Introduction to Game Theory:

Cooperative Game Theory

Version 10/29/17

The Two Branches of Game Theory

In **non-cooperative game theory**, a game model is a detailed description of all the moves available to the players (the matrix or the tree)

In **cooperative game theory**, a game model abstracts away from this level of detail and describes only the outcomes that result when players come together in different combinations

The terms are misleading!

Non-cooperative theory can study cooperation --- e.g., in the theory of repeated games

Cooperative theory can study competition --- e.g., in the theory of the core

Better (but non-standard) terms would be **procedural game theory** and **combinatorial game theory**

Another Way to Say It …

Non-cooperative theory studies **individual action** focused on **individual interests**

Cooperative theory studies **joint action** focused on **joint interests**

But it is not useful to spend too long on interpretation at this stage

Let's see some cooperative theory in action …

http://www.lifehack.org/articles/productivity/25-awesome-board-games-that-will-make-you-smarter-and-more-creative.html

Definition of a Cooperative Game

A **cooperative game** consists of

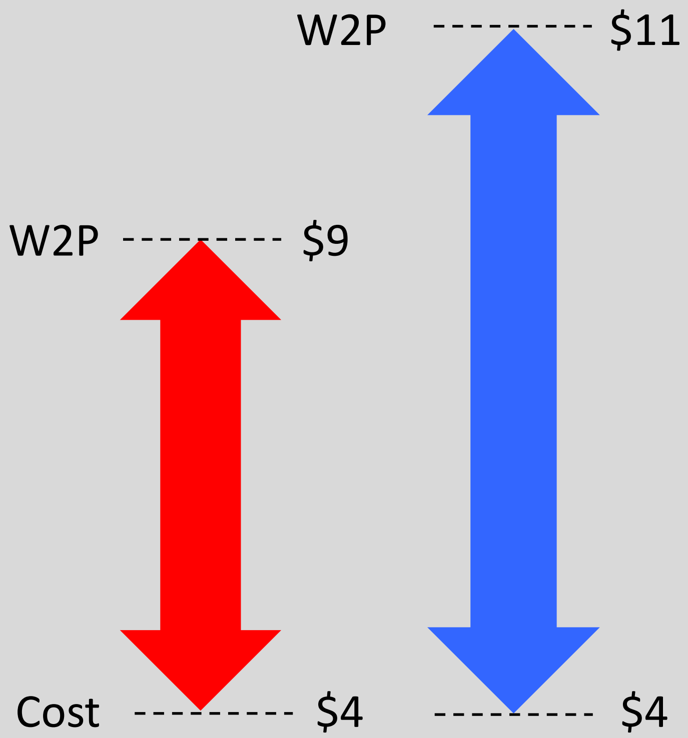
a **set of players** $N = \{1, 2, \ldots, n\}$

a **characteristic function** $v: 2^N \to \mathbb{R}$

where 2^N denotes the set of all subsets of N and $\mathbb{R}$ denotes the real numbers

For each subset S of N the number $v(S)$ is interpreted as the value created when the members of S come together and interact

Cooperative Games: Example #1



Player 1 is a seller with one unit to sell (cost $4)

Player 2 is a buyer interested in one unit (willingness-to-pay $9)

Player 3 is a buyer interested in one unit (willingness-to-pay $11)

$N = \{1,2,3\}$
$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$
$v(\{1,2\}) = 5, v\{(1,3\}) = 7, v(\{2,3\}) = 0$
$v(\{1,2,3\}) = 7$

Division of Value

Given a cooperative game (N, v), the quantity $v(N)$ specifies the overall amount of value created

We can then ask how this overall value is divided up among the various players

Intuition says that bargaining among the players in the game determines the division of overall value

Intuition also says that a player's 'power' in this bargaining depends on the extent to which the player needs other players to create value, as compared with the extent to which other players need this player

Marginal Contribution

Given the set of players N and a particular player i, let $N\setminus\{i\}$ denote the subset of N consisting of all the players except player i

