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Introduction To Logarithms

Logarithms were originally developed to simplify complex arithmetic calculations.

They were designed to transform multiplicative processes into additive ones.

If at first this seems like no big deal,
then try multiplying
2,234,459,912 and 3,456,234,459.

Without a calculator !

Clearly, it is a lot easier to add
these two numbers.

Today of course we have calculators and scientific notation to deal with such large numbers.

So at first glance, it would seem that logarithms have become obsolete.

Indeed, they would be obsolete except for one very important property of logarithms.

It is called
the power property and we
will learn about it in another lesson.

For now we need only to observe that
it is an extremely important part
of solving exponential equations.

Our first job is to try to make some sense of logarithms.

Our first question then must be:

What is a logarithm ?

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Of course logarithms have a precise mathematical definition just like all terms in mathematics. So let’s start with that.

Definition of Logarithm

Suppose b>0 and b≠1,
there is a number 'p'
such that:

$$\log_b n = p \quad if \quad and \quad only \quad if \quad b^p = n$$

Now a mathematician understands exactly what that means.

But, many a student is left scratching their head.

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The first, and perhaps the most important step, in understanding logarithms is to realize that they always relate back to exponential equations.

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You must be able to convert an exponential equation into logarithmic form and vice versa.

So let’s get a lot of practice with this !

Example 1:

Write $2^3 = 8$ in logarithmic form.

Solution: $\log_2 8 = 3$

We read this as: ”the log base 2 of 8 is equal to 3”.

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Example 1a:

$$Write\ \ 4^2 = 16\ \ in\ \ \text{logarithmic}\ \ form.$$

Solution: $\log_4 16 = 2$

Read as: “the log base 4 of 16 is equal to 2”.

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Example 1b:

$$Write \quad 2^{-3} = \frac{1}{8} \quad in \quad \log arithmic \quad form.$$

Solution: $$\log_2 \frac{1}{8} = -3$$

Read as: "the log base 2 of $\frac{1}{8}$ is equal to -3".

Okay, so now it's time for you to try some on your own.

1. $Write \;\; 7^2 = 49 \;\; in \;\; \log arithmic \;\; form.$

Solution: $\log_7 49 = 2$

2. *Write* $5^0 = 1$ *in logarithmic form.*

Solution: $\log_5 1 = 0$

3. *Write* $10^{-2} = \frac{1}{100}$ *in* log*arithmic form.*

Solution: $\log_{10} \frac{1}{100} = -2$

4. *Finally, write* $16^{\frac{1}{2}} = 4$ *in logarithmic form.*

Solution: $$\log_{16} 4 = \frac{1}{2}$$

It is also very important to be able to start with a logarithmic expression and change this into exponential form.

This is simply the reverse of what we just did.

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Example 1:

$$Write \quad \log_3 81 = 4 \quad in \quad \exp onential \quad form$$

Solution: $3^4 = 81$

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Example 2:

$$Write\ \ \log_2 \frac{1}{8} = -3\ \ in\ \ \exp onential\ \ form.$$

Solution: $$2^{-3} = \frac{1}{8}$$

Okay, now you try these next three.

1. $Write\ \ \log_{10} 100 = 2\ \ in\ \ \exp onential\ \ form.$

2. $Write\ \ \log_5 \dfrac{1}{125} = -3\ \ in\ \ \exp onential\ \ form.$

3. $Write\ \ \log_{27} 3 = \dfrac{1}{3}\ \ in\ \ \exp onential\ \ form.$

1. *Write* $\log_{10} 100 = 2$ *in exponential form.*

Solution: $10^2 = 100$

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2. *Write* $\log_5 \frac{1}{125} = -3$ *in exponential form.*

Solution: $5^{-3} = \frac{1}{125}$

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3. *Write* $\log_{27} 3 = \frac{1}{3}$ *in exponential form.*

Solution: $27^{\frac{1}{3}} = 3$

We now know that a logarithm is
perhaps best understood
as being
closely related to an
exponential equation.

In fact, whenever we get stuck
in the problems that follow
we will return to
this one simple insight.

We might even state a
simple rule.

When working with logarithms,
if ever you get “stuck”, try
rewriting the problem in
exponential form.

