Introduction to Fuzzy Logic

IJCAI2018

Tutorial

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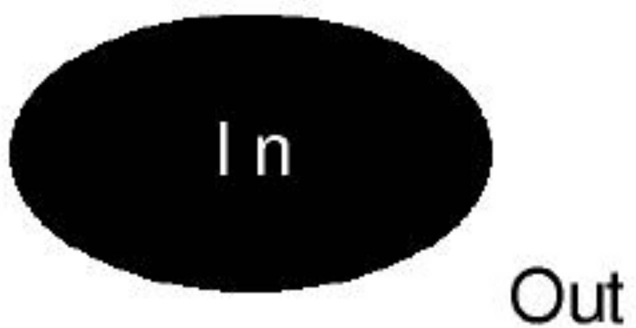
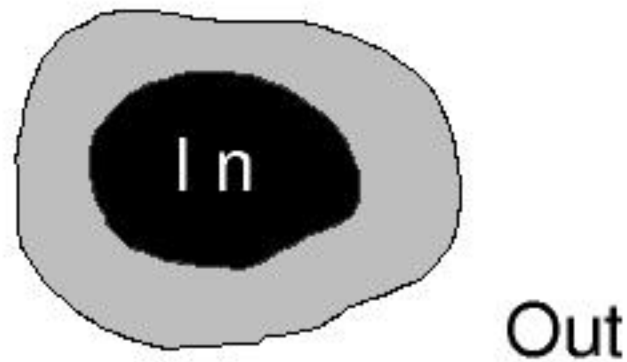
Crisp set vs. Fuzzy set

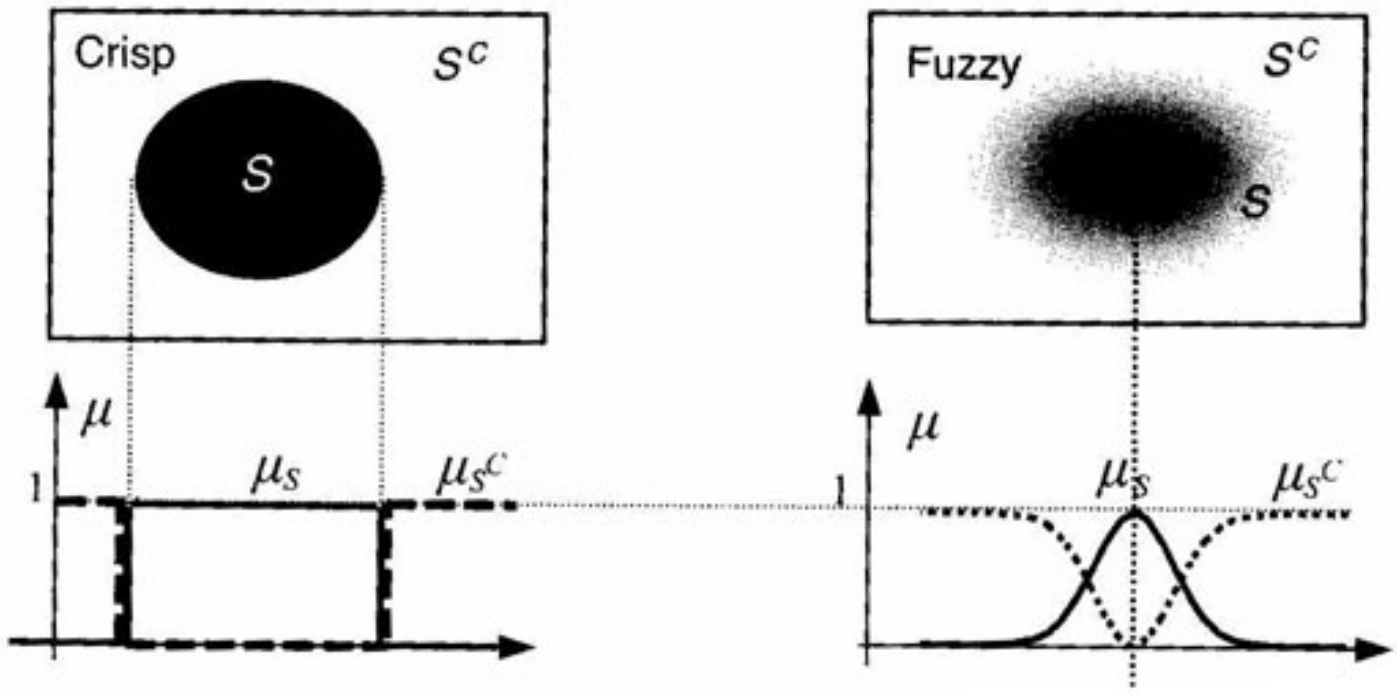


A traditional crisp set



A fuzzy set

Crisp set vs. Fuzzy set



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Crisp Logic Example I

- Crisp logic is concerned with absolutes-true or false, there is no in-between.
- Example:

Rule:

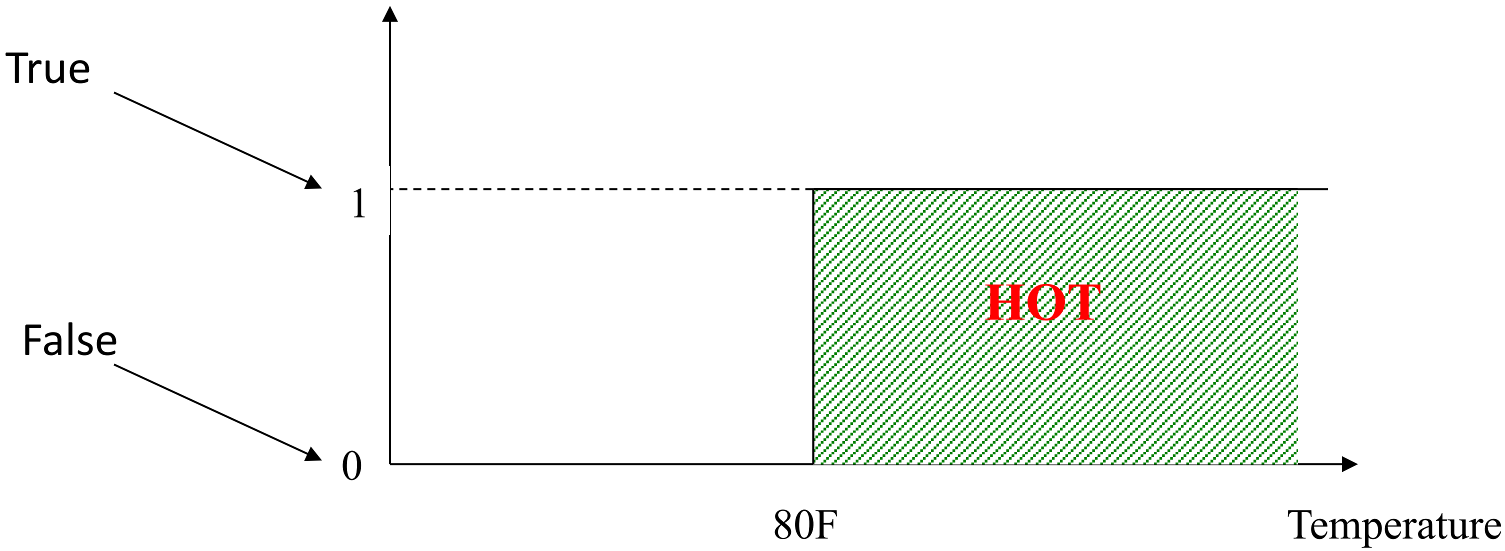
If the temperature is higher than 80F, it is hot; otherwise, it is not hot.

