Introduction to Fluid Mechanics

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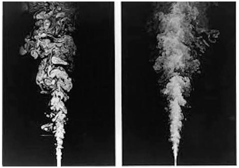
Lectures at IISER Pune

24-26 October, 2016



Acknowledgements

- I am grateful to Sudarshan Ananth for the kind invitation to deliver these lectures at IISER Pune and to all of you for your interest and participation.
- The illustrations in these slides are scanned from books mentioned or downloaded from the internet via Google images.
- Special thanks are due to PhD student Sonakshi Sachdev for her help in preparing these slides.

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Water, water, every where . . . (S T Coleridge, Rime of the Ancient Mariner)

- Whether we do physics, chemistry, biology, computation, mathematics, engineering or the humanities, we are likely to encounter fluids and be fascinated and challenged by their flows.
- Fluid flows are all around us: the air through our nostrils, tea stirred in a cup, eddies and turbulent flow in a river, plasmas in the ionosphere etc.
- From the standpoint of classical mechanics, a fluid is a continuum system with an essentially infinite number of degrees of freedom. A point particle has 3 translational degrees of freedom, a stone has three translational and 3 rotational degrees of freedom. On the other hand, to specify the state of a fluid, we must specify the velocity at each point!
- We believe that the basic physical laws governing fluid motion are those of mass conservation, Newton's laws and those of thermodynamics. The challenge lies in deducing the observed, often complex, patterns of flow from the known partial differential equations, boundary and initial conditions. This often requires a mix of physical insight, experimental data, mathematical techniques and computational methods.

Water water every where …

- Some of the best scientists have worked on fluid mechanics: I Newton, D Bernoulli, L Euler, J L Lagrange, Lord Kelvin, H Helmholtz, C L Navier, G G Stokes, N Y Zhukovsky, M W Kutta, O Reynolds, L Prandtl, von Karman, G I Taylor, J Leray, L F Richardson, A N Kolmogorov, L Onsager, R P Feynman, L D Landau, S Chandrasekhar, O Ladyzhenskaya, etc.
- Fluid dynamics finds application in numerous areas: flight of airplanes and birds, weather prediction, blood flow in the heart and blood vessels, waves on the beach, ocean currents and tsunamis, controlled nuclear fusion in a tokamak, jet engines in rockets, motion of charged particles in the solar corona and astrophysical jets, formation of clouds, melting of glaciers, global warming and sea level rise, traffic flow etc.
- Fluid flows can be regular (laminar) or chaotic (turbulent). Understanding turbulence is one of the great challenges of science.
- Fluid dynamics is among the 7 Clay Millenium Prize problems worth a million dollars. The other physics problem is from particle physics.

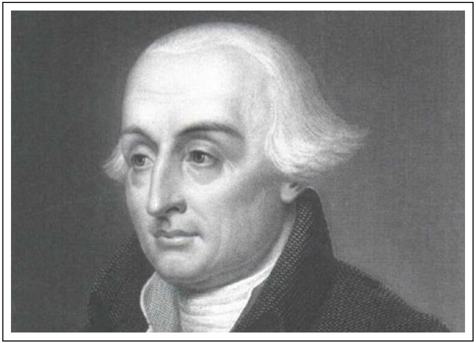
Continuum, Fluid element, Local thermal equilibrium

- In fluid mechanics we are not interested in microscopic positions and velocities of individual molecules. Focus instead on macroscopic fluid variables like velocity, pressure, density, energy and temperature that we can assign to a fluid element by averaging over it.
- By a fluid element, we mean a sufficiently large collection of molecules so that concepts such as 'volume occupied' make sense and yet small by macroscopic standards so that the velocity, density, pressure etc. are roughly constant over its extent. E.g.: divide a container with 10^{23} molecules into 10000 cells, each containing 10^{19} molecules.
- A flowing fluid is not in global thermal equilibrium. Collisions establish local thermodynamic equilibrium so that we can assign a local $T, p, \rho, E, \ldots$ to fluid elements, satisfying the laws of thermodynamics.
- Fluid description applies to phenomena on length-scale $\gg$ mean free path. On shorter length-scales, fluid description breaks down, but kinetic theory of molecules (Boltzmann transport equation) applies.

Eulerian and Lagrangian viewpoints

- In the Eulerian description, we are interested in the time development of fluid variables at a given point of observation $\vec{r} = (x, y, z)$. Interesting if we want to know how density changes, say, above my head. However, different fluid particles will arrive at the point $\vec{r}$ as time elapses.
- It is also of interest to know how the corresponding fluid variables evolve, not at a fixed location but for a fixed fluid element, as in a Lagrangian description.
- This is especially important since Newton's second law applies directly to fluid particles, not to point of observation!
- So we ask how a variable changes along the flow, so that the observer is always attached to the same fluid element.

Leonhard Euler and Joseph Louis Lagrange



Leonhard Euler (left) and Joseph Louis Lagrange (right).

Material derivative measures rate of change along flow

- Change in density of a fluid element in time dt as it moves from $\mathbf{r}$ to $\mathbf{r}+d\mathbf{r}$ is
$$d\rho = \rho(\mathbf{r}+d\mathbf{r}, t+dt) - \rho(\mathbf{r}, t) \approx \frac{\partial \rho}{\partial t} dt + d\mathbf{r}\cdot\nabla\rho. \tag{1}$$
- Divide by dt, let $dt \to 0$ and use $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ to get instantaneous rate of change of density of a fluid element located at $\mathbf{r}$ at time t:
$$\frac{D\rho}{Dt} \equiv \frac{\partial \rho}{\partial t} + \mathbf{v}\cdot\nabla\rho. \tag{2}$$
- $D\rho/Dt$ measures rate of change of density of a fluid element as it moves around. Material derivative of any quantity (scalar or vector) s in a flow field $\mathbf{v}$ is defined as $\frac{Ds}{Dt} = \partial_t s + \mathbf{v}\cdot\nabla s$.
- Material derivative of velocity $\frac{D\mathbf{v}}{Dt} = \partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}$ gives the instantaneous acceleration of a fluid element with velocity $\mathbf{v}$ located at $\mathbf{r}$ at time t.
- As a 1^{st} order differential operator it satisfies Leibnitz' product rule
$$\frac{D(fg)}{Dt} = f\frac{Dg}{Dt} + g\frac{Df}{Dt} \quad \text{and} \quad \frac{D(\rho\mathbf{v})}{Dt} = \rho\frac{D\mathbf{v}}{Dt} + \mathbf{v}\frac{D\rho}{Dt}. \tag{3}$$

Continuity equation and incompressibility

- Rate of increase of mass in a fixed vol V is equal to the influx of mass. Now, $\rho\mathbf{v}\cdot\hat{n}\,dS$ is the mass of fluid leaving a volume V through a surface element dS per unit time. Here $\hat{n}$ is the outward pointing normal. Thus,
$$\frac{d}{dt}\int_V \rho d\mathbf{r} = -\int_{\partial V} \rho\mathbf{v}\cdot\hat{n}\,dS = -\int_V \nabla\cdot(\rho\mathbf{v})\,d\mathbf{r} \quad \Rightarrow \int_V [\rho_t + \nabla\cdot(\rho\mathbf{v})]\,d\mathbf{r} = 0.$$
- As V is arbitrary, we get continuity equation for local mass conservation:
$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0 \qquad \text{or} \qquad \partial_t\rho + \mathbf{v}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{v} = 0. \tag{4}$$
- In terms of material derivative, $\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0$.
- Flow is incompressible if $\frac{D\rho}{Dt} = 0$: density of a fluid element is constant. Since mass of a fluid element is constant, incompressible flow preserves volume of fluid element.
- Alternatively imcompressible means $\nabla\cdot\mathbf{v} = 0$, i.e., $\mathbf{v}$ is divergence-free or solenoidal. $\nabla\cdot\mathbf{v} = \lim_{V,\delta t\to 0}\frac{1}{\delta t}\frac{\delta V}{V}$ measures fractional rate of change of volume of a small fluid element.
- Most important incompressible flow is constant ρ in space and time.

Sound speed, Mach number

- Incompressibility is a property of the flow and not just the fluid! For instance, air can support both compressible and incompressible flows.
- Flow may be approximated as incompressible in regions where flow speed is small compared to local sound speed $c_s = \sqrt{\frac{\partial p}{\partial \rho}} \sim \sqrt{\gamma p/\rho}$ for adiabatic flow of an ideal gas with $\gamma = c_p/c_v$.
- Compressibility $\beta = \frac{\partial \rho}{\partial p}$ measures increase in density with pressure. Incompressible fluid has $\beta = 0$, so $c^2 = 1/\beta = \infty$. An approximately incompressible flow is one with very large sound speed ($c_s \gg |\mathbf{v}|$).
- Common flows in water are incompressible. So study of incompressible flow is called hydrodynamics. High speed flows in air/gases tend to be compressible. Compressible flow is called aerodynamics/ gas dynamics.
- Incompressible hydrodynamics may be derived from compressible gas dynamic equations in the limit of small Mach number $M = |\mathbf{v}|/c_s \ll 1$.
- At high Mach numbers $M \gg 1$ we have super-sonic flow and phenomena like shocks.

