Introduction to Ito's Lemma

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Overview

1. Background

2. Ito Processes

3. Ito's Lemma

Background

- Proved by Kiyoshi Ito *(not Ito's theorem on group theory by Noboru Ito)*
- Used in Ito's calculus, which extends the methods of calculus to stochastic processes
- Applications in mathematical finance e.g. derivation of the Black-Scholes equation for option values

Ito Processes

Question

Want to model the dynamics of process $X(t)$ driven by Brownian motion $W(t)$.

Ito Processes: Discrete-time Construction

- Partition time interval [0,T] into N periods, each of length $\Delta t = \frac{T}{N}$; $t_n = n\Delta t$
- $X_{t_{n+1}} = X_{t_n} + \mu_{t_n}\Delta t + \sigma_{t_n}\Delta W_{t_n}$
  - drift μ
  - volatility σ
  - fluctuations $\Delta W_{t_n} = W_{t_{n+1}} - W_{t_n} \sim N(0, \Delta t)$

Ito Processes: Discrete-time Construction

- Summing the increments,

$$X_T = X_0 + \sum_{n=0}^{N-1} \mu_{t_n} \Delta t + \sum_{n=0}^{N-1} \sigma_{t_n} \Delta W_{t_n}$$

- Continuous-time analogue as $N \to \infty$,

$$X_T \stackrel{?}{\to} X_0 + \int_0^T \mu_t \mathrm{d}t + \int_0^T \sigma_t \mathrm{d}W_t$$

Ito Processes: Discrete-time Construction

Regularity conditions for μ_t and σ_t

- adapted to F_t^W
- continuous in t
- integrability conditions ($\sigma_t \in \mathcal{M}^2$ i.e. $\mathrm{E}\left(\int_0^T \sigma_t^2 \mathrm{d}t\right) < \infty$)

Riemann-Stieltjes integral

$$\sum_{n=0}^{N-1} \mu_{t_n} \Delta t \to \int_0^T \mu_t \mathrm{d}t$$

Ito integral

$$\sum_{n=0}^{N-1} \sigma_{t_n} \Delta W_{t_n} \xrightarrow{\mathcal{L}^2} \int_0^T \sigma_t \mathrm{d}W_t$$

Ito Integrals

Question

What is $\int_0^T \sigma_t \mathrm{d} W_t$?

Ito Integrals

Theorem (Existence and Uniqueness of Ito Integral)

Suppose that $v_t \in \mathcal{M}^2$ satisfies the following: For all $t \geq 0$,

A1) v_t is a.s. continuous

A2) v_t is adapted to F_t^W

Then, for any $T > 0$, the Ito integral $I_T(v) = \int_0^T v_t \mathrm{d}W_t$ exists and is unique a.e.

Steps for proof

1. Construct a sequence of adapted stochastic processes v_n such that
$$\|v - v_n\|_{\mathcal{M}^2} = \sqrt{E\left(\int_0^T |v_n(t) - v(t)|^2 \mathrm{d}t\right)} \to 0$$
2. Show that $\|I_T(v_n) - I_T(v)\|_{\mathcal{L}^2} \to 0$
3. Show the a.s. uniqueness of the limit $I_T(v)$

Ito Integrals: Example

Example (Ito Integral)

$$\int_0^T W_t \mathrm{d} W_t \quad \text{with approximating sums} \quad \sum_{n=0}^{N-1} W_{t_n} \Delta W_{t_n}$$

$$\begin{aligned}
\sum_{n=0}^{N-1} W_{t_n}(W_{t_{n+1}} - W_{t_n}) &= \sum_{n=0}^{N-1} \left[\frac{1}{2}(W_{t_{n+1}}^2 - W_{t_n}^2) - \frac{1}{2}(W_{t_{n+1}} - W_{t_n})^2 \right] \\
&= \frac{1}{2} W_T^2 - \frac{1}{2} \sum_{n=0}^{N-1} (W_{t_{n+1}} - W_{t_n})^2 \\
&\xrightarrow{\mathcal{L}^2} \frac{1}{2} W_T^2 - \frac{1}{2} T \quad \text{as } N \to \infty
\end{aligned}$$