Cases:

- Temperature = 100F **Hot**
- Temperature = 80.1F **Hot**
- Temperature = 79.9F **Not hot**
- Temperature = 50F **Not hot**

Crisp Logic Example I cont.

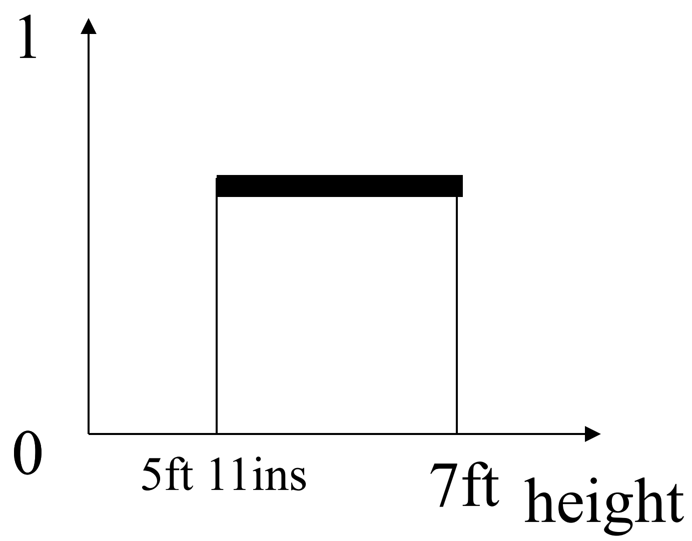
Membership function of crisp logic

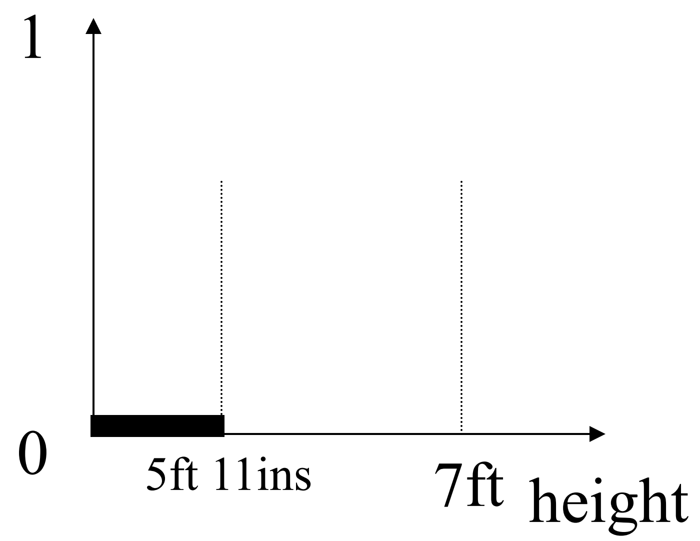


If temperature >= 80F, it is hot (1 or true);

If temperature < 80F, it is not hot (0 or false).