Newton's 2^{nd} law for fluid element: Inviscid Euler equation

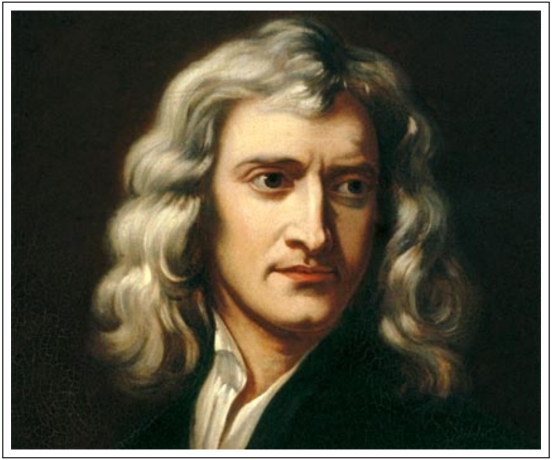
- Consider a fluid element of volume δV. Mass $\times$ acceleration is $\rho(\delta V)\frac{D\mathbf{v}}{Dt}$.
- Force on fluid element includes 'body force' like gravity derived from a potential ϕ. E.g. $\mathbf{F} = -\rho(\delta V)\nabla\phi$ where $-\nabla\phi$ is acceleration due to gravity.
- Also have surface force on a volume element, due to pressure exerted on it by neighbouring elements

$$F_{\text{surface}} = -\int_{\partial V} p\,\hat{n}\,dS = -\int_V \nabla p dV; \quad \text{if} \quad V = \delta V \quad \text{then} \quad F_{\text{surf}} \approx -\nabla p(\delta V).$$

- Newton's 2^{nd} law then gives the celebrated (inviscid) Euler equation

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\frac{\nabla p}{\rho} - \nabla\phi; \qquad \mathbf{v}\cdot\nabla\mathbf{v} \to \text{'advection term'} \tag{5}$$

- Continuity & Euler eqns. are first order in time derivatives: to solve initial value problem, must specify $\rho(\mathbf{r}, t=0)$ and $\mathbf{v}(\mathbf{r}, t=0)$.
- **Boundary conditions:** Euler equation is 1st order in space derivatives; impose BC on $\mathbf{v}$, not $\partial_i\mathbf{v}$. On solid boundaries normal component of velocity vanishes $\mathbf{v}\cdot\hat{n} = 0$. As $|\mathbf{r}| \to \infty$, typically $\mathbf{v} \to 0$ and $\rho \to \rho_0$.



Isaac Newton

Barotropic flow and specific enthalpy

- Euler & continuity are 4 eqns for 5 unknowns $\rho, \mathbf{v}, p$. Need another eqn.
- In local thermodynamic equilibrium, pressure may be expressed as a function of density and entropy. For isentropic flow it reduces to a barotropic relation $p = p(\rho)$. It eliminates p and closes the system of equations. E.g. $p \propto \rho^\gamma$ adiabatic flow of ideal gas; $p \propto \rho$ for isothermal.
- In barotropic flow, $\nabla p/\rho$ can be written as the gradient of an 'enthalpy'

$$h(\rho) = \int_{\rho_0}^{\rho} \frac{p'(\tilde{\rho})}{\tilde{\rho}} d\tilde{\rho} \quad \Rightarrow \quad \nabla h = h'(\rho)\nabla\rho = \frac{p'(\rho)}{\rho}\nabla\rho = \frac{\nabla p}{\rho}. \qquad (6)$$

  For example, $h = \frac{\gamma}{\gamma-1}\frac{p}{\rho}$ for adiabatic flow of an ideal gas.
- Reason for the name enthalpy: 1^{st} law of thermodynamics $dU = TdS - pdV$ becomes $dH = TdS + Vdp$ for enthalpy $H = U + pV$. For an isentropic process $dS = 0$, so $dH = Vdp$.
- Dividing by mass of fluid M we get $d(H/M) = (V/M)dp$. Defining enthalpy per unit mass $h = H/M$ and density $\rho = M/V$ gives $dh = dp/\rho$.

Barotropic flow and conserved energy

- In barotropic flow $p = p(\rho)$ and $\nabla p/\rho$ is gradient of enthalpy ∇h. So the Euler equation becomes

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla h. \tag{7}$$

- Using the vector identity $\mathbf{v}\cdot\nabla\mathbf{v} = \nabla(\frac{1}{2}\mathbf{v}^2) + (\nabla\times\mathbf{v})\times\mathbf{v}$, we get

$$\partial_t \mathbf{v} + (\nabla\times\mathbf{v})\times\mathbf{v} = -\nabla\left(h + \frac{1}{2}\mathbf{v}^2\right) \quad \text{where} \quad \nabla h = \frac{1}{\rho}\nabla p. \tag{8}$$

- Barotropic flow has a conserved energy: kinetic + compressional

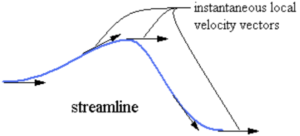
$$E = \int \left[\frac{1}{2}\rho\mathbf{v}^2 + U(\rho)\right] d^3r, \quad \text{where} \quad U'(\rho) = h(\rho). \tag{9}$$

  For adiabatic flow of ideal gas, $h = \frac{\gamma}{\gamma-1}\frac{p}{\rho}$ and $U = p/(\gamma-1)$. In the case of a monatomic ideal gas $\gamma = 5/3$ and compressional energy takes the familiar form $(3/2)pV = (3/2)NkT$.

- More generally, the Euler and continuity equations are supplemented by an equation of state and energy equation (1^{st} law of thermodynamics).

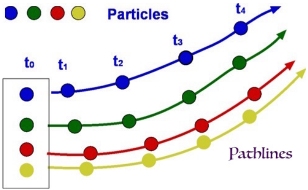
Flow visualization: Stream-, Streak- and Path-lines

- If $\mathbf{v}(\mathbf{r},t) = \mathbf{v}(\mathbf{r})$ is time-independent everywhere, the flow is steady.
- Stream, streak and pathlines coincide for steady flow. They are the integral curves (field lines) of $\mathbf{v}$, everywhere tangent to $\mathbf{v}$:



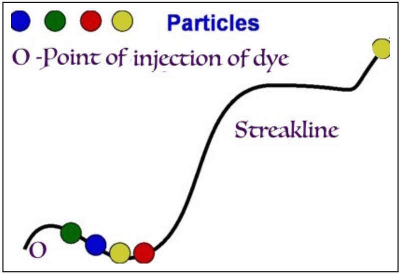
$$\frac{d\mathbf{r}}{ds} = \mathbf{v}(\mathbf{r}(s)) \text{ or } \frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z}; \quad \mathbf{r}(s_o) = \mathbf{r}_o.$$

- In unsteady flow, streamlines at time t_0 encode the instantaneous velocity pattern. Streamlines at a given time do not intersect.
- Path-lines are trajectories of individual fluid 'particles' (e.g. speck of dust stuck to fluid). At a point P on a path-line, it is tangent to $\mathbf{v}(P)$ at the time the particle passed through P. Pathlines can (self)intersect at $t_1 \neq t_2$.



Streak-lines

- Streak-line: Dye is continuously injected into a flow at a fixed point P. Dye particle sticks to the first fluid particle it encounters and flows with it. Resulting high-lighted curve is the streak-line through P. So at a given time of observation t_{obs}, a streak-line is the locus of all current locations of particles that passed through P at some time $t \leq t_{\text{obs}}$ in the past.

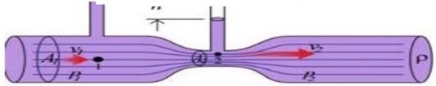


Steady Bernoulli principle

- Euler's equation for barotropic flow subject to a conservative body force potential Φ (e.g. $\Phi = gz$ for gravity at height z) is

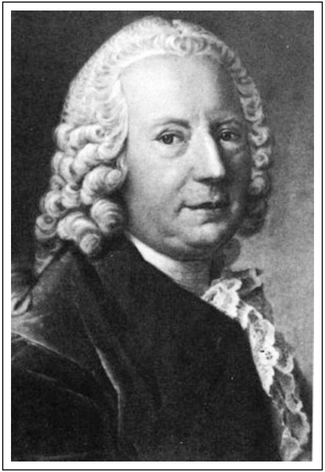
$$\frac{\partial \mathbf{v}}{\partial t} + (\nabla \times \mathbf{v}) \times \mathbf{v} = -\nabla \mathcal{B} \quad \text{where} \quad \mathcal{B} = \frac{1}{2}\mathbf{v}^2 + h + \Phi \tag{10}$$

- For steady flow $\partial_t \mathbf{v} = 0$. Dotting with $\mathbf{v}$ we find the Bernoulli specific energy $\mathcal{B}$ is constant along streamlines: $\mathbf{v} \cdot \nabla \mathcal{B} = 0$.
- For incompressible (constant density) flow, enthalpy $h = p/\rho$. Thus along a streamline $\frac{1}{2}\mathbf{v}^2 + p/\rho + gz$ is constant. For roughly horizontal flow, pressure is lower where velocity is higher.
- E.g. Pressure drops as flow speeds up at constrictions in a pipe. Try to separate two sheets of paper by blowing air between them!