Ito Integrals: Example

Example (Riemann-Stieltjes Integral)

$$\int_0^T G_t \mathrm{d} G_t \quad \text{with} \quad G \in \mathcal{C}^1, G(0) = 0$$

$$\begin{aligned} \int_0^T G_t \mathrm{d} G_t &= \int_0^T G_t G_t' \mathrm{d} t \\ &= \frac{1}{2} G_T^2 \end{aligned}$$

Ito Integral: Properties

- Linear in the integrand
- Time-additive
- Martingale

  Proof

  For $t < T$, increase the partition by an extra point $t_k = t$.

$$
\begin{aligned}
E[I_T(v_n) - I_t(v_n)|\mathcal{F}_t] &= E\left[\sum_{n=k}^{N-1} \sigma_{t_n} \Delta W_{t_n} | \mathcal{F}_t\right] \\
&= E\left[\sum_{n=k}^{N-1} E(\sigma_{t_n} \Delta W_{t_n} | \mathcal{F}_{t_n}) | \mathcal{F}_t\right] \\
&= E\left[\sum_{n=k}^{N-1} \sigma_{t_n} E(\Delta W_{t_n} | \mathcal{F}_{t_n}) | \mathcal{F}_t\right] \\
&= 0
\end{aligned}
$$

  $I_T(v_n)$ is a martingale. Martingales are preserved under $\mathcal{L}^2$ limits.

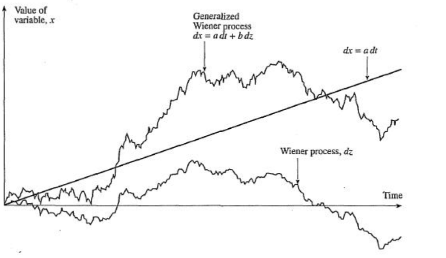
Ito Processes

$$X_t - X_0 = \int_0^t \mu_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s$$

SDE notation:

$$\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t$$

Figure 12.2 Generalized Wiener process with $a = 0.3$ and $b = 1.5$.



Ito's Lemma

Theorem (Ito's Lemma)

Suppose that $f \in \mathcal{C}^2$. Then with probability one, for all $t \geq 0$,

$$\mathrm{d}f(X_t) = \frac{\partial f}{\partial x}(X_t)\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(X_t)(\mathrm{d}X_t)^2$$

$$f(X_t) - f(X_0) = \int_0^t f'(X_s)\mathrm{d}X_s + \frac{1}{2}\int_0^t f''(X_s)\mathrm{d}s$$

Explicit statement:

$$\mathrm{d}f(X_t) = \left(\mu_t \frac{\partial f}{\partial x}(X_t) + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2}(X_t)\right)\mathrm{d}t + \sigma_t \frac{\partial f}{\partial x}(X_t)\mathrm{d}W_t$$

Ito's Lemma: Idea

- Can be obtained heuristically by second order Taylor expansion of f about X_t
- $(\mathrm{d}X_t)^2 = (\mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t)^2$ term cannot be dropped
    - $(\mathrm{d}W_t)^2 = \mathrm{d}t$
- drop terms $\ll \mathrm{d}t$

$$
\begin{aligned}
\int_0^T (\mathrm{d}t)^p &= \lim_{N\to\infty} \sum_{n=0}^{N-1} (\Delta t)^p \\
&= \lim_{N\to\infty} N \left(\frac{T}{N}\right)^p \\
&= 0 \quad \text{as } N \to \infty \text{ if } p > 1
\end{aligned}
$$

Ito's Lemma: Idea

$$(\mathrm{d}W_t)^2 = \mathrm{d}t \quad \text{since} \quad \int_0^T (\mathrm{d}W_t)^2 = \lim_{N\to\infty} \sum_{n=0}^{N-1} (\Delta W_{t_n})^2 \stackrel{\mathcal{L}^2}{=} T = \int_0^T \mathrm{d}t$$

$$E\left[\sum_{n=0}^{N-1} (\Delta W_{t_n})^2\right] = \sum_{n=0}^{N-1} E(\Delta W_{t_n})^2 = \sum_{n=0}^{N-1} \Delta t = T$$

$$\begin{aligned} E\left[\sum_{n=0}^{N-1} (\Delta W_{t_n})^2 - T\right]^2 &= Var\left[\sum_{n=0}^{N-1} (\Delta W_{t_n})^2\right] = \sum_{n=0}^{N-1} Var(\Delta W_{t_n})^2 \\ &= \sum_{n=0}^{N-1} (E(\Delta W)^4 - [E(\Delta W)^2]^2) = \sum_{n=0}^{N-1} (3(\Delta t)^2 - (\Delta t)^2) \\ &= \frac{2T^2}{N} \to 0 \quad \text{as } N \to \infty \end{aligned}$$

