

# Introduction to Nonlinear Dynamics, Fractals, and Chaos

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# Objective

The aim of the course is to give students an introduction to the new developments in nonlinear dynamics and fractals. Emphasis will be on the basic concepts of stability, bifurcations and intermittency, based on intuition rather than mathematical proofs. On successful completion of this course, students should understand and apply the theory to simple nonlinear dynamical systems and be able to evaluate the importance of nonlinearity in various environments.

# Plan of the Course

## 1. Introduction

- Dynamical and Geometrical View of the World
- Fractals
- Stability of Linear Systems

## 2. Nonlinear Dynamics

- Attracting and Stable Fixed Points
- Nonlinear Systems: Pendulum

## 3. Fractals and Chaos

- Strange Attractors and Deterministic Chaos
- Bifurcations

## 4. Strange Attractors

- Stretching and Folding Mechanism
- Baker's Map
- Logistic Map
- Hénon Map

## 5. Conclusion: importance of nonlinearity and fractals

## Dynamics - A Capsule History

1666	Newton	Invention of calculus, explanation of planetary motion
1700s		Flourishing of calculus and classical mechanics
1800s		Analytical studies of planetary motion
1890s	Poincaré	Geometric approach, nightmares of chaos
1920–1950		Nonlinear oscillators in physics and engineering, invention of radio, radar, laser
1920–1960	Birkhoff Kolmogorov Arnol'd Moser	Complex behavior in Hamiltonian mechanics
1963	Lorenz	Strange attractor in simple model of convection
1970s	Ruelle & Takens	Turbulence and chaos
	May	Chaos in logistic map
	Feigenbaum	Universality and renormalization, connection between chaos and phase transitions
		Experimental studies of chaos
	Winfrey	Nonlinear oscillators in biology
	Mandelbrot	Fractals
1980s		Widespread interest in chaos, fractals, oscillators, and their applications

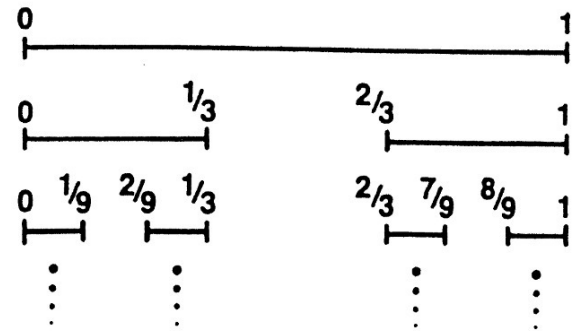


# Fractals

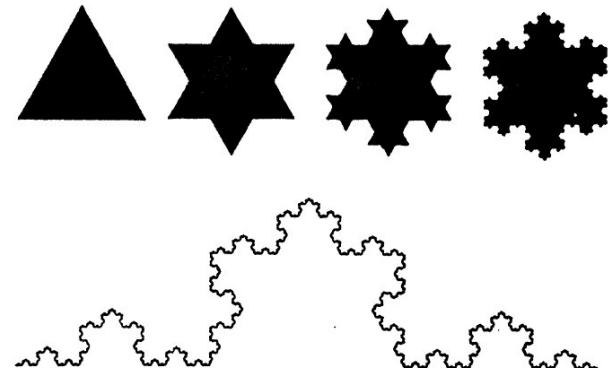
A **fractal** is a rough or fragmented geometrical object that can be subdivided in parts, each of which is (at least approximately) a reduced-size copy of the whole.

Fractals are generally *self-similar* and independent of scale (fractal dimension).

