Introduction to Mathematical Modeling

Difference Equations, Differential Equations, & Linear Algebra

(The First Course of a Two-Semester Sequence)

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Chapter 0

To the Student and the Instructor

This document contains lecture notes, classroom activities, examples, and challenge problems specifically designed for a first semester of differential equations and linear algebra taught with a focus on mathematical modeling. The content herein is written and maintained by Dr. Eric Sullivan of Carroll College. Problems were either created by Dr. Sullivan, the Carroll Mathematics Department faculty, part of NSF Project Mathquest, part of the Active Calculus text, or come from other sources and are either cited directly or cited in the LaTeX source code for the document (and are hence purposefully invisible to the student).

0.1 An Inquiry Based Approach

Problem 0.1 (Setting The Stage)**.**

- Get in groups of size 3-4.
- Group members should introduce themselves.
- For each of the questions that follow I will ask you to:
  1. **Think** about a possible answer on your own
  2. **Discuss** your answers with the rest of the group
  3. **Share** a summary of each group's discussion

Questions:

Question #1: What are the goals of a university education?

Question #2: How does a person learn something new?

Question #3: What do you reasonably expect to remember from your courses in 20 years?

Question #4: What is the value of making mistakes in the learning process?

Question #5: How do we create a safe environment where risk taking is encouraged and productive failure is valued?

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(The previous problem is inspired by Dana Ernst's first day activity in IBL activity titled: Setting the Stage.)

> "Any creative endeavor is built in the ash heap of failure."
> –Michael Starbird

This material is written with an Inquiry-Based Learning (IBL) flavor. In that sense, this document could be used as a stand-alone set of materials for the course but these notes are not a *traditional textbook* containing all of the expected theorems, proofs, examples, and exposition. The students are encouraged to work through problems and homework, present their findings, and work together when appropriate. You will find that this document contains collections of problems with only minimal interweaving exposition. It is expected that you do every one of the problems and then use other more traditional texts as a backup when you are stuck. Let me say that again: this is not the only set of material for the course. Your brain, your peers, and the books linked in the next section are your best resources when you are stuck.

To learn more about IBL go to http://www.inquirybasedlearning.org/about/. The long and short of it is that the students in the class are the ones that are doing the work; building models, proving theorems, writing code, working problems, leading discussions, and pushing the pace. The instructor acts as a guide who only steps in to redirect conversations or to provide necessary insight. If you are a student using this material you have the following jobs:

1. Fight! You will have to fight hard to work through this material. The fight is exactly what we're after since it is ultimately what leads to innovative thinking.

2. Screw Up! More accurately, don't be afraid to screw up. You should write code, work problems, and prove theorems then be completely unafraid to scrap what you've done and redo it from scratch. Learning this material is most definitely a non-linear path.* Embrace this!

3. Collaborate! You should collaborate with your peers with the following caveats: (a) When you are done collaborating you should go your separate ways. When you write your solution you should have no written (or digital) record of your collaboration. (b) The internet is not a collaborator. Use of the internet to help solve these problems robs you of the most important part of this class; the chance for original thought.

4. Enjoy! Part of the fun of IBL is that you get to experience what it is like to think like a true mathematician / scientist. It takes hard work but ultimately this should be fun!

*Pun intended: our goal, after all, is really to understand that linear algebra is the glue that holds mathematics together.

0.2 Online Texts and Other Resources

If you are looking for online textbooks for linear algebra and differential equations I can point you to a few. Some of the following online resources may be a good place to help you when you're stuck but they will definitely say things a bit differently. Use these resources wisely.

- The book *Differential Equations with Linear Algebra, An inquiry based approach to learning* is a nice collection of notes covering much of the material that we cover in our class. The order is a bit different but the notes are well done.
  content.byui.edu/file/664390b8-e9cc-43a4-9f3c-70362f8b9735/1/316-IBL%20(2013Spring).pdf
- The ODE Project by Thomas Juson is a nice online text that covers many (but not all) of the topics that we cover in differential equations.
  faculty.sfasu.edu/judsontw/ode/html/odeproject.html
- Elementary Differential Equations by William Trench. This book contains everything(!) you would ever want to look up for ordinary differential equations. It is a great resource to look up ODE techniques.
  ramanujan.math.trinity.edu/wtrench/texts/TRENCH_DIFF_EQNS_I.PDF
- A First Course in Linear Algebra by Robert Beezer. This book is very thorough and covers everything that we do in linear algebra and much more.
  linear.ups.edu/fcla/index.html
- Linear Algebra Workbook by TJ Hitchman. This is a workbook for Dr. Hitchman's class at U. Northern Iowa. Even though it is only a "workbook" it contains some nice explanations and it has embedded executable code for some problems.
  theronhitchman.github.io/linear-algebra/course-materials/workbook/LinAlgWorkbook.html

0.3 To the Instructor

If you are an instructor wishing to use these materials then I only ask that you adhere to the Creative Commons license. You are welcome to use, distribute, and remix these materials for your own purposes. Thanks for considering my materials for your course!

My typical use of these materials are to let the students tackle problems in small groups during class time and to intervene when more explanation appears to be necessary or if the students appear to be missing the deeper connections behind problems. The course that I have in mind for these materials is a first semester of differential equations and linear algebra taught from the standpoint of mathematical modeling. As such, this is not a complete collection of materials for either differential equations or linear algebra in isolation. We discuss matrix operations, Gaussian elimination, the eigenvalue problem, first order linear homogeneous and non-homogeneous differential equations, and second order homogeneous differential equations. In the second course we will expand upon these ideas and include more advanced topics.

Many of the theorems in the text come without a proof. If the theorem is followed by the statement "prove the previous theorem" then I expect the students to have the skill to prove that theorem and to do so with the help of their small group. However, this course is not intended to be a proof-based mathematics course so several theorems are stated without rigorous proof. If you are looking for a proof-based linear algebra or differential equations course then I believe that these notes will not suffice. I have, however, tried to give thought provoking problems throughout so that the students can engage with the material at a level higher than just the mechanics of differential equations and linear algebra. There are also several routine exercises throughout the notes that will allow students to practice mechanical skills.

There is a toggle switch in the LaTeX code that allows you to turn on and off the solutions to problems. The line of code

```
\def\ShowSoln{0}
```

is a switch that, when set to 0, turns the solutions off and when set to 1 turns the solutions on. Just re-compile (`pdflatex`) the document to display the solutions. I typically do not show the solutions to the students while they're learning the material.

Chapter 1

Fundamental Notions from Calculus

Welcome to the mathematical modeling class. We'll start the class with a brief review of the most basic ideas and notions from calculus. If you have never taken a Calculus course before then you should consider first taking a formal Calculus course before tackling this class. We only need a few of the main ideas for this class so what you'll find in this chapter is necessarily brief.

1.1 Sections from Active Calculus

The Active Calculus Textbook is a wonderful online resource that stands in place of this chapter. We will only discuss a few select sections and you can find the relevant links below. If you need further reading to brush up on Calculus you should use the Active Calculus text.

1. Active Calculus Section 1.1: How do we measure velocity?
2. Active Calculus Section 1.2: The notion of a limit
3. Active Calculus Section 1.4 The Derivative Function
4. Active Calculus Section 2.1: Elementary derivative rules
5. Active Calculus Section 2.2: The sine and cosine functions
6. Active Calculus Section 2.3: The product and quotient rules
7. Active Calculus Section 2.4: The chain rule
8. Active Calculus Section 4.3: The definite integral
9. Active Calculus Section 4.4: The Fundamental Theorem of Calculus

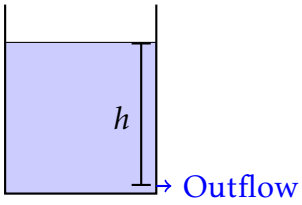
1.2 Modeling Explorations with Calculus

What remains in this chapter of these notes are several laboratory exercises meant to support your review of Calculus and to introduce you to the basic notions of mathematical modeling.

Problem 1.1. Water is being drained from a hole near the bottom of a cylindrical tank. According to Torrecelli's Law it can be shown that the rate at which the height h of the water changes is proportional to the square root of the height. This can be written with average rates of change as:

$$\text{average rate of change of the height} = \frac{\Delta h}{\Delta t} \approx K \cdot \sqrt{h} \tag{1.1}$$

where K is a constant that depends on gravity as well as the size of the hole and the shape of the tank.



(a) Using the video demonstration of Torrecelli's Law found here:
https://www.youtube.com/watch?v=gsNdsuQ1ZCo&app=desktop
and, using the pause button to your advantage, create a table of time vs. the height of the water in the container. Use as many data points as you feel necessary.

Time (sec)	Height (cm)
⋮	⋮

(b) Use your data to estimate the value of K in the experiment shown in the video. Be sure to include a discussion of units of K and discuss which parts of this experiment are being described by K?
(Hint: Given equation (1.1), what should a plot of $\sqrt{h}$ (on the x axis) vs $\frac{\Delta h}{\Delta t}$ (on the y axis) look like? How would you find K from this plot?)

(c) Two more experiments were performed with different cylinders and different sized drain holes. Find the values of K for each of these experiments, and from the data make comparisons between the sizes of the cylinders and the sizes of the holes for the three experiments.

Experiment #1			Experiment #2	
Time (sec)	Height (cm)		Time (sec)	Height (cm)
0	35		0	13
8	30		0.58	12
16	25		1.35	11
25	20		1.95	10
35.5	15		2.85	9
48.7	10		3.65	8
66.8	5		4.55	7
			5.55	6
			6.55	5
			7.55	4
			8.55	3
			10.45	2
			13.45	1

(d) Discussion: What would happen if this experiment were run on a different, non-cylindrical, tank? Provide a detailed explanation.

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Problem 1.2. A farmer with large land holdings has historically grown a wide variety of crops. With the price of ethanol fuel rising, he decides that it would be prudent to devote more and more of his acreage to producing corn. As he grows more and more corn, he learns efficiencies that increase his yield per acre. In the present year, he used 7000 acres of his land to grow corn, and that land had an average yield of 170 bushels per acre. At the current time, he plans to increase his number of acres devoted to growing corn at a rate of 600 acres/year, and he expects that right now his average yield is increasing at a rate of 8 bushels per acre per year. Use this information to answer the following questions.

(a) Say that the present year is $t = 0$, that $A(t)$ denotes the number of acres the farmer devotes to growing corn in year t, $Y(t)$ represents the average yield in year t (measured in bushels per acre), and $C(t)$ is the total number of bushels of corn the farmer produces. What is the formula for $C(t)$ in terms of $A(t)$ and $Y(t)$? Why?

(a) What is the value of $C(0)$? What does it measure?

(a) Write an expression for $C'(t)$ in terms of $A(t)$, $A'(t)$, $Y(t)$, and $Y'(t)$. Explain your thinking.

(a) What is the value of $C'(0)$? What does it measure?

(a) Based on the given information and your work above, estimate the value of $C(1)$.

(a) Assume that the annual yield decreases every year by 8 bushels per acre. Write expressions for $C(t)$ and $C'(t)$, find the approximate time and number of bushels when the total number of bushels is maximized, and discuss how the maximum value would change if the farmer were able to control the rate at which the yield decreased. Present your solution with thorough discussion and appropriate plots.

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Problem 1.3. Let $f(v)$ be the gas consumption (in liters/km) of a car going at velocity v (in km/hour). In other words, $f(v)$ tells you how many liters of gas the car uses to go one kilometer if it is traveling at v kilometers per hour. In addition, suppose that $f(80) = 0.05$ and $f'(80) = 0.0004$.

(a) Let $g(v)$ be the distance the same car goes on one liter of gas at velocity v. What is the relationship between $f(v)$ and $g(v)$? Hence find $g(80)$ and $g'(80)$.

(b) Let $h(v)$ be the gas consumption in liters per hour of a car going at velocity v. In other words, $h(v)$ tells you how many liters of gas the car uses in one hour if it is going at velocity v. What is the algebraic relationship between $h(v)$ and $f(v)$? Hence find $h(80)$ and $h'(80)$.

(c) How would you explain the practical meaning of these function and derivative values to a driver who knows no calculus? Include units on each of the function and derivative values you discuss in your response.

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Problem 1.4. The velocity (m/s) of an object dropped from a helicopter 1000 meters high is given in the table below. Use the velocity data to approximate the acceleration (m/s^2) and position (m) data for the object at each of the given times. Unfortunately the motion sensor broke 3 seconds into the experiment. Extrapolate from the data to approximate the velocity and acceleration when the object hit the ground. Obviously there is drag on this object. Make an argument to estimate when an object with 10% more drag would hit the ground.

In your writeup of this problem:

- Be sure to give enough detail about the decisions and assumptions that you made.
- Be sure to approximate the time that the object hit the ground in each instance.
- Be sure to thoroughly explain where calculus was (and wasn't) used in your solution.

Time (sec)	Velocity (meters/sec)
0.0	0
0.2	−1.96
0.4	−3.89
0.6	−5.70
0.8	−7.33
1.0	−8.76
1.2	−9.95
1.4	−10.92
1.6	−11.69
1.8	−12.29
2.0	−12.74
2.2	−13.08
2.4	−13.33
2.6	−13.52
2.8	−13.65
3.0	−13.75

▲

Problem 1.5. Consider the following letter:

11 Patinkin Way
First National Park of Drachma

Mathematics Students
Carroll College
Helena, MT, USA

Dear Calculus Students:

Things have finally quieted down around Drachma since the Prince was kicked out of office. The good news is that I've managed to find a government job as the head of the First National Park of Drachma. The bad news is that most of the Park consists of a Fire Swamp (google *fire swamp* if you need to). When I went looking for help with our long range planning, your enterprising and resourceful professor naturally referred me to you.

We have two species that have me worried about the future of the Park: the indigenous ROUS (rodents of unusual size) and the brown tree snake which entered the Park about 50 years ago as a stowaway on the Dread Pirate Roberts' ship. Fortunately, ROUS's eat brown tree snakes. Unfortunately, brown tree snakes reproduce very rapidly.

My predecessor at the Park was a meticulous census taker, so I have records of approximate populations for each species for a 30 year period (see Table 1.1).

It looks like the populations are following some sort of pattern, but I'm not sure what it is. My real problem is that when either population gets very large, I will need additional employees to make sure that both species stay within the park and don't escape in the neighboring farmland. This is where I need your expert help. Specifically, I need a

Year	Brown Tree Snakes	ROUS's
1982	15,300	415
1984	9,890	910
1986	2,860	950
1988	3,340	525
1990	9,340	250
1992	12,290	460
1994	9,050	830
1996	4,840	855
1998	5,130	545
2000	8,720	340
2002	10,490	500
2004	8,550	770
2006	6,030	790
2008	6,200	560
2010	8,350	410
2012	9,410	525

Table 1.1. Populations by year since 1982.

prediction for how large the populations will be in each of the next 20 years. I also need an estimate of the rate at which each of the populations are changing with time*. When the rates of change of the populations are largest the local witch doctors head into the woods to raid the snake nests and ROUS borrows for potion ingredients.

I believe that the populations are fluctuating less and less, and may eventually stabilize. I would like your expert opinion on whether or not the populations do stabilize, and if they do, I need to know how long it will take and what the eventual populations will be.

Once the populations stop fluctuating so drastically, we will be able to dramatically improve access to the Park by offering summer camps, establishing permanent camp grounds, and perhaps even adding a log ride. There are still some flame-retardant issues to be worked out and the 6-fingered man is terribly afraid of the log ride idea. This should all be possible when the ROUS population is fluctuating by less than 75 per year and the brown tree snake population is fluctuating by less than 500 per year. I need your expert recommendation on when this will occur.

I have a meeting with the Budget Advisory Committee in 8 days to propose our budget for the next two decades, so I would greatly appreciate your group's report in 1 week.

Gratefully yours,
Amigo Flamboya

*I'm thinking that a plot with time on the x-axis and average rate of change on the y-axis might be very informative

Notes from your professor:

- Work in groups of 2 or 3, but don't be afraid to bounce ideas off of other groups.
- To see the general trend of the populations, I would suggest plotting the points for each population separately (maybe in Excel), with time on the horizontal axis and population on the vertical axis. It may make things a bit easier if you let $t = 0$ be 1982.
- Hint: Once you plot the populations, what two major types of functions do you see controlling the behavior? You can fit the function by estimating things like the period (or frequency), the equilibrium value, and the function that controls the amplitude.
- Be sure to respond to Amigo Flamboya with an appropriate technical report. He understands mathematics quite well so don't be afraid to include all necessary detail (explanations, plots, functions, etc) in your report. Make sure that you answer every question.

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Chapter 2

An Introduction to Linear Algebra

2.1 Why Linear Algebra?

Of all the mathematical tools that an applied scientist has, linear algebra is the most important.

When a student first encounters linear algebra it may seem like a stretch to call it the *most* important mathematical tool. That is, until the student relizes that almost every mathematical operator (such as differentiation, integration, reflection, rotation) and mathematical process (such as finding a best fit line, solving a system of equations, or even revealing the frequencies of a sound wave) are all based on the concepts from linear algebra.

In this chapter we will take a brief tour of several of the large ideas from linear algebra to give the reader a flavor of the richness and depth of the topic. In this section we will present the *three main problems* from linear algebra to highlight the importance and impact of the field. Before launching into these problems, the following problem will give you an introduction to the organizational structure, called a *matrix*, as well as the basic concept of matrix multiplication.

Problem 2.1. Advertisements tend to change people's opinions about political issues. Suppose that on a certain political issue there are 3 different popular opinions (A, B, and C). A psychologist wants to study the shifts in people's opinions after viewing advertisements and hence gathers the data listed in the table below.

Previous Opinion	New Opinion After Viewing Advertisement	Percent Making This Switch
A	A	50%
A	B	20%
A	C	30%
B	A	10%
B	B	70%
B	C	20%
C	A	5%
C	B	5%
C	C	90%

(a) Create a visual representation of the psychologist's data*.

(b) Create a tabular representation of the psychologist's data.

	From A	From B	From C
To A			
To B			
To C			

(c) If there are currently 1200 people in a population with opinion A, 500 people with opinion B, and 800 with opinion C, then what would the psychologist's data predict about the numbers of people with each opinion after viewing the advertisements?

(d) The psychologist did the study twice, but in an unfortunate instance with a hot latte she lost her record of the numbers of people in each category before watching the advertisements. She knows that the end result was 1000 people with opinion A, 800 people with opinion B, and 700 people with opinion C. How many people were originally in each category?

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Systems of Linear Equations: $Ax = b$

Most people are familiar with systems of linear equations arising from problems in algebra, business, calculus, and a plethora of other fields. What is often overlooked in lower-level mathematics classes is that there is an immense amount of structure embedded inside a system of equations just waiting to be exploited.

Problem 2.2. Consider a long metal rod that is being heated at one end and held at a constant temperature at the other end. After some time the temperature profile throughout the rod will reach a steady state (independent of time). One way to estimate the steady state temperature of the rod is to partition the rod into several equally spaced points (see Figure 2.1) and then to observe that the temperature at a point is the average of the temperature of the point to the right and the point to the left.

(a) Write a system of equations associated with the steady state temperature profile shown in Figure 2.1.

(b) Solve the system of equations from part (a) to find the steady state temperatures on the rod. You should either use the substitution method or the elimination method (which do you think will be more efficient? more organized?).

(c) Now let's go back to the system of equations in part (a) and build a more organized description of the problem. Rearrange the equations so that all of the variables are on the left-hand side and all of the corresponding variables are aligned vertically.

*The type of visual representation that most people use here is called a *graph* in mathematics.

This organization lends itself nicely to a *matrix equation*, where all of the coefficients are gathered into one matrix, the variables into a column vector, and the right-hand sides of the equations in another column vector. Write the system of equations from part (a) as a matrix equation. Explain how this structure lends itself nicely to the elimination method.

$$\begin{pmatrix} \text{matrix} \\ \text{of} \\ \text{coefficients} \end{pmatrix} \cdot \begin{pmatrix} \text{column} \\ \text{of} \\ \text{variables} \end{pmatrix} = \begin{pmatrix} \text{right} \\ \text{hand} \\ \text{side} \end{pmatrix}$$

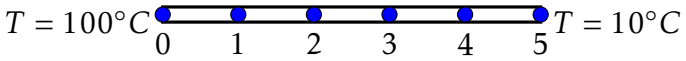

Figure 2.1. A metal rod partitioned into several discrete points.

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Fundamental Matrix Behavior: $Ax = \lambda x$

The second fundamental problem of linear algebra is to decompose a complicated system of equations to a collection of elements that are easily visualized. This is known as the eigenvector-eigenvalue problem. In the following activity we will set up a problem that, without the tools of linear algebra, is very difficult to answer. We will return to this example in future sections.

Problem 2.3. The female owls in a certain population can be classified as *juvenile*, *subadult*, and *adult*. In a given year, the number of new juvenile females in year $k+1$ is 0.33 times the number of adult females in year k, 18% of last year's juveniles become subadults, 71% of last year's subadults become adults, and only 94% of last year's adults survive.

(a) Write a discrete dynamical system for the populations of juveniles j_k, subadults s_k, and adults a_k where k is the year.

(b) Organize the discrete dyamical system into a matrix equation

$$\begin{pmatrix} \text{column} \\ \text{of} \\ \text{variables} \\ (\text{time} = k+1) \end{pmatrix} = \begin{pmatrix} \text{matrix} \\ \text{of} \\ \text{coefficients} \\ (\text{time} = k) \end{pmatrix} \cdot \begin{pmatrix} \text{column} \\ \text{of} \\ \text{variables} \\ (\text{time} = k) \end{pmatrix}$$

(c) The owl population will change over time, but it is very important to determine ahead of time if the owl popluation is in danger of going extinct. Use a spreadsheet program to predict the future of the owl population. The initial populations are not known.

(d) What is the long term behavior of the female owl population? If you used your spreadsheet model from part (c) to make this determination, then how do you know that your answer doesn't depend on your chosen intitial conditions?

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Least Squares: $A^TAx = A^Tb$

The final of the three big problems from linear algebra is that of least squares curve fitting. Instead of *least squares*, often times this is referred to as the *best fit line*. Of course, the word *best* is relative to how you measure the error. In the following activity you'll set up a best fit line problem with linear algebra. The techniques to solve the problem will not be covered in this text; this problem is presented here for completeness of the introduction.

Problem 2.4. Research on study time vs. exam scores yields the following data points

Time Studying	Exam Score (%)
0	75
25	70
30	92
45	88
15	90
30	70

(a) We would like to find a linear equation of the form $y = ax + b$ where x is the time spent studying and y is the exam score. Write 6 equations where the two unknowns are the parameters a and b.

(b) Organize your 6 equations into a matrix equation. The solution to this matrix equation is beyond the scope of this chapter, but if we could *solve*[†] this problem then we would know the slope and y-intercept that minimize the error between the predictor line and the data. This is a wide reaching topic that will unfortunately have to wait for a different course.

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[†]Even the word *solve* here is arbitrary since there are more equations than there are unknowns!

2.2 Matrix Operations and Gaussian Elimination

One of the first natural questions to ask when first encountering matrices is whether the regular operations of addition, subtraction, multiplication, division, and exponentiation make sense. In the cases of addition and subtraction the answer is simple: Yes! Addition and subtraction work in the simplest most natural way with matrices. The other operations, on the other hand, need a bit more care but their definitions are robust and immensely useful.

Problem 2.5. Consider the matrices

$$A = \begin{pmatrix} 1 & 7 & -3 \\ 2 & -3 & 5 \\ 2 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 3 & -7 & 0 \\ 0 & 0 & -2 \\ 2 & 4 & 5 \end{pmatrix}$$

(a) Calculate $A + B$.

(b) Calculate $A - B$.

(c) Calculate $2A$.

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Matrix Arithmetic

In this subsection we'll take a brief glimpse at each of the most fundamental matrix operations as well as some of the foundational definitions for linear algebra.

Definition 2.6 (Matrix Arithmetic)**.** Below are several definitions associated with matrices.

Size of a Matrix: If A is a matrix with m rows and n columns then we say that A has **size** (or dimensions) $m \times n$.

Equality: Two matrices are **equal** if their corresponding entries are equal. Matrices can only be equal if the sizes are equal.

Addition and Subtraction: Matrix **addition and subtraction** are done by regular addition and subtraction on the corresponding entries. Matrix addition and subtraction can only be performed on matrices of the same size.

Scalar Multiplication: If A is a matrix then cA is a **scalar multiple** of the matrix. Multiplying a matrix by a scalar multiplies every entry by the scalar.

Transposition: If A is a matrix then A^T is the **transpose** of the matrix found by interchanging the rows and columns of A. If A is $m \times n$ then A^T is $n \times m$.

The basic operations of addition, subtraction, scalar multiplication, and trasposition all follow our natural intuition, and we'll get a chance to play with them in the homework. The operation of multiplication, on the other hand, takes a bit more care to define.

Before giving a full description of matrix multiplication let us define some very common notation for matrices. The size of the matrix is stated by the number of rows then the number of columns. This lends itself to a system of double indices for keeping track of the entries in a matrix. For the matrix A of size $m \times n$ we denote the individual entry in row i and column j as a_{ij}. In the entire matrix, this becomes

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

So, for example, a_{37} is the entry in row 3 column 7. More concretely, in the matrix, $A = \begin{pmatrix} 7 & 4 & -2 \\ 3 & 1 & 4 \end{pmatrix}$, the size is 2×3 and the entry in row 2 column 1 is $a_{21} = 3$.

Definition 2.7. If A is an $m \times n$ matrix and B is an $n \times p$ matrix then the **product** of A and B is $C = AB$.

- The size of AB is $m \times p$. The number of columns in A must be the same as the number of rows of B.
- The entry in row i and column j of $C = AB$ is

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

It is very important to note that in general $AB \neq BA$.

Example 2.8. Consider the matrices A and B defined as

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 5 & -2 \\ -1 & 0 \\ 1 & 3 \end{pmatrix}.$$

Find AB and BA if they exist.
Solution: First note that A is a 2×3 matrix and B is a 3×2 matrix. Hence, AB will be

2×2 and BA will be 3×3. [a]

$$AB = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ -1 & 0 \\ 1 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \cdot 5 + 2 \cdot (-1) + (-3) \cdot 1 & 1 \cdot (-2) + 2 \cdot 0 + (-3) \cdot 3 \\ 2 \cdot 5 + 0 \cdot (-1) + \cdot 1 \cdot 1 & 2 \cdot (-2) + 0 \cdot 0 + 1 \cdot 3 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -11 \\ 11 & -1 \end{pmatrix}$$

To be very clear about the process, the -11 in row 1 column 2 of the answer came from multiplying the corresponding entries of row 1 of matrix A by column 2 of matrix B and finding the sum of the products.

$$BA = \begin{pmatrix} 5 & -2 \\ -1 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & -3 \\ 2 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 5 \cdot 1 + (-2) \cdot 2 & 5 \cdot 2 + (-2) \cdot 0 & 5 \cdot (-3) + (-2) \cdot 1 \\ (-1) \cdot 1 + 0 \cdot 2 & (-1) \cdot 2 + 0 \cdot 0 & (-1) \cdot (-3) + 0 \cdot 1 \\ 1 \cdot 1 + 3 \cdot 2 & 1 \cdot 2 + 3 \cdot 0 & 1 \cdot (-3) + 3 \cdot 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 10 & -17 \\ -1 & -2 & 3 \\ 7 & 2 & 0 \end{pmatrix}$$

[a] If you read this example without picking up your pencil and trying the example then you may want to pause and rethink your decisions. Mathematics is not a spectator's sport!

Problem 2.9. Consider the matrices

$$A = \begin{pmatrix} 2 & -1 & 4 \\ 3 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 0 & -3 \\ 4 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -1 \\ 3 & 2 \\ -2 & 1 \end{pmatrix}, \quad \text{and} \quad \mathrm{x} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}.$$

(a) Determine which products are possible: AB, AC, $A\mathrm{x}$, BA, CA, $\mathrm{x}A$, BC, $B\mathrm{x}$, CB, $C\mathrm{x}$. For each of the products that is possible, find the size of the result.

(b) Write the product AB and the product BA. Does $AB = BA$?

▲

2.3 Gaussian Elimination: A First Look At Solving Systems

A truly beautiful application of matrices, and the first real application of linear algebra, is the technique of solving systems of linear equations. The technique that we'll describe in the next several activities and examples is used to solve systems of equations in a very organized fashion. Most students are familiar with the *elimination method* from high school algebra, and the technique of *Gaussian Elimination* described herein is simply a more organized way to perform the exact same technique. You'll find that systems of linear equations arise naturally in all sorts of applications so we include this as one of the essential tools for mathematical modeling.

Example 2.10. Consider the following system of equations.

$$\begin{aligned} -x_1 + x_2 - x_3 &= 1 \\ 3x_2 + 2x_3 &= -8 \\ x_3 &= 2 \end{aligned} \tag{2.1}$$

Solve the system algebraically and reorganize the system using the powerful and beautiful structure of matrices [a]

Solution: Any technique for solving systems will suffice. This particular system is set up to reveal a solution quickly. Indeed, it is obvious that $x_3 = 2$. Using this fact, the second equation can be rewritten and solved as

$$3x_2 + 2(2) = -8 \quad \Longrightarrow \quad 3x_2 = -12 \quad \Longrightarrow \quad x_2 = -4.$$

Now that we have both x_2 and x_3 the first equation can be rewritten as solved as

$$-x_1 + (-4) - (2) = 1 \quad \Longrightarrow \quad -x_1 = 7 \quad \Longrightarrow \quad x_1 = -7$$

Now we'll leverage the organizational power of matrices:
The system of equations can be written in matrix form as

$$\begin{pmatrix} -1 & 1 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -8 \\ 2 \end{pmatrix} \tag{2.2}$$

Multiplying the 3×3 matrix on the left-hand side by the 3×1 vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ reveals that the matrix equation in (2.2) is indeed the same as the system of equations in (2.1). This important observation illustrated that we can take any system of linear equations and write it in such a way.

The matrix equation can be further reorganized into an *augmented system*:

$$\left(\begin{array}{ccc|c} -1 & 1 & -1 & 1 \\ 0 & 3 & 2 & -8 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

This is simply an organizational technique and we use it because the names of the variables are arbitrary and irrelevant to the solution. The first line of the augmented system can be read as $-x_1 + x_2 - x_3 = 1$ where the variables are inferred and only inserted when necessary. The last line of the augmented system can be read as $0x_1 + 0x_2 + x_3 = 2$, so this clearly reveals that $x_3 = 2$.

[a] If you haven't notice, the author loves matrices!

Problem 2.11. In this activity we wish to solve the system of equations.

$$\begin{aligned} -x_1 + x_2 - x_3 &= -6 \\ x_1 + x_3 &= 15 \\ 2x_1 - x_2 + x_3 &= 9 \end{aligned}$$

We will do so in a very structured and organized fashion to illustrate the *Gaussian Elimination* technique for solving systems.

(a) First write the system as a matrix equation.

$$\begin{pmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \_\_ \\ \_\_ \\ \_\_ \end{pmatrix}$$

(b) Now write the system as an *augmented system*

$$\left(\begin{array}{ccc|c} \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ \end{array}\right)$$

(c) Using the operations:

- multiply one row by a scalar quantity
- add a multiple of one row to another row
- interchange two rows

we wish to transform the augmented system you wrote in part (b) to something of the form

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \star \\ 0 & 1 & 0 & \star \\ 0 & 0 & 1 & \star \end{array}\right)$$

Discuss with your partners why the above operations are mathematically valid.

(d) Work with your partners to discuss the first and most logical operation to do that will move you toward that direction.

(e) Use the operations outlined in part (c) to solve the system. Pay particular attention to the order in which you perform the row reduction.

▲

Definition 2.12. The **Gaussian Elimination** technique (also called **row reduction**) is an algorithm used to perform the elimination method on a system of linear equations of virtually any size.

1. Write the system of equations in augmented form.
2. Perform row operations to get to a triangular system of equations. The row operations allowed are:
   - Multiply a row by any nonzero number.
   - Add a multiple of one row to another.
   - Interchange two rows.
3. Once the system is written in triangular form, either back substitute to solve the system or continue performing row operations to arrive at the form

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & \cdots & 0 & \star \\ 0 & 1 & 0 & \cdots & 0 & \star \\ 0 & 0 & \ddots & \cdots & 0 & \star \\ \vdots & \vdots & \cdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \star \end{array}\right)$$

   At which point, read the answer from the augmented form.

Next we will show a fully worked example of Gaussian Elimination in action to give some hints to the thought process that goes on behind the scenes.

Example 2.13. Solve the system of equations

$$\begin{aligned} x_1 + 0x_2 + 3x_3 + 2x_4 &= -20 \\ 0x_1 + x_2 - 4x_3 - 4x_4 &= 32 \\ 2x_1 - 3x_2 + 16x_3 + 16x_4 &= -120 \\ 0x_1 - x_2 + 4x_3 + 9x_4 &= -27 \end{aligned}$$

Solution: If we first write this as an augmented matrix we get

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & 2 & -20 \\ 0 & 1 & -4 & -4 & 32 \\ 2 & -3 & 16 & 16 & -120 \\ 0 & -1 & 4 & 9 & -27 \end{array}\right).$$

Next we start performing row operations with the goal of creating a triangular system. The observant reader will notice that, while this system of equations can be solved with any (non-graphical) technique from high school algebra, the Gaussian Elimination technique is far more organized.

Add (-2) times row 1 to row 3. Put the answer in row 3.

$$\xrightarrow{-2R_1+R_3} \left(\begin{array}{cccc|c} 1 & 0 & 3 & 2 & -20 \\ 0 & 1 & -4 & -4 & 32 \\ 0 & -3 & 10 & 12 & -80 \\ 0 & -1 & 4 & 9 & -27 \end{array}\right)$$

Add (3) times row 2 to row 3. Put the answer in row 3.

$$\xrightarrow{3R_2+R_3} \left(\begin{array}{cccc|c} 1 & 0 & 3 & 2 & -20 \\ 0 & 1 & -4 & -4 & 32 \\ 0 & 0 & -2 & 0 & 16 \\ 0 & -1 & 4 & 9 & -27 \end{array}\right)$$

Add (1) times row 2 to row 4. Put the answer in row 4.

$$\xrightarrow{(1)R_2+R_4} \left(\begin{array}{cccc|c} 1 & 0 & 3 & 2 & -20 \\ 0 & 1 & -4 & -4 & 32 \\ 0 & 0 & -2 & 0 & 16 \\ 0 & 0 & 0 & 5 & 5 \end{array}\right)$$

Divide row 3 by (-2) and put the answer in row 3.

$$\xrightarrow{R_3/(-2)} \left(\begin{array}{cccc|c} 1 & 0 & 3 & 2 & -20 \\ 0 & 1 & -4 & -4 & 32 \\ 0 & 0 & 1 & 0 & -8 \\ 0 & 0 & 0 & 5 & 5 \end{array}\right)$$

Divide row 4 by 5 to arrive at a triangular form.

$$\xrightarrow{R_4/5} \left(\begin{array}{cccc|c} 1 & 0 & 3 & 2 & -20 \\ 0 & 1 & -4 & -4 & 32 \\ 0 & 0 & 1 & 0 & -8 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

Now that this is in triangular form you can back substitute or simply continue performing row operations. We will choose to perform the row operations to determine the solution.

$$\xrightarrow{4R_3+R_2,\,4R_4+R_2}\left(\begin{array}{cccc|c}1&0&3&2&-20\\0&1&0&0&4\\0&0&1&0&-8\\0&0&0&1&1\end{array}\right)\xrightarrow{(-3)R_3+R_1,\,(-2)R_4+R_1}\left(\begin{array}{cccc|c}1&0&0&0&2\\0&1&0&0&4\\0&0&1&0&-8\\0&0&0&1&1\end{array}\right)$$

After all of this simplification, the final solution is

$$\boxed{x_1 = 2,\, x_2 = 4,\, x_3 = -8,\, x_4 = 1}$$

The process of performing Gaussian Elimination may take a lot of paper, but once you get the hang of the process it is far more organized than any other technique for solving systems of linear equations.

Technique 2.14 (Practical Tips for Gaussian Elimination)**.** You should use the following tips for doing Gaussian Elimination.

- First try to get a 1 in the upper left-hand corner of the augmented matrix.
- Next, use the new first row to eliminate all of the non-zero entries in the first column. By the time you're done with this you should have a column with a 1 on top and zeros below.
- Next get a 1 in row 2 column 2.
- Use your new second row to eliminate all of the non-zero entries in the second column.
- Proceed in a similar fashion until you have reached the final row.

Problem 2.15. Write the following system in augmented form and use Gaussian Elimination to solve for x_1, x_2, and x_3.

$$\begin{aligned}x_1 - 2x_2 + x_3 &= 0\\ 2x_2 - 8x_3 &= 8\\ -4x_1 + 5x_2 + 9x_3 &= -9\end{aligned}$$

▲

2.4 Systems of Linear Equations

In this section we further explore the notion of solving a linear system of equations. To begin our study consider the following Preview Activity.

Problem 2.16. Solve each of the three systems of two equations and two unknowns. One of the systems has infinitely many solutions, one of the systems has exactly one solution, and one of the systems has no solutions. In each case, use augmented matrices and Gaussian Elimination to solve the system.

(a) Solve the system

$$\begin{aligned} x_1 - 2x_2 &= 4 \\ -2x_1 + 4x_2 &= 5 \end{aligned}$$

(b) Solve the system

$$\begin{aligned} x_1 - 2x_2 &= 4 \\ -2x_1 + 4x_2 &= -8 \end{aligned}$$

(c) Solve the system

$$\begin{aligned} x_1 - 2x_2 &= 4 \\ 2x_1 + 4x_2 &= 5 \end{aligned}$$

▲

Systems, Matrix Equations, and Vector Equations

A system of linear equations can always be written in several different ways. The most familiar of which is the collection of equations themselves. The three other ways to write a system of equations are the **matrix form**, the **vector form**, and the **augmented matrix form**.

Consider the system of m linear equations with n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n &= b_3 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{2.3}$$

Definition 2.17. The **matrix form** of the system of equations (2.3) is

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{pmatrix}$$

In symbols, this is denoted $A\mathsf{x} = \mathsf{b}$.

Definition 2.18. The **vector form** of the system of equations (2.3) is

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \vdots \\ a_{m2} \end{pmatrix} + x_3 \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \\ \vdots \\ a_{m3} \end{pmatrix} + \cdots x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{pmatrix}$$

In symbols, this is denoted $x_1\mathsf{a}_1 + x_2\mathsf{a}_2 + x_3\mathsf{a}_3 + \cdots + x_n\mathsf{a}_n = \mathsf{b}$.

Definition 2.19. The **augmented form** of the system of equation (2.3) is

$$\left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right)$$

In symbols this is denoted $(A|\mathsf{b})$.

Solution Sets to Systems of Equations

It is not guaranteed that a system of equations will have a solution. Moreover, if there is a solution it is not guaranteed that the solution will be unique. Theorem 2.20 gives the conditions for which a system of linear equations will have no solutions, infinitely many solutions, or exactly one solution.

Theorem 2.20. 1. A system of linear equations has no solutions if after it is row

reduced it has a row of the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & | & \star \end{pmatrix}$$

where the number $\star$ is nonzero. This row in the reduced matrix is equivalent to saying that $0 = \star$; which is never true.

2. A system of linear equations has one unique solution if at the end of the row reduction one can determine every variable.