The **marginal contribution** of player i is $v(N) - v(N\setminus\{i\})$, to be denoted by MC_i

In words, the marginal contribution of a particular player is the amount by which the overall value created would change if the player in question were to leave the game

Example #1 cont'd: $\mathrm{MC}_1 = ?, \mathrm{MC}_2 = ?, \mathrm{MC}_3 = ?$

A Marginal Contribution Principle

An **allocation** is a collection $(x_1, x_2, \dots, x_n)$ of numbers

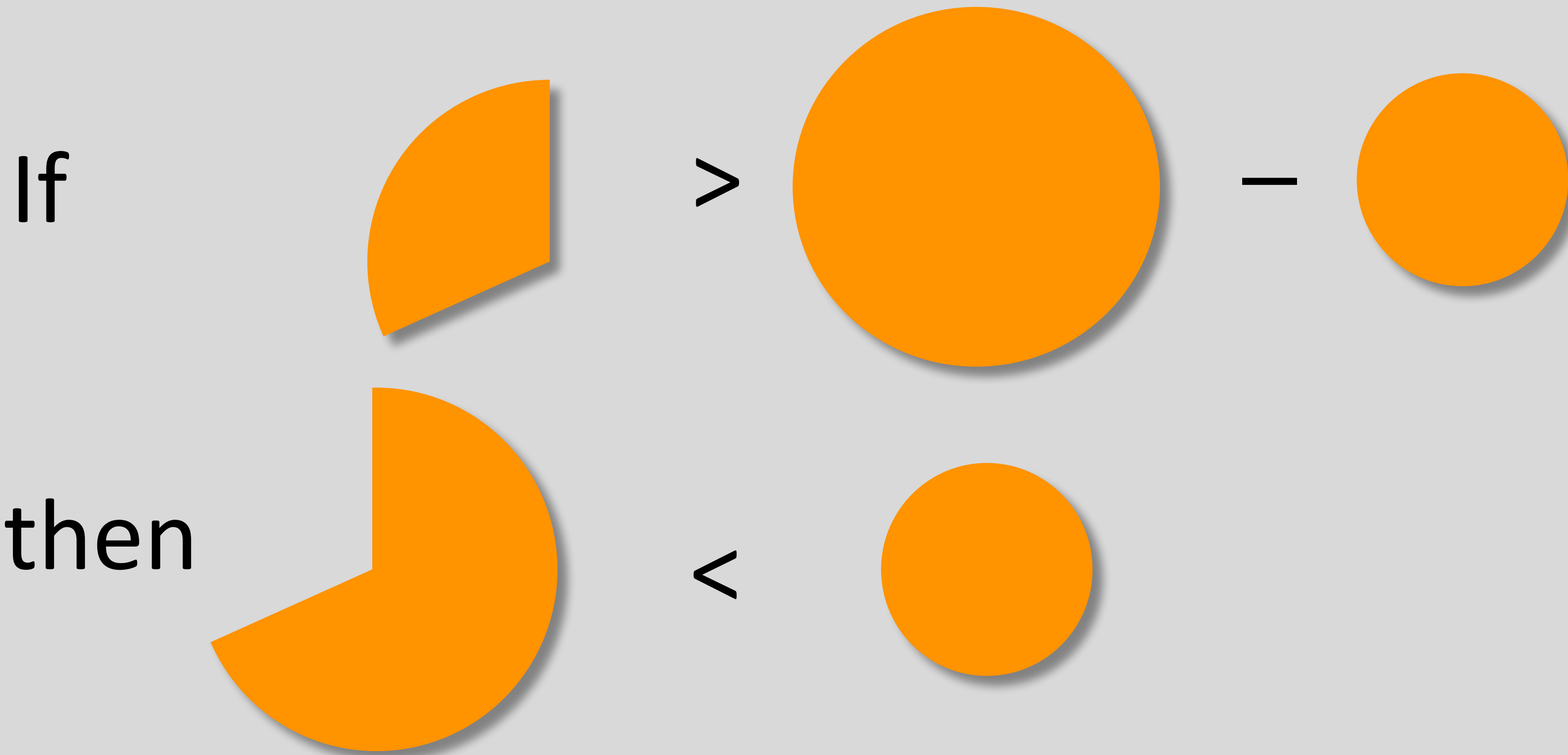
Here, the quantity x_i denotes the value received by player i