(a)



(b)



If  $N_n$  is the number of elements of size  $r_n$  needed to cover a set ( $C$  is a constant) is:

$$N_n = \frac{C}{r_n^D}, \quad (1)$$

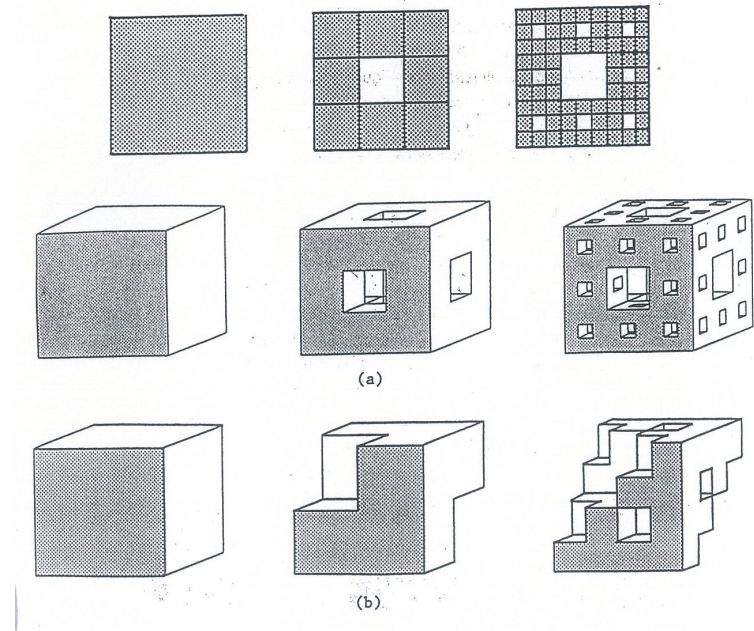
then in case of self-similar sets:

$$N_{n+1} = C/(r_{n+1})^D,$$

and hence the fractal similarity dimension  $D$  is

$$D = \ln(N_{n+1}/N_n) / \ln(r_n/r_{n+1}). \quad (2)$$

- Cantor set  $D = \ln 2 / \ln 3$
- Koch curve  $D = \ln 4 / \ln 3$
- Sierpinski carpet  $D = \ln 8 / \ln 3$
- Mengor sponge  $D = \ln 20 / \ln 3$
- Fractal cube  $D = \ln 6 / \ln 2$