Ito's Lemma: More details

More rigorously, for bounded continuous g,

$$\sum_{n=0}^{N-1} g(W_{t_n})(\Delta W_{t_n})^2 \xrightarrow{\mathcal{L}^2} \int_0^T g(W_t)\mathrm{d}t \quad \text{as } N \to \infty$$

<u>Proof</u>

Since $t \mapsto g(W_t)$ is a.s. continuous, $\sum_{n=0}^{N-1} g(W_{t_n})\Delta t \to \int_0^T g(W_t)\mathrm{d}t$.

WTS:

$$I_N = \sum_{n=0}^{N-1} g(W_{t_n}) \left[(\Delta W_{t_n})^2 - \Delta t\right] \xrightarrow{\mathcal{L}^2} 0$$

Ito's Lemma: More details

1. $E(I_N^2) = E\left\{\sum_{n=0}^{N-1}(g(W_{t_n}))^2[(\Delta W_{t_n})^2 - \Delta t]^2\right\}$

$$
\begin{aligned}
& E\left\{g(W_{t_n})g(W_{t_m})[(\Delta W_{t_n})^2 - \Delta t][(\Delta W_{t_m})^2 - \Delta t]\right\} \\
& = E\left\{E\left(g(W_{t_n})g(W_{t_m})[(\Delta W_{t_n})^2 - \Delta t][(\Delta W_{t_m})^2 - \Delta t]|\mathcal{F}_{t_m}\right)\right\} \\
& = E\left\{g(W_{t_n})g(W_{t_m})[(\Delta W_{t_n})^2 - \Delta t]E\left[(\Delta W_{t_m})^2 - \Delta t|\mathcal{F}_{t_m}\right]\right\} \\
& = 0 \quad \text{since } \{W_t^2 - t\} \text{ is martingale}
\end{aligned}
$$

2. $E(I_N^2) \leq \frac{CT^2}{N}$

$$
\begin{aligned}
E(I_N^2) & \leq \|g\|_\infty^2 \sum_{n=0}^{N-1} E[(\Delta W_{t_n})^2 - \Delta t]^2 \quad \text{since } g \text{ is bounded} \\
& = \|g\|_\infty^2 \sum_{n=0}^{N-1} (\Delta t)^2 E(W_1^2 - 1)^2 \quad \text{since } \Delta W_{t_n} \sim \sqrt{\Delta t} W_1 \\
& = C \sum_{n=0}^{N-1} \left(\frac{T}{N}\right)^2 = \frac{CT^2}{N}
\end{aligned}
$$

Ito's Lemma: Example

Example (Ito's Lemma)

Use Ito's Lemma, write $Z_t = W_t^2$ as a sum of drift and diffusion terms.

$$Z_t = f(X_t) \quad \text{with} \quad \mu_t = 0, \sigma_t = 1, X_0 = 0, f(x) = x^2$$

$$\begin{aligned}
\mathrm{d}Z_t &= \mathrm{d}f(X_t) \\
&= f'(X_t)\mathrm{d}X_t + \frac{1}{2}f''(X_t)(\mathrm{d}X_t)^2 \\
&= 2W_t\mathrm{d}W_t + \frac{1}{2}2(\mathrm{d}W_t)^2 \\
&= 2W_t\mathrm{d}W_t + \mathrm{d}t
\end{aligned}$$

Ito's Lemma: Higher dimensions

Ito's Lemma

If X_t and Y_t are Ito processes and $f : \mathbb{R}^2 \mapsto \mathbb{R}$ is sufficiently smooth, then

$$\begin{aligned} \mathrm{d}f(X_t, Y_t) &= \frac{\partial f}{\partial x}\mathrm{d}X_t + \frac{\partial f}{\partial y}\mathrm{d}Y_t \\ &\quad + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\mathrm{d}X_t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(\mathrm{d}Y_t)^2 + \frac{\partial^2 f}{\partial x \partial y}(\mathrm{d}X_t)(\mathrm{d}Y_t) \end{aligned}$$

References

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