An allocation $(x_1, x_2, \dots, x_n)$ is **individually rational** if $x_i \geq v(\{i\})$ for all i

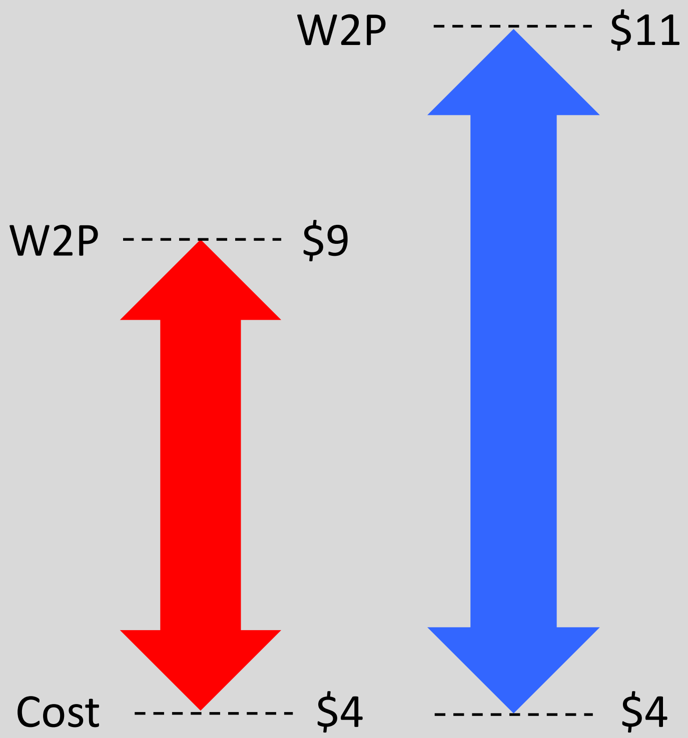
An allocation $(x_1, x_2, \dots, x_n)$ is **efficient** if $\sum_{i=1}^{n} x_i = v(N)$

An (individually rational and efficient) allocation $(x_1, x_2, \dots, x_n)$ satisfies the **Marginal Contribution Principle** if $x_i \leq \mathrm{MC}_i$ for all i