# Stability of Linear Systems

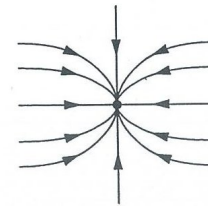
Two-Dimensional System

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

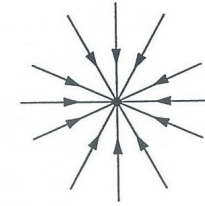
Solutions

$$x(t) = x_0 e^{at}$$

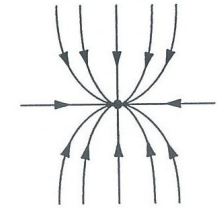
$$y(t) = y_0 e^{-t}$$



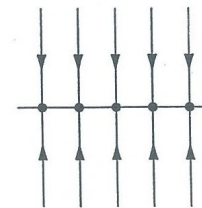
(a)  $a < -1$



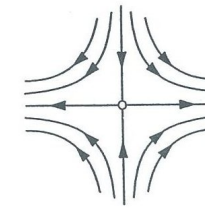
(b)  $a = -1$



(c)  $-1 < a < 0$



(d)  $a = 0$



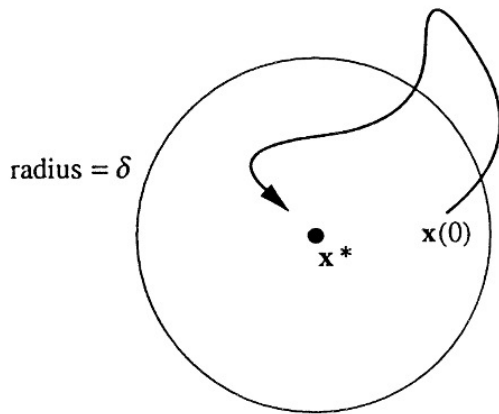
(e)  $a > 0$

# Attracting and Stable Fixed Points

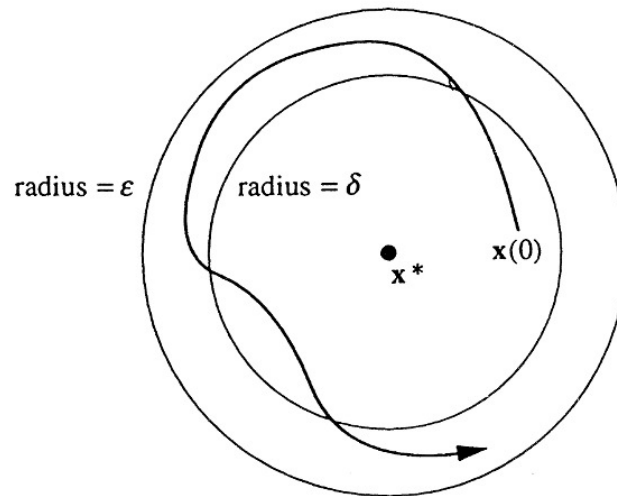
We consider a fixed point  $\mathbf{x}^*$  of a system  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ , where  $\mathbf{F}(\mathbf{x}^*) = \mathbf{0}$ .

We say that  $\mathbf{x}^*$  is *attracting* if there is a  $\delta > 0$  such that  $\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{x}^*$  whenever  $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$ : any trajectory that starts within a distance  $\delta$  of  $\mathbf{x}^*$  is guaranteed to converge to  $\mathbf{x}^*$ .

A fixed point  $\mathbf{x}^*$  is *Lyapunov stable* if for each  $\varepsilon > 0$  there is a  $\delta > 0$  such that  $\|\mathbf{x}(t) - \mathbf{x}^*\| < \varepsilon$  whenever  $t \geq 0$  and  $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$ : all trajectories that start within  $\delta$  of  $\mathbf{x}^*$  remain within  $\varepsilon$  of  $\mathbf{x}^*$  for all positive time.

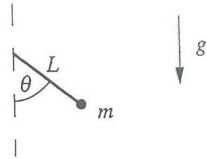


Attracting



Liapunov stable

# Nonlinear Systems: Pendulum



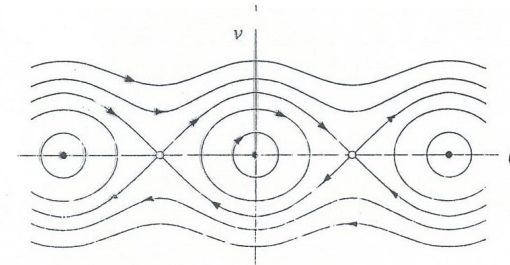
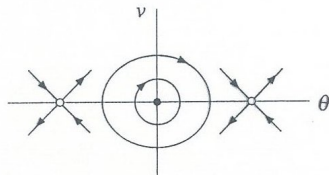
$$\frac{d^2\theta}{dt^2} + \frac{g}{L} \sin \theta = 0$$

$$\ddot{\theta} + \sin \theta = 0$$

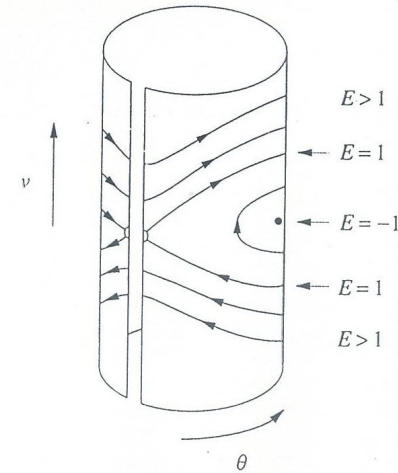
$$\begin{aligned} \dot{\theta} &= v \\ \dot{v} &= -\sin \theta \end{aligned}$$

The energy function

$$E(\theta, v) = \frac{1}{2} v^2 - \cos \theta$$



Cylindrical Phase Space



# Attractors

An **ATTRACTOR** is a *closed* set  $A$  with the properties:

1.  $A$  is an INVARIANT SET:  
any trajectory  $\mathbf{x}(t)$  that start in  $A$  stays in  $A$  for ALL time  $t$ .
2.  $A$  ATTRACTS AN OPEN SET OF INITIAL CONDITIONS:  
there is an open set  $U$  containing  $A$  ( $A \subset U$ ) such that if  $\mathbf{x}(0) \in U$ , then the distance from  $\mathbf{x}(t)$  to  $A$  tends to zero as  $t \rightarrow \infty$ .
3.  $A$  is MINIMAL:  
there is NO proper subset of  $A$  that satisfies conditions 1 and 2.

