Introduction to differential forms

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The calculus of differential forms give an alternative to vector calculus which is ultimately simpler and more flexible. Unfortunately it is rarely encountered at the undergraduate level. However, the last few times I taught undergraduate advanced calculus I decided I would do it this way. So I wrote up this brief supplement which explains how to work with them, and what they are good for, but the approach is kept informal. In particular, multlinear algebra is kept to a minimum, and I don't define manifolds or anything like that. By the time I got to this topic, I had covered a certain amount of standard material, which is briefly summarized at the end of these notes.

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1.1 1-forms

A differential 1-form (or simply a differential or a 1-form) on an open subset of $\mathbb{R}^2$ is an expression $F(x,y)dx + G(x,y)dy$ where F, G are $\mathbb{R}$-valued functions on the open set. A very important example of a differential is given as follows: If $f(x,y)$ is C^1 $\mathbb{R}$-valued function on an open set U, then its total differential (or exterior derivative) is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

It is a differential on U.

In a similar fashion, a differential 1-form on an open subset of $\mathbb{R}^3$ is an expression $F(x,y,z)dx + G(x,y,z)dy + H(x,y,z)dz$ where F, G, H are $\mathbb{R}$-valued functions on the open set. If $f(x,y,z)$ is a C^1 function on this set, then its total differential is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

At this stage, it is worth pointing out that a differential form is very similar to a vector field. In fact, we can set up a correspondence:

$$F\mathbf{i} + G\mathbf{j} + H\mathbf{k} \leftrightarrow F dx + G dy + H dz$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the standard unit vectors along the x, y, z axes. Under this set up, the gradient ∇f corresponds to df. Thus it might seem that all we are doing is writing the previous concepts in a funny notation. However, the notation is very suggestive and ultimately quite powerful. Suppose that that x, y, z depend on some parameter t, and f depends on x, y, z, then the chain rule says

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

Thus the formula for df can be obtained by canceling dt.

1.2 Exactness in $\mathbb{R}^2$

Suppose that $F dx + G dy$ is a differential on $\mathbb{R}^2$ with C^1 coefficients. We will say that it is *exact* if one can find a C^2 function $f(x, y)$ with $df = F dx + G dy$ Most differential forms are not exact. To see why, note that the above equation is equivalent to

$$F = \frac{\partial f}{\partial x},\ G = \frac{\partial f}{\partial y}.$$

Therefore if f exists then

$$\frac{\partial F}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial G}{\partial x}$$

But this equation would fail for most examples such as ydx. We will call a differential *closed* if $\frac{\partial F}{\partial y}$ and $\frac{\partial G}{\partial x}$ are equal. So we have just shown that if a differential is to be exact, then it had better be closed.

Exactness is a very important concept. You've probably already encountered it in the context of differential equations. Given an equation

$$\frac{dy}{dx} = \frac{F(x, y)}{G(x, y)}$$

we can rewrite it as

$$F dx - G dy = 0$$

If $F dx - G dy$ is exact and equal to say, df, then the curves $f(x, y) = c$ give solutions to this equation.

These concepts arise in physics. For example given a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ representing a force, one would like find a function $P(x, y)$ called the potential energy, such that $\mathbf{F} = -\nabla P$. The force is called *conservative* (see section 2.3) if it has a potential energy function. In terms of differential forms, $\mathbf{F}$ is conservative precisely when $F_1 dx + F_2 dy$ is exact.

1.3 Parametric curves

Before discussing line integrals, we have to say a few words about parametric curves. A parametric curve in the plane is vector valued function $C : [a, b] \to \mathbb{R}^2$. In other words, we let x and y depend on some parameter t running from a to b. It is not just a set of points, but the trajectory of particle travelling along the curve. To begin with, we will assume that C is C^1. Then we can define the the velocity or tangent vector $\mathbf{v} = (\frac{dx}{dt}, \frac{dy}{dt})$. We want to assume that the particle travels without stopping, $\mathbf{v} \neq 0$. Then $\mathbf{v}$ gives a direction to C, which we also refer to as its *orientation*. If C is given by

$$x = f(t),\ y = g(t), a \leq t \leq b$$

then

$$x = f(-u),\ y = g(-u), -b \leq u \leq -a$$

will be called $-C$. This represents the same set of points, but traveled in the opposite direction.

Suppose that C is given depending on some parameter t,

$$x = f(t),\ y = g(t)$$

and that t depends in turn on a new parameter $t = h(u)$ such that $\frac{dt}{du} \neq 0$. Then we can get a new parametric curve C'

$$x = f(h(u)),\ y = g(h(u))$$

It the derivative $\frac{dt}{du}$ is everywhere positive, we want to view the oriented curves C and C' as the equivalent. If this derivative is everywhere negative, then $-C$ and C' are equivalent. For example, the curves

$$C : x = \cos\theta,\ y = \sin\theta,\ 0 \leq \theta \leq 2\pi$$

$$C' : x = \sin t,\ y = \cos t,\ 0 \leq t \leq 2\pi$$

represent going once around the unit circle counterclockwise and clockwise respectively. So C' should be equivalent to $-C$. We can see this rigorously by making a change of variable $\theta = \pi/2 - t$.

It will be convient to allow piecewise C^1 curves. We can treat these as unions of C^1 curves, where one starts where the previous one ends. We can talk about parametrized curves in $\mathbb{R}^3$ in pretty much the same way.

1.4 Line integrals

Now comes the real question. Given a differential $Fdx + Gdy$, when is it exact? Or equivalently, how can we tell whether a force is conservative or not? Checking that it's closed is easy, and as we've seen, if a differential is not closed, then it can't be exact. The amazing thing is that the converse statement is often (although not always) true:

THEOREM 1.4.1 *If $F(x,y)dx + G(x,y)dy$ is a closed form on all of $\mathbb{R}^2$ with C^1 coefficients, then it is exact.*

To prove this, we would need solve the equation $df = Fdx + Gdy$. In other words, we need to undo the effect of d and this should clearly involve some kind of integration process. To define this, we first have to choose a parametric C^1 curve C. Then we define:

DEFINITION 1.4.2

$$\int_C Fdx + Gdy = \int_a^b \left[F(x(t), y(t))\frac{dx}{dt} + G(x(t), y(t))\frac{dy}{dt}\right] dt$$

If C is piecewise C^1, then we simply add up the integrals over the C^1 pieces. Although we've done everything at once, it is often easier, in practice, to do this in steps. First change the variables from x and y to expresions in t, then replace dx by $\frac{dx}{dt}dt$ etc. Then integrate with respect to t. For example, if we parameterize the unit circle $\mathbf{c}$ by $x = \cos\theta, y = \sin\theta$, $0 \le \theta \le 2\pi$, we see

$$-\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = -\sin\theta(\cos\theta)'d\theta + \cos\theta(\sin\theta)'d\theta = d\theta$$

and therefore

$$\int_C -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = \int_0^{2\pi} d\theta = 2\pi$$

From the chain rule, we get

LEMMA 1.4.3

$$\int_{-C} Fdx + Gdy = -\int_C Fdx + Gdy$$

If C and C' are equivalent, then

$$\int_C Fdx + Gdy = \int_{C'} Fdx + Gdy$$

While we're at it, we can also define a line integral in $\mathbb{R}^3$. Suppose that $Fdx + Gdy + Hdz$ is a differential form with C^1 coeffients. Let $C : [a,b] \to \mathbb{R}^3$ be a piecewise C^1 parametric curve, then

DEFINITION 1.4.4

$$\int_C Fdx + Gdy + Hdz =$$

$$\int_a^b \left[F(x(t), y(t), z(t))\frac{dx}{dt} + G(x(t), y(t), z(t))\frac{dy}{dt} + H(x(t), y(t), z(t))\frac{dz}{dt}\right] dt$$

The notion of exactness extends to $\mathbb{R}^3$ automatically: a form is exact if it equals df for a C^2 function. One of the most important properties of exactness is its path independence:

PROPOSITION 1.4.5 *If ω is exact and C_1 and C_2 are two parametrized curves with the same endpoints (or more acurately the same starting point and ending point), then*

$$\int_{C_1} \omega = \int_{C_2} \omega$$

It's quite easy to see why this works. If $\omega = df$ and $C_1 : [a,b] \to \mathbb{R}^3$ then

$$\int_{C_1} df = \int_a^b \frac{df}{dt} dt$$

by the chain rule. Now the fundamental theorem of calculus shows that the last integral equals $f(C_1(b)) - f(C_1(a))$, which is to say the value of f at the endpoint minus its value at the starting point. A similar calculation shows that the integral over C_2 gives same answer. If the C is closed, which means that the starting point is the endpoint, then this argument gives

COROLLARY 1.4.6 *If ω is exact and C is closed, then $\int_C \omega = 0$.*

Now we can prove theorem 1.4.1. If $Fdx + Gdy$ is a closed form on $\mathbb{R}^2$, set

$$f(x,y) = \int_C Fdx + Gdy$$

where the curve is indicated below:

We parameterize both line segments seperately by $x = t$, $y = 0$ and $x = x(constant)$, $y = t$, and sum to get

$$f(x,y) = \int_0^x F(t,0)dt + \int_0^y G(x,t)dt$$

Then we claim that $df = Fdx + Gdy$. To see this, we differentiate using the fundamental theorem of calculus. The easy calculation is

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\int_0^y G(x,t)dt \\ &= G(x,y)\end{aligned}$$

Slightly trickier is

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\int_0^x F(x,0)dt + \frac{\partial}{\partial x}\int_0^y G(x,t)dt \\ &= F(x,0) + \int_0^y \frac{\partial G(x,t)}{\partial x}dt \\ &= F(x,0) + \int_0^y \frac{\partial F(x,t)}{\partial t}dt \\ &= F(x,0) + F(x,y) - F(x,0) \\ &= F(x,y)\end{aligned}$$

The same proof works if if we replace $\mathbb{R}^2$ by an open rectangle. However, it will fail for more general open sets. For example,

$$-\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

is C^1 1-form on the open set $\{(x,y) \mid (x,y) \neq (0,0)\}$ which is closed. But it is not exact (see exercise 6). In more advanced treatments, this failure of closed forms to be exact can be measured by something called the *de Rham cohomology* of the set.

1.5 Work

Line integrals have many important uses. One very direct application in physics comes from the idea of work. If you pick up a rock off the ground, or perhaps roll it up a ramp, it takes energy. The energy expended is called work. If you're moving the rock in straight line for a short distance, then the displacement can be represented by a vector $\mathbf{d} = (\Delta x, \Delta y, \Delta z)$ and the force of gravity by a vector $\mathbf{F} = (F_1, F_2, F_3)$. Then the work done is simply

$$-\mathbf{F}\cdot\mathbf{d} = -(F_1\Delta x + F_2\Delta y + F_3\Delta z).$$

On the other hand, if you decide to shoot a rocket up into space, then you would have to take into account that the trajectory $\mathbf{c}$ may not be straight nor can the force $\mathbf{F}$ be assumed to be constant (it's a vector field). However as the notation suggests, for the work we would now need to calculate the integral

$$-\int_{\mathbf{c}} F_1dx + F_2dy + F_3dz$$

One often writes this as

$$-\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$$

(think of $d\mathbf{s}$ as the "vector" (dx, dy, dz).)

1.6 Green's theorem for a rectangle

Let R be the rectangle in the xy-plane with vertices $(0,0), (a,0), (a,b), (0,b)$. Let C be the boundary curve of the rectangle oriented counter clockwise. Given C^1 functions $P(x,y), Q(x,y)$ on R, the fundamental theorem of calculus yields

$$\iint_D \frac{\partial Q}{\partial x} dxdy = \int_0^b [Q(a,y)) - Q(0,y)]dy = \int_C Q(x,y)dy$$

$$\iint_D \frac{\partial P}{\partial y} dydx = \int_0^a [P(x,b) - P(x,0)]dx = -\int_C P(x,y)dx$$

Subtracting yields Green's theorem for R

THEOREM 1.6.1

$$\int_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Our goal is to understand, and generalize to 3 dimensions, the operation which takes the one form $Pdx+Qdy$ to the integrand on the right. In traditional vector calculus this is handled using the *curl* ($\nabla\times$) which a vector field defined so that

$$\nabla \times (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

is the integrand of the right in Green's theorem. In general, one can discover the formula for the other components of $\nabla \times (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k})$ by expressing the integrals of $P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ around the boundaries of rectangles in the xz and yz planes and rewriting them as double integrals. To make a long story short,

$$\nabla \times (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) = (R_y - Q_z)\mathbf{i} + (Q_x - P_y)\mathbf{k} + (P_z - R_x)\mathbf{j}$$

(In practice, this is often written as a determinant

$$\nabla \times (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

But this should really be treated as a memory aid and nothing more.)

1.7 Exercise Set 1

1. Determine which of the following 1-forms on $\mathbb{R}^2$ are exact. Express the exact 1-forms in the form df.

   (a) $3ydx + xdy$

   (b) $ydx + xdy$

   (c) $e^x ydx + e^x dy$

   (d) $-ydx + xdy$

2. It is sometimes possible make a 1-form exact by multiplying it by a nonzero function of x and y called an integrating factor. In each of the nonexact 1-forms of problem 1, find an integrating factor of the form x^n, for some integer n.