- $A_2 < A_1$; $V_2 > V_1$
- According to Bernoulli's Law, pressure at A_2 is lower.

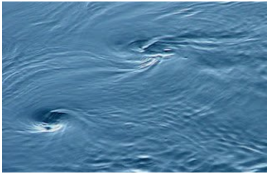
Daniel Bernoulli



Daniel Bernoulli

Vorticity and circulation

- *Vorticity* $\mathbf{w} = \nabla \times \mathbf{v}$ is a measure of local *rotation/angular momentum* in a flow. A flow without vorticity is called *irrotational*.
- Eddies and vortices are manifestations of vorticity in a flow. $[\mathbf{w}] = 1/T$, a frequecy.
- Given a closed contour C in a fluid, the circulation around the contour $\Gamma(C) = \oint_C \mathbf{v} \cdot d\mathbf{l}$ measures how much $\mathbf{v}$ 'goes round' C. By Stokes' theorem, it equals the flux of vorticity across a surface that spans C.

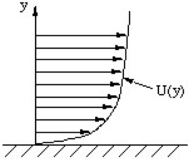


$$\Gamma(C) = \oint_C \mathbf{v} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{v}) \cdot dS = \int_S \mathbf{w} \cdot dS \quad \text{where} \quad \partial S = C.$$

- Enstrophy $\int \mathbf{w}^2 \, d\mathbf{r}$ measures global vorticity. It is conserved in ideal 2d flows, but not in 3d: it can grow due to 'vortex stretching' (see below).

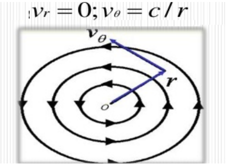
Examples of flow with vorticity $\mathbf{w} = \nabla \times \mathbf{v}$

- Shear flow with horizontal streamlines is an example of flow with vorticity: $\mathbf{v}(x,y,z) = (U(y),0,0)$. Vorticity $\mathbf{w} = \nabla \times \mathbf{v} = -U'(y)\hat{z}$.



- A bucket of fluid rigidly rotating at small angular velocity $\Omega\hat{z}$ has $\mathbf{v}(r,\theta,z) = \Omega\hat{z} \times \mathbf{r} = \Omega r\hat{\theta}$. The corresponding vorticity $\mathbf{w} = \nabla \times \mathbf{v} = \frac{1}{r}\partial_r(rv_\theta)\hat{z}$ is constant over the bucket, $\mathbf{w} = 2\Omega\hat{z}$.

- The planar azimuthal velocity profile $\mathbf{v}(r,\theta) = \frac{c}{r}\hat{\theta}$ has circular streamlines. It has no vorticity $\mathbf{w} = \frac{1}{r}\partial_r(r\frac{c}{r})\hat{z} = 0$ except at $r = 0$: $\mathbf{w} = 2\pi c\delta^2(\mathbf{r})\hat{z}$. The constant $2\pi c$ comes from requiring the flux of $\mathbf{w}$ to equal the circulation of $\mathbf{v}$ around any contour enclosing the origin



$$\oint \mathbf{v} \cdot d\mathbf{l} = \oint (c/r) r \, d\theta = 2\pi c.$$

Evolution of vorticity and Kelvin's theorem

- Taking the curl of the Euler equation $\partial_t \mathbf{v} + (\nabla \times \mathbf{v}) \times \mathbf{v} = -\nabla\left(h + \frac{1}{2}\mathbf{v}^2\right)$ allows us to eliminate the pressure term in barotropic flow to get

$$\partial_t \mathbf{w} + \nabla \times (\mathbf{w} \times \mathbf{v}) = 0. \tag{11}$$

- This may be interpreted as saying that vorticity is 'frozen' into $\mathbf{v}$.
- The flux of $\mathbf{w}$ through a surface moving with the flow is constant in time:

$$\frac{d}{dt}\int_{S_t} \mathbf{w} \cdot dS = 0 \quad \text{or by Stokes' theorem} \quad \frac{d}{dt}\oint_{C_t} \mathbf{v} \cdot d\mathbf{l} = \frac{d\Gamma}{dt} = 0. \tag{12}$$

- Here C_t is a closed material contour moving with the flow and S_t is a surface moving with the flow that spans C_t.
- The proof uses the Leibnitz rule for material derivatives $D_t \equiv \partial_t + \mathbf{v} \cdot \nabla$

$$\frac{d}{dt}\oint_{C_t} \mathbf{v} \cdot d\mathbf{l} = \oint_{C_t} D_t \mathbf{v} \cdot d\mathbf{l} + \oint_{C_t} \mathbf{v} \cdot D_t d\mathbf{l}. \tag{13}$$

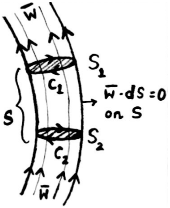
  Using the Euler equation $D_t \mathbf{v} = -\nabla h$ and $D_t d\mathbf{l} = d\mathbf{v}$ we get

$$\frac{d}{dt}\oint_{C_t} \mathbf{v} \cdot d\mathbf{l} = \oint_{C_t} d\left(\frac{1}{2}\mathbf{v}^2 - h\right) = 0. \tag{14}$$

Kelvin & Helmholtz theorems on vorticity

- $\frac{d}{dt}\oint_{C_t} \mathbf{v}\cdot d\mathbf{l} = 0$ is Kelvin's theorem: circulation around a *material* contour is constant in time. In particular, in the absence of viscosity, eddies and vortices cannot develop in an initially irrotational flow (i.e. $\mathbf{w} = 0$ at $t = 0$).

- Vortex tubes are cylindrical surfaces everywhere tangent to $\mathbf{w}$. So on a vortex tube, $\mathbf{w}\cdot dS = 0$.



- The circulation Γ around a vortex tube is independent of the choice of encircling contour. Consider part of a vortex tube S between two encircling contours C_1 and C_2 spanned by surfaces S_1 and S_2.
  - Applying Stokes' theorem to the closed surface $Q = S_1 \cup S \cup S_2$ we get

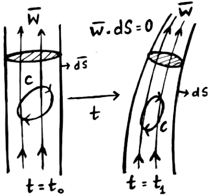
$$\int_Q \mathbf{w}\cdot dS = \int_{\partial Q} \mathbf{v}\cdot d\mathbf{l} = 0 \quad \text{as} \quad \partial Q \quad \text{is empty,}$$

$$\Rightarrow \int_{S_1} \mathbf{w}\cdot dS - \int_{S_2} \mathbf{w}\cdot dS = 0 \quad \text{or} \quad \Gamma(C_1) = \Gamma(C_2) \quad \text{since } \mathbf{w}\cdot dS = 0 \text{ on S.}$$

  - As a result, a vortex tube cannot abruptly end, it must close on itself to form a ring (e.g. a smoke ring) or end on a boundary.

Helmholtz's theorem: inviscid flow preserves vortex tubes

- Suppose we have a vortex tube at initial time t_0. Let the material on the tube be carried by flow till time t_1. We must show that the new tube is a vortex tube, i.e., that vorticity is everywhere tangent to it, or $\mathbf{w} \cdot d\mathbf{S} = 0$.



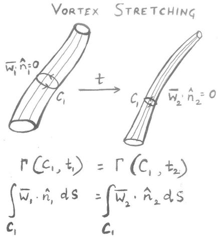
- Consider a contractible closed curve $C(t_0)$ lying on the initial vortex tube, the flow maps it to a contractible closed curve $C(t_1)$ lying on the new tube. By Kelvin's theorem, $\Gamma(C(t_0)) = 0 = \Gamma(C(t_1))$. Now suppose S is the surface on the new vortex tube enclosed by $C(t_1)$, $\partial S = C(t_1)$, then

$$0 = \Gamma(C(t_1)) = \int_S \mathbf{w} \cdot d\mathbf{S}.$$

- This is true for any contractible closed curve $C(t_1)$ on the new tube. Considering an infinitesimal closed curve, we conclude that $\mathbf{w} \cdot d\mathbf{S} = 0$ at every point of the new tube, i.e., it must be a vortex tube.