**STRANGE ATTRACTOR** is an attracting set that is a fractal: has zero measure in the embedding phase space and has FRACTAL dimension. Trajectories within a strange attractor appear to skip around randomly.

Dynamics on **CHAOTIC ATTRACTOR** exhibits sensitive (exponential) dependence on initial conditions (the 'butterfly' effect).

# Deterministic Chaos

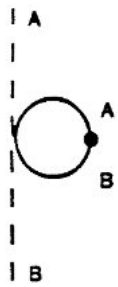


**CHAOS** ( $\chi\alpha\omicron\varsigma$ ) is

- NON-PERIODIC long-term behavior
- in a DETERMINISTIC system
- that exhibits SENSITIVITY TO INITIAL CONDITIONS.

We say that a bounded solution  $\mathbf{x}(t)$  of a given dynamical system is SENSITIVE TO INITIAL CONDITIONS if there is a finite fixed distance  $r > 0$  such that for any neighborhood  $\|\Delta\mathbf{x}(0)\| < \delta$ , where  $\delta > 0$ , there exists (at least one) other solution  $\mathbf{x}(t) + \Delta\mathbf{x}(t)$  for which for some time  $t \geq 0$  we have  $\|\Delta\mathbf{x}(t)\| \geq r$ .

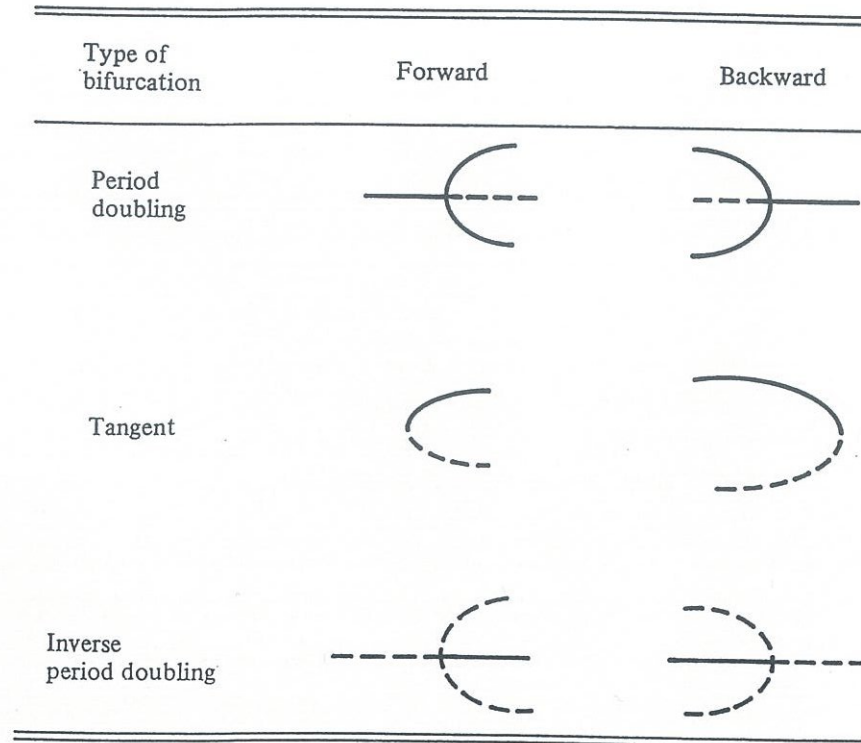
There is a fixed distance  $r$  such that no matter how precisely one specifies an initial state there is a nearby state (at least one) that gets a distance  $r$  away.

Given  $\mathbf{x}(t) = \{x_1(t), \dots, x_n(t)\}$  any positive finite value of Lyapunov exponents  $\lambda_k = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \left| \frac{\Delta x_k(t)}{\Delta x_k(0)} \right|$ , where  $k = 1, \dots, n$ , implies chaos.