3. An immediate consequence of Green's theorem is that the area of a rectangle enclosed by C is $\int_C xdy$. Check this by direct calculation.

4. Let C be a circle of radius r centered at 0 oriented counterclockwise. Check that $\int_C xdy$ gives the area of the circle.

5. Let C_1 be the helix $x = \cos\theta, y = \sin\theta, z = \theta$, $0 \le \theta \le 2\pi$, and C_2 be the line segment connecting $(0, 0, 0)$ to $(0, 0, 2\pi)$. Calculate $\int_{C_1} zdx + xdz$ and $\int_{C_2} zdx + xdz$.

6. Let C be the unit circle centered at 0 oriented counterclockwise. Calculate the integral $\int_C -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ and check that it is nonzero. Conclude that this 1-form is not exact.

7. Let r, θ be polar coordinates, so $x = r\cos\theta, y = r\sin\theta$. Convert dx and dy to polar coordinates. Use this to calculate $\int_C dx + dy$ where C is $r = \sin\theta$, $0 \le \theta \le \pi$.

8. Use the previous problem to solve for dr and $d\theta$. Observe that $d\theta$ is the same 1-form as in problem 6. So $d\theta$ isn't exact, in spite of the way it is written! How do you explain this?

2 2-forms

2.1 Wedge product

The cross product of vectors $\mathbf{u} \times \mathbf{v}$ is a very useful operation in 3 dimensional geometry. It determines the area of the parallelogram containing these vectors and the plane containing it. While there is no direct analogue of the cross product in higher dimensions, there is an operation which determines the last two things. Given (row) vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, define the matrix

$$\mathbf{u} \wedge \mathbf{v} = \frac{1}{2}(\mathbf{u}^T\mathbf{v} - \mathbf{v}^T\mathbf{u}) \tag{1}$$

($\wedge$ is pronounced "wedge"). In $\mathbb{R}^3$

$$(a, b, c) \wedge (d, e, f) = \frac{1}{2}\begin{pmatrix} 0 & ae - bd & af - cd \\ -ae + bd & 0 & bf - ce \\ -af + cd & -bf + ce & 0 \end{pmatrix}$$

We can see that the nonzero entries are basically the same as for the cross product,

$$(a, b, c) \times (d, e, f) = (bf - ce)\mathbf{i} + (-af + cd)\mathbf{j} + (ae - bd)\mathbf{k}$$

So these two operations are in some sense equivalent. The big difference is, of course, that the wedge product produces a matrix, but not just any matrix. Recall that a matrix $A = (a_{ij})$ is a skew symmetric if $A^T = -A$, i.e. $a_{ji} = -a_{ij}$. Let $\wedge^2\mathbb{R}^n$ denote the space of all skew symmetric $n \times n$ real matrices.

THEOREM 2.1.1 *The wedge product of two vectors lies in* $\wedge^2\mathbb{R}^n$.

This should be clear when $n = 3$ from the above formula. In general, we can use standard facts from linear algebra to see that

$$\frac{1}{2}(\mathbf{u}^T\mathbf{v} - \mathbf{v}^T\mathbf{u})^T = \frac{1}{2}(\mathbf{v}^T\mathbf{u}^{TT} - \mathbf{u}^T\mathbf{v}^{TT})$$

$$= \frac{1}{2}(\mathbf{v}^T\mathbf{u} - \mathbf{u}^T\mathbf{v}) = -\frac{1}{2}(\mathbf{u}^T\mathbf{v} - \mathbf{v}^T\mathbf{u})$$

The following properties are easy to check from the definition.

$$\mathbf{v} \wedge \mathbf{u} = -\mathbf{u} \wedge \mathbf{v} \tag{2}$$

$$\mathbf{u} \wedge \mathbf{u} = 0 \tag{3}$$

and

$$c(\mathbf{u} \wedge \mathbf{v}) = (c\mathbf{u}) \wedge \mathbf{v} = \mathbf{u} \wedge (c\mathbf{v}) \tag{4}$$

$$\mathbf{u} \wedge \mathbf{v} + \mathbf{u} \wedge \mathbf{w} = \mathbf{u} \wedge (\mathbf{v} + \mathbf{w}) \tag{5}$$

In practice, we will use these rules rather than the definition (1) for calculations. To really be convinced that the wedge captures the essential geometric features of the cross product, we note the following non-obvious fact.

THEOREM 2.1.2 *The product* $\mathbf{u} \wedge \mathbf{v}$ *determines the area of the parallelogram spanned by* $\mathbf{u}$ *and* $\mathbf{v}$ *and the plane containing these vectors, when there is a unique such plane.*

It will be useful to work in a bit more generality. Recall that a *vector space* V is an abstraction of $\mathbb{R}^n$, where elements of V, to be thought of as vectors, can be added and multiplied by numbers and these operations satisfy standard rules of algebra. Given a vector space V, we can construct a new vector space of 2-vectors

$$\wedge^2 V = \{\sum_i \mathbf{u}_i \wedge \mathbf{v}_i \mid \mathbf{u}_i, \mathbf{v}_i \in V\}$$

where the rules (2), (3), (4) and (5) are imposed, and only those rules. To see how this compares to the earlier description. We can expand a vector in $\mathbb{R}^n$ as

$$(a_1, \ldots, a_n) = a_1\mathbf{e}_1 + \ldots + a_n\mathbf{e}_n$$

uniquely, where

$$\mathbf{e}_1 = (1, 0, \ldots, 0), \mathbf{e}_2 = (0, 1, \ldots, 0), \ldots$$

In other words, the vectors $\mathbf{e}_i$ form a basis of $\mathbb{R}^n$. Any 2-vector can be expanded uniquely as an expression

$$\sum_{i=1}^{n}\sum_{j=1}^{i-1} a_{ij}\mathbf{e}_i \wedge \mathbf{e}_j = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\mathbf{e}_i \wedge \mathbf{e}_j$$

where the coefficients satisfy $a_{ij} = -a_{ji}$. The upshot is that we may identify $\wedge^2\mathbb{R}^n$ with the space of skew symmetric $n \times n$ matrices as we did above.

2.2 2-forms

A 2-form is an expression built using wedge products of pairs of 1-forms. On $\mathbb{R}^3$, this would be an expression:

$$F(x, y, z)dx \wedge dy + G(x, y, z)dy \wedge dz + H(x, y, z)dz \wedge dx$$

where F, G and H are functions defined on an open subset of $\mathbb{R}^3$. Any wedge product of two 1-forms can be put in this format. For example, using the above rules, we can see that

$$\begin{aligned}(3dx + dy) \wedge (e^x dx + 2dy) &= 3e^x dx \wedge dx + 6dx \wedge dy + e^x dy \wedge dx + 2dy \wedge dy \\ &= (6 - e^x)dx \wedge dy\end{aligned}$$

To be absolutely clear, we now allow c to be a function in (4).

The real significance of 2-forms will come later when we do surface integrals. A 2-form will be an expression that can be integrated over a surface in the same way that a 1-form can be integrated over a curve.

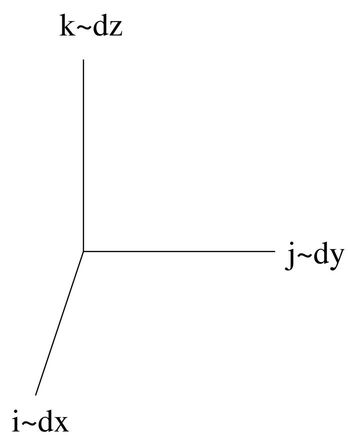
Earlier, we learned how to convert a vector field to a 1-form:

$$F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} \leftrightarrow F_1dx + F_2dy + F_3dz$$

We can also convert vector fields to 2-forms and back

$$F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} \leftrightarrow F_1dy \wedge dz + F_2dz \wedge dx + F_3dx \wedge dy$$

Here we are matching $\mathbf{i}$ with $dy \wedge dz$, $\mathbf{j}$ with $dz \wedge dx$, and $\mathbf{k}$ with $dx \wedge dy$. A rule that is easy to remember is as follows. To convert a 2-form to a vector, replace dx, dy, dz by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and take cross products.



To complete the picture, we can interchange 1-forms and 2-forms using the so called Hodge star operator.

$$*(F_1dx + F_2dy + F_3dz) = F_1dy \wedge dz + F_2dz \wedge dx + F_3dx \wedge dy$$

$$*(F_1dy \wedge dz + F_2dz \wedge dx + F_3dx \wedge dy) = F_1dx + F_2dy + F_3dz$$

Given a 1-form $F(x,y,z)dx + G(x,y,z)dy + H(x,y,z)dz$. We want to define its derivative $d\omega$ which will be a 2-form. The rules we use to evaluate it are:

$$d(\alpha + \beta) = d\alpha + d\beta$$
$$d(f\alpha) = (df) \wedge \alpha + fd\alpha$$
$$d(dx) = d(dy) = d(dz) = 0$$

where α and β are 1-forms and f is a function. Recall that

$$df = f_x dx + f_y dy + f_z dz$$

where $f_x = \frac{\partial f}{\partial x}$ and so on. Putting these together yields a formula

$$d(Fdx + Gdy + Hdz) = (G_x - F_y)dx \wedge dy + (H_y - G_z)dy \wedge dz + (F_z - H_x)dz \wedge dx$$

If we start with a vector field $\mathbf{V} = F\mathbf{i} + G\mathbf{j} + H\mathbf{k}$, replace it by a 1-form $Fdx + Gdy + Hdz$, apply d, then convert it back to a vector field, we end up with the curl of $\mathbf{V}$

$$\nabla \times \mathbf{V} = (H_y - G_z)\mathbf{i} + (G_x - F_y)\mathbf{k} + (F_z - H_x)\mathbf{j}$$

2.3 Exactness in $\mathbb{R}^3$ and conservation of energy

A C^1 1-form $\omega = Fdx + Gdy + Hdz$ is called exact if there is a C^2 function (called a potential) such that $\omega = df$. A 1-form ω is called closed if $d\omega = 0$, or equivalently if

$$F_y = G_x,\ F_z = H_x,\ G_z = H_y$$

These equations must hold when

$$F = f_x,\ G = f_y,\ H = f_z$$

Therefore:

THEOREM 2.3.1 *Exact* 1*-forms are closed.*

We have a converse statement which is sometimes called "Poincaré's lemma".

THEOREM 2.3.2 *If $\omega = Fdx + Gdy + Hdz$ is a closed form on $\mathbb{R}^3$ with C^1 coefficients, then ω is exact. In fact if $f(x_0, y_0, z_0) = \int_C \omega$, where C is any piecewise C^1 curve connecting $(0,0,0)$ to (x_0, y_0, z_0), then $df = \omega$.*

This can be rephrased in the language of vector fields. If $\mathbf{F} = F\mathbf{i} + G\mathbf{j} + H\mathbf{k}$ is C^1 vector field representing a force, then it is called conservative if there is a C^2 real valued function P, called potential energy, such that $\mathbf{F} = -\nabla P$. The theorem implies that a force $\mathbf{F}$, which is C^1 on all of $\mathbb{R}^3$, is conservative if and only if $\nabla \times \mathbf{F} = 0$. $P(x, y, z)$ is just the work done by moving a particle of unit mass along a path connecting $(0,0,0)$ to (x, y, z).

To appreciate the importance of this concept, recall from physics that the kinetic energy of a particle of constant mass m and velocity

$$\mathbf{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

is

$$K = \frac{1}{2}m||\mathbf{v}||^2 = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v}.$$

Also one of Newton's laws says

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F}$$

If $\mathbf{F}$ is conservative, then we can replace it by $-\nabla P$ above, move it to the other side, and then dot both sides by $\mathbf{v}$ to obtain

$$m\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} + \mathbf{v}\cdot\nabla P = 0 \qquad (6)$$

which can be simplified (exercise 6) to

$$\frac{d}{dt}(K + P) = 0. \qquad (7)$$

This implies that the total energy $K + P$ is constant.