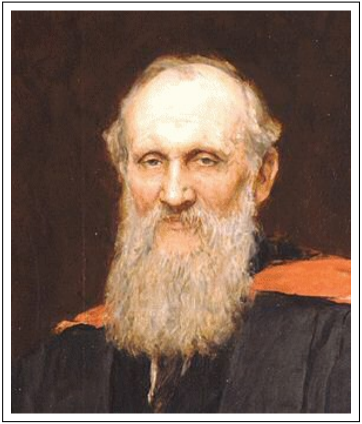
Vortex rings and vortex stretching

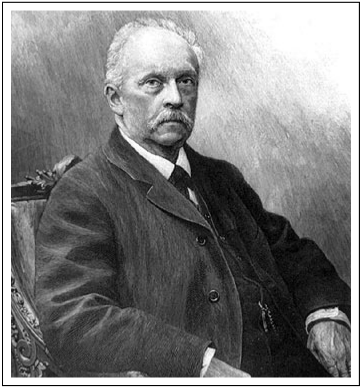
- Smoke rings are examples of vortex tubes. Dolphins blow vortex rings in water and chase them.



- Kelvin's theorem implies that the strength Γ of a vortex tube is independent of time.
- Fluid flow tends to stretch and bend vortex tubes.
- Since $\Gamma = \int \mathbf{w} \cdot dS$ is independent of time for a vortex tube, if the cross section of a vortex tube decreases, the vorticity must increase.
- This typically leads to growth of enstrophy $\int \mathbf{w}^2 d\mathbf{r}$.

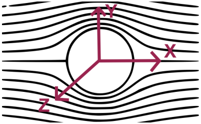
Lord Kelvin and Hermann von Helmholtz





Lord Kelvin (left) and Hermann von Helmholtz (right).

Irrotational incompressible inviscid flow around cylinder

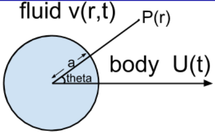


- When flow is irrotational ($\mathbf{w} = \nabla \times \mathbf{v} = 0$) we may write $\mathbf{v} = -\nabla\phi$. Velocity potential ϕ is like the electrostatic potential in $\mathbf{E} = -\nabla\phi$ which guarantees $\nabla \times \mathbf{E} = 0$.
- Incompressibility $\nabla \cdot \mathbf{v} = 0 \Rightarrow \phi$ satisfies Laplace's equation $\nabla^2\phi = 0$.
- We impose impenetrable boundary conditions: normal component of velocity vanishes on solid surfaces: $\frac{\partial\phi}{\partial\hat{n}} = 0$ on boundary (Neumann BC).
- For flow with asymptotic velocity $-U\hat{x}$ past a fixed ∞ cylinder of radius a, translation invariance along z-axis makes this a 2d problem in r,θ plane. The BCs are $\frac{\partial\phi}{\partial r} = 0$ at $r = a$ and $\phi \to Ur\cos\theta$ as $r \to \infty$ (so $\mathbf{v} \to -U\hat{x}$).
- Separating variables, gen. soln. to $\nabla^2\phi = (1/r)\partial_r(r\partial_r\phi) + (1/r^2)\partial_\theta^2\phi = 0$ is

$$\phi = (A_0 + B_0 \ln r) + \sum_{n=1}^{\infty}\left(A_n r^n + \frac{B_n}{r^n}\right)(C_n \cos n\theta + D_n \sin n\theta). \tag{15}$$

- Imposing BC at ∞ we get $A_0 = B_0 = A_n = C_n = D_n = 0$ except for $A_1 = U$ and $C_1 = 1$. The BC at $r = a$ gives $B_1 = Ua^2$. Thus $\phi = U\cos\theta\left(r + \frac{a^2}{r}\right)$. The corresponding velocity field is $\mathbf{v} = -\nabla\phi = -U\hat{x} + U\frac{a^2}{r^2}(\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$.

Potential flow and the added mass effect



- Velocity field for potential flow ($\mathbf{v} = -\nabla\phi$) past a cylinder is $\mathbf{v} = -U\hat{x} + U\frac{a^2}{r^2}(\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$.
- Now consider problem of a cylinder moving with velocity $U\hat{x}$ through a fluid asymptotically at rest.
- By a Galilean transformation, the velocity field around the cylinder is $\mathbf{v}' = \mathbf{v} + U\hat{x} = U\frac{a^2}{r'^2}(\cos\theta'\,\hat{r}' + \sin\theta'\,\hat{\theta}')$ where r', θ' are relative to the center of the cylinder.
- This example can be used to illustrate the added mass effect. The force required to accelerate a body (of mass M at $\dot{U}$) through potential flow exceeds $M\dot{U}$, since part of the force applied goes to accelerate the fluid.
- Indeed the flow KE $\frac{1}{2}\rho\iint_a^\infty (\mathbf{v}')^2 r'\,dr'\,d\theta' = \frac{1}{2}\rho\iint \frac{U^2a^4}{r'^3}dr'\,d\theta' = \frac{1}{2}\rho\pi a^2U^2$ $\equiv \frac{1}{2}M'U^2$ is quadratic in U just like the KE of cylinder itself. Thus the total KE of cylinder + fluid is $K_{\text{total}} = \frac{1}{2}(M+M')U^2$.
- The associated power to be supplied is $\dot{K}_{\text{total}} = F\cdot U$. So a force $F = (M+M')\dot{U}$ is required to accelerate the body at $\dot{U}$. Body behaves as if it has an effective mass $M+M'$. M' is its added or virtual mass. Ships must carry more fuel than expected after accounting for viscosity.

Sound waves in compressible flow

- Sound waves are excitations of the ρ or p fields. Arise in compressible flows, where regions of compression and rarefaction can form.
- Notice first that a fluid at rest ($\mathbf{v} = 0$) with *constant* pressure and density ($p = p_0$, $\rho = \rho_0$) is a static solution to the continuity and Euler equations

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \text{and} \quad \rho(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla p. \tag{16}$$

- Now suppose the stationary fluid suffers a small disturbance resulting in small variations $\delta \mathbf{v}, \delta p$ and $\delta \rho$ in velocity, pressure and density

$$\mathbf{v} = 0 + \mathbf{v}_1(\mathbf{r},t), \quad \rho = \rho_0 + \rho_1(\mathbf{r},t) \quad \text{and} \quad p = p_0 + p_1(\mathbf{r},t). \tag{17}$$

  What can the perturbations $\mathbf{v}_1(\mathbf{r},t), p_1(\mathbf{r},t)$ and $\rho_1(\mathbf{r},t)$ be? They must be such that $\mathbf{v}, p$ and ρ satisfy the continuity and Euler equations with $\mathbf{v}_1, p_1, \rho_1$ treated to linear order (as they are assumed small).
- It is found empirically that the small pressure and density variations are proportional i.e., $p_1 = c^2 \rho_1$. We will derive the simplest equation for sound waves by linearizing the continuity and Euler eqns around the static solution. It will be possible to interpret c as the speed of sound.

Sound waves in static fluid with constant p_0, ρ_0

- Ignoring products of small quantities $\mathbf{v}_1, p_1$ and ρ_1, the continuity equation $\partial_t(\rho_0+\rho_1)+\nabla\cdot((\rho_0+\rho_1)\mathbf{v}_1)=0$ becomes $\partial_t\rho_1+\rho_0\nabla\cdot\mathbf{v}_1=0$.
- Similarly, the Euler equation $(\rho_0+\rho_1)(\partial_t\mathbf{v}_1+\mathbf{v}_1\cdot\nabla\mathbf{v}_1)=-\nabla(p_0+p_1)$ becomes $\rho_0\partial_t\mathbf{v}_1=-\nabla p_1$ upon ignoring products of small quantities.
- Now we assume pressure variations are linear in density variations ($p_1=c^2\rho_1$) and take a divergence to get $\rho_0\partial_t(\nabla\cdot\mathbf{v}_1)=-c^2\nabla^2\rho_1$.
- Eliminating $\nabla\cdot\mathbf{v}_1$ using continuity eqn we get the wave equation for density variations $\partial_t^2\rho_1=c^2\nabla^2\rho_1$.
- Why is c called the sound speed? Notice that any function of $\xi=x-ct$ solves the 1D wave equation: $\partial_t^2\rho_1=c^2\partial_x^2\rho_1$ for $\rho_1(x,t)=f(x-ct)$

$$\partial_t\rho_1=-cf', \quad \partial_t^2\rho_1=c^2f'' \quad \text{while} \quad \partial_x\rho_1=f' \quad \text{and} \quad \partial_x^2\rho_1=f''. \qquad (18)$$

  $f(x-ct)$ is a *traveling wave* that retains its shape as it travels at speed c to the right. Plot $f(x-ct)$ vs x at $t=0$ and $t=1$ for $f(\xi)=e^{-\xi^2}$ and $c=1$.
- For incompressible flow ($\rho=\rho_0, \rho_1=0$) $c^2=\frac{p_1}{\rho_1}=\frac{\delta p}{\delta\rho}\to\infty$ as the density variation is vanishingly small even for large pressure variations.

