

# Introduction to Differential Equations

## MATH 2400 – Fall 2014

### Suggested Homework Problems

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Problems labelled with capital letters are suggested for all; the quizzes will be chosen from these problems and you must know how to solve these problems if you wish to perform well on the exams. Problems labelled with Greek letters are suggested for those who wish to learn differential equations at a somewhat deeper level or practice using Maple or other computational software; being able to solve these problems is not necessary for success with the exams. The 10th edition problem number is indicated after the alphabetic label. To find the solutions:

- If the problem letter is followed by a plus (+), the solution is in the accompanying solution file.
- If the letter W followed by the 7th edition problem number is given and italicized, then the solution can be found by following the link to the collection of 7th edition solutions, originally developed in Prof. William Siegmann's class.
- If the letter V followed by the 7th edition problem number is given and italicized, then a video solution can be found by following the link to the collection of 7th edition video solutions prepared by Dr. David Schmidt.
- If the problem label is followed by an italicized label of the form  $G\#$ , then the solution can be found in the solution files of Prof. Gregor Kovačič (with the corresponding problem number).

Note that for any plots requested in the standard (capital letter) problems, you must understand how to sketch the plots by hand (as in on an exam where you do not have access to Maple).

# 1 Introduction

## 1.1 Basic Mathematical Models and Direction Fields

### Problems for Everyone

(A; 10th 1.1#4, 6) [VW 7th: 1.1#4, 6]

Draw a direction field for each of the given differential equations. Based on the direction field, determine the behavior of  $y$  as  $t \rightarrow \infty$ . If this behavior depends upon the initial value of  $y$  at  $t = 0$ , describe this dependency.

a.  $y' = -1 - 2y$

b.  $y' = y + 2$

(B; 10th: 1.1#9) [W 7th: 1.1#9]

Create a differential equation of the form

$$\frac{dy}{dt} = ay + b$$

which has a solution  $y = 2$ , and for which all other solutions diverge from  $y = 2$ .

(C; 10th: 1.1#12) [7th: 1.1#12]

Draw a direction field for the differential equation  $y' = -y(5 - y)$ . Based on the direction field, determine the behavior of  $y$  as  $t \rightarrow \infty$ . If this behavior depends on the initial value of  $y$  at  $t = 0$ , describe this dependency. Note that this equation is not of the form  $y' = ay + b$ , and the behavior of its solution is somewhat more complicated than for the equations in the text.

(D+; 10th: 1.1#15-20)

Below is a series of direction fields and a list of differential equations. Match each direction field to the appropriate differential equation.

a.  $\frac{dy}{dt} = 2 - y$

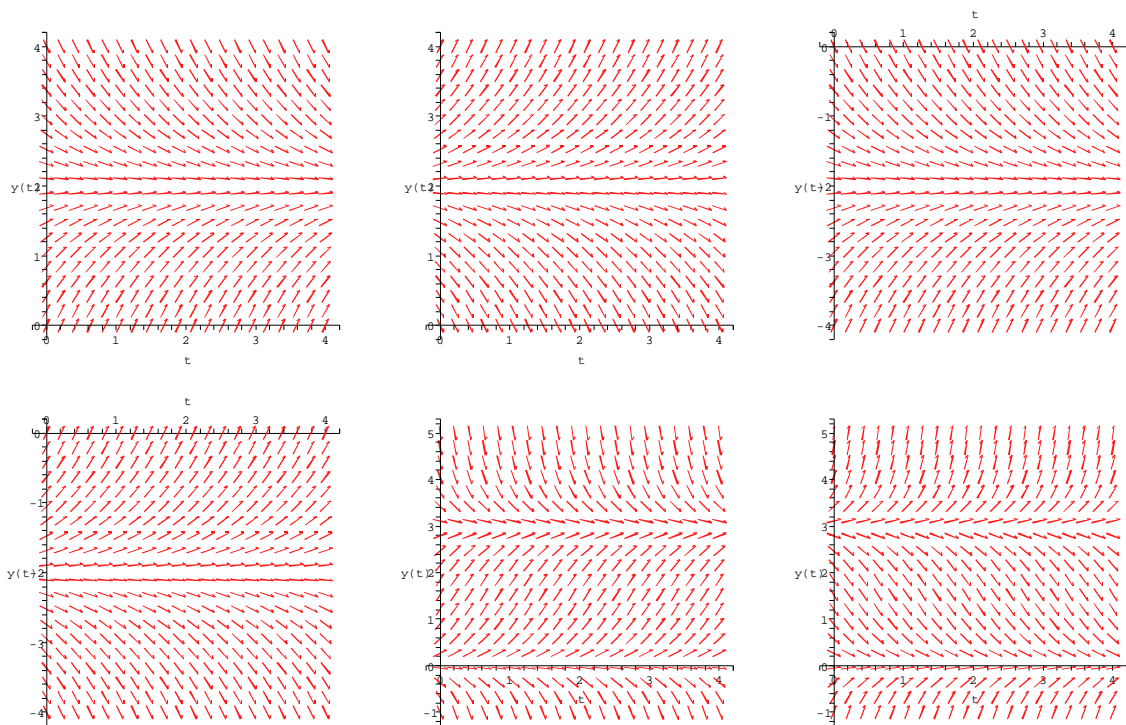
b.  $\frac{dy}{dt} = y - 2$

c.  $\frac{dy}{dt} = y(y - 3)$

d.  $\frac{dy}{dt} = -2 - y$

e.  $\frac{dy}{dt} = -y(y - 3)$

f.  $\frac{dy}{dt} = 2 + y$



(E; 10th:1.1 #22)[W 7th:1.1#16]

A spherical raindrop evaporates at a rate proportional to its surface area. Write a differential equation for the volume of the raindrop as a function of time.

(F; 10th 1.1#24)[W 7th:1.1#17]

A certain drug is being administered intravenously to a hospital patient. Fluid containing  $5 \frac{\text{mg}}{\text{cm}^3}$  of the drug enters the patient's bloodstream at a rate of  $100 \frac{\text{cm}^3}{\text{hr}}$ . The drug is absorbed by the body's tissues or otherwise leaves the bloodstream at a rate proportional to the amount present, with a rate constant of  $0.4 \text{hr}^{-1}$ .

- Assuming the drug is always uniformly distributed throughout the bloodstream, write a differential equation for the amount of drug present in the bloodstream at any time.
- How much of the drug is present in the bloodstream after a long time?

## 1.2 Solutions to Some Differential Equations

### Problems for everyone

(A; 10th: 1.2 #1)[W 7th:1.2 #1]

Solve each of the following initial value problems and plot solutions for several values of  $y_o$ . Describe how the different solutions differ from one another.

a.  $\frac{dy}{dt} = -y + 5 \quad y(0) = y_o$

b.  $\frac{dy}{dt} = -2y + 5 \quad y(0) = y_o$

c.  $\frac{dy}{dt} = -2y + 10 \quad y(0) = y_o$

(B; 10th: 1.2#7)[W 7th:1.2#6]

Assume that a particular field mouse population obeys the following differential equation:

$$\frac{dp}{dt} = 0.5p - 450$$

- a. Find the time the population goes extinct if  $p(0) = 850$ .
- b. Find the time to extinction if  $p(0) = p_o$  where  $0 < p_o < 900$
- c. Find the initial population  $p_o$  which will become extinct in one year.

(C+; 10th: 1.2#15)

According to Newton's Law of Cooling, the temperature,  $u(t)$  of an object satisfies the differential equation:

$$\frac{du}{dt} = -k(u - T)$$
$$k, T \in \mathfrak{R}$$

where  $T$  is the constant ambient temperature and  $k$  is a positive constant. Suppose the temperature is initially  $u(0) = u_o$ .

- a. Find an expression for the temperature at any time.
- b. Let  $\tau$  be the time at which the initial temperature difference  $u_o - T$  has been reduced by half. Find the relation between  $k$  and  $\tau$ .

(D; 10th: 1.2#14)[W 7th: 1.2 #12]

Radium-226 has a half life of 1620 years. Find the time period during which a given amount of this material is reduced by one quarter.

(E+; 10th: 1.2#17)

Consider an electric circuit containing a capacitor, resistor and battery; The charge  $Q(t)$  on the capacitor satisfies the following differential equation:

$$R \frac{dQ}{dt} + \frac{Q}{C} = V$$
$$R, C, V \in \mathfrak{R}$$

where  $R$  represents the resistance,  $C$  is the capacitance, and  $V$  is the constant voltage provided by the battery.

- a. If  $Q(0) = 0$ , find  $Q(t)$  at any time  $t$  and sketch a graph of  $Q$  vs.  $t$ .
- b. Find the limiting value  $Q_L$  that  $Q(t)$  approaches after a long time (i.e. take  $\lim_{t \rightarrow \infty} Q(t)$ )
- c. Suppose that  $Q(t_1) = Q_L$  and that at time  $t = t_1$  the battery is removed and the circuit is closed again. Find  $Q(t)$  for  $t > t_1$  and sketch its graph.

### More challenging problems

( $\alpha$ ; 10th: 1.2 #3)[W 7th:1.2 #3]

Consider the differential equation:

$$\frac{dy}{dt} = -ay + b$$

- a. Solve the differential equation
- b. Sketch the solution for several different initial conditions.
- c. Describe how the solutions change under the following circumstances:
  - (a)  $a$  increases.
  - (b)  $b$  increases.
  - (c) Both  $a$  and  $b$  increase, however the ration  $\frac{b}{a}$  stays constant.