Example II : Crisp set

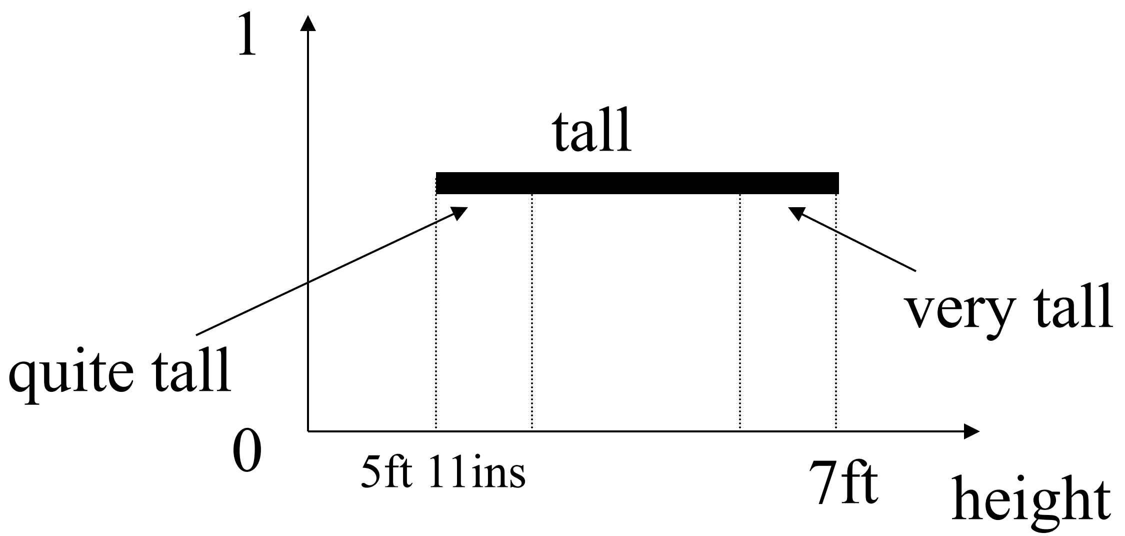




A crisp way of modelling tallness

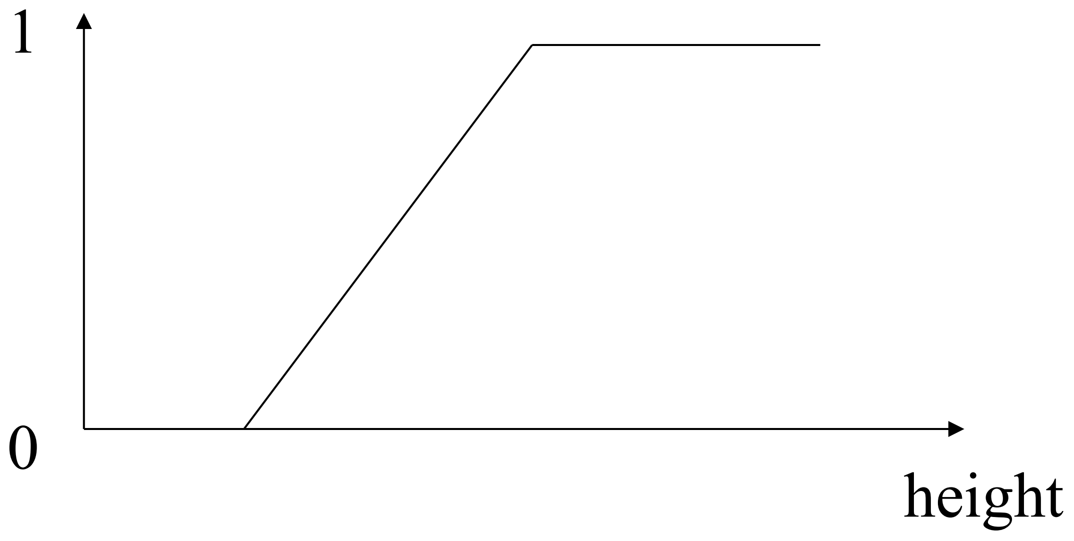
A crisp version of short

Example II : Crisp set

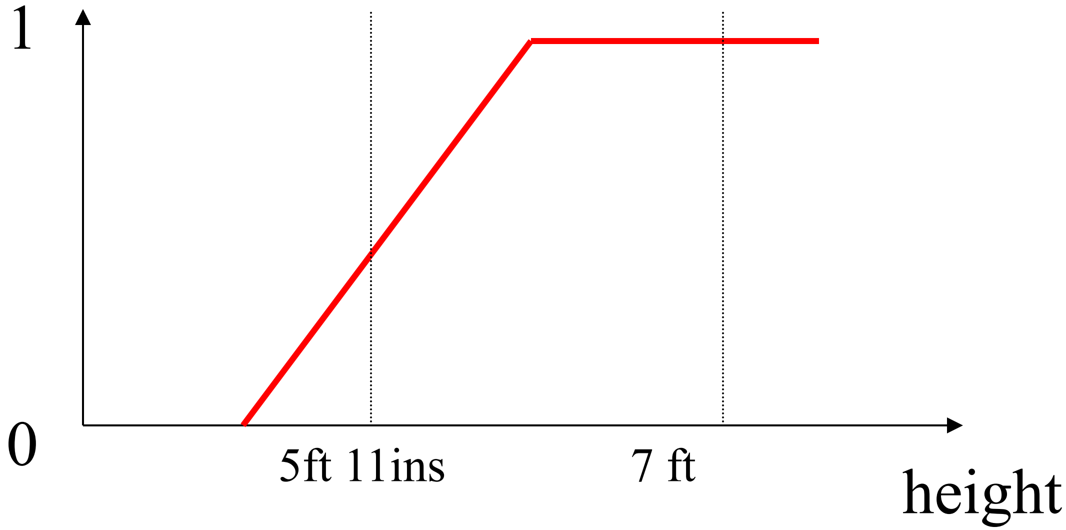


crisp definitions for tallness

Example II : Fuzzy set

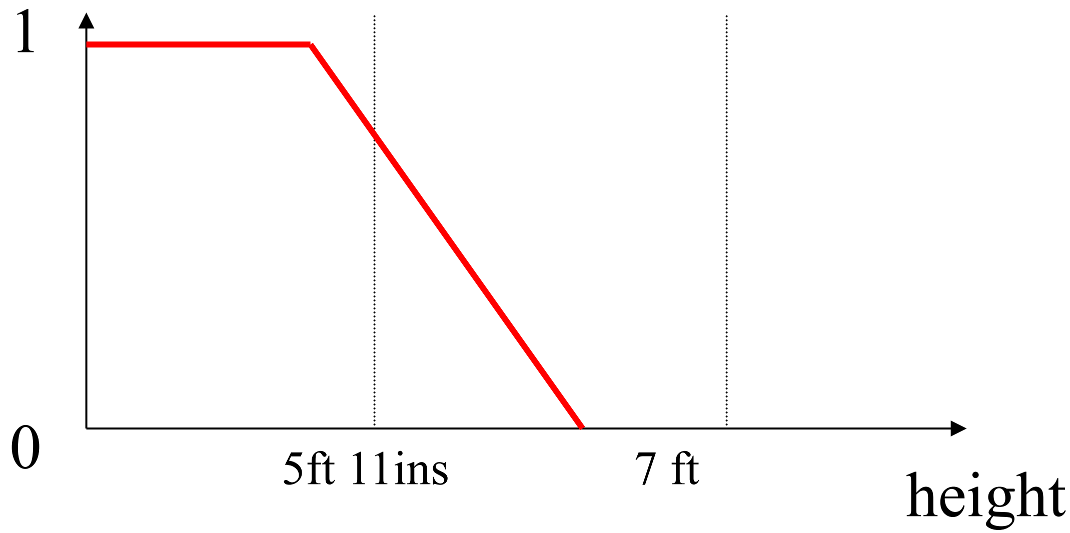


Example II : Fuzzy set



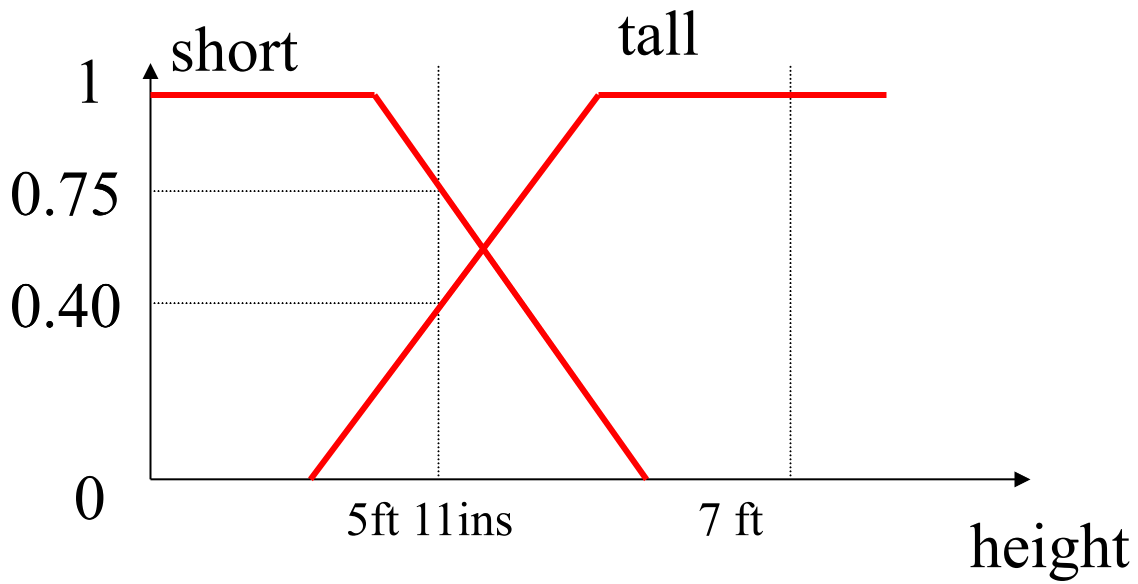
A possible fuzzy set tall

Example II : Fuzzy set



A possible fuzzy set short

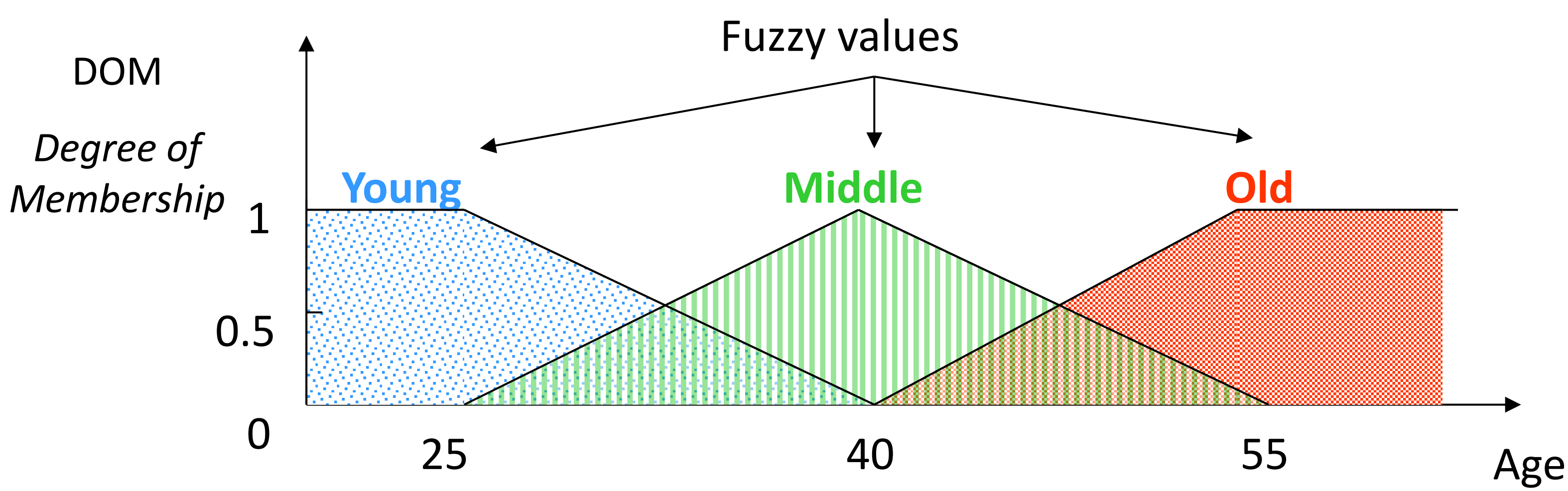
Example II : Fuzzy set



Membership functions that represent tallness and short

Fuzzy Sets

- **Fuzzy sets** is fully defined by its membersh
functions.
- **Membership function** is a function in [0,1] that represents the degree of belonging.



Some maths! Formal definitions of a fuzzy set

- For any fuzzy set, (let's say) A, the function μ_A represents the membership function for which $\mu_A(x)$ indicates the degree of membership that x, of the universal set X, belongs to set A and is, usually, expressed as a number between 0 and 1