Heat diffusion equation

- Empirically it is found that the heat flux between bodies grows with the temperature difference. Fourier's law of heat diffusion states that the heat flux density vector (energy crossing unit area per unit time) is proportional to the negative gradient in temperature

$$\mathbf{q} = -k\nabla T \quad \text{where} \quad k = \text{thermal conductivity.} \tag{19}$$

- Consider gas in a fixed volume V. The increase in internal energy $U = \int_V \rho c_v T d\mathbf{r}$ must be due to the influx of heat across its surface S.

$$\int_V \partial_t(\rho c_v T) d\mathbf{r} = -\int_S \mathbf{q}\cdot\hat{n}\, dS = \int_S k\nabla T\cdot\hat{n}\, dS = k\int_V \nabla\cdot\nabla T\, d\mathbf{r}. \tag{20}$$

  c_v = specific heat/mass (at constant volume, no work) and ρ = density.

- V is arbitrary, so integrands must be equal. Heat equation follows:

$$\frac{\partial T}{\partial t} = \alpha\nabla^2 T \quad \text{where} \quad \alpha = \frac{k}{\rho c_v} \quad \text{is thermal diffusivity.} \tag{21}$$

- Heat diffusion is dissipative, temperature differences even out and heat flow stops at equilibrium temperature. It is not time-reversal invariant.

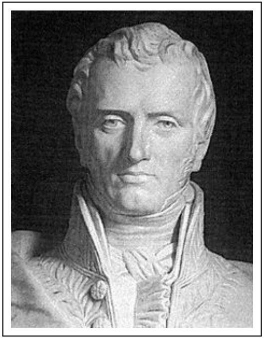
Including viscosity: Navier-Stokes equation

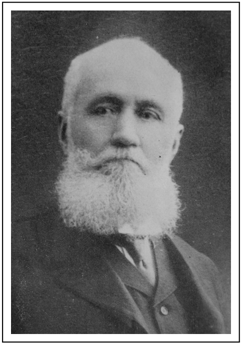
- Heat equation $\partial_t T = \alpha \nabla^2 T$ describes diffusion from hot → cold regions.
- (Shear) viscosity causes diffusion of velocity from a fast layer to a neighbouring slow layer of fluid. The viscous stress is $\propto$ velocity gradient. If a fluid is stirred and left, viscosity brings it to rest.
- By analogy with heat diffusion, velocity diffusion is described by $\nu \nabla^2 \mathbf{v}$.
- Kinematic viscosity ν has dimensions of diffusivity (areal velocity L^2/T).
- *Postulate* the Navier-Stokes equation for viscous *incompressible* flow:

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v} \qquad \text{(NS)}. \tag{22}$$

- NS has not been derived from molecular dynamics except for dilute gases. It is the simplest equation consistent with physical requirements and symmetries. It's validity is restricted by experiment.
- NS is second order in space derivatives unlike the inviscid Euler eqn. Experimentally relevant boundary condition is impenetrability $\mathbf{v} \cdot \hat{n} = 0$ and 'no-slip' $\mathbf{v}_{||} = 0$ on fixed solid surfaces.

Claude Louis Navier, Saint Venant and George Stokes





Claude Louis Navier (left), Saint Venant (middle) and George Gabriel Stokes (right).

Reynolds number $\mathcal{R}$ and similarity principle

- Incompressible flows with same $\mathcal{R}$ have similar (rescaled) flow patterns.
- Suppose U and L are a typical speed and length associated to a flow (e.g. asymptotic flow speed U past a sphere of size L). Define

$$\text{dim. less variables} \quad \mathbf{r}' = \frac{\mathbf{r}}{L}, \;\; \nabla' = L\nabla \;\; t' = \frac{U}{L}t, \;\; \mathbf{v}' = \frac{\mathbf{v}}{U}, \;\; \mathbf{w}' = \frac{\mathbf{w}L}{U}.$$

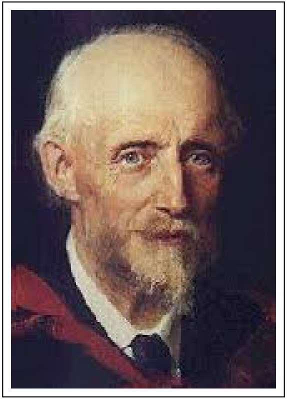
- Then incompressible NS vorticity eqn in non-dimensional variables is

$$\frac{\partial \mathbf{w}'}{\partial t'} + \nabla' \times (\mathbf{w}' \times \mathbf{v}') = \frac{\nu}{LU}\nabla'^2 \mathbf{w}'. \quad \text{We define } \frac{1}{\mathcal{R}} = \frac{\nu}{LU}. \tag{23}$$

- ν enters only through $\mathcal{R}$. If 2 flows expressed in scaled variables have same $\mathcal{R}$ and BCs, then flow patterns are similar. Flow around aircraft is simulated in wind tunnel using a scaled down aircraft with same $\mathcal{R}$.
- $\mathcal{R}$ is a measure of ratio of inertial to viscous forces

$$\frac{F_{\text{inertial}}}{F_{\text{viscous}}} = \frac{|\mathbf{v}\cdot\nabla\mathbf{v}|}{|\nu\nabla^2\mathbf{v}|} \sim \frac{U^2/L}{\nu U/L^2} \sim \frac{LU}{\nu} = \mathcal{R}. \tag{24}$$

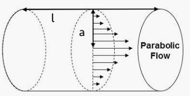
- When $\mathcal{R}$ is small (e.g. in slow creeping flow), viscous forces dominate inertial forces and vice versa.



Osbourne Reynolds

Laminar viscous flow through a pipe: Poiseuille flow

- Consider slow ($\mathcal{R} \ll 1$) steady logitudinal incompressible flow between two ends of a long cylindrical pipe of length l, radius a, generated by pressure drop Δp down the pipe.



  - Assume pressure drops linearly along axis of cylinder (chosen along $\hat{z}$) and that velocity $\mathbf{v} = v_z(r)\hat{z}$ depends only on the radial distance form axis. Continuity equation $\nabla \cdot \mathbf{v} = 0$ is identically satisfied. Ignoring time derivatives and non-linear advection, the z-component of the NS equation $\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p/\rho + \nu \nabla^2 \mathbf{v}$ in cylindrical coordinates becomes:

$$\frac{-\Delta p}{\rho l} = \frac{\nu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) \quad \text{where} \quad \nu = \text{kinematic viscosity.} \qquad (25)$$

  - This is a 2^{nd} order ODE requiring 2 BCs. We impose no-slip $[v_z = 0]$ on the boundary $r = a$ and smoothness on the axis $\left[\frac{dv_z}{dr} = 0 \text{ at } r = 0\right]$.
  - Integrating and imposing BCs we get a parabolic velocity profile:

$$\int_0^r d\left(r'\frac{dv_z}{dr'}\right) = \frac{-\Delta p}{\rho l \nu}\int_0^r r'\,dr' \quad \Rightarrow \quad \frac{dv_z}{dr} = \frac{-\Delta p}{\rho l \nu}\frac{r}{2} \quad \Rightarrow \quad v_z(r) = \frac{\Delta p}{4\rho\nu l}\left(a^2 - r^2\right).$$

  - Mass flowing through pipe/time: $Q = \int \rho \mathbf{v} \cdot d\mathbf{S} = \int_0^a \rho v_z \, 2\pi r dr = \frac{\pi \Delta p}{8\nu l} a^4$.

Stokes flow: drag on a sphere in steady creeping flow

- Stokes studied incompressible (constant ρ) flow around a sphere of radius a moving through a viscous fluid with velocity $\mathbf{U}$

$$\mathbf{v}'_t + \mathbf{v}' \cdot \nabla' \mathbf{v}' = -\frac{1}{\rho}\nabla' p + \frac{1}{\mathcal{R}}\nabla'^2 \mathbf{v}, \quad \frac{1}{\mathcal{R}} = \frac{\nu}{aU} \tag{26}$$

- For steady flow $\partial_t \mathbf{v}' = 0$. For creeping flow ($\mathcal{R} \ll 1$) we may ignore advection term and take a curl to eliminate pressure to get

$$\nabla'^2 \mathbf{w}' = 0. \tag{27}$$

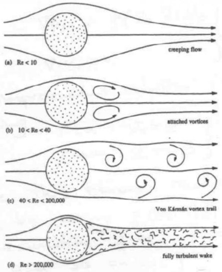
- By integrating the stress over the surface Stokes found the drag force

$$F_i = -\int \sigma_{ij} n_j dS \quad \Rightarrow \quad \mathbf{F}_{\text{drag}} = -6\pi\rho\nu a \mathbf{U}. \tag{28}$$

- Upto 6π factor, this follows from dimensional analysis! Magnitude of drag force is $F_D = \frac{12}{\mathcal{R}} \times \frac{1}{2}\pi a^2 U^2$. $12/\mathcal{R}$ is the drag coefficient for stokes flow.