Argument for this Marginal Contribution Principle



Example #1 cont'd



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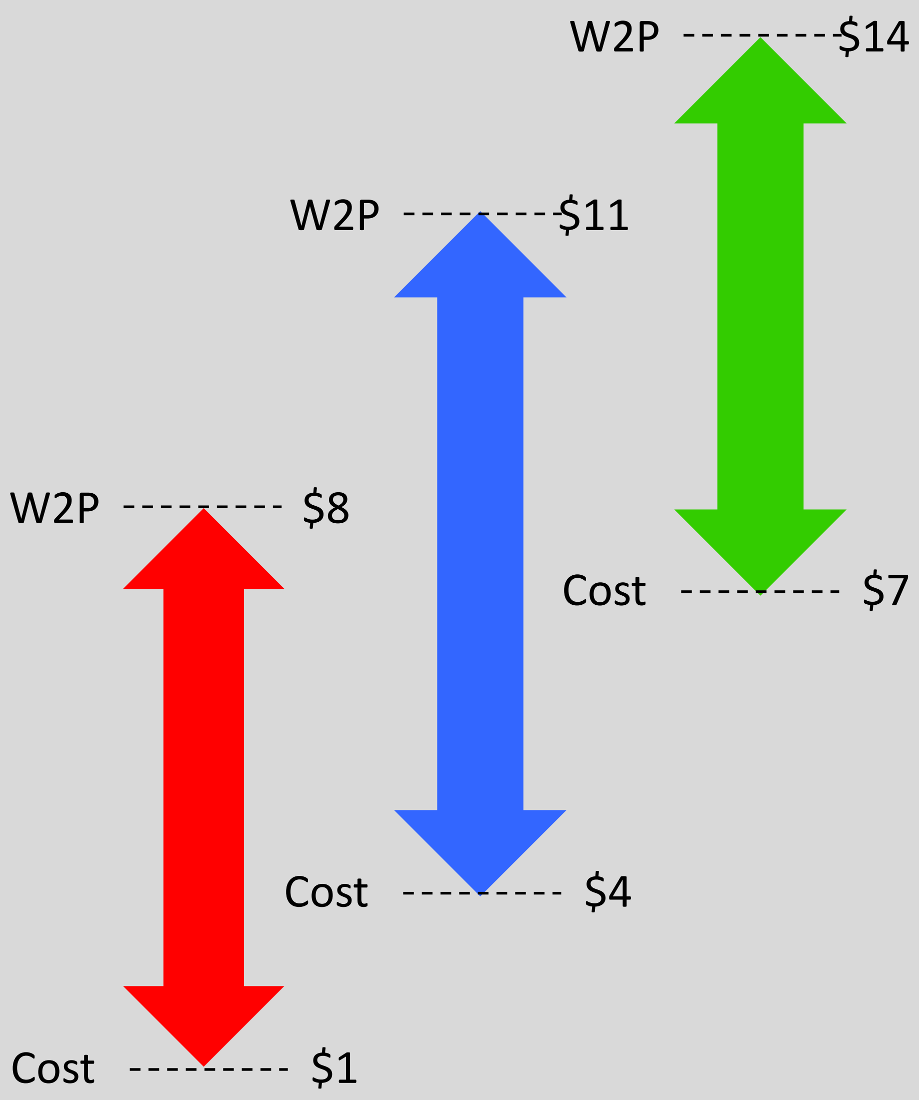
What does the Marginal Contribution Principle say about how the overall value of $7 gets divided among the players?

Cooperative Games: Example #2

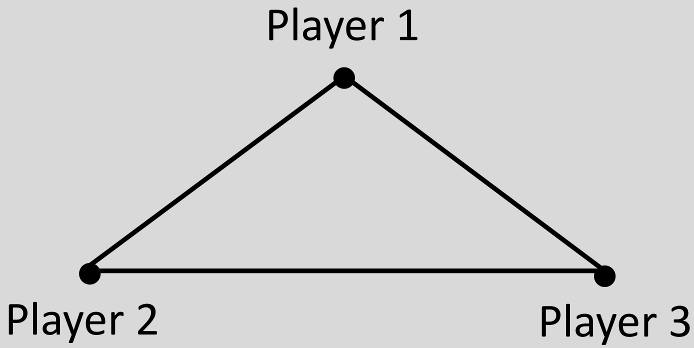


There are three firms, each with one unit to sell

There are two identical buyers, each interested in one unit of product from some firm

The blue firm can spend $1 to raise W2P to $12 and lower Cost to $3

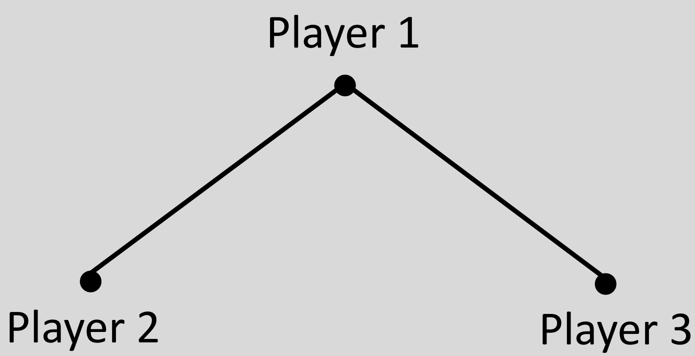
An Application: Game-Theoretic Analysis of Hierarchy



$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$$
$$v(\{1,2\}) = v\{(1,3\}) = v(\{2,3\}) = 3$$
$$v(\{1,2,3\}) = 4$$

What divisions of the overall value satisfy the Marginal Contribution Principle?