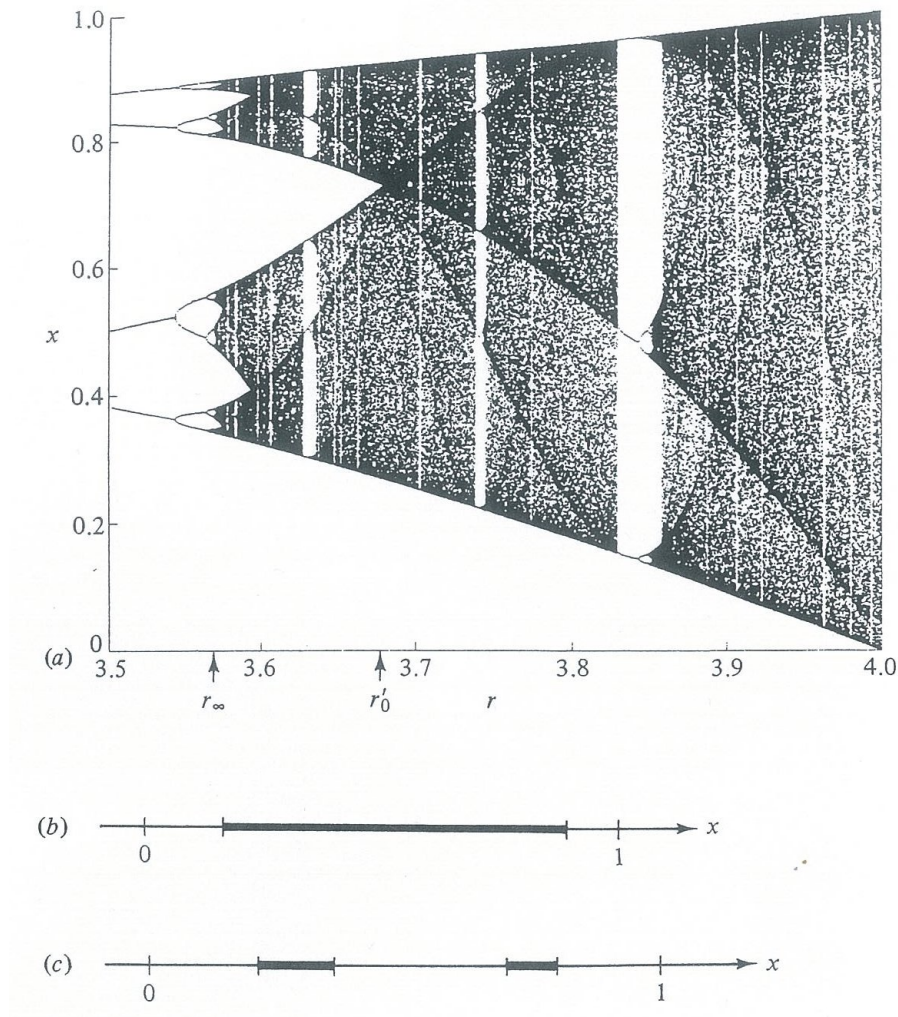
	ATTRACTOR	LYAPUNOV EXPONENT SPECTRUM
C	<p>LIMIT CYCLE</p> 	(0,-,-)
D	<p>POINT</p> 	(-,-,-)
E	<p>STRANGE ATTRACTOR</p> 	(+,0,-)

# Types of Bifurcations

Figure 2.15 Generic bifurcations of differentiable one-dimensional maps. The system parameter  $r$  increases toward the right. The vertical scale represents the value of the map variable. Dashed lines represent unstable orbits. Solid lines represent stable orbits.



