CARL VON OSSIETZKY universität OLDENBURG

University of Oldenburg and BSU Minsk

Introduction to Solitons

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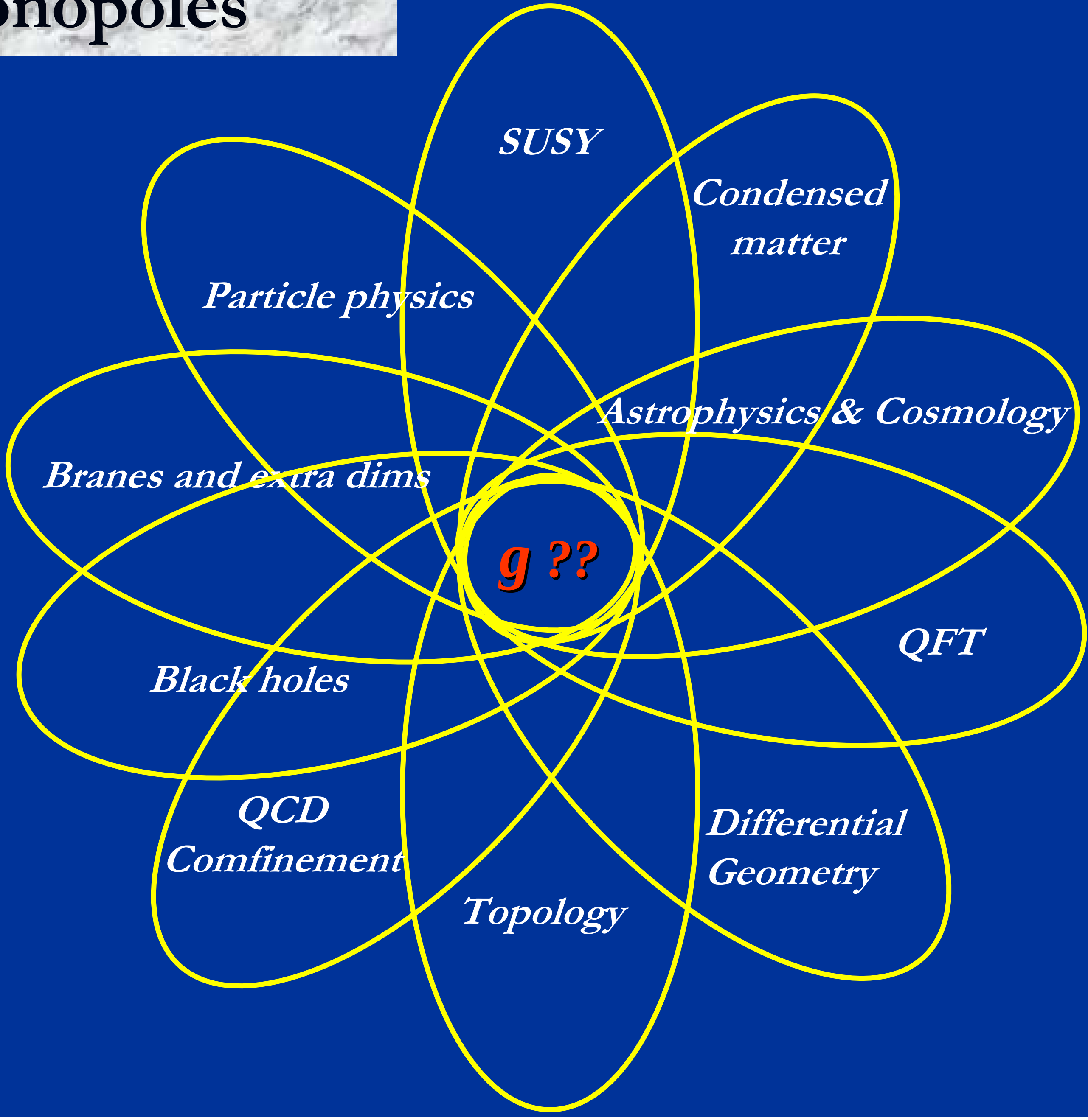
Vilnius, 2013

Monopoles


SUSY
Condensed matter
Particle physics
Astrophysics & Cosmology
Branes and extra dims
g ??
QFT
Black holes
QCD Comfinement
Differential Geometry
Topology