## 1.3 Classification of Differential Equations

### Problems for everyone

(A; 10th: 1.3#1,2,6) [VW 7th: 1.3#1,2,6]

In each of the following differential equations, determine its order and state whether the equation is linear or non-linear (in  $y$ ).

a.  $t^2 \frac{d^2 y}{dt^2} + t \frac{dy}{dt} + 2y = \sin t$

b.  $(1 + y^2) \frac{d^2 y}{dt^2} + t \frac{dy}{dt} + y = e^t$

c.  $\frac{d^3 y}{dt^3} + t \frac{dy}{dt} + (\cos^2 t)y = t^3$

(B; 10th: 1.3#11) [VW 7th: 1.3#11]

Verify that each of the functions  $y_1(t) = \sqrt{t}$  and  $y_2(t) = \frac{1}{t}$  is a solution to the differential equation  $2t^2 y'' + 3ty' - y = 0$ .

## 2 First Order Differential Equations

### 2.1 Linear Equations; Method of Integrating Factors

#### Problems for everyone

(A; 10th: 2.1#2,4,7) [W 7th: 2.1#2,4; G#3]

Find the general solution of the following differential equations:

a.  $y' - 2y = t^2 e^{2t}$

b.  $y' + (1/t)y = 3 \cos 2t, \quad t > 0$

c.  $y' + 2ty = 2te^{-t^2}$

(B; 10th: 2.1#18,20) [W 7th: 2.1#18,20]

Find the solution of the following initial value problems:

a.  $ty' + 2y = \sin t, \quad y(\pi/2) = 1, \quad t > 0$

b.  $ty' + (t+1)y = t, \quad y(\ln 2) = 1, \quad t > 0$

(C; 10th: 2.1#22) [V 7th: 2.1#22]

- (a) Draw a direction field for the given differential equation. How do solutions appear to behave as  $t$  becomes large? Does the behavior depend on the choice of initial value  $a$ ? Let  $a_0$  be the value of  $a$  for which the transition from one type of behavior to another occurs. Estimate the value of  $a_0$ .
- (b) Solve the initial value problem and find the critical value  $a_0$  exactly.
- (c) Describe the behavior of the solution corresponding to the initial value of  $a_0$ .

$$2y' - y = e^{t/3}, \quad y(0) = a$$

(D; 10th: 2.1#30) [W 7th: 2.1#28]

Find the value of  $y_0$  for which the solution of the initial value problem

$$y' - y = 1 + 3 \sin t, \quad y(0) = y_0$$

remains finite as  $t \rightarrow \infty$ .

### More challenging problems

(α; 10th: 2.1#28) [W 7th: 2.1#26]

Consider the initial value problem

$$y' + \frac{2}{3}y = 1 - \frac{1}{2}t, \quad y(0) = y_0$$

Find the value of  $y_0$  for which the solution touches, but does not cross, the  $t$ -axis.

(β; 10th: 2.1#35) [W 7th: 2.1#32]

Construct a first order linear differential equation whose solutions have the required behavior as  $t \rightarrow \infty$ . Then solve your equation and confirm that the solutions do indeed have the specified property: **All solutions are asymptotic to the line  $y = 3 - t$  as  $t \rightarrow \infty$ .**



## 2.2 Separable Equations

### Problems for everyone

(A; 10th: 2.2#2) [W 7th: 2.2#2]

Solve the given differential equation

$$y' = x^2/y(1 + x^3)$$

(B; 10th: 2.2#11,14,15) [W 7th: 2.2#11,14,15]

- Find the solution of the given initial value problem in explicit form.
- Plot the graph of the solution (optional).
- For those labeled with “determine interval,” determine the interval over which the solution is defined. (For the others you probably would need numerical assistance, and those interested should pursue it, but this skill will not be assessed on exams.)

- $x dx + ye^{-x} dy = 0, \quad y(0) = 1$
- $y' = xy^3(1 + x^2)^{-1/2}, \quad y(0) = 1$  (determine interval)
- $y' = 2x/(1 + 2y), \quad y(2) = 0$  (determine interval)

(C; 10th: 2.2#22) [VW 7th: 2.2#22]

Solve the initial value problem

$$y' = 3x^2/(3y^2 - 4), \quad y(1) = 0$$

and determine the interval in which the solution is valid.

*Hint:* To find the interval of definition, look for points where the integral curve has a vertical tangent.

### More challenging problems

(α; 10th: 2.2#24) [V 7th: 2.2#24]

Solve the initial value problem

$$y' = (2 - e^x)/(3 + 2y), \quad y(0) = 0$$

and determine where the solution attains its maximum value.

( $\beta$ ; 10th: 2.2#30)

Consider the equation

$$\frac{dy}{dx} = \frac{y - 4x}{x - y} \quad (1)$$

(a) Show that Eq.(1) can be rewritten as

$$\frac{dy}{dx} = \frac{(y/x) - 4}{1 - (y/x)} \quad (2)$$

thus Eq.(1) is homogeneous.

(b) Introduce a new dependent variable  $v$  so that  $v = y/x$ , or  $y = xv(x)$ . Express  $dy/dx$  in terms of  $x, v$ , and  $dv/dx$ .

(c) Replace  $y$  and  $dy/dx$  in Eq.(2) by the expressions from part (b) that involve  $v$  and  $dv/dx$ . Show that the resulting differential equation is

$$v + x \frac{dv}{dx} = \frac{v - 4}{1 - v}$$

or

$$x \frac{dv}{dx} = \frac{v^2 - 4}{1 - v} \quad (3)$$

Observe that Eq.(3) is separable.

(d) Solve Eq.(3), obtaining  $v$  implicitly in terms of  $x$ .

(e) Find the solution of Eq.(1) by replacing  $v$  by  $y/x$  in the solution in part(d).

(f) Draw a direction field and some integral curves for Eq.(1). Recall that the right side of Eq.(1) actually depends only on the ratio  $y/x$ . This means that integral curves have the same slope at all points on any given straight line through the origin, although the slope changes from one line to another. Therefore the direction field and the integral curves are symmetric with respect to the origin. Is this symmetry property evident from your plot?

## 2.3 Modeling with First Order Equations

### Problems for everyone

(A; 10th: 2.3#3) [*V 7th: 2.3#3*]

A tank originally contains 100 gal of fresh water. Then water containing  $\frac{1}{2}$  lb of salt per gallon is poured into the tank at a rate of 2gal/min, and the mixture is allowed to leave at the same rate. After 10 min the process is stopped, and fresh water is poured into the tank at a rate of 2gal/min, with the mixture again leaving at the same rate. Find the amount of salt in the tank at the end of an additional 10 min.

(B; 10th: 2.3#9) [W 7th: 2.3#9]

A certain college graduate borrows \$8000 to buy a car. The lender charges interest at an annual rate of 10%. Assuming that interest is compounded continuously and that the borrower makes payments continuously at a constant annual rate  $k$ , determine the payment rate  $k$  that is required to pay off the loan in 3 years. Also determine how much interest is paid during the 3-year period.

(C; 10th: 2.3#16) [W 7th: 2.3#18]

Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that the temperature of a cup of coffee obeys Newton's law of cooling. If the coffee has a temperature of  $200^\circ F$  when freshly poured, and 1 min later has cooled to  $190^\circ F$  in a room at  $70^\circ F$ , determine when the coffee reaches a temperature of  $150^\circ F$ .

(D; 10th: 2.3#19) [W 7th: 2.3#20]

Consider a lake of constant volume  $V$  containing at time  $t$  an amount  $Q(t)$  of pollutant, evenly distributed throughout the lake with a concentration  $c(t)$ , where  $c(t) = Q(t)/V$ . Assume that water containing a concentration  $k$  of pollutant enters the lake at a rate  $r$ , and that water leaves the lake at the same rate. Suppose that pollutants are also added directly to the lake at a constant rate  $P$ . Note that the given assumptions neglect a number of factors that may, in some cases, be important—for example, the water added or lost by precipitation, absorption, and evaporation; the stratifying effect of temperature differences in a deep lake; the tendency of irregularities in the coastline to produce sheltered bays; and the fact that pollutants are not deposited evenly throughout the lake but (usually) at isolated points around its periphery. The results below must be interpreted in the light of neglect of such factors as these.

(a) If at time  $t = 0$  the concentration of pollutant is  $c_0$ , find an expression for the concentration  $c(t)$  at any time. What is the limiting concentration as  $t \rightarrow \infty$ ?

(b) If the addition of pollutants to the lake is terminated ( $k = 0$  and  $P = 0$  for  $t > 0$ ), determine the time interval  $T$  that must elapse before the concentration of pollutants is reduced to 50% of its original value; to 10% of its original value.

(c) Table 2.2.3 on page 64 (Reproduced below) contains data for several of the Great Lakes. Using these data, determine from part(b) the time  $T$  necessary to reduce the contamination of each of these lakes to 10% of the original value.

