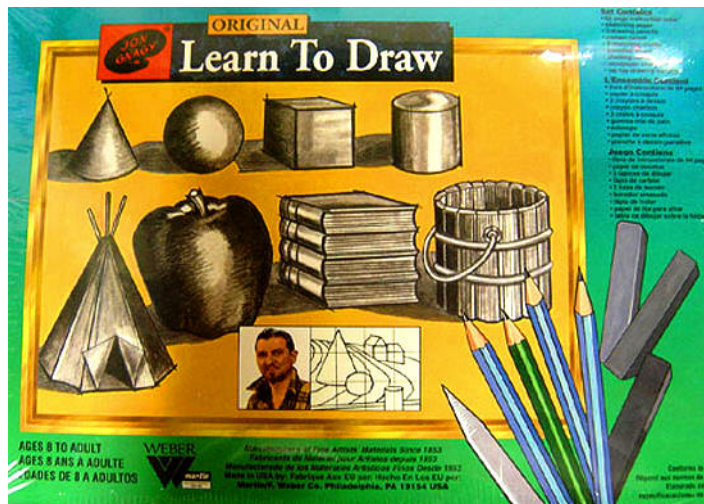


Introduction to Fractals: The Geometry of Nature

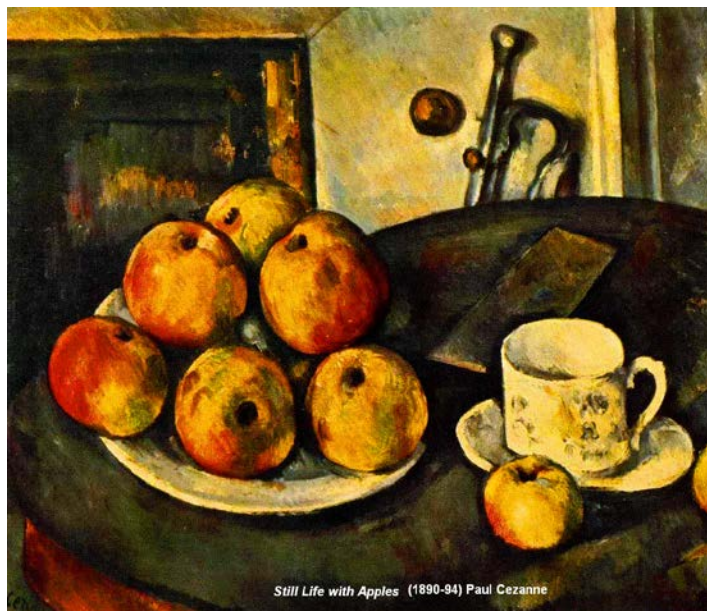
Written by Dave Didur

July 17, 2014 - Classical geometry, which traces its origins back to Euclid, is concerned with figures, shapes and the properties of space. For centuries, this branch of mathematics was used to describe the physical world. Today, fractal geometry is used instead.

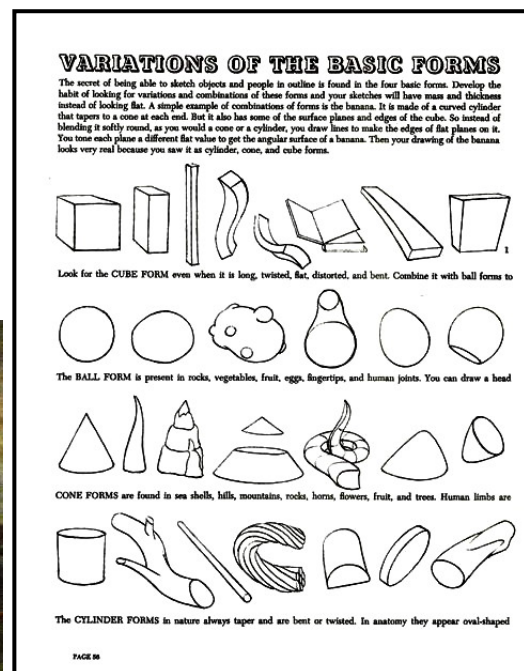


When I was a young boy in the 1950s, I used to avidly watch a half hour black and white TV show called “Learn to Draw,” featuring artist Jon Gnagy. Our local library also had some of his instructional books. The four basic shapes – the cone, the sphere, the cube, and the cylinder – were the fundamental building blocks for his course on drawing. An advertisement for one of his kits and a page from one of his books are shown here.