3. A system of linear equations has infinitely many solutions if at the end of the row reduction there are variables that you cannot determine uniquely.

Problem 2.21. (a) Consider the following augmented matrices in reduced row echelon form. Determine the number of solutions. If there is one unique solution then find it.

(i)
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -5 \end{array}\right)$$

(ii)
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & -5 \end{array}\right)$$

(iii)
$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(b) Determine the value of h such that the matrix is the augmented matrix of a linear system with infinitely many solutions

$$\left(\begin{array}{cc|c} 3 & -4 & 4 \\ 9 & h & 12 \end{array}\right)$$

▲

In Activity 2.21, problem (a) part (iii) has infinitely many solutions. In order to find a complete description of those solutions we can rewrite the problem in terms of the variables to get

$$\begin{aligned} x_1 + x_3 &= 3 \\ x_2 &= 2. \end{aligned}$$

It is obvious from this description that x_2 is fixed at 2, but the values of x_1 and x_3 depend on each other. In cases like this we choose one variable to be a *parameter* and express the

other variable in terms of that parameter. In this case, we let $x_3 = t$ and we can write the first equation as

$$x_1 = 3 - t$$

Since t can take on any real value we finally write the solution as

$$x_1 = 3 - t \quad \text{and} \quad x_2 = 2 \quad \text{where} \quad x_3 = t \quad \text{and} \quad -\infty < t < \infty.$$

Written as a vector, the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} t.$$

In two and three dimensions there are nice geometric interpretations for these types of solutions.

Example 2.22. Consider the following three systems of equations and their row reduced forms. Describe their solution sets geometrically.

$$\text{System \#1:} \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 2 & 1 & 0 \end{array}\right) \to \cdots \to \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \end{array}\right)$$

$$\text{System \#2:} \left(\begin{array}{cc|c} 1 & -1 & 3 \\ -1 & 1 & 0 \end{array}\right) \to \cdots \to \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 0 & 3 \end{array}\right)$$

$$\text{System \#3:} \left(\begin{array}{cc|c} 1 & -1 & 3 \\ -1 & 1 & -3 \end{array}\right) \to \cdots \to \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 0 & 0 \end{array}\right)$$

Solution: Figure 2.2 shows the graphical interpretation for each system. Clearly if there is a unique solution then there is one unique point where the lines cross. If there are no solutions then the lines are parallel. In the case where there are infinitely many solutions (system #3) we see that we can write $x_2 = x_1 - 3$. Letting $x_1 = t$ we have $x_2 = t - 3$. This is clearly the line with y-intercept -3 and slope 1.

Example 2.23. Solve the system of equations. If there are infinitely many solution, express them as a parameterization.

$$-4x_1 + x_2 = 0$$
$$-12x_1 + 3x_2 = 0$$

Solution: First we write the system of equations as an augmented system. Then we row reduce as much as possible.

$$\left(\begin{array}{cc|c} -4 & 1 & 0 \\ -12 & 3 & 0 \end{array}\right) \overset{(-3)R_1+R_2}{\longrightarrow} \left(\begin{array}{cc|c} -4 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

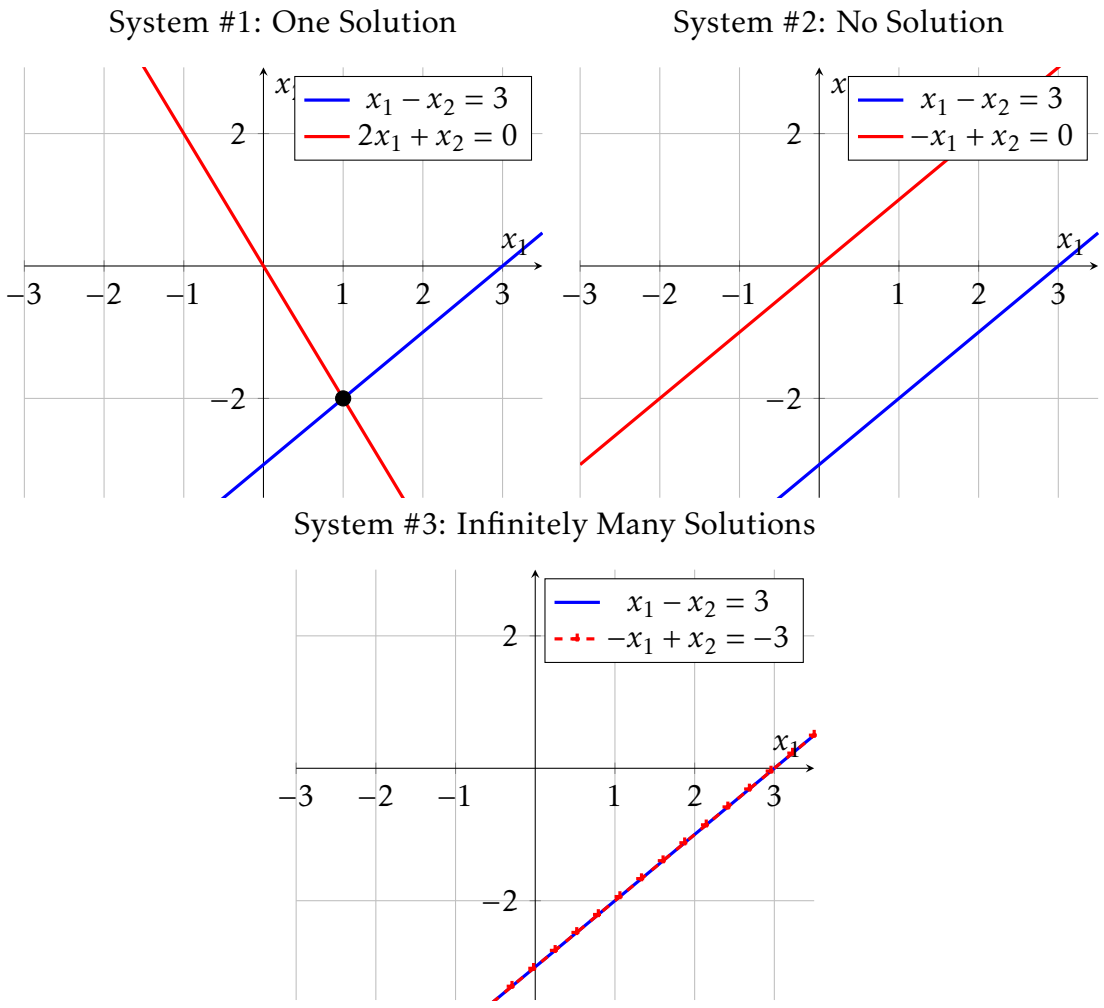


Figure 2.2. Three possible solution sets in two spatial dimensions

In this particularly simple example, this means that $-4x_1 + 1x_2 = 0$. Written another way, $x_2 = 4x_1$.

If we write $x_1 = t$ then the solution is $x_1 = t, x_2 = 4t$ for $-\infty < t < \infty$. If, on the other hand, we write $x_2 = t$ then the solution is $x_2 = t, x_1 = t/4$ for $-\infty < t < \infty$.

In vector form, this solution can be written as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} t$$

or

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ 1 \end{pmatrix} t.$$

Example 2.24. Solve the system of equations $A\mathsf{x} = \mathsf{b}$ where

$$A = \begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{pmatrix} \quad \text{and} \quad \mathsf{b} = \begin{pmatrix} 7 \\ -1 \\ -4 \end{pmatrix}.$$

Solution: Writing the augmented matrix $(A|\mathsf{b})$ and doing several steps of row reduction gives

$$\left(\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This implies that $x_2 = 2$ and $x_1 = -1 + \frac{4}{3}t$ for some parameter t.
Written in vector form, the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} \frac{4}{3} \\ 0 \\ 1 \end{pmatrix}.$$

Example 2.25. Solve the system of equations

$$\begin{aligned} 4x_1 + x_2 + 5x_3 + 7x_4 &= 0 \\ 8x_1 + x_2 - 5x_3 + 4x_4 &= 0 \end{aligned}$$

Solution: Written in augmented form and row reduced we see that

$$\left(\begin{array}{cccc|c} 4 & 1 & 5 & 7 & 0 \\ 8 & 1 & -5 & 4 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 4 & 1 & 5 & 7 & 0 \\ 0 & -1 & -15 & -18 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 4 & 0 & -10 & -11 & 0 \\ 0 & 1 & 15 & 18 & 0 \end{array}\right)$$

In this case there are two variables that cannot be solve for: x_3 and x_4. These are now both parameters. If $x_3 = s$ and $x_4 = t$ then

$$x_1 = \frac{10s + 11t}{4} \quad \text{and} \quad x_2 = -15s - 18t \quad \text{where} \quad x_3 = s \quad \text{and} \quad x_4 = t.$$

In vector form, this solution can be written as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 10/4 \\ -15 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 11/4 \\ -18 \\ 0 \\ 1 \end{pmatrix} t.$$

Problem 2.26. Each of the following systems has been row reduced as much as possible. If there is a unique solution to the system then find it. If there are infinitely many solutions

then express them as a vector equation. If there are no solutions then indicate the reason.

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{2.4}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right) \tag{2.5}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{2.6}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{2.7}$$

▲

2.5 Linear Combinations

One of the most beautiful parts of linear algebra is the richness of the structure of matrices. As we showed earlier, every system of linear equations can be written several different ways (as a system, as a matrix equation, as a vector equation, or as an augmented system). In this subsection we'll look in particular at the vector equation. Hiding below a vector equation is one of the most fundamental ideas behind all of linear algebra: the linear combination.

Definition 2.27. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ be vectors in n-dimensional space and let $c_1, c_2, \ldots, c_p$ be scalar quantities. The vector $\mathbf{u}$ defined by

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_p\mathbf{v}_p$$

is called a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ with weights $c_1, c_2, \ldots, c_p$.

In the system of equations

$$2x_1 + 3x_2 = 5 \tag{2.8}$$
$$4x_1 - 6x_2 = 6, \tag{2.9}$$

we can rephrase the underlying question as: *find the weights which solve the vector equation*

$$x_1\begin{pmatrix}2\\4\end{pmatrix} + x_2\begin{pmatrix}3\\-6\end{pmatrix} = \begin{pmatrix}5\\6\end{pmatrix}. \tag{2.10}$$

Notice that this is simply stating that a system of equations is nothing more than a linear combination with unknown weights!

There is also a nice graphical interpretation of linear combination (2.10): How many $\begin{pmatrix}2\\4\end{pmatrix}$ plus how many $\begin{pmatrix}3\\-6\end{pmatrix}$ do we need to create $\begin{pmatrix}5\\6\end{pmatrix}$? Solving for x_1 and x_2 (using Gaussian elimination) we find that $x_1 = 2$ and $x_2 = 1/3$. Hence, $\begin{pmatrix}5\\6\end{pmatrix} = 2\begin{pmatrix}2\\4\end{pmatrix} + (1/3)\begin{pmatrix}3\\-6\end{pmatrix}$ as seen in Figure 2.3.

Problem 2.28. Open the GeoGebra applet http://tube.geogebra.org/student/m1254137 in a browser window.

(a) Move the vectors $\mathbf{u}$ and $\mathbf{v}$ to

$$\mathbf{u} = \begin{pmatrix}1\\2\end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix}2\\-1\end{pmatrix}.$$

We would like to determine all of the combinations possible forming the vector

$$\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}.$$

Use the sliders in the GeoGebra applet to answer the following questions:

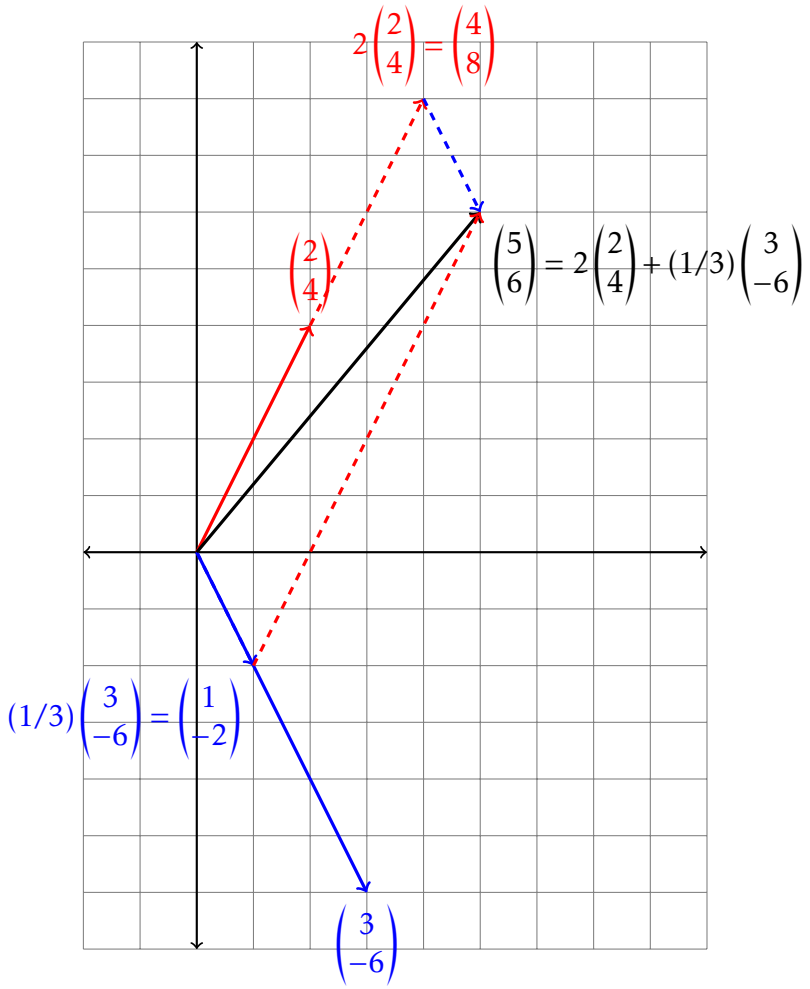


Figure 2.3. A graphical example of a linear combination.

(i) Describe all of the possible vectors $\mathtt{w}$ if $c_1 = 0$.

(ii) Describe all of the possible vectors $\mathtt{w}$ if $c_2 = 0$.

(iii) Which vector results if $c_1 = c_2 = 0$?

(iv) Is it possible to find c_1 and c_2 such that $\mathtt{w} = \begin{pmatrix}-6\\0.5\end{pmatrix}$? If so, what are c_1 and c_2. If not, why not?

(b) Move $\mathtt{u}$ and $\mathtt{v}$ so they are parallel. Form the vector $\mathtt{w} = c_1\mathtt{u} + c_2\mathtt{v}$ and describe all of the possible values of the vector $\mathtt{w}$.

(c) Find c_1 and c_2 both algebraically and graphically so that

$$\begin{pmatrix}1\\-12\end{pmatrix} = c_1\begin{pmatrix}1\\-2\end{pmatrix} + c_1\begin{pmatrix}0\\2\end{pmatrix}.$$

(d) Express the vector $\mathtt{w} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$ as a linear comination of $\mathtt{u} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$ and $\mathtt{v} = \begin{pmatrix} 1 \\ -6 \end{pmatrix}$.

▲

2.6 Inverses and Determinants

Division is always a bit of a touchy subject. In the real numbers division is well defined except when the denominator is zero. The same story is true in the rational numbers: a fraction divided by a fraction is another fraction so long as the divisor is not zero. What if we wanted to stay only in the integers? Can we divide two integers and get another integer? Of course you can always divide by 1, but in most other cases division will move you into the rational numbers. Hence, division on the integers doesn't really make sense.[‡]

Similarly, if we try to define division on matrices we run into trouble. What does it mean to *divide by a matrix*? In general, that phrase is meaningless! Let's expand our view a bit.

When considering the operation of addition, we call 0 the **additive identity** and we call $(-a)$ the *additive inverse* of a since $a + (-a) = 0$. When considering multiplication, we call 1 the **multiplicative identity** and $1/a$ is the *multiplicative inverse* of a (when $a \neq 0$) since $a \cdot \frac{1}{a} = 1$.

Problem 2.29. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -4 & 3 \end{pmatrix}.$$

(a) Find a matrix B such that $A + B = A$ and $B + A = A$.

(b) Find a matrix I such that $I \cdot A = A$ and $A \cdot I = A$.

(c) Find a matrix C such that $C \cdot A = I$ and $A \cdot C = I$.

▲

One way to tackle the third part of the preceding problem is let let C be a matrix filled with unknowns and then to build the associated system of equations. More specifically, if we let

$$C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and then observe that the equation $AC = I$ becomes

$$\begin{pmatrix} 1 & 2 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

After multiplying the left-hand side we get the equation

$$\begin{pmatrix} a + 2c & b + 2d \\ -4a + 3c & -4b + 3d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

[‡]The mathematician would say that the integers are not closed under division.

This results in a system of four equations with four unknowns:

$$\begin{cases} a+2c &= 1 \\ b+2d &= 0 \\ -4a+3c &= 0 \\ -4b+3d &= 1 \end{cases}$$

which can be solved using Gaussian Elimination (row reduction):

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ -4 & 0 & 3 & 0 & 0 \\ 0 & -4 & 0 & 3 & 1 \end{array}\right) \to \cdots \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3/11 \\ 0 & 1 & 0 & 0 & -2/11 \\ 0 & 0 & 1 & 0 & 4/11 \\ 0 & 0 & 0 & 1 & 1/11 \end{array}\right).$$

Hence,

$$C = \frac{1}{11}\begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}.$$

You should fill in all of the missing row reduction to verify this answer. Also, to check this answer you should multiply AC and CA to be sure that you get the identity matrix with both multiplications.

Definition 2.30. In matrices we define the following:

- The **additive identity** of an $m \times n$ matrix is

$$0_{m\times n} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

- The **additive inverse** of an $m \times n$ matrix A is $(-A)$ since $A + (-A) = (-A) + A = 0$.
- The **multiplicative identity** of an $n \times n$ matrix A is the matrix

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

- The **multiplicative inverse** of an $n \times n$ matrix A is an $n \times n$ matrix C such that $AC = CA = I$.

The zero matrix act's like the zero integer; adding the zero matrix doesn't change the sum. Similarly, the identity matrix (with ones down the main diagonal and zeros elsewhere) acts like the integer 1; when multiplying by the identity matrix the product doesn't change.

2.6.1 Inverses

The following activity defines the matrix inverse for a 2×2 matrix. The reader should carefully work this problem and take careful note of the result since it shows the general process for finding inverses.

Problem 2.31. Consider the matrix $A = \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$. We would like to find the inverse of A so that when we multiply the inverse by A we get the identity I.

(a) Let's first try a na ive *inverse*. Let

$$B = \begin{pmatrix} 1/2 & 1/3 \\ 1/2 & 1/4 \end{pmatrix}.$$

Find the products AB and BA and verify that $AB \neq I$ and $BA \neq I$.

We really want to find the matrix B such that $AB = BA = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The following parts of this activity will guide you toward that goal.

(b) Create an augmented matrix $(A|I)$.

(c) Use elementary row operations to reduce the matrix in part (a) to an augmented matrix of the form $(I|\star)$.

(d) Check that the matrix on the right-hand side of your answer in part (b) is actually the inverse of A.

(e) Now consider the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Repeat parts (a) and (b) to find the inverse of a general 2×2 matrix. You should have a factor of $\frac{1}{ad-cb}$ in your matrix. The denominator of this fraction is called the *determinant* of the matrix A.

▲

The previous problem illustrated the method for finding the inverse of a matrix:

Technique 2.32 (Process for finding A^{-1} if it exists)**.** The following is the technique for find the inverse of a matrix (if it exists).

1. augment the matrix with the identity, then
2. row reduce to get the identity on the left-hand side of the augmented matrix.

Example 2.33. Let's find the inverse of the matrix from the preview activity using this method instead. Let $A = \begin{pmatrix} 1 & 2 \\ -4 & 3 \end{pmatrix}$. We want to find A^{-1} such that $AA^{-1} = I$ and $A^{-1}A = I$.

Solution: Using the equation $AA^{-1} = I$ and knowing that we are seeking A^{-1} we can write the augmented system $(A|I)$ and row reduce until we get $\left(I|A^{-1}\right)$:

$$\begin{aligned}(A|I) &= \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -4 & 3 & 0 & 1 \end{array}\right) \\ &\to \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 11 & 4 & 1 \end{array}\right) \\ &\to \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 4/11 & 1/11 \end{array}\right) \\ &\to \left(\begin{array}{cc|cc} 1 & 0 & 3/11 & -2/11 \\ 0 & 1 & 4/11 & 1/11 \end{array}\right) = \left(I|A^{-1}\right)\end{aligned}$$

Hence, the inverse of A is $A^{-1} = \frac{1}{11}\begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$

Example 2.34. In part (e) of the previous activity you also worked to find the inverse for the generic 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If you did all of your work correctly you will have found that

$$A^{-1} = \frac{1}{ad - bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Let's use this formula to verify (for a third time) the inverse of the matrix from the preview.

Solution: Since $A = \begin{pmatrix} 1 & 2 \\ -4 & 3 \end{pmatrix}$ we can apply the 2×2 inverse formula to get

$$A^{-1} = \frac{1}{(1)(3) - (2)(-4)} \cdot \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$$

The reader should be cautious here. The formula that you derived for 2×2 matrices only makes sense for that size. The only true method for finding the inverse (with the tools we have) is to augment your matrix with the identity and to row reduce.

One other trouble comes when it is impossible to get the identity matrix to appear on the right. When this happens it is an indication that the matrix does not have a multiplicative inverse. In the following activity you will practice this technique on a few matrices.

Problem 2.35. Find the inverse for each of the following matrices if it exists. If it does not exist then determine why not.

(a) $\begin{pmatrix} 1 & -3 \\ 4 & -9 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & 6 \\ 4 & 7 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 4 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

▲

2.6.2 Determinants

Finding a matrix inverse is often a tedious task. As it turns out, there is a very handy number associated with a square matrix that one can use to determine if a matrix is invertible[§]. As we saw previously, if the quantity $ad - bc$ is zero in the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the matrix cannot have an inverse. This is not a peculiarity of 2×2 matrices! The value $ad - bd$ is called the determinant of the 2×2 matrix.

What we need is a way to define the **determinant** of a square matrix of any size. In order to do so it is easiest to observe how the formula works by following a pattern rather than reading the technical definition. The next problem will walk you through the process.

Problem 2.36. Follow these instructions to see how to find a determinant of a square matrix.

(a) Find the determinant of the 2×2 matrix $\begin{pmatrix} 4 & -1 \\ -2 & 1 \end{pmatrix}$ using the fact that

$$\det(A) = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

(b) Now consider the matrix

$$A = \begin{pmatrix} 1 & 5 & 3 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{pmatrix}.$$

(i) Cross out row 1 and column 1. Call the resulting 2×2 matrix A_{11}.

(ii) Cross out row 1 and column 2. Call the resulting 2×2 matrix A_{12}.

(iii) Cross out row 1 and column 3. Call the resulting 2×2 matrix A_{13}.

[§]This is actually one of many tests that you can be used to determine if a matrix is invertible. A discussion of such techniques will wait until you know a bit more linear algebra.

(iv) The determinant of A is

$$\det(A) = 1 \cdot \det(A_{11}) - 5 \cdot \det(A_{12}) + 3 \cdot \det(A_{13}).$$

Perform this computation.

(c) Find the determinant of the matrix

$$A = \begin{pmatrix} 2 & 3 & -4 \\ 4 & 0 & 5 \\ 5 & 1 & 6 \end{pmatrix}.$$

(d) Fill in the blanks below to set up the determinant calculation for the following 4×4 matrix.

$$A = \begin{pmatrix} 2 & 3 & 5 & 7 \\ 0 & 3 & 2 & 9 \\ 3 & -2 & 6 & 1 \\ 2 & 0 & 0 & 4 \end{pmatrix}$$

$$\det(A) = 2\cdot\begin{vmatrix} \_ & \_ & \_ \\ \_ & \_ & \_ \\ \_ & \_ & \_ \end{vmatrix} - 3\cdot\begin{vmatrix} \_ & \_ & \_ \\ \_ & \_ & \_ \\ \_ & \_ & \_ \end{vmatrix} + 5\cdot\begin{vmatrix} \_ & \_ & \_ \\ \_ & \_ & \_ \\ \_ & \_ & \_ \end{vmatrix} - 7\cdot\begin{vmatrix} \_ & \_ & \_ \\ \_ & \_ & \_ \\ \_ & \_ & \_ \end{vmatrix}$$

(e) In your notes, write a few sentences describing the process for finding determinants of square matrices.

▲

Example 2.37. In this example we will work through the determinant of the 3×3 matrix

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & -5 \\ 0 & 0 & 3 \end{pmatrix}$$

Solution: Let's expand along the first row:

$$\begin{aligned} \det(A) &= 1 \cdot \begin{vmatrix} 2 & -5 \\ 0 & 3 \end{vmatrix} - 0 \cdot \begin{vmatrix} 0 & -5 \\ 0 & 3 \end{vmatrix} + 3 \cdot \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} \\ &= 1 \cdot ((2)(3) - (0)(-5)) - 0 \cdot ((0)(3) - (0)(-5)) + 3 \cdot ((0)(0) - (0)(2)) \\ &= 1 \cdot 6 - 0 \cdot 0 + 3 \cdot 0 \\ &= 6. \end{aligned}$$

Also notice in this example that the entire lower triangle of the matrix is filled with zeros. When this is the case you may observe the nice pattern that the determinant is actually just the product of the entries on the main diagonal (you should prove that

this is true). Hence, in this problem we know that $\det(A) = 1 \cdot 2 \cdot 3 = 6$. Be careful! If you don't have an entire triangle of zeros then this little *trick* will not work.

Problem 2.38. Find the determinant of the following matrices. Is there anything special that you can say about these matrices? Do you notice any ways to make the determinant computation faster on these matrices?

$$A = \begin{pmatrix} 1 & 3 \\ 6 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & 3 & 2 \\ 4 & 7 & 3 \\ 1 & 0 & 5 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 3 & 2 \\ 0 & 0 & 3 \\ 1 & 0 & 5 \end{pmatrix}$$

$$E = \begin{pmatrix} 2 & 3 & 2 \\ 0 & 7 & 3 \\ 0 & 0 & 5 \end{pmatrix}$$

▲

Problem 2.39. Find the value of k so that the matrix A is not invertible.

$$A = \begin{pmatrix} 2 & 4 \\ 3 & k \end{pmatrix}$$

▲

Problem 2.40. Given the matrix

$$B = \begin{pmatrix} 2-x & 1 \\ 4 & 2-x \end{pmatrix}$$

find all of the values of x that are solutions to the equation $\det(B) = 0$. ▲

The following Theorem states several properties of determinants. While these are all very useful in their own right, we will not take the time here to expand upon their proofs.

Theorem 2.41 (Important Properties of Determinants)**.** Let A be a square matrix.

1. The determinant of the identity matrix is 1.
2. The $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$
3. $\det(A^T) = \det(A)$.
4. $\det(AB) = \det(A)\det(B)$

5. $\det(A^{-1}) = 1/\det(A)$
6. If A is an $n \times n$ matrix and c is a real number then $\det(cA) = c^n \det(A)$
7. If a multiple of one row of A is added to another row to produce matrix B then $\det(A) = \det(B)$.
8. If two rows are interchanged in matrix A to produce matrix B then $\det(B) = -\det(A)$.
9. If one row of A is multiplied by k to product matrix B then $\det(B) = k\det(A)$.
10. The absolute value of the determinant of a matrix A is the volume of the parallelepiped formed by the column vectors of A. (In 2D this means that the determinant is the area of the associated parallelogram.)

In Theorem 2.41, every property listed has significant impact on the computation of determinants and inverses. The final activity in this section illustrated the time savings and possible pitfalls that can occur.

Problem 2.42. In this activity we'll explore a few properties of the determinant.

(a) How do $\det(A)$ and $\det(A^{-1})$ relate to each other? Hint: we know that $AA^{-1} = I$.

(b) Determine if the matrix B is invertible without trying to calculate the inverse.

$$B = \begin{pmatrix} 2 & 4 & -3 \\ 0 & 0 & 7 \\ 0 & 0 & 3 \end{pmatrix}$$

(c) If the entries in a 10×10 matrix A are known to within a 5% error, what is the maximum error in the determinant computation?
Hint: Let $c = 1.05$. What is $\det(cA)$?

▲

The last problem in the previous problem should serve as a warning! When doing a determinant computation on a computer there is natural roundoff error due to the fact that any number stored in a computer can only be stored with finite bits. Hence, every determinant computation on a large matrix should be immediately suspect! For this reason, it is often preferred to avoid the use of the determinant if at all possible when dealing with large matrices.

Problem 2.43. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\det(A) = 8$ then what is $\det(B)$ where $B = \begin{pmatrix} a & b \\ 3c & 3d \end{pmatrix}$ ▲

Problem 2.44. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\det(A) = 8$ then what is $\det(B)$ where $B = \begin{pmatrix} a & b \\ 2a+c & 2b+d \end{pmatrix}$
▲

Problem 2.45. True or False: The determinant of A is the same as the determinant of the row reduced form of A. Explain your answer. ▲

The last statement of Theorem 2.41 give a bit of a deeper insight to the geometry of determinants and why invertible matrices must have a non-zero determinant. Think about a 2×2 matrix with non-zero determinant. Under matrix multiplication by this matrix, a shape with non-zero area will be transformed to another shape with non-zero area. Hence, if we were to reverse the mapping then we have all of the vectors accounted for and can reverse the transformation. If, on the other hand, a matrix has a determinant of zero then a shape with non-zero area will be transformed into a shape with zero area. Naturally, in this case, many vectors will be mapped on top of each other and any hope of reversing the transformation is lost – hence the fact that a matrix with a zero determinant is not invertible.

For example, if we consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, the determinant is $\det(A) = 4 - 1 = 3 \neq 0$ and we know that A^{-1} exists. To see the action on the vectors $u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ see Figure 2.4. Notice that before the multiplication by A the parallelogram (square) formed by u_1 and u_2 is 1 (since the determinant of the identity is 1). After the multiplication by A the original square is stretched into the parallelogram on the right of Figure 2.4 and with some geometry we can see that the area of the parallelogram is 3. If we were to imagine reversing the transformation, morphing the red parallelogram back into the blue square, we can see visually how each vector in the plane gets stretched and rotated – hence giving a good meaning to *reversing* the transformation and giving us a visual sense that the inverse of A exists.

Now consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 0.5 \end{pmatrix}$. The determinant of this matrix is clearly zero, and geometrically in Figure 2.5 we see that the square gets *squished* into a single line segment with zero area. In fact, both of the vectors u_1 and u_2 are mapped to exactly the same line segment and figuring out how to reverse these actions for every vector in the plane is impossible – hence giving us a sense that if the determinant is zero then the inverse of the matrix A must not exist.

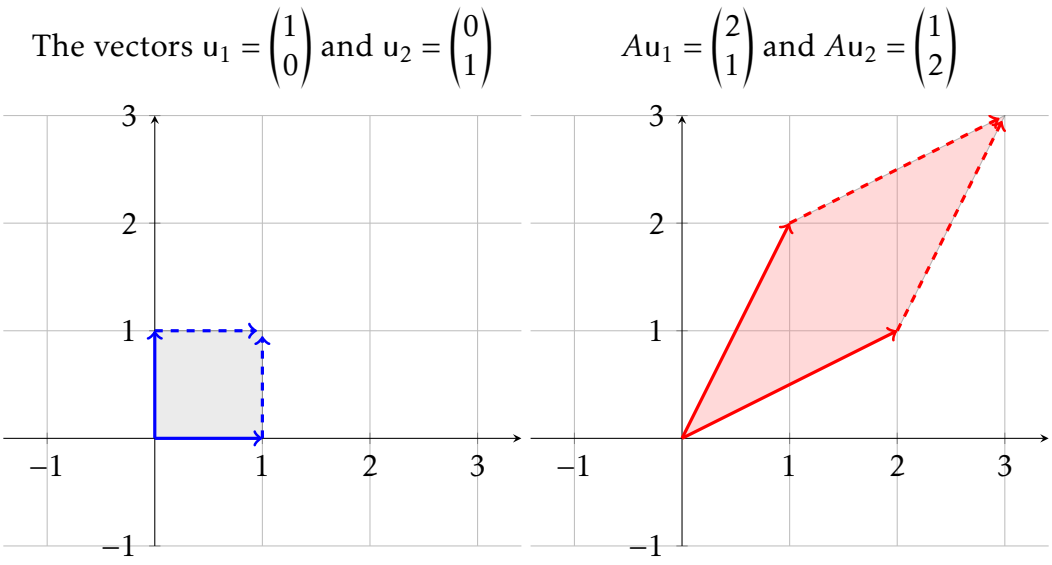


Figure 2.4. A 2D mapping from region with non-zero area to a region non-zero area.

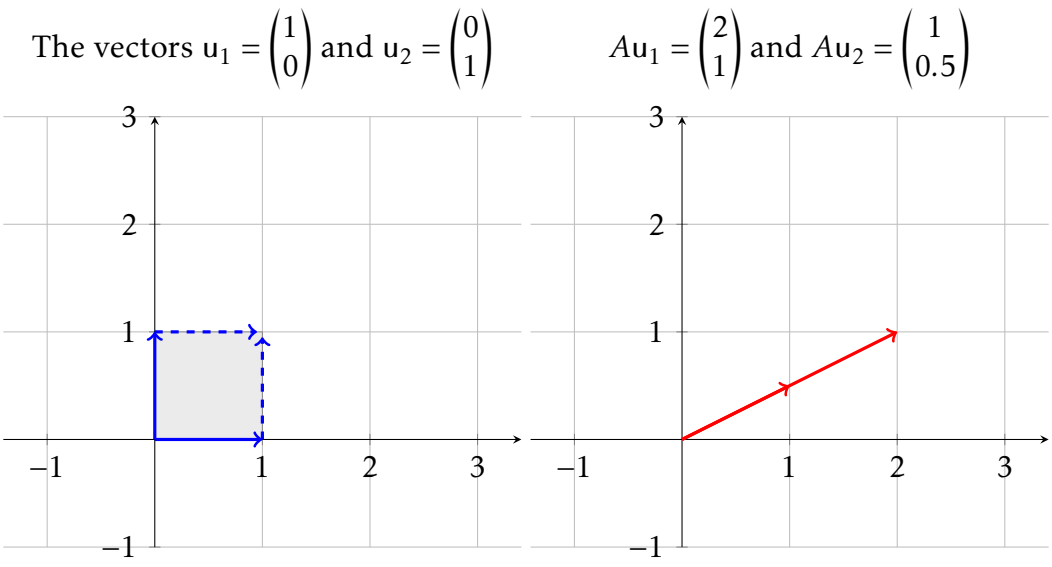


Figure 2.5. A 2D mapping from a region non-zero area to a region with zero area.

2.7 Technology For Linear Algebra

There are many technological tools designed to efficiently perform linear algebra operations. At this stage in many student's academic careers they are very familiar with the TI calculators (TI-89, TI-Nspire, TI-Voyage 200, etc). The computer software MatLab is also very prevalent in many colleges and industries as well as Maple, Mathematica, Sage, MS Excel, and many many others. The following is a very brief guide to using the TI Calculator and MatLab for common operations. If the reader wants more information about how

to use these tools they are only an internet search away.

Entering matrices: Consider the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

MatLab: A matrix is written in square brackets with columns separated by commas and rows separated by semicolons.

```
A = [1 , 2 , 3 ; 4 , 5 , 6 ; 7 , 8 , 9]
```

TI Calculator: A matrix is written with curly brackets around the entire matrix and curly brackets around each row. The rows and entries are separated by commas.

```
{{1,2,3},{4,5,6},{7,8,9}} -> A
```

The right arrow is the STO button. Storing the matrix as a letter will allow you to do computations with the matrix.

Arithmetic: Addition, subtraction, and multiplication all work as expected in both pieces of software.

Transpose: The transpose switches the rows and columns.

MatLab: The apostrophe is the transpose operator in MatLab. For example, $B = A^T$ in MatLab is

```
B = [1 , 2 , 3 ; 4 , 5 , 6 ; 7 , 8 , 9]'
```

TI Calculator: After entering your matrix and storing it as a variable, the additional matrix operations are found under the MATH – Matrix menu. The MATH menu is 2ND – 5.

Reduced Row Echelon Form: Both pieces of software will perform elementary row operations with the `rref` command.

Solving Systems in MatLab: The `rref` command will give the reduced form of the matrix, but to solve a system, MatLab has a very powerful system solver: the \. As an example, consider the matrix equation

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 7 \end{pmatrix} \quad \text{symbolically:} \quad A\mathbf{x} = \mathbf{b}$$

```
A = [1 , 2 , 3 ; 4 , 5 , 6 ; 7 , 8 , -2];
b = [-3 ; 2 ; 7];
x = A\b
```

2.8 The Magic Carpet Ride

Problem 2.46 (The Magic Carpet Ride 1)**.** You are a young traveler leaving home for the first time. Your parents want to help you on your journey, so just before your departure they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

- If you traveled "forward" on the hover board for one hour it wold move along a diagonal path that would result in a displacement of 3 miles East and 1 mile North of the starting location. Mathematically, the hover board's motion is restricted to the vector $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$
- If you traveled "forward" on the magic carpet for one hour it wold move along a diagonal path that would result in a displacement of 1 mile East and 2 miles North of the starting location. Mathematically, the magic carpet's motion is restricted to the vector $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Your Uncle Euler suggests that your first adventure should be to go visit the wise man, Old Man Gauss. Uncle Euler tells you that Old Man Gauss lives in a cabin that is 107 miles East and 64 miles North of your home. Can you use the hover board and the magic carpet to get to Old Man Gauss' cabin? Be able to defend your answer. ▲

Problem 2.47 (Magic Carpet Ride 2)**.** Old Man Gauss wants to move to a cabin in a different location. You are not sure whether Gauss is just trying to test your wits at finding him or if he actually wants to hide somewhere that you can't visit him.

Are there some locations that he can hide and you cannot reach him with using the hover board and the magic carpet? Describe the places that you can reach using a combination of the hover board and the magic carpet and those you cannot. Be able to support your answers. ▲

Problem 2.48 (Magic Carpet Ride 3)**.** Suppose now that you get a third mode of transporation: a jet pack!. In this new scenario assume that your three modes of transportation work as follows:

- The hover board's motion is restricted to the vector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.
- The magic carpet's motion is restricted to the vector $\mathbf{v}_2 = \begin{pmatrix} 4 \\ 1 \\ 6 \end{pmatrix}$.
- The jet pack's motion is restricted to the vector $\mathbf{v}_3 = \begin{pmatrix} 6 \\ 3 \\ 8 \end{pmatrix}$.

You are allowed to use each mode of transportation **EXACTLY ONCE** (in the forward or backward direction) for a fixed amount of time (c_1 on v_1, c_2 on v_2, and c_3 on v_3). Find the amounts of time on each mode of transportation (c_1, c_2, and c_3 respectively) need t to go on a journey that starts and ends at home $(0, 0, 0)$ OR explain why it is not possible to do so. ▲

Problem 2.49 (Magic Carpet Ride 4)**.** Modify the jet pack's restriction so that it is not possible to ride each mode of transportation exactly once and end up back at home. ▲

2.9 Span

The importance of linear algebra cannot be under stated. Indeed, most every mathematician will agree that linear algebra is the most important mathematical subject that a mathematical scientist can possibly learn. Let's say that again

> **Linear algebra is the most important mathematical subject a student can learn!**

As such, this section and the next few are dedicated to a few of the fundamental theoretical ideas behind linear algebra. The presentation in these section is restricted to matrices and vectors but we'll soon see that the ideas presented herein are applicable in a very wide variety of areas.