2.4 Derivative of a 2-form and divergence

Earlier we defined wedge products of pairs of vectors. Now we extend it to triples. Given three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, we may think of the 3-vector $\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w}$ as the oriented volume of parallelopiped with $\mathbf{u}, \mathbf{v}, \mathbf{w}$ as the first, second and third sides. Oriented volume is the usual volume if $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is right-handed, otherwise it is minus the usual volume. With these rules, we see that

$$\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w} = -\mathbf{v}\wedge\mathbf{u}\wedge\mathbf{w} = \mathbf{v}\wedge\mathbf{w}\wedge\mathbf{u} = \ldots$$

and that

$$\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w} = 0$$

if any two of the vectors are equal. In addition, we have the distributive law

$$(\mathbf{u}_1 + \mathbf{u}_2)\wedge\mathbf{v}\wedge\mathbf{w} = \mathbf{u}_1\wedge\mathbf{v}\wedge\mathbf{w} + \mathbf{u}_2\wedge\mathbf{v}\wedge\mathbf{w} = 0$$

A 3-form is simply an expression

$$f(x,y,z)dx\wedge dy\wedge dz = -f(x,y,z)dy\wedge dx\wedge dz = f(x,y,z)dy\wedge dz\wedge dx = \ldots$$

These are things that will eventually get integrated over solid regions. The important thing for the present is an operation which takes 2-forms to 3-forms once again denoted by "d".

$$\begin{aligned} d(Fdy\wedge dz + Gdz\wedge dx + Hdx\wedge dy) &= (F_x dx + F_y dy + F_z dz)\wedge dy\wedge dz \\ &\quad + (G_x dx + G_y dy + G_z dz)\wedge dz\wedge dx \\ &\quad + (H_x dx + H_y dy + H_z dz)\wedge dx\wedge dy \end{aligned}$$

This simplifies to

$$d(Fdy\wedge dz + Gdz\wedge dx + Hdx\wedge dy) = (F_x + G_y + H_z)dx\wedge dy\wedge dz$$

It's probably easier to understand the pattern after converting the above 2-form to the vector field $\mathbf{V} = F\mathbf{i} + G\mathbf{j} + H\mathbf{k}$. Then the coefficient of $dx\wedge dy\wedge dz$ is the divergence

$$\nabla\cdot\mathbf{V} = F_x + G_y + H_z$$

So far we've applied d to functions (also called 0-forms) to obtain 1-forms, and then to 1-forms to get 2-forms, and finally to 2-forms.

$$\begin{tikzcd} \{\text{0-forms}\} \arrow[r, "d"] \arrow[d] & \{\text{1-forms}\} \arrow[r, "d"] \arrow[d] & \{\text{2-forms}\} \arrow[r, "d"] \arrow[d] & \{\text{3-forms}\} \arrow[d] \\ \{\text{functions}\} \arrow[r, "\nabla"] & \{\text{vector fields}\} \arrow[r, "\nabla\times"] & \{\text{vector fields}\} \arrow[r, "\nabla\cdot"] & \{\text{functions}\} \end{tikzcd}$$

The real power of this notation is contained in the following simple-looking formula

PROPOSITION 2.4.1 $d^2 = 0$

What this means is that given a C^2 real valued function defined on an open subset of $\mathbb{R}^3$, then $d(df) = 0$, and given a 1-form $\omega = Fdx + Gdy + Hdz$ with C^2 coefficents defined on an open subset of $\mathbb{R}^3$, $d(d\omega) = 0$. Both of these are quite easy to check:

$$d(df) = (f_{yx} - f_{xy})dx \wedge dy + (f_{zy} - f_{yz})dy \wedge dz + (f_{xz} - f_{zx})dz \wedge dx = 0$$

$$d(d\omega) = [G_{xz} - F_{yz} + H_{yx} - G_{zx} + F_{zy} - H_{xy}]dx \wedge dy \wedge dz = 0$$

In terms of standard vector notation this is equivalent to

$$\nabla \times (\nabla f) = 0$$

$$\nabla \cdot (\nabla \times \mathbf{V}) = 0$$

2.5 Poincaré's lemma for 2-forms

The analogue of theorem 2.3.2 holds:

THEOREM 2.5.1 *If ω is a 2-form on $\mathbb{R}^3$ such that $d\omega = 0$, then there exists a 1-form ξ such that $d\xi = \omega$.*

In terms of vector calculus, this says that a vector field on $\mathbb{R}^3$ is a gradient if it satisfies $\nabla \times \mathbf{F} = 0$. We will outline the proof, which gives a method for finding ξ. Let

$$\omega(x, y, z) = f(x, y, z)dx \wedge dy + g(x, y, z)dz \wedge dx + h(x, y, z)dz \wedge dy$$

The equation $d\omega = 0$ implies

$$f_z = h_x - g_y \tag{8}$$

The first step is to integrate out z. We define an operation called a *homotopy* by

$$H_z(\omega) = \left(\int_0^z g(x, y, z)dz\right) dx + \left(\int_0^z h(x, y, z)dz\right) dy$$

Differentiating, using the fundamental theorem of calculus and (8) yields

$$\begin{aligned}
dH_z\omega &= \left(\int_0^z (h_x - g_y)dz\right) dx \wedge dy \\
&\quad + [g(x, y, z) - g(x, y, 0)]dz \wedge dy + [h(x, y, z) - h(x, y, 0)]dz \wedge dx \\
&= \left(\int_0^z f_z dz\right) dx \wedge dy + [g(x, y, z) - g(x, y, 0)]dz \wedge dy + [h(x, y, z) - h(x, y, 0)]dz \wedge dx \\
&= [f((x, y, z) - f(x, y, 0)]dx \wedge dy + [g(x, y, z) - g(x, y, 0)]dz \wedge dy + [h(x, y, z) - h(x, y, 0)]dz \wedge dx
\end{aligned}$$

We can write this as

$$dH_z\omega = \omega - \omega(x, y, 0)$$

This doesn't solve the problem but it simplifies it, because $\omega(x, y, 0)$ doesn't depend on z. Differentiating both sides of the last equation, shows that $\omega(x, y, 0)$ is also closed. So if we can find ξ_1 so that

$$\omega(x, y, 0) = d\xi_1$$

then

$$\xi = \xi_1 + H_z\omega$$

solves the original problem. To find ξ_1, we reduce the problem further by integrating out y and then x as above, to get

$$dH_y\omega(x, y, 0) = \omega(x, y, 0) - \omega(x, 0, 0)$$

$$dH_z\omega(x, 0, 0) = \omega(x, 0, 0) - \omega(0, 0, 0)$$

So now we have reduced the problem to finding ξ_2 such that $\omega(0, 0, 0) = d\xi_2$; this is easy (exercise 7). Then using the above equations, we see that

$$\xi = \xi_2 + H_z\omega + H_y\omega(x, y, 0) + H_x\omega(x, 0, 0)$$

gives a solution.

2.6 Exercise Set 2

Let $\alpha = xdx + ydy + zdz$, $\beta = zdx + xdy + ydz$ and $\gamma = xydz$ in the following problems.

1. Calculate
   (a) $\alpha \wedge \beta$
   (b) $\alpha \wedge \gamma$
   (c) $\beta \wedge \gamma$
   (d) $(\alpha + \gamma) \wedge (\alpha + \gamma)$

2. Calculate
   (a) $d\alpha$
   (b) $d\beta$
   (c) $d(\alpha + \gamma)$
   (d) $d(x\alpha)$

3. Given $\omega = fdx + gdy + hdz$ such that $\omega \wedge dz = 0$, what can we conclude about f, g and h?

4. (a) Let $\omega = Fdx + Gdy + Hdz$ and let C be the straight line connecting $(0, 0, 0)$ to (x_0, y_0, z_0) show that

$$\int_C \omega = \int_0^1 (F(x_0t, y_0, z_0) + G(x_0, y_0t, z_0) + H(x_0, y_0, z_0t))dt$$

(b) Use this to prove theorem 2.3.2

5. Check that the following 1-forms are exact, and express them as df.

   (a) $dx + 2ydy + 3z^2dz$

   (b) $zy\cos(xy)dx + zx\cos(xy)dy + \sin(xy)dz$.

6. Check that equations (6) and (7) in the text are equivalent.

7. Prove the last step of the proof of theorem 2.5.1 that a 2-form with constant coefficients is exact. Hint: observe that $d(xdy) = dx \wedge dy$ etc.

8. Find a solution of $d\xi = \omega$, when $\omega = zdx \wedge dz + dy \wedge dz$.

3 Surface integrals

3.1 Parameterized Surfaces

Recall that a parameterized curve is a C^1 function from an interval $[a, b] \subset \mathbb{R}^1$ to $\mathbb{R}^3$. If we replace the interval by subset of the plane $\mathbb{R}^2$, we get a parameterized surface. Let's look at a few of examples

1) The upper half sphere of radius 1 centered at the origin can be parameterized using cartesian coordinates

$$\begin{cases} x = u \\ y = v \\ z = \sqrt{1 - u^2 - v^2} \\ u^2 + v^2 \leq 1 \end{cases}$$

2) The upper half sphere can be parameterized using spherical coordinates

$$\begin{cases} x = \sin(\phi)\cos(\theta) \\ y = \sin(\phi)\sin(\theta) \\ z = \cos(\phi) \\ 0 \leq \phi \leq \pi/2,\ 0 \leq \theta < 2\pi \end{cases}$$

(Since there is more than one convention, we should probably be clear about this. For us, θ is the "polar" angle, and ϕ is the angle measured from the positve z-axis.)

3) The upper half sphere can be parameterized using cylindrical coordinates

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = \sqrt{1 - r^2} \\ 0 \leq r \leq 1,\ 0 \leq \theta < 2\pi \end{cases}$$

An orientation on a curve is a choice of a direction for the curve. For a surface an orientation is a choice of "up" or "down". The easist way to make this precise is to view an orientation as a choice of (an upward, downward, outward or inward pointing) unit normal vector field $\mathbf{n}$ on S. A parameterized surface S

$$\begin{cases} x = f(u, v) \\ y = g(u, v) \\ z = h(u, v) \\ (u, v) \in D \end{cases}$$

is called *smooth* provided that f, g, h are C^1, the function that they define from $D \to \mathbb{R}^3$ is one to one, and the tangent vector fields

$$\mathbf{T}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$$

$$\mathbf{T}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$

are linearly independent. In this case, once we pick an ordering of the variables (say u first, v second) an orientation is determined by the normal

$$\mathbf{n} = \frac{\mathbf{T}_u \times \mathbf{T}_v}{||\mathbf{T}_u \times \mathbf{T}_v||}$$

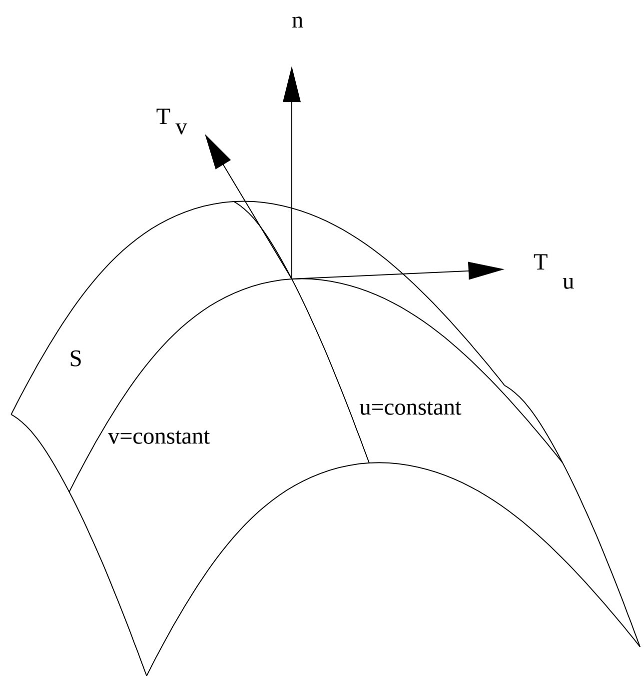


FIGURE 1

If we look at the examples given earlier. (1) is smooth. However there is a slight problem with our examples (2) and (3). Here $\mathbf{T}_\theta = 0$, when $\phi = 0$ in example (2) and when $r = 0$ in example (3). To deal with scenario, we will consider a surface smooth if there is at least one smooth parameterization for it.

Let C be a closed C^1 curve in $\mathbb{R}^2$ and D be the interior of C. D is an example of a surface with a boundary C. In this case the surface lies flat in the plane, but more general examples can be constructed by letting S be a parameterized surface

$$\begin{cases} x = f(u,v) \\ y = g(u,v) \\ z = h(u,v) \\ (u,v) \in D \subset \mathbb{R}^2 \end{cases}$$

then the image of C in $\mathbb{R}^3$ will be the boundary of S. For example, the boundary of the upper half sphere S

$$\begin{cases} x = \sin(\phi)\cos(\theta) \\ y = \sin(\phi)\sin(\theta) \\ z = \cos(\phi) \\ 0 \leq \phi \leq \pi/2, \ 0 \leq \theta < 2\pi \end{cases}$$

is the circle C given by

$$x = \cos(\theta), \ y = \sin(\theta), \ z = 0, \ 0 \leq \theta \leq 2\pi$$

In what follows, we will need to match up the orientation of S and its boundary curve. This will be done by the *right hand rule:* if the fingers of the right hand point in the direction of C, then the direction of the thumb should be "up".

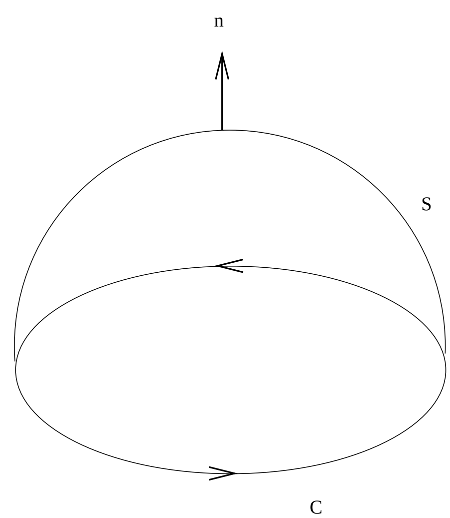


FIGURE 2

3.2 Surface Integrals

Let S be a smooth parameterized surface

$$\begin{cases} x = f(u,v) \\ y = g(u,v) \\ z = h(u,v) \\ (u,v) \in D \end{cases}$$

with orientation corresponding to the ordering u, v. The symbols dx etc. can be converted to the new coordinates as follows

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$$

$$dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv$$

$$\begin{aligned}
dx \wedge dy &= \left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv\right) \wedge \left(\frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv\right) \\
&= \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) du \wedge dv \\
&= \frac{\partial(x, y)}{\partial(u, v)} du \wedge dv
\end{aligned}$$

In this way, it is possible to convert any 2-form ω to uv-coordinates.