A century earlier Paul Cézanne (1839 – 1906), a famous French post-impressionist painter, wrote, "everything in nature is modeled according to the sphere, the cone, and the cylinder. You have to learn to paint with reference to these simple shapes; then you can do anything."



Still Life with Apples (1890-94) Paul Cezanne



The mathematician Benoit Mandelbrot (1924 – 2010) turned the world on its head with his essays and books on the fractal geometry of nature. His publisher, W. H. Freeman and Company (1977) wrote, “The complexity of nature’s shapes differs in

kind, not merely degree, from that of the shapes of ordinary geometry. To describe such shapes, Benoit Mendlebrot conceived and developed a new geometry, the geometry of fractal

shapes.” Mandelbrot (pictured here) begins the first chapter of his book *The Fractal Geometry of Nature* as follows:



“Why is geometry often described as ‘cold’ and ‘dry’? One reason lies in its inability to describe the shape of a cloud, a mountain, a coastline, or a tree. Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.”

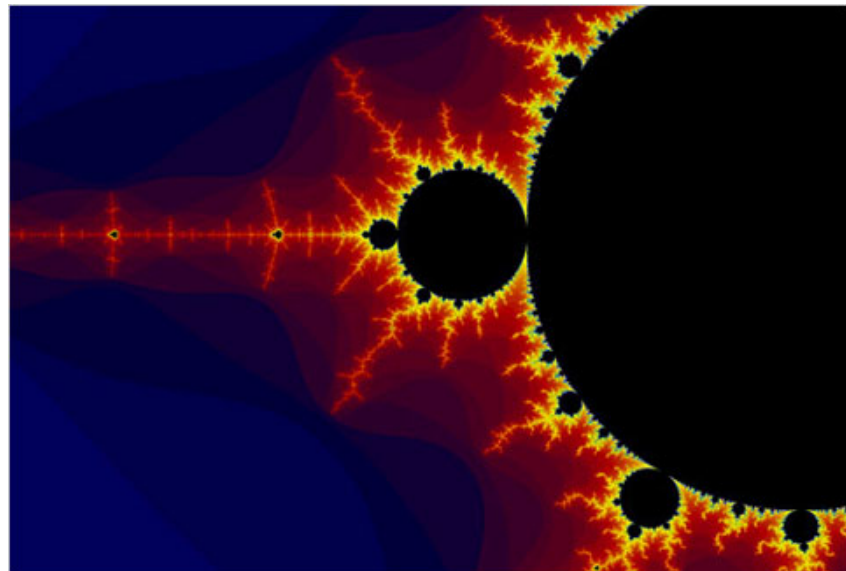
“More generally, I claim that many patterns of Nature are so irregular and fragmented, that, compared to *Euclid* – a term used in this work to denote all of standard geometry – Nature exhibits not simply a higher degree but an altogether different level of complexity. The number of distinct scales of length of natural patterns is for all practical purposes infinite.”

“The existence of these patterns challenges us to study those forms that Euclid leaves aside as being ‘formless,’ to investigate the morphology of the ‘amorphous.’ Mathematicians have disdained this challenge, however, and have increasingly chosen to flee from nature by devising theories unrelated to anything we can see or feel.”

“Responding to this challenge, I conceived and developed a new geometry of nature and implemented its use in a number of diverse fields. It describes many of the irregular and fragmented patterns around us, and leads to full-fledged theories; by identifying a family of shapes I call *fractals*.”

