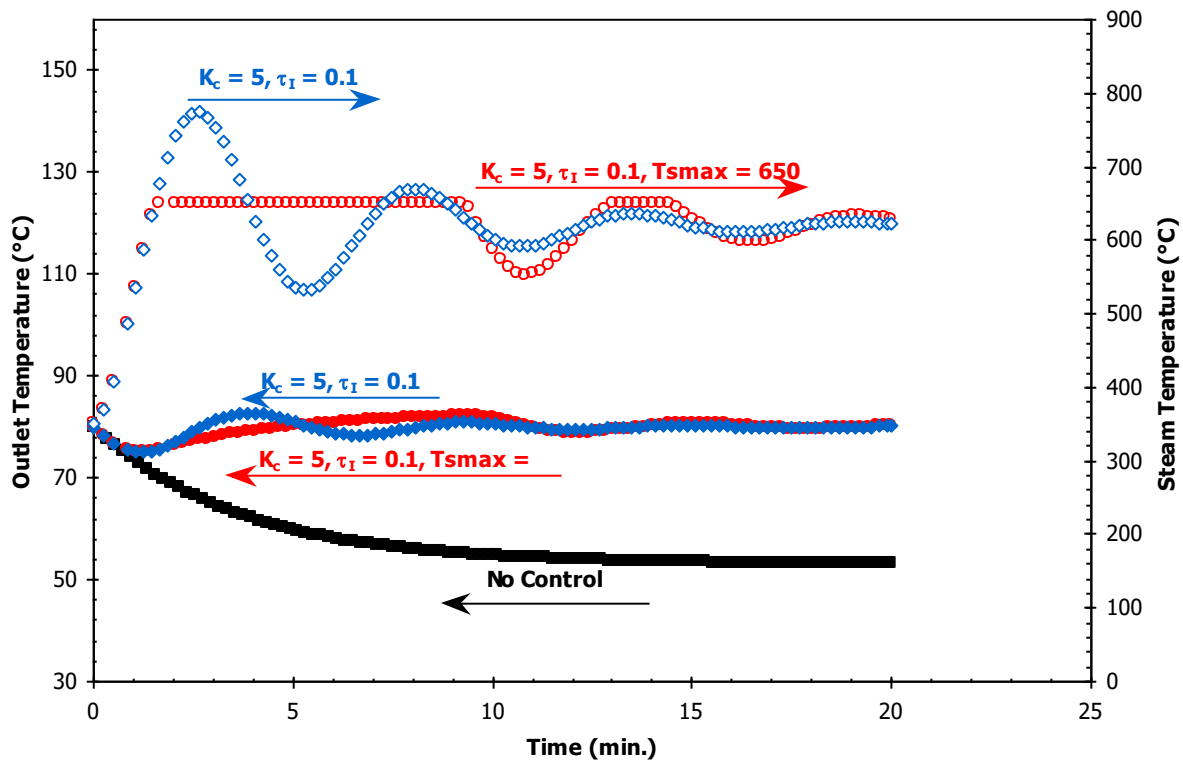
☒ Comments

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| 6  | $F0 = 0.50$  | n.a.          | Inlet flow rate  |
| 7  | $T0 = T0Star$  | n.a.          | Inlet temperature  |
| 8  | $UA = \rho \cdot Cp \cdot F0Star \cdot (T1Star - T0Star) / (TsStar - T1Star)$                        | n.a.          | Back-calculate product of heat transfer coefficient & area |
| 9  | $Tm = T1$  | n.a.          | Measured temperature                                       |
| 10 | $Error = 80 - Tm$  | n.a.          | Controller error - value is set point                      |
| 11 | $TsMan = TsStar + Kc \cdot (Error + ErrorSum \cdot InvIntTime)$                                      | n.a.          | Manipulated steam temperature                              |
| 12 | $Kc = 5.$  | n.a.          | Controller Gain  |
| 13 | $InvIntTime = 1. / .1$   | n.a.          | Inverse of Integral Time                                   |
| 14 | $F0Star = 0.25$  | n.a.          | Initial Steady State                                       |
| 15 | $T0Star = 20$  | n.a.          | Initial Steady State                                       |
| 16 | $TsStar = 350$   | n.a.          | Initial Steady State                                       |
| 17 | $T1Star = 80.$   | n.a.          | Initial Steady State                                       |
| 18 | $Ts = \text{if } (TsMan < 650) \text{ then } (TsMan) \text{ else } (650.)$                           | n.a.          | Limit the value of the steam temperature                   |

Differential Equations: 2 Auxiliary Equations: 16

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One other thing that can easily be done in a numerical solution is to put in non-traditional constraints. For example, what if our steam system is limited to 650°C? This screen shot shows how we can easily add this constraint. Now we have a variable  $TsMan$  which is the steam temperature that the controller would like us to use & the actual value from the system is  $Ts$ .



This figure shows the true response when steam temperature cannot get above 650°C. This constraint on the steam temperature actually helps the response, not in the time that it takes to get to the new steady state but it prevents the response from getting too far away from the set point.