Lake	$V \text{ (km}^3 \times 10^3\text{)}$	$r \text{ (km}^2/\text{year)}$
<i>Superior</i>	12.2	65.2
<i>Michigan</i>	4.9	158
<i>Erie</i>	0.46	175
<i>Ontario</i>	1.6	209

## More challenging problems

( $\alpha^*$ ; 10th: 2.3#14)

Suppose that a certain population has a growth rate that varies with time and that this population satisfies the differential equation

$$dy/dt = (0.5 + \sin t)y/5 \quad (4)$$

- (a) If  $y(0) = 1$ , find (or estimate) the time  $\tau$  at which the population has doubled. Choose other initial conditions and determine whether the doubling time  $\tau$  depends on the initial population.
- (b) Suppose that the growth rate is replaced by its average value  $1/10$ . Determine the doubling time  $\tau$  in this case.
- (c) Suppose that the term  $\sin t$  in the differential equation is replaced by  $\sin 2\pi t$ ; that is, variation in the growth rate has a substantially higher frequency. What effect does this have on the doubling time  $\tau$ ?
- (d) Plot the solutions obtained in parts (a),(b),and (c) on a single set of axes.

## 2.5 Autonomous Equations and Population Dynamics

### Problems for everyone

(A; 10th: 2.5#2,3) [*W 7th: 2.5#2 and V 7th: 2.5#3*]

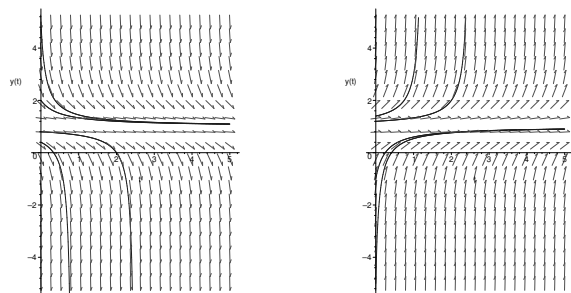
Problem a and b involve equations of the form  $dy/dt = f(y)$ . In each problem sketch the graph of  $f(y)$  versus  $y$ , determine the critical (equilibrium) points, and classify each one as asymptotically stable or unstable. Draw the phase line, and sketch several graphs of solutions in the  $ty$ -plane.

a.  $dy/dt = ay + by^2$ ,  $a > 0$ ,  $b > 0$ ,  $-\infty < y_0 < \infty$

b.  $dy/dt = y(y - 1)(y - 2)$ ,  $y_0 \geq 0$

(B; 10th: 2.5#7) [W 7th: 2.5#7]

**Semistable Equilibrium Solutions.** Sometimes a constant equilibrium solution has the property that solutions lying on one side of the equilibrium solution tend to approach it, whereas solutions lying on the other side depart from it (See figure 2.5.9). In this case the equilibrium solution is said to be **semistable**.



(a) Consider the equation

$$dy/dt = k(1 - y)^2, \quad (i)$$

where  $k$  is a positive constant. Show that  $y = 1$  is the only critical point, with the corresponding equilibrium solution  $\phi(t) = 1$ .

(b) Sketch  $f(y)$  versus  $y$ . Show that  $y$  is increasing as a function of  $t$  for  $y < 1$  and also for  $y > 1$ . The phase line has upward-pointing arrows both below and above  $y = 1$ . Thus solutions below the equilibrium solution approach it, and those above it grow farther away. Therefore  $\phi(t) = 1$  is semistable.

(c) Solve Eq.(i) subject to the initial condition  $y(0) = y_0$  and confirm the conclusions reached in part (b).

(C; 10th: 2.5#8,9) [W 7th: 2.5#8,9]

The following 2 problems involve equations of the form  $dy/dt = f(y)$ . In each problem sketch the graph of  $f(y)$  versus  $y$ , determine the critical (equilibrium) points, and classify each one as asymptotically stable, unstable or semistable (see Problem 7). Draw the phase line, and sketch several graphs of solutions in the  $ty$ -plane.

a.  $dy/dt = -k(y - 1)^2, \quad k > 0, \quad -\infty < y_0 < \infty$

b.  $dy/dt = y^2(y^2 - 1), \quad -\infty < y_0 < \infty$

(D; 10th: 2.5#22) [W 7th: 2.5#20]

Suppose that a given population can be divided into two parts: those who have a given disease and can infect others, and those who do not have it but are susceptible. Let  $x$  be the proportion of susceptible individuals and  $y$  the proportion of infectious individuals; then  $x + y = 1$ . Assume that the disease spreads by contact between sick and well members of the population and that the rate of spread  $dy/dt$  is proportional to the number of such contacts. Further, assume that members of both groups move about freely among each other, so the number of contacts is proportional to the product of  $x$  and  $y$ . Since  $x = 1 - y$ , we obtain the initial value problem

$$dy/dt = \alpha y(1 - y), \quad y(0) = y_0, \quad (i)$$

where  $\alpha$  is a positive proportionality factor, and  $y_0$  is the initial proportion of infectious individuals.

- (a) Find the equilibrium points for the differential equation (i) and determine whether each is asymptotically stable, semistable, or unstable.
- (b) Solve the initial value problem (i) and verify that the conclusions you reached in part (a) are correct. Show that  $y(t) \rightarrow 1$  as  $t \rightarrow \infty$ , which means that ultimately the disease spreads through the entire population.

### More challenging problems

( $\alpha$ ; 10th: 2.5#18) [V 7th: 2.5#18]

A pond forms as water collects in a conical depression of radius  $a$  and depth  $h$ . Suppose that water flows in at a constant rate  $k$  and is lost through evaporation at a rate proportional to the surface area.

- (a) Show that the volume  $V(t)$  of water in the pond at time  $t$  satisfies the differential equation

$$dV/dt = k - \alpha\pi(3a/\pi h)^{2/3}V^{2/3}, \quad (i)$$

where  $\alpha$  is the coefficient of evaporation.

- (b) Find the equilibrium depth of water in the pond. Is the equilibrium asymptotically stable?
- (c) Find a condition that must be satisfied if the pond is not to overflow.

## 3 Second Order Linear Equations

### 3.1 Homogeneous Equations with Constant Coefficients

#### Problems for everyone

(A; 10th: 3.1 #1, 5 ,7)[W 7th: 3.1#1,7 and G#26]

Find the general solution to the following differential equations:

a.  $y'' + 2y' - 3y = 0$

b.  $y'' + 5y' = 0$

c.  $y'' - 9y' + 9y = 0$

(B; 10th: 3.1 #9, 11, 12)[V 7th: 3.1#9 and W 7th: 3.1#11,12]

Find the specific solution to the following initial value problems. Sketch a simple graph of the solution and determine the behavior for large  $t$ .

a.  $y'' + y' - 2y = 0 \quad y(0) = 1; y'(0) = 1$

b.  $6y'' - 5y' + y = 0 \quad y(0) = 4; y'(0) = 0$

c.  $y'' + 3y' = 0 \quad y(0) = -2; y'(0) = 3$

(C; 10th: 3.1 #17)[VW 7th: 3.1#17]

Find a differential equation whose general solution is given by the following:

$$y = c_1 e^{2t} + c_2 e^{-3t}$$

(D; 10th: 3.1 #20)[W 7th: 3.1#20]

Find the solution to the initial value problem:

$$2y'' - 3y' + y = 0 \quad y(0) = 2; \quad y'(0) = \frac{1}{2}$$

After finding the solution, determine the maximum value of the solution and find the point where the solution is zero.

(E; 10th: 3.1 #21)[VW 7th: 3.1#21]

Solve the following initial value problem and then determine the value for  $\alpha$  for which the solution approaches 0 as  $t \rightarrow \infty$ .

$$y'' - y' - 2y = 0 \quad y(0) = \alpha, y'(0) = 2$$

## 3.2 Solutions of Linear Homogenous Equations; the Wronskian

### Problems for everyone

(A; 10th: 3.2#3)[W 7th: 3.2#3]

Find the Wronskian of the following pair of functions:

$$y_1 = e^{-2t}, \quad y_2 = te^{-2t}$$

(B; 10th: 3.2#22)[W 7th: 3.2#21]

Find a fundamental set of solutions,  $y_1(t)$  and  $y_2(t)$  for the following differential equation:

$$y'' + y' - 2y = 0$$

which satisfies the initial conditions:

$$\begin{aligned} y_1(0) &= 1 & y_1'(0) &= 0 \\ y_2(0) &= 0 & y_2'(0) &= 1 \end{aligned}$$

(This is an illustration of Theorem 3.2.5 (in the 10th edition).)

(C; 10th: 3.2#26)[V 7th: 3.2#25]

Verify that the given functions  $y_1$  and  $y_2$  are indeed solutions to the given differential equation, and determine whether they constitute a fundamental set of solutions,

$$x^2 y'' - x(x+2)y' + (x+2)y = 0; \quad x > 0 \quad y_1(x) = x; \quad y_2(x) = xe^x$$

### More challenging problems

(α; 10th: 3.2#11, 12)[W 7th: 3.2#11 and V 7th: 3.2#12]

For each of the following differential equations determine the longest interval over which the initial value problem is guaranteed to have a twice differentiable solution, (Do not attempt to find the solutions.)

- a.  $(x-3)y'' + xy' + (\ln|x|)y = 0 \quad y(1) = 0; \quad y'(1) = 1$
- b.  $(x-2)y'' + y' + (x-2)(\tan x)y = 0 \quad y(3) = 1; \quad y'(3) = 2$

(β; 10th: 3.2#17)[VW 7th: 3.2#18]

If the Wronskian of  $f$  and  $g$  is  $3e^{4t}$  and if  $f(t) = e^{2t}$ , find  $g(t)$ .

(γ; 10th: 3.2 #30)[W 7th: 3.3#16]

Find the Wronskian of two solutions of the given differential equation without solving the equation.

$$(\cos t)y'' + (\sin t)y' - ty = 0$$



( $\delta$ ; 10th: 3.2 #34)[W 7th: 3.3#20]

If  $y_1$  and  $y_2$  are a fundamental set of solutions of  $ty'' + 2y + te^ty = 0$  and if  $W(y_1, y_2)(1) = 2$ , find the value of  $W(y_1, y_2)(5)$ .