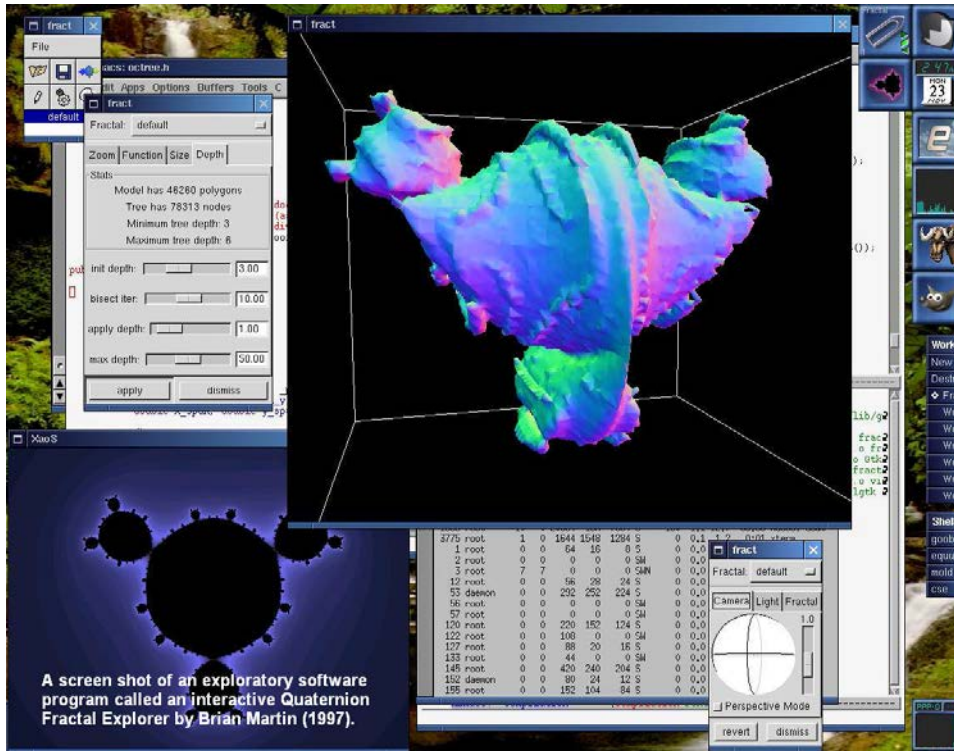
Mandelbrot coined the word ‘fractal’ from the Latin adjective ‘fractus’. The corresponding Latin verb *fragere* means “to break” – to create irregular fragments.

Because of his work with IBM (which began in 1958), Mandelbrot had access to some of the earliest computers. His early work with computer graphics allowed him to devise techniques for generating complex images from a series of simple steps. Many of those early B&W images have been described as ‘psychedelic art forms’. In a documentary with science writer Arthur C. Clarke, Mandelbrot stated:



“I certainly never had the feeling of invention. I never had the feeling that my imagination was rich enough to invent all those extraordinary things on

discovering them. They were there, even though nobody had seen them before. It's marvellous; a very simple formula explains all these very complicated things. So the goal of science is starting with a mess, and explaining it with a simple formula, a kind of dream of science.”



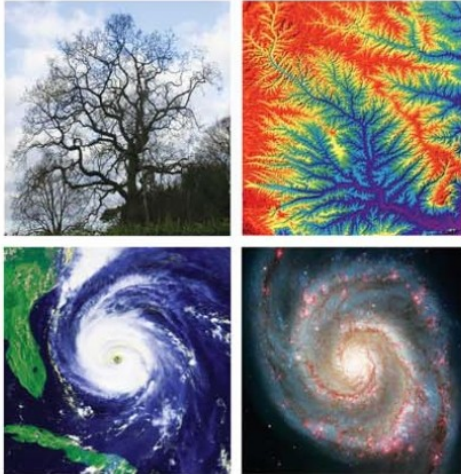
A screen shot of an exploratory software program called an interactive Quaternion Fractal Explorer by Brian Martin (1997).

Computers play a vital role in developing fractal imagery. As technology has evolved over the past fifty years, improvements in processing speed, memory, and graphics have made the field of study explode with new developments. Most people know little about fractal geometry, but everybody is aware of the fantastic CGI (computer-generated images) in today's Hollywood movies.

The mountains on the right were CGI-generated using a technique called 'midpoint displacement' (more can be found in the Reference section in the article 'Clouds are Not Spheres...'). Other links to fractal art can be found in the same section.