Now let's formalize a few of the ideas that we explored in the Magic Carpet Ride problems.

Definition 2.50. The **span** of a collection of vectors $\{u_1, u_2, \ldots, u_n\}$ is the set

$$\{c_1u_1 + c_2u_2 + \cdots c_nu_n : c_j \in \mathbb{R}\}$$

This is the set of all linear combinations of the vectors $u_1, \ldots, u_n$.

Problem 2.51. In this activity we will explore the concept of span. The formal definition will be delayed slightly. Instead we will use technology in this activity to build intuition. Open the GeoGebra applet http://tube.geogebra.org/student/m1254137 in a browser window and use the applet to help answer the following questions.

Loosely speaking, the **span** of a set of vectors is the collection of all vectors that can be formed by taking linear combinations.

(a) If $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ then the span of u and v is the set of all vectors of the form $c_1u + c_2v$. Use the applet to geometrically describe span($\{u, v\}$).

(b) Describe the span of $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v = \begin{pmatrix} -3 \\ -6 \end{pmatrix}$ geometrically.

(c) Is the zero vector always part of the span of a collection of vectors? Explain?

(d) What must be true (geometrically) about two 2-dimensional vectors if their span is the entire two-dimensional plane?

(e) What must be true (geometrically) about two 2-dimensional vectors if their span is a line in the two-dimensional plane?

▲

Now we'll officially define the span of a collection of vectors. In order to do that we'll define a few symbols that are very useful in mathematics.

Definition 2.52. The space $\mathbb{R}^n$ is the set of order n-tuples with real entries. For example, $\mathbb{R}^2$ is the set of all ordered pairs $\mathbb{R}^2 = \{(x, y)$ such that x and y are real numbers$\}$ that make up 2D space (you're probably used to this set from prior math classes). The set $\mathbb{R}^3$ is the set of ordered triples $\mathbb{R}^3 = \{(x, y, z)$ such that x, y and z are real numbers$\}$ that make up 3D space.

Example 2.53. What is the span of the vectors $\mathsf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\mathsf{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$ in $\mathbb{R}^3$? In other words, what is

$$\text{span}\left(\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}\right)?$$

Is the vector $\mathsf{u} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ in the span of v_1 and v_2?

Solution: The span is the collection of all linear combinations of the two vectors. Therefore, any vector w that is in the span is a linear combination of v_1 and v_2:

$$\mathsf{w} = c_1 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}.$$

Since there are only two vectors in this three-dimensional space the span will be a plane in $\mathbb{R}^3$ (shown in Figure 2.6). To test if u is in $\text{span}(\{\mathsf{v}_1, \mathsf{v}_2\})$ we need to see if there are constants c_1 and c_2 such that $c_1\mathsf{v}_1 + c_2\mathsf{v}_2 = \mathsf{u}$. Setting this up as a system of equations we see that we want to solve

$$c_1 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

for c_1 and c_2. This vector equation can be rewritten as an augmented system and row reduced as follows:

$$\left(\begin{array}{cc|c} 1 & 0 & 2 \\ 2 & -1 & -1 \\ -1 & 3 & 3 \end{array}\right) \to \cdots \to \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & -10 \end{array}\right).$$

Given that the system of equations does not have a unique solution we see that u is not in the span of v_1 and v_2.

Problem 2.54. In each of the following, determine if w is in the span of the other vectors. If it is then express w as a linear combination of the other vectors.

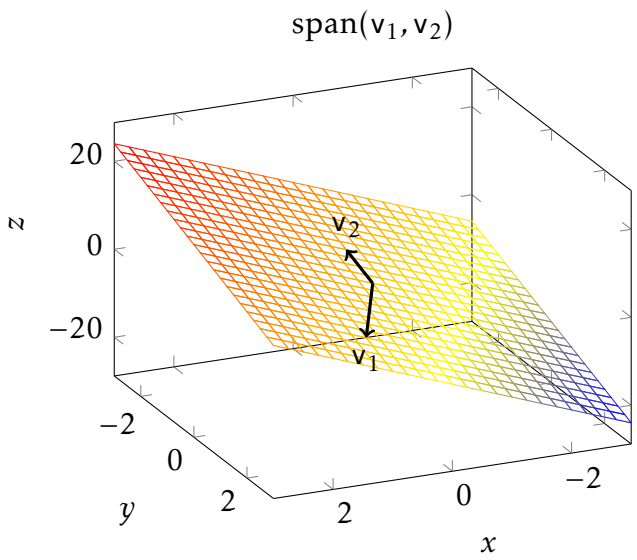


Figure 2.6. The span of $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbb{R}^3$.

(a) $\mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and $\mathbf{w} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$. (Use the applet http://tube.geogebra.org/student/m1254137 to visualize this problem)

(b) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 5 \\ -6 \\ 8 \end{pmatrix}$, and $\mathbf{w} = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$.

▲

2.10 Linear Independence, Linear Dependence, and Basis

The span of the vectors $\mathsf{u} = (1,0)^T$ and $\mathsf{v} = (0,1)^T$ is the entire two dimensional plane $\mathbb{R}^2$. This means that if we have a vector in $\mathbb{R}^2$, say $\mathsf{x} = (3,5)^T$ then we know that it can be written as a linear combination of the vectors u and v:

$$\begin{pmatrix}3\\5\end{pmatrix} = 3\begin{pmatrix}1\\0\end{pmatrix} + 5\begin{pmatrix}0\\1\end{pmatrix}.$$

The vector $\mathsf{x} = (3,5)^T$ can be built from the vectors u and v so, in some sense, x *depends* on u and v. We would say that the set of vectors $\{\mathsf{u},\mathsf{v},\mathsf{x}\}$ is *linearly dependent* since one of the vectors in the set can be built from linear combinations of the others. The set $\{\mathsf{u},\mathsf{v}\}$ is called *linearly independent* since none of the vectors in the set can be built from linear combinations of the other vectors in the set.

Definition 2.55. A collection of vectors $\{\mathsf{v}_1,\mathsf{v}_2,\ldots,\mathsf{v}_p\}$ in $\mathbb{R}^n$ is called **linearly independent** if the vector equation

$$c_1\mathsf{v}_1 + c_2\mathsf{v}_2 + \cdots c_p\mathsf{v}_p = 0$$

has *only* the trivial solution $c_1 = 0, c_2 = 0, \ldots, c_p = 0$.

Definition 2.56. A collection of vectors $\{\mathsf{v}_1,\mathsf{v}_2,\ldots,\mathsf{v}_p\}$ is called **linearly dependent** if it is not linearly independent.

Example 2.57. Determine whether or not the vectors v_1, v_2, and v_3 are linearly independent or dependent.

$$\mathsf{v}_1 = \begin{pmatrix}1\\2\\3\end{pmatrix} \quad \mathsf{v}_2 = \begin{pmatrix}3\\-2\\0\end{pmatrix} \quad \mathsf{v}_3 = \begin{pmatrix}4\\1\\1\end{pmatrix}$$

Solution: To determine if $\mathsf{v}_1, \mathsf{v}_2$, and v_3 are linearly independent or not we need to consider the equation

$$c_1\mathsf{v}_1 + c_2\mathsf{v}_2 + c_3\mathsf{v}_3 = 0.$$

If the solution to this equation is that $c_1 = c_2 = c_3 = 0$ then the vectors will be linearly independent. Otherwise they will be linearly dependent. This can be viewed as a question about the solutions to the following system of equations (which we will write in several different ways):

$$c_1\begin{pmatrix}1\\2\\3\end{pmatrix} + c_2\begin{pmatrix}3\\-2\\0\end{pmatrix} + c_3\begin{pmatrix}4\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix},$$

which is the same as

$$\begin{cases} c_1 + 3c_2 + 4c_3 &= 0 \\ 2c_1 - 2c_2 + c_3 &= 0 \\ 3c_1 + c_3 &= 0 \end{cases}$$

which is the same as the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & -2 & 1 & 0 \\ 3 & 0 & 1 & 0 \end{array}\right)$$

Row reducing the augmented form gives us

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

Hence, we can see that the solution to the system of equations is $c_1 = c_2 = c_3 = 0$. This implies that $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent.

Now let's look at one more example.

Example 2.58. Determine whether or not the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent or dependent.

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \quad \mathbf{v}_3 = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$$

Solution: We'll start by writing the system of equations $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = 0$ in multiple ways just as we did before.

$$c_1\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ or } \quad \begin{cases} c_1 + 3c_2 + 4c_3 &= 0 \\ 2c_1 - 2c_2 &= 0 \\ 3c_1 + 3c_3 &= 0 \end{cases} \text{ or } \quad \left(\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & -2 & 0 & 0 \\ 3 & 0 & 3 & 0 \end{array}\right)$$

Row reducing the augmented form gives us

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Now we can see that there are infinitely many solutions to the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = 0$ and so the vectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent.

Moreover, we can see that $c_1 + c_3 = 0$ and $c_2 + c_3 = 0$. We can rewrite these equations

as $c_1 = -c_3$ and $c_2 = -c_3$, and therefore c_1, c_2 and c_3 must satisfy the equation

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} t, \quad \text{for} \quad t \in \mathbb{R}.$$

The fact that there are infinitely many solutions to the system of equations further illustrates the idea that the vectors $\mathsf{v}_1, \mathsf{v}_2$ and v_3 are linearly dependent: the coefficients of c_1 and c_2 depend on the value for c_3.

Problem 2.59. Use the definitions of linearly independent and linearly dependent to answer the following questions.

(a) Are the vectors $\mathsf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathsf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ linearly independent or dependent? Explain.

(b) Are the vectors $\mathsf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathsf{v}_2 = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$ linearly independent or dependent? Explain.

(c) Are the vectors $\mathsf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \mathsf{v}_2 = \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}, \mathsf{v}_3 = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$, and $\mathsf{v}_4 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ linearly independent or dependent? Explain.

(d) If nonzero vectors v_1 and v_2 in $\mathbb{R}^2$ only span a line in $\mathbb{R}^2$ are they linearly independent or linearly dependent?

(e) If nonzero vectors v_1, v_2 and v_3 in $\mathbb{R}^3$ span all of $\mathbb{R}^3$ are they linearly independent or linearly dependent?

▲

Loosely speaking, linear independence means that no vector in the collection can be *built* from the other vectors in the collection. Linearly dependence, on the other hand, means that at least one vector can be built from the other vectors in the collection. The real beauty of linearly independent sets is that they can serve as very simple descriptors of a much larger set. For example, the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are linearly independent (as seen in the previous activity) and the span of these two simple vectors is all of $\mathbb{R}^2$. These two facts together mean that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the simplest building blocks for the entire xy-plane; there is no unneeded information and there is enough to build every point. In some sense, these two vectors are the DNA of the xy-plane.

There are many other linearly independent spanning sets for $\mathbb{R}^2$ but the amazing fact is that they all have exactly 2 vectors in them. Similarly, all of the linearly independent spanning sets of $\mathbb{R}^3$ will all contain exactly 3 vectors. In fact, this is where our intuitive notion of dimensionality comes from; the number of vectors in the linearly independent spanning set.

Definition 2.60. If the set of vectors $\mathcal{B} = \{b_1, b_2, \ldots, b_p\}$ spans a space V then $\mathcal{B}$ is said to be a **basis** of V if it is linearly independent. In other words, a basis is a linearly independent spanning set.

Example 2.61. Show that $\mathcal{B} = \left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}\right\}$ is a basis for $\mathbb{R}^2$.

Solution: We first need to show that $\mathcal{B}$ is linearly independent. This means that we need to show that the equation $c_1 b_1 + c_2 b_2 = 0$ has only the trivial solution $c_1 = c_2 = 0$. Indeed,

$$\begin{aligned}
&c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}\\
&\implies \begin{pmatrix}1 & -1\\2 & 1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}\\
&\implies \left(\begin{array}{cc|c}1 & -1 & 0\\2 & 1 & 0\end{array}\right) \to \cdots \to \left(\begin{array}{cc|c}1 & 0 & 0\\0 & 1 & 0\end{array}\right)\\
&\implies c_1 = c_2 = 0.
\end{aligned}$$

Hence, $\mathcal{B}$ is a linearly independent set.

To show that $\mathcal{B}$ spans $\mathbb{R}^2$ we need to show that any vector $\begin{pmatrix}x\\y\end{pmatrix}$ can be *built* as a linear combination of the vectors in $\mathcal{B}$. Indeed,

$$\begin{aligned}
&c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}\\
&\implies \begin{pmatrix}1 & -1\\2 & 1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}\\
&\implies \left(\begin{array}{cc|c}1 & -1 & x\\2 & 1 & y\end{array}\right) \to \cdots \to \left(\begin{array}{cc|c}1 & 0 & \frac{x+y}{3}\\0 & 1 & \frac{-2x+y}{3}\end{array}\right)\\
&\implies c_1 = \frac{x+y}{3} \quad \text{and} \quad c_2 = \frac{-2x+y}{3}.
\end{aligned}$$

Hence, given **any** point in the xy-plane we can find c_1 and c_2 that will build the point as a linear combination of vectors in $\mathcal{B}$. Therefore, $\mathcal{B}$ is a basis for $\mathbb{R}^2$.

Example 2.62. Is $\mathcal{B} = \left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\4\end{pmatrix}\right\}$ a basis for $\mathbb{R}^2$?

Solution: No. The reader should verify that the two vectors in $\mathcal{B}$ are linearly dependent so they cannot possible span all of $\mathbb{R}^2$.

Problem 2.63. (a) Find a basis for the set of vectors in $\mathbb{R}^2$ on the line $y = 3x$.

(b) Find a basis for the space spanned by the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_5$.

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 2 \\ -2 \\ -8 \\ 0 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 2 \\ -1 \\ 10 \\ 3 \end{pmatrix}, \quad \mathbf{v}_5 = \begin{pmatrix} 3 \\ -1 \\ -6 \\ 9 \end{pmatrix}.$$

(Use technology to complete this exercise to save a bit of time)

(c) Let $\mathbf{v}_1 = \begin{pmatrix} 4 \\ -3 \\ 7 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 9 \\ -2 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 7 \\ 11 \\ 6 \end{pmatrix}$, and also let $H = \text{span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$. It can be verified that $4\mathbf{v}_1 + 5\mathbf{v}_2 - 3\mathbf{v}_3 = \mathbf{0}$. Use this information to find a basis for H. (There is more than one answer)

▲

Problem 2.64. Mark each statement as True or False. Justify each answer.

(a) A single vector by itself is linearly dependent.

(b) If $H = \text{span}(\{\mathbf{b}_1, \ldots, \mathbf{b}_p\})$ then $\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a basis for H.

(c) The columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$.

(d) A basis is a spanning set that is as large as possible.

(e) Any linearly independent set in a space V is a basis for V.

(f) If a finite set S of nonzero vectors spans a space V, then some subset of S is a basis for V.

(g) A basis is a linearly independent set that is as large as possible.

▲

2.11 The Column and Null Spaces of a Matrix

For any matrix there are several fundamental spaces that fully describe the actions that a matrix can have on a vector. The two that we focus on here are the column space and the null space.

Definition 2.65. The **column space** of a matrix A is the space that is spanned by the columns of the matrix.

Definition 2.66. The **null space** of an $m \times n$ matrix A, written $\text{Nul}(A)$, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Example 2.67. Find the null and column spaces of the matrix

$$A = \begin{pmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{pmatrix}$$

and determine if $\mathbf{u} = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}$ belongs to the null space of A.

Solution: The column space of A is the space spanned by the columns of A. We need to determine if the three columns are linearly independent. Row reduction shows us that

$$\begin{pmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{pmatrix} \to \cdots \to \begin{pmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & \frac{3}{2} \end{pmatrix}$$

so the third column vector depends on the other two. Hence, there are two linearly independent vectors that form a basis for the column space. Since each vector is in $\mathbb{R}^2$ we see that the column space is all of $\mathbb{R}^2$.

The null space of A is the set of solutions to $A\mathbf{x} = \mathbf{0}$. We have already row reduced the homogeneous system so we know that $x_1 = -5/2x_3$ and $x_2 = -3/2x_3$. Hence, if $x_3 = t$ then

$$\mathbf{x} = \begin{pmatrix} -5/2t \\ -3/2t \\ t \end{pmatrix} = \begin{pmatrix} -5/2 \\ -3/2 \\ 1 \end{pmatrix} t$$

where t is any real number.

Clearly, if we take $t = -2$ then $\mathbf{x} = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}$ showing that $\mathbf{u}$ is in the null space of A.

Another (simpler) way to see that u is in the null space of A is to observe that

$$A\mathbf{u} = \begin{pmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 5-9+4 \\ -25+27-2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Problem 2.68. Find the null and column spaces for the matrix

$$A = \begin{pmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{pmatrix}.$$

The matrix A is 3×4. Be sure to specify where the null and column spaces

▲

2.12 Modeling Explorations with Linear Algebra

In 1973, Wassily Leontief was awarded the Nobel prize for his work in economics. Part of his work was to apply the basic concepts of linear algebra to model supply and demand within simple economies. His theory, called the Leontief Input Output model, serves as a simplified model to predict production.

Problem 2.69. Suppose that an economy is divided into three sectors: manufacturing, agriculture, and services. Let $\mathbf{x}$ be a 3×1 vector (called the *production vector*) that lists the outputs of each sector for one year. At the same time, let $\mathbf{d}$ be a 3×1 vector (called the *demand vector*) that lists any external demands on the goods and services from the non productive part of the economy (consumer demand, government consumption, exports, etc.)

As the three sectors produce goods to meet consumer demand, the producers themselves create additional *intermediate demand* for goods they need as inputs for their own production. For example, the agriculture sector will use equipment made by the manufacturing sector. The basic question Leontief asked is "is there a production level $\mathbf{x}$ such that the amounts produced will exactly balance the total demand for the production?" In other words, find $\mathbf{x}$ so that

$$\underbrace{\text{amount produced}}_{\mathbf{x}} = \underbrace{\text{intermediate demand}}_{C\mathbf{x}} + \underbrace{\text{external demand}}_{\mathbf{d}}.$$

Here we take C to be a matrix defining unit consumptions of the various sectors (see the table).

	Inputs per Unit of Output		
Purchased From:	**Manufacturing**	**Agriculture**	**Services**
Manufacturing	0.50	0.40	0.20
Agriculture	0.20	0.30	0.10
Services	0.10	0.10	0.30

In the table, the columns read as follows:

- To produce 1 unit, manufacturing needs 0.50 units from other parts of manufacturing, 0.20 units from agriculture, and 0.10 units from services.
- To produce 1 unit, agriculture needs 0.40 units from manufacturing, 0.30 units from other parts of agriculture, and 0.10 units from services.
- To produce 1 unit, services need 0.20 units from manufacturing, 0.10 units from agriculture, and 0.30 units from other services.

Suppose that the external demand is 50 units for manufacturing, 30 units for agriculture, and 20 units for services.

(a) If the production is

$$\mathbf{x} = \begin{pmatrix} 140 \\ 20 \\ 50 \end{pmatrix}$$

then what is the sum of the intermediate demand and the external demand? Did the economy under or over produce in each of the sectors?

(b) We want to find the production $\mathbf{x}$ that gives a perfectly balance economy: $\mathbf{x} = C\mathbf{x}+\mathbf{d}$. How do we solve this matrix equation?

▲

2.12.1 Input-Output Economies

In the previous problem we explored situations where a simple economy over or under produces. Theoretically, a perfectly functioning economy has no surplus or shortage. We would like to explore Leontief's original question and determine if there is a perfect production level that will yield no surplus or shortage.

The matrix equation associated with the Leontif Input Output model is

$$\mathbf{x} = C\mathbf{x} + \mathbf{d}. \tag{2.11}$$

The trouble is that the production level, $\mathbf{x}$, shows up on both sides of the matrix equation. To overcome this fact we do a few simple steps of matrix arithmetic to equation (2.11):

$$\mathbf{x} = C\mathbf{x} + \mathbf{d} \quad \Longrightarrow \quad \mathbf{x} - C\mathbf{x} = \mathbf{d} \quad \Longrightarrow \quad (I - C)\,\mathbf{x} = \mathbf{d} \quad \Longrightarrow \quad \mathbf{x} = (I - C)^{-1}\,\mathbf{d}. \tag{2.12}$$

The right-hand equation of (2.12) shows that there is a ready-made formula that allows us to solve for the production level.

Example 2.70. Solve for the optimal production level using the internal and external demands from Problem 2.69.
Solution: We want to find the production level $\mathbf{x}$ given that the consumption matrix C and the demand vector $\mathbf{d}$ are

$$C = \begin{pmatrix} 0.5 & 0.4 & 0.2 \\ 0.2 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.3 \end{pmatrix} \quad \text{and} \quad \mathbf{d} = \begin{pmatrix} 50 \\ 30 \\ 20 \end{pmatrix}.$$

From equation (2.12) we need to solve the equation $(I - C)\mathbf{x} = \mathbf{d}$ by evaluating $\mathbf{x} = (I - C)^{-1}\mathbf{d}$.

It is impractical to actually compute the inverse[a], so instead we can create an augmented matrix $(I - C\,|\,\mathbf{d})$ and use row reduction.

$$(I - C\,|\,\mathbf{d}) = \left(\begin{array}{ccc|c} 0.5 & -0.4 & -0.2 & 50 \\ -0.2 & 0.7 & -0.1 & 30 \\ -0.1 & -0.1 & 0.7 & 20 \end{array}\right) \to \cdots \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 225.9 \\ 0 & 1 & 0 & 118.5 \\ 0 & 0 & 1 & 77.8 \end{array}\right).$$

Hence, in this economy, manufacturing must produce approximately 226 units, agriculture must produce approximately 119 units, and services must produce approximately 78 units. This will lead to a balanced economy where this is no surplus or shortage.

[a] In the vast majority of linear algebra applications the inverse is never computed.

Problem 2.71. The consumption matrix C below is based on input-output data for the U.S. economy in 1958, with data for 81 sectors grouped into 7 larger sectors:

1. nonmetal household and personal products,
2. final metal products (such as motor vehicles),
3. basic metal products and mining,
4. basic nonmetal products and agriculture,
5. energy,
6. services, and
7. entertainment and miscellaneous products.

$$C = \begin{pmatrix} 0.1588 & 0.0064 & 0.0025 & 0.0304 & 0.0014 & 0.0083 & 0.1594 \\ 0.0057 & 0.2645 & 0.0436 & 0.0099 & 0.0083 & 0.0201 & 0.3414 \\ 0.0264 & 0.1506 & 0.3557 & 0.0139 & 0.0142 & 0.0070 & 0.0236 \\ 0.3299 & 0.0565 & 0.0495 & 0.3636 & 0.0204 & 0.0483 & 0.0649 \\ 0.0089 & 0.0081 & 0.0333 & 0.0295 & 0.3412 & 0.0237 & 0.0020 \\ 0.1190 & 0.0901 & 0.0996 & 0.1260 & 0.1722 & 0.2368 & 0.3369 \\ 0.0063 & 0.0126 & 0.0196 & 0.0098 & 0.0064 & 0.0132 & 0.0012 \end{pmatrix}, \quad \mathrm{d} = \begin{pmatrix} 74000 \\ 56000 \\ 10500 \\ 25000 \\ 17500 \\ 196000 \\ 5000 \end{pmatrix}$$

(a) Use technology to find the production levels needed to satisfy the final demand d.

(b) The demand in 1964 was

$$\mathrm{d} = \begin{pmatrix} 99640 & 75548 & 14444 & 33501 & 23527 & 263985 & 6526 \end{pmatrix}^T$$

Find the production levels needed to satisfy this demand.

(c) In the six years between 1958 and 1964 the demand changed drastically in several categories leading to changes in the production levels. Use the data in this problem to extrapolate to current demands and production levels. Give several reasons why your extrapolation will ultimately give poor estimates.

▲

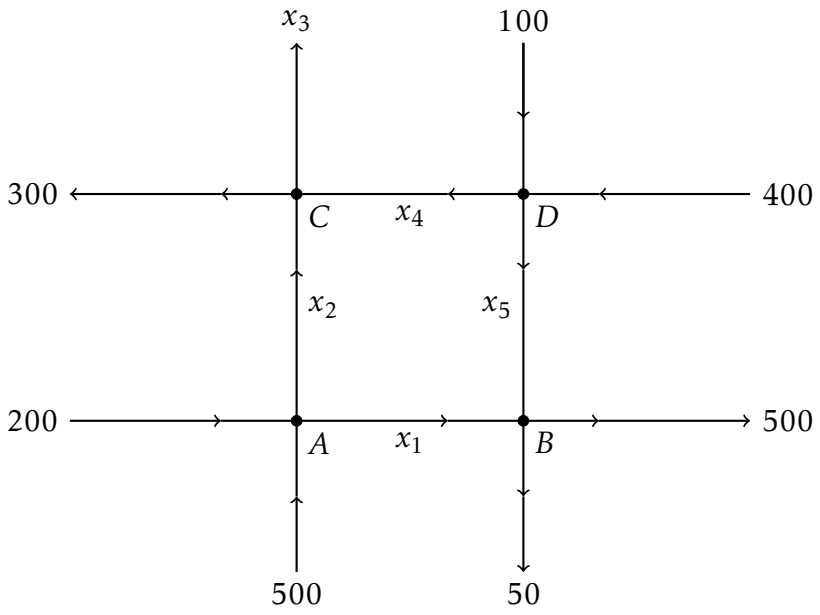
2.12.2 Traffic Networks

A traffic network can be viewed as a collection of lines (streets) and intersections. When modeling a large street network, the influx from certain suburbs and other cities is generally well known. To estimate the number of cars on given streets, traffic engineers often put counters at the intersections. This allow them to calculate flow given that

flow into an intersection = flow out of an intersection.

This simple *conservation law* creates a system of equations for all of the streets that can be solved to find the flow on each individual street.

Problem 2.72. Consider the collection of one-way streets in a downtown area.



(a) Fill in the table for each intersection. The first line has been done to get you started.

Intersection	Flow In	Flow Out
A	$200 + 500$	$x_1 + x_2$
B		
C		
D		

(b) Write the equation

total flow into the network = total flow out of the network.

This equation describes the fact that the network is not closed but the number of cars in the network must be conserved (just like conservation of mass or momentum from physics).

(c) Use parts (a) and (b) to write a system of equations. Solve the system to determine the traffic flow at $x_1, x_2, \ldots, x_5$.

(d) **Sensitivity Analysis:** There are 7 different given values in this network. What is the percent change in each of the variables $x_1, \ldots, x_5$ if each of the given values were to increase or decrease by 10%? Use a table like the one below to help organize the results, and use your results to write a short report to the traffic division.
Hint: Test each value one at a time.

		Percent Change				
Old Value	New Value	x_1	x_2	x_3	x_4	x_5
100	110					
100	90					
400	440					
400	360					
500	550					
500	450					
50	55					
50	45					
500	550					
500	450					
200	220					
200	180					
300	330					
300	270					

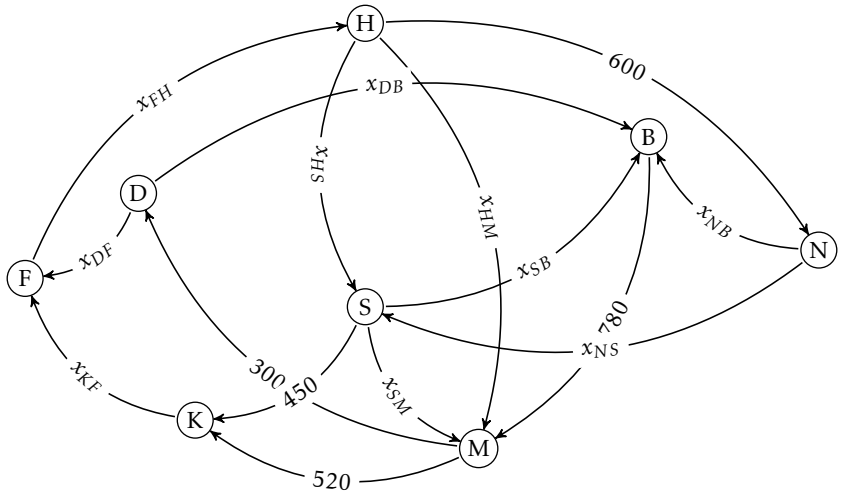
▲

Problem 2.73. Your boss at the trucking company wants you to solve the following traffic flow problem. In this problem, the lettered nodes are distribution centers for your trucking company and the numbers and variables along the arcs are the truck flows in a given year. Remember that for each distribution center

yearly flow into the distribution center = yearly flow out of the distribution center.

For example, at node H we must have $x_{FH} = x_{HS} + x_{HM} + 600$.
Hint: write an equation for each node and then rearragne into an augmented system of equations.

Your goal is to find the roads with the maximum truck volume and to perform a sensitivity analysis on the known information in the problem (i.e. if each known value were to change by as much as 10%, how much would the maximum road volume change). Give advice to your boss based on your analysis keeping in mind that the highway department would ultimately like to minimize the maximum truck volume on the roads.



▲

2.12.3 Balancing Chemical Equations

The final application that we will discuss here is that of balancing chemical equations.

Consider the chemical reaction: Nitrogen Dioxide plus water yields Nitrous acid and Nitric acid.

$$aNO_2 + bH_2O \to cHNO_2 + dHNO_3$$

The coefficients a, b, c, and d are unknown positive integers. The reaction must be balanced; that is, the number of atoms of each element must the same before and after the reaction. Because the number of atoms must remain the same we end up with the following system of equations:

Oxygen:	$2a + b = 2c + 3d$
Nitrogen:	$a = c + d$
Hydrogen:	$2b = c + d$

Rearranging this into an augmented system gives the homogeneous system

$$\left(\begin{array}{cccc|c} 2 & 1 & -2 & -3 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 2 & -1 & -1 & 0 \end{array}\right)$$

Performing row reductions on this system gives

$$\left(\begin{array}{cccc|c} 2 & 1 & -2 & -3 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 2 & -1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 2 & -1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \quad \Longrightarrow \quad \begin{array}{l} a = 2d \\ b = d \\ c = d \end{array}$$

In this chemical equation we see that d, the coefficient of Nitric acid, is a free variable. Mathematically this means that we can freely choose d to be any value we like, but scientifically we choose the smallest positive integer: $d = 1$. Hence

$$a = 2, \quad b = 1, \quad c = 1, \quad \text{and} \quad d = 1$$

so the balanced chemical equation becomes

$$2NO_2 + H_2O \to HNO_2 + HNO_3$$

Problem 2.74. Find the lower case values that balance each of the chemical equations.

(a) Aluminum oxide and carbon react to create elemental aluminum and carbon dioxide:

$$aAl_2O_3 + bC \to cAl + dCO_2.$$

(b) Limestone, $CaCO_3$, netralizes the acid, H_3O, in acid rain by following the equation

$$aH_3O + bCaCO_3 \to cH_2O + dCa + eCO_2.$$

(c) Boron sulfide reacts violently with water for form boric acid and hydrogen sulfide gas (the smell of rotten eggs).

$$aB_2S_3 + bH_2O \to cH_3BO_3 + dH_2S$$

▲

Chapter 3

First Order Models

3.1 Birth, Death, and Immigration Exploration

Supplies

1 partner, 1 cup, 1 handfull or bag of M&M's, and 1 computer with access to Moodle and Excel.

Scenarios

The M&M's in this activity represent a very fragile species with a high death rate. To determine which M&M's survive and which die in a given year we will shake the M&M's and pour them on the table. The M&M's that land with the M up unfortunately die. The remaining M&M's survive for another year.

Scenario #1 (Deaths) Count the number of M&M's in your cup and track the number of M&M's remaining until the species goes extinct. Add your data to the Google Sheet linked from Moodle.

Scenario #2 (Deaths and Immigration): In this scenario, every M&M that lands M-up will die, but at the end of the year there are 5 new M&M's added to the population due to immigration. Track the number of M&M's remaining until you believe that the population has reached a steady state. Add your data to the second tab of the Google Sheet linked from Moodle.

Scenario #3 (Births, Deaths, and Immigration): In this final scenario, every M&M that lands M-up will die, and at the end of the year there are 5 new M&M's added to the population due to immigration. After the immigration occurs, 10% of the remaining M&M's give birth to new M&M's. Add these births to the population. Track the number of M&M's (after the death, immigration, and the births) until you think the population has reached a steady state. input your data into the Google Sheet, and move to the next year.

Mathematical Modeling

Our goal is to find a function that models each scenario. Copy your data sets from the Google Sheet to your own machines and try to determine the type of function that best models this scenario. When you write your functions, use the variable n for the time.

Remember, we are trying to find a mathematical function that best models the scenario. That does not necessarily mean that you should be doing any data fitting! Instead, you should try two methods:

1. Can you find an algebraic function that appears to fit your data?

$$M(n) = \_\_\_\_\_$$

2. Or, can you write an expression for how the population changes? In other words,

$$\text{number of } \underline{\text{new}} \text{ M\&M's at year } n = \_\_\_\_\_$$

3.2 Models for Birth, Death, and Immigration

Problem 3.1. In the birth, death, and immigration exploration you should have generated data with your class for the three scenarios. Go back to the Google Sheet with all of the data, copy your group's data to Excel and make the following plots:

- A plot with population on the horizontal axis and average rate of change of the population for the birth model on the vertical axis. You will need to calculate the average rate of change in Excel first.
- A plot with population on the horizontal axis and average rate of change of the population for the birth & death model on the vertical axis. You will need to calculate the average rate of change in Excel first.
- A plot with population on the horizontal axis and average rate of change of the population for the birth, death, & death model on the vertical axis. You will need to calculate the average rate of change in Excel first.

▲

Problem 3.2. All of the plots that you made in the previous problem *should* appear to follow a linear trend. Use Excel to fit a line to each of these plots. Use your results from Excel to fill in the following blanks.

$$\text{Death Model:} \quad \frac{\Delta P}{\Delta t} = \_\_\_\_\_$$

$$\text{Death \& Immigration Model:} \quad \frac{\Delta P}{\Delta t} = \_\_\_\_\_$$

$$\text{Birth, Death, \& Immigration Model:} \quad \frac{\Delta P}{\Delta t} = \_\_\_\_\_$$

(a) Explain why the slope of each of these linear functions is negative.

(b) Explain what the "y-intercept" means in the context of each model.

▲

In the *Birth* model you should have found a model similar to

$$\frac{\Delta P}{\Delta t} = -0.5P,$$

but notice that in our exploration of the problem with the M&M's we were taking Δt to be 1. Hence we can rewrite our model more simply as

$$\Delta P = -0.5P.$$

When we're dealing with discrete time steps as in this problem we often use the notation P_n to talk about the population at time $t = n$ and P_{n+1} to talk about the population at time $t = n + 1$. With this new notation the model becomes

$$P_{n+1} - P_n = -0.5P_n,$$

and this is known as a **difference equation** since the left-hand side of the equation gives the difference between two time steps.

Problem 3.3. Write a difference equation model for the *Death & Immigration* model and for the *Birth, Death, & Immigration model*.

Death Model: $P_{n+1} - P_n = -0.5P_n$

Death & Immigration Model: $P_{n+1} - P_n =$ _____

Birth, Death, & Immigration Model: $P_{n+1} - P_n =$ _____

▲

In a difference equation model it is often convenient to solve for P_{n+1} and rewrite as $P_{n+1} = P_n - 0.5P_n$ (in the death model). This clearly simplifies to $P_{n+1} = 0.5P_n$, and furthermore this allows for an easy implementation into a spreadsheet program.

Problem 3.4. Using a spreadsheet program like Excel, implement the difference equation models to build a table and graph that predicts the population at any discrete time. ▲

Problem 3.5. In the M&M exploration we were treating time as if it occurred only in discrete steps. What is wrong with this assumption? Specifically, if the M&M's are supposed to represent a population then what is wrong with using discrete time in our model? ▲

Let's return to the models in Problem 3.2 and now consider that time really shouldn't be taken in discrete steps. Instead, time really happens on a continuous scale and, as such, taking $\Delta t = 1$ doesn't really make sense. If we take $\Delta t \to 0$ the average rate of change $\Delta P/\Delta t$ becomes the derivative $\frac{dP}{dt}$. For example, in the *Birth* model $\frac{\Delta P}{\Delta t} = -0.5P$ becomes $\frac{dP}{dt} = -0.5P$ as $\Delta t \to 0$. This new model is called a **differential equation**.

Problem 3.6. Write a differential equation modeling the *Death & Immigration* and the *Birth, Death, & Immigration* scenarios.

Death Model: $\frac{dP}{dt} = -0.5P$

Death & Immigration Model: $\frac{dP}{dt} =$ _____

Birth, Death, & Immigration Model: $\frac{dP}{dt} =$ _____

▲

The differential equation models are most likely more realistic if we are thinking of the M&M exploration as modeling population. Populations just simply don't change all at once; they change continuously in time.

For the *Death* model

$$\frac{dP}{dt} = -0.5P$$

we can still use Excel to create a visual model of the solution to the differential equation. Recall that

$$\frac{dP}{dt} \approx \frac{\Delta P}{\Delta t}$$

and if we take Δt to be *really small* then the approximation isn't *too bad*. Hence, the differential equation $\frac{dP}{dt} = -0.5P$ can be approximated by

$$\frac{\Delta P}{\Delta t} \approx -0.5P \quad \text{for } \Delta t \text{ small,}$$

and using the notation of difference equations we can rewrite as

$$\frac{P_{n+1} - P_n}{\Delta t} \approx -0.5P_n.$$

Now if we solve for P_{n+1} we can get an equation that can be implemented in a spreadsheet program:

$$P_{n+1} \approx P_n - \Delta t\,(0.5P_n).$$

Problem 3.7. Use a spreadsheet program to get a graphical approximation to the *Death* model

$$\frac{dP}{dt} = -0.5P.$$

Compare your results to the data and to the solution to the difference equation that you built in Problem 3.4. ▲

Problem 3.8. Now use a spreadsheet program to build approximations to the differential equations for the *Death & Immigration* and *Birth, Death, & Immigration* models that you built in Problem 3.6. Compare your graphical solutions to the graphical solutions of the discrete time models and to the data. ▲

3.3 Difference Equations and Differential Equations

In Calculus, change is measured over infinitesimally small increments. Unfortunately, this is not always appropriate. For example, if a bank compounds the interest on a savings account every month then it may not make sense to examine the value of the account during the middle of the month; it will be the same as after the interest was last computed. On the other hand, if the temperature of a cup of coffee is being modeled over time then it makes sense to talk about the temperature at any given instant in time. Time in the bank account example is taken in discrete steps (one per month), while time in the coffee example is continuous.

Definition 3.9. A **discrete time model** is a mathematical model where time is measured in individual steps that can be enumerated. The model either does not make sense at intermediate steps or is considered constant.

Definition 3.10. A **continuous time model** is a mathematical model where time can take any non-negative real number quantity. In continuous time models it makes sense to discuss the meaning of the model at any non-negative value of time.

Problem 3.11. In the following scenarios decide whether you should use a discrete time model or a continuous time model.

(a) Water draining from a tank drains at a rate proportional to the square root of the height of the water.