( $\epsilon$ ; 10th: 3.2#38)[VW 7th: 3.3#24]

Prove that if  $y_1$  and  $y_2$  are zero at the same point in  $I$ , then they cannot be a fundamental set of solutions on that interval.

### 3.3 Complex Roots of the Characteristic Equation

#### Problems for everyone

(A; 10th: 3.3#8, 10, 11)[W 7th: 3.4#8, 10, 11]

Find the general solution of the given differential equation.

a.  $y'' - 2y' + 6y = 0$

b.  $y'' + 2y' + 2y = 0$

c.  $y'' + 6y' + 13y = 0$

(B; 10th: 3.3#18)[VW 7th: 3.4#18]

Find the solution of the given initial value problem and describe the behavior of the solution for increasing  $t$ .

$$y'' + 4y' + 5y = 0, \quad y(0) = 1; \quad y'(0) = 0$$

(C; 10th: 3.3# 27)[VW 7th: 3.4#27] Show that the Wronskian of  $(e^{\lambda t} \cos(\mu t), e^{\lambda t} \sin(\mu t)) = \mu e^{2\lambda t}$

#### More challenging Problems

(α; 10th: 3.3#24)[W 7th: 3.4#24] Consider the initial value problem:

$$5u'' + 2u' + 7u = 0, \quad u(0) = 2, \quad u'(0) = 1$$

- Find the solution  $u(t)$  of this problem.
- Find the smallest  $T$  such that  $|u(t)| \leq 0.1$  for all  $t > T$ .

(β; 10th: 3.3 #36)[G#49] Use the substitution  $x = \ln t$  to transform the following equation into an equation with constant coefficients, then solve. This is an example of an Euler equation, discussed in Problem 34 in Section 3.3 (9th edition).

$$t^2 y'' + 4ty' + 2y = 0$$

### 3.4 Repeated Roots, Reduction of Order

#### Problems for everyone

(A; 10th: 3.4#1, 4)[G#33, V 7th: 3.5#1, and W 7th: 3.5#4]

Find the general solution of the differential equations:

a.  $y'' - 2y' + y = 0$

b.  $4y'' + 12y' + 9y = 0$

(B; 10th: 3.4#11)[W 7th: 3.5#11]

Solve the given initial value problem:

$$9y'' - 12y' + 4y = 0 \quad y(0) = 2 \quad y'(0) = -1$$

Sketch the graph of the solution and describe its behavior for increasing  $t$ .

(C; 10th: 3.4#16)[W 7th: 3.5#16]

Consider the following modification of the initial value problem in Example 2 of Section 3.5 (9th edition):

$$y'' - y' + \frac{1}{4}y = 0 \quad y(0) = 2 \quad y'(0) = b$$

Find the solution as a function of  $b$  and then determine the critical value of  $b$  which separates solutions that grow positively from those that eventually grow negatively.

#### More challenging Problems

(α; 10th: 3.4 #28, 30)[W 7th: 3.5#28 and G#48]

Use the method of Reduction of Order to find a second solution to the differential equation.

a.  $(x - 1)y'' - xy' + y = 0 \quad x > 0 \quad y_1(x) = e^x$

b.  $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0 \quad x > 0 \quad y_1(x) = x^{-1/2}\sin(x)$

(β; 10th: 3.4#37)[W 7th: 3.5#38]

If  $a, b, c$  are all positive constants show that all solutions of:

$$ay'' + by' + cy = 0$$

approach zero as  $t \rightarrow \infty$ .

### 3.5 Nonhomogeneous Equations; Method of Undetermined Coefficients

#### Problems for everyone

(A; 10th: 3.5#9,11)[VW 7th: 3.6#7 and W 7th: 3.6#9]

In each of the following problems find the general solution of the given differential equation.

a.  $2y'' + 3y' + y = t^2 + 3 \sin t$

b.  $u'' + \omega_0^2 u = \cos \omega t, \quad \omega^2 \neq \omega_0^2$

(B; 10th: 3.5#15,16,17)[G#35, W 7th: 3.6#14,15]

In each of the following problems find the solution of the given initial value problem.

a.  $y'' + y' - 2y = 2t, \quad y(0) = 0, \quad y'(0) = 1$

b.  $y'' + 4y = t^2 + 3e^t, \quad y(0) = 0, \quad y'(0) = 2$

c.  $y'' - 2y' + y = te^t + 4, \quad y(0) = 1, \quad y'(0) = 1$

(C; 10th: 3.5#21, 23, 25)[W 7th: 3.6#19,21,23]

In each of the following problems:

(a) Determine a suitable form for a particular solution  $Y(t)$  if the method of undetermined coefficients is to be used.

(b) (Optional) Use a computer algebra system to find a particular solution of the given equation.

a.  $y'' + 3y' = 2t^4 + t^2 e^{-3t} + \sin 3t$

b.  $y'' - 5y' + 6y = e^t \cos 2t + e^{2t}(3t + 4) \sin t$

c.  $y' - 4y' + 4y = 2t^2 + 4te^{2t} + t \sin 2t$

#### More challenging Problems

(α; 10th: 3.5#37, 39)[G#42, 41]

In each of the following problems use the factorization method of problem 3.6#33 in the 9th edition of the textbook to solve the given differential equation.

a.  $2y'' + 3y' + y = t^2 + 3 \sin t$

b.  $y'' + 2y' = 3 + 4 \sin 2t$

### 3.6 Variation of Parameters

#### Problems for everyone

(A; 10th: 3.6#5, 6, 8)[W 7th: 3.7#5, 6, 8]

In each of the following problems find the general solution of the given differential equation.

- a.  $y'' + y = \tan t, \quad 0 < t < \pi/2$
- b.  $y'' + 9y = 9 \sec^2 3t, \quad 0 < t < \pi/6$
- c.  $y'' + 4y = 3 \csc 2t, \quad 0 < t < \pi/2$

(B; 10th: 3.6#17,18)[W 7th: 3.7#17, G#57]

In each of the following problems verify that the given functions  $y_1$  and  $y_2$  satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation.

- a.  $x^2 y'' - 3xy' + 4y = x^2 \ln x, \quad x > 0, \quad y_1(x) = x^2, \quad y_2(x) = x^2 \ln x$
- b.  $x^2 y'' + xy' + (x^2 - 0.25)y = 3x^{3/2} \sin x, \quad x > 0, \quad y_1(x) = x^{-1/2} \sin x, \quad y_2(x) = x^{-1/2} \cos x$

### 3.7 Mechanical and Electrical Vibrations

#### Problems for everyone

(A; 10th: 3.7#2)[VW 7th: 3.8#2]

In the following problem determine  $\omega_0$ ,  $R$ , and  $\delta$  so as to write the given expression in the form  $u = R \cos(\omega_0 t - \delta)$ .

- a.  $u = -\cos t + \sqrt{3} \sin t$

(B; 10th: 3.7#6)[W 7th: 3.8#6]

A mass of 100g stretches a spring 5cm. If the mass is set in motion from its equilibrium position with a downward velocity of 10cm/sec, and if there is no damping, determine the position  $u$  of the mass at any time  $t$ . When does the mass first return to its equilibrium position?

**(C; 10th: 3.7#9)**[*W 7th: 3.8#9*]

A mass of 20g stretches a spring 5cm. Suppose that the mass is also attached to a viscous damper with a damping constant of 400 dyne-sec/cm. If the mass is pulled down an additional 2cm and then released, find its position  $u$  at any time  $t$ . Plot  $u$  versus  $t$ . Determine the quasi frequency and the quasi period. Determine the ratio of the quasi period to the period of the corresponding undamped motion. Also find the time  $\tau$  such that  $|u(t)| < 0.05\text{cm}$  for all  $t > \tau$ .

**(D; 10th: 3.7#13)**[*W 7th: 3.8#13*]

A certain vibrating system satisfies the equation  $u'' + \gamma u' + u = 0$ . Find the value of the damping coefficient  $\gamma$  for which the quasi period of the damped motion is 50% greater than the period of the corresponding undamped motion.

**(E; 10th: 3.7#17)**[*W 7th: 3.8#17*]

A mass weighing 8 lb stretches a spring 1.5in. The mass is also attached to a damper with coefficient  $\gamma$ . Determine the value of  $\gamma$  for which the system is critically damped; be sure to give the units for  $\gamma$ .

**(F; 10th: 3.7#24)**[*W 7th: 3.8#24*]

The position of a certain spring-mass system satisfies the initial value problem

a.  $\frac{3}{2}u'' + ku = 0, \quad u(0) = 2, \quad u'(0) = v$

If the period and amplitude of the resulting motion are observed to be  $\pi$  and 3, respectively, determine the values of  $k$  and  $v$ .