In the References section, there is a video entitled "The CGI Talking Animals – 'Babe' to 'Life of Pi' by Rhythm & Hues Studios". Although the video focuses on showing us the evolution of CGI-generated animals, you will hear the studio say that the total effect is enhanced when the complete environment is also created by CGI. Water, mountains, clouds, trees and more are created by computer programs that use fractal algorithms. We've come a long way from Euclid, Cézanne and Jon Gnagy!

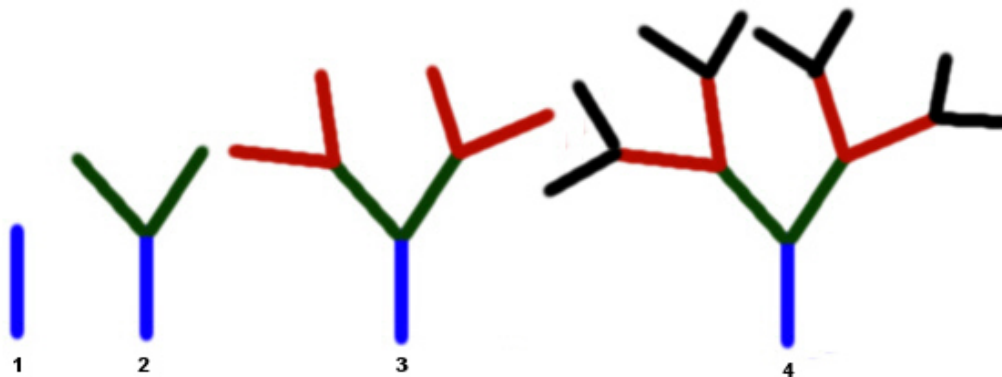


Michael Barnsley (Georgia Institute of Technology), in his book *'Fractals Everywhere'* says, "the observation by Mandelbrot of the existence of a 'Geometry of Nature' has led us to think in a new scientific way about the edges of clouds, the profiles of the tops of forests on the horizon, and the intricate moving arrangement of the feathers on the wings of a bird as it flies. Geometry is concerned with making our spatial intuitions objective. Classical geometry provides a first approximation to the structure of physical objects; it is the language which we use to communicate the designs of technological products, and, very approximately, the forms of natural creations. Fractal geometry is a new language. Once you can speak it, you can describe the shape of a cloud as precisely as an architect can describe a house."

What is a Fractal?

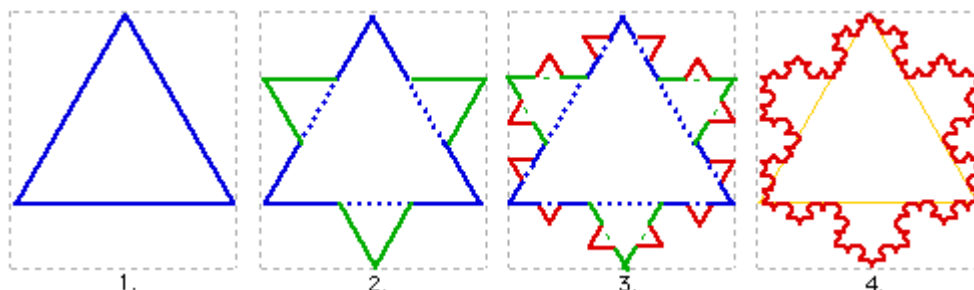
A fractal is a pattern that repeats itself forever, in ever decreasing size, within the same figure (a characteristic that is known as 'self-similarity'). Consider the following:

- Starting with a stick, at each stage add a 'V-shape' at the top end of each line segment.
The resultant figure resembles a plant or tree in the natural world.
 As the process continues for many more stages, the top gets bushier & bushier.

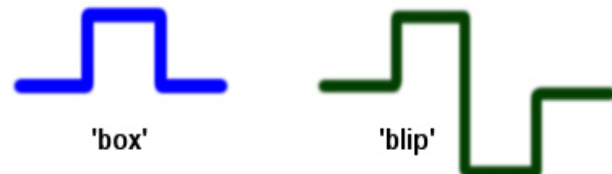


The Fractal Foundation website link listed in the References section will allow you to manipulate a 'fractal tree' in order to see many possible and interesting variations of this type of pattern.