(b) The rate of change of the money in an interest-bearing savings account is proportional to the amount of money that is in the account.

(c) The Department of Fish, Wildlife, and Parks proposes a restocking plan where they restock the mountain lakes in Montana once per year.

(d) The displacement of a spring-mass oscillator is proportional to the acceleration of the mass.

▲

As we've already seen in the M&M exploration, it was easier to model the change in the population that it was to *guess* the algebraic function that modeled the populations. The two primary types of mathematical models that model *change* are **difference equations** and **differential equations.**

Definition 3.12. A **difference equation** is a discrete-time model that relates discrete

rates of change.

$$\text{(new value)} - \text{(old value)} = \text{(change in value)}. \tag{3.1}$$

Definition 3.13. A **differential equation** is a continuous-time model that relates instantaneous rates of change.

Problem 3.14. When money is placed into a bank savings account, the bank pays interest into the account once a month, thereby increasing the value of the account, provided no money is withdrawn!

Consider a bank account that has been opened with an initial deposit of \$4,500 and which receives monthly interest equal to 1.1% of the account value. The account is to be held for two years before any withdrawals are made.

(a) should we be modeling this situation with discrete time or continuous time?

(b) For the given initial deposit and interest rate, how much money would the bank add to the savings account after one month?

(c) How much money would the savings account be worth after 1 month?

(d) What is the value of the savings account after one and a half months?

(e) How much money would there be in the savings account at the beginning of the second month?

(f) How much interest does the bank add to the account at the end of the second month?

(g) How much money is in the savings account after two months?

(h) In words, describe how the money in your savings account changes from one month to the next.

(i) Use everything that you just did to write a difference equation modeling the amount of money in the bank account.

$$A_{n+1} - A_n = \_\_\_\_\_$$

(j) Use a spreadsheet program to build a numerical and graphical solution to the difference equation.

▲

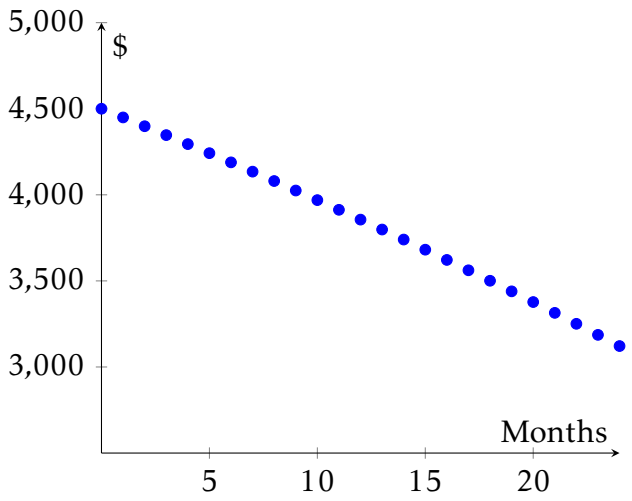
Example 3.15. Consider the savings account situation from Problem 3.14 but include a deduction of \$100 per month. Find a difference equation and find a numerical solution to model this situation over a two-year horizon.

The difference equation is

$$\underbrace{\text{new amount in account}}_{a_{n+1}} - \underbrace{\text{old amount in account}}_{a_n} = \underbrace{\text{interest}}_{0.011a_n} - \underbrace{\text{withdrawal}}_{100}$$

More simply: $a_{n+1} = a_n + 0.011a_n - 100$. Or, after some algebra, $a_{n+1} = 1.011a_n - 100$. A numerical solution is simply a table of values (see this video) but is much simpler to visualize as a plot (see this video and this video).

Month	Amount
0	\$4500.00
1	\$4449.50
2	\$4398.44
3	\$4346.83
4	\$4294.64
5	\$4241.88
6	\$4188.54
7	\$4134.62
8	\$4080.10
9	\$4024.98
10	\$3969.25
11	\$3912.92
12	\$3855.96
13	\$3798.37
14	\$3740.16
15	\$3681.30
16	\$3621.79
17	\$3561.63
18	\$3500.81
19	\$3439.32
20	\$3377.15
21	\$3314.30
22	\$3250.76
23	\$3186.52
24	\$3121.57



Example 3.16. A student has a college loan of \$$A$ with an interest rate of r% APR compounded monthly. She pays \$$p$ per month. Model the amount that the student owes on the loan with a discrete dynamical system, examine the behavior of the system based on different values of the parameter, and give a graph of several numerical solutions.

This situation calls for a discrete dynamical system because the interest and payments are made in discrete time steps. Let a_n be a sequence that represents the amount owed on the loan after n months. The initial loan amount, \$$A$, is the ini-

tial condition, so

$$a_0 = A.$$

The interest is compounded monthly, so each month the new interest is $r/12\%$ of the amount owed in the previous month.

$$\text{new interest} = \frac{r}{12} a_n.$$

The payments are subtracted from the amount owed. Hence, the mathematical model for the amount owed on the loan is

$$\underbrace{a_{n+1} - a_n}_{\text{change in monthly balance}} = \underbrace{\frac{r}{12} a_n}_{\text{amount accumulated from interest}} \underbrace{-p}_{\text{payment}}, \quad \text{where} \quad \underbrace{a_0 = A}_{\text{initial condition}}$$

More simply,

$$a_{n+1} = a_n + \frac{r}{12} a_n - p \quad \text{where} \quad a_0 = A.$$

Even simpler yet (after a little algebra)

$$a_{n+1} = \left(1 + \frac{r}{12}\right) a_n - p.$$

This is a mathematical model for any such situation, and there are many unknown parameters. The fact that you don't know the parameters is what makes the problem interesting!

To discuss the parameters one must think like a scientist; vary one parameter at a time and try to cover all of the possible bases. Let us first consider a loan of $A = \$5000$ at 15% APR. Figure 3.1 shows numerical solutions for several different payments. Clearly if the payment is larger then the amount of time to pay off the loan decreases. Next let us discuss a different situation. Assume that the initial loan is still \$5000 but now assume that the payments are fixed at \$200 and we don't know the rate. Figure 3.2 shows several numerical solutions to the discrete dynamical system at various rates.

Again, there aren't any real surprises here; as the rate goes down the time to pay off the loan goes down. In order to fully explore the parameter space of a discrete dynamical system (i.e. difference plus initial conditions) it is easy to implement the model in Excel and to use absolute cell references for the parameters.

Problem 3.17. For each of the following situations, create a mathematical model using a difference equation or a differential equation.

(a) The population of a town grows at an annual rate of 1.25%.

(b) A radioactive sample losses 5.6% of its mass every day.

(c) You have a bank account that earns 4% interest every year. At the same time, you withdraw money continually from the account at the rate of \$1000 per year.

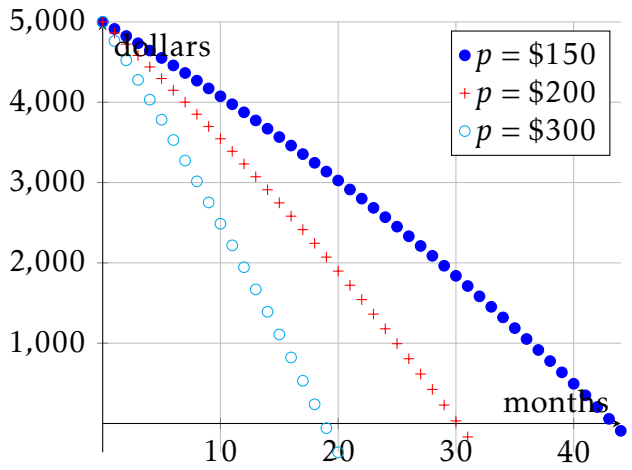


Figure 3.1. Numerical solutions to $a_{n+1} = a_n + \frac{r}{12}a_n - p$ with $a_0 = 5000$ and $r = 0.1$ and different values of p

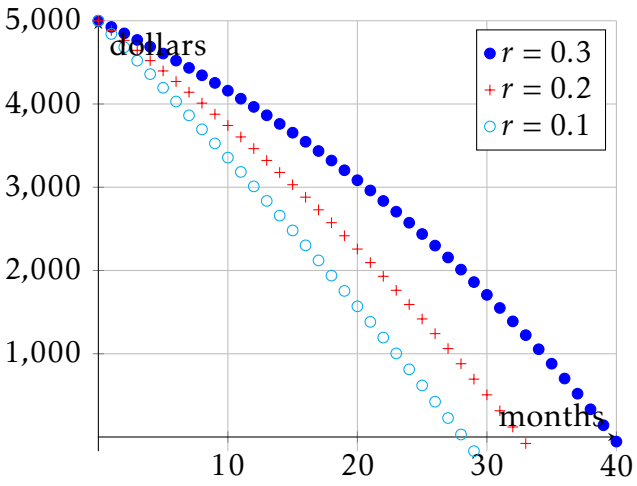


Figure 3.2. Numerical solutions to $a_{n+1} = a_n + \frac{r}{12}a_n - p$ with $a_0 = 5000$ and $p = \$200$ and different values of r

(d) A cup of hot chocolate is sitting in a 70° room. The temperature of the hot chocolate cools by 10% of the difference between the hot chocolate's temperature and the room temperature every minute.

(e) A can of cold soda is sitting in a 70° room. The temperature of the soda warms at the rate of 10% of the difference between the soda's temperature and the room's temperature every minute.

(f) Suppose that there are 400 college students living in a dormitory and that one or more students has a severe case of the flu. Assume some interaction between those infected and those not infected is required to pass the flue. Assume that a student is either infected or susceptible (no immunities). Further assume the number of new infected students each hour is proportional to the product of the infected and

the susceptible students. Create a mathematical model for the number of infected students over the course of a day.

Hint: If there are 400 students in the dorm, then the number of susceptible students is 400 minus the number of infected students.

(g) A town has a large reservoir of water that currently contains 450,000 gallons. Each week during the spring W gallons are used by the town, while M gallons flow in from snow melt. If r% of the water evaporates each week, formulate a mathematical model for the amount of water in the reservoir and explore the parameter space with the help of Excel.

Estimates of parameters are as follows: Last year the average water usage during the spring was $W = 50,000$ gallons and the average intake from snow melt was $M = 62,500$ gallons. During the spring the water is estimated to evaporate at approximately 1%.

▲

3.4 Stability and Equilibria

Problem 3.18. Consider the differential equation

$$\frac{dA}{dt} = -0.5A + 0.1.$$

This model comes from a drug dosing problem where your body removes 50% of the drug every hour and an IV drip administers 0.1mg of drug each hour. The value of A is the number of milligrams of drug in the body at time t. If done correctly, the amount of drug in your system will stabilize at a constant level. What is that level? Be sure to fully explain your thinking. ▲

Problem 3.19. Consider the difference equation

$$A_{n+1} - A_n = -0.2A_n + 100.$$

This model comes from a car sales problem where 20% of the cars on the lot are sold in a given month and 100 new cars are delivered each month. If you've done things correctly as the manager of the lot, the number of cars will stabilize at a certain amount. How many cars is this? Be sure to fully explain your thinking. ▲

Problem 3.20. For each of the two previous problems make a plot with "change" on the vertical axis and "amount" on the horizontal axis. Each of these should be a linear function just like what we saw in the *Birth, Death, and Immigration* models. What is the meaning of the "x intercept" on each of these plots? ▲

Problem 3.21. Return to the *Birth, Death and Immigration* models. You already made a plot with "change" on the vertical axis and "population" on the horizontal axis. Find the point where the linear functions intersect the horizontal axis and interpret these points in the context of the problem. ▲

Definition 3.22 (Equilibrium Point)**.** An **equilibrium point** in a difference or differential equation is the point where the change is 0.

Example 3.23. Consider the differential equation $\frac{dy}{dt} = -\frac{1}{2}(y-4)$. If we set the change to zero then we find that

$$0 = -\frac{1}{2}(y_{eq} - 4) \quad \Longrightarrow \quad y_{eq} = 4.$$

The equilibrium is at $y = 4$. Furthermore, if we have a y value a bit less than 4 we see that dy/dt is positive so y will be going up. Also, if we have a y value that is a bit more than 4 then dy/dt is negative so y will be going down (see Figure 3.3). We call this type of equilibrium point **stable**.

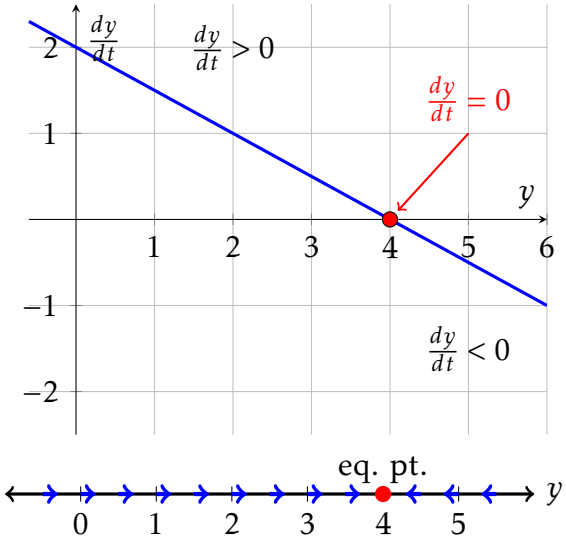


Figure 3.3. A plot of $\frac{dy}{dt}$ vs y (top) and a phase line diagram (bottom) for $\frac{dy}{dt} = -\frac{1}{2}(y-4)$

Example 3.24. Consider the differential equation $\frac{dy}{dt} = \frac{1}{2}(y-4)$. If we set the change to zero then we find that

$$0 = \frac{1}{2}(y_{eq} - 4) \quad \Longrightarrow \quad y_{eq} = 4.$$

The equilibrium is at $y = 4$. Furthermore, if we have a y value a bit less than 4 we see that dy/dt is negative so y will be going down. Also, if we have a y value that is a bit more than 4 then dy/dt is positive so y will be going up (see Figure 3.4). We call this type of equilibrium point **unstable**.

Problem 3.25. Find and classify the equilibrium points for each of the following difference or differential equations.

(a) $\dfrac{dy}{dt} = y(1-y)$

(b) $A_{n+1} - A_n = 2A_n + 6$

(c) $x_{n+1} - x_n = -3x_n - 2$

(d) $\dfrac{dP}{dt} = -0.5P + 2$

▲

Problem 3.26. Find and classify the equilibrium point for the difference equations modeling the three *Birth, Death, and Immigration* problems. ▲

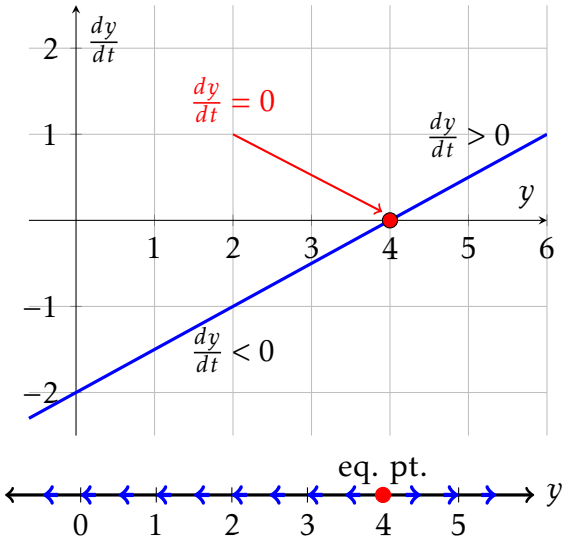


Figure 3.4. A plot of $\frac{dy}{dt}$ vs y (top) and a phase line diagram (bottom) for $\frac{dy}{dt} = -\frac{1}{2}(y-4)$

3.5 Euler's Method: Numerical Solutions for Differential Equations

One of the fastest ways to get an approximate solution to a difference or a differential equation is to use technology. You have already used technology such as MS Excel to build numerical solutions to difference equations, but what about differential equations? Let's look at the two in parallel. Consider the difference and differential equations

$$\text{difference equation: } a_{n+1} - a_n = -0.5a_n + 0.1, \text{ and} \tag{3.2}$$

$$\text{differential equation: } \frac{dy}{dt} = -0.5y + 0.1. \tag{3.3}$$

We can turn the differential equation (3.3) into a difference equation by recalling from Calculus that

$$\frac{dy}{dt} \approx \frac{\Delta y}{\Delta t}.$$

Taking $\Delta y = y_{n+1} - y_n$ we arrive at an approximate form of the differential equation (3.3)

$$\frac{y_{n+1} - y_n}{\Delta t} \approx -0.5y_n + 0.1. \tag{3.4}$$

In (3.4) we can then rearrange algebraically to get

$$y_{n+1} \approx y_n + \Delta t\,(-0.5y_n + 0.1). \tag{3.5}$$

All we have done at this point is to approximate the derivative $\frac{dy}{dt}$

Problem 3.27. In MS Excel we are going to solve the difference equation (3.2) alongside the differential equation (3.3). For both we will use the initial condition $a_0 = y(0) = 1$.

Notice that these two equations model the same behavior with the difference being that the difference equation models in discrete time and the differential equations models in continuous time.

(a) In MS Excel label column A "Discrete Time" and label column B "Difference Equation". Fill in the times starting at 0 and increasing by 1 in column A. Put the initial condition in cell B2 and build the difference equation starting in cell B3. See the left side of Table 3.1.

(b) In the same Excel worksheet label column D "Continuous Time" and label column E "Differential Equation". Put in "0" for the initial time, choose a time step, and fill in column D with times progressing by that time step. Put the initial condition in cell E2 and build the approximation to the differential equation (3.5) starting at cell E3. See the right side of Table 3.1.

(c) Fill both numerical solutions so that they end at the same time (the continuous time model will obviously be 10 times longer if you use $\Delta t = 0.1$).

(d) Make a plot showing both solutions on top of each other.

▲

	A	B	C	D	E
1	Discrete Time	Difference Equation		Continuous Time	Differential Equation
2	0	1		0	1
3	1	=B2-0.5*B2+0.1		0.1	=E2+(0.1)*(-0.5*E2+0.2)
4	2			0.2	
5	⋮	⋮		⋮	⋮

Table 3.1. Excel setup for a difference equation (left) and differential equation (right).

You should have noticed in the previous problem that the solutions to the difference and differential equations are slightly different than each other. This is no surprise since we are taking time steps of "1" in the difference equation and we are taking much finer time steps in the differential equation. Note further that the only way to make the approximate solution to the differential equation accurately portray the actual dynamics of the differential equation we would need to take $\Delta t \to 0$. Doing so is clearly infeasible on a computer since we would need infinite memory to do so. We settle, instead, for taking Δt to be small (0.1 or 0.01 or smaller).

Definition 3.28 (Euler's Method)**.** To numerically approximate the solutions to the differential equation

$$\frac{dy}{dt} = f(t, y)$$

we approximate the derivative with

$$\frac{dy}{dt} \approx \frac{y_{n+1} - y_n}{\Delta t}$$

and use the difference equation

$$y_{n+1} = y_n + \Delta t f(t_n, y_n)$$

to build an approximate solution.

Remember that the only reasonable choice for Δt is to make it *very small*. The trade off to choosing Δt small is that it will take more computer memory to approximate the problem.

A way to think about Euler's method is that at a given point, the slope is approximated by the value of the right-hand side of the differential equation and then we step forward Δt units in time following that slope. Figure 3.5 shows a depiction of the idea. Notice in the figure that in regions of high curvature Euler's method will overshoot the exact solution to the differential equation. However, taking $\Delta t \to 0$ theoretically gives the exact solution at the trade off of needing infinite computational resources.

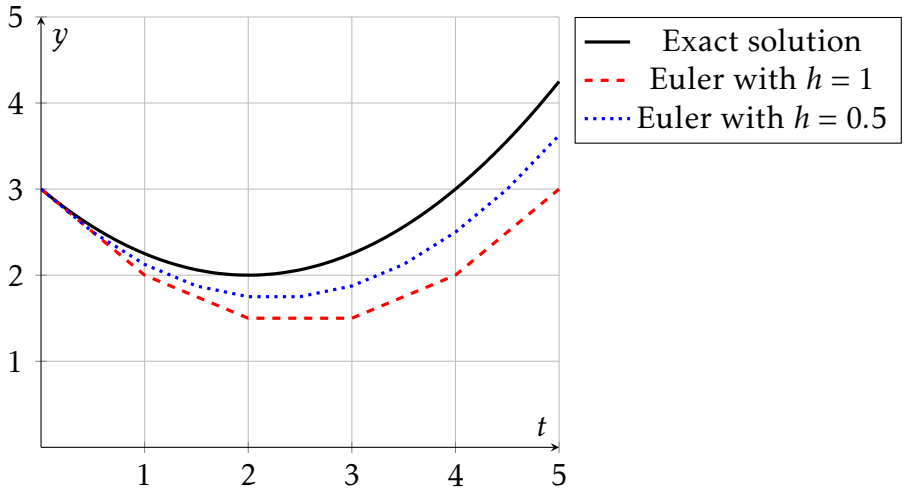


Figure 3.5. A depiction of Euler's method with step size $h = 1$ (red) and $h = 0.5$ (blue).

In Figure 3.6 we see a graphical depiction of how Euler's method works on the differential equation $y' = y$ with $\Delta t = 1$ and $y(0) = 1$. The exact solution to the differential equation is $y(t) = e^t$ and at $t = 1$ the solution is $y(1) = e^1 \approx 2.718$ and is shown in red in the figure. We can see that for large step sizes Euler's method will drastically miss the true dynamics of the differential equation.

In Excel the process of building an Euler solver is relatively simple. In MATLAB it takes a bit more work the very first time, but trust me, the work will pay off in the long run! Think carefully about the computer resources necessary to truly capture the dynamics of a differential equation with Euler's method: very small step size and lots of computer

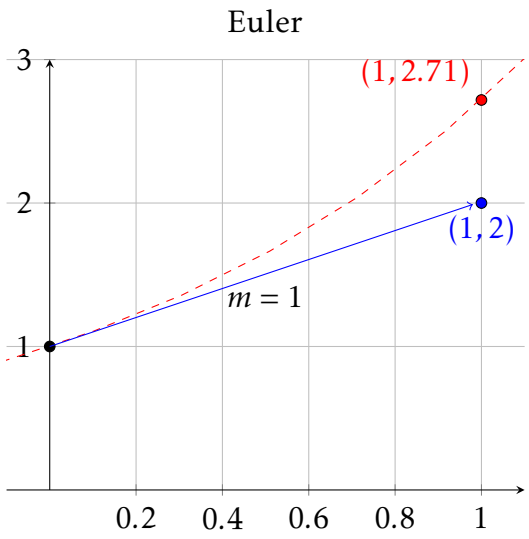


Figure 3.6. Graphical depiction of Euler's method. Here we use the simple differential equation $y' = y$ with $y(0) = 0.5$ and $\Delta t = 1$. The exact solution is shown in red.

memory. In Excel this means lots of dragging with the mouse which is slow, cumbersome, and annoying. Let's build the same problem in MATLAB.

Problem 3.29 (Euler's Method in MATLAB)**.** Open MATLAB and create a new script. The code given below sets up and plots the numerical solution to the problem

$$\frac{dy}{dt} = -0.5y + 0.1.$$

Explain what each line of code does using comments. Some of the lines of code are likely new to you so I suggest you either use the help command or you do some basic experimentation.

```
clear; clc; clf;
tmin=0; % define the minimum time
tmax=5; % define the maximum time
dt = 0.1; % what is this line doing?
IC = 1; % define the initial condition for the problem
t=tmin : dt : tmax;  % what does this line do?
y=zeros(size(t)); % what does this line do?
y(1) = IC; % what does this line do?
for n=1:length(t)-1 % what does this line do?
    y(n+1) = y(n) + dt*(-0.5*y(n)+0.1);
end
plot(t,y,'b'), grid on, xlabel('time'), ylabel('y(t)')
title('Euler Approximation')
```

▲

Problem 3.30. Experiment with different values of Δt in your MATLAB code. I hope that you realize that this code is now FAR more flexible than using Excel even though it takes a bit more thought to set up at the beginning. ▲

Problem 3.31. Modify your Euler's method MATLAB code to get an approximate solution to the differential equation

$$y' = -y \cdot \left(1 - \frac{y}{5}\right) + 0.1t \quad \text{where} \quad y(0) = 2$$

▲

Problem 3.32. Modify your Euler's method MATLAB code to get an approximate solution to the differential equation

$$y' = -y\left(1 - \frac{y}{5}\right)^2 + \sin(t)$$

for several different initial conditions. Save your plot in an appropriate place with appropriate labels and title. ▲

Problem 3.33. Solve both of the problems below with MATLAB. For the difference equation your code should be absolutely identical to that of Euler's method with the exception that $\Delta t = 1$. Put the plots on top of each other so you can easily see the differences and similarities.

$$a_{n+1} - a_n = 0.25a_n\left(1 - \frac{a_n}{10}\right)$$
$$\frac{dy}{dt} = 0.25y\left(1 - \frac{y}{10}\right).$$

▲

Problem 3.34. Go back to every difference and differential equation that we have build thus far in this chapter and solve it with both Excel and with MATLAB. ▲

Save all of the code from both Excel and MATLAB. We will be using numerical approximations to solve both difference and differential equations throughout the remainder of this course. You will need to be able to build the Euler code in MATLAB without looking back at old code so take some time in the next few days and practice writing this code.

3.6 Classifying Difference and Differential Equations

In order to properly analyze difference equations it is useful to develop a classification system for all difference and differential equations. The following definitions describe several broad classes of difference and differential equations that will be studied in detail. These classifications are not merely just names, but they give a hint to the behavior of the system and the solution techniques.

Definition 3.35 (Linear Difference Equations)**.** In a **linear** difference equation, the terms that involve sequence variables (a_{n+1} or a_n) involve no products, no powers, nor functions of sequences variables such as exponentials, logarithmic, or trigonometric.

Definition 3.36 (Linear Differential Equations)**.** In a **linear** differential equation, the terms that involve output function $y(t)$ involve no products, no powers, nor nonlinear functions of $y(t)$ such as exponentials, logarithmic, or trigonometric.

Definition 3.37 (Nonlinear Difference and Differential Equations)**.** A difference or differential equation is called *non linear* if it is not linear.

Definition 3.38 (Homogeneous Difference and Differential Equations)**.** A difference or differential equation is called **homogeneous** if the only terms that appear in the relationship are ones which involve the output variable (e.g. a_n or a_{n+1} in difference equations or $y(t)$ in differential equations).

Definition 3.39 (Nonhomogeneous Equations)**.** A difference or differential equation is called **nonhomogeneous** if it is not homogeneous.

Example 3.40. Classify each of the following first order models.

$$\frac{dy}{dt} = y^2 \tag{3.6}$$

$$b_{n+1} - b_n = b_n^2 \tag{3.7}$$

$$x_{n+1} - x_n = 0.5x_n + \sin(n) \tag{3.8}$$

$$\frac{dx}{dt} = x\sin(t) \tag{3.9}$$

Solution:

- Equation (3.6) is a first order nonlinear homogeneous differential equation. It is first order because it only contains first derivatives. It is nonlinear because of the square on the y on the right-hand side. It is homogeneous because there is nothing else being added to the y term on the right-hand side.
- Equation (3.7) is a first order nonlinear homogeneous difference equation. It is first order because it only contains a difference between b_{n+1} and b_n. It is nonlinear because of the square on the b_n on the right-hand side. It is homogeneous because there is nothing else being added to the b_n term on the right-hand side.
- Equation (3.8) is a first order linear non-homogeneous difference equation. It is first order because it only contains a difference between x_{n+1} and x_n. It is linear because the only thing happening to the x_n on the right-hand side is multiplication by a scalar. It is non-homogeneous since there is a $\sin(n)$ term added to the right-hand side.
- Equation (3.9) is a first order linear homogeneous (non-autonomous) differential equation. This equation is linear because it only contains the first derivative. It is linear since no non-linear functions act on the x on the right-hand side. This equation is called "non-autonomous" since there is a "t" on the right-hand side – that is, the rate depends not only on x but also on t.

Problem 3.41. Classify the following using the words

- "difference equation" or "differential equation"
- "linear" or "nonlinear"
- "homogeneous" or "non-homogeneous"

If an equation is nonhomogeneous then give the type of function for the nonhomogeneity. For example, the equation $a_{n+1} - a_n = 0.5a_n + 10$ is a linear nonhomogeneous difference equation with a constant nonhomogeneity.

$$a_{n+1} - a_n = 3a_n + n^2 \tag{3.10}$$
$$P_{n+1} - P_n = 5P_n \tag{3.11}$$
$$P'(t) = 5P(t) \tag{3.12}$$
$$b'(t) = (b(t))^2 + 1 \tag{3.13}$$
$$b_{n+1} - b_n = (b_n)^2 + 1 \tag{3.14}$$
$$a_{n+2} = a_n a_{n+1} \tag{3.15}$$
$$f'(t) = \frac{2}{f(t)} + t \tag{3.16}$$
$$a_{n+2} = a_{n+1} + 3a_n \tag{3.17}$$
$$y'(t) = 3y(t) + t^2 \tag{3.18}$$

▲

3.7 Technique: Solving 1^{st} Order Linear Homogeneous Equations

In this section we will solve the most basic difference and differential equations.

$$a_{n+1} - a_n = ra_n \quad \text{with initial condition} \quad a_0 \tag{3.19}$$
$$y'(t) = ry(t) \quad \text{with initial condition} \quad y(0) = y_0. \tag{3.20}$$

The reader should note that (3.19) is a linear homogeneous difference equation and (3.20) is a linear homogeneous differential equation. These solution techniques will be the basis for all future solution techniques so take careful note of the techniques . . . we'll use them a lot!!

Problem 3.42. We wish to solve the difference equation $a_{n+1} - a_n = ra_n$ with initial condition a_0. Fill in the blanks in the following table to arrive at an analytic solution to this difference equation.

n	a_n
0	a_0
1	$a_1 =$ _____a_0
2	$a_2 =$ _____$a_1 =$ _____a_0
3	$a_3 =$ _____$a_2 =$ _____a_0
4	$a_4 =$ _____$a_3 =$ _____a_0
$\vdots$	$\vdots$
k	$a_k =$ _____a_0

▲

Your work in the previous problem has just provided sufficient evidence to believe following theorem.

Theorem 3.43. If $a_{n+1} - a_n = ra_n$ with initial condition a_0 then the analytic solution to the discrete dynamical system is

$$a_k = \_\_\_\_\_ \cdot a_0. \tag{3.21}$$

Problem 3.44. Returning now to the original difference equation from this chapter, find the solution to the difference equation

$$P_{n+1} - P_n = -0.5P_n$$

with initial condition $P_0 = 50$. Use Excel to make a plot of the analytic solution along with a numerical solution. ▲

Now let's solve equation (3.20). To do this we'll need to make a simple observation about calculus.

Problem 3.45. There are two functions that have the property that they are equal to their derivative. Both of these equations solve the differential equation $y'(t) = y(t)$. The first solution is the function $y(t) = 0$ since $\frac{d}{dt}(0) = 0$. What is the other solution function? ▲

Now let's solve almost the exact same differential equation: $y'(t) = ry(t)$.

Problem 3.46. Based on your answer to the previous problem, what is the solution to the differential equation $y'(t) = ry(t)$? ▲

You should notice that in both the difference equation, $a_{n+1} - a_n = ra_n$, and the differential equation, $y(t) = ry(t)$, the solution turns out to be exponential. The only big difference is the base of the exponential.

Theorem 3.47. The solution to the differential equation $y'(t) = ry(t)$ with initial condition $y(0) = y_0$ is

$$y(t) = \_\_\_\_\_.$$

Problem 3.48. Solve the following first order homogeneous difference and differential equations.

$$\begin{aligned} a_{n+1} - a_n &= 3a_n \quad \text{with} \quad a_0 = 7 \\ y'(t) &= 3y(t) \quad \text{with} \quad y(0) = 7 \\ b_{n+1} - b_n &= -0.3b_n \quad \text{with} \quad b_0 = 7 \\ x'(t) &= -0.3x(t) \quad \text{with} \quad x(0) = 2 \end{aligned}$$

▲

Problem 3.49. Write a mathematical model associated with each of the previous equations. ▲

3.8 Technique: Solving 1^{st} Order Linear Nonhomogeneous Equations

We now turn to finding analytic solutions to the difference equations and differential equation that we've encountered thus far. We will start with a technique, called *separation of variables*, that only applies to differential equations since in differential equation we can leverage the ideas of calculus. We will then look at a powerful technique called *the method of undetermined coefficients* that works for both differential equations and difference equations. With practice you'll get the hang of which one to use in which instances.

3.8.1 Solution Technique: Separation of Variables

Problem 3.50. Consider the differential equation

$$\frac{dy}{dt} = y$$

with the initial condition $y(0) = 1$.

(a) Putting the differential equation into words:
the derivative of some unknown function is equal to the function itself.
what is the function?

(b) Allow me to abuse some notation:
If you multiply both sides by dt and divide both sides by y we end up with

$$\frac{dy}{y} = dt.$$

Integrate both sides and solve for y. Don't forget the arbitrary constants resulting from indefinite integrals.

(c) Compare your answers to parts (b) and (c).

▲

Problem 3.51. In part (b) of the previous problem we said that we were "abusing notation". What does that mean? What notation is being abused? ▲

Technique 3.52 (Separation of Variables)**.** To solve a differential equation of the form

$$\frac{dy}{dx} = f(y) \cdot g(x)$$

Separate and integrate by treating the "dy/dt" as a fraction[a]

$$\int \frac{dy}{f(y)} = \int g(x)dx$$

Notice that the right-hand side of the differential equation factors perfectly hence separating the variables into the functions f and g.

[a] Technically speaking the "dy/dt" is not a fraction it is a shorthand notation for a limit. More technically there is some sneaky chain rule happening behind the scenes here . . . can you find it.

Problem 3.53. With your partner, write a differential equation that can be solved via separation of variables. Once you have your equation trade with a different group and solve their equation. ▲

Example 3.54. Solve the differential equation

$$\frac{dy}{dt} = 0.5y \quad \text{with} \quad y(0) = 7$$

using the method of separation of variables.
Solution: Notice that this differential equation is separable since we can separate the functions of y and the functions of t

$$\frac{dy}{y} = 0.5dt.$$

Integrating both sides of this equation we get

$$\int \frac{1}{y} dy = \int 0.5dt \quad \Longrightarrow \quad \ln(y) + C_1 = 0.5t + C_2.$$

Notice that if we subtraction the constant C_1 from both sides we actually just get a new arbitrary constant on the right-hand side. For this reason it is customary to only write one of the two constants when showing the work for this method

$$\ln(y) = 0.5t + C.$$

Exponentiating both sides of this equation gives

$$y(t) = e^{0.5t+C}$$

and we can now recognize that this is the same, algebraically, as

$$y(t) = e^{0.5t} e^{C}.$$

Furthermore, e^C is just another constant so we write the general solution as

$$y(t) = Ce^{0.5t}.$$

To get the value of C we substitute $t = 0$ into the equation to get $7 = Ce^0$ which implies that $C = 7$ and the solution is

$$y(t) = 7e^{0.5t}.$$

Example 3.55. Solve the differential equation

$$\frac{dy}{dt} = 3y + 12 \quad \text{with} \quad y(0) = 2$$

using separation of variables.
Solution: We're first going to factor the right-hand side of the differential equation so that the integration that we run in to is not so hard.

$$\frac{dy}{dt} = 3(y + 4).$$

Separating and integrating gives

$$\int \frac{1}{y+4} dy = \int 3dt \quad \Longrightarrow \quad \ln(y + 4) = 3t + C.$$

Exponentiating both sides and repeating the same type of algebra as in the previous example we get

$$y + 4 = Ce^{3t}.$$

Finally, we can subtract 4 from both sides of the equation to get the general solution

$$y(t) = Ce^{3t} - 4.$$

Using the initial condition we see that

$$2 = C - 4 \quad \Longrightarrow \quad C = 6$$

which implies that

$$y(t) = 6e^{3t} - 4.$$

Problem 3.56. A drug is eliminated from the body via natural metabolism. Assume that there is an initial amount of A_0 drug in the body. Which of the following is the best differential equation model for the drug removal? Once you have the model solve it with the appropriate technique.

1. $A' = -kt$
2. $A' = -kA$

3. $A' = -kA\left(1 - \frac{A}{A_{max}}\right)$
4. $A' = -kAt$

▲

Problem 3.57. An ideal ice cube that is exactly cube shaped (perfect squares on all sides) melts in your drink. A differential equation that models the melting of the ice is

$$\frac{dV}{dt} = kS(t)$$

where $S(t)$ is a function describing the surface area of the ice cube.

(a) Explain why this differential equation is physically reasonable.

(b) Write down the geometric formulas for the volume and the surface area of a cube in terms of the length of the side, x:

$$V(x) = \_\_\_\_\_ \qquad S(x) = \_\_\_\_\_$$

(c) Solve the surface area function for x as a function of S.

(d) Substitute your answer from part (c) into the x in the volume equation in part (b).

(e) Rewrite the differential equation

$$\frac{dV}{dt} = kS$$

in terms of V using your answer from part (d).

(f) Solve the differential equation resulting from part (e) using separation of variables.

▲

Problem 3.58. Suppose that a cup of coffee is initially at a temperature of 105° F and is placed in a 75° F room. Newton's law of cooling says that

$$\frac{dT}{dt} = -k(T - 75)$$

where k is a constant related to the insulation properties of the coffee mug.

(a) Suppose you measure that the coffee is cooling at one degree per minute at the time the coffee is brought into the room. Use the differential equation to determine the value of the constant k.

(b) Find all solutions to this differential equation.

(c) What happens to all of the solutions as $t \to \infty$? Explain how this agrees with your intuition.

(d) What is the temperature of the cup of coffee after 20 minutes?

(e) How long does it take for the coffee to cool to 80°F?

▲

Problem 3.59. In a dog, an intravenous dose of 30 mg of pentobarbital sodium per kilogram of body weight will usually produce surgical anesthesia. Also in the dog, pentobarbital has a biological halflife of about 4.5 hours, due almost entirely to metabolism. You anesthetize a 14-kg dog with the above dose of pentobarbital. Two hours later the anesthesia is obviously beginning to lighten and you want to restore the original depth of anesthesia. How many milligrams of pentobarbital sodium should you inject? Write and solve a differential equation to answer this question. ▲

Problem 3.60. In a local pine forest the Pine Beetle is killing the trees at a rate proportional to the number of available trees in the forest. A conservation group is attempting to curb the problem by planting 5 live trees per week. Write a differential equation describing this scenario, classify the differential equation, and determine if it can be solved with separation of variables. ▲

Problem 3.61. Solve each of the following differential equations. If an initial condition is given then use it to find any unknown constants. (Some of these may require advanced integration techniques)

$$\frac{dy}{dt} - (2-t)y = 2-t$$

$$\frac{1}{t}\frac{dy}{dt} = e^{t^2-2y}$$

$$\frac{dy}{dt} = 2y+2,\ y(0)=2$$

$$\frac{dy}{dt} = 2y^2,\ y(-1)=2$$

▲

We'll wrap up this subsection with a few more examples.