Drag on a sphere at higher Reynolds number $\mathcal{R} = \frac{Ua}{\nu}$

- At higher speeds ($\mathcal{R} \gg 1$), naively expect viscous term to be negligible. However, experimental flow is far from ideal (inviscid) flow!
- At higher $\mathcal{R}$, flow becomes unsteady, vortices develop downstream and eventually a turbulent wake is generated.
- Dimensional analysis implies drag force on a sphere is expressible as $F_D = \frac{1}{2} C_D(\mathcal{R})\, \pi a^2 \rho U^2$, where $C_D = C_D(\mathcal{R})$ is the dimensionless drag coefficient, determined by NS equation.
- F can only depend on ρ, U, a, ν. To get mass dimension correctly, $F \propto \rho U^b \nu^c a^d$. Dimensional analysis $\Rightarrow b = d$ and $c = 2 - d$, so $F \propto \rho \left(\frac{Ua}{\nu}\right)^d \nu^2$. Thus, $F = C'_D(\mathcal{R})\, (\rho a^2 U^2)/\mathcal{R}^2 = \frac{1}{2} C_D(\mathcal{R})\, \pi a^2 \rho U^2$.
- Comparing with Stokes' formula for creeping flow $F = 6\pi a \rho \nu U$ we get $C_D \sim 12/\mathcal{R}$ as $\mathcal{R} \to 0$.
- Significant experimental deviations from Stokes' law: enhancement of drag at higher $1 \leq \mathcal{R} \leq 10^5$, then drag drops with increasing U!



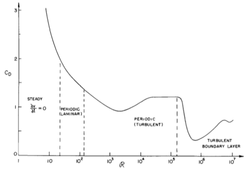
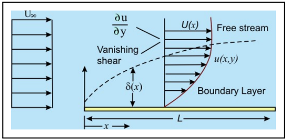


Fig. 41–4. The drag coefficient C_D of a circular cylinder as a function of the Reynolds number.

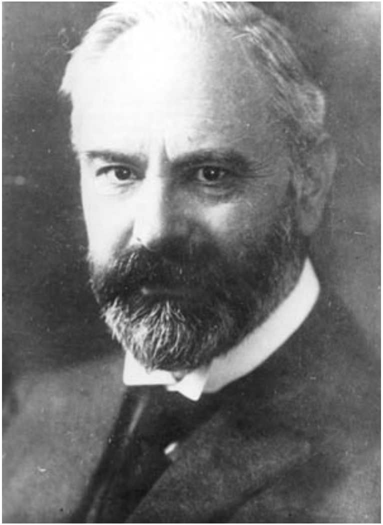
Drag crisis clarified by Prandtl's boundary layers

- In inviscid flow (Euler equation) tangential velocity on solid surfaces is unconstrained, can be large.
- For viscous NS flow, no slip BC implies tangential $\mathbf{v} = 0$ on solid surfaces.
- Even for low viscosity, there is a thin boundary layer where tangential velocity drops rapidly to zero. In the boundary layer, cannot ignore $\nu \nabla^2 \mathbf{v}$.



- Though upstream flow is irrotational, vortices are generated in the boundary layer due to viscosity. These vortices are carried downstream in a (turbulent) wake.
- Larger vortices break into smaller ones and so on, due to inertial forces. Small vortices (at the Taylor microscale) dissipate energy due to viscosity increasing the drag for moderate $\mathcal{R}$.

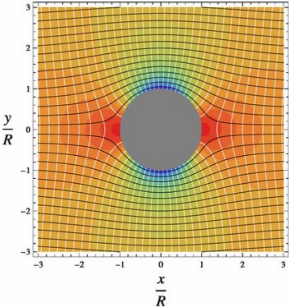
Ludwig Prandtl



Ludwig Prandtl

2D Incompressible flows: stream function

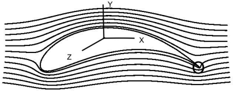
- In 2D incompressible flow the velocity components are expressible as derivatives of a stream function: $\mathbf{v} = (u, v) = (-\psi_y, \psi_x)$. Incompressibility condition $\nabla \cdot \mathbf{v} = -\psi_{yx} + \psi_{xy} = 0$ is identically satisfied.
- Streamlines defined by $\frac{dx}{ds} = u(x(s), y(s))$ and $\frac{dy}{ds} = v(x(s), y(s))$ or $\frac{dx}{u} = \frac{dy}{v} = ds$ are level curves of ψ. For, along a streamline $d\psi = (\partial_x \psi) dx + (\partial_y \psi) dy = v dx - u dy = 0$.



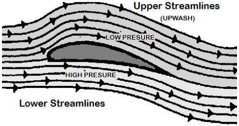
- If in addition, flow is irrotational ($\mathbf{w} = \nabla \times \mathbf{v} = 0$), then $\mathbf{v}$ admits a velocity potential $\mathbf{v} = -\nabla \phi$ so that $(u, v) = (-\phi_x, -\phi_y)$.
- So ϕ & ψ satisfy the Cauchy Riemann equations: $\phi_x = \psi_y$, $\phi_y = -\psi_x$ and the complex velocity potential $f = \phi + i\psi$ is analytic! ϕ and ψ are harmonic: $\nabla^2 \phi = 0 \Rightarrow$ incompressible and $\mathbf{w} = \nabla^2 \psi \hat{z} = 0 \Rightarrow$ irrotational.
- The level curves of ψ and ϕ are orthogonal : $\nabla \phi \cdot \nabla \psi = -(u, v) \cdot (v, -u) = 0$.
- Complex velocity $g = u - iv$ is the derivative $-f'(z)$ of complex potential.

Lift on an airfoil

- Consider an infinite airfoil of uniform cross section (axis along z). Airflow around it can be treated as 2-dimensional, i.e. on x, y plane.

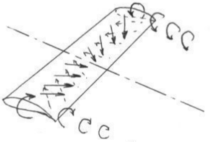


- Airfoil starts from rest moves left with zero initial circulation. Ignoring $\nu\nabla^2\mathbf{v}$, Kelvin's theorem precludes any circulation developing around wing. Streamlines of potential flow have a singularity as shown in Fig 1.



- Viscosity at rearmost point due to large $\nabla^2\mathbf{v}$ regularizes flow pattern as shown in Fig.2.

- In fact, circulation Γ develops around airfoil (Fig. 3). In frame of wing, we have an infinite airfoil with circulation Γ placed perpendicularly in a rightward velocity field $v_\infty \hat{x}$.



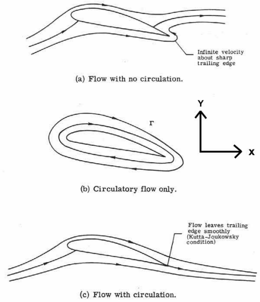
- Situation is analogous to infinite wire carrying current I placed perpendicularly in a $\mathbf{B}$ field!

Circulation around and airfoil

- Current $\mathbf{j}$ in $\mathbf{B}$ field feels Lorentz force/Vol. $\mathbf{j}\times\mathbf{B}$ where $\mathbf{j} = \nabla\times\mathbf{B}/\mu_0$ by Ampere's law. Analogue of Lorentz force is vorticity force in Euler equation

$$\rho\partial_t\mathbf{v} + \rho\mathbf{w}\times\mathbf{v} = -\rho\nabla\sigma + \rho\nu\nabla^2\mathbf{v}$$

- $\mathbf{B}\leftrightarrow\rho\mathbf{v}$, $\mathbf{j}\leftrightarrow\mathbf{w}$, $\mu_0\leftrightarrow\rho$, $I\leftrightarrow\Gamma$. Current carrying wire feels transverse force BI/length. Expect airfoil to feel force $\rho v_\infty\Gamma$/length upwards ($\hat{y}$).

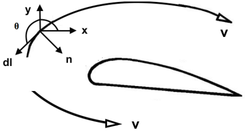


(a) Flow with no circulation.
(b) Circulatory flow only.
(c) Flow with circulation.

- Outside the boundary layer flow can be approximated as ideal irrotational flow which can be represented by a complex velocity $g = u - iv$. Since g is analytic outside the airfoil, we can expand it in a Laurent series, $g = v_\infty + \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \cdots$.
- Circulation around a closed streamline enclosing airfoil just outside boundary layer is $\Gamma = \oint \mathbf{v}\cdot d\mathbf{l} = \oint g\,dz = \oint (u\,dx + v\,dy) + i(u\,dy - v\,dx)$ since $(u\,dy - v\,dx) = 0$ along a streamline. Thus by Cauchy's residue theorem, $\Gamma = 2\pi i a_{-1}$.