DEFINITION 3.2.1 *The integral of a 2-form on S is given by*

$$\iint_S F dx \wedge dy + G dy \wedge dz + H dz \wedge dx = \iint_D \left[F \frac{\partial(x, y)}{\partial(u, v)} + G \frac{\partial(y, z)}{\partial(u, v)} + H \frac{\partial(z, x)}{\partial(u, v)}\right] dudv$$

In practice, the integral of a 2-form can be calculated by first converting it to the form $f(u, v) du \wedge dv$, and then evaluating $\iint_D f(u, v)\, dudv$.

Let S be the upper half sphere of radius 1 oriented with the upward normal parameterized using spherical coordinates, we get

$$dx \wedge dy = \frac{\partial(x, y)}{\partial(\phi, \theta)} d\phi \wedge d\theta = \cos(\phi) \sin(\phi) d\phi \wedge d\theta$$

So

$$\iint_S dx \wedge dy = \int_0^{2\pi} \int_0^{\pi/2} \cos(\phi) \sin(\phi) d\phi d\theta = \pi$$

On the other hand if use the same surface parameterized using cylindrical coordinates

$$dx \wedge dy = \frac{\partial(x, y)}{\partial(r, \theta)} dr \wedge d\theta = r dr \wedge d\theta$$

Then

$$\iint_S dx \wedge dy = \int_0^{2\pi} \int_0^1 r dr d\theta = \pi$$

which leads to the same answer as one would hope. The general result is:

THEOREM 3.2.2 *Suppose that an oriented surface S has two different smooth C^1 parameterizations, then for any 2-form ω, the expression for the integrals of ω calculated with respect to both parameterizations agree.*

(This theorem needs to be applied to the half sphere with the point $(0, 0, 1)$ removed in the above examples.) Let us give the proof. Suppose that u, v is a parameterization as above, and $u = p(s, t), v = q(s, t)$ for one to one C^1-functions p, q on a domain E in pq-plane. We also want to assume that $\frac{\partial(u,v)}{\partial(s,t)} > 0$ on E. This will ensure that s, t gives a new parameterization of S with the correct orientation. Let us assume for simplicity that we have $\omega = F dx \wedge dy$. Then computing $\iint_S \omega$ in two ways gives either

$$\iint_D F \frac{\partial(x,y)}{\partial(u,v)} dudv$$

or

$$\iint_E F \frac{\partial(x,y)}{\partial(s,t)} dsdt$$

We have to prove that these integrals are equal. The chain rule can be expressed as a matrix equation

$$\begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{pmatrix}$$

Take the determinant of both sides to obtain

$$\frac{\partial(x,y)}{\partial(s,t)} = \frac{\partial(x,y)}{\partial(u,v)} \frac{\partial(u,v)}{\partial(s,t)} \tag{9}$$

The change of variables formula for double integrals tells us that

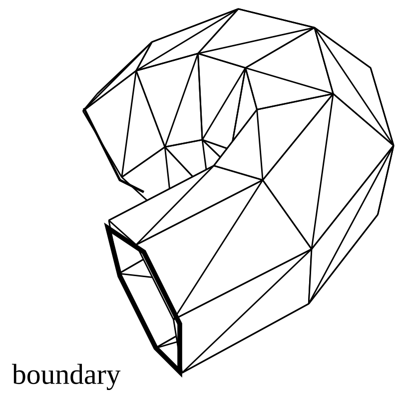
$$\iint_D F \frac{\partial(x,y)}{\partial(u,v)} dudv = \iint_E F \frac{\partial(x,y)}{\partial(u,v)} \left| \frac{\partial(u,v)}{\partial(s,t)} \right| dsdt$$

We can drop the absolute value sign on the second Jacobian because it is positive. Now substitute (9) to obtain

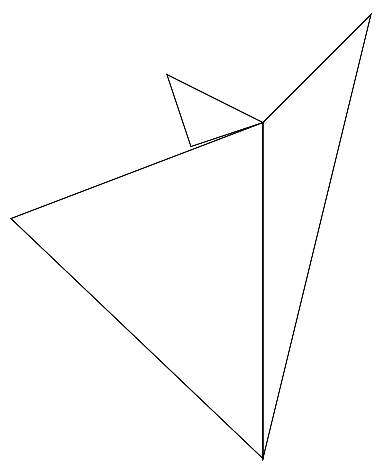
$$\iint_D F \frac{\partial(x,y)}{\partial(u,v)} dudv = \iint_E F \frac{\partial(x,y)}{\partial(s,t)} dsdt$$

3.3 Surface Integrals (continued)

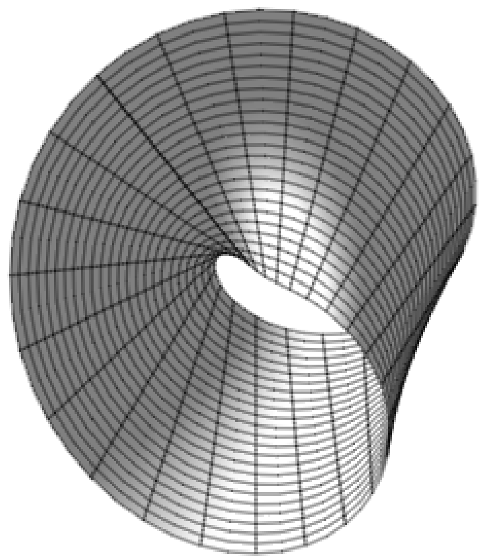
Complicated surfaces may be divided up into nonoverlapping patches which can be parameterized separately. The simplest scheme for doing this is to *triangulate* the surface, which means that we divide it up into triangular patches as depicted below. Each triangle on the surface can be parameterized by a triangle on the plane.



We will insist that if any two triangles touch, they either meet only at a vertex, or they share an entire edge. In addition, any edge lies on at most two triangles. So the picture below is not a surface from our point of view. We define the *boundary* of a surface to be the union of all edges which are not shared. The surface is called *closed* if the boundary is empty.



Given a surface which has been divided up into patches, we can integrate a 2-form on it by summing up the integrals over each patch. However, we require that the orientations match up, which is possible if the surface has “two sides”. Below is a picture of a one sided, or nonorientable, surface called the Mobius strip.



Once we have picked an orientation of S, we get one for the boundary using the right hand rule.

In many situations arising in physics, one needs to integrate a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ over a surface. The resulting quantity is often called a *flux*, which is usually written as $\iint_S \mathbf{F} \cdot d\mathbf{S}$ or $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$. As a typical example, consider a fluid such as air or water. Associated to this, there is a scalar field $\rho(x, y, z)$ which measures the density, and a vector field $\mathbf{v}$ which measures the velocity of the flow (e.g. the wind velocity). Then the rate at which the fluid passes through a surface S is given by the flux integral $\int\int_S \rho\mathbf{v} \cdot d\mathbf{S}$.

So how does one actually define or compute the flux? For our purposes, we can simply define it as

DEFINITION 3.3.1

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$$

It is probably easier to view this as a two step process, first convert $\mathbf{F}$ to a 2-form as follows:

$$F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} \leftrightarrow F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy,$$

then integrate. We will say more about the traditional way of defining the flux in the next section.

3.4 Length and Area

It is important to realize some line and surface integrals are not expressible as integrals of differential forms in general. Two notable examples are the arclength and area integrals.

DEFINITION 3.4.1 *The arclength of $C : [a, b] \to \mathbb{R}^3$ is given by*

$$\int_C ds = \int_a^b \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2 + \frac{dz}{dt}^2}\, dt$$

The symbol ds is **not** a 1-form in spite of the notation. For example

$$\int_{-C} ds = \int_C ds$$

whereas for 1-forms the integral would change sign. Nevertheless, ds (or more accurately its square) is a sort of generalization of a differential form called a *tensor*. To get a sense what this means, let us calculate the arclength of a curve lying on a surface. Suppose that S is a parameterized surface given by

$$\begin{cases} x = f(u, v) \\ y = g(u, v) \\ z = h(u, v) \\ (u, v) \in D \end{cases}$$

and suppose that C lies on S. This means that there are functions $k, \ell : [a, b] \to \mathbb{R}$ such that $x = f(k(t), \ell(t)), \ldots$ determines C. We can calculate the arclength of C by applying the chain to the above integral all at once. Instead, we want to break this down into a series of steps.

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$$

$$dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv$$

$$dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv$$

For the next step, we introduce a new product (indicated by juxtaposition) which is distributive and unlike the wedge product is commutative. We square the previous formulas and add them up. (The objects that we are getting are tensors.)

$$dx^2 = \left(\frac{\partial x}{\partial u}\right)^2 du^2 + 2\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} dudv + \left(\frac{\partial x}{\partial v}\right)^2 dv^2$$

$$dy^2 = \left(\frac{\partial y}{\partial u}\right)^2 du^2 + 2\frac{\partial y}{\partial u}\frac{\partial y}{\partial v} dudv + \left(\frac{\partial y}{\partial v}\right)^2 dv^2$$

$$dz^2 = \left(\frac{\partial z}{\partial u}\right)^2 du^2 + 2\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} dudv + \left(\frac{\partial z}{\partial v}\right)^2 dv^2$$

$$dx^2 + dy^2 + dz^2 = Edu^2 + 2Fdudv + Gdv^2 \tag{10}$$

where

$$E = ||\mathbf{T}_u||^2,\ F = \mathbf{T}_u \cdot \mathbf{T}_v,\ G = ||\mathbf{T}_v||^2$$

in the notation of section 3.1. The expression in (10) is called the metric tensor of the surface. We can easily deduce a formula for arclength in terms of it:

$$\int_C ds = \int_a^b \sqrt{E\frac{du}{dt}^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\frac{dv}{dt}^2}\,dt$$

The area of S can also be expressed in terms of the metric tensor. First recall that

DEFINITION 3.4.2 *The area of S is given by*

$$\iint_S dS = \iint_D ||\mathbf{T}_u \times \mathbf{T}_v||dudv$$

THEOREM 3.4.3 *The area is given by*

$$\iint_S dS = \iint_D \sqrt{EG - F^2}dudv$$

The proof is as follows

$$||\mathbf{T}_u \times \mathbf{T}_v||^2 = ||\mathbf{T}_u||^2||\mathbf{T}_v||^2 \sin^2\theta$$

$$= ||\mathbf{T}_u||^2||\mathbf{T}_v||^2(1 - \cos^2\theta)$$

$$= ||\mathbf{T}_u||^2||\mathbf{T}_v||^2 - (\mathbf{T}_u \cdot \mathbf{T}_v)^2$$

$$= EG - F^2$$

If S is sphere of radius 1 parameterized by spherical coordinates, a straight forward calculation gives the metric tensor as

$$\sin^2\phi\, d\theta^2 + d\phi^2$$

which yields

$$area(S) = \int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\phi d\theta = 4\pi$$

as expected.

More generally, if $f(x, y, z)$ is a scalar valued function, we can define

DEFINITION 3.4.4

$$\iint_S f(x,y,z)dS = \iint_D f(x,y,z)||\mathbf{T}_u \times \mathbf{T}_v||dudv$$

We can now explain the usual method of introducing the flux. Given a vector field $\mathbf{F}$, the dot product $\mathbf{F}\cdot\mathbf{n}$ with the unit normal field gives a scalar field. This can now be integrated over S as above to obtain $\iint_S \mathbf{F}\cdot\mathbf{n}dS$. This coincides with flux.

THEOREM 3.4.5 $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}dS$.

If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, then can compute

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left[F_1 \frac{\partial(x,y)}{\partial(u,v)} + F_2 \frac{\partial(y,z)}{\partial(u,v)} + F_3 \frac{\partial(z,x)}{\partial(u,v)} \right] dudv$$

$$= \iint_D \mathbf{F} \cdot \left(\frac{\partial(x,y)}{\partial(u,v)}\mathbf{i} + \frac{\partial(y,z)}{\partial(u,v)}\mathbf{j} + \frac{\partial(z,x)}{\partial(u,v)}\mathbf{k} \right) dudv$$

$$= \iint_D \mathbf{F} \cdot (\mathbf{T}_u \times \mathbf{T}_v) dudv = \iint_D \mathbf{F} \cdot \left(\frac{\mathbf{T}_u \times \mathbf{T}_v}{||\mathbf{T}_u \times \mathbf{T}_v||} \right) ||\mathbf{T}_u \times \mathbf{T}_v|| dudv$$

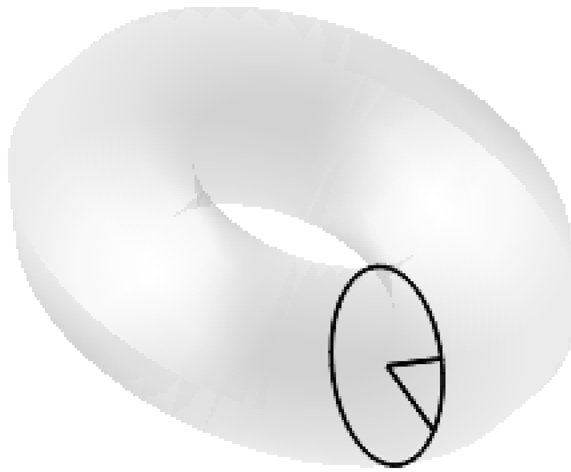
$$= \iint_S \mathbf{F} \cdot \mathbf{n} dS$$

In practice, however, this is usually not a very efficient method for computing the flux.