## More challenging problems

( $\alpha$ ; 10th: 3.7#21)[W 7th: 3.8#21]

- a. For the damped oscillations described by the following equation:

$$u(t) = Re^{-\gamma t/2m} \cos(\mu t - \delta)$$

show that the time between successive maxima is  $T = \frac{2\pi}{\mu}$

- b. Show that the ratio of displacements at two successive maxima is given by  $\exp(\gamma T_d/2m)$ . Observe that this ratio does not depend on which pair of maxima is chosen. The natural logarithm of this ratio is called the logarithmic decrement and is denoted by  $\Delta$ .
- c. Show that  $\Delta = \pi\gamma/m\mu$ . Since  $m$ ,  $\mu$  and  $\Delta$  are quantities that can be measured easily for a mechanical system, this result provides a convenient and practical method for determining the damping constant,  $\gamma$ , which is very difficult to measure directly. In particular, for the motion of a vibrating mass in a viscous fluid, the damping constant depends on the viscosity of the fluid; for simple geometric shapes the form of this dependence is known, and the preceding relation allows the experimental determination of the viscosity. This is one of the most accurate ways of determining the viscosity of a gas at high pressure.

( $\beta$ ; 10th: 3.7#27)[W 7th: 3.8#27]

A cubic block of side length  $l$  and mass density  $\rho$  per unit volume is floating in a fluid of mass density  $\rho_0$  per unit volume, where  $\rho_0 > \rho$ . If the block is slightly depressed and released, it oscillates in the vertical direction. assuming that the viscous damping of the fluid and the air can be neglected, derive the differential equation of motion and determine the period of the motion.

*HINT:* Use Archimedes' Principle: An object that is completely or partially submerged in a fluid is acted upon by an upward buoyant force equal to the mass of the displaced fluid.

( $\gamma$ ; 10th: 3.7#28)[G#61]

The position of a certain undamped spring-mass system satisfies the initial value problem

a.  $u'' + 2u = 0, \quad u(0) = 0, \quad u'(0) = 2$

- (a) Find the solution of this initial value problem.
- (b) Plot  $u$  versus  $t$  and  $u'$  versus  $t$  on the same axes.
- (c) Plot  $u'$  versus  $u$ ; that is, plot  $u(t)$  and  $u'(t)$  parametrically with  $t$  as the parameter. This plot is known as a phase plot, and the  $uu'$ -plane is called the phase plane. Observe that a closed curve in the phase plane corresponds to a periodic solution  $u(t)$ . What is the direction of motion on the phase plot as  $t$  increases?

( **$\delta$** ; 10th: 3.7#29)[*G*#62]

The position of a certain undamped spring-mass system satisfies the initial value problem

a.  $u'' + \frac{1}{4}u' + 2u = 0, \quad u(0) = 0, \quad u'(0) = 2$

(a) Find the solution of this initial value problem.

(b) Plot  $u$  versus  $t$  and  $u'$  versus  $t$  on the same axes.

(c) Plot  $u'$  versus  $u$  in the phase plane(see above problem  $\gamma$ ). Identify several corresponding points on the curves in parts (b) and (c). What is the direction of motion on the phase plot as  $t$  increases?

### 3.8 Forced Vibrations

(**A**; 10th: 3.8#6, 8)[*W* 7th: 3.9#6, 8]

A mass of 5kg stretches a spring 10cm. The mass is acted upon by an external force of  $10 \sin(\frac{t}{2})$  N (Newtons) and moves in a medium that imparts a viscous force of 2N when the speed of the mass is 4cm/sec. If the mass is set in motion from its equilibrium position with an initial velocity of 3 cm/sec, formulate the initial value problem describing the motion of the mass.

a. Find the solution of the initial value problem formulated above.

b. Identify the transient and steady-state portions of the solution.

c. Plot a graph of the steady state solution.

d. If the given force is replaced by a force of  $2 \cos(\omega t)$  of frequency  $\omega$ , determine the value of  $\omega$  for which the amplitude of the forced response is at a maximum.

(**B**; 10th: 3.8#11)[*W* 7th: 3.9#11]

A spring is stretched 6in by a mass with a weight of 8 lbs. The mass is attached to a dashpot mechanism that has a damping constant of  $\frac{1}{4}$  lb-sec/ft and is acted on by an external force of  $4 \cos(2t)$ .

a. Determine the steady state response of this system. (Be careful with units.)

b. If the given mass is replaced by a mass of  $m$ , determine the value of  $m$  for which the amplitude of the steady state response is at a maximum.



(C; 10th: 3.8#17)[W 7th: 3.9#17]

Consider a vibrating system governed by the following initial value problem:

$$u'' + \frac{1}{4}u' + 2u = 2\cos(\omega t)$$
$$u(0) = 0 \quad u'(0) = 2$$

- Determine the steady state portion of the solution to this problem.
- Find the amplitude  $A$  of the steady state in terms of  $\omega$
- Plot  $A$  vs.  $\omega$
- Find the maximum value of  $A$  and the frequency  $\omega$  for which it occurs.

## 5 Series Solutions of Second Order Linear Equations

### 5.4 Euler Equations

More challenging problems

( $\alpha$ ; 10th: 5.4#6, 9)[G#50, 52]

Determine the general solution of the given differential equation that is valid away from the singular point.

- $(x-1)^2y'' + 8(x-1)y' + 12y = 0$
- $x^2y'' - 5xy' + 9y = 0$

( $\beta$ ; 10th: 5.4#14)[G#53]

Find the solution to the given initial value problem, plot a graph of the solution and describe its behavior as  $t \rightarrow \infty$ .

$$4x^2y'' + 8xy' + 17y = 0 \quad y(1) = 2; \quad y'(1) = -3$$

## 7 Systems of First Order Linear Equations

### 7.1 Introduction

#### Problems for everyone

(A; 10th: 7.1#2,3)[W 7th: 7.1#2,3]

In each of the following problems, transform the given equation into a system of first order equations.

a.  $u'' + 0.5u' + 2u = 3 \sin t$

b.  $t^2 u'' + tu' + (t^2 - 0.25)u = 0.$

#### More challenging problems

( $\alpha$ ; 10th: 7.1#7)[W 7th: 7.1#7]

Systems of first order equations can sometimes be transformed into a single equation of higher order. Consider the system

$$x_1' = -2x_1 + x_2, \quad x_2' = x_1 - 2x_2$$

(a) Solve the first equation for  $x_2$  and substitute into the second equation, thereby obtaining a second order equation for  $x_1$ . Solve this equation for  $x_1$  and then determine  $x_2$  also.

(b) Find the solution of the given system that also satisfies the initial conditions  $x_1(0) = 2$ ,  $x_2(0) = 3$ .

(c) Sketch the curve, for  $t \geq 0$ , given parametrically by the expressions for  $x_1$  and  $x_2$  obtained in part (b).

( $\beta$ ; 10th: 7.1#14)[W 7th: 7.1#14]

Show that if  $a_{11}$ ,  $a_{12}$ ,  $a_{21}$  and  $a_{22}$  are constants with  $a_{11}$  and  $a_{21}$  not both zero, and if the functions  $g_1$  and  $g_2$  are differentiable, then the initial value problem

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + g_1(t), & x_1(0) &= x_1^0 \\ x_2' &= a_{21}x_1 + a_{22}x_2 + g_2(t), & x_2(0) &= x_2^0 \end{aligned}$$

can be transformed into an initial value problem for a single second order equation. Can the same procedure be carried out if  $a_{11}, \dots, a_{22}$  are functions of  $t$ ?

## 7.2 Review of Matrices

### Problems for everyone

(A; 10th: 7.2#1)[V 7th: 7.2#1]

If  $A = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ -2 & 1 & 3 \end{pmatrix}$  and

$B = \begin{pmatrix} 4 & -2 & 3 \\ -1 & 5 & 0 \\ 6 & 1 & 2 \end{pmatrix}$ , find

- (a)  $2\mathbf{A} + \mathbf{B}$  (b)  $\mathbf{A} - 4\mathbf{B}$   
(c)  $\mathbf{AB}$  (d)  $\mathbf{BA}$

(B; 10th: 7.2#22)[V 7th: 7.2#22]

In the following problem, verify that the given vector satisfies the given differential equation.

$$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}, \mathbf{x} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t}$$

## 7.3 System of Linear Algebraic Equations; Linear Independence, Eigenvalues, Eigenvectors

### Problems for everyone

(A; 10th: 7.3#16,17)[VW 7th: 7.3 #15; W 7th: 7.3 #16]

In each of the following problems find all eigenvalues and eigenvectors of the given matrix.

a.  $\begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$

b.  $\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$

## More challenging problems

( **$\alpha$** ; 10th: 7.3#25)[W 7th: 7.3#24]

Find all eigenvalues and eigenvectors of the given matrix.

$$\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

## 7.5 Homogeneous Linear Systems with Constant Coefficients

### Problems for Everyone

(**A**; 10th: 7.5#3,5)[VW 7th: 7.5#3,5]

Find the general solution to the given system of differential equations and describe the behavior as  $t \rightarrow \infty$ . Also plot a direction field (**optional**) and plot representative trajectories of the system.

a.

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x}$$

b.

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x}$$

(**B**; 10th: 7.5#16)[W 7th: 7.5#16]

Solve the given initial value problem and describe the behavior of the solution as  $t \rightarrow \infty$ .

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

(C; 10th: 7.5#24, 25 [W 7th: 7.5#24, 25])

In the problems below you are given the eigenvalues and eigenvectors of a matrix  $A$ . Consider the system  $\mathbf{x}' = A\mathbf{x}$ .

- Sketch the phase portrait for this system
- Sketch the trajectory passing through the point  $(2, 3)$ .
- For the trajectory in (b) sketch the graph of  $x_1$  vs  $t$  and  $x_2$  vs  $t$ .

$$r_1 = -1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \quad r_2 = -2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (5)$$

$$r_1 = 1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \quad r_2 = -2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (6)$$

### More challenging problems:

( $\alpha$ ; 10th: 7.5#7) [W 7th: 7.5#7]

Find the general solution of the system of equations:

$$\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} \mathbf{x}$$

Also draw a direction field and a few of the trajectories. In this problem, the coefficient matrix has a zero eigenvalue. As a result, the pattern of trajectories is different from those in the examples in the text.