Example 3.62. Solve the differential equation

$$\frac{dy}{dt} = \frac{y}{t^2} \quad \text{with} \quad y(1)=5$$

using separation of variables.
Solution: We can separate variables, integrate, and do some algebra to get

$$\int \frac{dy}{y} = \int \frac{dt}{t^2} \quad \Longrightarrow \quad \ln(y) = -\frac{1}{t} + C \quad \Longrightarrow \quad y(t) = Ce^{-1/t}.$$

Using the condition $y(1)=5$ we see that $5 = Ce^{-1}$ which implies that $C = 5e$ and the

solution to the differential equation is

$$y(t) = 5e^{1-1/t}.$$

Example 3.63. Solve the differential equation

$$\frac{dy}{dt} = \frac{2ty}{t^2+1}$$

using separation of variables.
Solution: If we separate the variables and integrate we see that

$$\int \frac{dy}{y} = \int \frac{2t}{t^2+1}dt \quad \Longrightarrow \quad \ln(y) = \ln(t^2+1) + C.$$

The right-hand integral used the idea of u-substitution (you should stop now and work out the u-substitution by hand). Exponentiating both sides gives

$$y(t) = e^{\ln(t^2+1)+C} = Ce^{\ln(t^2+1)} = C(t^2+1).$$

For more problems related to separation of variables see Active Calculus Section 7.3.

3.8.2 Solution Technique: Undetermined Coefficients

Now let's focus on a solution technique that works for both difference and differential equations. When faced with a separable differential equation you'll find that separation of variables will always be easier. However, not all differential equations are separable! For example, the differential equation $y' = 0.5y + t$ cannot be separated. Furthermore, separation of variables makes no sense on difference equations since there are no derivatives.

You can think of this new method, *the method of undetermined coefficients*, as *mathematical detective work*. We're going to guess the form of the solution and then follow our noses to find the missing coefficients – hence the name *undetermined coefficients*.

Technique 3.64 (Undetermined Coefficients)**.** The method of *undetermined coefficients* is roughly outlined as:

1. find a solution for the homogeneous equation,
2. conjecture a particular solution for the non-homogeneous equation,
3. use the initial condition to find a relationship between some of the coefficients, and

4. use the particular solution in the equation to find the remaining coefficients.

We'll highlight this technique with a sample problem. Work through all of these steps to get the idea of how to use undetermined coefficients.

Problem 3.65. Solve the following first order linear non-homogeneous differential equation by following the steps outlined.

$$\frac{dy}{dt} = -0.2y + 3 \quad \text{with} \quad y(0) = 5$$

(a) First solve the homogeneous part of the equation: $y' = -0.2y$. Don't use the initial condition yet.

$$y_{hom}(t) = \_\_\_\_\_$$

(b) Next conjecture that a *particular* solution has the same functional form as the non-homogeneity. In this case the non-homogeneity is a constant function so we guess that the particular function is a generic constant function

$$y_{particular}(t) = C.$$

(c) The full analytic solution to the differential equation is the sum of the homogeneous and particular solutions: $y(t) = y_{hom}(t) + y_{part}(t)$. Note that this is only the case for linear difference or differential equations.

$$y(t) = \_\_\_\_\_$$

(d) Substitute the particular solution into the differential equation and see what equation comes out

$$\_\_\_\_\_ = \_\_\_\_\_$$

(e) Substitute the initial condition into the analytic solution and see what equation comes out

$$\_\_\_\_\_ = \_\_\_\_\_$$

(f) Determine the final solution

▲

Problem 3.66. Leave all of your work from the previous problem somewhere you can see it. We're now going to solve a difference equation of the exact same form using the method of undetermined coefficients.
Solve the difference equation

$$a_{n+1} - a_n = -0.2a_n + 3 \quad \text{with} \quad a_0 = 5.$$

(a) First solve the homogeneous part of the equation: $a_{n+1} - a_n = -0.2a_n$. Don't use the initial condition yet.

$$a_{hom} = \_\_\_\_\_$$

(b) Next conjecture that a *particular* solution has the same functional form as the non-homogeneity. In this case the non-homogeneity is a constant function so we guess that the particular function is a generic constant function

$$a_{particular} = C.$$

(c) The full analytic solution to the differential equation is the sum of the homogeneous and particular solutions: $a_n = a_{hom} + a_{part}$. Note that this is only the case for linear difference or differential equations.

$$a_n = \_\_\_\_\_$$

(d) Substitute the particular solution into the difference equation and see what equation comes out

$$\_\_\_\_\_ = \_\_\_\_\_$$

(e) Substitute the initial condition into the analytic solution and see what equation comes out

$$\_\_\_\_\_ = \_\_\_\_\_$$

(f) Determine the final solution

▲

Problem 3.67. Time for a compare and contrast. Put the solutions to both of the previous two problems somewhere you can see both of them. Compare and contrast the steps. Once you've convinced yourself that there is essentially no difference in the technique for either difference or differential equations, plot both equations on the same coordinate plane and discuss the similarities and differences. Where did the differences come from? ▲

Technique 3.68 (Undetermined Coefficients)**.** To solve a non-homogenous linear difference or differential equation:

1. Solve the associated homogeneous equation.
2. Conjecture a *particular solution* that has the same functional form as the non-homogeneity.
3. Build the full analytic solution as a linear combination of the homogeneous and particular solutions.
4. Substitute the particular solution into the difference or differential equation.
5. Substitute the initial condition(s) into the analytic solution.
6. Use the equations that you found in steps 4 and 5 to find the constants.

Problem 3.69. For each of the following linear non-homogeneous difference or differential equations write the homogeneous solution and the particular solution.

(a) $y' = 3y + 4$ $y_{hom}(t) =$ _____ and $y_{part}(t) =$ _____

(b) $a_{n+1} - a_n = 3a_n + 4$ $a_{hom} =$ _____ and $a_{part} =$ _____

(c) $y' = 3y + 4t$ $y_{hom}(t) =$ _____ and $y_{part}(t) =$ _____

(d) $a_{n+1} - a_n = 3a_n + 4t$ $a_{hom} =$ _____ and $a_{part} =$ _____

(e) $y' = 3y + 4\sin(t)$ $y_{hom}(t) =$ _____ and $y_{part}(t) =$ _____

(f) $a_{n+1} - a_n = 3a_n + 4\sin(n)$ $a_{hom} =$ _____ and $a_{part} =$ _____

(g) $y' = 3y + 4e^{-t}$ $y_{hom}(t) =$ _____ and $y_{part}(t) =$ _____

(h) $a_{n+1} - a_n = 3a_n + 4e^{-n}$ $a_{hom} =$ _____ and $a_{part} =$ _____

▲

Problem 3.70. Solve all of the differential equations in the previous problem using either separation of variables (if possible) or undetermined coefficients. For each one use $y(0) = 2$. ▲

Example 3.71. Solve the difference equation $a_{n+1} - a_n = a_n - 1$ with $a_0 = 5$.
Solution: Notice that this is a linear non-homogeneous difference equation so we can use the method of undetermined coefficients.

1. The homogeneous difference equation is $a_{n+1} - a_n = a_n$ so we know that the general homogeneous solution is $a_{hom} = C_0 \cdot 2^n$.
2. The non-homogeneity is constant so we guess that the particular solution will be constant: $a_{part} = C_1$.
3. The analytic solution takes the form $a_n = C_0 \cdot 2^n + C_1$.
4. Putting the initial condition into the analytic solution gives $5 = C_0 + C_1$.
5. Putting the particular solution into the original difference equation gives $C_1 - C_1 = C_1 - 1$ which implies that $C_1 = 1$. Together with the information from the previous step we also now know that $C_0 = 4$.
6. The analytic solution to the difference equation is $a_n = 4 \cdot 2^n + 1$.

Example 3.72. Solve the differential equation $y' = y - 1$ with $y(0) = 5$.
Solution: We know that the steps from the previous example will be exactly the same with the exception of the homogeneous solution. Therefore we know that the general solution is $y(t) = C_0 e^t + C_1$, the value of C_1 is $C_1 = 1$, and the value of C_0 is $C_0 = 4$. Therefore the analytic solution to this differential equation is $y(t) = 4e^t + 1$.

Example 3.73. Solve the differential equation $y' = 2y + 7t$ with $y(0) = 3$.
Solution: We will use the method of undetermined coefficients.

1. The homogeneous differential equation is $y' = 2y$ and the associated solution is $y_{hom}(t) = Ce^{2t}$.
2. The non-homogeneity is linear so we assume that the particular solution is a linear function. The most general form of a linear function is: $y_{part}(t) = C_1 t + C_2$.
3. The full general solution is
$$y(t) = C_0 e^{2t} + C_1 t + C_2.$$
4. Putting the initial condition into the analytic solution gives
$$3 = C_0 + C_1 \cdot 0 + C_2.$$
5. Putting the particular solution into the differential equation gives
$$C_1 = 2(C_1 t + C_2) + 7t.$$
We can rewrite this as
$$0t + C_1 = 2C_1 t + 2C_2 + 7t,$$
and now we match coefficients to get
$$0 = 2C_1 + 7$$
and
$$C_1 = 2C_2.$$
From this we see that $C_1 = -7/2$ and $C_2 = -7/4$. Furthermore, we can use the result from the previous step to get $C_0 = 3 - C_2 = 3 + 7/4 = 19/4$.
6. The solution is
$$y(t) = \frac{19}{4}e^{2t} - \frac{7}{2}t - \frac{7}{4}.$$

Problem 3.74. Solve the difference equation $a_{n+1} - a_n = 2a_n + 7n$ with $a_0 = 3$. ▲

Example 3.75. Solve the differential equation $y'(t) = \frac{1}{3}y + \cos(t)$ with $y(0) = 1$.
Solution: We will use the method of undetermined coefficients.

1. The homogeneous equation is $y' = \frac{1}{3}y$ and the solution is $y_{hom}(t) = C_0 e^{t/3}$.

2. The non-homogeneity is a trigonometric function so we use a linear combination of both sine and cosine for the particular solution:

$$y_{part}(t) = C_1 \cos(t) + C_2 \sin(t).$$

3. The full general solution is

$$y(t) = C_0 e^{t/3} + C_1 \cos(t) + C_2 \sin(t).$$

4. Putting the initial condition into the analytic solution gives

$$1 = C_0 + C_1 + 0C_2 \quad \Longrightarrow 1 = C_0 + C_1.$$

5. Putting the particular solution into the differential equation gives

$$-C_1 \sin(t) + C_2 \cos(t) = \frac{1}{3}(C_1 \cos(t) + C_2 \sin(t)) + \cos(t).$$

   At this point we do some fun algebra. The "equal sign" must be true for all t so we can match the like terms and write two equations. For the coefficients of the sine functions we must have

$$-C_1 = \frac{1}{3}C_2.$$

   for the coefficients of the cosine functions we must have

$$C_2 = \frac{1}{3}C_1 + 1.$$

6. We now have three equations with three unknowns. I'll write them very carefully so we can switch to matrices and solve using Gaussian Elimination easily.

$$1C_0 + 1C_1 + 0C_2 = 0$$
$$0C_0 + 1C_1 + \frac{1}{3}C_2 = 0$$
$$0C_0 + \frac{1}{3}C_1 - 1C_2 = -1$$

Switching to matrix notation we have the augmented system and resulting row reduction

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 1/3 & -1 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1/3 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & -10/9 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3/10 \\ 0 & 1 & 0 & -3/10 \\ 0 & 0 & 1 & 9/10 \end{array}\right).$$

7. The analytic solution is

$$\frac{3}{10}e^{t/3} - \frac{3}{10}\cos(t) + \frac{9}{10}\sin(t).$$

3.9 Modeling Explorations with Difference and Differential Equations

Problem 3.76. A whiffle ball is dropped from a height with no initial velocity. From Newton's laws of motion we know that the sum of the forces on the wiffle ball, $\sum F$, is given by the product of the mass and the acceleration: $\sum F = ma$. The two primary forces are gravity and drag so

$$ma = F_{gravity} + F_{drag}.$$

The force due to gravity is $F_{gravity} = mg$ and the force due to drag is proportional to the square of the velocity: $F_{drag} = kv^2$. Also note that acceleration is the derivative of velocity. Therefore the differential equation modeling this scenario is

$$mv' = mg - kv^2$$

which can be rearranged to

$$v' = g - \frac{k}{m}v^2.$$

The data given in the table below give the distance (in feet) measured from the drop point as a function of time (in seconds). Use the data and the differential equation to find the ratio of the drag coefficient to the mass, k/m (this is best done with technology). Then use the differential equation and your value of k/m to determine the terminal velocity of the whiffle ball.
Note that 1 meter $\approx 3.28 feet$.

Time (sec)	Distance (feet)
0	0
0.32601	1.6154
0.72587	8.073
0.86624	11.3219
0.97305	14.232
1.18558	20.0783
1.28771	23.4167
1.41354	27.3056
1.61894	34.1013
1.76001	39.3958

▲

Problem 3.77. In the Great Lakes region, rivers flowing into the lakes carry a great deal of pollution in the form of small pieces of plastic averaging 1 millimeter in diameter. In order to understand how the amount of plastic in Lake Michigan is changing, construct a model for how this type pollution has built up in the lake. We'll walk through the process.

You'll need the following basic facts:

- The volume of the lake is 5×10^{12} cubic meters.

- Water flows into the lake at a rate of 5×10^{10} cubic meters per year. It flows out of the lake at the same rate.
- Each cubic meter flowing into the lake contains roughly 3×10^{-8} cubic meters of plastic pollution.

(a) Let's denote the amount of pollution in the lake by $P(t)$ where P is measured in cubic meters of plastic and t is in years. We want to build a differential equation for the amount of pollution:

$$\frac{dP}{dt} = \text{Rate that Pollution Enters} - \text{Rate that Pollution Exits}$$

- What is the rate that pollution is entering the lake? This will be a number.
- What is the rate that pollution is exiting the lake? This will be a function of P since we don't know how much pollution is in each cubic meter of water in the lake.

(b) Write the differential equation.

(c) Find and classify all equilibrium points for the differential equation. What do the equilibrium points mean in context?

(d) Solve the differential equation using an appropriate technique assuming that $P(0) = 0$ (what does this initial condition mean?).

(e) Make a plot of your solution.

▲

Problem 3.78. For each of the following situations,

- write a first order nonhomogeneous linear differential equation,
- find the general solution to the differential equation,
- use the given information to find all of the constants and parameters,
- Identify any equilibrium solutions and determine whether they're stable or unstable,
- make a conjecture for a physically realistic initial condition,
- solve the differential equation,
- build an Euler approximation of the differential equation in MATLAB, and
- finally plot the analytic solution and the Euler approximation on the same coordinate axes.

Once you've done all of these mathematical tasks write a detailed technical paper outlining your solutions to all of the scenarios.

(a) A patient is hooked to an IV which infuses 3mg of morphine per hour. The patient's body absorbs the morphine at a rate proportional to the amount in the bloodstream. Let $M(t)$ be the amount of morphine in the patient's bloodstream and let k be the constant of proportionality. Assume that the patient starts with no morphine in their blood stream and 1 hour later they have 2mg of morphine in their bloodstream.

(b) When a skydiver jumps from a plane, gravity causes her downward velocity to increase at a rate of $g \approx 9.8$ meters per second per second. At the same time, wind resistance causes her downward velocity to decrease at a rate proportional to the square of the velocity. Let $v(t)$ be the velocity of the skydiver. If there is no initial downward velocity then $v(0) = 0$. Assume that 1 second after the jump the sky diver is traveling $8m/s$.

(c) Suppose that you have a water tank that holds 100 gallons of water. A briny solution, which contains 20 grams of salt per gallon, enters the take at a rate of 3 gallons per minute. At the same time, the solution is well mixed, and water is pumped out of the tank at the rate of 3 gallons per minute (obviously the water level in the tank remains constant). Let $S(t)$ denote the number of grams of salt in the tank at minute t.

▲

Problem 3.79. Watch the following YouTube videos

- This video describes how to use Excel for calculating a sequence from a discrete dynamical system.
  https://youtu.be/kVx7bZTP9L4
- This video shows how to graph a discrete dynamical system in Excel.
  https://youtu.be/k3bwXAeQsKo
- This video shows a bit more about plotting multiple simumlations of a discrete dynamical system with Excel. https://youtu.be/g6OGCmvxCHQ

▲

Now consider the following problems related to discrete dynamical systems. For each problem you need to fully explore the situation by examining all of the possible parameters and hypothetical situations. A good mathematical modeler will consider every possible case and then present their arguments in a thorough, logical, and complete way.

Problem 3.80. A classmate in MA141 has a credit card that charges 1.25% interest per month. Suppose he can only afford to pay \$75 per month, and he does not make any other charges on this credit card. Your classmate has already racked up \$1,200 on the card and he seems pretty freaked out about this debt! After all, it is very early in the school year!

Model the amount owed on the credit card using a discrete dynamical system (DDS). Use your DDS model to give your classmate advice about how to handle credit cards. Use appropriate mathematics and graphics to make your arguments. Scare tactics are absolutely allowed (and encouraged)!

Some good points might be (but are certainly not limited to):

- If he makes the $75 payments each month, when will he pay off the card and what will the last payment be? Furthermore, how much will he have paid for this debt?
- If the initial debt was doubled what would happen?
- Is there an initial debt where his $75 payments will only pay the interest (this is called an equilibrium state)? What happens if he has more on the card that this amount? What about less? Would you classify this equilibrium as *stable* or *unstable*?
- What happens if he changes his payments?
- . . . there are many other scenarios that you should explore. Write up your arguments in a way that will better educate your classmate. Be sure to include all relevant mathematics and graphics with thorough explanations.

▲

Problem 3.81. Presume that there is a population of about 150 moose in the Bob Marshall Wilderness in northwest Montana (often called "The Bob" . . . if you've never been up there, you absolutely must go!). The Fish, Wildlife, and Parks department in Montana estimates that in each year about 25 moose permanently migrate out of The Bob to other places like Glacier National Park, Canada, or further west into the Flathead Wilderness. They also estimate that about 0.5% of the population die off due to old age and other natural causes. Finally, they estimate that new moose calves are born equal to about 9% of the total population in The Bob.

Write a discrete dynamical system for the yearly moose population, model it in Excel, and then consider the fact that every parameter given in the paragraph above is subject to variability. Perform a sensitivity analysis on your model by allowing the parameters to vary by as much as 10%, and use your results to write an informative paragraph to the Fish, Wildlife, and Parks department of Montana about the moose population in The Bob.

▲

Problem 3.82. How long does it take an ant to build a tunnel? That seems like a reasonable question. If you ever had an ant colony purchased by a well-meaning aunt for you in grade school you may have watched ants building industriously and you just might have an idea on this. To answer the question we might need some narrowing of scope, some simplification, and certainly some identification of terms and variables before we can get a nice answer. Let us identify some variables and then together make some assumptions which will lead to a mathematical model.

- Let x be the length of the tunnel in feet that an ant builds.

- Let $T(x)$ be the time in hours it takes the ant to build the tunnel of length x.
- We can get some idea of our situation by making a sketch.

1. If we are going to create a model for the time T as a function of the length of the tunnel x should we use a difference equation or a differential equation? Explain.

2. Maybe we can circumvent building a difference or differential equation by simply writing down an algebraic equation for $T(x)$.

   (a) Write down several candidate functions for $T(x)$ and give one or two statements in each's defense and one or two statements against each.

   (b) You may not have gotten very far with part (a), so how about we try some graphical intuition. Make several sketches of $T(x)$ (tunnel length (x) on the x-axis and total time (T) on the y-axis). Give one or two statements in each's defense and one or two statements against each.

3. Hopefully you see that attempting to jump right on top of $T(x)$ can be hard. So, instead of going after $T(x)$ directly let us examine

   (a) List some assumptions which reflect the reality of such a situation and might make the model simple in a first attempt.

   (b) Modeling *change* is often times much simpler than trying to create an algebraic model from scratch. For the present tunnel-building situation, $T(x+h)-T(x)$ models the amount of time it might take an ant to *extend* a tunnel from distance x to distance $x+h$.

   Below are several possible mathematical models for equation $T(x+h)-T(x)$. Defend or reject each and offer your reasons. Perhaps modify one or two and make it better. When trying to reject a model consider some trivial cases and see if it makes sense, e.g., $h=0$ or $x=0$ or either h or x very large.

   i) $T(x+h)-T(x)=x+h$.
   ii) $T(x+h)-T(x)=x-h$.
   iii) $T(x+h)-T(x)=x^h$.
   iv) $T(x+h)-T(x)=x\cdot h$.
   v) $T(x+h)-T(x)=h^x$.
   vi) $T(x+h)-T(x)=c$.

4. At this point you are ready to write your own equation.

$$T(x+h)-T(x)=\_\_\_\_\_$$

   (a) List the variables and parameters with all of their units. Also list any assumption on which your equation depends.

   (b) Convert your model difference equation to a differential equation with appropriate initial conditions.

(c) Solve the differential equation you create in (d) for $T(x)$. Hint: What initial condition $T(0)$ will you use?

(d) Use your solution from (c) to determine how much longer it takes to build a tunnel which is twice as long as an original tunnel of length L. What would some of your original function models you set forth in 2(a) have told you here?

5. Suppose we had two ants digging from either side of our sand hill along the same straight line. How would this alter the total time for digging the tunnel?

6. Of course, we can apply these same principles of our model to real tunnel building for engineers. If we were considering #5 as related to engineering construction of a long tunnel of length L, outline some of the issues we should be aware of when having two crews (one from each end of the tunnel) working on the tunnel.

▲

Problem 3.83. Consider a situation in which we are studying Helicobacter Pyloria; an antiobiotic resistant organism that gives people an upset stomach. Assume that there are 50 Helicobacter Pyloria initially in a Petri dish. We lose 35% of the population each hour due to "forces of death," but through a one way hatch, 2 microorganisms per hour can enter our Petri dish in the first hour, 4 microorganisms per hour can enter our Petri dish in the second hour, 6 microorganisms per hour can enter our Petri dish in the third hour, 8 in the fourth hour, etc.

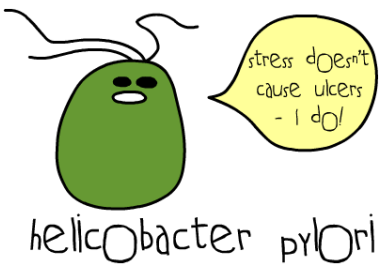
1. Model this situation with (a) a discrete difference equation model **and** (b) a continuous differential equation model.

2. State all of your assumptions used in the model building process.

3. Is there an equilibrium for your model? If so, what it is? If not, why not?

4. Solve each of your models numerically. Use Excel for both the difference and differential equation models. You will need to use Euler's method for the differential equation model (choose a small time step!). Create the plots for the first 12 hours of the experiment.

5. Compare your models and comment on differences and similarities.



6. Your models should take the form

Difference Equation:	$a_{n+1} - a_n = r \cdot a_n + m \cdot n + b$
Difference Equation (after simplifying):	$a_{n+1} = (1+r) \cdot a_n + m \cdot n + b$
Differential Equation:	$\frac{dy}{dt} = r \cdot y + m \cdot t + b$

   What are the values of r, m, and b for this modeling scenario?

7. We would like to find an analytic solution for each of these models, but we haven't encountered these types of difference or differential equations yet. One technique is to *guess* the form of the solution and then use the difference equation or differential equation along with the intial conditions to find the coefficients.

   The guesses for this model are:

   - Difference Equation:
   $$a_n = C_1(1+r)^n + C_2 t + C_3$$
   - Differential Equation:
   $$y(t) = C_1 e^{rt} + C_2 t + C_3$$

   Work with your team to find C_1, C_2, and C_3 for each of the two models.

8. Finally, plot the analytic solutions along side your numerical solutions.

▲

Problem 3.84. A lake in northern Montana is dominated by Arctic Grayling (henceforth called "species A") but the Department of Fish, Wildlife, and Parks is planning to slowly introduce Bull Trout ("species B"). The lake is popular with sport fishermen who remove both species of fish from the lake regularly.

The Department of Fish, Wildlife, and Parks has carefully estimated the number of fish taken by sport fishing each week, and they have decided to keep the fish population as constant as possible by replacing the fish lost by an equal number of Arctic Grayling and Bull Trout. For example, if there are $N = 50$ fish in the lake at the beginning of the week and fishermen remove $M = 10$ fish during that week, then the fish and wildlife people will restock the lake with 5 Arctic Grayling and 5 Bull Trout. Hence the population of the lake will remain $N = 50$ fish at the end of each week, assuming no new fish are born. Both fish species swim freely throughout the lake and both are targeted by similar bait used by sport fisherman.

In your lake you will use $N =$ _____ and $M =$ _____.

In summary:

- The week starts with $N =$ _____ fish.
- The fish swim freely around the lake.

- $M =$ _____ fish are removed from the lake at random during the week.
- $M =$ _____ fish are restocked at the end of the week. $M/2 =$ _____ of those fish are Arctic Grayling and $M/2 =$ _____ of those fish are Bull Trout.

1. **Conjecture:**

   (a) What do you think will happen to the populations of species A and B over a long period of time?

   (b) Is it possible that species A will be eliminated from the lake with the restocking plan? Explain.

2. **Simulate:**

   (a) Use pennies to represent your N fish and decide with your partner(s) which coin face represents which species. Start your lake with 100% species A.

   (b) Decide with your partner(s) how to simulate the swimming of fish, the fishermen, and the Department of Fish, Wildlife, and Parks' restocking plan. Simulate roughly 15 weeks of the fish population representing species A and B with coins. Be sure to let the fish swim thoroughly around the lake and keep track of the proportions made up by species A and B.

Week #	**Number in population**		**Proportion of population**	
	species A	species B	species A	species B
0			1	0
1				
2				
3				
4				
⋮	⋮	⋮	⋮	⋮

3. **Model:**

   (a) Propose a verbal model for the rate of change of species B in the lake.

   rate at which species B changes = _____

   (b) Explicitly state any assumptions that you are using in your verbal model.

   (c) Introduce mathematical notation for your proposed model and write your verbal model mathematically. Be sure to include any necessary condition(s).

   model: _____

   condition(s): _____

4. **Analyze:**

(a) According to your model, what is the long term effect on the fish population in the lake? Use your model to justify your answer algebraically and graphically.

(b) Solve your mathematical model (either numerically or analytically) and compare with your data.

(c) (extension) Suppose now that the Department of Fish, Wildlife, and Parks does not attempt to keep the population in the lake constant. That is, suppose that fishing reduces the population by M_1 fish each week and the Department of Fish, Wildlife, and Parks restocks M_2 fish each week. Fully explore this scenario.

▲

Problem 3.85.

Niedjatu Elpmeyout
MT Environmental Law Partners
101 Park St.
Helena, MT 59625

O.D.E. Consulting
1601 N. Benton Ave.
Helena, MT 59625

Dear sir or madam:

I have been assigned a case here at my law offices defending a client who got himself into a quite a sticky situation (or rather, a slippery one). My firm would like to secure your services to help us understand the physical aspects and data surrounding the event. In order to protect our client's anonymity, we will request your discretion in sharing this information with the press.

Our client allegedly caused an oil spill over some open water while transporting some cargo. There seems to be some dispute with respect to the amount of oil spilled, and the EPA (those tree-huggers!) has assigned massive fines, which we dispute. While we concede that there was a small amount of oil spilled, we contend that the amount is really not nearly as much as they claim. In fact, our client actually improved the local economy by hiring local workers to assist with containing and cleaning the oil. They should be thanking our client, really. But I digress.

Here's where we need your help. We know that as soon as the resulting oil slick was detected, the Coast Guard wanted to document the size of the oil slick. From time to time, but irregularly, a helicopter was dispatched to photograph the oil slick. On each trip, it arrived over the slick, the pilot took a picture, waited 10 minutes, took another, and then headed home. On each of seven trips the size (in area) of the slick was measured from

both photographs, as below.

Area of oil slick (in miles):

Initial Obs.	10 min. later
1.047	1.139
2.005	2.087
3.348	3.413
5.719	5.765
7.273	7.304
8.410	8.426
9.117	9.127

We would like to request the following information from you.

- Build a model for the growth of the oil slick at time t.
- Predict the size of the oil slick, say at $t = 10$, $t = 20$, and $t = 120$ minutes from the start of the oil spill.
- Plot your model of the size of the oil slick as a function of time.
- Find the time at which the oil slick was 8 square miles.
- Determine the time of each of the observations.

Please help us help our client (who, despite what you might have heard in the news, was definitely *not* under the influence of an illegal substance–not at the time of the incident, anyway). We will have to present your argument in court, so please fully explain your work in a clear and concise fashion.

Your company was suggested by one Professor Sullivan of Carroll College, whose services we have used before. He has promised to be available to you, but cannot himself commit to this work because he is teaching some talented and motivated students techniques in mathematical modeling this semester.

Looking forward to seeing your results soon.

Sincerely,

Niedjatu Elpmeyout
MT Environmental Law Partners ▲

Problem 3.86. A beaker of warm water is placed in a room with an ambient temperature of 72°F. The data for this experiment can be found in the `Newton.xlsx` Excel file on Moodle.

1. Below are 5 proposed differential equation models for the temperature of the water in the beaker.

$$\frac{dT}{dt} = a \tag{3.22}$$
$$\frac{dT}{dt} = a + bt \tag{3.23}$$
$$\frac{dT}{dt} = \frac{A}{B + Ct} \tag{3.24}$$
$$\frac{dT}{dt} = -k(T - T_{env}) \tag{3.25}$$
$$\frac{dT}{dt} = -kT \tag{3.26}$$
$$\frac{dT}{dt} = Ae^{-kt} \tag{3.27}$$

   Spend a few minutes critiquing each of these models. For each model that seems unreasonable, be sure to give a brief explanation.

2. Choose the most appropriate model from the above list (only 1 of them is the *right* one!) and do the following:

   (a) Find any equilibrium points and determine their stability

   (b) solve the differential equation using an appropriate technique. Your answer will have some unknown parameters.

3. Use the Solver in Excel to find the value(s) of the parameter(s) in your model so that your model best fits the data.

4. How would the data (and your solution) change if the beaker had been insulated?

▲

Chapter 4

Second Order Models

4.1 Modeling Oscillations

To begin our study of linear second-order differential equations we consider a very simple physical system: a mass hanging from a spring that is oscillating in time. Figure 4.1 shows the basic setup for the situation. In Figure 4.1 the mass is oscillating up and down in the y direction.

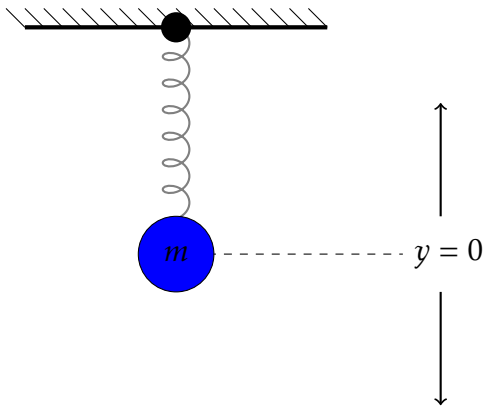


Figure 4.1. A mass and spring oscillating system.

Problem 4.1. Consider the mass and spring system in Figure 4.1 Assuming that the motion is always in the vertical direction, the displacement of the mass at time t is $y(t)$, the instantaneous velocity of the mass at the time t is $y'(t)$, and the acceleration is $y''(t)$. Newton's second law: "mass times acceleration equals the sum of the forces" can be used to write

$$my'' = F_r + F_d + f(t) \tag{4.1}$$

where m is the mass of the object, F_r is the restoring force due to the spring, F_d is the force due to the damping in the system, and $f(t)$ represents any external forces on the system.

(a) Hooke's Law states that the restoring force of the spring is proportional to its displacement. Use the statement of Hooke's law to propose an expression for the restoring force F_r. The constant of proportionality is called the *spring constant*. Keep in mind that the restoring force works opposite the displacement so be sure to get the sign correct.

(b) The damping force F_d is assumed to be proportional to the velocity and acts in the direction opposite the direction of motion. Use this statement to propose an expression for the damping force F_d. The constant of proportionality is called the *damping constant*. Keep in mind that the damping force works opposite the velocity so be sure to get the sign correct.

(c) If $f(t)$ is any external force acting on the system then we can finally write a differential equation describing the motion of the mass and spring system. Write this system and give a full description of each of the coefficients.

(d) What are the units of the coefficients given that the units of force are Newtons and

$$1 \text{ Newton} = \frac{1\text{kg} \cdot \text{m}}{\text{s}^2}.$$

▲

In previous problem we saw a system of the form

$$my'' = -ky - by' + f(t). \tag{4.2}$$

This equation can be rearranged to

$$my'' + by' + ky = f(t). \tag{4.3}$$

The dynamics of this equation are both very interesting and complex. To get started consider the next problem.

Problem 4.2. Go to the GeoGebra applet
http://www.geogebratube.org/student/m217165
This applet is designed to allow you to explore the mass spring system

$$my'' + by' + ky = f(t)$$

(a) We will start with an un-driven mass spring system where the forcing function is zero. In each of the following cases, sketch a plot of the typical behavior seen.

1. Pick several m, b, and k values that generate an over damped system. An over damped system has the feature that $b^2 - 4mk > 0$. What physical situation would this scenario model?

2. Pick several m, b, and k values that generate a critically damped system. A critically damped system has the feature that $b^2 - 4mk = 0$. What physical situation would this scenario model?
3. Pick several m, b, and k values that generate an under damped system. An under damped system has the feature that $b^2 - 4mk < 0$. What physical situation would this scenario model?

(b) Now experiment with a forced spring mass system. Get a feel for what different forcing terms do to control the behavior of the system.

▲

4.2 Homogeneous Linear 2^{nd} Order Differential Equations

To begin our study of linear second order equations we need to first examine the homogeneous equation

$$my'' + by' + ky = 0 \tag{4.4}$$

where m, b, k are real numbers and $m \neq 0$. Taking a clue from the method of undetermined coefficients we can guess the type of solution to be some sort of exponential function:

$$\text{Guess: } y(t) = e^{rt}.$$

Under this guess we can observe that $y'(t) = re^{rt}$ and $y''(t) = r^2e^{rt}$ to rewrite equation (4.4) as

$$mr^2e^{rt} + bre^{rt} + ke^{rt} = 0.$$

After some algebra we see that

$$e^{rt} \cdot \left(mr^2 + br + k\right) = 0. \tag{4.5}$$

The exponential function is never zero when r is a real number so equation (4.5) only has a solution if $mr^2 + br + k = 0$. The left-hand side of this equation is called the *characteristic polynomial* of the differential equation.

Definition 4.3. If $my'' + by' + ky = 0$ then the *characteristic polynomial* associated with the differential equation is

$$p(r) = mr^2 + br + k. \tag{4.6}$$

Since equation (4.6) is a quadratic equation the solutions can be found via the quadratic formula

$$r = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

There are typically two solutions, r_1 and r_2, to the quadratic equation (or two repeated roots), and using the guess that $y(t) = e^{rt}$ we can write the solutions to (4.4) as

$$y(t) = C_1e^{r_1t} + C_2e^{r_2t}.$$

Recall from high school algebra that it is is possible that there are imaginary solutions or repeated solutions to a quadratic equation like $p(r)$. In these cases we take a slightly different form of the solution. To classify the solutions to the differential equation (4.4) recall that the **discriminant** of the quadratic function is $D = b^2 - 4mk$, and this corresponds to the possibilities listed in the following theorem.

Theorem 4.4. If $my'' + by' + ky = 0$ then a typical solution takes the form $y(t) = e^{rt}$ and the characteristic polynomial is $p(r) = mr^2 + br + k$. There are three cases for the solutions that each depend on the discriminant $D = b^2 - 4mk$ of the characteristic polynomial.

Discriminant: $b^2 - 4mk$	Roots of $p(r)$	General Solution
$b^2 - 4mk > 0$	2 Roots: $r_1 \neq r_2$	$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$
$b^2 - 4mk = 0$	Single root: r	$y(t) = C_1 e^{rt} + C_2 t e^{rt}$
$b^2 - 4mk < 0$	Complex roots: $\alpha \pm \beta i$	$y(t) = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t)$

In each case of Theorem 4.4 we see that there are two unknown constants. In order to find both constants there need to be 2 conditions: an initial displacement $y(0)$ and an initial velocity $y'(0)$. The following examples show the typical solutions of various homogeneous linear second order differential equations.

Example 4.5. Consider the second order linear homogeneous differential equation $y'' + 4y' + 3y = 0$. Find the general solution to this differential equation.
Solution: If we tie this example to the mass and spring system we have a mass of $m = 1$kg, a damping force of $b = 4$kg/s, and a spring constant of $k = 3$N/m. The damping force is *rather high* in comparison to the restoring force so it is expected that the spring mass system will lose oscillations rather quickly.

The discriminant is $D = b^2 - 4ac = 16 - 4(1)(3) = 4$ so the two roots of the characteristic polynomial are

$$r_1 = \frac{-4 + \sqrt{4}}{2} = -1 \quad \text{and} \quad r_2 = \frac{-4 - \sqrt{4}}{2} = -3$$

and by Theorem 4.4 we see that the general solution to $y'' + 4y' + 3y = 0$ is

$$y(t) = C_1 e^{-1t} + C_2 e^{-3t}$$

This is an infinite collection of possible solutions that depend on two constants C_1 and C_2. In order to have a single solution we must specify an initial condition $y(0)$ and an initial velocity $y'(0)$.

Figure 4.3 shows several solutions to the differential equation $y'' + 4y' + 3y = 0$ with various initial displacements and initial velocities. With a damping force of $b = 4$kg/s this "mass and spring system" is working in an environment where the motion is damped rather quickly. Imagine that we are running the experiment in honey!

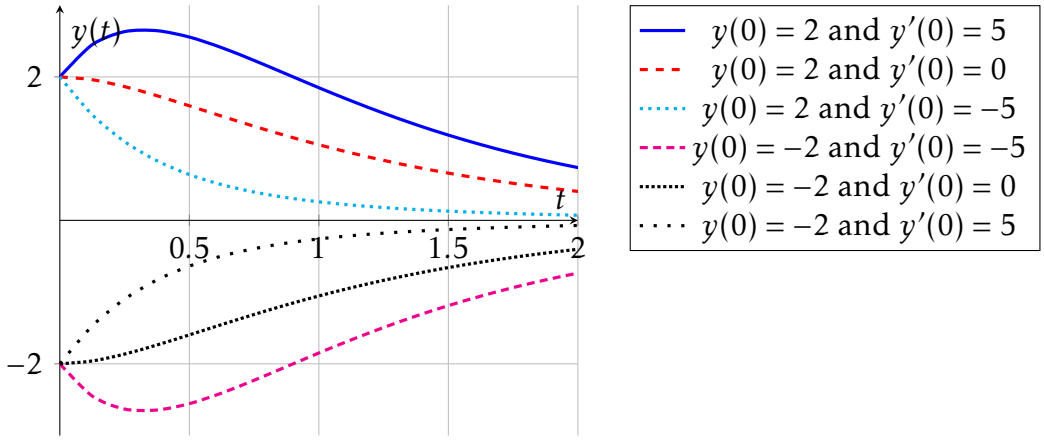


Figure 4.2. Several solutions to $y'' + 4y' + 3y = 0$ shown in example 4.6.

Example 4.6. Consider the second order linear homogeneous differential equation $y'' + 1y' + 1y = 0$. Find the general solution to this differential equation.
Solution: If we tie this example to the mass and spring system we have a mass of $m = 1\text{kg}$, a damping force of $b = 1\text{kg/s}$, and a spring constant of $k = 1\text{N/m}$. In this case the spring constant (the restoring force) and the damping force will play against each other to create a damped oscillator.