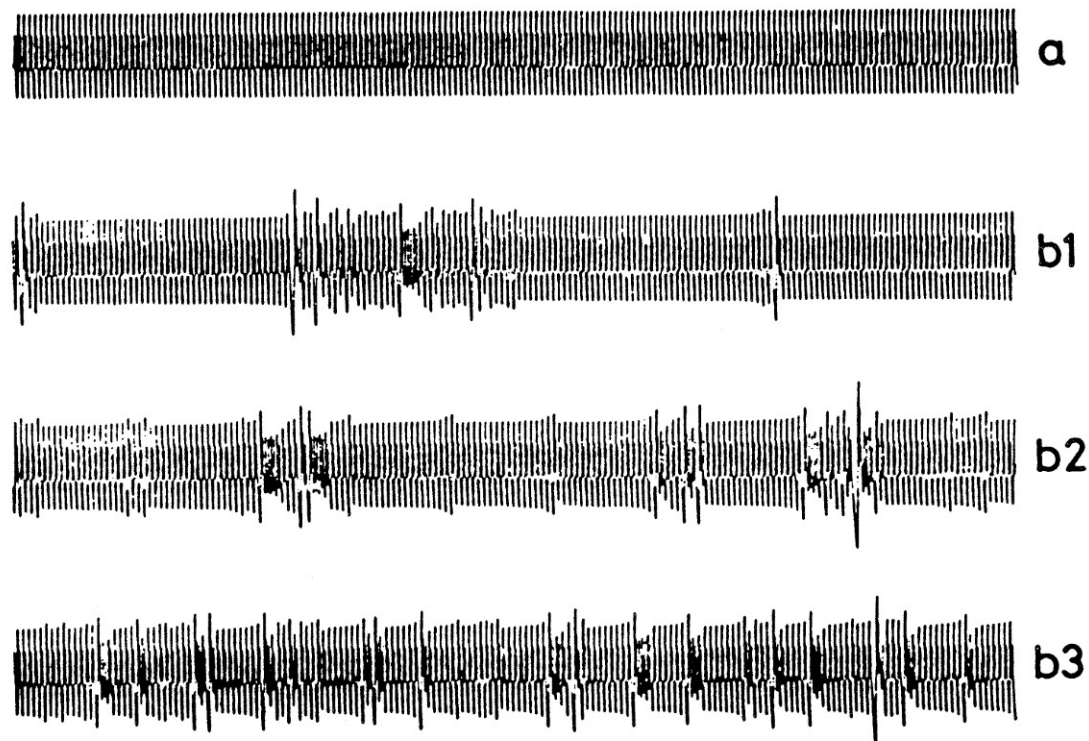
# Bifurcation Diagram for the Logistic Map





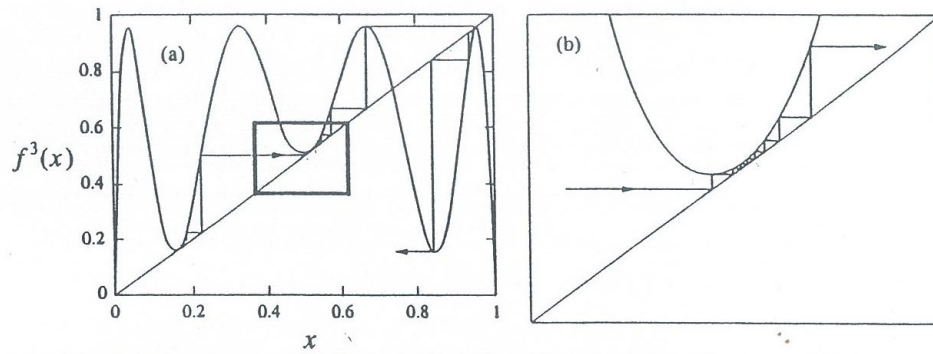
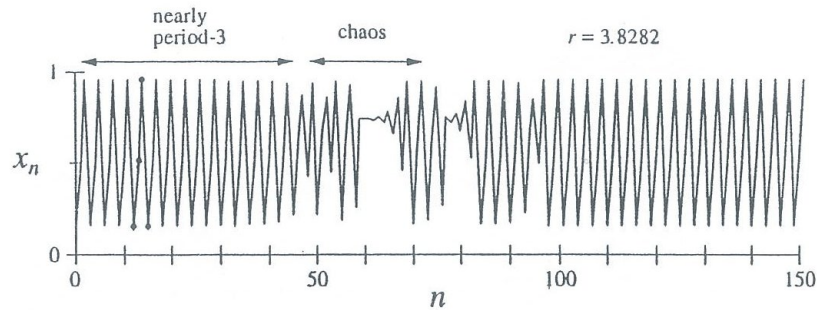
# Intermittency

In dynamical systems theory: occurrence of a signal that alternates randomly between long periods of regular behavior and relatively short irregular bursts. In other words, motion in intermittent dynamical system is nearly periodic with occasional irregular bursts.