$$\mu_A(x) : X \rightarrow [0,1]$$

- Fuzzy sets can be either discrete or continuous

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- **The notation for fuzzy sets**: for the member, *x*, of a discrete set with membership *μ*, we use the notation μ/x . In other words, x is a member of the set to degree μ.
- **Discrete sets** are defined as:

$$A = \mu_1 / x_1 + \mu_2 / x_2 + \ldots\ldots + \mu_n / x_n$$

- or (in a more compact form)

$$A = \sum_{i=1 \ldots n} \mu_i / x_i$$

$x_1, x_2, \ldots\ldots x_n$: members of the set A

$\mu_1, \mu_2, \ldots\ldots \mu_n$: $x_1, x_2 \ldots\ldots x_n$'s degree of membership.

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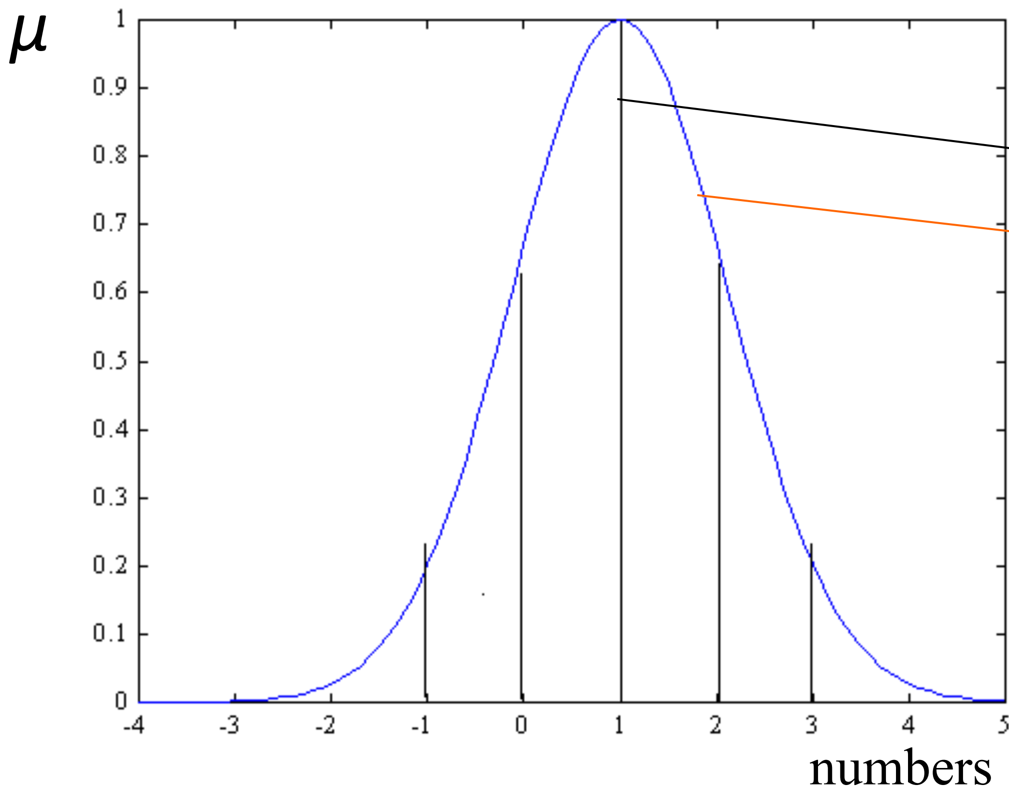
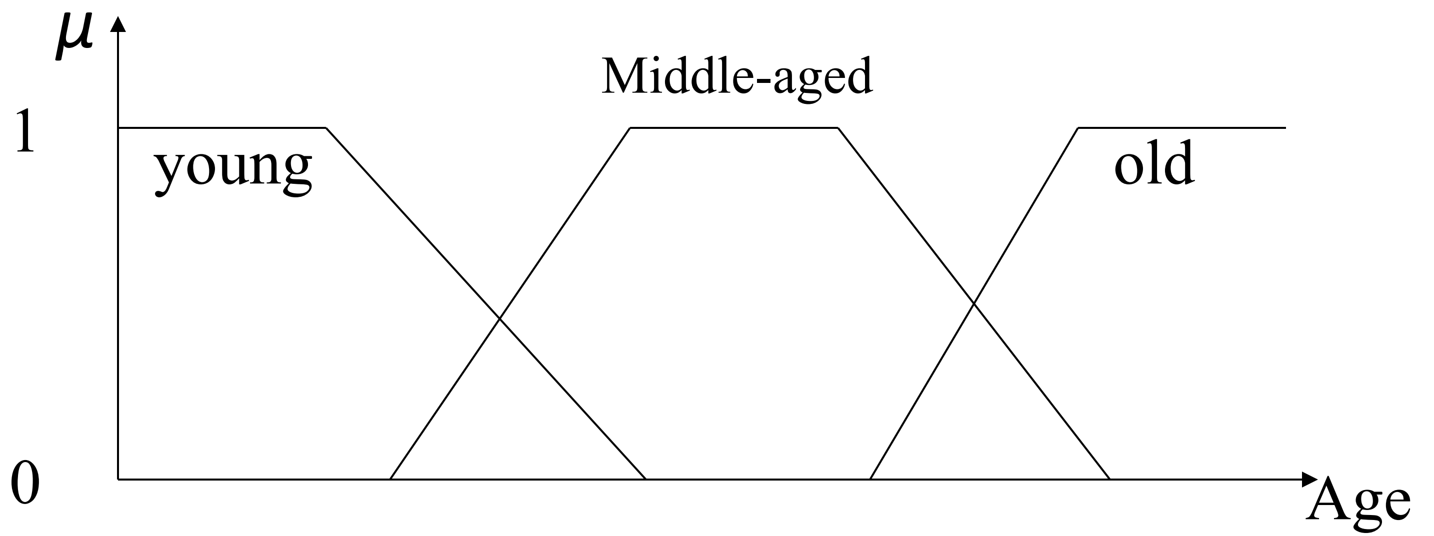
- **A continuous fuzzy set** *A can be* defined as:

$$A = \int_X \mu(x) / x$$



Example:
Discrete and Continuous fuzzy sets to represent the set of numbers "close to 1"