Electromagnetic duality and Dirac monopole

- System of generalized Maxwell equations

$$\nabla \cdot \vec{E} = 4\pi e; \qquad \nabla \cdot \vec{B} = 4\pi g;$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = j_g \qquad \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = j_e$$

is invariant with respect to the transformations of electromagnetic duality:

$$E \to E\cos\vartheta - B\cos\vartheta; \quad B \to E\sin\vartheta + B\cos\vartheta$$

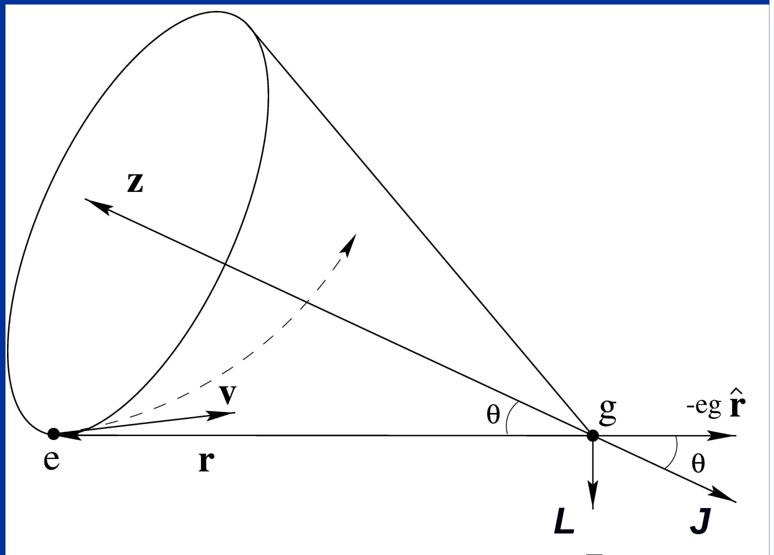
$$e \to e\cos\vartheta - g\cos\vartheta; \quad g \to e\sin\vartheta + g\cos\vartheta$$

- Classical motion in the monopole Coulomb magnetic field:

$$m\frac{d^2\vec{r}}{dt^2} = e[\vec{v} \times \vec{B}] = \frac{eg}{r^3}\left[\frac{d\vec{r}}{dt} \times \vec{r}\right]$$

- Generalized angular momentum is conserved:

$$\vec{J} = [\vec{r} \times m\vec{v}] - eg\frac{\vec{r}}{r} = \vec{L} - eg\hat{r}$$



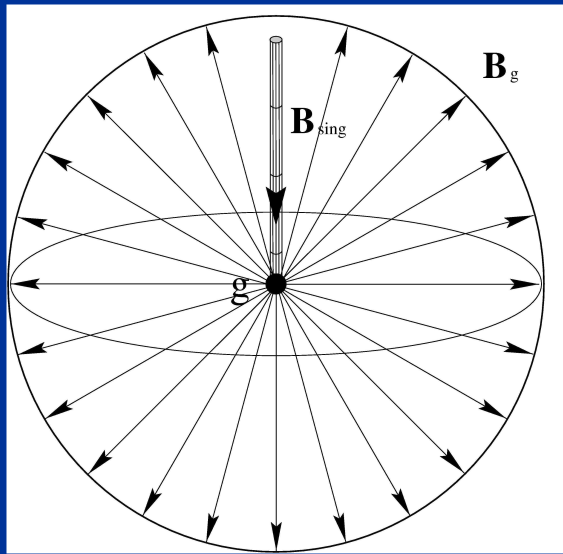
Dirac's monopole: Charge quantization

$$\vec{B} = g\frac{\vec{r}}{r^3} = \nabla \times \vec{A}; \qquad \nabla \cdot \vec{B} = 4\pi g \ ?$$

$$\vec{A} = \frac{g}{r}\frac{\vec{r}\times\vec{n}}{r-(\vec{r}\cdot\vec{n})}$$ **- Dirac monopole potential**

$$\vec{B} = \vec{B}_g + \vec{B}_{string} = g\frac{\vec{r}}{r^3} - 4\pi g\ \vec{n}\ \theta(z)\delta(x)\delta(y)$$

Gauge invariance: $\vec{A} \longrightarrow \vec{A} + \nabla\lambda(\vec{n},\vec{n}')$