3.5 Exercise Set 3

Let $\alpha = xdx + ydy + zdz$, $\beta = zdx + xdy + ydz$ and $\gamma = xydz$ in the following problems.

1. Let D be the square $0 \le x \le 1$, $0 \le y \le 1$, $z = 1$ oriented with the upward normal. Calculate
   (a) $\iint_D \alpha \wedge \beta$
   (b) $\iint_D \alpha \wedge \gamma$
   (c) $\iint_D \beta \wedge \gamma$
2. Let S be the surface of the unit sphere $x^2 + y^2 + z^2 = 1$ oriented with the outward normal. Calculate
   (a) $\iint_S z dx \wedge dy$
   (b) $\iint_S (\mathbf{i} + \mathbf{j}) \cdot \mathbf{n} dS$
   (c) $\iint_S d\gamma$
3. Let S be the hyperboloid $z^2 - x^2 - y^2 = 1$, $-1 \le z \le 1$ oriented with outward normal. Calculate
   $$\iint_S \frac{xdx \wedge dz}{\sqrt{x^2 + y^2}} + \frac{ydy \wedge dz}{\sqrt{x^2 + y^2}}$$
   (Try using cylindrical coordinates.)
4. Calculate the area of the intersection of the sphere $x^2 + y^2 + z^2 = 4$ and $z \ge 1$.
5. Fix numbers $a > b > 0$. Draw a circle of radius b in the xz-plane centered at $(a, 0, 0)$, and rotate it about the z-axis to get a *torus* or doughnut shape T.



Calculate the surface area of T. It may help to parameterize it by the angle $0 \leq \theta \leq 2\pi$ which has the same meaning as in cylindrical coordinates, and the angle $0 \leq \psi \leq 2\pi$ indicated in the picture. A little trigonometry gives

$$x = (a + b\cos\psi)\cos\theta$$

$$y = (a + b\cos\psi)\sin\theta$$

$$z = b\sin\psi$$

6. With T as in the last problem, compute $\iint_T z dx \wedge dy$.

4 Stokes' Theorem

4.1 Green and Stokes

Stokes' theorem is really the fundamental theorem of calculus for surface integrals. We assume that the surfaces can be triangulated.

THEOREM 4.1.1 (Stokes' theorem) *Let S be an oriented smooth surface with smooth boundary curve C. If C is oriented using the right hand rule, then for any C^1 1-form ω on an open set of $\mathbb{R}^3$ containing S,*

$$\iint_S d\omega = \int_C \omega$$

If the surface lies in the plane, it is possible make this very explicit:

THEOREM 4.1.2 (Green's theorem) *Let C be a closed C^1 curve in $\mathbb{R}^2$ oriented counterclockwise and D be the interior of C. If $P(x,y)$ and $Q(x,y)$ are both C^1 functions then*

$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

As an application, Green's theorem shows that the area of D can be computed as a line integral on the boundary

$$\iint_D dxdy = \int_C y dx$$

If S is a closed oriented surface in $\mathbb{R}^3$, such as the surface of a sphere, Stoke's theorem shows that any exact 2-form integrates to 0, where a 2-form is exact if it equals $d\omega$ for some 1-form ω. To see this write S as the union of two surfaces S_1 and S_2 with common boundary curve C. Orient C using the right hand rule with respect to S_1, then orientation coming from S_2 goes in the opposite direction. Therefore

$$\iint_S d\omega = \iint_{S_1} d\omega + \iint_{S_2} d\omega = \iint_C \omega - \iint_C \omega = 0$$

In vector notation, Stokes' theorem is written as

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, dS = \int_C \mathbf{F} \cdot d\mathbf{s}$$

where $\mathbf{F}$ is a C^1-vector field.

In physics, there a two fundamental vector fields, the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$. They're governed by Maxwell's equations, one of which is

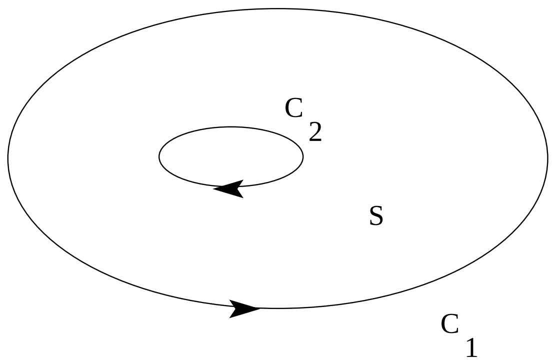
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

where t is time. If we integrate both sides over S, apply Stokes' theorem and simplify, we obtain Faraday's law of induction:

$$\int_C \mathbf{E} \cdot d\mathbf{s} = -\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot \mathbf{n}\, dS$$

To get a sense of what this says, imagine that C is wire loop and that we are dragging a magnet through it. This action will induce an electric current; the left hand integral is precisely the induced voltage and the right side is related to the strength of the magnet and the rate at which it is being dragged through.

Stokes' theorem works even if the boundary has several components. However, the inner an outer components would have opposite directions.



THEOREM 4.1.3 (Stokes' theorem II) *Let S be an oriented smooth surface with smooth boundary curves $C_1, C_2 \ldots$. If C_i is oriented using the right hand rule, then for any C^1 1-form ω on an open set of $\mathbb{R}^3$ containing S,*

$$\iint_S d\omega = \int_{C_1} \omega + \int_{C_2} \omega + \ldots$$

4.2 Proof of Stokes' theorem

Suppose that

$$\omega = f dx + g dy + h dz$$

is a 1-form in $\mathbb{R}^3$, and S is a surface parameterized by u, v. Let us denote by

$$\pi^*\omega = (f\frac{\partial x}{\partial u} + g\frac{\partial y}{\partial u} + h\frac{\partial z}{\partial u})du + (f\frac{\partial x}{\partial v} + g\frac{\partial y}{\partial v} + h\frac{\partial z}{\partial v})dv$$

the conversion of ω to uv-coordinates. Up to now, we have not used any special notation for this process, but it is easy to get very confused at this point if we don't.

THEOREM 4.2.1

$$d(\pi^*\omega) = \pi^* d\omega$$

ie. converting ω to uv-coordinates and differentiating is the same as differentiating and then converting to uv-coordinates.

Both sides of the putative equation decompose over sums. So it is enough to treat the case where ω consists of a single term, say fdx. Even then, it's a bit messy, but we only have to do this once. We have

$$d\omega = -f_y dx \wedge dy - f_z dx \wedge dz$$

So

$$\pi^* d\omega = -f_y(x_u du + x_v dv) \wedge (y_u du + y_v dv) - f_z(x_u du + x_v dv) \wedge (z_u du + z_v dv)$$

$$= (f_y(x_v y_u - x_u y_v) + f_z(x_v z_u - x_u z_v)) du \wedge dv$$

On the other hand,

$$\begin{aligned} d(\pi^*\omega) &= d(f x_u du + f x_v dv) \\ &= (-f_v x_u - \cancel{f x_{uv}} + f_u x_v + \cancel{f x_{vu}}) du \wedge dv \\ &= (-(f_x x_v + f_y y_v + f_z z_v) x_u + (f_x x_u + f_y y_u + f_z z_u) x_v) du \wedge dv \\ &= (\cancel{-f_x x_v x_u} - f_y y_v x_u - f_z z_v x_u + \cancel{f_x x_u x_v} + f_y y_u x_v + f_z z_u x_v) du \wedge dv \\ &= (f_y(x_v y_u - x_u y_v) + f_z(x_v z_u - x_u z_v)) du \wedge dv \end{aligned}$$

which is the same.

We are now ready to prove Stokes' theorem. We first assume that S consists of a single triangle.

PROPOSITION 4.2.2 *Stokes' theorem holds when S has a single triangular patch.*

We parameterize S by a triangle D in the uv-plane. After a translation, we can assume that one of the vertices is the origin. Then after a linear change of variables

$$u' = au + bv$$

$$v' = cu + dv$$

we can assume that D is the triangle with vertices $(0,0), (1,0), (0,1)$.

Let us start with a 1-form

$$\omega = f dx + g dy + h dz$$

and write

$$\pi^*\omega = P(u,v)du + Q(u,v)dv$$

Let C denote the boundary of D, which consists of line segments $u = 0, 0 \leq v \leq 1$; $v = 0, 0 \leq u \leq 1$ and $u + v = 1, 0 \leq u, v \leq 1$. Therefore

$$\int_C \omega = \int_C P du + Q dv$$

$$= \int_0^1 P(u,0)du - \int_0^1 P(u,1-u)du - \int_0^1 Q(0,v)dv + \int_0^1 Q(1-v,v)dv$$

On the other hand,

$$\iint_S d\omega = \iint_D (Q_u - P_v)dudv = \int_0^1 \int_0^{1-v} Q_u dudv - \int_0^1 \int_0^{1-u} P_v dvdu$$
$$= \int_0^1 Q(1-v,v) - Q(0,v)dv - \int_0^1 P(u,1-u) - P(u,0)du$$

So these are equal.

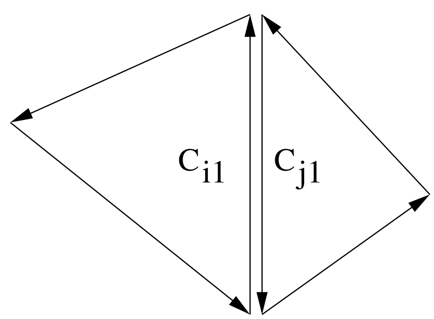
We are now ready to prove the full version of Stokes' theorem. We triangulate the surface S. Thus S is covered by triangular patches S_i. We have

$$\iint_S d\omega = \sum_i \iint_{S_i} d\omega$$

By the last proposition, we can write this as

$$\sum_i \int_{C_{i1}} \omega + \int_{C_{i2}} \omega + \int_{C_{i3}} \omega$$

where C_{i1}, C_{i2}, C_{i3} are the edges of S_i. We separate these into exterior curves, which are contained in the boundary C, and interior curves. A curve C_{i1} is an interior curve if it also occurs as an edge, say C_{j1} of an adjacent triangle S_j. If we take into account the orientations, then we see that $C_{j1} = -C_{i1}$.



Therefore the sum of integrals along the interior curves cancel. The conclusion is that

$$\iint_S d\omega = \sum_{\text{exterior } C_{ik}} \int_{C_{ik}} \omega = \int_C \omega$$

4.3 Cauchy's theorem*

Recall that a *complex number* is an expression $z = a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$, so that $i^2 = -1$. The components a and b are called the real and imaginary parts of z. We can identify the set of complex numbers $\mathbb{C}$ with the plane $\mathbb{R}^2 = \{(a,b) \mid a, b \in \mathbb{R}\}$. Addition and subtraction of complex numbers correspond to the usual vector operations:

$$(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i$$

However, we can do more, such as multiplication and division:

$$(a+bi)(c+di) = (ac-bd)+(ad+bc)i$$

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$$

The next step is calculus. The power of complex numbers is evident in the beautiful formula of Euler

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

which unifies the basic functions of calculus. Given a function $f : \mathbb{C} \to \mathbb{C}$, we can write it as

$$f(z) = f(x+yi) = u(x,y) + iv(x,y)$$

where x, y are real and imaginary parts of $z \in \mathbb{C}$, and u, v are the real and imaginary parts of f. f is continuous at $z = a + bi$ if u and v are continuous at (a, b) in the usual sense. So far there are no surprises. However, things get more interesting when we define the complex derivative

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

Notice that h is a complex number. For the limit to exist, we should get the same value no matter how it approaches 0. If $h = \Delta x$ approaches along the x-axis, we get

$$\begin{aligned} f'(z) &= \lim_{\Delta y\to 0} \frac{[u(x+\Delta x,y) - u(x,y)] + i[v(x+\Delta x,y) - v(x,y)]}{\Delta x} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \end{aligned}$$

If $h = \Delta y\, i$ approaches along the y-axis, then

$$\begin{aligned} f'(z) &= \lim_{\Delta y\to 0} \frac{[u(x,y+\Delta y) - u(x,y)] + i[v(x,y+\Delta y) - v(x,y)]}{i\Delta y} \\ &= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \end{aligned}$$

Setting these equal leads to the *Cauchy-Riemann* equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \; \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

These equations have to hold when the complex derivative $f'(z)$ exists, and in fact $f'(z)$ exists when they do. f is called *analytic* at $z = a + bi$ when these hold at that point. For example, $z^2 = x^2 + y^2 + 2xyi$ and

$$e^z = e^x\cos(y) + ie^x\sin(x)$$

are analytic everywhere. But $f(z) = \bar{z} = x - iy$ is not analytic anywhere.