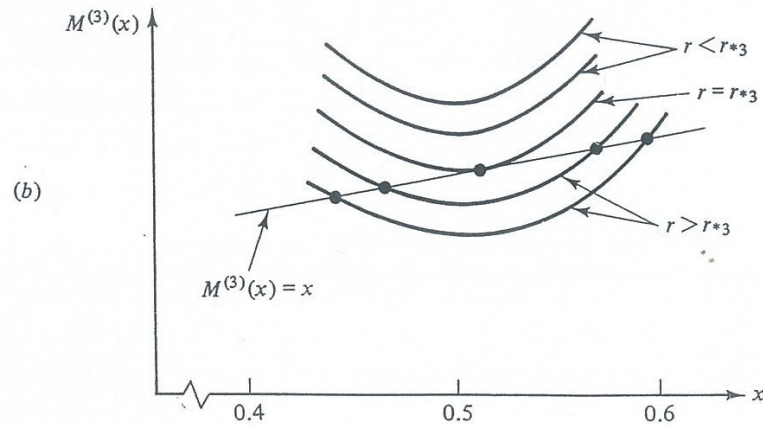
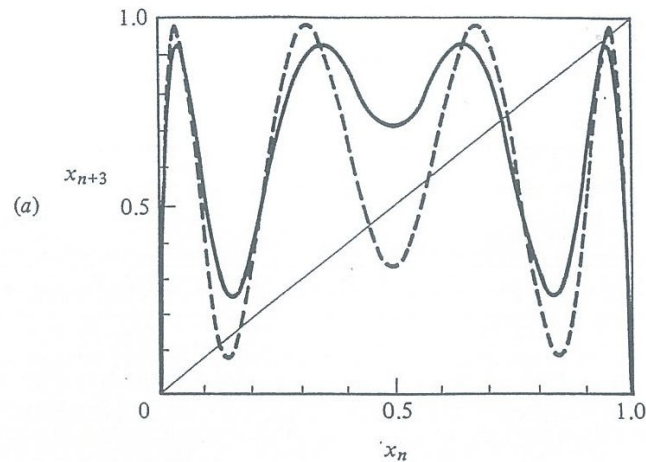
Pomeau & Manneville, 1980

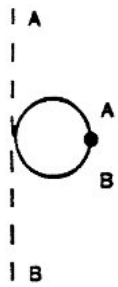


# Intermittent Behavior



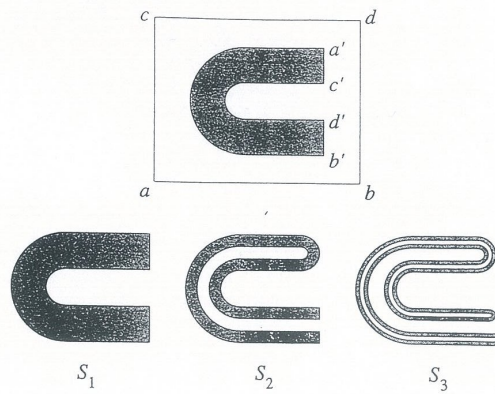
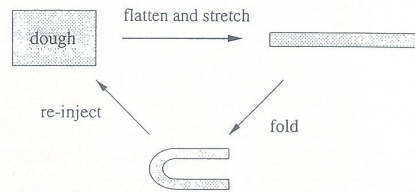
# Bifurcation and Intermittency

Figure 2.13 (a)  $x_{n+3}$  versus  $x_n$  from  $r = 3.7 < r_{*3}$  (solid curve) and for  $r = 3.9 > r_{*3}$  (dashed curve). (b) Schematic of  $M^3(x)$  versus  $x$  near  $x = 0.5$ .