Conversely, when working
with exponential expressions,
if ever you get “stuck”, try
rewriting the problem
in logarithmic form.

Let’s see if this simple
rule
can help us solve some
of the following
problems.

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Example 1

Solve for x: $\log_6 x = 2$

Solution:

Let's rewrite the problem in exponential form.

$$6^2 = x$$

We're finished !

Example 2

Solve for y: $\log_5 \frac{1}{25} = y$

Solution: Rewrite the problem in exponential form.

$$5^y = \frac{1}{25} \qquad Since \left(\frac{1}{25} = 5^{-2}\right)$$

$$5^y = 5^{-2}$$

$$y = -2$$

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Example 3

$$\textit{Evaluate } \log_3 27.$$

Solution:

Try setting this up like this:

$$\log_3 27 = y$$

Now rewrite in exponential form.

$$3^y = 27$$

$$3^y = 3^3$$

$$y = 3$$

These next two problems tend to be some of the trickiest to evaluate.

Actually, they are merely identities and
the use of our simple rule
will show this.

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Example 4

$$Evaluate: \quad \log_7 7^2$$

Solution:

$$\log_7 7^2 = y$$

First, we write the problem with a variable.

$$7^y = 7^2$$

Now take it out of the logarithmic form and write it in exponential form.

$$y = 2$$

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Example 5

$$Evaluate: \quad 4^{\log_4 16}$$

Solution:

$$4^{\log_4 16} = y$$

First, we write the problem with a variable.

$$\log_4 y = \log_4 16$$

Now take it out of the exponential form and write it in logarithmic form.

$Just \quad like \quad 2^3 = 8 \quad converts \quad to \quad \log_2 8 = 3$

$$y = 16$$

Ask your teacher about the last two examples.

They may show you a nice shortcut.

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Finally, we want to take a look at the Property of Equality for Logarithmic Functions.

$$\text{Suppose} \quad b > 0 \quad \text{and} \quad b \neq 1.$$

$$\text{Then} \quad \log_b x_1 = \log_b x_2 \quad \text{if and only if} \quad x_1 = x_2$$

Basically, with logarithmic functions,
if the bases match on both sides of the equal sign , then simply set the arguments equal.

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Example 1

$$Solve: \quad \log_3(4x+10) = \log_3(x+1)$$

Solution:

Since the bases are both ‘3’ we simply set the arguments equal.

$$4x+10 = x+1$$
$$3x+10 = 1$$
$$3x = -9$$
$$x = -3$$

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Example 2

$$Solve: \quad \log_8(x^2 - 14) = \log_8(5x)$$

Solution:

Since the bases are both ‘8’ we simply set the arguments equal.

$$x^2 - 14 = 5x$$

$$x^2 - 5x - 14 = 0$$ Factor

$$(x-7)(x+2) = 0$$

$$(x-7) = 0 \quad or \quad (x+2) = 0$$

$$x = 7 \quad or \quad x = -2$$

continued on the next page

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Example 2 continued

$$Solve: \quad \log_8(x^2 - 14) = \log_8(5x)$$

Solution:

$$x = 7 \quad or \quad x = -2$$

It appears that we have 2 solutions here.
If we take a closer look at the definition of a logarithm however, we will see that not only must we use positive bases, but also we see that the arguments must be positive as well. Therefore -2 is not a solution.
Let's end this lesson by taking a closer look at this.

Our final concern then is to determine why logarithms like the one below are undefined.

$$\log_2(-8)$$

Can anyone give us an explanation ?

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$$\log_2(-8) = \text{ undefined} \qquad \text{WHY?}$$

One easy explanation is to simply rewrite this logarithm in exponential form.
We'll then see why a negative value is not permitted.

$$\log_2(-8) = y$$

First, we write the problem with a variable.

$$2^y = -8$$

Now take it out of the logarithmic form and write it in exponential form.

What power of 2 would gives us -8 ?

$$2^3 = 8 \quad and \quad 2^{-3} = \frac{1}{8}$$

Hence expressions of this type are undefined.

That concludes our introduction to logarithms. In the lessons to follow we will learn some important properties of logarithms.

One of these properties will give us a very important tool which we need to solve exponential equations. Until then let's practice with the basic themes of this lesson.