A complex differential form is an expression $\alpha + i\beta$ where α, β are differential forms in the usual sense. Complex 1-forms can be integrated by the rule

$$\int_C \alpha + i\beta = \int_C \alpha + i \int_C \beta$$

Suppose that f is analytic. Then expanding

$$f(z)dz = (u + iv)(dx + idy) = [udx - vdy] + i[vdx + udy]$$

Differentiating and applying the Cauchy-Riemann equations shows

$$d(f(z)dz) = -\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) dx \wedge dy + i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx \wedge dy = 0$$

Therefore Stokes' theorem implies what may be thought of as the fundamental theorem of complex analysis:

THEOREM 4.3.1 (Cauchy's theorem) *If $f(z)$ is analytic on a region with boundary C then*

$$\int_C f(z)dz = 0$$

Suppose we replace $f(z)$ by $g(z) = \frac{f(z)}{z}$. This is analytic away from 0. Therefore the theorem applies to the boundary of any region not containing 0. If C is a closed positively oriented curve whose interior U contains 0, then applying Cauchy's theorem to a $U - D_r$,where D_r is a disk of small radius r in U, shows that

$$\int_C g(z)dz = \int_{C_r} g(z)dz \tag{11}$$

Here C_r is a circle of radius r around 0. We can parameterize this with the help of Euler's formula by

$$z = r\cos(\theta) + ri\sin(\theta) = re^{i\theta},\ 0 \le \theta \le 2\pi$$

Then $dz = rie^{i\theta}d\theta$, so that

$$\int_{C_r} g(z)dz = ri\int_0^{2\pi} \frac{f(re^{i\theta})}{re^{i\theta}} e^{i\theta} d\theta = i\int_0^{2\pi} f(re^{i\theta})d\theta$$

As $r \to 0$, $f(re^{i\theta}) \to f(0)$, therefore the above integral approaches $2\pi r$. Since (11) holds for all small r, it follows that this equality holds on the nose. Therefore:

THEOREM 4.3.2 (Cauchy's Integral Formula) *If $f(z)$ is analytic in the interior of positively oriented closed curve C, then*

$$f(0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z} dz$$

Using a change of variable $z \to z - a$, we get a more general formula:

COROLLARY 4.3.3 (Cauchy's Integral Formula II) *With the same assumptions, for any point a in the interior of C*

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - a} dz$$

This formula has many uses. Among other things, it can be used to evaluate complicated definite integrals. For example, even if we didn't know that $\int \frac{dx}{x^2+1} = \tan^{-1} x$, we could still integrate this from $-\infty$ to ∞ as follows. Let C be the union of the line segment $[-R, R]$ with the semicircle C_2 of radius R in the upper half of the complex plane. Here we choose R large, and in particular $R > 1$. We can factor

$$\frac{1}{z^2 + 1} = \left(\frac{1}{z + i}\right)\left(\frac{1}{z - i}\right)$$

Set $f(z)$ to the first factor on the right, and apply Cauchy's formula with $a = i$ to obtain

$$\int_C \frac{1}{z^2 + 1} = 2\pi i f(i) = \frac{2\pi i}{i + i} = \pi$$

The integral can be rewritten as

$$\int_{-R}^{R} \frac{dx}{x^2 + 1} + \int_{C_2} \frac{1}{z^2 + 1}$$

Now let $R \to \infty$, the first integral goes to what we want to evaluate, and the second goes to 0 because its absolute value is bounded by a constant times $1/R$. Therefore

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} = \pi$$

4.4 Exercise Set 4

1. Given a region $D \subset \mathbb{R}^2$ with a smooth boundary C.
   (a) Show that the area of D is $\frac{1}{2} \int_C x dy - y dx$.
   (b) Show that the area is $\frac{1}{2} \int_C r^2 d\theta$ in polar coordinates.
2. Use the formula of 1(b) to compute the area contain in $r = \sin\theta$, $0 \leq \theta \leq \pi$.
3. Calculate $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} dS$, where S is the paraboloid $z = x^2 + y^2$, $z \leq 1$, and $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + x\mathbf{k}$.
4. Let S be the part of the sphere of radius 2 centered at the origin below the plane $z = 1$. We orient this using the outward normal. Calculate $\iint_S z dy \wedge dz$. (You can either do this directly, or try to use Stokes' theorem which might be easier.)

5. T be the half torus $0 \leq \theta \leq \pi$, $0 \leq \psi \leq 2\pi$ (see problem 5 of exercise set 3 for the notation). Calculate $\int\int_T z dy \wedge dz$ (same hint as above).

6. Calculate $\int_{-\infty}^{\infty} \frac{dx}{x^2+x+1}$ as above.

5 Gauss' theorem

5.1 Triple integrals

Recall that a 3-form is an expression $\omega = f(x,y,z)dx \wedge dy \wedge dz$. Given a solid region $V \subset \mathbb{R}^3$, we define

$$\iiint_V f(x,y,z)dx \wedge dy \wedge dz = \iiint_V f(x,y,z)dxdydz$$

Orientations are also implicit here. To be consistent with earlier rules, we have to take

$$\iiint_V f(x,y,z)dy \wedge dx \wedge dz = -\iiint_V f(x,y,z)dxdydz$$

for example. The rationale may be clearer if we distinguish left and right handed coordinate systems. The system (x,y,z) is right handed because the thumb of the right hand points in the direction of the third coordinate axis, when the fingers are curled from first to second. If we reorder, say to (y,x,z) then we get a left handed system. For more general coordinate systems, such as spherical, it may be harder to visualize what right handed means. So let us give a more precise mathematical definition. Suppose that u,v,w is a new ordered system of coordinates, we will say that it is right handed if

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} > 0$$

holds everywhere; left handed means that this strictly negative. Now given ω, we can convert it to an expression $g(u,v,w)du \wedge dv \wedge dw$. Let W be the region corresponding to V in uvw-space. Then

$$\iiint_V \omega = \pm \iiint_W g(u,v,w)dudvdw$$

where we choose a plus sign if these are right handed coordinates, and a minus sign if these are left handed.

5.2 Gauss' theorem

THEOREM 5.2.1 (Gauss' theorem) *Let V be the interior of a smooth closed surface S oriented with the outward pointing normal. If ω is a C^1 2-form on an open subset of $\mathbb{R}^3$ containing V, then*

$$\iiint_V d\omega = \iint_S \omega$$

This is also called the divergence theorem. The reason for this name becomes clear if we express this in standard vector notation, where it reads

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

where $\mathbf{F}$ is a C^1 vector field.

As an application, consider a fluid with density ρ and velocity $\mathbf{v}$. If S is the boundary of a solid region V with outward pointing normal $\mathbf{n}$, then the flux $\iint_S \rho\mathbf{v}\cdot\mathbf{n}\,dS$ is the rate at which matter flows out of V. In other words, it is minus the rate at which matter flows in, and this equals $-\partial/\partial t\iiint_V \rho dV$. On the other hand, by Gauss' theorem, the above flux integral equals $\iiint_S \nabla\cdot(\rho\mathbf{v})\,dV$. Therefore

$$\iiint_V \nabla\cdot(\rho\mathbf{v})dV = -\frac{\partial}{\partial t}\iiint_V \rho dV$$

which yields

$$\iiint_V \left[\nabla\cdot(\rho\mathbf{v}) + \frac{\partial\rho}{\partial t}\right] dV = 0.$$

The only way this can hold for all possible regions V is that the integrand

$$\nabla\cdot(\rho\mathbf{v}) + \frac{\partial\rho}{\partial t} = 0 \tag{12}$$

This is one of the basic laws of fluid mechanics.

We can extend Gauss' theorem to solids with disconnected boundary. Suppose that S_2 is a smooth closed oriented surface contained inside another such surface S_1. We use the outward pointing normal S_1 and the inner pointing normal on S_2. Let V be region in between S_1 and S_2. Then,

THEOREM 5.2.2 (Gauss' theorem II) *If ω is a C^1 2-form on an open subset of $\mathbb{R}^3$ containing V, then*

$$\iiint_V d\omega = \iint_{S_1}\omega + \iint_{S_2}\omega$$

5.3 Proof for the cube

We will just give the proof for the easiest case, where we can choose right handed coordinates u, v, w, so that V corresponds to the the unit cube $W = \{(u, v, w) \mid 0 \le u, v, w \le 1\}$. The boundary consist of 6 pieces, the top

$$T = \{(1, v, w) \mid 0 \le v, w \le 1\}$$

the bottom

$$B = \{(0, v, w) \mid 0 \le v, w \le 1\}$$

and 4 sides that we won't label.

The proof is similar to the proof of Stokes' theorem. Let denote by $\pi^*\omega$ the conversion of ω to uvw-coordinates.

THEOREM 5.3.1 $\pi^* d\omega = d\pi^*\omega$

We skip the proof which would involve a messy computation. Now let us write

$$\pi^*\omega = Pdv \wedge dw + Qdw \wedge du + Rdu \wedge dv$$

Then

$$\iint_S \omega = \iint_T Pdv \wedge dw - \iint_B Pdv \wedge dw + \dots$$
$$= \int_0^1 \int_0^1 P(1, u, v)dudv - \int_0^1 \int_0^1 P(0, u, v)dudv + \dots \tag{13}$$

By the previous theorem

$$\pi^* d\omega = (P_u + Q_v + R_w)du \wedge dv \wedge dw$$

The integral

$$\iiint_V d\omega = \iiint_W (P_u + Q_v + R_w)dudvdw$$

By the fundamental theorem of calculus,

$$\iiint_W P_u dudvdw = \int_0^1 \int_0^1 P(1, u, v)dudv - \int_0^1 \int_0^1 P(0, u, v)dudv$$

and similar expression for the integrals of Q_u and R_w. Adding these up gives us (13).

5.4 Gravitational Flux

Place a "point particle" of mass m at the origin of $\mathbb{R}^3$, then this generates a force on any particle of unit mass at $\mathbf{r} = (x, y, z)$ given by

$$\mathbf{F} = \frac{-m\mathbf{r}}{r^3}$$

where $r = ||\mathbf{r}|| = \sqrt{x^2 + y^2 + z^2}$. This has singularity at 0, so it is a vector field on $\mathbb{R}^3 - \{0\}$. The corresponding 2-form is given by

$$\omega = -m\,\frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{r^3} \tag{14}$$

Let B_R be the ball of radius R around 0, and let S_R be its boundary. Since the outward unit normal to S_R is just $\mathbf{n} = \mathbf{r}/r$. One should expect that the flux

$$\iint_{S_R} \mathbf{F} \cdot \mathbf{n} dS = -\frac{m}{R^2} \iint_{S_R} dS$$
$$= -\frac{m}{R^2}\, area(S_R) = -\frac{m}{R^2}(4\pi R^2)$$

This can be justified by theorem 3.4.5. This can also be checked directly. We work in spherical coordinates. S_R is given by

$$\begin{cases} x = \rho\sin(\phi)\cos(\theta) \\ y = \rho\sin(\phi)\sin(\theta) \\ z = \rho\cos(\phi) \\ \rho = R \\ 0 \le \phi \le \pi,\ 0 \le \theta < 2\pi \end{cases}$$

Rewriting ω in these coordinates and simplifying:

$$\omega = -m\sin(\phi)d\phi \wedge d\theta$$

Therefore,

$$\iint_{S_R} \mathbf{F}\cdot\mathbf{n}dS = \iint_{S_R}\omega = -\int_0^{2\pi}\int_0^{\pi}\sin(\phi)d\phi d\theta = 4\pi m$$

as hoped. We claim that if S is any closed surface not containing 0

$$\iint_S \omega = \begin{cases} -4\pi m & \text{if 0 lies in the interior of } S \\ 0 & \text{otherwise} \end{cases}$$

We can see that $d\omega = 0$ (exercise 5). Therefore, Gauss' theorem yields

$$\iint_S \omega = \iiint_V d\omega = 0$$

if the interior V does not contain 0. On the other hand, if V contains 0, let B_R be a small ball contained in V, and let $V - B_R$ denote part of V lying outside of B_R. We use the second form of Gauss' theorem

$$\iint_S \omega - \iint_{S_R}\omega = \iiint_{V-B_R} d\omega = 0$$

We are subtracting the second surface integral, since we are supposed to use the inner normal for S_R. Thus

$$\iint_S \omega = -4\pi m$$

From here, we can easily extract an expression for the flux for several particles or even a continuous distribution of matter. If $\mathbf{F}$ is the force of gravity associated to some mass distribution, for any closed surface S oriented by the outer normal, then the flux

$$\iint_S \mathbf{F}\cdot\mathbf{n}dS$$

is -4π times the mass inside S. For a continuous distribution with density μ, this is given by $\iiint \mu dV$. Applying Gauss' theorem again, in this case, yields

$$\iiint_V (\nabla\cdot\mathbf{F} + 4\pi\mu)dV = 0$$

for all regions V. Therefore

$$\nabla\cdot\mathbf{F} = -4\pi\mu \tag{15}$$

5.5 Laplace's equation*

The *Laplacian* is a partial differential operator defined by

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

This can expressed using previous operators as $\Delta f = \nabla \cdot (\nabla f)$. As an example of where this arises, suppose that $\mathbf{F}$ is a gravitational force, this is known to be conservative so that $\mathbf{F} = -\nabla P$. Substituting into (15) yields the Poisson equation

$$\Delta P = 4\pi\mu$$

In a vacuum, this reduces to Laplace's equation

$$\Delta P = 0$$

A solution to Laplace's equation is called a *harmonic* function. These are of fundamental importance both in pure and applied mathematics. If we write $r = \sqrt{x^2 + y^2 + z^2}$, then

$$P = -\frac{m}{r} + Const.$$

is the potential energy associated to particle of mass m at $\mathbf{0}$. This is harmonic away from the singularity $\mathbf{0}$.