The discriminant is $D = b^2 - 4ac = 1 - 4(1)(1) = -3$. Since the discriminant is negative we will have the sine and cosine solution on the third line of the table in Theorem 4.4.

$$r_1 = \frac{-1 + \sqrt{-3}}{2} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \quad \text{and} \quad r_2 = \frac{-1 - \sqrt{-3}}{2} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

If we define $\alpha = -1/2$ and $\beta = -\frac{\sqrt{3}}{2}$ we get the general solution to $y'' + y + 1 = 0$ as

$$y(t) = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t)$$

$$\implies y(t) = C_1 e^{-1/2t}\cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 e^{-1/2t}\sin\left(\frac{\sqrt{3}}{2}t\right).$$

Since there are two constants this is an infinite collection of solutions that depend on the initial displacement $y(0)$ and the initial velocity $y'(0)$. Figure 4.3 shows several solutions.

Example 4.7. Consider the second order linear homogeneous differential equation $y'' + 6y' + 9y = 0$. Find the general solution to this differential equation.
Solution: The discriminant is $D = b^2 - 4ac = 6^2 - 4(1)(9) = 36 - 36 = 0$. This is the

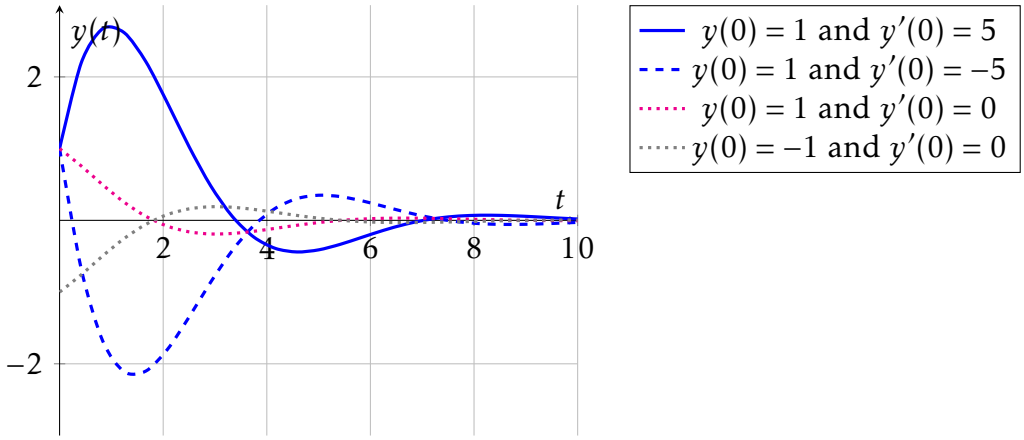


Figure 4.3. Several solutions to $y'' + y' + y = 0$ shown in example 4.6. Notice that this equation models an underdamped oscillator where some oscillations occur.

second case in the table in Theorem 4.4; a repeated root

$$r = \frac{-6 \pm \sqrt{0}}{2} = -3.$$

The solution is therefore

$$y(t) = C_1 e^{-3t} + C_2 t e^{-3t}.$$

As in the previous two examples there are infinitely many solutions that depend on the initial condition $y(0)$ and initial velocity $y'(0)$. Figure 4.4 shows several solutions.

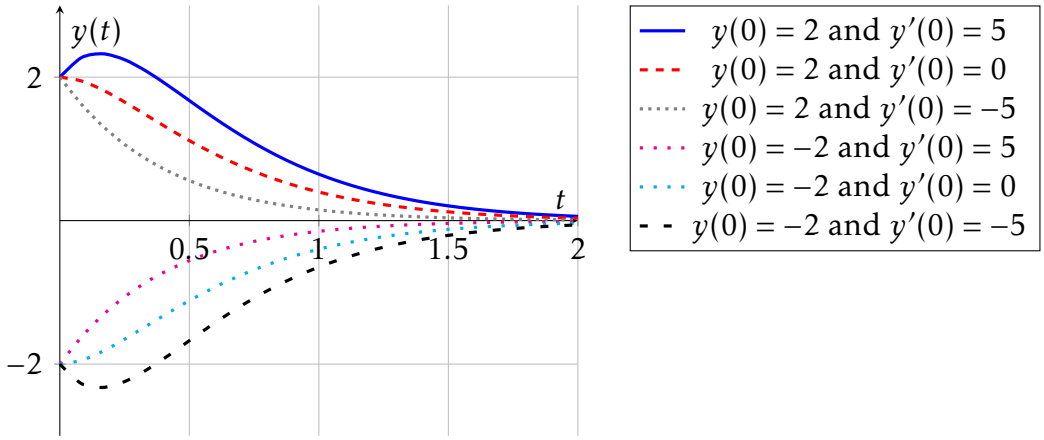


Figure 4.4. Several solutions to $y'' + 6y' + 9y = 0$ shown in example 4.7. This is a model for a critically damped oscillator where no oscillations can occur..

The mass spring system can be written as $my'' + by' + ky = 0$ when there is no external forcing. This has the same form as the second order linear homogeneous differential

equation in Theorem 4.4. In the mass spring system, the discriminant is

$$D = b^2 - 4(m)(k)$$

and we can classify all such systems with the following definitions.

Definition 4.8.
- $D = b^2 - 4mk > 0$ The system is *over damped*
- $D = b^2 - 4mk = 0$ The system is *critically damped*
- $D = b^2 - 4mk < 0$ The system is *under damped*

Theorem 4.9. For the homogeneous mass spring oscillator equation

$$my'' + by' + ky = 0$$

with $m, k, b > 0$ there are four primary solution types.

Un-Damped ($b = 0$):

$$y(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$$

where $\omega = \sqrt{\frac{k}{m}}$ is called the natural frequency of the oscillator.

Under Damped (two complex roots):

$$y(t) = e^{\alpha t}(C_1 \cos(\omega t) + C_2 \sin(\omega t))$$

where $r = \alpha \pm i\omega$

Over Damped (two real roots):

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Critically Damped (one repeated real root):

$$y(t) = C_1 e^{rt} + C_2 t e^{rt}$$

Problem 4.10. Use the equation derived in this chapter to change the descriptions of the mass spring systems to a second order linear homogeneous differential equation. Then solve the equation with the aid of Theorem 4.4 and the given descriptions of the initial displacement and initial velocity. State whether each situation is an over- or under-damped oscillator.

(a) An object with a mass of $m = 1$kg is suspended from a spring with a spring constant $k = 4$N/m. The system is submerged in a liquid causing it to have a large damping constant $b = 5$kg/s. The object is lifted up 1meter and let go with no initial velocity.

(b) An object with a mass of $m = 1$kg is suspended from a spring with a spring constant $k = 10$N/m. The system is submerged in a liquid causing it to have a large damping constant $b = 2$kg/s. The object is pulled down 1meter and given an initial velocity of 1m/s.

(c) An object with a mass of $m = 10$kg is suspended on a spring with spring constant $k = 20$N/m. The damping coefficient is $b = 30$kg/s. The mass is initially held at equilibrium and is given an initial velocity of 2m/s in the downward direction.

▲

Problem 4.11. For each of the following, use the applet https://www.geogebra.org/m/S4ktuMbX to show the dynamics of situation before you find the analytic solution.

1. Consider a mass-spring system with mass $m = 1kg$ and restoring force $k = 4N/m$. Let $y(0) = 1$ and $y'(0) = 0$. Find the position function $y(t)$.

2. Consider a mass-spring system with mass $m = 1kg$ and restoring force $k = 4N/m$. Let $y(0) = 0$ and $y'(0) = 1$. Find the position function $y(t)$.

3. Consider a mass-spring system with mass $m = 9kg$ and restoring force $k = 1N/m$. Let $y(0) = 3$ and $y'(0) = 0$. Find the position function $y(t)$.

4. Consider a mass-spring system with mass $m = 2kg$ and restoring force $k = 18N/m$. Let $y(0) = 1$ and $y'(0) = -1$. Find the position function $y(t)$.

5. Consider the motion of a brick with a mass of $m = 6kg$ that is hung from the end of a spring. When the brick is at rest, the weight of the brick stretches the spring by $0.1m$, so that the force of gravity down is equal to the force of the spring pulling up.

   (a) The weight the force of gravity on the brick, is equal to the brick's mass multiplied by g, the acceleration of gravity. Use $g = 9.8$ meters per second squared to calculate what the spring constant k must be.

   (b) Set up a differential equation for the motion of the brick.

   (c) The spring is then stretched 0.11 m away from equilibrium and released. To describe the motion, set up a differential equation with initial conditions.

▲

Problem 4.12. Recall that in a spring-mass system Newton's second law gives:

$$my'' = F_r + F_d$$

where

- m is the mass,
- F_r is the restoring force (which is proportional to the displacement), and
- F_d is the damping force (which is proprotion to the velocity).

Hence, the motion is governed by the equation

$$my'' = -ky - by'$$

and after some algebra we get

$$my'' + by' + ky = 0$$

1. Consider spring-mass system with mass $m = 1kg$, damping force $b = 3\,kg/s$, and restoring force $k = 2\,N/m$. If $y(0) = 1$ and $y'(0) = 0$ then find the function modeling the position: $y(t)$.

2. Consider spring-mass system with mass $m = 1kg$, damping force $b = 2\,kg/s$, and restoring force $k = 1\,N/m$. If $y(0) = 0$ and $y'(0) = 1$ then find the function modeling the position: $y(t)$.

3. Consider spring-mass system with mass $m = 2kg$, damping force $b = 4\,kg/s$, and restoring force $k = 4\,N/m$. If $y(0) = 1$ and $y'(0) = 0$ then find the function modeling the position: $y(t)$.

▲

Problem 4.13. Classify each of the scenarios from the previous two problems as "undamped", "under damped", "critically damped", or "over damped". ▲

4.3 Forced Oscillations

In the previous Section we encountered the mass spring system

$$my'' + by' + ky = 0$$

where m is the mass of the object, b is the damping term, and k is the restoring force called the spring constant. In the present situation we will consider what happens with the right-hand side is not zero, but instead if there is an external force acting driving (or working in opposition to) the oscillations. The following Preview Activity will get you started.

Problem 4.14. For a nonhomogeneous linear differential equation, the general solution takes the form

$$y(t) = y_h(t) + y_p(t)$$

where $y_h(t)$ is the homogeneous solution and $y_p(t)$ is the particular solution given the nonhomogeneity. For each of the following second order linear nonhomogeneous differential equations, write the homogeneous solution and a possible particular solution.

(a) $y'' + 5y' + 6y = \sin(2t)$

(b) $y'' + 4y = e^{-t}$

(c) $y'' + 6y' + 9y = 2 + t$

▲

Resonance

Consider an undamped mass spring system forced by an oscillating term with amplitude R and a frequency ω.

$$my'' + ky = R\sin(\omega t).$$

The homogeneous solution can be found by solving $my'' + ky = 0$ and the particular solution will take the form of a sinusoidal function with frequency ω. The **natural frequency** of the homogeneous equation is $\omega_0 = \sqrt{k/m}$ by Theorem 4.4. When the natural frequency of the homogeneous solution and the natural frequency of the forcing term match we get the phenomenon called **resonance.**

The following activity will walk you through solving problems with resonance.

Problem 4.15. Consider differential equation $y'' + 4y = \sin(2t)$. This can be viewed as a mass spring system with a restoring force of $k = 4$, no damping force $b = 0$ and a forcing term $f(t) = \sin(2t)$.

(a) Use the ideas from the previous Section to write a general solution to the homogeneous equation $y'' + 4y = 0$.

(b) Conjecture the form of the particular solution $y_p(t)$ that matches the form of the nonhomogeneity. In this case the homogeneous solution and the particular solution have exactly the same form. The fix for this is to multiply the particular solution by t. Write the particular solution.

(c) Write the solution as the sum of the homogeneous and particular solutions $y(t) = y_h(t) + y_p(t)$.

(d) Use the initial conditions $y(0) = 0$ and $y'(0) = 0$ and the differential equation to find all of the coefficients. State how these initial conditions relate to the mass spring system.

(e) Plot the solution for $0 < t \leq 10$ and explain the behavior you see in relation to the mass spring system.

(f) If the differential equation were changed to $y'' + 3y = \sin(2t)$ (same forcing term but different spring constant), what would you expect from the behavior of the model?

▲

4.4 Energy in Mass Spring Systems – A Lab Exploration

Background

Consider a simple mass-spring system depicted in Figure 4.5 where m is the mass of an object suspended by a spring. Given some initial energy or displacement in the vertical direction the mass will oscillate vertically. Using Newton's second law of motion we note immediately that the sum of the forces acting on the mass will be balanced by the product of the mass and the acceleration:

$$ma = \sum F. \tag{4.7}$$

There are three primary forces driving the oscillations in the mass-spring system:

- the restoring force due to the spring: F_r,
- the damping force working against the motion of the mass: F_d, and
- any external forces that may depend on time: $f(t)$.

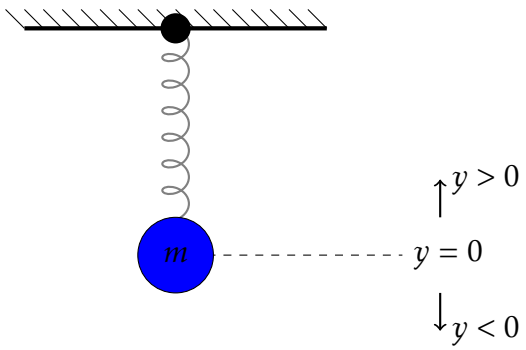


Figure 4.5. A mass-spring oscillating system connected to a rigid body above with mass m. The coordinate system uses $y = 0$ as the rest position of the mass with $y > 0$ indicating positions above equilibrium and $y < 0$ indicating position below equilibrium.

For an ideal linear spring, Hooke's Law states that the restoring force is proportional to the displacement of the mass from equilibrium: $F_r = -ky(t)$. The proportionality constant k is called the spring constant. In simple terms, Hooke's Law states that if the mass-spring system has been stretched a long way from equilibrium then the restoring force will be large. If, however, the mass-spring system has been stretched only a short way from equilibrium then the restoring force will be small. The negative sign indicates that the force will pull in the opposite direction of the position and, hence, back toward equilibrium. Since force is measured in Newtons, the spring constant k has units of Newtons per meter.

For an ideal linear spring, the force due to drag will oppose the motion in a manner that is approximately proportional to the velocity of the mass: $F_d = -by'(t)$. That is to say,

if the mass is moving quickly then the force due to drag will be large and if the mass is moving slowly then the force due to drag will be small. The damping constant has units of Newtons per meter per second.

External forces, $f(t)$, are any other forces that act on the system. Examples of such forces would be the presence of a magnetic field, the presence of upward or downward air currents, a periodic forcing term such as pushes or pulls on the mass or spring, etc.

Using Newton's second law (4.7) we can write the balanced forces as

$$ma = F_r + F_d + f(t). \tag{4.8}$$

Substituting the restoring force, the damping force, and $a = y''$ into (4.8) gives the linear second-order differential equation

$$my''(t) = -ky(t) - by'(t) + f(t). \tag{4.9}$$

Rearranging (4.9) algebraically gives the standard form for a linear second-order non-homogeneous differential equation:

$$my'' + by' + ky = f(t). \tag{4.10}$$

It should be noted that the forms of F_r and F_d used to build (4.10) are idealizations. If the spring is stretched *too far*, if the speeds are *too high*, or if the materials used are atypical in some way then different forms of the restoring and damping forces may be necessary.

In this problem we investigate how the mass-spring system (4.10) can be described in terms of potential and kinetic energy. We begin with a few definitions:

- **Kinetic Energy**, the energy of motion, is defined as

$$E_{kinetic}(t) = \frac{\text{mass} \times \text{velocity}^2}{2} = \frac{1}{2} m [y'(t)]^2.$$

- **Potential Energy** in a mass-spring system, also called the *elastic potential*, is defined as

$$E_{potential}(t) = \frac{\text{spring constant} \times \text{position}^2}{2} = \frac{1}{2} k [y(t)]^2.$$

- The **Total Energy** in a mechanical system is the sum of the kinetic energy and the potential energy.

$$E_{total}(t) = E_{kinetic}(t) + E_{potential}(t)$$

The units of energy are Newton-Meters or Joules. In terms of a mass-spring system, kinetic energy is the energy that the mass has due to its motion. If the mass is at rest then the kinetic energy is zero. Also, if the mass has reached a maximum displacement (and is just about to turn around and move in the opposite direction) the kinetic energy will be zero. The potential energy in a mass-spring system is the energy that the mass has relative to its equilibrium position. If the mass is at equilibrium then it has no potential energy but if the mass is far from equilibrium it will have a large amount of potential energy.

Student Tasks:

The following tasks ask you to explore the mass-spring system by examining the total energy of the system. The tasks are necessarily open ended meaning that each group could (and should) get different answers for each task. You should use the MATLAB file provided on the Moodle page for this exploration. At the end of the explorations you will write your results in a formal lab report.

1. **Make a conjecture**: In what cases (related to m, b, k, and $f(t)$) do you think that the total energy will be constant? Give a few sentences to support your claim and then create plots of position, potential energy, kinetic energy, and total energy to graphically verify your conjecture.

2. **More conjectures**: In what cases (related to m, b, k, and $f(t)$) do you think that the total energy will be decreasing, increasing, or oscillating in a mass-spring system? Give a few sentences to support your claims.

3. **Exploration:** Fully explore how the energy behaves in the mass-spring system. To make your exploration somewhat easier let's assume the following:

   $$\text{mass} = m = 1\text{kg}, \quad \text{initial position } = y(0) = 0\text{m}, \quad \text{initial velocity} = y'(0) = 1\text{m/sec}.$$

   This way you only have the damping constant b, the restoring constant k, and the forcing function $f(t)$ to experiment with. The given initial conditions will start the mass at equilibrium and given it an initial upward velocity.

   Use the background information presented earlier in this document to conjecture and test combinations of b, k, and $f(t)$ that result in the following situations. You must find the situations listed, and the last item in the following list gives you a chance to look for situations that are not listed.

   (a) Find a combination of parameters where the total energy drops slowly to zero.

   (b) Find a combination of parameters where the total energy drops very quickly to zero.

   (c) Find a combination of parameters where the total energy oscillates but never reaches zero and does not increase for all time.

   (d) Find a combination of parameters where the total energy increases for all time.

   (e) Find a combination of parameters where the total energy changes initially but eventually finds a nonzero equilibrium.

   (f) Find a combination of parameters where the system exhibits resonance (where the unforced frequency matches the frequency of the forcing term).

   (g) Find a combination of parameters where the total energy oscillates with two frequencies: a slow frequency and a faster frequency (hint: get the resonant system first and then change the frequency of the forcing term).

   (h) Now go find several other combinations of parameters that give behaviors different than the ones listed above.

4. **Summary:** Summarize all of your findings into a well-formated lab report clearly showing the mathematical and graphical representations all of the cases used in your experimentations. Your initial conjectures (from problems 1 and 2) may have been incorrect so take this chance to clarify what you've found. Your summary must include general descriptions of the following four general scenarios.

   (a) The total energy remains constant.

   (b) The total energy drops to zero.

   (c) The total energy increases without bound.

   (d) The total energy oscillates.

4.5 Modeling Explorations with 2^{nd} Order Differential Equations

Problem 4.16. A large water tower holds 3 million gallons of water, which has a mass of about 11 million kilograms. When the wind blows, this causes the steel structure to sway back and forth due to the force. An engineer studying the tower observes that a steady wind at a speed of 35 mph exerts a force of 1.45 million Newtons on the tower, causing it to lean 0.27 meters away from equilibrium. The engineer begins by assuming that the restoring force is proportional to the displacement $F = -kx$, so that the motion of the system can be modeled by the differential equation $ma = -kx$. Here m is the mass of the tower, a is the tower's acceleration, k is the spring constant, and x is the displacement of the tower away from its equilibrium position.

(a) What is the spring constant of the steel structure? (Be sure to use the right units!)

(b) What is the angular frequency ω at which the tower will tend to oscillate? (Note that angular frequency is measured in radians per second.)

(c) Write down the general solution to the differential equation. (This is the version with the two arbitrary constants that we will have to figure out from the initial conditions.)

(d) Suppose that the tower is sitting comfortably in equilibrium when a sudden brief gust of wind gives the tower a velocity of +0.24 meters per second. What function will describe how the position of the tower changes after this?

(e) Make a plot of this function.

(f) What is the maximum displacement away from equilibrium that the water tower goes? (Use your graph as an aid, but use your function from (d) to get the exact value.)

(g) Reading from your graph, what is the period of oscillation? That is, how much time does it take the tower to go through one complete back-and-forth cycle?

(h) How would your plot be different if there had been a stronger gust of wind? What would be the same and what would be different?

(i) What function describes the velocity of the tower?

(j) Make a plot of velocity versus time in.

(k) What is the maximum speed that the water tower attains? (Speed is the absolute value of velocity.) Again, use both your graph and your velocity function.

(l) Where is the water tower when it is moving the fastest?

(m) Compare your plot of position versus time with your plot of velocity versus time. You will find that first the velocity reaches a positive peak, then position follows, then velocity reaches a negative peak, then position follows, etc. Why is this? Explain in terms of the motion of the physical object.

(n) What function describes the acceleration of the tower?

(o) Make a plot of acceleration versus time.

(p) What is the greatest acceleration experienced by the tower?

(q) Where is the tower when it is accelerating the most?

(r) Another way to analyze motion is to create a phase plot, which puts velocity on the y-axis and position on the x-axis. Create a plot like this.

(s) This is a strange looking plot! What point on this plot represents our initial condition?

(t) What is going on when the tower is at a point on the far left side of this curve?

(u) Would the motion of the water tower cause this curve to be traversed in a clockwise or a counterclockwise direction? Explain your thinking.

(v) Suppose that a month later, the tower is holding 21.5 million kilograms of water, when it experiences a gust of wind that again gives it a speed of 0.24 m/s. What function will describe the displacement of the tower?

(w) Make a plot of the resulting position as a function of time.

(x) Now what is the period of oscillation? (Estimate from graph and confirm with position function.)

(y) The steel framework will experience catastrophic structural failure if it sways more than 1.2 meters. What is the maximum speed that a gust of wind can give the tower while it's holding 21.5 million kg of water before this causes unpleasant results?

(z) Another engineer studies a similar tower in the neighboring town, finding that while this contains only 9.5 million kilograms of water, it tends to sway back and forth with an angular frequency of = 0.55 rad/s. What must be the spring constant of the structure holding up this water tower?

▲

Problem 4.17. If we construct an electrical circuit with a capacitor and an inductor in series we find that the amount of charge on the capacitor $Q(t)$ (with charge measured in Coulombs) can be modeled by the following differential equation:

$$L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0,$$

where C is the capacitance of the capacitor as measured in Farads, and L is the inductance of the inductor as measured in Henrys, and the second derivative has units of Coulombs/second^2.

(a) If we have $C = 2 \times 10^{-6}$F and $L = 2$ Henrys, what is the general solution to this differential equation?

(b) Your answer to the first question should include the constant "500." What units does this number have? This is called the natural frequency of the circuit.

(c) What is the period of these oscillations?

(d) Suppose we begin with 8 nanocoulombs of charge on the capacitor and no current flowing through the circuit ($Q'(t) = 0$). What solution function corresponds to this set of initial conditions?

(e) Create a graph showing how the charge on the capacitor varies over time. Your graph should begin at $t = 0$ and should show only a few periods of oscillation.

(f) How much charge is on the capacitor at $t = 10$ milliseconds?

(g) New scenario: Suppose we begin with no charge on the capacitor, but charge flowing through the circuit at a rate of 25 millicoulombs per second. (This is the same as 25 milliamps.) What solution function corresponds to this set of initial conditions?

(h) In this scenario: What is the maximum amount of charge on the capacitor? Give your answer in microcoulombs.

(i) Create a graph showing how the charge on the capacitor varies over time. Your graph should begin at $t = 0$ and should show only a few periods of oscillation.

(j) Now, suppose we begin with 10 microcoulombs of charge on the capacitor and 10 milliamps of charge flowing through the circuit. What solution function corresponds to this set of initial conditions?

(k) What is the maximum amount of charge on the capacitor?

(l) If we add an antenna to this circuit, then the voltage from the antenna $V(t)$ will be added to the differential equation like this:

$$L\frac{d^2Q}{dt^2} + \frac{Q}{C} = V(t).$$

Find a function that will serve as a particular solution to this differential equation with an antenna signal of

(m) What is the maximum amount of charge on the capacitor produced by this particular solution?

(n) What is the period of oscillation found in your particular solution? State your answer in milliseconds.

(o) Create a graph showing how the charge on the capacitor varies over time. Your graph should begin at $t = 0$ and should show only a few periods of oscillation.

(p) A radio uses a circuit like this in order to amplify one frequency, the frequency of the station that you want to listen to, while ignoring the frequencies of the other stations. The circuit has a variable capacitor and when the natural frequency of the circuit matches the antenna frequency that you want amplified, then the circuit produces a behavior called resonance. If the inductor remains constant at $L = 2$ Henrys, then what capacitance C do we need in order for the circuit to have resonance with an antenna signal at a frequency of 600 radians per second?

▲

Chapter 5

Systems of Difference and Differential Equations

Thus far we have only discussed difference and differential equations that involved a single unkonwn. Unfortunately the world is much more complicated than that! It is more common to have several variables interacting with each other in a model, and that is the type of problem that we'll attack in this chapter. The first three sections are dedicated to one modeling problem each. The subsequent sections focus on linear systems of equations and some extremely powerful linear algebra tools that we use to analyze such systems.

5.1 Spread of Disease

Problem 5.1. We're going to try a social experiment.

(a) Everyone in the class get a random number between 1 and 5.

(b) I'm sorry, but if your random number is "1" then you just got infected with the horribly contagious disease ODEbola. Raise your hand if you are infected.

(c) For the next 15 seconds everyone needs to walk aimlessly around the classroom (move the tables out of the way and don't be afraid to bump into each other). This step is supposed to simulate *homogeneous mixing* so . . . mix homogeneously!

(d) At the end of the 15 seconds stop and stand still. Reach your arms out. If someone within arm's reach is infected with ODEbola then you now are too! Raise your hand if you are infected.

(e) Repeat steps (c) and (d) again. At the end of every step 10% of the infected people that are infected will recover and are removed from the experiment. Keep track of the number of people that are infected and recovered at each step. Run the experiment for several iterations

▲

Problem 5.2. Watch the video https://youtu.be/NSNWDUXN2p4 to see a simulation where a 150 person population has an outbreak and the virus is spread via close proximity contact. Notice, in particular, the homogeneous mixing. ▲

Problem 5.3. In the previous problems there were three distinct populations: Susceptible (S), Infected and contagious (I), and Recovered (R).

(a) Does this problem call for difference equations or differential equations? Why?

(b) The three populations depend on each other? That is, at any given time step do we need to know all three populations to propagate forward in time or does it suffice to only know one of them?

(c) Write a system of difference or differential equations for the experiment that we ran and remember to keep in mind that we were homogeneously mixing the population the entire time. Think very carefully about how a susceptible person is actually infected.

Susceptible Population: _____ = _____ with $S(0) =$ _____
Infective Population: _____ = _____ with $I(0) =$ _____
Recovered Population: _____ = _____ with $R(0) =$ _____

A model equation would read

rate at which pop is changing = an expression for how the pop is changing

▲

Problem 5.4. Consider the system from the previous problem.

(a) Is the system linear or nonlinear? Why?

(b) What is the expected long-term behavior of this system? Why?

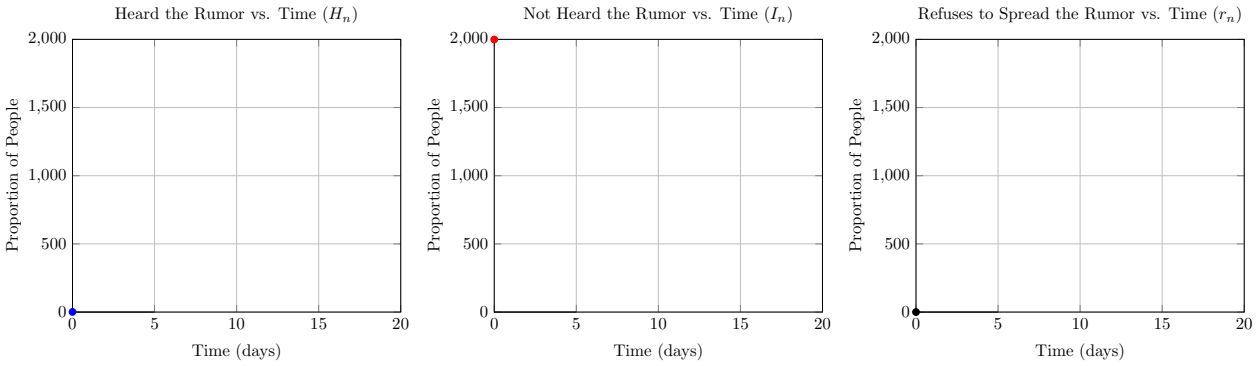
▲

5.2 Spreading a Juicy Rumor

A particularly juicy rumor has cropped up on your college campus. The goal of this activity is to model the spread of the rumor through the student body by classifying students into three compartments:

- people who have heard the rumor and spread it to other people (H),
- people who are ignorant of the rumor (I), and
- people who have heard the rumor but refuse to spread it (R).

Problem 5.5. Assume that in a population of 2000 student there is 1 person who knows the rumor initially so $H_0 = 1$. This means that $I_0 = 1999$ and $R_0 = 0$. Using the plots below, make several conjectures about how the three sub-populations change over time.



▲

Problem 5.6. The three sub-populations clearly trade individuals as the rumor spreads. Since this is a closed system we can treat this as a *compartment model* where each state (H, I, and R) is connected to the other states based on how the transfer occurs.

There are several ways to organize the model. Use Table 5.1, Table 5.2, the graphical representation (Figure 5.1), and partially complete difference equations to complete the mathematical model. Some of the model has been completed for you to, but note that only one possibility is presented (feel free to change it later). The biggest challenge here is making conjectures about the way in which the transfers happens.

▲

New Pop.	=	Old Pop.	+	Influx From	−	Output To
Heard (new)	=	Heard (old)	+	Influx from Ignorant	−	Output to Refuse
Ignorant (new)	=	Ignorant (old)	+		−	
Refuse (new)	=	Refuse (old)	+		−	

Table 5.1. Table to organize transfer between compartments. Fill in the blanks.

		From		
		Heard	Ignorant	Refuse
To	Heard	–	proportional to $H_n \cdot I_n$	_____
	Ignorant	_____	–	_____
	Refuse	proportional to H_n	_____	–

Table 5.2. Organization table for how the populations might transfer people. Fill in the blanks.

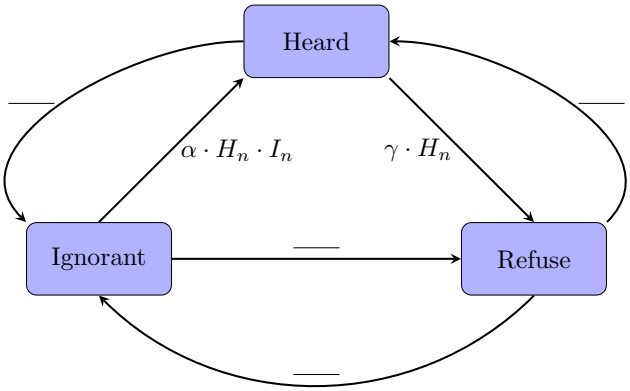


Figure 5.1. A graphical representation of how the sub-populations trade people. Fill in the blanks.

$$H_{n+1} - H_n = \_\_\_\_\_$$
$$I_{n+1} - I_n = \_\_\_\_\_$$
$$R_{n+1} - R_n = \_\_\_\_\_$$

Problem 5.7. Now we get to think about the modeling process:

- We have a mathematical model.
- Explicitly state your assumptions and simplifications.
- Determine how you might estimate each parameter in the model.
- Solve the mathematical model numerically using Excel (based on your estimated parameters) and compare with your classmates.
- Analyze the results and see if they seem reasonable.
- Tweak the model to loosen assumptions or to more accurately model reality. This can include:
  - allowing people to forget the rumor;

  - changing the way the rumor transfers;
  - adding more compartments; or
  - many other possibilities.

- Iterate this process until you think that the model approximates reality.

▲

Problem 5.8. The data found HERE contains the proportion of the population that has heard the rumor at each given time step. Use Excel's solver along with your numerical model to estimate the parameters in your model. Your end result should be a plot of all three population proportions. One modeling step that may simplify things is that we have a closed population so we know that the sum of the three proportions must be 1 – hence eliminating one of the populations from the model. ▲

5.3 The H1N1 Virus

The H1N1 Virus (commonly known as Bird Flu) hit your college back in 2010. H1N1 is a particularly interesting virus since a infected person stays infectious for exactly 7 days after the initial infection. After that time the person may still be ill but is recovering and is no longer infectious.

Problem 5.9. Consider a closed population of 1500 students, faculty, and staff at a small college campus. On a particular day in 2010, 1 person was diagnosed with H1N1. We'll assume that person was on the second day of their infectious period. An announcement went out to everyone on campus to come to the health center the moment they started feeling symptoms. Three days later, two new people showed up at the health center with symptoms.

Using the rather small amount of data given in the paragraph above, create a model for spread of the H1N1 virus through the campus. Consider modeling the population as 9 different compartments:

1. Susceptible,
2. Infectious Day 1,
3. Infectious Day 2,
4. Infectious Day 3,
5. Infectious Day 4,
6. Infectious Day 5,
7. Infectious Day 6,
8. Infectious Day 7, and
9. Recovered.

Use Excel's Solver to find any parameters in your model. Model first with a system of difference equations, then discuss with your partners whether you think it would be scientifically and mathematically more appropriate to use a system of differential equations instead. ▲

5.4 Writing Systems of Difference Equations

In order to get a great feel for setting up systems of difference equations this section will present several examples and activities to simply get your hands dirty. The idea being that it takes some practice reading these problems and identifying the key components. Each example will include some sensitivity discussion as well as a peak at the numerical solutions. Analytical solutions are possible for some classes of systems but we will delay this discussion until later.

Example 5.10. In the Bob Marshall Wilderness of Montana, black bears and grizzly bears compete with each other for food. Suppose that in the absence of any competition or hunting, the black bear population will grow by 10% per year, while the grizzly bear population will grow by 4%. Each year the competition between the two types of bears leads to the death of a certain number of each type of bear (due to fighting and food shortages). The number of black bears that die is equal to the product of the black and grizzly bear populations multiplied by 0.0003. The number of grizzly bears that die is equal to the product of the black and grizzly populations multiplied by 0.0001. Assume that there are 1500 black bears and 730 grizzly bears currently present in the Bob Marshall Wilderness.

(a) Write and numerically solve a pair of difference equations to model the bear populations.

(b) Modify the model from part (a) to include the unfortunate fact that poachers kill 30 black bears and 10 grizzly bears per year.

Solution:
Let B_n represent the population of black bears at year n and G_n represent the population of grizzly bears at year n.

First let us consider the black bear population.

- The population of black bears increases by 10% per year
- The population of black bears decreases from competition

With these two key features in mind the model for the black bear population is

$$B_{n+1} = B_n + 0.10 \cdot B_n - 0.0003 B_n G_n.$$

Next let us consider the grizzly bear population.

- The population of grizzly bears increases by 4% per year
- The population of grizzly bears decreases from competition

With these two key features in mind the model for the bear population is

$$G_{n+1} = G_n + 0.04 \cdot G_n - 0.0001 B_n G_n.$$

Obviously these two equations depend on each other because of the competition terms. Systems that display this property are called *coupled*. Putting the equations together in one place and simplifying the algebra we see that the coupled system of difference equations is

$$\begin{aligned} B_{n+1} &= 1.10 \cdot B_n - 0.0003 \cdot B_n \cdot G_n \\ G_{n+1} &= 1.04 \cdot G_n - 0.0001 \cdot B_n \cdot G_n \end{aligned}$$

A numerical solution of this model will result in three plots: bear population vs. time, grizzly population vs. time, and bear population vs. grizzly population. The last of these is called the *phase plot* for the system. Figure 5.2 shows the population dynamics for the individual bear populations. The right-hand plot in Figure 5.2 shows the phase plot for the two bear populations. To read the phase plot start at the initial condition and follow the trajectory of the points through the phase space.

If poaching is considered then the model becomes

$$\begin{aligned} B_{n+1} &= 1.10 \cdot B_n - 0.0003 \cdot B_n \cdot G_n - 30 \\ G_{n+1} &= 1.04 \cdot G_n - 0.0001 \cdot B_n \cdot G_n - 10 \end{aligned}$$

Figure 5.3 shows the new population dynamics when poaching occurs.

In the non-poaching scenario it appears that the black bear population is on the rise (over 3000 bears) and the grizzly bear population is driven to extinction. In the poaching scenario it appears that the black bear population reaches 600 bears after 36 years and the grizzly bears are driven to extinction within the wilderness area. Comparing the phase plots (the right-hand figures of Figure 5.2 and 5.3) we see very different dynamics with the addition of poaching.

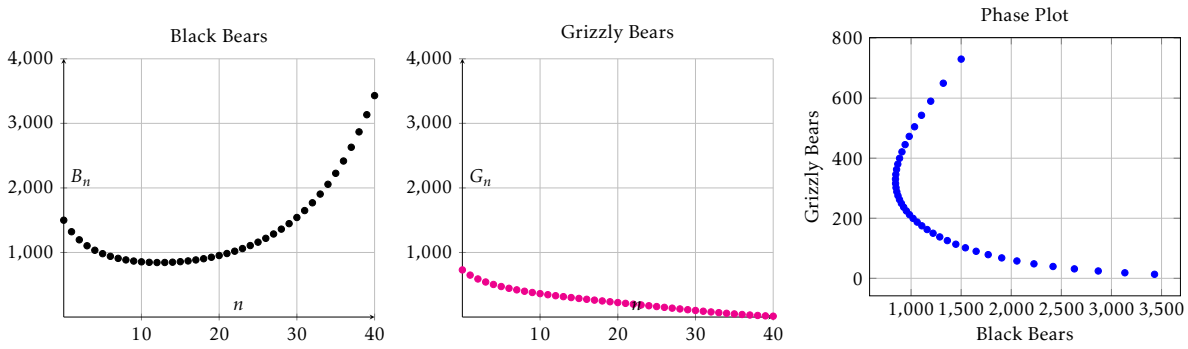


Figure 5.2. Phase plane showing the bear population dynamics

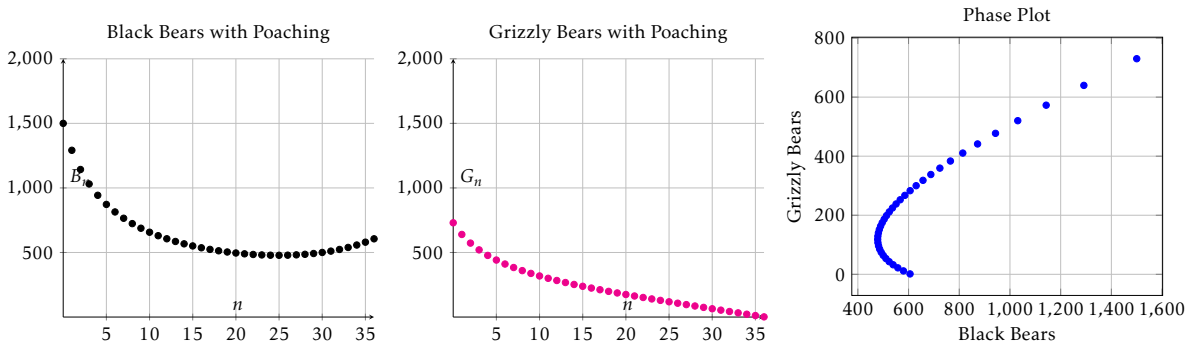


Figure 5.3. Phase plane showing the bear population dynamics

Example 5.11. Suppose that the past voting results of congressional elections at a certain voting precinct are represented by the three sequences D_n (Democrats), R_n (Republicans), and T_n (Tea Party). The voting behaviors of people change according to the following observations:

Last Year's Vote	Prediction of This Year's Vote	Percent Making This Switch
Democrat	Democrat	70%
Democrat	Republican	20%
Democrat	Tea Party	10%
Republican	Republican	50%
Republican	Democrat	20%
Republican	Tea Party	30%
Tea Party	Tea Party	50%
Tea Party	Democrat	20%
Tea Party	Republican	30%

For example, the second line of the table means that 20% of people that voted democratic last time will vote republican this time. Model this scenario with a system of difference equations. Presume that there are 3,000 people in the precinct that are initially split equally among the voting categories. Is the dynamical system stable over many years of voting supposing that the transition percentages stay the same?