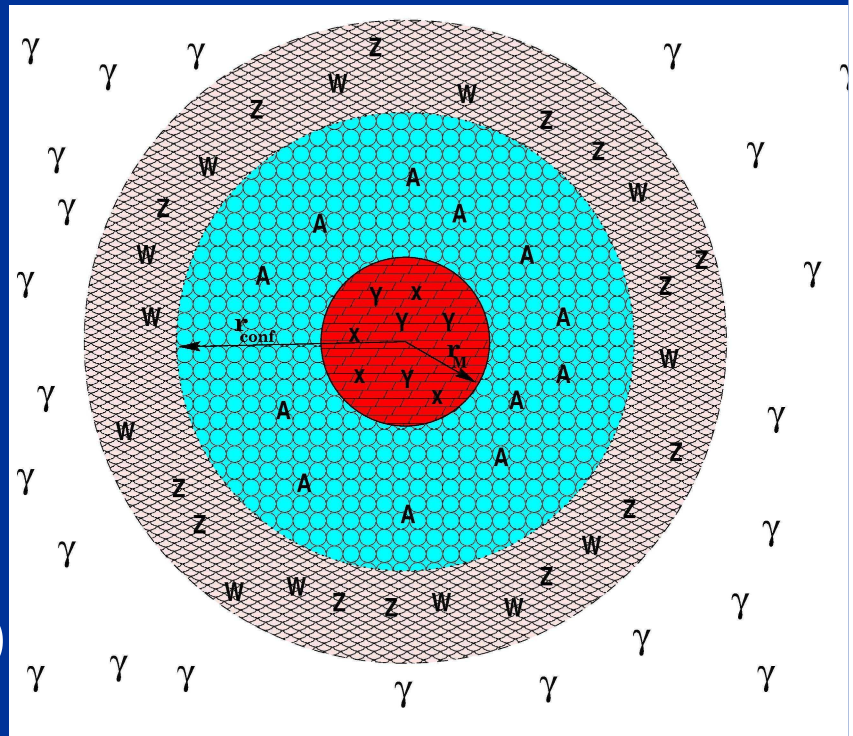
Dirac's string is invisible if the charge quantization condition is imposed:

$$eg = \frac{n}{2}$$

Non-Abelian monopoles

Break-through of 1974:

`t Hooft-Polyakov monopole solution



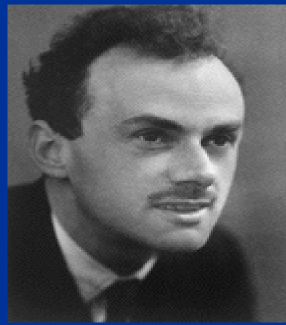
While a Dirac monopole *could be* incorporated in an Abelian theory, some non-Abelian models *inevitably contain* monopole solutions

Non-Abelian monopole is a non-linear system of coupled gauge and scalar (Higgs) fields, its energy is finite and the fields are regular everywhere in space. The gauge symmetry is spontaneously broken via Higgs mechanism

$$L = \frac{1}{2}\ \mathrm{Tr}\ F_{\mu\nu}^2 + \ \mathrm{Tr}\ (D_\mu \Phi)^2 + V(|\Phi|)$$

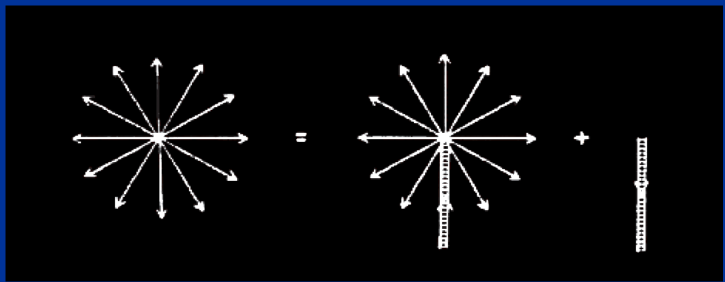
Magnetic monopoles

$$\vec{B} = g\frac{\vec{r}}{r^3}, \qquad \vec{E} = Q\frac{\vec{r}}{r^3}$$

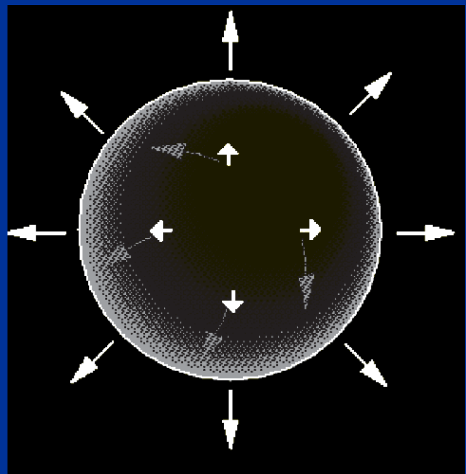


P.A.M.Dirac
1902-1984

Dirac monopole (1931)