We express Δ in terms of forms as

$$\Delta f dx \wedge dy \wedge dz = d * df$$

or simply

$$\Delta f = *d * df$$

once we define $*(gdx \wedge dy \wedge dz) = g$. This last formula also works in the *plane* provided we define

$$*(fdx + gdy) = fdy - gdx$$

$$*(fdx \wedge dy) = f$$

(The $*$-operator in n dimensions always takes p-forms to $(n-p)$-forms.)

As an exercise, let us work out the Laplace equation in polar coordinates, and use this to determine the radially symmetric harmonic functions on the plane. The key is the determination of the $*$-operator:

$$dr = \frac{\partial r}{\partial x}dx + \frac{\partial r}{\partial y}dy = \frac{x}{r}dx + \frac{y}{r}dy$$

Similarly

$$d\theta = -\frac{y}{r^2}dx + \frac{x}{r^2}dy$$

So that

$$*dr = \frac{x}{r}dy - \frac{y}{r}dx = rd\theta$$

$$
\begin{aligned}
*d\theta &= -\frac{y}{r^2}dy - \frac{x}{r^2}dx = -\frac{1}{r}dr \\
*(dr \wedge d\theta) &= *(\frac{1}{r}dx \wedge dy) = \frac{1}{r}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\Delta f &= *d*\left(\frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta\right) \\
&= *d\left(r\frac{\partial f}{\partial r}d\theta - \frac{1}{r}\frac{\partial f}{\partial \theta}dr\right) \\
&= \frac{1}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}
\end{aligned}
$$

If f is radially symmetric, then it depends only on r so we obtain

$$
\frac{1}{r}\frac{df}{dr} + \frac{d^2 f}{dr^2} = \frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) = 0
$$

This differential equation can be solved using standard techniques to get

$$
f(r) = C + D\log r
$$

for constants C, D. By a similar, but more involved, calculation we find that

$$
f(r) = C + \frac{D}{r}
$$

are the only radially symmetric harmonic functions in $\mathbb{R}^3$, where as above we write r instead of ρ for the distance from the origin. These are precisely the physical solutions written at the beginning of this section.

5.6 Exercise Set 5

1. Calculate $\iiint_E dx \wedge dy + 2xdy \wedge dz$ on the elliposoid $x^2 + y^2 + \frac{1}{4}z^2 = 1$.
2. Give *two* proofs that if S is a closed surface then $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} dS = 0$.
3. Show that if S is a closed surface then $\iint_S zdx \wedge dy$ is the volume of the region enclosed by S. Use this to calculate the area of the tetrahedron bounded by planes $x = 0, y = 0, z = 0$ and $x + y + z = h$, where $h > 0$.
4. Show that if S is a closed surface then $\iint_S zrdr \wedge d\theta$, using cylindrical coordinates, is the volume of the region enclosed by S. Calculate the volume of the solid $x^2 + y^2 \leq 1$, $0 \leq z \leq 4 - x^2 - y^2$.
5. Prove that the volume of a region enclosed by S is
   (a) $\frac{1}{3}\iint_S xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$
   (b) $\frac{1}{3}\iint_S \rho^3 \sin(\phi)d\phi \wedge d\theta$ in spherical coordinates.

6. Show that for any fixed value of $m \neq 0$, ω given in (14) is closed i.e. $d\omega = 0$ but not exact. Therefore theorem 2.5.1 fails for $\mathbb{R}^3 - \{0\}$.

7. Prove Green's identities

   (a) $\iint_S f * dg = \iiint_V (df \wedge *dg + f d * dg)$

   (b) $df \wedge *dg = (f_x g_x + f_y g_y + f_z g_z) dx \wedge dy \wedge dz = dg \wedge *df$

   (c) $\iint_S (f * dg - g * df) = \iiint_V (f d * dg - g d * df)$

   Conclude that $\iint_S (f * dg - g * df) = 0$ if both f and g are harmonic.

6 Beyond 3 dimensions*

6.1 Beyond 3D

It is possible to do calculus in $\mathbb{R}^n$ with $n > 3$. Here the language of differential forms comes into its own. While it would be impossible to talk about the curl of a vector field in, say, $\mathbb{R}^4$, the derivative of a 1-form or 2-form presents no problems; we simply apply the rules we've already learned. For example, if x, y, z, t are the coordinates of $\mathbb{R}^4$, then a 1-form is a linear combination of the 4 basic 1-forms

$$dx,\ dy,\ dz,\ dt$$

a forms is a linear combination of the 6 basic 2-forms

$$dx \wedge dy = -dy \wedge dx$$

$$dx \wedge dz = -dz \wedge dx$$

$$\ldots$$

$$dz \wedge dt = -dt \wedge dz$$

and a 3 form is a linear combination of

$$dx \wedge dy \wedge dz = -dy \wedge dx \wedge dz = -dx \wedge dz \wedge dy = dy \wedge dz \wedge dx = \ldots$$

$$\ldots$$

$$dy \wedge dz \wedge dt = -dz \wedge dy \wedge dt = \ldots$$

The higher dimensional analogue of a surface is a *k-manifold*. A parameterized k-manifold M in $\mathbb{R}^n$ is given by a collection of C^1 functions

$$\begin{cases} x_1 = f_1(u_1, \ldots u_k) \\ x_2 = f_2(u_1, \ldots u_k) \\ \ldots \\ x_n = f_n(u_1, \ldots u_k) \\ (u_1, \ldots u_k) \in D \subseteq \mathbb{R}^k \text{ open} \end{cases}$$

such that the map $D \to \mathbb{R}^n$ is one to one and the tangent vectors $(\frac{\partial x_1}{\partial u_1}, \ldots \frac{\partial x_n}{\partial u_1}), \ldots (\frac{\partial x_1}{\partial u_k}, \ldots \frac{\partial x_n}{\partial u_k})$ are linearly independent for all values of the coordinates $(u_1, \ldots u_k)$. Given a k-form α on $\mathbb{R}^n$, we can express it as a linear combination of k-fold wedges of $dx_1 \ldots dx_n$, and then rewrite it as $g(u_1, \ldots u_k) du_1 \wedge \ldots \wedge du_k$. The "surface" integral is defined as

$$\int_M \alpha = \int \ldots \int_D g(u_1 \ldots u_k) du_1 \ldots du_k \tag{16}$$

Notice that number of integrations is usually suppressed in the notation on the left, since it gets too cumbersome after a while. In practice, for computing integrals, it's convenient to relax the conditions a bit by allowing D to be nonopen and allowing some degenerate points where the map $D \to \mathbb{R}^n$ isn't one to one.

More generally, a k-manifold M is obtained by gluing several parameterized manifolds as we did for surfaces. To be more precise, a closed set $M \subset \mathbb{R}^n$ is a k-manifold, if each point of M lies in the image of a parameterized k-manifold called a chart. As with curves and surfaces, it is important to specify orientations. Things are a little trickier since we can no longer rely on our geometric intuition to tell us which way is "up" or "down". Instead we can think that an orientation is a rule for specifying whether a coordinate system on a chart is right or left handed. We'll spell this out in an example below. The integral $\int_M \alpha$ can be defined by essentially summing up (16) over various non-overlapping right handed charts (we can use left handed charts provided we use the opposite sign). A k-manifold with boundary M is a closed set which can be decomposed as a union of a $(k-1)$-manifold ∂M, called the boundary, and a k-manifold $M - \partial M$. We orient this by the rule that a coordinate system $u_2, \ldots u_k$ of ∂M is right handed if it can be completed to right handed coordinate system $u_1, u_2, \ldots u_k$ of M such that the tangent vector $(\frac{\partial x_1}{\partial u_1}, \ldots \frac{\partial x_n}{\partial u_1})$ "points out".

Then the ultimate form of Stokes' theorem is:

THEOREM 6.1.1 (Generalized Stokes' theorem) *If M is an oriented k manifold with boundary ∂M and if α is a $(k-1)$-form defined on (an open set containing) M, then*

$$\int_M d\alpha = \int_{\partial M} \alpha$$

In order to get a feeling for how this works, let's calculate the "volume" V of the 4-dimensional ball $B = \{(x,y,z,t) \mid x^2 + y^2 + z^2 + t^2 \leq R\}$ of radius R in $\mathbb{R}^4$ in two ways. This is a 4-manifold with boundary $S = \{(x,y,z,t) \mid x^2 + y^2 + z^2 + t^2 = R\}$.

V can be expressed as the integral

$$V = \iiiint_B dxdydzdt = \int_B dx \wedge dy \wedge dz \wedge dt$$

We use extended spherical coordinates $\sigma, \psi, \phi, \theta$, where σ measures the distance of (x,y,z,t) to the origin in $\mathbb{R}^4$, and ψ the angle to the t-axis. So that

$$t = \sigma \cos\psi$$

and

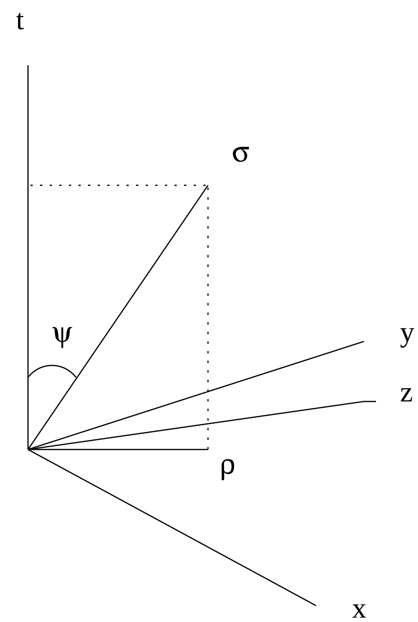
$$\rho = \sigma \sin\psi$$

is the distance from the projection (x,y,z) to the origin. Then letting ϕ, θ be the remaining spherical coordinates gives

$$x = \rho \sin\phi \cos\theta = \sigma \sin\psi \sin\phi \cos\theta$$

$$y = \rho \sin\phi \sin\theta = \sigma \sin\psi \sin\phi \sin\theta$$

$$z = \rho \cos\phi = \sigma \sin\psi \cos\phi$$



In these coordinate B is described as

$$\begin{cases} 0 \leq \psi \leq \pi \\ 0 \leq \phi \leq \pi \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \sigma \leq R \end{cases}$$

To simply computations, we note that form will get multiplied by the Jacobian when we change coordinates:

$$dx \wedge dy \wedge dz \wedge dt = \frac{\partial(x,y,z,t)}{\partial(\sigma,\psi,\theta,\phi)} d\sigma \wedge d\psi \wedge d\theta \wedge d\phi$$

$$= \sigma^3 \sin^2 \psi \sin \phi \, d\sigma \wedge d\psi \wedge d\theta \wedge d\phi$$

Note that the Jacobian is positive, and this what it means to say the coordinate system $\sigma, \psi, \theta, \phi$ is right handed or positively oriented. The volume is now easily computed

$$\int_0^R \sigma^3 d\sigma \int_0^\pi \sin^2 \psi d\psi \int_0^{2\pi} d\theta \int_0^\pi \sin \phi d\phi = \frac{1}{2}\pi^2 R^4$$

Alternatively, we can use Stokes' theorem, to see that

$$V = \int_B dx \wedge dy \wedge dz \wedge dt = -\int_S t dx \wedge dy \wedge dz$$

The parameter x, y, z gives a left hand coordinate system on the upper hemisphere $U = S \cap \{t > 0\}$. It is left handed because t, x, y, z is left handed on $\mathbb{R}^4$.