- **Example: describing people as "young", "middle-aged", and "old"**



- **Fuzzy Logic allows modelling of linguistic terms using linguistic variables and linguistic values. The fuzzy sets "young", "middle-aged", and "old" are fully defined by their membership functions. The linguistic variable "Age" can then take linguistic values.**

- **Key Points for a fuzzy set:**
  - The members of a fuzzy set are members to some degree, known as a ***membership grade*** or ***degree of membership***
  - A fuzzy set is fully determined by the membership function
  - The membership grade is the degree of belonging to the fuzzy set. The larger the number (in [0,1]) the more the degree of belonging.
  - The translation from x to $\mu A(x)$ is known as ***fuzzification***
  - A fuzzy set is either continuous or discrete.
  - Fuzzy sets are NOT probabilities
  - Graphical representation of membership functions is very useful.

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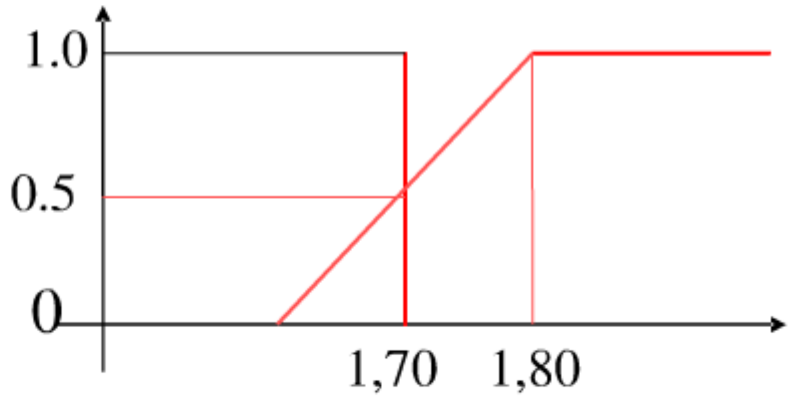
Fuzzy Sets

- The membership function of elements in a fuzzy set A is characterised by

$$\mu(.) : X \rightarrow [0,1]$$

that maps every element from the set *X* into a real number in the interval [0,1].



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Fuzzy Sets – definition

- A fuzzy set can be represented by an ordered set of pairs:

Value

$$A = \{(x, \mu_A(x)) \mid x \in X\}$$

Fuzzy Set

Membership function

Universe of Discourse

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Membership Degree

- Membership degree: an element belongs to a given set by a degree of certainty.
- Some elements are more representative of the set main idea than others.
  - *Excellent students={(Pedro,0.8), (Ana,0.9), (Paulo,0.9), (Marta,1.0)}*
  - *Very high = {(Oscar,0.95), (Michael Jordan, 0.95), (Junior Baiano,0.8)}*

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Fuzzy Sets Representation I

- **Ordered Pairs**: A fuzzy set may represented by a set ordered pairs
- The first element is the element itself and the second its membership degree in the set.
- João = 1.65 m; Ana = 1.70 m; Oscar= 1.80 m

High = {(João, 0.25), (Ana,0.5), (Oscar,1.0)}

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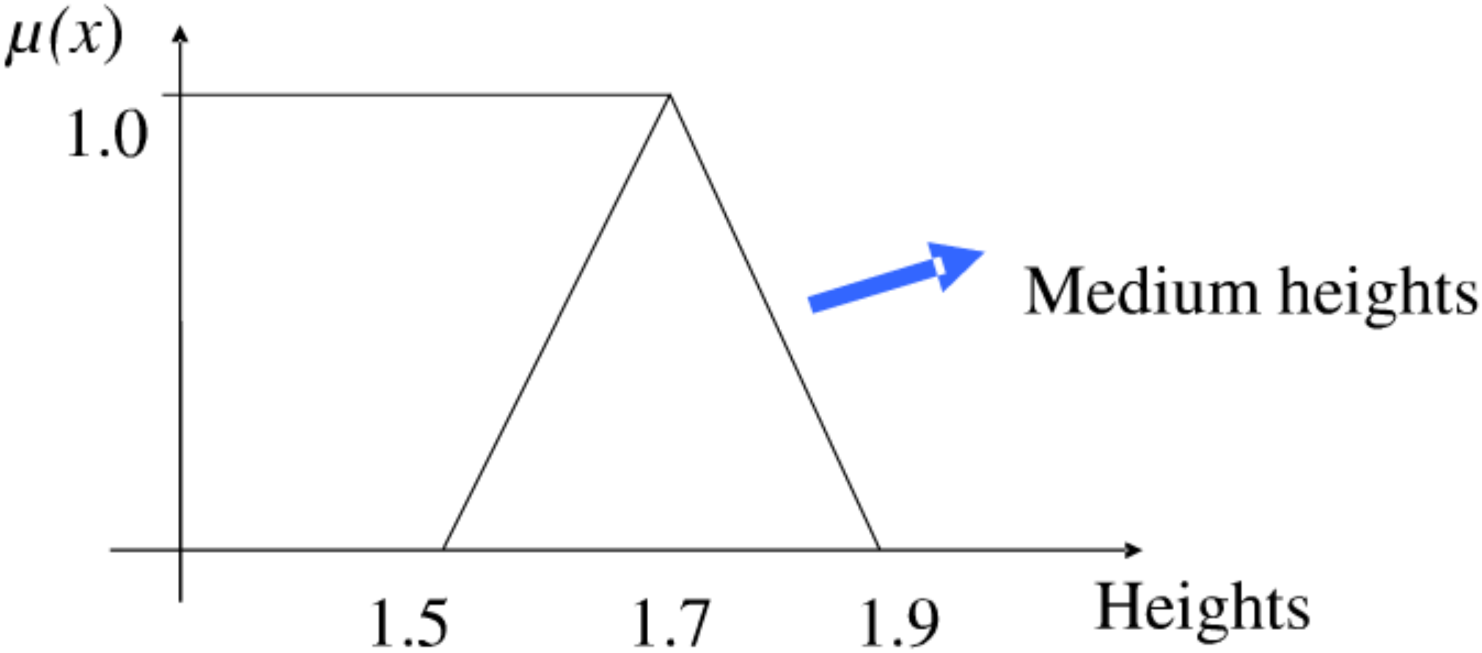
Fuzzy Sets Representation II