( $\beta$ ; 10th: 7.5#12) [W 7th: 7.5#12]

Find the general solution to the system of equations given below:

$$\mathbf{x}' = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \mathbf{x}$$

## 7.6 Complex Eigenvalues

### Problems for everyone

(A; 10th: 7.6# 1, 3, 4)[VW 7th: 7.6#1,3; W 7th: 7.6#4]

Express the general solutions of the systems below in terms of real valued functions. Also draw a direction field for the system (**optional**) and sketch a few trajectories. Describe the behavior of the solutions as  $t \rightarrow \infty$

a.

$$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \mathbf{x}$$

b.

$$\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}$$

c.

$$\mathbf{x}' = \begin{pmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{pmatrix} \mathbf{x}$$

(B; 10th: 7.6#9)[W 7th: 7.6#9] Find the solution to the initial value problem.

$$\mathbf{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(\mathbf{0}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Describe the behavior as  $t \rightarrow \infty$ .

### More challenging problems

( $\alpha$ ; 10th: 7.6#13,16)[W 7th: 7.6#13, 16]

In each of the problems below the matrix contains a 'tunable' parameter  $\alpha$ . In each problem:

- Determine the eigenvalues as functions of the parameter  $\alpha$ .
- Find the critical value or values of  $\alpha$  where the qualitative nature of the phase portrait for the system changes.
- Draw a phase portrait for an  $\alpha$  slightly below and another phase portrait for  $\alpha$  slightly above each critical value.

$$\mathbf{x}' = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix} \mathbf{x}$$

$$\mathbf{x}' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \alpha & \frac{5}{4} \end{pmatrix} \mathbf{x}$$

## 9 Nonlinear Differential Equations and Stability

### 9.1 The Phase plane, Linear systems

(A; 10th: 9.1#1,5,6,7)[VW 7th: 9.1#1,5; W 7th: 9.1#6,7]

For each of the problems below:

- Find the eigenvalues and eigenvectors.
- Classify the critical point  $(0, 0)$  as to type, and determine its stability (stable, asymptotically stable, or unstable)
- Sketch several trajectories of the system in the phase plane and also sketch some typical graphs of  $x_1$  vs  $t$ .
- (Optional) Use a computer to plot accurately the curves requested in part (c).

(a)

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x} \quad (7)$$

(b)

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{x} \quad (8)$$

(c)

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x} \quad (9)$$

(d)

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \mathbf{x} \quad (10)$$



(B; 9th: 9.1#17)[W 7th: 9.1#17]

The equation of motion for a spring-mass system with damping is given by:

$$mu'' + \gamma u' + ku = 0$$

with  $m, \gamma, k > 0$ . Write this second order equation as a system of two first order equations for  $x = u, y = u'$ . Show that  $x = y = 0$  is a critical point, and analyze the nature and stability of this critical point as a function of the parameters  $m, \gamma, k$ . A similar analysis can be applied to the electric circuit equation.

$$L \frac{d^2 I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C} I = 0$$

## 9.2 Autonomous Systems and Stability

### Problems for Everyone

(A; 10th: 9.2 # 1,4)[VW 7th: 9.2 #1; W 7th: 9.2 #4]

In the problems below sketch the trajectory corresponding to the solution satisfying the specified initial conditions, and indicate the direction of motion for increasing  $t$ .

a.  $\frac{dx}{dt} = -x, \quad \frac{dy}{dt} = -2y, \quad x(0) = 4, \quad y(0) = 2$

b.  $\frac{dx}{dt} = ay, \quad \frac{dy}{dt} = -bx, \quad x(0) = \sqrt{a}, \quad y(0) = 0$  with  $a > 0$  and  $b > 0$

## More Challenging problems

( **$\alpha$** ; 10th: 9.2 #11, 13)[W 7th: 9.2# 11, 13]

For each of the systems below:

- Find all of the critical points in the system (equilibrium solutions).
- Use a computer to draw a direction field and phase portrait of the system.
- From the plots in part (b) determine the stability of each critical point.
- Describe the basin of attraction for each asymptotically stable critical point.

(a)

$$\frac{dx}{dt} = -x + 2xy, \quad \frac{dy}{dt} = y - x^2 - y^2$$

(b)

$$\frac{dx}{dt} = (2+x)(y-x), \quad \frac{dy}{dt} = (4-x)(y+x)$$

## 9.3 Locally Linear Systems

(**A**; 10th: 9.3#2)[W 7th: 9.3#2]

Verify that (0,0) is a critical point, show that the system is almost linear, and discuss the type and stability of the critical point by examining the corresponding linear system.

$$\frac{dx}{dt} = -x + y + 2xy, \quad \frac{dy}{dt} = -4x - y + x^2 - y^2$$

(B; 10th: 9.3#6, 7)[W 7th: 9.3#6, 7]

In the problems below do all of the following:

- Determine all critical points of the system.
- Find the corresponding linear system about each critical point
- Find the eigenvalues of each linear system. What conclusions can you then draw about the non-linear system?
- (Optional) Draw a phase portrait of the nonlinear system to confirm your conclusions or extend them in those cases where the linear system does not provide definite information about the non-linear system.

(a)

$$\frac{dx}{dt} = x - x^2 - xy, \quad \frac{dy}{dt} = 3y - xy - 2y^2$$

(b)

$$\frac{dx}{dt} = 1 - y, \quad \frac{dy}{dt} = x^2 - y^2$$

### More challenging problem

(α; 10th: 9.3#27)[W 7th: 9.3#25]

In this problem, we demonstrate how small changes in the coefficients of a system on linear equations can affect a critical point that is a center. Consider the system:

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x}$$

Show that the eigenvalues of this system are  $\pm i$  so that the critical point is a center. Now consider:

$$\mathbf{x}' = \begin{pmatrix} \epsilon & 1 \\ -1 & \epsilon \end{pmatrix} \mathbf{x}$$

where  $\epsilon \ll 1$  is arbitrarily small. Show that the eigenvalues are  $\epsilon \pm i$ . Thus, no matter how small we make epsilon ( $\neq 0$ ), the center becomes a spiral point. If  $\epsilon < 0$  the spiral point is asymptotically stable, but if  $\epsilon > 0$  the point is unstable.

## 9.4 Competing Species

### Problems for Everyone

(A; 10th: 9.4# 3, 4, 5)[W 7th: 9.4#3, 4, 5]

Each of the systems below can be interpreted as a description of the interaction of two species with populations  $x$  and  $y$ . In each of these problems carry out the following steps:

- (Optional) Draw a direction field and describe how solutions seem to behave. (Use Maple for this.)
- Find all of the critical points
- For each critical point find the corresponding linear system. Find the eigenvalues and eigenvectors of the linear system, and classify each critical point as to both type and stability.
- Sketch the trajectories in the neighborhood of each critical point.
- (Optional) Compute and plot enough trajectories to clearly show the behavior of the solutions.
- Determine the limiting behavior of  $x$  and  $y$  as  $t \rightarrow \infty$  and interpret the results in terms of the populations of the two species (e.g. coexistence, mutual extinction etc.)

(a)  $\frac{dx}{dt} = x(1.5 - 0.5x - y), \quad \frac{dy}{dt} = y(2 - y - 1.125x)$

(b)  $\frac{dx}{dt} = x(1.5 - 0.5x - y), \quad \frac{dy}{dt} = y(0.75 - y - 0.125x)$

(c)  $\frac{dx}{dt} = x(1 - x - y), \quad \frac{dy}{dt} = y(1.5 - y - x)$

## 9.5 Predator-Prey Equations

(A; 10th: 9.5#1,3,4)[W 7th: 9.5#1,3,4]

Each of the following systems can be interpreted as describing the interaction of two populations with densities  $x$  and  $y$ . In each of the problems carry out the following steps:

- (Optional) Draw a direction field and describe how the solutions seem to behave (Use Maple for this).
- Find all critical points.
- For each critical point linearize and find the appropriate, approximate linear system. Find the eigenvalues and eigenvectors of this system; classify each critical point as to type and stability.
- Sketch the trajectories in a neighborhood of each critical point.
- (Optional) Draw a phase portrait of the system.
- (Optional) Determine the limiting behavior of  $x$  and  $y$  as  $t \rightarrow \infty$  and interpret the results in terms of the populations of two species.

(a)

$$\frac{dx}{dt} = x(1.5 - 0.5y) \quad \frac{dy}{dt} = y(-0.5 + x)$$

(b)

$$\frac{dx}{dt} = x(1 - 0.5x - 0.5y) \quad \frac{dy}{dt} = y(-0.25 + 0.5x)$$

(c)

$$\frac{dx}{dt} = x(1.125 - x - 0.5y) \quad \frac{dy}{dt} = y(-1 + x)$$

**More challenging problems:**

( $\alpha$ ; 10th: 9.5 #8)[W 7th: 9.5#8]

- a. Find the period of oscillations for the predator and prey populations. using the approximation below which is valid for small oscillations:

$$x = \frac{c}{\gamma} + \frac{c}{\gamma} K \cos(\sqrt{act} + \phi)$$
$$y = \frac{c}{\alpha} + \frac{c}{\alpha} K \sin(\sqrt{act} + \phi)$$

Note that the period is independent of the amplitude of the oscillations.