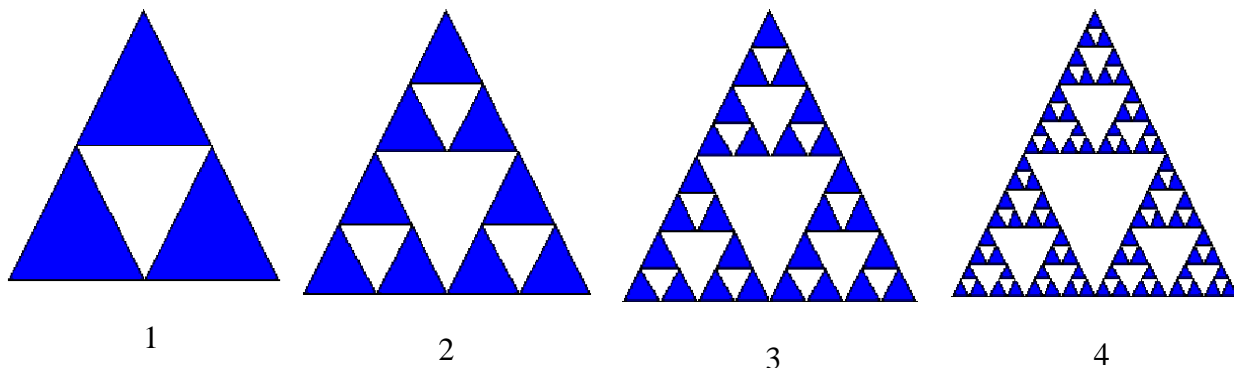
- Starting with a triangle, at each stage replace the middle of each straight line with a 'peak'.
The resultant figure resembles a snowflake. It is called a 'Koch Snowflake'.
 As the process continues for many more stages, the edges get infinitely frillier.



This same procedure – selecting a shape such as a ‘box’ or a ‘blip’ – and inserting it into the middle of each line segment of an initial figure at every stage will produce a variety of patterns of ever-increasing complexity. If small rotations are included in each transformation, even more impressive shapes result. Such iterative processes are easily done by computers.



3. Begin with a solid equilateral triangle that has a white inverted triangle in its centre. At each stage replace the remaining solid triangles with one that has the inverted white triangle in its centre. *The resultant figure maintains its exterior shape, but becomes more “airy” in its interior. The resultant image is called the Sierpinski Triangle.*



Any fractal will have one or more of the following properties:

1. Self-similarity at any scale
2. Irregular or fragmented forms at any scale
3. Distinct elements at any level of magnification

Fractals are described as having *non-integer dimensions*. Edyta Patrzalek (Eindhoven University of Technology, The Netherlands) explains this property nicely in the article “Fractals: Useful Beauty”:

“Classical geometry deals with objects of integer dimensions: zero dimensional points, one dimensional lines and curves, two dimensional plane figures such as squares and circles, and three dimensional solids such as cubes and spheres. However, many natural phenomena are better described using a dimension between two whole numbers. So while a straight line has a dimension of one, a fractal curve will have a dimension between one and two, depending on how much space it takes up as it twists and curves. The more the flat fractal fills a plane, the closer it approaches two dimensions. Likewise, a “hilly fractal scene” will reach a dimension somewhere between two and three. So a fractal landscape made up of a large hill covered with tiny mounds would be close to the second dimension, while a rough surface composed of many medium-sized hills would be close to the third dimension.”

It can be fun to examine some of the strange properties of fractals. Consider the Koch Snowflake and the Sierpinski Triangle. The former keeps getting more ‘bumps’ added to its outer edge at every stage (so its perimeter keeps getting larger) while its area seems to only increase marginally. In fact, the area approaches a limiting value that it never exceeds, while its