- Indicated as the union of all elements

$$A = \sum_{x_i \in A} \mu_A(x_i) / x_i$$

High = 0.25/João + 0.5/Ana + 1.0/Oscar

Example of Membership Functions



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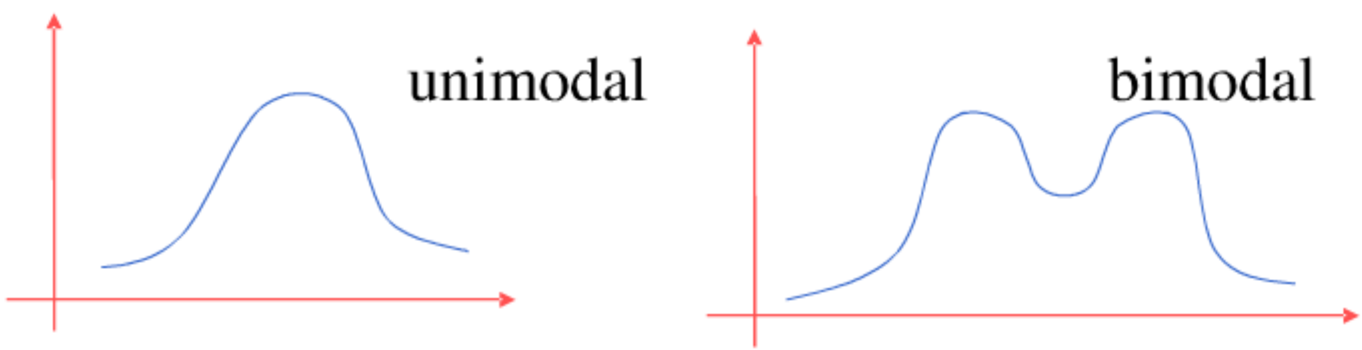
Unimodal Functions

- A function is **unimodal** if

$$\forall x_1, x_2 \in X, \forall \lambda \in [0,1] : \mu(\lambda x_1 + (1-\lambda)x_2) \geq \min[\mu(x_1), \mu(x_2)]$$

- An unimodal function implies that whenever $\mu(x)>\mu(y)$ for a given set *A* then *x* is closer to the ideal definition of *A* than *y*.



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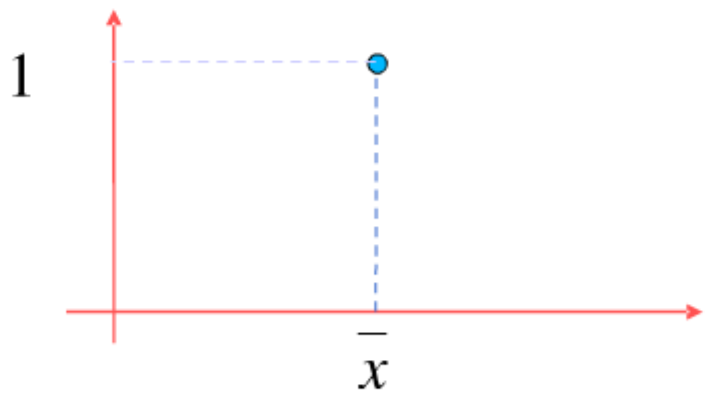
Singletons

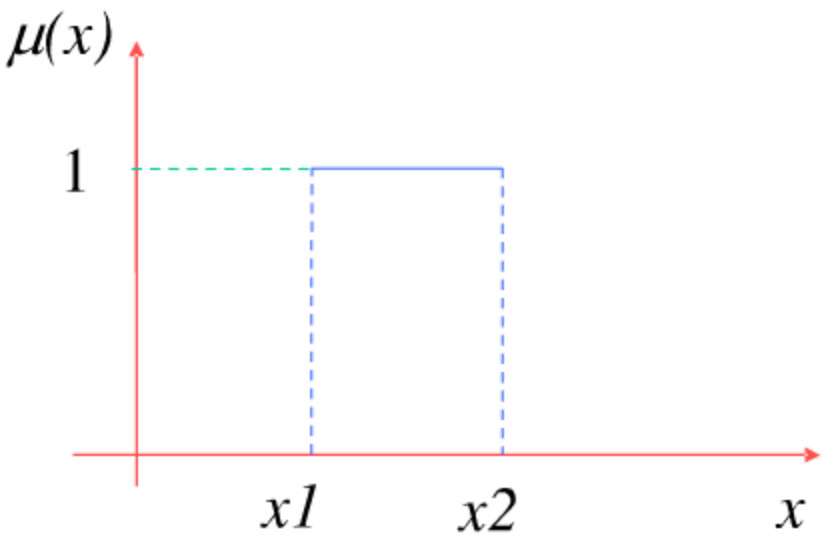
- A Function is a **singleton** whenever

$$\mu_{\bar{x}}(x) = \begin{cases} 1 & x = \bar{x} \\ 0 & x \neq \bar{x} \end{cases}$$



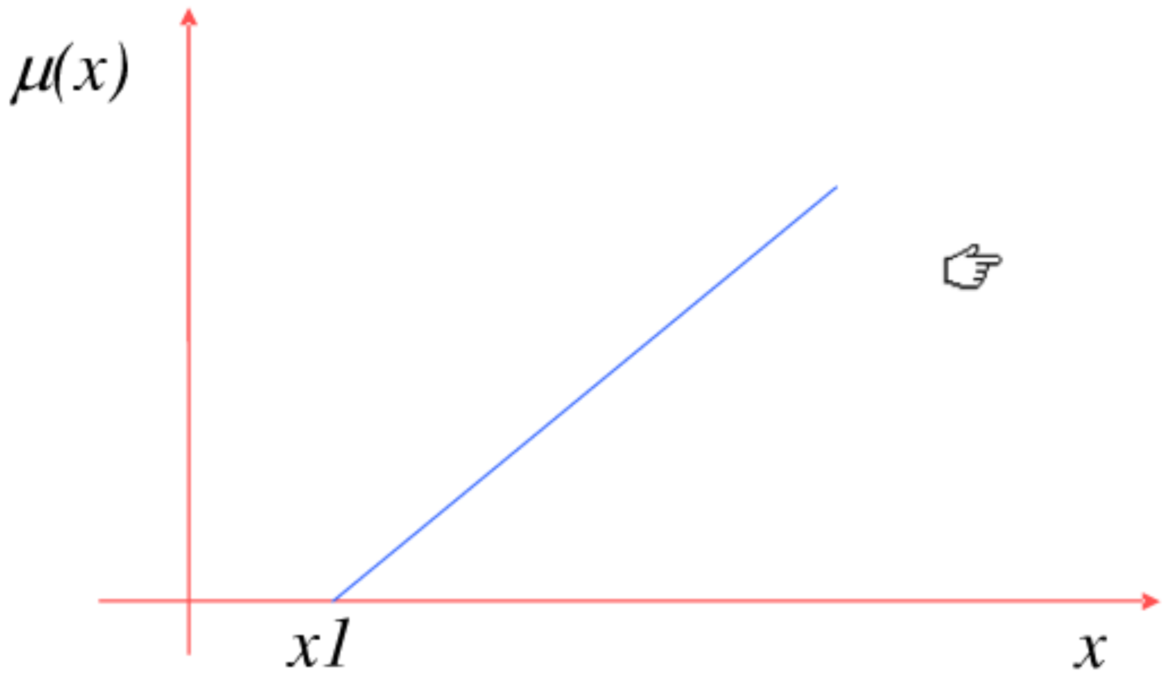
Classical Set Function

- Classical set functions are used to define Classical sets.