- b. For the solution to the nonlinear solution in (2) shown in Figure 9.5.3 (Boyce and DiPrima 9th: p. 532) estimate the period as well as possible. Is the result the same as the linear approximation above?
- c. Calculate other solutions to (2), that is solutions satisfying other initial conditions, and determine their periods. Are the periods the same for all initial conditions?

## 10 Partial Differential Equations and Fourier Series

### 10.1 Two-Point Boundary Value Problems

#### Problems for everyone

(A; 10th: 10.1#2,7)[W 7th: 10.1#2 and VW 7th: 10.1#7]

In the following problems, either solve the given boundary value problem or else show that it has no solution.

a.  $y'' + 2y = 0, \quad y'(0) = 1, \quad y'(\pi) = 0$

b.  $y'' + 4y = \cos x, \quad y(0) = 0, \quad y(\pi) = 0.$

(MEMO: Solution for 10.1#2 is incorrect.)

(B; 10th: 10.1#14,19)[VW 7th: 10.1#11 and W 7th: 10.1#16]

In the following problems find the eigenvalues and eigenfunctions of the given boundary value problem. Assume that all eigenvalues are real.

a.  $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0$

b.  $y'' - \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0$

### 10.2 Fourier Series

#### Problems for everyone

(A; 10th: 10.2#10)[W 7th: 10.2#10]

If  $f(x) = \begin{cases} x+1, & -1 \leq x < 0, \\ x, & 0 \leq x < 1, \end{cases}$  and if  $f(x+2) = f(x)$ , find a formula for  $f(x)$  in the interval  $1 < x < 2$ ; in the interval  $8 < x < 9$ .

(B; 10th: 10.2#13,14,16)[W 7th: 10.2#13, VW 7th: 10.2#14, G #70, V 7th: 10.2#16]

In each of the following problems,

- (a) Sketch the graph of the given function for three periods;
- (b) Find the Fourier series corresponding to the given function:

- a.  $f(x) = -x \quad -L \leq x < L; \quad f(x+2L) = f(x)$
- b.  $f(x) = \begin{cases} 1, & -L \leq x < 0, \\ 0, & 0 \leq x < L, \end{cases} \quad f(x+2L) = f(x).$
- c.  $f(x) = \begin{cases} x+1, & -1 \leq x < 0, \\ 1-x, & 0 \leq x < 1, \end{cases} \quad f(x+2) = f(x).$

(C; 10th: 10.2#19,22)[VW 7th: 10.2#19, W 7th: 10.2#22]

In the following problems:

- (a) Sketch the graph of the given function for three periods.
- (b) Find the Fourier series for the given function.
- (c) (**Optional**) Plot  $s_m(x)$  versus  $x$  for  $m = 5, 10$ , and  $20$ .
- (d) (**Optional**) Describe how the Fourier series seems to be converging.

- a.  $f(x) = \begin{cases} -1, & -2 \leq x < 0, \\ 1, & 0 \leq x < 2, \end{cases} \quad f(x+4) = f(x).$
- b.  $f(x) = \begin{cases} x+2, & -2 \leq x < 0, \\ 2-2x, & 0 \leq x < 2, \end{cases} \quad f(x+4) = f(x).$

### 10.3 The Fourier Convergence Theorem

#### Problems for everyone

(A; 10th: 10.3#2,4,5)[VW 7th: 10.3#2,5, G#74]

In the following problems assume that the given function is periodically extended outside the original interval.

- (a) Find the Fourier series for the extended function.
- (b) Sketch the graph of the function to which the series converges for three periods.

- a.  $f(x) = \begin{cases} 0, & -\pi \leq x < 0, \\ x, & 0 \leq x < \pi, \end{cases}$
- b.  $f(x) = 1 - x^2, \quad -1 \leq x < 1$
- c.  $f(x) = \begin{cases} 0, & -\pi \leq x < -\pi/2, \\ 1, & -\pi/2 \leq x < \pi/2, \\ 0, & \pi/2 \leq x < \pi, \end{cases}$



### Challenge Problem

( $\alpha$ ; 10th: 10.3#10)[W 7th: 10.3#10]

In the following problem, assume that the given function is periodically extended outside the original interval.

- (a) Find the Fourier series for the extended function.
- (b) Let  $e_n(x) = f(x) - s_n(x)$ . Find the least upper bound or the maximum value (if it exists) of  $|e_n(x)|$  for  $n = 10, 20$ , and  $40$ .
- (c) If possible, find the smallest  $n$  for which  $|e_n(x)| \leq 0.01$  for all  $x$ .

$$f(x) = \begin{cases} x + 2, & -2 \leq x < 0, \\ 2 - 2x, & 0 \leq x < 2, \end{cases} \quad f(x + 4) = f(x).$$

## 10.4 Even and Odd Functions

### Problems for everyone

(A; 10th: 10.4#1,2)[V 7th: 10.4#1,2]

In the following problems, determine whether the given function is even, odd or neither.

a.  $x^3 - 2x$

b.  $x^3 - 2x + 1$

(B; 10th: 10.4#7)[VW 7th: 10.4#7]

In the following problem, a function  $f$  is given on an interval of length  $L$ . In each case sketch the graphs of the even and odd extensions of  $f$  of period  $2L$

$$f(x) = \begin{cases} x, & 0 \leq x < 2, \\ 1, & 2 \leq x < 3, \end{cases}$$

(C; 10th: 10.4#15,21)[VW 7th: 10.4#15,21]

In the following problems, find the required Fourier series for the given function, and sketch the graph of the function to which the series converges over three periods.

a.  $f(x) = \begin{cases} 1, & 0 \leq x < 1, \\ 0, & 1 \leq x < 2, \end{cases}$  cosine series, period 4

b.  $f(x) = L - x, \quad 0 \leq x \leq L;$  cosine series, period  $2L$

(D; 10th: 10.4#23)[W 7th: 10.4#23]

In the following problem:

(a) Find the cosine series of period  $4\pi$  for the function:

$$f(x) = \begin{cases} x, & 0 < x < \pi, \\ 0, & \pi < x < 2\pi, \end{cases}$$

(b) Sketch the graph of the function to which the series converges for three periods.

(c) (**Optional**) Plot one or more partial sums of the series.

(E; 10th: 10.4#27)[W 7th: 10.4#27]

In the following problem, the function  $f(x) = 3 - x$  is given on the interval  $0 < x < 3$ .

- (a) Sketch the graphs of the even extension  $g(x)$  and the odd extension  $h(x)$  of the given function of period  $2L$  over three periods.
- (b) Find the Fourier cosine and sine series for the given function.
- (c) (**Optional**) Plot a few partial sums of each series.
- (d) (**Optional**) For each series investigate the dependence on the number of terms in the partial sum of the maximum error on  $[0, L]$ .

### More challenging problems

(α; 9th: 10.4#38)[W 7th: 10.4#38]

Let  $f$  be extended into  $(L, 2L)$  in an arbitrary manner. Then extend the resulting function into  $(-2L, 0)$  as an odd function and elsewhere as a periodic function of period  $4L$  (see Figure 10.4.6 (9th edition)). Show that this function has a Fourier sine series in terms of the functions  $\sin(n\pi x/2L)$ ,  $n = 1, 2, 3 \dots$  that is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/2L),$$

where

$$b_n = \frac{1}{L} \int_0^{2L} f(x) \sin(n\pi x/2L) dx$$

This series converges to the original function on  $(0, L)$ .

(β; 10th: 10.4#39)[W 7th: 10.4#39]

Let  $f$  first be extended into  $(L, 2L)$  so that it is symmetric about  $x = L$ ; that is, so as to satisfy  $f(2L - x) = f(x)$  for  $0 \leq x < L$ . Let the resulting function be extended into  $(-2L, 0)$  as an odd function and elsewhere (see Figure 10.4.7 (9th edition)) as a periodic function of period  $4L$ . Show that this function has a Fourier series in terms of the functions  $\sin(\pi x/2L)$ ,  $\sin(3\pi x/2L)$ ,  $\sin(5\pi x/2L)$ , ...; that is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{(2n-1)\pi x}{2L}$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{(2n-1)\pi x}{2L} dx. \quad (11)$$

This series converges to the original function on  $(0, L]$ .

## 10.5 Separation of Variables, Heat Conduction in a Rod

### Problems for everyone

(A; 10th: 10.5#2, 5)[VW 7th: 10.5#2 and W 7th: 10.5#5]

Determine whether the following partial differential equations can be solved using the method of separation of variables. If separation of variables works, find the two resulting ordinary differential equations which must be solved.

a.  $tu_{xx} + xu_t = 0$

b.  $u_{xx} + (x + y)u_{yy} = 0$

(B; 10th: 10.5#8)[W 7th: 10.5#8]

Find the solution to the following heat conduction problem:

$$u_{xx} = 4u_t, \quad 0 < x < 2, \quad t > 0$$
$$u(0, t) = 0 \quad u(2, t) = 0 \quad u(x, 0) = 2 \sin\left(\frac{\pi x}{2}\right) - \sin(\pi x) + 4 \sin(2\pi x)$$

(C; 10th: 10.5#9)[VW 7th: 10.5#9]

Consider heat conduction in a rod of 40 cm length whose ends are maintained at  $0^\circ C$  for all  $t > 0$ . Find an expression for the temperature  $u(x, t)$  at any time if the initial temperature distribution is  $u(x, 0) = 50$  for  $0 < x < 40$ . Suppose further that  $\alpha^2 = 1$ .