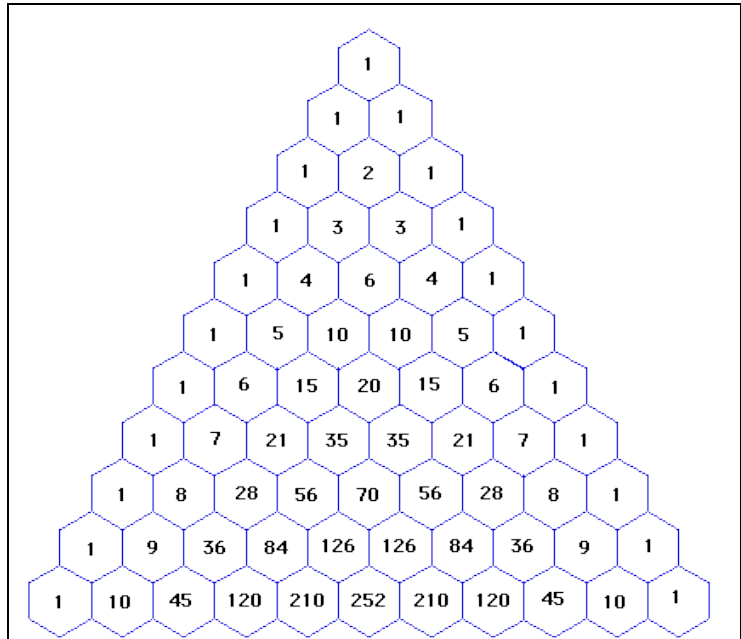
perimeter continues to grow infinitely larger! What seems to be happening in the Sierpinski Triangle at each stage? I'll leave that for you to ponder.

Mathematics has interesting connections between areas of study. In my last article on [Continued Fractions](#), I showed a connection to the Fibonacci Sequence, the Golden Ratio, and Pascal's Triangle. Pascal's Triangle can also have a link to fractals! The first eleven rows of Pascal's Triangle are shown here. Try the following activity: colour in all of the odd-numbered cells. What do you see?

Here's another activity. We will alter the entries by using *modulo 3* arithmetic. Divide each entry by 3, take note of the remainder, and write that remainder in the triangle in place of the original entry. For example:

- 1 divided by 3 = 0, rem 1
- 2 divided by 3 = 0, rem 2
- 3 divided by 3 = 1, rem 0
- 4 divided by 3 = 1, rem 1
- ⋮
- 8 divided by 3 = 2, rem 2

The original triangle will be replaced by the new one shown below. Now we have some colouring fun! If the entry is 0, we leave it blank; if the entry is 1 or 2, we shade it. Notice anything?



The resultant triangle in each previous activity is a duplication of the Sierpinski Triangle! Interesting?

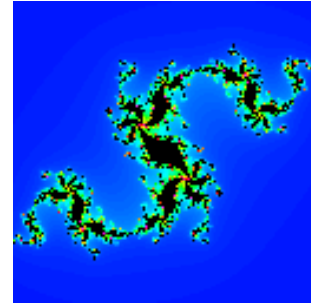
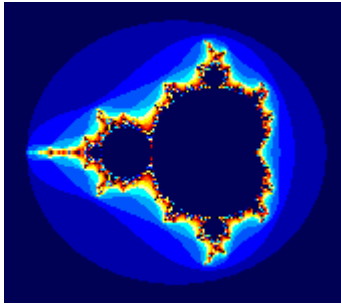
Try investigating this even further. Start with an even larger version of Pascal's Triangle – but this time use *modulo 9* arithmetic. Divide every entry by 9 – and shade in every cell that has a remainder of zero (otherwise, leave the cell unshaded).

Pascal's Triangle, as pointed out last month, contains numbers that evolve from an algebraic process. Colouring the cells according to different rules reveals various patterns or structures. When the rules are changed, new structures appear. Processes of this kind are

called *cellular automata*. There are many relevant applications of this process, including the simulation of how fluid flows around obstacles. The book "*Fractals for the Classroom: Strategic Activities - Volume Three*" by Hartmut Jurgens, Dietmar Saupe, Evan Maletsky & Terry Perciante (1999) contains many investigations like these.

There are many more processes for producing fractals, but even though the steps are simple the mathematics involved is more complicated. The results are quite spectacular. If you look

into some of the resources listed in the References section, you will find some of these, including the Mandelbrot Set and the Julia Set.