Solution:

The dynamical system needs only to keep track of how many people are in a particular group. Therefore the dynamical system is

$$\begin{aligned} D_{n+1} &= 0.7D_n + 0.2R_n + 0.1T_n \\ R_{n+1} &= 0.2D_n + 0.5R_n + 0.3T_n \\ T_{n+1} &= 0.1D_n + 0.3R_n + 0.5T_n \end{aligned}$$

Written in a more convenient matrix notation, this three dimensional linear system

can be written as

$$\begin{pmatrix} D_{n+1} \\ R_{n+1} \\ T_{n+1} \end{pmatrix} = \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.3 & 0.5 \end{pmatrix} \cdot \begin{pmatrix} D_n \\ R_n \\ T_n \end{pmatrix}$$

Obviously this is a three dimensional dynamical system so visualization of the numerical solution is challenging. There are naturally 6 two-dimensional plots that come out of the numerical solution: D_n vs n, R_n vs n, T_n vs n, D_n vs R_n, D_n vs T_n, and R_n vs T_n. This can also be visualized in a three dimensional phase plot of D_n vs R_n vs T_n (we leave this to the curious reader). For simplicity we only present the three time plots here. It is clear from Figure 5.4 that over the next 20 years the political parties will stabilize at equilibrium values and the Democratic party will eventually win. Given the obvious volatile nature of political parties, the flawed assumption in this model is that the transition percentages will stay fixed over the next several years.

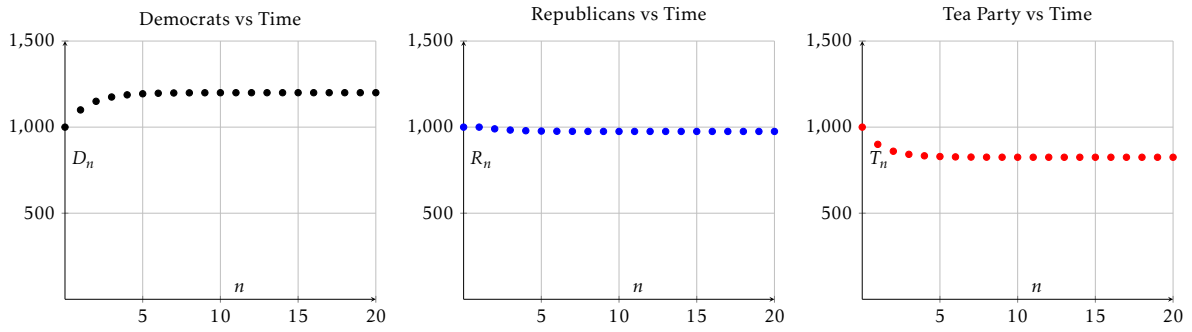


Figure 5.4. Numerical visualization of the voting problem. The equilibrium state appears to be $D_n \to 1200$, $R_n \to 975$, and $T_n \to 825$; hence giving the democrats a clear advantage.

Problem 5.12. Bull trout and rainbow trout are prevalent fish species in Montana's rivers. They have very similar feeding, migrating, and spawning habits, and as such they are in competition for the same food sources. Suppose that in the absence of the other species in the Blackfoot River, both species would exhibit unconstrained growth. Since the effect of the presence of the competitor diminishes the growth rate of the other species, we will consider that this decrease in growth is proportional to the product of the size of the two competing species. Bull trout are protected in Montana so no fish can be legally taken from the river. Rainbow trout, on the other hand, are a popular sport fish and roughly 155 rainbows are taken from the river each year by fishermen. Establish a model for this interaction of the two species of fish in the Blackfoot River. Let n be the number of years and evaluate the model over a 30 year span. Based on this information should fishing for bull trout be allowed?

Use the following data for the competing trout:

- Assume equal initial populations of 2500 fish for each species.

- In the absence of competition, the bull trout population increases by approximately 10% per year at small populations and follow a logistic growth pattern otherwise with a carrying capacity of 8000 fish.
- In the absence of competition, the rainbow trout population increases by approximately 23% per year per year at small populations and follow a logistic growth pattern otherwise with a carrying capacity of 8000 fish.
- The constant of proportionality decreasing the bull trout population is estimated to be approximately 0.0001.
- The constant of proportionality decreasing the rainbow trout population is estimated to approximately 0.0001

▲

Problem 5.13. In a distant country, people have only two types of breakfast cereal to eat: Corn Flakes and Raisin Bran. The people in this country are very loyal to their cereal and eat their one type of cereal every morning. Every year 200 new people join the ranks of Corn Flakes eaters, and 650 new people become Raisin Bran eaters every year. Of course sometimes people get sick of eating Corn Flakes all the time, and they switch to Raisin Bran. Each year about 5% of Corn Flakes eaters switch to Raisin Bran. Write and solve a system of difference equations to represent the number of people eating each type of cereal after n years.

The following system of difference equations models the populations of two types of animals, A and B.

$$A_{n+1} = 0.8A_n + 0.00035A_nB_n$$
$$B_{n+1} = 1.14B_n - 0.00055A_nB_n.$$

In the absence of animal B what is the rate of change of animal A? In the absence of animal A what is the rate of change of animal B? How does the presence of animal A affect the population of animal B? How does the presence of animal B affect the population of animal A. How do these animals interact with each other? Answer these questions first without solving the system numerically. Check your answers against the numerical solution.

▲

5.5 Linear Systems

The previous sections had us building models for systems of difference or differential equations. In this section (and for the remainder of this chapter) will will focus on a particular type of system – a linear system of difference or differential equations. As you may have guessed from the name, we will be leveraging Linear Algebra to do the analysis on these systems. Before we get to that let's at least build a few linear systems.

Problem 5.14. The little town of Pudunk is filled with people who cannot make up their minds whether to vote Democratic or Republican at the upcoming election. Each day, 30% of the Republicans switch to the Democratic line, while 50% of the Democratic voters decide to go with the Republicans. Assume that the population of 1000 people is initial split 50/50.

$$D_{n+1} - D_n = \_\_\_\_\_ D_n + \_\_\_\_\_ R_n$$
$$R_{n+1} - R_n = \_\_\_\_\_ D_n + \_\_\_\_\_ R_n$$

(a) Write the system of difference equations algebraically

(b) Write the system of difference equations using matrices

(c) Use Excel to find a numerical solutions for the system (discuss stability if applicable)

▲

Problem 5.15. Deep in the redwood forest of California, dusky-footed wood rats provide up to 80% of the diet for the spotten owl, the main predator of the wood rat. Assume that initially there are 1000 rats and 10 owls.

- In the absence of wood rats, the owl population would decay by 50% each year.
- If rats are plentiful then the change in owl population from year to year will increase by 40% of the total rat population.
- In the absence of owls, the population of the rats will grow by 10% per year.
- The owls hunt the rats, so if owls are present the rate population will change by, let's say, 10.4% per year.

$$O_{n+1} - O_n = \_\_\_\_\_ O_n + \_\_\_\_\_ R_n$$
$$R_{n+1} - R_n = \_\_\_\_\_ O_n + \_\_\_\_\_ R_n$$

(a) Write the system of difference equations algebraically

(b) Write the system of difference equations using matrices

(c) Use Excel to find a numerical solutions for the system (discuss stability if applicable)

▲

Problem 5.16. The female owls in a certain population can be classified as *juvenile*, *subadult*, and *adult*. Assume that right now there are 10 juveniles, 30 sub-adults, and 20 adults in a small owl population. In a given year,

- the number of new juvenile females in year $n+1$ is 0.33 times the number of adult females in year n,
- 18% of least year's juveniles becomes subadults,
- 71% of last year's subadults become adults,
- and 94% of last year's adults survive.

(a) Write the system of difference equations algebraically

(b) Write the system of difference equations using matrices

(c) Use Excel to find a numerical solutions for the system (discuss stability if applicable)

▲

Problem 5.17. Two cell phone service providers, Alpha and Beta, are constantly competing for the largest market share. Each month 5% of Alpha customers switch their service to Beta, and each month 7% of Beta customers switch their service to Alpha. The total number of customers served by the two companies stays fixed.

(a) Write the system of difference equations algebraically

(b) Write the system of difference equations using matrices

(c) Use Excel to find a numerical solutions for the system (discuss stability if applicable)

▲

You should now have noticed that every one of the systems we've built thus far in this section can be rewritten in the form $\mathbf{x}_{n+1} = A\mathbf{x}_n$. This form is almost exactly what we experienced from difference equations in previous chapters. In fact, we know how to find the solution to the difference equation $a_{n+1} = ra_n$ with initial condition a_0. Perhaps knowing how to solve the problem in one variable can give us hints about how to solve the more general linear system.

Problem 5.18. Work with your partner to remember how to solve the difference equation $a_{n+1} = ra_n$ with initial condition a_0. ▲

Problem 5.19. In modeling exercises in this section we wrote the system of difference equations as a matrix equation in the form: $\mathbf{x}_{n+1} = A\mathbf{x}_n$. Using the same logic as with the previous problem, what is the analytic solution to the system of difference equations? ▲

Wouldn't it be nice if we had a way to *easily* raise a matrix to a power ...

5.6 The Eigenvalue / Eigenvector Problem

To answer the question posed at the end of the previous section we now turn our attention to the second fundamental question of linear algebra: $A\mathsf{x} = \lambda\mathsf{x}$. We are going to jump right in to the key definitions and we'll spend the remainder of this section exploring these definitions. The following two sections of the book give details about where these ideas arise in application.

Definition 5.20. Let A be an $n \times n$ square matrix

- An **eigenvector** of the matrix A is a nonzero $n \times 1$ vector x such that $A\mathsf{x} = \lambda\mathsf{x}$.
- The scalar λ is called the **eigenvalue** of A associated with the eigenvector x.

Note that an eigenvector MUST be nonzero, but the eigenvalue MAY be zero.

In simple language, If x is an eigenvector of A then $A\mathsf{x}$ only scales x. The result of $A\mathsf{x}$ will be a vector parallel to x, so no rotations (other than 180°) result.

Problem 5.21. Consider the matrix $A = \begin{pmatrix} -7 & 9 \\ -6 & 8 \end{pmatrix}$.

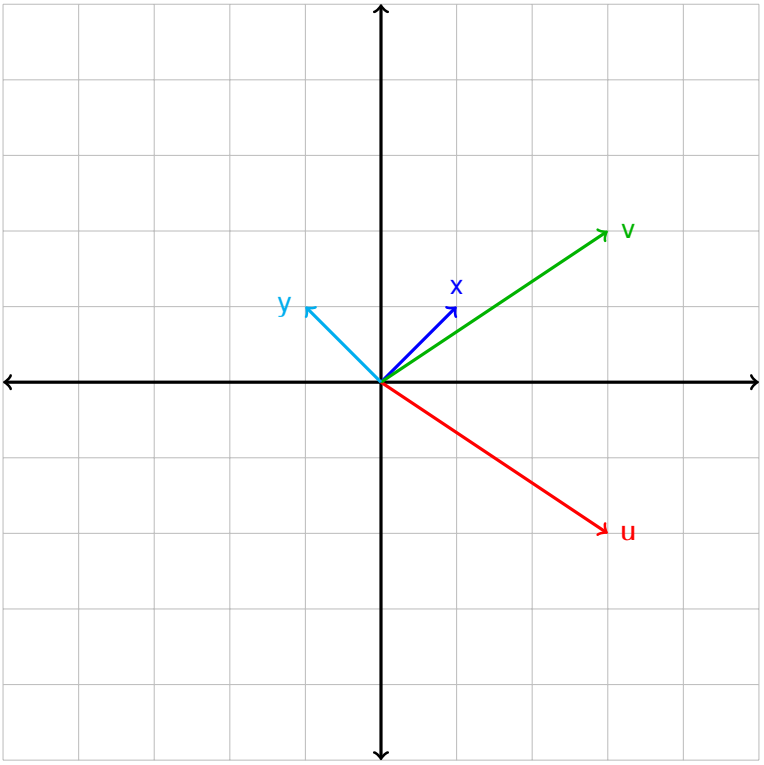
(a) Is $\mathsf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ an eigenvector for A? If so, what is the associated eigenvalue? If not, why not?

(b) Is $\mathsf{u} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ an eigenvector for A? If so, what is the associated eigenvalue? If not, why not?

(c) Is $\mathsf{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ an eigenvector for A? If so, what is the associated eigenvalue? If not, why not?

(d) Is $\mathsf{y} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ an eigenvector for A? If so, what is the associated eigenvalue? If not, why not?

(e) The plot below shows the four vectors considered in parts (a) - (d) of this problem. Sketch $A\mathsf{x}$, $A\mathsf{u}$, $A\mathsf{v}$, and $A\mathsf{y}$ on the same coordinate plane, indicate which vectors are eigenvectors, and indicate the eigenvalues geometrically.



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Problem 5.22. Use the GeoGebra applet http://www.geogebratube.org/student/m334841 to explore the following questions.

(a) Approximate the eigenvalue-eigenvector pairs for the matrix $\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$.

(b) Approximate the eigenvalue-eigenvector pairs for the matrix $\begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}$.

(c) Approximate the eigenvalue-eigenvector pairs for the matrix $\begin{pmatrix} 0.25 & 0.1 \\ 0.75 & 0.9 \end{pmatrix}$.

▲

5.6.1 Geometry of Eigenvectors and Eigenvalues

In the introduction to this section we saw that an eigenvector of a matrix is only scaled by the matrix. This simple geometric fact leads us to the fact that if you know all of the eigenvalue / eigenvector information about a matrix, you can determine its full behavior on any vector. This leads us to the following theorem.

Theorem 5.23. Let A be an $n \times n$ matrix with eigenvectors $\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_n$ and associated eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. If u is an n-dimensional vector and we can write u as a linear

combination of the eigenvectors

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_n\mathbf{v}_n,$$

then we know the action of A on $\mathbf{u}$

$$A\mathbf{u} = A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_n\mathbf{v}_n) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + \cdots c_n\lambda_n\mathbf{v}_n.$$

The following example illustrates this idea.

Example 5.24. Let $\mathbf{v}_1 = \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ be eigenvectors of the matrix A with corresponding eigenvalues $\lambda_1 = -2$, $\lambda_2 = -1$, and $\lambda_3 = 0$. If $\mathbf{u} = \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}$ then what is $A\mathbf{u}$?

Solution:
Since we know all of the information about the eigenvectors, if we can write $\mathbf{u}$ as a linear combination of the eigenvectors then we can easily determine $A\mathbf{u}$.

We need to find c_1, c_2, and c_3 such that $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$. This is a system of equation, and we can write the matrix form of this system as

$$\begin{pmatrix} 0 & 1 & 2 \\ -3 & 3 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}.$$

Augmenting this system and row reducing gives

$$\left(\begin{array}{ccc|c} 0 & 1 & 2 & 3 \\ -3 & 3 & 0 & -3 \\ 1 & 0 & -1 & 1 \end{array}\right) \to \cdots \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Therefore,

$$\mathbf{u} = 2\mathbf{v}_1 + 1\mathbf{v}_2 + 1\mathbf{v}_3$$

and

$$A\mathbf{u} = 2\lambda_1\mathbf{v}_1 + 1\lambda_2\mathbf{v}_2 + 1\lambda_3\mathbf{v}_3 = -4\mathbf{v}_1 - \mathbf{v}_2 = \begin{pmatrix} -1 \\ 9 \\ -4 \end{pmatrix}$$

Notice that this was done with NO knowledge of A itself.

In the previous example it wasn't necessary to find the matrix A. In some applications the eigenvalues and eigenvectors are well known but the matrix is actually needed. The following example illustrates how to work backwards from eigen-information to the matrix.

Example 5.25. Assume that $\mathsf{v}_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\mathsf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are the eigenvectors of a matrix A with eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -2$. What is the matrix A?
Solution:
We know that A is a 2×2 matrix so we can write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We know that $A\mathsf{v}_1 = \lambda_1 \mathsf{v}_1$ and $A\mathsf{v}_2 = \lambda_2 \mathsf{v}_2$, and out of these equations we can write a system of four equations with unknowns a, b, c, and d.

$$A\mathsf{v}_1 = \lambda_1 \mathsf{v}_1 \quad \Longrightarrow \quad \begin{cases} 2a - 3b &= 6 \\ 2c - 3d &= -9 \end{cases}$$

$$A\mathsf{v}_2 = \lambda_2 \mathsf{v}_2 \quad \Longrightarrow \quad \begin{cases} 1a - 1b &= -2 \\ 1c - 1d &= 2 \end{cases}$$

The augmented 4×4 system of equations is

$$\left(\begin{array}{cccc|c} 2 & -3 & 0 & 0 & 6 \\ 0 & 0 & 2 & -3 & -9 \\ 1 & -1 & 0 & 0 & -2 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right) \to \cdots \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -12 \\ 0 & 1 & 0 & 0 & -10 \\ 0 & 0 & 1 & 0 & 15 \\ 0 & 0 & 0 & 1 & 13 \end{array}\right)$$

Therefore,

$$A = \begin{pmatrix} -12 & -10 \\ 15 & 13 \end{pmatrix}$$

Another advantage to eigenvalue / eigenvector information is the ability to easily examine powers of a matrix. This will be especially important when examining time-dependent matrix systems where the exponent represents steps in time.

Theorem 5.26. If A is an $n\times n$ matrix, λ is an eigenvalue of A and v is the associated eigenvector, then

$$A^k\mathsf{v} = \lambda^k \mathsf{v}$$

Furthermore, If A has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and eigenvectors $\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_n$ then if u is an n-dimensional vector with $\mathsf{u} = c_1\mathsf{v}_1 + c_2\mathsf{v}_2 + \cdots c_n\mathsf{v}_n$, then

$$A^k\mathsf{u} = c_1\lambda_1^k\mathsf{v}_1 + c_2\lambda_2^k\mathsf{v}_2 + \cdots c_n\lambda_n^k\mathsf{v}_n.$$

Example 5.27. Let A have eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 0.1$ with eigenvectors $\mathsf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\mathsf{v}_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$. Explore the quantities $A^k\mathsf{v}_1$ and $A^k\mathsf{v}_2$ for various values of k.

Solution:
From Theorem 5.26 we know that $A^k \mathsf{v}_j = \lambda_j^k \mathsf{v}_j$. Therefore

$$A^k \mathsf{v}_1 = 4^k \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 4^k \\ -2 \cdot 4^k \end{pmatrix}.$$

Similarly,

$$A^k \mathsf{v}_2 = (0.1)^k \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \cdot (0.1)^k \\ 5 \cdot (0.1)^k \end{pmatrix}.$$

For larger and larger values of k, the entries of $A^k \mathsf{v}_1$ grow exponentially and the values of $A^k \mathsf{v}_2$ decay exponentially to zero.

Problem 5.28. Use the geometry and definitions of eigenvectors and eigenvalues to answer the following questions.

(a) Given that $\mathsf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\mathsf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ are the eigenvectors of the matrix $A = \begin{pmatrix} -2 & 1 \\ -2 & -5 \end{pmatrix}$, what are the corresponding eigenvalues?

(b) Find the 2×2 matrix A such that $\mathsf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathsf{v}_2 = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$ are the eigenvectors of A with eigenvalues 6 and 3 respectively.

(c) If $\mathsf{v}_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\mathsf{v}_2 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ are the eigenvectors of a matrix A with corresponding eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$, then what is $A(2\mathsf{v}_1 + 5\mathsf{v}_2)$?

(d) Let $\mathsf{v}_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\mathsf{v}_2 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ be the eigenvectors of a matrix A with corresponding eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$. If $\mathsf{u} = \begin{pmatrix} 17 \\ 0 \end{pmatrix}$, what is $A\mathsf{u}$? Hint: Find c_1 and c_2 such that $\mathsf{u} = c_1 \mathsf{v}_1 + c_2 \mathsf{v}_2$.

(e) If $\mathsf{v}_1 = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$ and $\mathsf{v}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ are the eigenvectors of A with eigenvalues $\lambda_1 = 3$ and $\lambda_2 = \frac{1}{2}$, find $A^5 \mathsf{v}_1$ and $A^5 \mathsf{v}_2$.

(f) If $\mathsf{v}_1 = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$ and $\mathsf{v}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ are the eigenvectors of A with eigenvalues $\lambda_1 = 3$ and $\lambda_2 = \frac{1}{2}$, find $\lim_{k \to \infty} \left(A^k \mathsf{v}_1 \right)$ and $\lim_{k \to \infty} \left(A^k \mathsf{v}_2 \right)$.

▲

5.6.2 The Eigenvalue Eigenvector Problem

Given a matrix and a vector it is easy to determine if the vector is an eigenvector. If, on the other hand, we try to extract the eigenvectors and eigenvalues from a given matrix the

problem is a bit more interesting. We want to solve $A\mathsf{x} = \lambda\mathsf{x}$ for x, but the unknown is on both sides of the equation. Doing a bit of (linear) algebra we can rearrange the equation to get

$$A\mathsf{x} = \lambda\mathsf{x} \quad \Longrightarrow \quad A\mathsf{x} - \lambda\mathsf{x} = 0 \quad \Longrightarrow \quad (A - \lambda I)\,\mathsf{x} = 0. \tag{5.1}$$

The last equation of (5.1) says something very profound about the matrix $A - \lambda I$ when λ is an eigenvalue: this matrix must NOT have an inverse! From our properties of determinants, this means that we must have

$$\det(A - \lambda I) = 0.$$

This determinant will be an n^{th}-order polynomial equation in λ.

Definition 5.29. The **characteristic polynomial** of a square matrix A is given by

$$p(\lambda) = \det(A - \lambda I).$$

If A is $n \times n$ then

$$p(\lambda) = c_n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + p_1\lambda + p_0.$$

We now have a way to find the eigenvalues of a matrix. The process is outlined in Theorem 5.30.

Theorem 5.30. Let A be an $n \times n$ matrix. To find the eigenvalues and eigenvectors of A

1. Find the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$.
2. Set the characteristic polynomial to zero to get the characteristic equation
$$\det(A - \lambda I) = 0.$$
3. Solve the characteristic equation for all values of λ. These are the eigenvalues of A.
4. For each eigenvalue λ, find x such that $(A - \lambda I)\mathsf{x} = 0$. The nonzero solution to this system of equations is the eigenvector of A associated with the eigenvalue λ.

The observant reader should note that step 3 of the eigenvalue process is often very difficult or simply impossible to do by hand. This is the step that makes the eigenvalue / eigenvector problem so difficult! For a 2×2 matrix, the characteristic equation is quadratic and there is a convenient formula to find the roots. For a 3×3 matrix the characteristic polynomial is cubic. There is a formula to solve the general cubic similar to the quadratic formula (due to Gerolamo Cardano in the early 1500's), but it is quite cumbersome to use.

For a 4×4 matrix the characteristic polynomial is quartic. Like the previous two cases, there is a general formula (due to Lodovico Ferrari in the 1500's) that solves the general quartic. At this point, the reader may think that there are general formulas to solve most polynomial equations. Unfortunately, that is simply false! In the 1800's, Évariste Galois showed that there can not be any such formula for the general quintic polynomial![*] This amazing fact has a huge impact on the search for eigenvalues: If you are dealing with a 5×5 matrix there is not way to get a general solution to the resulting characteristic equation, and a 5×5 matrix is very small!!

The preceding paragraph leads us to two conclusions:

1. To find the eigenvalues of a matrix by hand we need to be dealing with small matrices.

2. To find the eigenvalues of a larger matrix, we need computational methods that will allow us to approximate the roots of polynomials.

For now, we'll stick with small examples just to get the hang of the algorithm. Let's examine a few examples of the eigenvector / eigenvalue problem. Work along with these examples so that you understand every step.

Example 5.31. Find the eigenvalue eigenvector pairs for the matrix $A = \begin{pmatrix} 2 & 7 \\ 7 & 2 \end{pmatrix}$.

Solution:
The characteristic polynomial of A is

$$p(\lambda) = \det\left(\begin{pmatrix} 2 & 7 \\ 7 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\left(\begin{pmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{pmatrix}\right) = (2-\lambda)^2 - 49$$

If we now set the characteristic polynomial to zero and solve we get

$$\begin{aligned} & p(\lambda) = 0 \\ \implies & (2-\lambda)^2 - 49 = 0 \\ \implies & (2-\lambda)^2 = 49 \\ \implies & 2-\lambda = \pm 7 \\ \implies & -\lambda = -2 \pm 7 \\ \implies & \lambda = 2 \pm 7 \\ \implies & \lambda_1 = 9 \quad \text{and} \quad \lambda_2 = -5. \end{aligned}$$

Now we need to find the two associated eigenvectors.

For $\lambda_1 = 9$: We need to solve $(A - \lambda_1 I)\mathrm{x} = 0$. First, $A - \lambda_1 I = \begin{pmatrix} -7 & 7 \\ -7 & 7 \end{pmatrix}$, so augmenting

[*] Galois' discovery paved the way for an entire field of study called Abstract Algebra.

this with the zero vector and row reducing gives

$$(A-\lambda_1 I\,|\,0)=\left(\begin{array}{cc|c} -7 & 7 & 0 \\ 7 & -7 & 0 \end{array}\right)\to\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Therefore if $\mathsf{x}=\begin{pmatrix}x_1\\x_2\end{pmatrix}$ then $x_1=x_2$ and the simplest nonzero vector describing this is $\mathsf{x}=\begin{pmatrix}1\\1\end{pmatrix}$. The reader should verify that $A\begin{pmatrix}1\\1\end{pmatrix}=9\begin{pmatrix}1\\1\end{pmatrix}$

For $\lambda_2=-5$: We need to solve $(A-\lambda_2 I)\mathsf{x}=0$. First, $A-\lambda_2 I=\begin{pmatrix}7&7\\7&7\end{pmatrix}$, so augmenting this with the zero vector and row reducing gives zero vector and row reducing gives

$$(A-\lambda_2 I\,|\,0)=\left(\begin{array}{cc|c} 7 & 7 & 0 \\ 7 & 7 & 0 \end{array}\right)\to\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Therefore if $\mathsf{x}=\begin{pmatrix}x_1\\x_2\end{pmatrix}$ then $x_1=-x_2$ and the simplest nonzero vector describing this is $\mathsf{x}=\begin{pmatrix}1\\-1\end{pmatrix}$. The reader should verify that $A\begin{pmatrix}1\\-1\end{pmatrix}=(-5)\begin{pmatrix}1\\-1\end{pmatrix}$

The eigenvalue / eigenvector pairs for $\begin{pmatrix}2&7\\7&2\end{pmatrix}$ are

$$\lambda_1=9 \text{ with } \mathsf{x}_1=\begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad \lambda_2=-5 \text{ with } \mathsf{x}_2=\begin{pmatrix}1\\-1\end{pmatrix}.$$

In the next example we illustrate a convenient shortcut when the matrix has triangular structure.

Example 5.32. For $A=\begin{pmatrix}5&5&0&5\\0&2&-3&6\\0&0&3&-2\\0&0&0&5\end{pmatrix}$, find the eigenvalues and eigenvectors.

Solution:
Notice that the matrix A is upper triangular. This means that $A-\lambda I$ is upper triangular and the determinant is the product of the diagonal entries. Hence, the characteristic polynomial is

$$p(\lambda)=(5-\lambda)(2-\lambda)(3-\lambda)(5-\lambda).$$

The resulting characteristic equation $p(\lambda)=0$ is very easy to solve giving $\lambda_1=5$, $\lambda_2=2$, and $\lambda_3=3$.

For $\lambda_1=5$: We form the homogeneous system $(A-5I)\mathsf{x}=0$, augment, and row reduce

to get

$$\left(\begin{array}{cccc|c} 0 & 5 & 0 & 5 & 0 \\ 0 & -3 & -3 & 6 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \to \cdots \to \left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Hence, if $\mathsf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ then $x_2 = x_3 = x_4 = 0$ and x_1 is free. Taking $x_1 = 1$ we get $\mathsf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. The reader should verify that $\lambda_1 = 5$ and $\mathsf{x} = (1,0,0,0)^T$ is indeed an eigenvalue /eigenvector pair for the matrix A.

For $\lambda_2 = 2$: We form the homogeneous system $(A-2I)\mathsf{x} = 0$, augment, and row reduce to get

$$\left(\begin{array}{cccc|c} 3 & 5 & 0 & 5 & 0 \\ 0 & 0 & -3 & 6 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right) \to \cdots \to \left(\begin{array}{cccc|c} 1 & \frac{5}{3} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Therefore, $x_3 = x_4 = 0$, x_2 is free, and $x_1 = -\frac{5}{3}x_2$. Picking $x_4 = 3$ (for convenience) we see that $\mathsf{x} = \begin{pmatrix} -5 \\ 3 \\ 0 \\ 0 \end{pmatrix}$ is the eigenvector associated with $\lambda_2 = 2$.

For $\lambda_3 = 3$: We form the homogeneous system $(A-3I)\mathsf{x} = 0$, augment, and row reduce to get

$$\left(\begin{array}{cccc|c} 2 & 5 & 0 & 5 & 0 \\ 0 & -1 & -3 & 6 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right) \to \cdots \to \left(\begin{array}{cccc|c} 1 & 0 & -\frac{15}{2} & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Hence, $x_4 = 0$, x_3 is free, $x_2 = -3x_3$, and $x_1 = \frac{15}{2}x_3$. Picking $x_3 = 2$ (for convenience) gives $\mathsf{x} = \begin{pmatrix} 15 \\ -6 \\ 2 \\ 0 \end{pmatrix}$.

Therefore, the eigenvalue / eigenvector pairs are

$$\lambda_1 = 5 \text{ with } \mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \lambda_2 = 2 \text{ with } \mathbf{x}_2 = \begin{pmatrix} -5 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \lambda_3 = 3 \text{ with } \mathbf{x}_3 = \begin{pmatrix} 15 \\ -6 \\ 2 \\ 0 \end{pmatrix}$$

Notice that in this problem the matrix was 4×4 but there are only three eigenvalue / eigenvector pairs. This occasionally occurs with repeated roots in the characteristic polynomial.

Problem 5.33. Find the eigenvectors and eigenvalues for each of the following matrices.

(a) $A = \begin{pmatrix} -4 & -1 \\ 6 & 1 \end{pmatrix}$

(b) $B = \begin{pmatrix} 5 & 3 \\ 2 & 0 \end{pmatrix}$

(c) $C = \begin{pmatrix} 3 & 0 & 0 \\ 6 & 2 & 0 \\ 0 & 3 & 6 \end{pmatrix}$

(d) $D = \begin{pmatrix} 4 & 0 & -1 \\ 0 & 4 & -1 \\ 1 & 0 & 2 \end{pmatrix}$

▲

5.6.3 Technology for the Eigenvalue Eigenvector Problem

Linear algebra software is designed to handle the eigenvalue-eigenvector problem with refined efficiency. Based on the length of the computations in this section it is clear that some technological aid is necessary for many eigvalue-eigenvector computations. In this final subsection we will illustrate how MatLab computes eigenvalues and eigenvectors.

Example 5.34. Consider the matrix

$$A = \begin{pmatrix} 26 & -18 & 6 \\ 18.5 & -11 & 5.5 \\ -26 & 22 & -2 \end{pmatrix}.$$

Use MatLab to find the eigenvalues and eigenvectors.
Solution:
The MatLab code is shown below. The function `eig` is the command that finds the eigen-pairs.

```
A = 26 , -18 , 6 ;
```

```
    18.5 , -11 , 5.5 ;
    -26 , 22 , -2];
[vectors , values] = eig(A)
```

The output to the MatLab command line is:

```
vectors =
 0.5774    0.3906    0.4851
 0.5774    0.1302    0.7276
-0.5774   -0.9113    0.4851

values =
2.0000         0         0
0              6.0000    0
0              0         5.0000
```

The eigenvalue and eigenvector pairs in the matrix are

$$\lambda_1 = 2 \text{ with } \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \lambda_2 = 6 \text{ with } \mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \\ -7 \end{pmatrix}, \quad \text{and } \lambda_3 = 5 \text{ with } \mathbf{v}_3 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}.$$

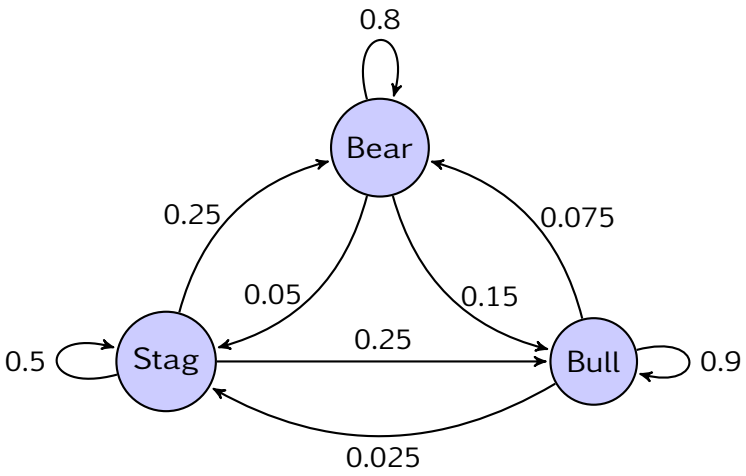
Reading the values from the eigenvectors is a bit tricky. MatLab scales the vectors so that the magnitude is 1 so to get integer values out of the eigenvectors (if that is possible) it is a matter of looking at the relative weights of the values. For example, in $\mathbf{v}_1$, the numbers are all the same so they might as well be 1. In $\mathbf{v}_2$, on the other hand, taking the smallest value to be 1 we see that the middle value is three times as large and the negative values is seven times as large.

5.7 Markov Chains

Problem 5.35. The stock market can be divided into three behaviors:

- A bull market is a period of generally rising prices.
- A bear market is a period of generally declining prices.
- A stagnant market is a period of generally no rise or decline of prices.

The graph at right shows how the markets evolve over time. For example, a bull week is followed by a bull week 90% of the time, a bear week 7.5% of the time, and a stagnant week 2.5% of the time.



(a) Fill in the table below to create a *transition matrix* indicating how the market transitions over time.

	From Bull	From Bear	From Stagnant
To Bull			
To Bear			
To Stagnant			

(b) Verify that the sum of every column is exactly 1. Why is this important?

(c) If we know that we are in a bull market this week, what is the probability of being in each type of market next week? What if this week were a bear market? What about a stagnant market?
Hint: This can be calculated by simple matrix multiplication.

(d) If we are in a bull market right now, what is the probability that we'll be in a bull market 10 weeks from now?

▲

In Problem 5.35 we saw our first example of a stochastic matrix and a Markov chain. The formal definitions are as follows.

Definition 5.36.
- A **probability vector** is a vector with nonnegative entries that add to 1.
- A **stochastic matrix** is a matrix whose columns are probability vectors.
- A **Markov chain** is a sequence of probability vectors $x_0, x_1, x_2, \ldots$ together with

a stochastic matrix P such that

$$\mathbf{x}_1 = P\mathbf{x}_0, \quad \mathbf{x}_2 = P\mathbf{x}_1, \quad \mathbf{x}_3 = P\mathbf{x}_2, \quad \ldots.$$

Note that a Markov chain is most simply written as the first order linear difference equation

$$\mathbf{x}_{k+1} = P\mathbf{x}_k \quad \text{for } k = 0, 1, 2, \ldots.$$

A Markov chain is a tool used in many fields to understand long-term probability in a closed system. In Definition 5.36 we see that the Markov chain of probabilities can be stated as a discrete dynamical system $\mathbf{x}_{k+1} = P\mathbf{x}_k$. Hence we can determine the probability of being in any state at a given time by observing Table 5.3. This proves the following

k	$\mathbf{x}_{k+1}$	$P\mathbf{x}_k$
0	$\mathbf{x}_1$	$P\mathbf{x}_0$
1	$\mathbf{x}_2$	$P\mathbf{x}_1 = P(P\mathbf{x}_0) = P^2\mathbf{x}_0$
2	$\mathbf{x}_3$	$P\mathbf{x}_2 = P(P^2\mathbf{x}_0) = P^3\mathbf{x}_0$
3	$\mathbf{x}_4$	$P\mathbf{x}_3 = P(P^3\mathbf{x}_0) = P^4\mathbf{x}_0$
$\vdots$	$\vdots$	$\vdots$
$n-1$	$\mathbf{x}_n$	$P^n\mathbf{x}_0$

Table 5.3. Table showing the evolution of a Markov chain.

theorem.

Theorem 5.37. If P is a transition matrix for the first order discrete dynamical system $\mathbf{x}_{n+1} = P\mathbf{x}_n$ then the analytic solution to this system is

$$\mathbf{x}_n = P^n\mathbf{x}_0$$

where $\mathbf{x}_n$ is the state vector at time n and $\mathbf{x}_0$ is the initial state vector.

The example of the stock market in Problem 5.35 is only one simple example of a closed system. From Theorem 5.37 we know that we can progress this model in time by taking powers of the transition matrix P. We can take this example a bit further by examining the eigenvalue structure of the transition matrix and observing what happens to the eigen-structure under exponentiation of the transition matrix.

Example 5.38. Examine the eigen-structure of the transition matrix from Problem 5.35 and use it to predict the long-term behavior of the stock market system.
Solution:

The transition matrix of interest is

$$P = \begin{pmatrix} 0.90 & 0.15 & 0.25 \\ 0.075 & 0.80 & 0.25 \\ 0.025 & 0.05 & 0.50 \end{pmatrix}.$$

where the first column is the probability vector for bull market, the second column is the probability vector for the bear market, and the third column is the probability vector for the stagnant market. Using linear algebra technology, the eigenvalue-eigenvector pairs are

$$\lambda_1 = 1 \text{ with } \mathbf{v}_1 \approx \begin{pmatrix} 0.89 \\ 0.44 \\ 0.09 \end{pmatrix}, \quad \lambda_2 \approx 0.74 \text{ with } \mathbf{v}_2 \approx \begin{pmatrix} 0.74 \\ -0.67 \\ -0.06 \end{pmatrix}, \quad \text{and} \quad \lambda_3 \approx 0.46 \text{ with } \mathbf{v}_3 = \begin{pmatrix} -0.28 \\ -0.53 \\ 0.80 \end{pmatrix}.$$

Let's assume that $\mathbf{x}_0$ is a given initial state. We can write $\mathbf{x}_0$ as a linear combination of the eigenvectors of P

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3,$$

and now if we use Theorems 5.37 and 5.26 we know that

$$\mathbf{x}_n = P^n\mathbf{x}_0 = P^n(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = c_1\lambda_1^n\mathbf{v}_1 + c_2\lambda_2^n\mathbf{v}_2 + c_3\lambda_3\mathbf{v}_3.$$

Since $\lambda_1 = 1$ we know that $\lambda_1^n = 1$ for all n. On the other hand, both λ_2 and λ_3 are less than 1 so as we consider longer time spans the coefficients of $\mathbf{v}_2$ and $\mathbf{v}_3$ drop exponentially to zero! This proves something amazing!!!!