	ATTRACTOR	LYAPUNOV EXPONENT SPECTRUM
C	<p>LIMIT CYCLE</p> 	(0,-,-)
D	<p>POINT</p> 	(-,-,-)
E	<p>STRANGE ATTRACTOR</p> 	(+,0,-)

# Horseshoe Map

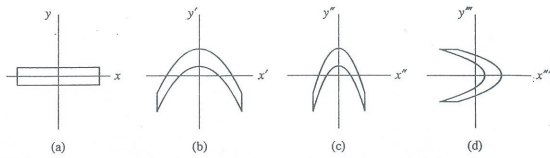


# Henon Map

The Hénon map is given by

$$x_{n+1} = y_n + 1 - ax_n^2, \quad y_{n+1} = bx_n,$$

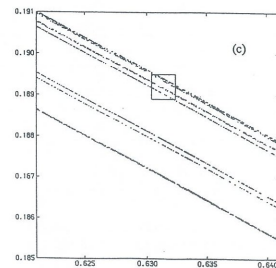
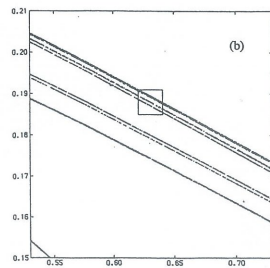
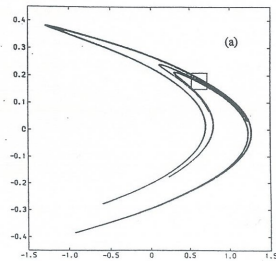
where  $a$  and  $b$  are adjustable parameters.



$$T' : x' = x, \quad y' = 1 + y - ax^2.$$

$$T'' : x'' = bx', \quad y'' = y'$$

$$T''' : x''' = y'', \quad y''' = x''.$$

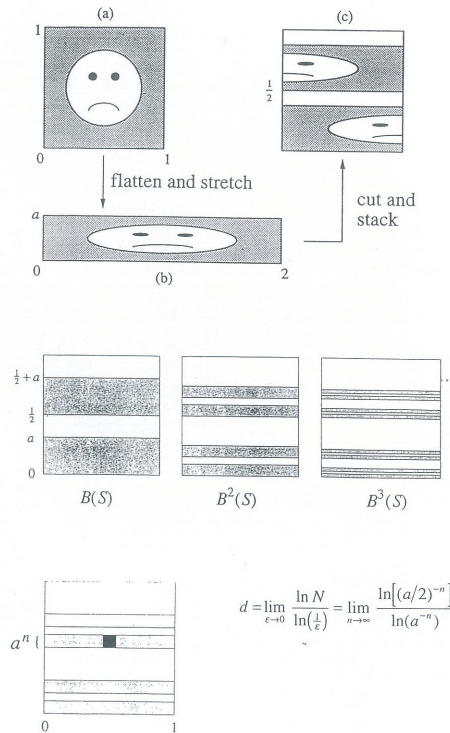


# Baker's Map

The *baker's map*  $B$  of the square  $0 \leq x \leq 1, 0 \leq y \leq 1$  to itself is given by

$$(x_{n+1}, y_{n+1}) = \begin{cases} (2x_n, ay_n) & \text{for } 0 \leq x_n < \frac{1}{2} \\ (2x_n - 1, ay_n + \frac{1}{2}) & \text{for } \frac{1}{2} \leq x_n \leq 1 \end{cases}$$

where  $a$  is a parameter in the range  $0 < a \leq \frac{1}{2}$ .



# Conclusions

- Fractal structure can describe complex shapes in the real world.
- Nonlinear systems exhibit complex phenomena, including bifurcation, intermittency, and chaos.
- Strange chaotic attractors has fractal structure and are sensitive to initial conditions.



# Bibliography

- S. H. Strogatz, Nonlinear Dynamics and Chaos, Addison-Wesley, Reading, 1994.
- E. Ott, Chaos in Dynamical Systems, Cambridge University Press, Cambridge, 1993.
- H. G. Schuster, Deterministic Chaos: An Introduction, VCH Verlagsgesellschaft, Weinheim 1988.