Applications

“With fractal geometry scientists had a new way to describe, classify, and measure pieces of the scientific world. By the early 1980's fractals had gathered much attention. Fractal geometry provided a new set of tools that could be used by physicists, chemists, seismologists, metallurgists, and many other scientists. Scientists at Exxon examined fractal problems. Fractals became important in the study of polymers and even the study of nuclear safety. Geologists used fractal geometry to describe the particular bumpiness of earth's surface and metallurgists found the same for the surfaces of different kinds of steel. Fractals found their way into computer graphics. They were a way of modeling real aspects of the world. Hollywood used fractals in CGI. Fractals found their way into image and video compression and even music.”

Anthony Howe, University of Victoria

See: “Clouds are Not Spheres...” in Ref. section

Edyta Patrzalek describes other applications as well – in the study of interstellar gases (in astrophysics) and in the studies of chromosomes and DNA, with possible applications in the study of evolution (in the biological sciences). More applications occur in clothing designs, economics, communications antennas, medicine, etc. Fractals are everywhere! Watch the excellent NOVA video in the References section to learn about them.

Conclusion

You can watch a video of Jon Gnagy giving a lesson on how to draw an old oak tree in the References section below. He uses geometric figures, like a circle to sketch the moon and cylinders to sketch the tree trunk, but for those ‘fuzzy’ parts of nature – like the trees on the horizon or the upper tiers of leaves – he makes rough swathes with the sides of his chalk.

Back in the 1950s when this was filmed, there were barely a handful of computers in the entire world. Today computers are everywhere, and with the aid of mathematical algorithms they are able to quickly churn out spectacular graphical images based on fractal geometry: stunningly realistic views of tree-lined horizons, leafy treetops, fluffy clouds, churning waves, and craggy mountains. Sixty years ago who would have imagined that machines would be producing art that imitated nature? I didn't. Jon Gnagy didn't. But Benoit Mandelbrot was starting work at IBM – and his imagination and intelligence would take us into new directions of thought, and into the wonderful world of fractals!

“Fractal geometry is a new language. Once you can speak it, you can describe the shape of a cloud as precisely as an architect can describe a house.”

Michael Barnsley,
Georgia Institute of
Technology
‘Fractals Everywhere’
1988



This article is the fifth of a series of mathematics articles published by CHASA.

[Marvellous Mathematics](#) – Introduction

[Euclidian Geometry](#) – Article # 1

[Non-Euclidean Geometry](#) – Article #2

[Rational Numbers – Fractions, Decimals and Calculators](#) – Article #3

[Continued Fractions](#) – Article #4

CHASA has received many communications from concerned parents about the difficulties their children are having with the math curriculum in their schools as well as their own frustration in trying to understand the concepts - so that they can help their children. The intent of these articles is to not only help explain specific areas of history, concepts and topics in mathematics, but to also show the beauty and majesty of the subject.

***Dave Didur** is a retired secondary school mathematics teacher with a B. Sc. degree from the University of Toronto majoring in Mathematics and Physics. He was Head of Mathematics for over twenty years, as well as the Computer Co-ordinator and consultant for the Board of Education for the City of Hamilton. He served with the Ontario Ministry of Education for three years as an Education Officer.*

References

[Jon Gnagy Learn-To-Draw Lesson \(Video\)](#) – The Old Oak Tree

[A Fractal Unit for Elementary and Middle School Students by Cynthia Lanius](#) – a great web site with lots of information for everyone, with printable notes for all lessons

[Fractal Fraction Fun](#) – suggested activities for grades K-3, 3-8, and 4-8

[Fractals: Unit Study Ideas](#) – Diane Manuel lists a variety of web sites (some of which are listed in this 'References' section) to investigate

[Fractals: Hunting The Hidden Dimension \(Video\)](#) – a 53-minute NOVA program that is **GREAT!!** It provides an excellent survey of the topic of fractals, including historical facts and applications.