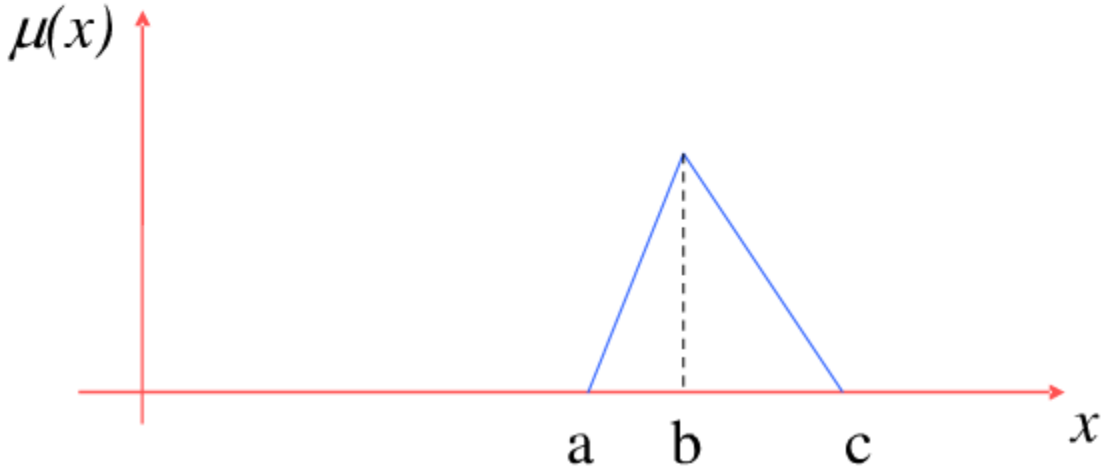
Linear Functions

- Very simple fuzzy set.



Triangular Functions

- Easy to implement, allowing representation of very complex fuzzy sets.
- May be generated from 3 real values.



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Triangular Functions

- May be generated from 3 real values.

$$tri(x;a,b,c)=\begin{cases} 0 & x<a \\ (x-a)/(b-a) & a\le x\le b \\ (c-x)/(c-b) & b\le x\le c \\ 0 & x>c \end{cases}$$

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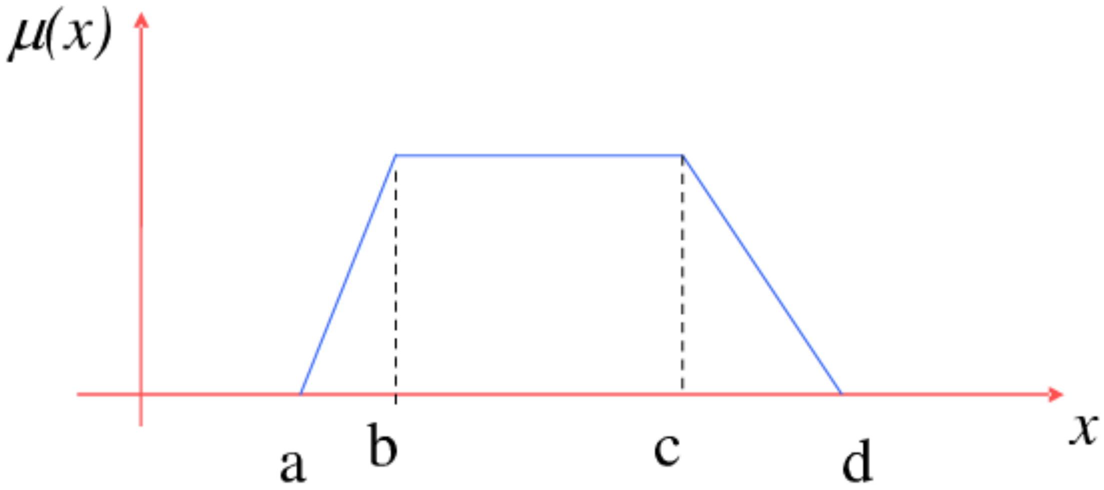
Trapezoidal Functions

- Easy to implement, allowing representation of very complex fuzzy sets.
- May be generated from 4 real values.



Trapezoidal Functions

- May be generated from 4 real values.

$$trap(x:a,b,c,d)=\begin{cases}0 & x<a\\ \dfrac{x-a}{b-a} & a\le x<b\\ 1 & b\le x<c\\ \dfrac{d-x}{d-c} & c\le x<d\\ 0 & x\ge d\end{cases}$$

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Support of Fuzzy Sets

- The **support of a fuzzy set** *A*, defined in the universe of discourse *X*, is the classical set defined as

$$S_A = \{x \in X \mid \mu_A(x) > 0\}$$

Empty Fuzzy Set

- A fuzzy set (A=Ø) is **empty** if its membership function is zero everywhere in its universe of discourse.

$$A = \varnothing \quad if \;\; \mu_A(x) = 0, \forall x \in X$$

An empty fuzzy set has an empty support

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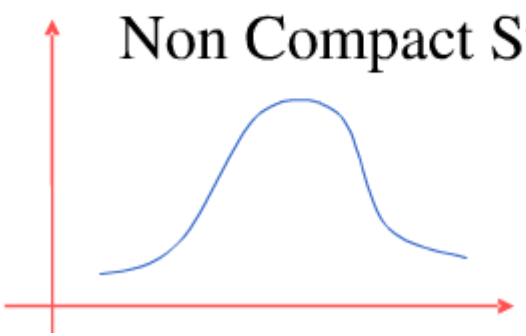
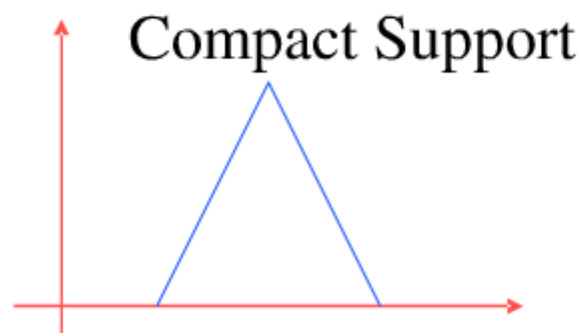
Compact Support

- **The support is compact when the set is smaller than the size of the universe of discourse.**

- **If the support were not compact then several rules would be activated at every input causing an increase in the system load.**





Alpha-cuts Sets

- The **classical set A_α, called alpha-cut set**, is the set of elements whose degree of membership in *A* is no less than α. It is defined as:

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$$

- The **classical set A'_α is called strong alpha-cut** set. It is defined as:

$$A_\alpha = \{x \in X \mid \mu_A(x) > \alpha\}$$

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Level Set

- The set of all levels alpha that represent distinct alpha-cuts of a given fuzzy set *A* is called a level set of *A*.

$$\Lambda_A = \{\alpha \mid \mu_A(x) = \alpha \ \ for\ some\ x \in X\}$$

$Ex.$

$$\Lambda_{young} = \{0.2, 0.4, 0.6, 0.8, 1.0\}$$

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Resolution Principle

- The membership function of a fuzzy set *A* can be expressed in terms of its alpha cuts as (+ is union):

$$\Lambda = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$$

$$A = \alpha_0 \times A_{\alpha_0} + \alpha_1 \times A_{\alpha_1} + \cdots + \alpha_n \times A_{\alpha_n}$$

$$\mu_{\alpha_0 \times A_{\alpha 0}} = \begin{cases} \alpha_i & if\ \mu_A(x) \geq \alpha_i \\ 0 & otherwise \end{cases}$$

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Cardinality

- The cardinality $|A|$ of a fuzzy set A is defined as

$$| A | = \sum_{x \in X} \mu(x)$$

- The relative cardinality of A is defined as

$$\|A\| = \frac{|A|}{|X|}$$

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Cardinality – cont.