For similar reasons, x, y, z gives a right handed system on the lower hemisphere L where $t < 0$. Therefore

$$\begin{aligned} V &= -\int_U t dx \wedge dy \wedge dz - \int_L t dx \wedge dy \wedge dz \\ &= 2 \iiint_{x^2+y^2+z^2 \le R} \sqrt{R^2 - x^2 - y^2 - z^2} dx dy dz \\ &= 2 \int_0^R \rho^2 \sqrt{R^2 - \rho^2} d\rho \int_0^\pi \sin\phi \, d\phi \int_0^{2\pi} d\theta \\ &= 8\pi \int_0^{\pi/2} R^4 \sin^2 \alpha \cos^2 \alpha d\alpha \\ &= \frac{1}{2}\pi^2 R^4 \end{aligned}$$

6.2 Maxwell's equations in $\mathbb{R}^4$

As exotic as higher dimensional calculus sounds, there are many applications of these ideas outside of mathematics. For example, in relativity theory one needs to treat the electric $\mathbf{E} = E_1\mathbf{i} + E_2\mathbf{j} + E_3\mathbf{k}$ and magnetic fields $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$ as part of a single "field" on space-time. In mathematical terms, we can take space-time to be $\mathbb{R}^4$ - with the fourth coordinate as time t. The electromagnetic field can be represented by a 2-form

$$F = B_3 dx \wedge dy + B_1 dy \wedge dz + B_2 dz \wedge dx + E_1 dx \wedge dt + E_2 dy \wedge dt + E_3 dz \wedge dt$$

If we compute dF using the analogues of the rules we've learned:

$$dF = \left(\frac{\partial B_3}{\partial x} dx + \frac{\partial B_3}{\partial y} dy + \frac{\partial B_3}{\partial z} dz + \frac{\partial B_3}{\partial t} dt\right) \wedge dx \wedge dy + \dots$$

$$= \left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\right) dx \wedge dy \wedge dz + \left(\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y} + \frac{\partial B_3}{\partial t}\right) dx \wedge dy \wedge dt + \dots$$

Two of Maxwell's equations for electromagnetism

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

can be expressed very succintly in this language as $dF = 0$. The analogue of theorem 2.5.1 holds for $\mathbb{R}^n$, and shows that

$$F = d(A_1 dx + A_2 dy + A_3 dz + A_4 dt)$$

for some 1-form called the potential. Thus we've reduced the 6 quantites to just 4. In terms of vector analysis this amounts to the more complicated looking equations

$$\mathbf{B} = \nabla \times (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}), \quad \mathbf{E} = \nabla A_4 - \frac{\partial A_1}{\partial t}\mathbf{i} + \frac{\partial A_2}{\partial t}\mathbf{j} + \frac{\partial A_3}{\partial t}\mathbf{k}$$

There are two remaining Maxwell equations

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + 4\pi \mathbf{J}$$

where ρ is the electric charge density, and $\mathbf{J}$ is the electric current. The first law is really an analog of (15) for the electric field. After applying the divergence theorem, it implies that the electric flux through a closed surface equals (-4π) times the electric charge inside it. These last two Maxwell equations can also be replaced by the single equation $d{*F} = 4\pi J$ of 3-forms. Here

$$*F = E_3 dx \wedge dy + E_1 dy \wedge dz + E_2 dz \wedge dx - B_1 dx \wedge dt - B_2 dy \wedge dt - B_3 dz \wedge dt$$

and

$$J = \rho dx \wedge dy \wedge dz - J_3 dx \wedge dy \wedge dt - J_1 dy \wedge dz \wedge dt - J_2 dz \wedge dx \wedge dt$$

(We have been relying on explicit formulas to avoid technicalities about the definition of the $*$-operator. In principle however, it involves a metric, and in this case we use the so called Lorenz metric.)

Let's see how the calculus of differential forms can be used to extract a physically meaningful consequence of these laws. Proposition 2.4.1 (in extended form) implies that $dJ = \frac{1}{4\pi} d^2{*F} = 0$. Expanding this out yields

$$\frac{\partial \rho}{\partial t} dt \wedge dx \wedge dy \wedge dz - \frac{\partial J_3}{\partial z} dz \wedge dx \wedge dy \wedge dt + \ldots =$$

$$-\left(\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J}\right) dx \wedge dy \wedge dz \wedge dt = 0$$

Thus the expression in brackets is zero. This really an analog of the equation (12). To appreciate the meaning integrate $\frac{\partial \rho}{\partial t}$ over a solid region V with boundary S. Then this equals

$$-\iiint_V \nabla \cdot \mathbf{J} dV = -\iint_S \mathbf{J} \cdot \mathbf{n} dS$$

In other words, the rate of change of the electric charge in V equals minus the flux of the current accross the surface. This is the law of conservation of electric charge.

7 Further reading

For more information about differential forms, see the books [Fl, S, W]. Most the physics background – which is only used here for illustration, in any case – can be found in [Fe]. A standard reference for complex analysis is [A]. The material of the appendix can be found in any book on advanced calculus. For a rigorous treatment, see [R, S].

References

[A] L. Ahlfors, *Complex Analysis*

[Fe] R. Feynman et. al., *Lectures on Physics*

[Fl] H. Flanders, *Differential forms and applications to the physical sciences*

[R] W. Rudin, *Principles of mathematical analysis*

[S] M. Spivak, *Calculus on manifolds*

[W] S. Weintraub, *Differential forms: theory and practice*

A Review of multivariable calculus

A.1 Differential Calculus

To simplify the review, we'll stick to two variables, but the corresponding statements hold more generally. Let $f(x, y)$ be a real valued function defined on open subset of $\mathbb{R}^2$. Recall that the limit

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

means that $f(x, y)$ is approximately L whenever (x, y) is close to (a, b). The precise meaning is as follows. If we specified $\epsilon > 0$ (say $\epsilon = 0.0005$), then we could pick a tolerance $\delta > 0$ which would guarantee that $|f(x, y) - L| < \epsilon$ (i.e. $f(x, y)$ agrees with L up to the first 3 digits for $\epsilon = 0.0005$) whenever the distance between (x, y) and (a, b) is less than δ. A function $f(x, y)$ is *continuous* at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y)$$

exists and equals $f(a, b)$. It is continuous if it is so at each point of its domain.

We say that f is *differentiable*, if near any point $p = (x_0, y_0, f(x_0, y_0))$ the graph $z = f(x, y)$ can approximated by a plane passing through p. In other words, there exists quantities A, B such that we may write

$$f(x, y) = f(x_0, y_0) + A(x - x_0) + B(y - y_0) + remainder$$

with

$$\lim_{(x,y)\to(x_0,y_0)} \frac{|remainder|}{|(x, y) - (x_0, y_0)|} = 0$$

The last condition says that as the distance $|(x, y) - (x_0, y_0)|$ goes to zero, the remainder goes to zero at an even faster rate. We can see that the coefficients are nothing but the partial derivatives

$$A = \frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$B = \frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

There is a stronger condition which is generally easier to check. f is called continuously differentiable or C^1 if it and its partial derivatives exist and are continuous. Consider the following example

$$f(x, y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

This is continuous, however

$$\frac{\partial f}{\partial x} = \frac{3x^2}{x^2 + y^2} - \frac{2x^4}{(x^2 + y^2)^2}$$

has no limit as $(x,y)\to(0,0)$. To see this, note that along the x-axis $y=0$, we have $\frac{\partial f}{\partial x}=1$. So the limit would have to be 1 if it existed. On the other hand, along the y-axis $x=0$, $\frac{\partial f}{\partial x}=0$, which shows that there is no limit. So $f(x,y)$ is not C^1.

Partial derivatives can be used to determine maxima and minima.

THEOREM A.1.1 *If (a,b) is local maximum or minimum of a C^1 function $f(x,y)$, then (a,b) is a critical point, i.e.*

$$\frac{\partial f}{\partial x}(a,b)=\frac{\partial f}{\partial y}(a,b)=0$$

THEOREM A.1.2 (Chain Rule) *If $f,g,h:\mathbb{R}^2\to\mathbb{R}$ are C^1 functions, then $f(g(u,v),h(u,v))$ is also C^1 and if $z=f(x,y)$, $x=g(u,v)$, $y=h(u,v)$ then*

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

A function $f(x,y)$ is twice continously differentiable or C^2 if is C^1 and if its partial derivatives are also C^1. We have the following basic fact:

THEOREM A.1.3 *If $f(x,y)$ is C^2 then the mixed partials*

$$\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}\frac{\partial f}{\partial x}$$
$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\frac{\partial f}{\partial y}$$

are equal.

If f is C^2, then we have a Taylor approximation

$$\begin{aligned}f(x,y)\quad\approx\quad & f(a,b)+\left[\frac{\partial f}{\partial x}(a,b)\right](x-a)+\left[\frac{\partial f}{\partial y}(a,b)\right](y-b)+\frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2}(a,b)\right](x-a)^2\\ &+\left[\frac{\partial^2 f}{\partial y\partial x}(a,b)\right](x-a)(y-b)+\frac{1}{2}\left[\frac{\partial^2 f}{\partial y^2}(a,b)\right](y-b)^2\end{aligned}$$

More precisely, the remainder, which is the difference of left and right, should go to zero faster than $|(x,y)-(a,b)|^2$ goes to zero. Since it is relatively easy to determine when quadratic polynomials have maxima or minima, this leads to the second derivative test.

THEOREM A.1.4 *A critical point (a,b) of a C^2 function $f(x,y)$ is a local minimum (respectively maximum) precisely when the matrix*

$$\begin{pmatrix}\frac{\partial^2 f}{\partial x^2}(a,b) & \frac{\partial^2 f}{\partial y\partial x}(a,b)\\ \frac{\partial^2 f}{\partial y\partial x}(a,b) & \frac{\partial^2 f}{\partial y^2}(a,b)\end{pmatrix}$$

is positive (respectively negative) definite.

The above conditions are often formulated in a more elementary but ad hoc way in calculus books. Positive definiteness is equivalent to requiring

$$\frac{\partial^2 f}{\partial x^2}(a,b) > 0$$

$$\left[\frac{\partial^2 f}{\partial x^2}(a,b)\right]\left[\frac{\partial^2 f}{\partial y^2}(a,b)\right] - \left[\frac{\partial^2 f}{\partial y \partial x}(a,b)\right]^2 > 0$$

A.2 Integral Calculus

Integrals can be defined using Riemann's method. This has some limitations but it's the easiest to explain. Given a rectangle $R = [a,b] \times [c,d] \subset \mathbb{R}^2$, choose integers $m, n > 0$ and let $\Delta x = \frac{b-a}{m}$ $\Delta y = \frac{d-c}{n}$. Choose a set of sample points $P = \{(x_1, y_1), \ldots (x_m, y_n)\} \subset R$ with

$$(x_i, y_j) \in R_{ij} = [a + (i-1)\Delta x, a + i\Delta x] \times [b + (j-1)\Delta y, b + j\Delta y]$$

The Riemann sum

$$S(m,n,P) = \sum_{i,j} f(x_i, y_j)\Delta x \Delta y$$

Then the double integral is

$$\iint_R f(x,y)dxdy = \lim_{m,n\to\infty} S(m,n,P)$$

This definition is not really that precise because we need to choose P for each pair m, n. For the integral to exist, we really have to require that the limit exists for any choice of P, and that any two choices lead to the same answer.

The usual way to resolve the above issues is to make the two extreme choices. Define upper and lower sums

$$U(m,n) = \sum_{i,j} M_{ij}\Delta x \Delta y$$

$$L(m,n) = \sum_{i,j} m_{ij}\Delta x \Delta y$$

where

$$M_{ij} = \max\{f(x,y) \mid (x,y) \in R_{ij}\}$$
$$m_{ij} = \min\{f(x,y) \mid (x,y) \in R_{ij}\}$$

In the event that the maxima or minima don't exist, we should use the greatest lower bound and least upper bound instead. As $m, n \to \infty$ the numbers $L(m,n)$ tend to increase. So their limit can be understood as the least upper bound, i.e. the smallest number $L \geq L(m,n)$. Likewise we define the limit U as the largest number $U \leq U(m,n)$. If these limits coincide, the common value is taken to be

$$\iint_R f(x,y)dxdy = L = U$$

otherwise the (Riemann) integral is considered to not exist.

THEOREM A.2.1 *$\iint_R f(x,y)dxdy$ exists if f is continuous.*

The integral of

$$f(x,y) = \begin{cases} 1 & \text{if } (x,y) \text{ has rational coordinates} \\ 0 & \text{otherwise} \end{cases}$$

would be undefined from the present point of view, because $L = 0$ and $U = 1$. Although, in fact the integral can be defined using the more powerful Lebesgue theory [R]; in this example the Lebesgue integral is 0.

For more complicated regions $D \subset R$, set

$$\iint_D f(x,y)dxdy = \iint_R f(x,y)\chi_D(x,y)dxdy$$

where $\chi_D = 1$ inside D and 0 elsewhere. The key result is

THEOREM A.2.2 (Fubini) *If $D = \{(x,y) \mid a \le x \le b,\ g(x) \le y \le h(x)\}$ with f, g, h continuous. Then the double integral exists and*

$$\iint_D f(x,y)dxdy = \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)dy \right) dx$$

A similar statement holds with the roles of x and y interchanged.

This allows one to compute these integrals in practice.

The final question to answer is how double integrals behave under change of variables. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation given by C^1 functions

$$x = f(u,v),\ y = g(u,v)$$

We think of the first $\mathbb{R}^2$ as the uv-plane and the second as the xy-plane. Given a region D in the uv-plane, we can map it to the xy-plane by

$$T(D) = \{(f(u,v), g(u,v)) \mid (u,v) \in D\}$$

The Jacobian

$$\begin{aligned} \frac{\partial(x,y)}{\partial(u,v)} &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \\ &= \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}. \end{aligned}$$

THEOREM A.2.3 *If T is a one to one function, h is continuous and D a region of the type occurring in Fubini's theorem, then*

$$\iint_{T(D)} h(x,y)dxdy = \iint_D h(f(u,v), g(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv$$