(D; 10th: 10.5#18)[W 7th: 10.5#18]

Let a metallic rod 20cm long be heated to a uniform temperature of  $100^\circ C$ . Suppose that at  $t = 0$  the ends of the bar are plunged into an ice bath at  $0^\circ C$  and thereafter maintained at that temperature, but no heat is allowed to escape along the lateral surface of the bar. Find an expression for the temperature at any point in the bar at any later time. Determine the temperature of the center of the bar at time  $t = 30$  sec if the bar is made of:

a. silver ( $\alpha^2 = 1.71 \text{ cm}^2/\text{sec}$ )

b. aluminum ( $\alpha^2 = 0.86 \text{ cm}^2/\text{sec}$ )

c. cast iron ( $\alpha^2 = 0.12 \text{ cm}^2/\text{sec}$ )

### More challenging problems:

( $\alpha$ ; 10th: 10.5#13)[W 7th: 10.5#13]

Consider the rod in part C. For  $t = 5$  and  $x = 20$  determine how many terms  $m$  are needed to find the solution correct to three decimal places. A reasonable way to do this is find  $m$  such that adding one more term does not change the first three decimal places for the approximate solution of  $u(20, 5)$ . Repeat for  $t = 20$  and  $t = 80$ . Form a conclusion about the speed of convergence of the series for  $u(x, t)$ .

( $\beta$ ; 10th: 10.5#14)[W 7th: 10.5#14]

For the same rod as in part C.

- Plot  $u$  versus  $x$  for  $t = [5, 10, 20, 40, 100, 200]$ . Put all of the graphs on the same set of axes and thereby obtain a picture of the way that the temperature distribution changes with time.
- Plot  $u$  versus  $t$  for  $x = [5, 10, 15, 20]$ .
- Draw a three dimensional plot of  $u$  versus both  $x$  and  $t$ .
- How long (approximately) does it take for the whole rod to cool off to a temperature of no more than  $1^\circ\text{C}$ .

( $\gamma$ ; 10th: 10.5#20)[W 7th: 10.5#20]

In solving differential equations the computations can almost always be simplified by the use of dimensionless variables. Show that if the dimensionless variable:  $\xi = \frac{x}{L}$  is introduced, the heat conduction equation becomes:

$$\frac{\partial^2 u}{\partial \xi^2} = \frac{L^2}{\alpha^2} \frac{\partial u}{\partial t}, \quad 0 < \xi < 1, \quad t > 0$$

Since  $L^2/\alpha^2$  has the units of time, it is convenient to use this quantity to define a dimensionless time variable  $\tau = (\alpha^2/L^2)t$ . Then show that the heat conduction equation reduces to

$$\frac{\partial^2 u}{\partial \xi^2} = \frac{\partial u}{\partial \tau}, \quad 0 < \xi < 1, \quad \tau > 0.$$

( $\delta$ ; 10th: 10.5#22)[W 7th: 10.5#22]

The heat conduction equation in two space dimensions is:

$$\alpha^2(u_{xx} + u_{yy}) = u_t$$

Assuming that  $u(x, y, t) = X(x)Y(y)T(t)$ , find the ordinary differential equations which are satisfied by these component functions.

## 10.6 Other Heat Conduction Problems

### Problems for everyone

(A; 10th: 10.6#12)[G#90]

Consider a uniform rod of length,  $L$  with an initial temperature given by  $u(x, 0) = \sin(\pi x/L)$ ,  $0 \leq x \leq L$ . Assume that both sides of the bar are insulated.

- Find the temperature,  $u(x, t)$
- What is the steady-state temperature as  $t \rightarrow \infty$ ?
- (Optional) Let  $\alpha^2 = 1$  and  $L = 40$  plot  $u$  versus  $x$  for several values of  $t$ . Also plot  $u$  versus  $t$  for several values of  $x$ .
- (Optional) Describe briefly how the temperature in the rod changes as time progresses.

(B; 10th: 10.6#15)[W 7th: 10.6#15]

Consider a uniform bar of length  $L$  having an initial temperature distribution given by  $f(x)$ ,  $0 \leq x \leq L$ . Assume that the temperature at  $x = 0$  is held at  $0^\circ\text{C}$  while the end  $x = L$  is insulated so that no heat passes through it,

- Show that the fundamental solutions of this partial differential equation and boundary value problem are given by:

$$u_n(x, t) = e^{-(2n-1)^2\pi^2\alpha^2/4L^2t} \sin\left(\frac{(2n-1)\pi x}{2L}\right)$$

- (Optional) Find a formal series expansion for the temperature:

$$u(x, t) = \sum_{n=1}^{\infty} c_n u_n(x, t)$$

that also satisfies the given initial condition. (HINT: even though the fundamental solution only involves odd sines, it is still possible to represent  $f$  with a Fourier series only using these functions. See Problem  $\beta$  of Section 10.4.)

### More challenging problems

( $\alpha$ ; 10th: 10.6#1,4,7)[*V 7th: 10.6#1; W 7th: 10.6#4,7*]

Find the steady-state solution to the heat conduction problem  $u_t = \alpha^2 u_{xx}$  that satisfies the given set of boundary conditions:

- a.  $u(0, t) = 10 \quad u(50, t) = 40$
- b.  $u_x(0, t) = 0 \quad u(L, t) = T$
- c.  $u_x(0, t) - u(0, t) = 0 \quad u(L, t) = T$

( $\beta$ ; 10th: 10.6#9)[*G#89; V 7th: 10.6#9*]

Let an aluminum rod of length 20cm be initially at the uniform temperature of  $25^\circ C$ . suppose that at time  $t = 0$ , the end  $x = 0$  is cooled to  $0^\circ C$  while the end  $x = 20$  is heated to  $60^\circ C$  and both are thereafter maintained at those temperatures.

- a. Find the temperature distribution in the rod at any time,  $t$ .
- b. Plot the initial temperature distribution, the final, steady state distribution and the distribution at two representative intermediate times on the same set of axes.
- c. Plot  $u$  vs  $t$  for  $x = 5, x = 10, x = 15$ .
- d. Determine how much time must elapse before the temperature at  $x = 5$  comes within 1% of its steady state value.

## 10.7 The Wave Equation: Vibrations of an Elastic String

### Problems for everyone

(A; 10th: 10.7#1)[W 7th: 10.7#1]

Consider an elastic string of length  $L$  whose ends are held fixed. The string is set in motion with no initial velocity from an initial position  $u(x, 0) = f(x)$ , where

$$f(x) = \begin{cases} \frac{2x}{L} & 0 \leq x \leq \frac{L}{2} \\ \frac{2(L-x)}{L} & \frac{L}{2} \leq x \leq L. \end{cases} \quad (12)$$

Let  $L = 10$  and  $a = 1$  in parts (b) through (d).

- Find the displacement of the string,  $u(x, t)$ .
- (Optional) Plot  $u(x, t)$  versus  $x$ ,  $0 \leq x \leq 10$  for several values of  $t$ ,  $0 \leq t \leq 20$ .
- (Optional) Plot  $u(x, t)$  versus  $t$  for several values of  $x$ .
- (Optional) Construct an animation of the solution for at least one period.
- (Optional) Describe the motion of the string in a few sentences.

(B; 10th: 10.7#5)[W 7th: 10.7#5]

Consider an elastic string of length  $L$  whose ends are held fixed. The string is set in motion from its equilibrium position with an initial velocity  $u_t(x, 0) = g(x)$ , where

$$g(x) = \begin{cases} \frac{2x}{L} & 0 \leq x \leq \frac{L}{2} \\ \frac{2(L-x)}{L} & \frac{L}{2} \leq x \leq L \end{cases} \quad (13)$$

Let  $L = 10$  and  $a = 1$  in parts (b) through (d).

- Find the displacement of the string,  $u(x, t)$ .
- (Optional) Plot  $u(x, t)$  versus  $x$ ,  $0 \leq x \leq 10$  for several values of  $t$ ,  $0 \leq t \leq 20$ .
- (Optional) Plot  $u(x, t)$  versus  $t$  for several values of  $x$ .
- (Optional) Construct an animation of the solution for at least one period.
- (Optional) Describe the motion of the string in a few sentences.



(C; 10th: 10.7#21)[W 7th: 10.7#22]

The motion of a circular elastic membrane such as a drumhead, is governed by the two dimensional wave equation in polar coordinates:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = \alpha^{-2}u_{tt}$$

Assuming that  $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$  find ordinary differential equations satisfied by  $R$ ,  $\Theta$  and  $T$ .

### More challenging problems

( $\alpha$ ; 10th: 10.7#9)[W 7th: 10.7#9]

If an elastic string is free at one end, the boundary condition to be satisfied there is:  $u_x = 0$ . Find the displacement of  $u(x, t)$  in an elastic string of length  $L$ , fixed at  $x = 0$  and free at  $x = L$ , set in motion with no initial velocity from the initial position  $u(x, 0) = f(x)$  where  $f(x)$  is a known function.

HINT: show the fundamental solution to this problem satisfying everything except the inhomogeneous initial condition is given by:

$$u_n(x, t) = \sin(\lambda_n x) \cos(\lambda_n \alpha t)$$
$$\lambda_n = \frac{(2n-1)\pi}{2L} \quad n \in \{1, 2, 3, \dots\}$$

Compare this problem with Problem B of the previous subsection. Pay special attention to the extension of the initial data outside the original interval  $(0, L)$ .