Now, let's impose a **hierarchy**, by which we mean that players 2 and 3 cannot interact (no superadditivity!) without player 1's involvement



$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$$
$$v(\{1,2\}) = v\{(1,3\}) = 3$$
$$v(\{2,3\}) = v(\{2\}) + v(\{3\}) = 0$$
$$v(\{1,2,3\}) = 4$$

What divisions of the overall value satisfy the MCP now?

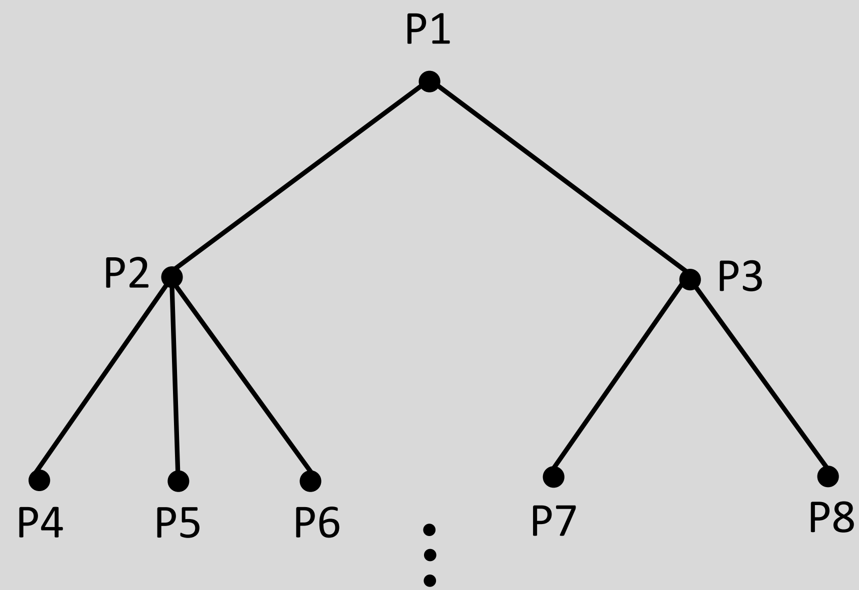
Game-Theoretic Analysis of Hierarchy cont'd

We see that hierarchy can create stability by allocating power

(But hierarchy would be costly if players 2 and 3 could create a lot of value together)

How general is this stability effect?

Theorem: For a cooperative game defined on a finite tree, there is always an allocation satisfying the Marginal Contribution Principle



Method of proof: Give each player its marginal contribution to the subtree starting at its node

A stronger stability property is true: There is always an allocation lying in the **core**

 Demange, G., "On Group Stability in Hierarchies and Networks," *Journal of Political Economy*, 112, 2004, 754-778; Brandenburger, A., H.J. Keisler, and P. Miret, "Cooperative Games on Infinite Trees," August 2017

Diagram from https://commons.wikimedia.org/wiki/File:Social_order_continuum.png, author Cameroncrazies; photo of spotted hyena in Sabi Sabi Game Reserve, South Africa, August 2017 (by Adam Brandenburger)

The Core

An allocation $(x_1, x_2, \dots, x_n)$ is in the **core** of the game if it is efficient and is such that for every subset S of N we have

$$\sum_{i \in S} x_i \geq v(S)$$

The **marginal contribution** of subset S of N is $v(N) - v(N \backslash S)$, to be denoted by MC_S

Theorem: An efficient allocation $(x_1, x_2, \dots, x_n)$ lies in the core if and only if for every subset S of N we have

$$\sum_{i \in S} x_i \leq \mathrm{MC}_S$$

This shows that the core is a strengthening of the Marginal Contribution Principle

Example #3: There are two sellers, each with two units to sell where Cost = \$0. There are three buyers, each interested in buying one unit where W2P = \$1 for either seller's product.

Example adapted from Postlewaite, A., and R. Rosenthal, "Disadvantageous Syndicates," *Journal of Economic Theory*, 9, 1974, 324-326