Kutta-Joukowski lift formula for incompressible flow

- Force exerted by flow on airfoil is $\mathbf{F} = \oint p\,\hat{n}\,dl$ where p is the air pressure along the boundary and $\hat{n}$ is the inward normal. By Bernoulli's theorem, $\oint p\,\hat{n}\,dl = -\frac{1}{2}\rho \oint \mathbf{v}^2\,\hat{n}\,dl$.

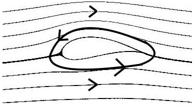


- If the line element $d\mathbf{l}$ along the streamline makes an angle θ with $\hat{x}$ then $(dx, dy) = (dl\cos\theta, dl\sin\theta)$ and the inward normal $\hat{n} = (-\sin\theta, \cos\theta)$. Thus, $F_x = \frac{1}{2}\rho \oint \mathbf{v}^2 \sin\theta\,dl = \frac{1}{2}\rho \oint \mathbf{v}^2\,dy$ and $F_y = -\frac{1}{2}\rho \oint \mathbf{v}^2 \cos\theta\,dl = -\frac{1}{2}\rho \oint \mathbf{v}^2\,dx$.
- The complex force $Z = F_y + iF_x = -\frac{\rho}{2}\oint \mathbf{v}^2(dx - idy)$ may be expressed in terms of the circulation Γ using the complex velocity g. As $udy - vdx = 0$,

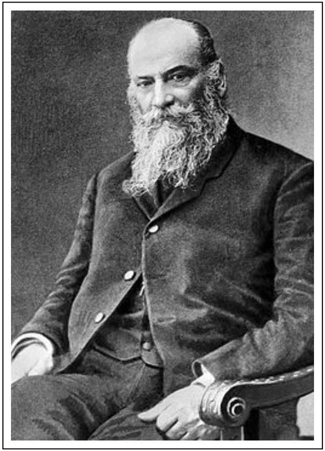
$$\begin{aligned} Z &= -\frac{\rho}{2}\oint\left[\mathbf{v}^2(dx - idy) + 2i(udy - vdx)(u - iv)\right] = \frac{\rho}{2}\oint(v^2 - u^2 - 2iuv)(dx + idy) \\ Z &= -\frac{\rho}{2}\oint g^2 dz = -\frac{\rho}{2}\oint\left[v_\infty^2 + (2v_\infty a_{-1})/z + \cdots\right]dz = -(\rho/2)[2\pi i(2v_\infty a_{-1})] = -\rho v_\infty \Gamma. \end{aligned}$$

by Cauchy's theorem. So $F_x = 0$ and $F_y = -\rho v_\infty \Gamma$.

- $F_y > 0$ and generates lift if the counter-clockwise circulation Γ is negative, which is the case if speed above airfoil is more than below.



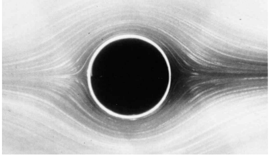
Nikolay Yegorovich Zhukovsky and Martin Wilhelm Kutta



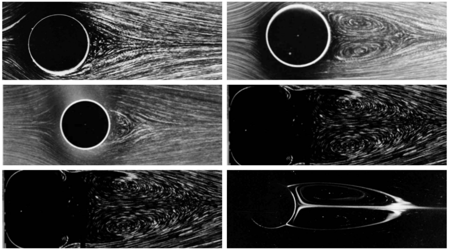
Nikolay Yegorovich Zhukovsky (left) and Martin Wilhelm Kutta (right).

Transition from laminar to turbulent flow past a cylinder

- Consider flow with asymptotic velocity $U\hat{x}$ past a fixed cylinder of diameter L and axis along $\hat{z}$. The components of velocity are (u,v,w).
- At very low $\mathcal{R} \approx .16$, the symmetries of the (steady) flow are (a) $y \to -y$ (reflection in $z-x$ plane), (b) time and z translation-invariance (c) left-right symmetry w.r.t. center of cylinder ($x \to -x$ and $(u,v,w) \to (u,-v,-w)$).



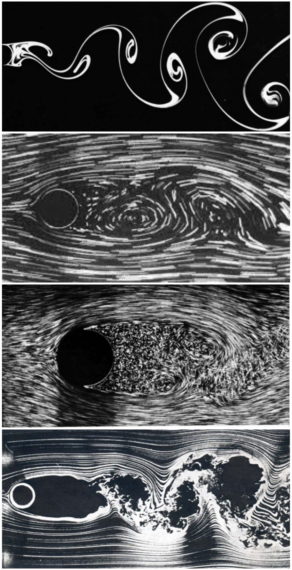
- All these are symmetries of Stokes flow (ignoring the non-linear advection term).
- At $\mathcal{R} \approx 1.5$ a marked left-right asymmetry develops.



- At $\mathcal{R} \approx 5$, change in topology of flow: flow separates and recirculating standing eddies (from diffusion of vorticity) form downstream of cylinder.
- At $\mathcal{R} \approx 40$, flow ceases to be steady, but is periodic in time.

Transition to turbulence in flow past a cylinder

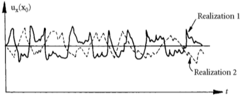
- At $\mathcal{R} \gtrsim 40$, recirculating eddies are periodically (alternatively) shed to form the celebrated von Karman vortex street.
- The z-translation invariance is spontaneously broken when $\mathcal{R} \sim 40 - 75$.
- At higher $\mathcal{R} \sim 200$, flow becomes chaotic with turbulent boundary layer.
- At $\mathcal{R} \sim 1800$, only about two vortices in the von Karman vortex street are distinct before merging into a quasi uniform turbulent wake.
- At much higher $\mathcal{R}$, many of the symmetries of NS are restored in a statistical sense and turbulence is called fully-developed.



von Karman

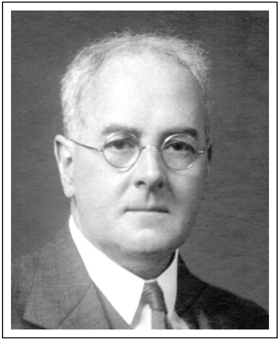
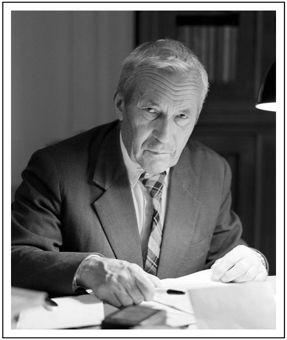
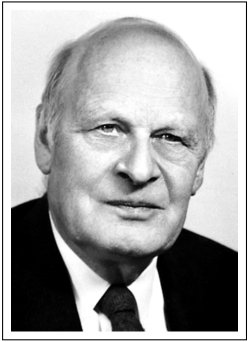
What is turbulence? Key features.

- Slow flow or very viscous fluid flow tends to be regular & smooth (laminar). If viscosity is low or speed sufficiently high ($\mathcal{R}$ large enough), irregular/chaotic motion sets in.
- Turbulence is chaos in a driven dissipative system with many degrees of freedom. Without a driving force (say stirring), the turbulence decays.
- $\mathbf{v}(\mathbf{r}_0, t)$ appears random in time and highly disordered in space.
- Turbulent flows exhibit a wide range of length scales: from the system size, size of obstacles, through large vortices down to the smallest ones at the Taylor microscale (where dissipation occurs).
- $\mathbf{v}(\mathbf{r}_0, t)$ are very different in distinct experiments with approximately the same ICs/BCs. But the time average $\bar{\mathbf{v}}(\mathbf{r}_0)$ is the same in all realizations.



- Unlike individual flow realizations, statistical properties of turbulent flow are reproducible and determined by ICs and BCs.
- As $\mathcal{R}$ is increased, symmetries (rotation/reflection/translation) are broken, but can be restored in a statistical sense in fully developed turbulence.

Lewis Richardson, Andrei Kolmogorov and Lars Onsager

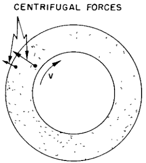
Lewis Richardson (L), Andrei Kolmogorov (C) and Lars Onsager (R).

Big whirls have little whirls that feed on their velocity,
and little whirls have lesser whirls and so on to viscosity.

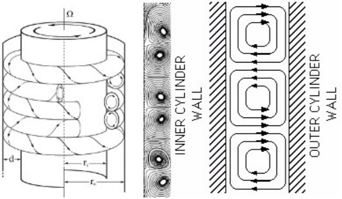
– L F Richardson, Weather Prediction by Numerical Process (1922).

Taylor experiment: flow between rotating cylinders

- Oil with Al powder between concentric cylinders $a \leq r \leq b$. Inner cylinder rotates slowly at ω_a with outer cylinder fixed. Oil flows steadily with azimuthal v_ϕ dropping radially outward from $\omega_a r_a$ to zero at $r = b$.
- Shear viscosity transmits v_ϕ from inner cylinder to successive layers of fluid. Centrifugal force tends to push inner layers outwards, but inward pressure due to wall and outer layers balance it. So pure azimuthal flow is stable.