- When the fuzzy set is continuous the cardinality is defined as

$$|A| = \int_{x} \mu_A(x)dx$$

Height of a fuzzy set

- The **height** of a fuzzy set is the **highest membership** value of its membership function

$$H_A = \max_{x \in X} \{\mu_A(x)\}$$

- A fuzzy set is defined as normal when H_A=1 and subnormal when it is not.

Fuzzy subsets

- If the membership grade of each element of the fuzzy set A is less than or equal its membership grade in fuzzy set B, then A is called a subset of B.

$$A \subseteq B \quad if \quad \mu_A(x) \leq \mu_B(x) \quad \forall x \in X$$

Equal Fuzzy Sets

- If the membership grade of each element of the fuzzy set A is equal its membership grade in fuzzy set B, then A is equal to B.

$$A = B \quad if \quad \mu_A(x) = \mu_B(x)$$

$$A = B \;\; if \;\; A \subseteq B \;\; and \;\; B \subseteq A$$

Fuzzy Membership Functions

- One of the key issues in all fuzzy sets is how to determine fuzzy membership functions
- The membership function fully defines the fuzzy set
- A membership function provides a measure of the degree of similarity of an element to a fuzzy set
- Membership functions can take any form, but there are some common examples that appear in real applications

- Membership functions can
  - either be chosen by the user arbitrarily, based on the user’s experience (MF chosen by two users could be different depending upon their experiences, perspectives, etc.)
  - Or be designed using machine learning methods (e.g., artificial neural networks, genetic algorithms, etc.)
- There are different shapes of membership functions; triangular, trapezoidal, piecewise-linear, Gaussian, bell-shaped, etc.

Fuzzy Operations
(Fuzzy Union, Intersection, and Complement)

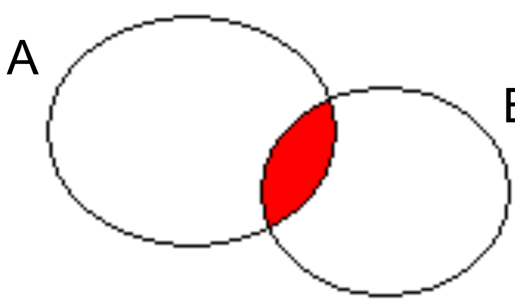
- Fuzzy logic begins by borrowing notions from crisp logic, just as fuzzy set theory borrows from crisp set theory. As in our extension of crisp set theory to fuzzy set theory, our extension of crisp logic to fuzzy logic is made by replacing membership functions of crisp logic with fuzzy membership functions [J.M. Mendel, *Uncertain Rule-Based Fuzzy Logic Systems*, 2001]
- In Fuzzy Logic, intersection, union and complement are defined in terms of their membership functions
- This section concentrates on providing enough of a theoretical base for you to be able to implement computer systems that use fuzzy logic
- Fuzzy intersection and union correspond to 'AND' and 'OR', respectively, in classic/crisp/Boolean logic
- These two operators will become important later as they are the building blocks for us to be able to compute with fuzzy if-then rules

Classic/Crisp/Boolean Logic

- Logical AND (∩)

Truth Table

A	B	A ∩ B
0	0	0
0	1	0
1	0	0
1	1	1



Crisp Intersection

- Logical OR (**U**)

Truth Table

A	B	A **U** B
0	0	0
0	1	1
1	0	1
1	1	1

Crisp Union

Fuzzy Union

- The union (OR) is calculated using t-conorms
- t-conorm operator is a function s(.,.)
- Its features are
  - s(1,1) = 1, s(a,0) = s(0,a) = a (boundary)
  - s(a,b) ≤ s(c,d) if a ≤ c and b ≤ d (monotonicity)
  - s(a,b) = s(b,a) (commutativity)
  - s(a,s(b,c)) = s(s(a,b),c) (associativity)
- The most commonly used method for fuzzy union is to take the maximum. That is, given two fuzzy sets *A* and *B* with membership functions $\mu_A(x)$ and $\mu_B(x)$

$$\mu_{AUB}(x) = \max(\mu_A(x), \mu_B(x))$$

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Fuzzy Intersection

- The intersection (AND) is calculated using t-norms.
- t-norm operator is a function t(.,.)
- Its features
  - t(0,0) = 0, t(a,1) = t(1,a) = a (boundary)
  - t(a,b) ≤ t(c,d) if a ≤ c and b ≤ d (monotonicity)
  - t(a,b) = t(b,a) (commutativity)
  - t(a, t(b,c)) = t(t(a,b),c) (associativity)
- The most commonly adopted t-norm is the minimum. That is, given two fuzzy sets *A* and *B* with membership functions $\mu_A(x)$ and $\mu_B(x)$

$$\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$$

Fuzzy Complement

- To be able to develop fuzzy systems we also have to deal with NOT or complement.
- This is the same in fuzzy logic as for Boolean logic
- For a fuzzy set A, $\overline{A}$ denotes the fuzzy complement of A
- Membership function for fuzzy complement is

$$\mu_{\overline{A}}(x) = 1 - \mu_A(x)$$

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Product of Two Fuzzy Sets

Product of two fuzzy sets A and B defined on the same universe of discourse X is a new fuzzy set A·B, with membership function that equals to the algebraic product of the membership function of A and B:

$$\mu_{A \cdot B}(x) \equiv \mu_A(x) \cdot \mu_B(x)$$

Sum of Two Fuzzy Sets

Summing of two fuzzy (a.k.a Probabilistic sum) sets A and B defined on the same universe of discourse X is a new fuzzy set (A+B) -A·B, with membership function that equals to the algebraic sum of the membership function of A and B:

$$\mu_{A+B}(x) \equiv \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)$$

Linguistics Hedges and Operators

- A fuzzy set can be regarded as corresponding to a linguistic value such as “tall”, and a linguistic variable “height” can be regarded as ranging over such linguistic values.

- One powerful aspect of fuzzy sets in this context is the ability to deal with linguistic quantifiers or “hedges”.

- Hedges such as more or **“less”, “very”, “not very”, “slightly”** etc correspond to modifications in the membership function of the fuzzy set involved.

- The fuzzy set operations such CON, DIL, INT etc (see the table of hedges and Operators) can be used to modify the fuzzy set.

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Hedges and Operators

Table of Hedges and Operators

Hedge	Operator definition
Very F	$CON=F^{2}$
More or Less F	$DIL=F^{0.5}$
Plus F	$F^{1.25}$
Not F	1-F
Not very F	1-CON(F)
Slightly F	INT [Plus F AND Not (Very F)]

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