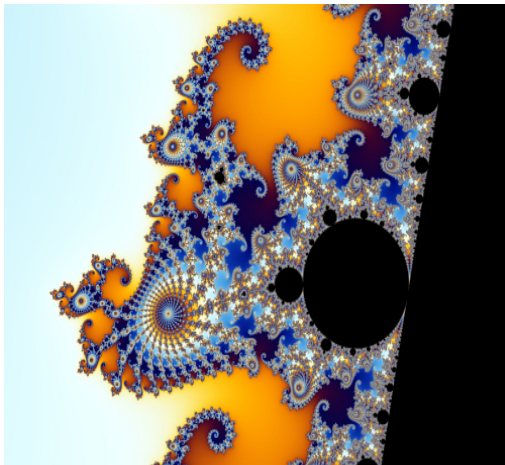


[Arthur C. Clarke: Fractals – The Colors of Infinity \(Video\)](#) – another 53 minute program on fractals, but more emphasis is placed on the M-Set (the Mandelbrot Set)

[A Radical Mind](#) – A NOVA-Online question-and-answer interview with Benoit Mandelbrot by Bill Jersey, conducted on April 24, 2005

[What Are Fractals?](#) – The Fractal Foundation provides notes for an introduction to fractals and a more detailed 20-page study guide for educators. Fascinating videos, downloadable software, and interactions allow you to explore the world of fractals in more depth.

[Fractals and Chaos Theory](#) – an introduction, at a slightly higher level



[Clouds are Not Spheres, Mountains are Not Cones](#) – Anthony Howe (Univ. of Victoria) writes about fractals and chaos

[Fractals: Useful Beauty](#) -- Edyta Patrzalek, Stan Ackermans Institute, IPO, Centre for User-System Interaction, Eindhoven University of Technology

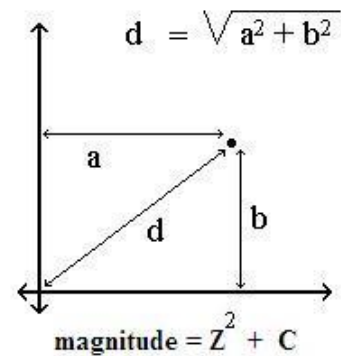
[Fractal Geometry](#) – background, history, applications

[Fractal Geometry](#) – excellent content on a wide variety of topics to support a first course in fractals (Yale University)

[How to Plot the Mandelbrot Set by Hand](#) – WikiHow illustrates the process of plotting points to make a graphic plot of an M-Set using the recursion formula $z = z^2 + c$. Imaginary numbers, coloured pencils, and tips to help explain parts of the process are involved.

[Understanding Complex Numbers in Order to Understand M-Sets](#)

[Mandelbrot Plotter](#) – **TRY the Flash application below to explore the Mandelbrot Set** on PCs (just CTRL-click on the “screen shot”). The source code can be downloaded for those who are knowledgeable. More info is available at this web site.



Mandelbrot Plot
By Dan Gries flashandmath.com

gradient editor

smooth auto

smooth auto

smooth auto

choose a preset gradient

input specific colors by value

After changing the gradient, click on 'update' on the left panel to color the fractal.

export: png jpg show plot info

export side length: 400 (max length = 800)

color period 0.25 coloring method log log

color phase 0.58

max iter 5000 4x oversampling

To get started, draw a rectangle on the fractal with the mouse and click on 'zoom selection' to zoom in. For more information on additional controls and parameters, click on 'help.'

[The CGI Talking Animals: Babe to Life of Pi by Rhythm & Hues Studios](#) (Video) – CGI in the movies

[How Fractals Can Explain What's Wrong With Wall Street](#) – a reprint of a Scientific American article first published in 1999

Fractal Art: Stills and Videos



"Fractal Scene I," by Anne M. Burns (Long Island University, Brookville, NY)

[Coolmath's Fractal Gallery](#) – look at various fractals, and try creating your own

[Mathematics – Art – Nature](#) – Mathematical art by Anne Burns (Long Island University)

[Panorama of Fractals and their Uses](#) – Nature and Fractals (Yale University)

[Fractal Geometry \(Galleries and Explorations\)](#) – in French, but you can get the pages translated

[Mandelbrot Fractal tour Guide \(Video\)](#) – 27 minutes of video, examining different sections of the M-Set with zoom-ins, set to music, for a wonderful psychedelic experience for the artist that is within us all

[Morphy's World: Mandelbulb 3D Fractal Animation \(Video\)](#) – a 16-minute voyage into a 3D fractal world, created by Arthur Stammet in Oct. 2013 (once again, set to music)



[Before They Were Fractals](#) -- Katsushika Hokusai used the fractal concept of self-similarity in his painting "The Great Wave Off Kanagawa" in the early 1800s.