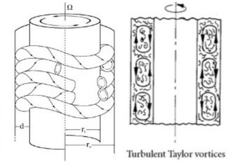


- When $\omega_a > \omega_{critical}$, flow is unstable to formation of toroidal Taylor vortices superimposed on the circumferetial flow. Translation invariance with z is lost. Fluid elements trace helical paths.
- Above $\omega_{critical}$, inward pressure and viscous forces can no longer keep centrifugal forces in check. The outer layer of oil prevents the whole inner layer from moving outward, so the flow breaks up into horizontal Taylor bands.

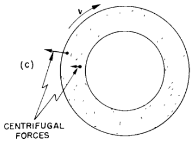


Taylor experiment: flow between rotating cylinders

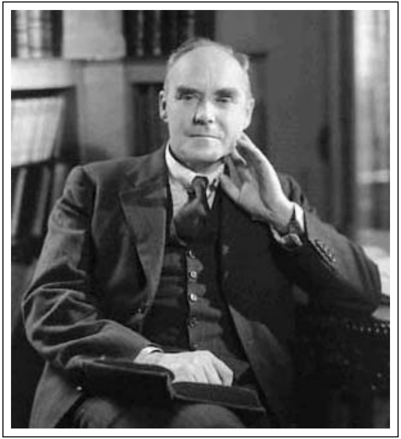
- If ω_a is further increased, keeping $\omega_b = 0$ then # of bands increases, they become wavy and go round at $\approx \omega_a/3$. Rotational symmetry is further broken though flow remains laminar.



- At sufficiently high ω_a, flow becomes fully turbulent but time average flow displays approximate Taylor vortices and cells.
- There are 3 convenient dimensionless combinations in this problem: $(b-a)/a$, L/a and the Taylor number $Ta = \omega_a^2 a(b-a)^3/\nu^2$.
- For small annular gap and tall cylinders, Taylor number alone determines the onset of Taylor vortices at $Ta = 1.7 \times 10^4$.
- If the outer cylinder is rotated at ω_b holding inner cylinder fixed ($\omega_a = 0$), no Taylor vortices appear even for high ω_b. Pure azimuthal flow is stable.
- When outer layers rotate faster than inner ones, centrifugal forces build up a pressure gradient that maintains equilibrium.



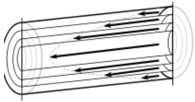
Geoffrey Ingram Taylor



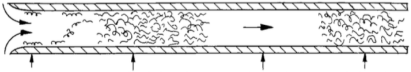
Geoffrey Ingram Taylor

Reynolds' expt (1883): Pipe flow transition to turbulence

- Consider flow in a pipe with a simple, straight inlet. Define the Reynolds number $\mathcal{R} = Ud/\nu$ where pipe diameter is d and U is flow speed.
- At very low $\mathcal{R}$ flow is laminar: steady Poiseuille flow (parabolic vel. profile).
- In general, turbulence in the pipe seems to originate in the boundary layer near the inlet or from imperfections in the inlet.



- If $\mathcal{R} \lesssim 2000$, any turbulent patches formed near the inlet decay.

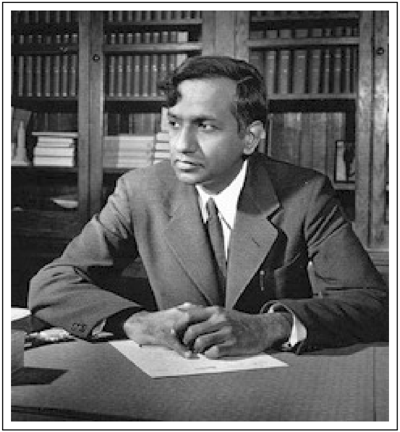


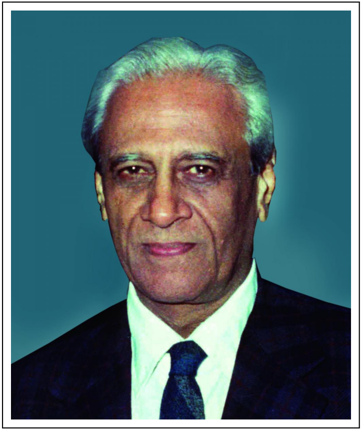
- When $\mathcal{R} \gtrsim 10^4$ turbulence first begins to appear in the annular boundary layer near the inlet. Small chaotic patches develop and merge until turbulent 'slugs' are interspersed with laminar flow regions.
- For $2000 \lesssim \mathcal{R} \lesssim 10,000$, the boundary layer is stable to small perturbations. But finite amplitude perturbations in the boundary layer are unstable and tend to grow along the pipe to form fully turbulent flow.

Shocks in compressible flow

- A shock is usually a surface of small thickness across which $\mathbf{v}, p, \rho$ change significantly: modelled as a surface of discontinuity.
- Shock moves faster than the speed of sound. Roughly, if shock propagates sub-sonically, it could emit sound waves ahead of the shock that eliminate the discontinuity. Mach number $M = v_1/c > 1$.
- Sudden localized explosions like supernovae or bombs often produce spherical shocks called blast waves. Nature of spherical blast wave from atom bomb was worked out by Sedov and Taylor in the 1940s.
- Material from undisturbed medium in front of shock (ρ_1) moves behind the shock and gets compressed to ρ_2.
- Fluxes of mass, momentum and energy are equal in front of and behind the shock. This may be used to relate ρ_1, v_1, p_1 to ρ_2, v_2, p_2. These lead to the Rankine-Hugoniot 'jump' conditions.
- Viscous term $\nu \nabla^2 \mathbf{v}$ is often important in a shock since $\mathbf{v}$ changes rapidly. Leads to heating of the gas and entropy production.

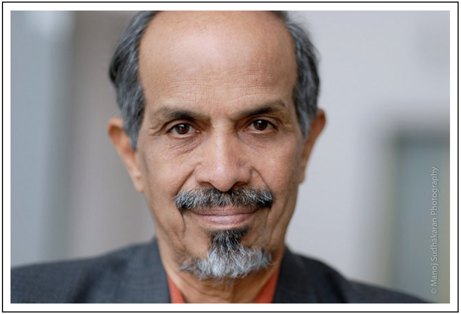
Prominent Indian fluid dynamicists

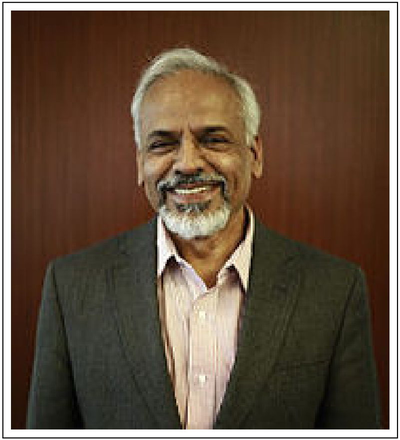




Subrahmanyan Chandrasekhar (left) and Satish Dhawan (right).

Prominent Indian fluid dynamicists



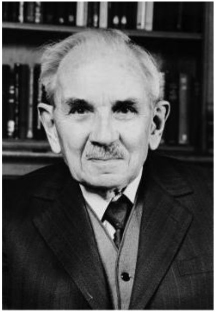


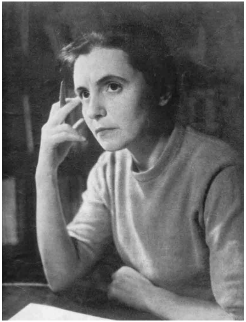
Roddam Narasimha (left) and Katepalli Sreenivasan (right).

Existence & Regularity: Clay Millenium Problem

- Either prove the existence and regularity of solutions to incompressible NS subject to smooth initial data [in $\mathbb{R}^3$ or in a cube with periodic BCs] OR show that a smooth solution could cease to exist after a finite time.
- J Leray (1934) proved that weak solutions to NS exist, but need not be unique and could not rule out singularities.
- Hausdorff dim of set of space-time points where singularities can occur in NS cannot exceed one. So hypothetical singularities are rare!
- O Ladyzhenskaya (1969) showed existence and regularity of classical solutions to NS regularized with hyperviscosity $-\mu(-\nabla^2)^\alpha \mathbf{v}$ with $\alpha \geq 2$. J-L Lions (1969) extended it to $\alpha \geq 5/4$.
- A proof of existence/uniqueness/smoothness of solutions to NS or a demonstration of finite time blow-up is mathematically important.
- Physically, it is know that for large enough $\mathcal{R}$, most laminar flows are unstable, they become turbulent and seem irregular. Methods to calculate/predict features of turbulent flows would also be very valuable.

Jean Leray, Olga Ladyzhenskaya and Jacques Louis Lions


Jean Leray - Royal society (1991)



Jean Leray (left), Olga Ladyzhenskaya (middle) and Jacques Louis Lions (right).

von Karman vortex street in the clouds

von Karman vortex street in the clouds above Yakushima Island

Thank you!