Over a long time period, the only behavior that matters is that of $\mathbf{v}_1$!! Mathematically, this means that

$$\lim_{n\to\infty}(\mathbf{x}_n) \approx c_1\begin{pmatrix} 0.89 \\ 0.44 \\ 0.09 \end{pmatrix}.$$

Notice that we haven't found the coefficient c_1, but it turns out that we don't have to. We know that the vector should be a probability vector so the sum must be 1. Therefore, $c \approx 1/(0.88 + 0.44 + 0.09)$ and the long-term state vector is

$$\lim_{n\to\infty}(\mathbf{x}_n) \approx \frac{1}{1.42}\begin{pmatrix} 0.89 \\ 0.44 \\ 0.09 \end{pmatrix} \approx \begin{pmatrix} 0.63 \\ 0.31 \\ 0.06 \end{pmatrix}.$$

In the long run, there is a 63% chance of ending up as a bull market, a 31% chance of ending up as a bear market, and a 6% chance of ending up as a stagnant market.

The previous examples suggest proofs to the following theorems.

Theorem 5.39. If P is a stochastic matrix (column sums are all 1) then all of the eigenvalues are positive and the largest eigenvalue is 1. Furthermore, there is only

one eigenvector associated with the eigenvalue $\lambda = 1$.

Theorem 5.40. Let P be a stochastic matrix in the Markov chain $\mathbf{x}_{k+1} = P\mathbf{x}_k$. Let $\mathbf{v}$ be the eigenvector of P with eigenvalue $\lambda = 1$. The long-term behavior of the Markov chain is

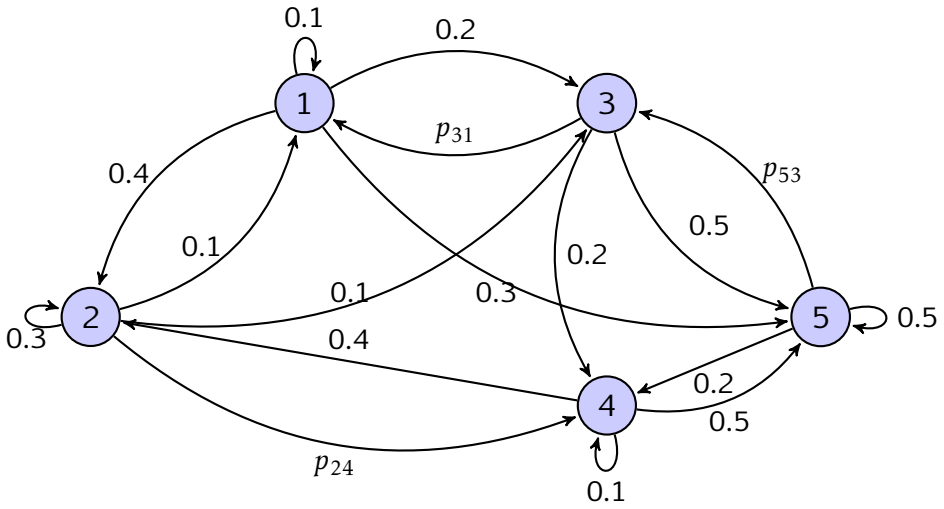
$$\lim_{n\to\infty}(\mathbf{x}_n) = C\mathbf{v} \quad \text{where} \quad C = \frac{1}{\text{sum}(\mathbf{v})}.$$

For the remainder of this section we will consider several problems illustrating more examples of Markov chains. In each activity, your goal is to find the steady state (long-term) behavior of the system.

Problem 5.41. The five nodes in the graph below show rental car centers. At node 1 we can read the graph as follows:

- a car has a 10% chance of being rented and then returned to node 1,
- a 20% chance of being rented and dropped off at node 3,
- a 40% chance of being rented and dropped off at node 2, and
- a 30% chance of being rented and dropped off at node 5.

The other nodes are read similarly.



(a) Find p_{24}, p_{31} and p_{53}.

(b) Write a stochastic transition matrix P for the transitions between nodes.

(c) Use Theorems 5.39 and 5.40 and linear algebra software to determine the long-term behavior of this system. Based on your answer, how would you advise the owner of the rental car agency?

▲

Problem 5.42. A lab rat is put in a maze with six rooms. At intervals of 1 minute, the rat is given visual stimulation which causes it to move according to the following permutation matrix.

	From					
	Room 1	Room 2	Room 3	Room 4	Room 5	Room 6
To Room 1	0.33	0.25	0.0	0.33	0.0	0.0
To Room 2	0.33	0.25	0.25	0.0	0.25	0.0
To Room 3	0.0	0.25	0.25	0.0	0.0	0.5
To Room 4	0.34	0.0	0.0	0.33	0.25	0.0
To Room 5	0.0	0.25	0.0	0.34	0.25	0.25
To Room 6	0.0	0.0	0.5	0.0	0.25	0.25

(a) Draw the graph of this Markov chain.

(b) If the rat starts in room 1, what is the probability that it is in room 6 after two moves? three moves? four moves?

(c) What is the long-term behavior of the Makov chain? Use Theorems 5.39 and 5.40 as well as linear algebra software to answer this question.

▲

Problem 5.43. Advertisements tend to change people's opinions about political issues. Suppose that on a certain political issue there are 3 different popular opinions (A, B, and C). A psychologist wants to study the shifts in people's opinions after viewing advertisements and hence gathers the data listed in the table below.

Previous Opinion	New Opinion After Viewing Advertisement	Percent Making This Switch
A	A	50%
A	B	20%
A	C	30%
B	A	10%
B	B	70%
B	C	20%
C	A	5%
C	B	5%
C	C	90%

During the weeks leading up to an election the advertisements are run continuously. Let's make the very simplistic assumption that every time someone views the advertisements that their opinion shifts according to the table above. After several weeks of advertising, which is the popular opinion. ▲

5.8 Analysis of Linear Systems

The concluding section of this chapter gives a thorough analysis of systems of linear difference equations. The Markov chains discussed in the previous section were but one type of example possible. In coming subsections we will investigate questions of equilibria and stability for linear systems.

5.8.1 Homogeneous Systems of Linear Difference Equations

If a system of difference equations is linear then the system can be written as $\mathbf{x}_{n+1} = A\mathbf{x}_n$ where A is the matrix of coefficients, $\mathbf{x}_n$ is the past state vector, and $\mathbf{x}_{n+1}$ is the new state vector. Writing this as a matrix equation greatly reduces the amount of writing and adds a layer of organization to the description of the system of difference equations.

In the previous section on Markov chains we examined a particular type of difference equation where A is a stochastic matrix. This is a very special case, and in many instances the transition matrix is not stochastic. Even so, we can still observe a very nice solution to a homogeneous system of difference equations.

k	$\mathbf{x}_{k+1}$	$A\mathbf{x}_k$
0	$\mathbf{x}_1$	$A\mathbf{x}_0$
1	$\mathbf{x}_2$	$A\mathbf{x}_1 = A^2\mathbf{x}_0$
2	$\mathbf{x}_3$	$A\mathbf{x}_2 = A^3\mathbf{x}_0$
3	$\mathbf{x}_4$	$A\mathbf{x}_3 = A^4\mathbf{x}_0$
$\vdots$	$\vdots$	$\vdots$
$n-1$	$\mathbf{x}_n$	$A^n\mathbf{x}_0$

Table 5.4. Tabular solution to a homogeneous system of linear difference equations.

This table proves the following theorem

Theorem 5.44. The solution to a homogeneous system of linear difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ is

$$\mathbf{x}_n = A^n\mathbf{x}_0$$

where $\mathbf{x}_0$ is the initial condition of the system.

As before, when exponentiating a matrix, the eigenvalues play the champion role! Remember that if $\mathbf{x}_0$ is written as a linear combination of the eigenvectors

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_n\mathbf{v}_n$$

then

$$A^n\mathbf{x}_0 = c_1\lambda_1^b\mathbf{v}_n + c_2\lambda_2^n\mathbf{v}_2 + \cdots c_n\lambda_n^2\mathbf{v}_n.$$

This is a huge time saver if the eigen-structure of the matrix is readily available. Furthermore, realizing that the eigenvalues and eigenvectors play such a role proves the following theorem.

Theorem 5.45. If A is an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and associated eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, then the solution to the system of difference equations $\mathbf{x}_{k+1} = A\mathbf{x}_k$ is

$$\mathbf{x}_k = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + \cdots c_n\lambda_n^k\mathbf{v}_n.$$

Example 5.46. Long ago, in a galaxy far, far away…there are two cell-phone companies serving a town: the Evil Empire and the Rebel Alliance. The Evil Empire has terrible service, so each week 25% of their customers switch to the Rebel Alliance and 2% give up their cell phone service entirely. The Rebel Alliance loses only 5% of their customers to the Evil Empire every week due to the advertising. If there are currently 100 customers in the Evil Empire and 75 customers in the Rebel Alliance, what is the long-term enrollment in the two plans?

Solution:

This is a system of difference equations that we can write as

$$\begin{aligned} e_{k+1} &= 0.73e_k + 0.05r_k \\ r_{k+1} &= 0.25e_k + 0.95r_k. \end{aligned}$$

Written as a matrix equation we see that

$$\mathbf{x}_{k+1} = A\mathbf{x}_k \quad \Longrightarrow \quad \begin{pmatrix} e_{k+1} \\ r_{k+1} \end{pmatrix} = \begin{pmatrix} 0.73 & 0.05 \\ 0.25 & 0.95 \end{pmatrix}\begin{pmatrix} e_k \\ r_k \end{pmatrix}.$$

The solution to this system of equations depends on the eigenvalues of A. After some computation we see that the eigenvalue / eigenvector pairs are $\lambda_1 \approx 0.997$ with $\mathbf{v}_1 \approx \begin{pmatrix} 0.18 \\ 0.98 \end{pmatrix}$ and $\lambda_2 \approx 0.68$ with $\mathbf{v}_2 \approx \begin{pmatrix} -0.73 \\ 0.68 \end{pmatrix}$. Therefore, the general solution to the system is

$$\mathbf{x}_k \approx c_1(0.997)^k\begin{pmatrix} 0.18 \\ 0.98 \end{pmatrix} + c_2(0.68)^k\begin{pmatrix} -0.73 \\ 0.68 \end{pmatrix}.$$

Using the initial condition $\mathbf{x}_0 = \begin{pmatrix} 100 \\ 75 \end{pmatrix}$ we see that

$$\begin{pmatrix} 100 \\ 75 \end{pmatrix} = c_1\begin{pmatrix} 0.18 \\ 0.98 \end{pmatrix} + c_2\begin{pmatrix} -0.73 \\ 0.68 \end{pmatrix}.$$

This is a system of equations in c_1 and c_2

$$\begin{pmatrix} 0.18 & -0.73 \\ 0.98 & 0.68 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 100 \\ 75 \end{pmatrix}.$$

After augmenting and row reducing we see that $c_1 \approx 146.5$ and $c_2 \approx -100.9$. In Figure 5.5 one can see the very slow convergence to a steady state of 0 customers in both plans.

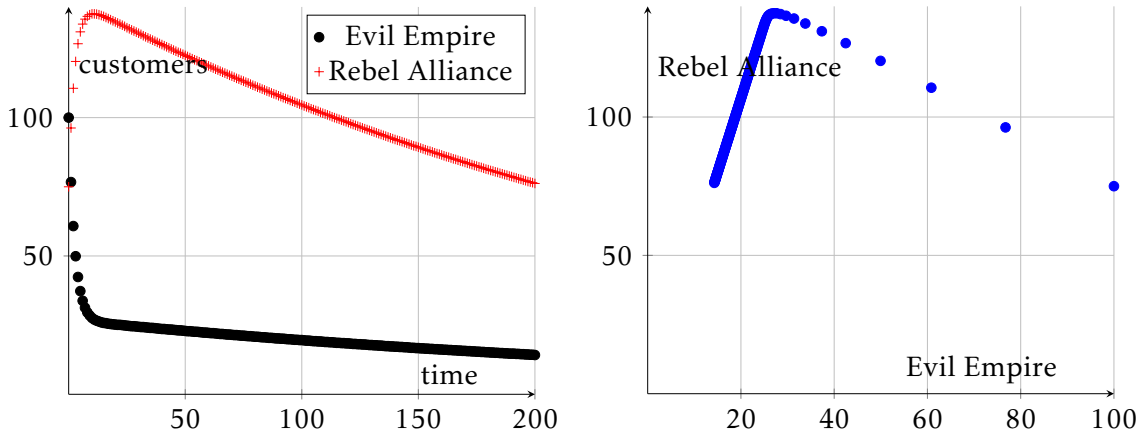


Figure 5.5. Evil empire customers and Rebel Alliance customers over a 200 week time period.

Problem 5.47. Find the general solution to the two problems from Section 5.5. ▲

5.8.2 Analysis of Equilibrium Behavior

The equilibrium solution of a difference equation occurs when the new term x_{k+1} is equal to the old term x_k. If the system of difference equations is $x_{k+1} = Ax_k$ then to find the equilibrium solution we need to solve $x_e = Ax_e$. The reader should recognize this as either the hunt for the eigenvalue $\lambda = 1$ and its associated eigenvector, or the zero vector. Remember that not every matrix will have a unit eigenvalue and we don't call the zero vector an eigenvector. Even so, the zero vector will be an equilibrium point since it is the trivial solution to $x_e = Ax_e$.

Theorem 5.48. To determine the stability of the origin in the discrete dynamical system $x_{k+1} = Ax_k$ we have several cases related to the two eigenvalues λ_1 and λ_2:

1. If $|\lambda_1| < 1$ and $|\lambda_2| < 1$ then the origin is stable.
2. If either $|\lambda_1| > 1$ or $|\lambda_2| > 1$ then the origin is unstable.
3. If $\lambda_1 = 1$ and
   - $\lambda_2 = 1$ then there is no equilibrium solution
   - $|\lambda_2| < 1$ then the origin is unstable and the equilibrium is a multiple of the eigenvector associated with $\lambda_1 = 1$.

- $\lambda_2 = -1$ then there may or may not be an equilibrium. If there is an equilibrium then it is unstable.

Example 5.49. Consider the system of difference equations

$$\begin{aligned} a_{n+1} &= 2a_n - b_n \\ b_{n+1} &= -4a_n - b_n \\ a_0 &= 1 \\ b_0 &= -6 \end{aligned}$$

Write the system as a matrix equation and write the solution as a linear combination of the eigenvectors.
Solution:
The matrix form of the equation is

$$\underbrace{\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix}}_{\mathsf{x}_{n+1}} = \underbrace{\begin{pmatrix} 2 & -1 \\ -4 & -1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a_n \\ b_n \end{pmatrix}}_{\mathsf{x}_n} \quad \text{with} \quad \mathsf{x}_0 = \begin{pmatrix} 1 \\ -6 \end{pmatrix}$$

Recall that the analytic solution to this difference equation is

$$\mathsf{x}_n = A^n \mathsf{x}_0.$$

In order to efficiently evaluate the matrix power we find the eigenvalue-eigenvector pairs. After some work, the eigenvalues and eigenvectors of the coefficient matrix A are $\lambda_1 = 3$ with $\mathsf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\lambda_2 = -2$ with $\mathsf{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. Therefore

$$\mathsf{x}_n = C_1 \cdot 3^n \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 \cdot (-2)^n \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

Given that $\lambda_1 = 3$ and $\lambda_2 = -2$ we know that the equilibrium solution $\mathsf{x} = 0$ will be unstable.

It only remains to find the constants. The constants come from observing that $\mathsf{x}_0 = C_1\mathsf{v}_1 + C_2\mathsf{v}_2$. This is a system of equations, and in augmented form we see that

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 4 & -6 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right) \Longrightarrow \begin{cases} c_1 = 2 \\ c_2 = -1 \end{cases}$$

Therefore, the final solution to the discrete dynamical system is

$$\mathbf{x}_n = \begin{pmatrix} a_n \\ b_n \end{pmatrix} = 2 \cdot 3^n \begin{pmatrix} 1 \\ -1 \end{pmatrix} + (-1) \cdot (-2)^n \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \cdot 3^n - (-2)^n \\ -2 \cdot 3^n - 4 \cdot (-2)^n \end{pmatrix}.$$

This solution can easily be plotted with a graphing utility (we leave this to the reader).

Problem 5.50. For each of the problems below, find the general solution to the difference equation and discuss the stability of the origin.

(a) $A = \begin{pmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{pmatrix}$ with $\mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The eigen-pairs for A are

$$\lambda_1 = 0.75 \text{ with } \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \lambda_2 = 0.25 \text{ with } \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

(b) $A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$ with $\mathbf{x}_0 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$. The eigen-pairs are

$$\lambda_1 = 3 \text{ with } \mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \lambda_2 = 2 \text{ with } \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(c) $A = \begin{pmatrix} 7 & 0 & 5 \\ 0 & 5 & 0 \\ -4 & 0 & -2 \end{pmatrix}$ with $\mathbf{x}_0 = \begin{pmatrix} 9 \\ -2 \\ -8 \end{pmatrix}$. Hint: The eigen-pairs are

$$\lambda_1 = 3 \text{ with } \mathbf{v}_1 = \begin{pmatrix} 5 \\ 0 \\ -4 \end{pmatrix}, \quad \lambda_2 = 2 \text{ with } \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \lambda_3 = 5 \text{ with } \mathbf{v}_3 = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}.$$

▲

5.8.3 Non Homogeneous Systems of Linear Difference Equations

Now that we have an intuitive sense of how complicated the dynamics of a linear homogeneous system can be, let's dive into the least complicated non-homogeneous dynamical system:

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + \mathbf{b}$$

An equilibrium solution to occurs when $\mathbf{x}_{k+1} = \mathbf{x}_k$. In other words, we need to solve

$$\mathbf{x}_e = A\mathbf{x}_e + \mathbf{b}.$$

Unlike the homogeneous system, in this case we can use (linear) algebra to rearrange:

$$\mathbf{x}_e = A\mathbf{x}_e + \mathbf{b} \quad \Longrightarrow \quad \mathbf{x}_e - A\mathbf{x}_e = \mathbf{b} \quad \Longrightarrow \quad (I - A)\mathbf{x}_e = \mathbf{b}.$$

On paper, the equilibrium vector is $\mathbf{x}_e = (I - A)^{-1}\mathbf{b}$. As discussed previously, inverting a matrix is often not a great ideal so it may be necessary to lean on some linear algebra technology to solve the system for $\mathbf{x}_e$. The eigenvalue structure of A tells you complete information about the stability of the equilibrium point as it did in the homogeneous case. In general, if $\mathbf{b}$ is not constant then the system will likely not have an equilibrium point.

Theorem 5.51. Let $\mathbf{x}_{n+1} = A\mathbf{x}_n + \mathbf{b}$ be a non-homogeneous system of linear difference equations. If A has eigen-pairs $\lambda_1, \mathbf{v}_1$, $\lambda_2, \mathbf{v}_2$, $\ldots$, $\lambda_k, \mathbf{v}_k$ and the initial condition $\mathbf{x}_0$ can be written as $\mathbf{x}_0 = C_1\mathbf{v}_1 + C_2\mathbf{v}_2 + \cdots + C_k\mathbf{v}_k + \mathbf{x}_e$ then the general solution to the non-homogeneous system of difference equations is

$$\mathbf{x}_n = \left(C_1\lambda_1^n\mathbf{v}_1 + C_2\lambda_2^n\mathbf{v}_n + \cdots C_k\lambda_k^n\mathbf{v}_k\right) + \mathbf{x}_e$$

where $\mathbf{x}_e$ is the equilibrium solution $\mathbf{x}_e = (I - A)^{-1}\mathbf{b}$.

Example 5.52. Consider the system of equation

$$\mathbf{x}_{n+1} = \begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 0.75 \\ -0.5 & -0.25 \end{pmatrix}\begin{pmatrix} a_n \\ b_n \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Find the general solution to the system of equation with the initial condition $\mathbf{x}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Solution:
The eigen-pairs for the matrix A are

$$\lambda_1 = 0.25 \text{ with } \mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \lambda_2 = 0.5 \text{ with } \mathbf{v}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

Furthermore, the equilibrium point is $\mathbf{x}_e = (I - A)^{-1}\mathbf{b}$ where $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Hence,

$$\mathbf{x}_e = \begin{pmatrix} 10/3 \\ -4/3 \end{pmatrix}.$$

Therefore, the general solution to the system of difference equations is

$$\mathbf{x}_n = C_1(0.25)^n\begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2(0.5)^n\begin{pmatrix} 3 \\ -2 \end{pmatrix} + \begin{pmatrix} 10/3 \\ -4/3 \end{pmatrix}.$$

Since both eigenvalues are less than 1 and greater than zero the long term behavior of the system will tend toward the equilibrium, so

$$\lim_{n\to\infty} \mathbf{x}_n = \begin{pmatrix} 10/3 \\ -4/3 \end{pmatrix}.$$

In other words, $\mathbf{x}_e$ is a stable equilibrium.

To solve the system of equation with the given initial condition we observe the system of equations $\mathbf{x}_0 = C_1\mathbf{v}_1 + C_2\mathbf{v}_2 + \begin{pmatrix} 10/3 \\ -4/3 \end{pmatrix}$, augment, and row reduce to find $C_1 = 16/3$ and $C_2 = 1$. Therefore,

$$\mathbf{x}_n = \left(\frac{16}{3}\right)(0.25)^n \begin{pmatrix} -1 \\ 1 \end{pmatrix} + 1(0.5)^n \begin{pmatrix} 3 \\ -2 \end{pmatrix} + \begin{pmatrix} 10/3 \\ -4/3 \end{pmatrix}.$$

Problem 5.53. For each of the problems below, find the general solution to the difference equation and discuss the stability of any equilibrium points. Use linear algebra software where appropriate.

(a) Consider the system of difference equations

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 0.2 & 0.6 \\ 0.3 & 0.1 \end{pmatrix}\begin{pmatrix} a_n \\ b_n \end{pmatrix} + \begin{pmatrix} -18 \\ 27 \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 28 \end{pmatrix}$$

(b) Consider the system of difference equations

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} 0.1 & 0.1 & 0.3 \\ 0.1 & 0.3 & 0.1 \\ 0.3 & 0.1 & 0.1 \end{pmatrix}\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} + \begin{pmatrix} -96 \\ -24 \\ -120 \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} = \begin{pmatrix} -226 \\ -45 \\ -218 \end{pmatrix}$$

▲

5.9 Modeling Explorations with Systems

Problem 5.54 (Deer and Cougar Problem)**.** We are going to build a difference equation model that predicts the populations of cougars C and deer D (predators and prey) at year t

1. **The Mathematical Model:**

   (a) **Deer Population:** The change in deer population can be modeled as

   $D'(t)$ = (logistic population growth)–(deer removed by interactions with cougars)

   - a generic logistic growth difference equation takes the form:

   $$y'(t) = ky\left(1 - \frac{y}{N}\right)$$

   where k is the growth rate parameter and N is the carrying capacity of the population.
   - Assume that the carrying capacity for the deer is $N = 4,500$, and assume that $k \approx 0.3$
   - The rate at which the deer population is removed due to interactions with cougars is proportional to the product of the deer and cougar populations. Assume that the proporationality constant is -0.15.
   - Write the differential equation for the deer population.

   (b) **Cougar Population:** The change in cougar population can be modeled as

   $C'(t)$ = (natural death rate) + (gain due to abundance of food source)

   - Assume that the natural death rate for the cougar population is proportional to the current number of cougars. Let the proportionality constant be -0.30.
   - Assume that the gain in population is proportional to the product of the deer and cougar populations with proportionality constant 0.00015.
   - Write the difference equation for the cougar population.

   (c) **The Model:** The two difference equations that you've written along with the initial conditions $D_0 = 500$ and $C_0 = 2$ are now your mathematical model! Summarize them in one place in your lab.

2. **Model Description:** Describe some features of this model by answering the following questions. (please restate the questions in your lab writeup)

   (a) Which animal is the predator and which is the prey?

   (b) What is the significance of the + sign in the equation for cougars?

   (c) What is the significance of the − sign in the equation for deer?

(d) In the absence of deer, does the cougar population increase or decrease? At what rate?

(e) In the absence of cougars, does the deer population increase or decrease? At what rate?

(f) How many deer does each cougar consume when there are 1000 deer vs when there are 2000 deer?

3. **Numerical Solution:** Construct a numerical solution to this system of difference equations. Recall that a numerical solution for a differential equation model will use Euler's method. This time you'll need to build both equations simultaneously. Your numerical approximation will give you both the deer population and the cougar population. For a system of differential equations there are three appropriate plots that you need to generate.

   Deer vs Time: Plot the number of deer on the y axis and time (t) on the x axis.

   Cougars vs Time: Plot the number of cougars on the y axis and time (t) on the x axis.

   Deer vs Cougars: Plot the number of deer on the y axis and the number of cougars on the x axis (or visa versa). This is called the phase plot.

   Write a paragraph describing the behavior of the deer and cougar populations over the course of 100 years. Identify each of the key features[†] of the population dynamics.

4. **Analysis:**

   (a) There are several assumptions in this model, but in particular we have assumed that the cougar population will only go up via direct interactions with prey (deer). Change the death rate term in the cougar population to a plausible model that depends on the current number of cougars. Repeat the numerical solutions for several possible parameters.

   (b) Is either model (the original or the new from part (a)) sensitive to initial conditions? Vary each initial condition slightly and discuss how the dynamical system behaves. Support your disucssion with appropriate plots or tables.

   (c) There are *interaction terms* in each equation of the model (the terms involving both C and D). How sensitive is the model to the interaction rate 0.15? Vary this interaction term slightly and discuss the behavior of the dynamical system. Support your discussion with appropriate plots or tables.

   (d) Attack several other assumptions built into this model, make the appropriate changes to the model, and determine what the impact is on the resulting populations.

[†] *Key features* would include maximums, minimums, regions of increase or decrease, extinction, overpopulation, equilibrium, etc.

5. Write an Executive Summary describing what you did in this lab. This is meant to be a reference to you when you need to re-use the ideas (both mathematical and computational) from this lab. Your executive summary should be immediately under the title on the front page of the lab. It should be no more than a paragraph or two, and it should make for easy reference later.

▲

Problem 5.55 (The Bedridden Boys Problem)**.** A boarding school is a relatively closed community in which all students live on campus, teachers tend to live on or near campus, and students do not regularly interact with people not in the boarding school community. The data set in Table 5.5 gives data for an influenza outbreak at a boarding school in England during which there were no fatalities. There were 763 boys at the English boarding school from which the data was obtained. This is only one instance of the types of influenza outbreaks that occur every year at boarding schools and on college campuses. **Create a mathematical model for the number of infected people in a closed community (such as a dorm or boarding school) as a function of time.** Assume there is 1 person initially infected.

What makes this situation interesting:

- People don't generally show symptoms right away after they are exposed to a virus.
- Some people may have been previously exposed to the virus and are therefore immune.
- In a closed community it may or may not be possible to implement a quarantine.
- Some people will not admit themselves for medical care even though they are infected. Hence, those people will likely not be counted in medical records for the outbreak.
- . . . and many others . . .

Your Tasks:

(a) create, solve, and analyze a simple model (with as many assumptions as necessary),

(b) refine your model to either reduce the number of assumptions or to increase the complexity (and hence make the model more realistic), and

(c) refine the model a second time to create an even more realistic model.

At each step your group must do a thorough analysis of the mathematical model. This includes a sensitivity analysis for any model parameters. The modeling process will require your group to research the real problem to determine the assumptions and any necessary terminology. Be sure to properly cite any sources used and to check the reliability of those sources. ▲

Time (in days)	Number of Bedridden Boys
0	1
1	3
2	25
3	72
4	222
5	282
6	256
7	233
8	189
9	123
10	70
11	25
12	11
13	4

Table 5.5. Bedridden Boys Data

Problem 5.56 (The Ibuprofen Problem)**.** Ibuprofen, an analgesic pain reliever, is ingested into the gastrointestinal tract by swallowing a pill of the substance or drinking a solution. The drug then moves to the plasma or serum where it travels to sites to do its work of pain relief. **Create a mathematical model for the concentration of the Ibuprofen (in μg/ml) in the plasma as a function of time.** The data set in Table 5.6 gives one patient's Ibuprofen concentration over time with a 400 mg initial dose.

What makes this situation interesting:

- The Ibuprofen doesn't actually move directly into the plasma.
- The units of the dose are milligrams (typically, 400mg) but the units of the concentration are micro grams per milliliter.
- The volume of the gastrointestinal tract and the plasma are different.
- Every person's body will metabolize Ibuprofen at different rates.
- As the effects of the drug wear off the patient usually takes more Ibuprofen.
- . . . and many others . . .

Your Tasks:

(a) create, solve, and analyze a simple model (with as many assumptions as necessary),

(b) refine your model to either reduce the number of assumptions or to increase the complexity (and hence make the model more realistic), and

(c) refine the model a second time to create an even more realistic model.

At each step your group must do a thorough analysis of the mathematical model. This includes a sensitivity analysis for any model parameters. The modeling process will require your group to research the real problem to determine the assumptions and any necessary terminology. Be sure to properly cite any sources used and to check the reliability of those sources. ▲

Time (hours)	Ibuprofen Concentration in Plasma (micro grams / ml)
0	0
0.0833	12.5
0.167	26.5
0.25	33
0.33	37.4
0.5	36.2
1	33.2
1.5	30.2
2	23.9
3	16.3
4	11.7
6	5.82
8	3.09

Table 5.6. Ibuprofen Data

Chapter 6

Infinite Series

To wrap up this course we discuss an incredibly useful tool in mathematics: the infinite series. It may come as a surprise, but it is entire possible to add infinitely many things together and get a finite sum. For a deep exploration of this idea we return to the Active Calculus Textbook.

6.1 Sections from Active Calculus

The Active Calculus Textbook is a wonderful online resource that stands in place of this chapter. We will only discuss a few select sections and you can find the relevant links below.

1. Active Calculus Section 8.1: Sequences
2. Active Calculus Section 8.2: Geometric Series
3. Active Calculus Section 8.3: Series of Real Numbers
4. Active Calculus Section 8.4: Alternating Series
5. Active Calculus Section 8.5: Taylor Polynomials and Taylor Series
6. Active Calculus Section 8.6: Power Series

Appendices

Appendix A

MATLAB Basics

In this appendix we'll go through a few of the basics in MATLAB. This is by no means meant to be an all-encompassing resource for MATLAB programming. A few more thorough resources for MATLAB are listed here.

- https://www.mathworks.com/help/pdf_doc/matlab/matlab_prog.pdf
- https://www.mathworks.com/products/matlab/examples.html
- https://en.wikibooks.org/wiki/MATLAB_Programming
- http://gribblelab.org/scicomp/scicomp.pdf (this is a personal favorite)

In this appendix we'll give examples of some of the more common coding practices that the reader will run into while working through the exercises and problems in these notes.

A.1 Vectors and Matrices

Example A.1. Write the vectors $\mathtt{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathtt{w} = \begin{pmatrix} 4 & 5 & 6 & 7 \end{pmatrix}$ using MATLAB.

Solution:

```
v = [1 ; 2 ; 3]
w = [4 , 5 , 6 , 7]
w = 4:7  % this is shorthand for writing a sequence as a row vector
```

Example A.2. Consider the matrices and vectors

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 5 & 7 \\ 9 & 1 & 3 \\ 5 & 7 & 11 \end{pmatrix} \quad \text{and} \quad \mathrm{b} = \begin{pmatrix} 4 & 3 & -1 \end{pmatrix}$$

- Calculate the product AB using regular matrix multiplication

```
A = [1 , 2 , 3;
     4 , 5 , 6;
     7 , 8 , 0]
B = [3 , 5 , 7;
     9 , 1 , 3;
     5 , 7 , 11]
Product = A*B
```

- Calculate the element-by-element multiplication of A and B

```
ElementWiseProduct = A .* B
```

- Calculate the inverse of A

```
Ainv = A^(-1)
Ainv = inverse(A) % alternative
```

- Calculate the transpose of B

```
Atranspose = transpose(A)
% or as an alternative:
Atranspose = A'  % actually the conjugate transpose but if A is real then ok
```

- Solve the system of equations $A\mathrm{x} = \mathrm{b}$

```
b = [4 ; 3 ; -1]
x = A \ b
```

Example A.3. Code for a matrix of zeros

```
Z = zeros(5,5) % 5 x 5 matrix of all zeros
```

Example A.4. Code for an identity matrix

```
Ident = eye(5,5) % 5 x 5 identity matrix
```

Example A.5. Code for random matrices.

- random matrix from a uniform distribution on $[0,1]$

```
R = rand(5,5) % random 5 x 5 matrix
```

- random matrix from the standard normal distribution

```
R = randn(5,5) % random 5 x 5 matrix
```

Example A.6. A linearly spaced sequence

```
List = linspace(0,10,100)
% a list of 100 equally spaced numbers from 0 to 10
```

A.2 Looping

A loop is used when a process needs to be repeated several times.

A.2.1 For Loops

A `for` loop is code that repeats across a pre-defined sequence.

Example A.7. Write a loop that produces the squares of the first 10 integers.

```
for j = 1:10
    j^2
end
```

The output of this code will be

```
1
4
9
16
25
36
49
64
81
100
```

Example A.8. Plot the functions $f(x) = \sin(kx)$ for $k = 1, 1.5, 2, 2.5, \ldots, 5$ on the domain $x \in [0, 2\pi]$.

```
x = linspace(0,2*pi,1000);
for k = 1:0.5:5
    plot(x , sin(k*x) )
    hold on
end
```

A.2.2 The While Loop

A `while loop` is a process that only repeats while a conditional statement is true. Be careful with `while loops` since it is possible to create a loop that runs forever.

Example A.9. Build the Fibonnaci sequence up until the last term is greater than 1000.

```
F(1) = 1; % first term
F(2) = 1; % second term
n = 3;
while F(end)<1000
    F(n) = F(n-1) + F(n-2);
    n=n+1;
end
```

Example A.10. An example of a while loop that runs forever.

```
a = 1;
```

```
while a>0
    a=a+1;
end
```

Example A.11. An example of a while loop that runs forever but with a failsafe step that stops the loop after 1000 steps.

```
a = 1;
counter=1;
while a>0
    a=a+1;
    if counter >= 1000
        break
    end
    counter = counter+1;
end
```

A.3 Conditional Statements

Conditional statements are used to check if something is true or false. The output of a conditional statement is a boolean value; true (1) or false (0).

A.3.1 If Statements

Example A.12. Loop over the integers up to 100 and output only the multiples of three.

```
for j = 1:100
    if mod(j,3) == 0
        j
    end
end
```

Example A.13. Check the signs of two function values and determine if they are opposite.

```
f = @(x) x^3*(x-3);
a = 2;
b = 4;
if f(a)*f(b) < 0
```

```
    fprintf('The function values are opposite sign\n')
elseif f(a)*f(b) >0
    fprintf('The function values are the same sign\n')
else
    fprintf('The function values are both zero\n')
end
```

A.3.2 Case-Switch Statements

Example A.14. Evaluate over several cases.

```
n = 3
switch n
    case 1 % if n == 1
        fprintf('n is 1\n')
    case 2 % if n == 2
        fprintf('n is 2\n')
    case 3 % if n == 3
        fprintf('n is 3\n')
end
```

A.4 Functions

A mathematical function has a single output for every input, and in some sense a computer function is the same: one single executed process for each collection of inputs.

Example A.15. Define the function $f(x) = \sin(x^2)$ so that it can accept any type of input (symbol, number, or list of numbers).

```
f = @(x) sin(x.^2) % defines the function
f(3) % evaluates the function at x=3
x=linspace(0,pi,100);
f(x) % evaluates f at 100 points equally spaced from 0 to pi
```

Example A.16. Write a computer function that accepts two numbers as inputs and outputs the sum plus the product of the two numbers.
First write a file with the following contents.

```
function MyOutput = MyFunctionName(a,b)
    MyOutput = a + b + a*b;
end
```

Be sure that the file name is the same as the function name.
Then you can call the function by name in a script or another function.

```
SumPlusProduct = MyFunctionName(3,4)
```

which will output the number 19.

Example A.17. Write a function with three inputs that outputs the sum of the three. The third input should be optional and the default should be set to 5.

```
function AwesomeOutput = SumOfThree(a,b,c)
    if nargin < 3
        c = 5;
    end
    AwesomeOutput = a+b+c;
end
```

You can call this function with

```
SumOfThree(17,23)
```

which will output $17+25+5=47$. Notice that the third input was left off and a 5 was used in its place.

A.5 Plotting

In numerical analysis we are typically plotting numerically computed lists of numbers so as such we will give a few examples of this type of plotting here. We will not, however, give examples of symbolic plotting.

The `plot` command in MATLAB accepts a list of x values followed by a list of y values then followed by color and symbol options.

```
plot(xlist , ylist , color options)
```

Example A.18. Plot the function $f(x) = \sin(x^2)$ on the interval $[0, 2\pi]$ with 1000 equally spaced points. Make the plot color blue.

```
x = linspace(0,2*pi,1000);
f = @(x) sin(x.^2);
plot(x , f(x) , 'b')
```

Alternatively

```
x = linspace(0,2*pi,1000);
y = sin(x.^2);
plot(x, y, 'b')
```

Example A.19. Make a 2×2 array of 4 plots of $f(x) = \sin(kx^2)$ for $k = 1, 2, 3, 4$.

```
x = linspace(0,2*pi,1000);
for k=1:4
    subplot(2,2,k)
    plot(x , sin(k*x.^2) , 'b')
end
```

Example A.20. Plot $f(x) = \sin(kx^2)$ for $k = 1, 2, \ldots, 10$ all on the same plot.

```
x = linspace(0,2*pi,1000);
for k=1:10
    plot(x, sin(k*x.^2))
    hold on % this holds the figure window open so you can write on top of it
end
```

Example A.21. Plot the function $f(x) = e^{-x}\sin(x)$ and put a mark at the local max at $x = \pi/4$.

```
x = linspace(0,2*pi,1000); % set up the domain
f = @(x) exp(-x) .* sin(x);
plot(x,f(x),'b',pi/4,f(pi/4),'ro')
```

A.6 Animations

Example A.22. Plot $f(x) = \sin(kx^2)$ for $k = 1$ to $k = 10$ by small increments with a short pause in between each step.

```
x = linspace(0,2*pi,1000);
for k=1:0.01:10 % 1 to 10 by 0.01
    plot(x, sin(k*x.^2))
    hold on % this holds the figure window open so you can write on top of it
    drawnow % draws the plot
```

```
6       % the last line gives the illusion of animation
7   end
```