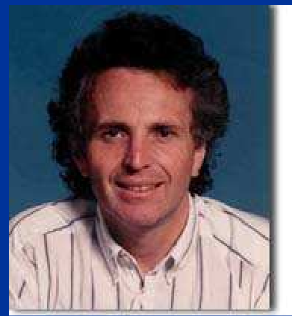
Non-Abelian monopole



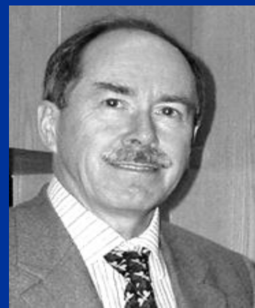
Wu-Yang monopole (1975)

$$\begin{cases} \mathbf{A}^N = g\dfrac{1-\cos\theta}{r\sin\theta}\hat{\mathbf{e}} \\ \mathbf{A}^S = -g\dfrac{1+\cos\theta}{r\sin\theta}\hat{\mathbf{e}} \end{cases}$$

- Regular static configuration
- Gauge group SU(2)
- Magnetic charge is the topological number: *Qg=n/2*
- The monopole is *very* heavy, *M~m_v/e*



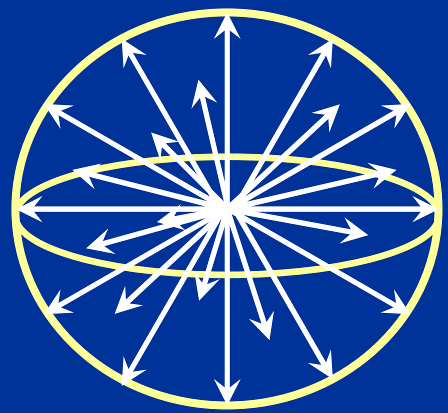
A.M.Polyakov
*1945



Gerard 't Hooft
*1946

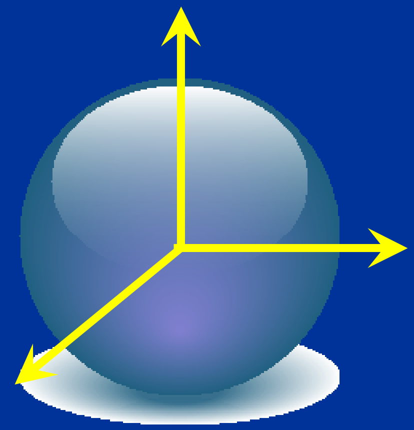
Properties of non-Abelian monopoles [SU(5)]

- Monopole has a core of radius $r_m \sim m_x^{-1} \sim 10^{-29}$ cm
- Monopole is superheavy: $M \sim m_x/\alpha \sim 10^{17}$ GeV $\sim 10^{-7}$ g
- Magnetic charge of the monopole has topological roots:



$$\vec{\Phi} \rightarrow v\vec{r}$$

$$S^2 \rightarrow S^2$$



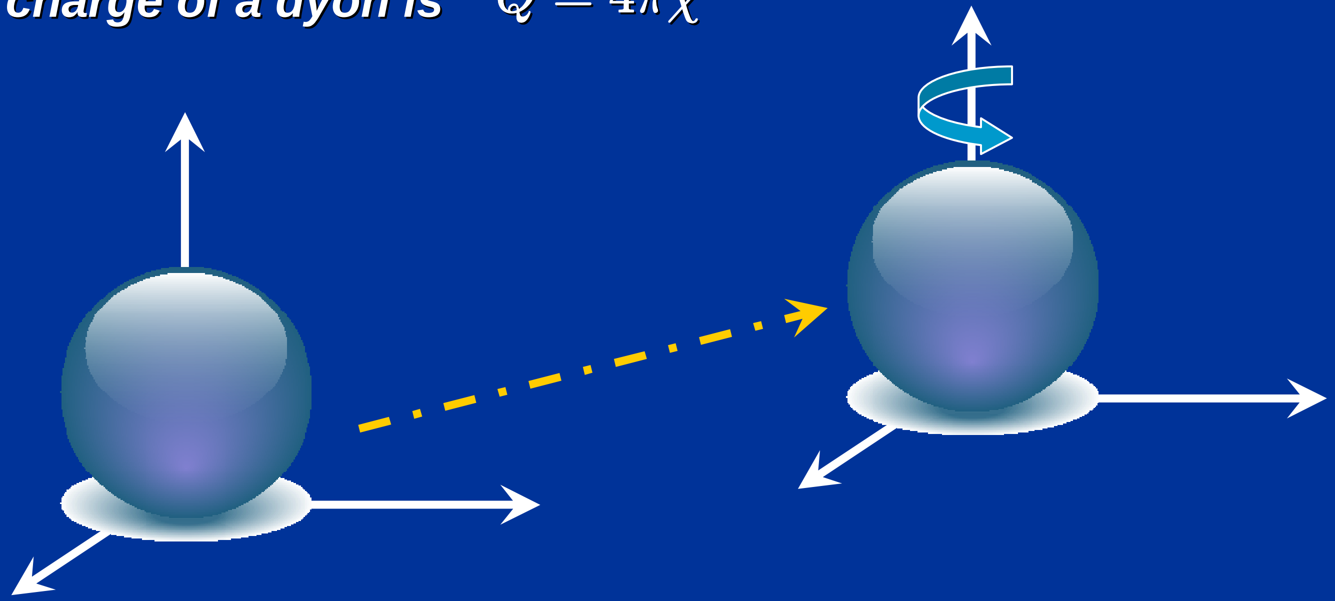
- Electromagnetic subgroup is associated with rotations about direction of the Higgs field
- Monopole solution mixes the spacial and group rotations:

$$\vec{J} = \vec{L} + \vec{T} + \vec{S}$$

Dyons

Monopole has 4 collective coordinates: R_k and $\chi(t)$

Electric charge of a dyon is $Q = 4\pi\dot{\chi}$



Charge quantization condition for a pair of dyons: $Q_1 g_2 - Q_2 g_1 = \frac{n}{2}$

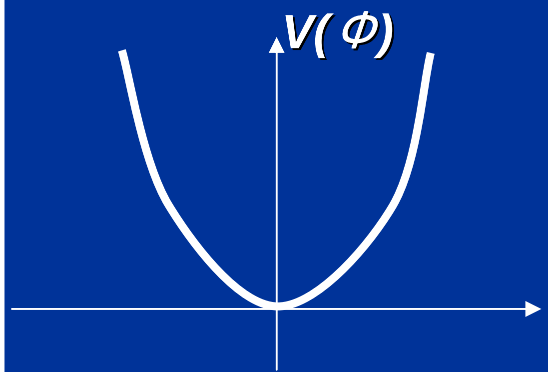
Consequence: ***Spin-statistic theorem admits both Bose-Einstein and Fermi-Dirac statistics.***

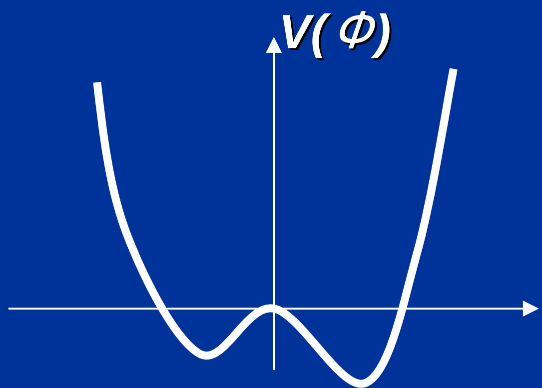
Relic Monopoles

Monopoles should have been produced in the very early Universe:

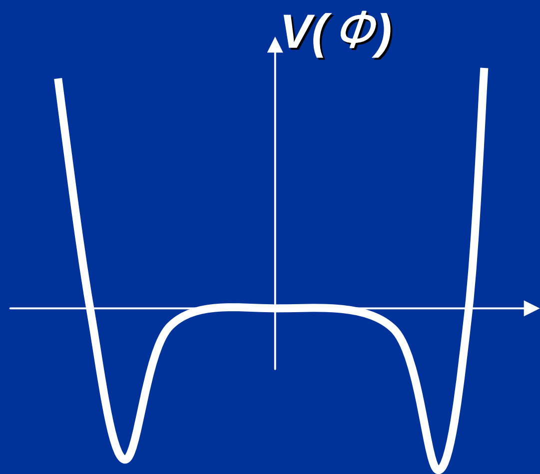
$$SU(5) \rightarrow SU(3) \times SU(2) \times U(1) \rightarrow SU(3) \times U(1)_{em}$$

As $T < T_c \sim 10^{15}$ GeV the Higgs field acquires a non-zero v.e.v.





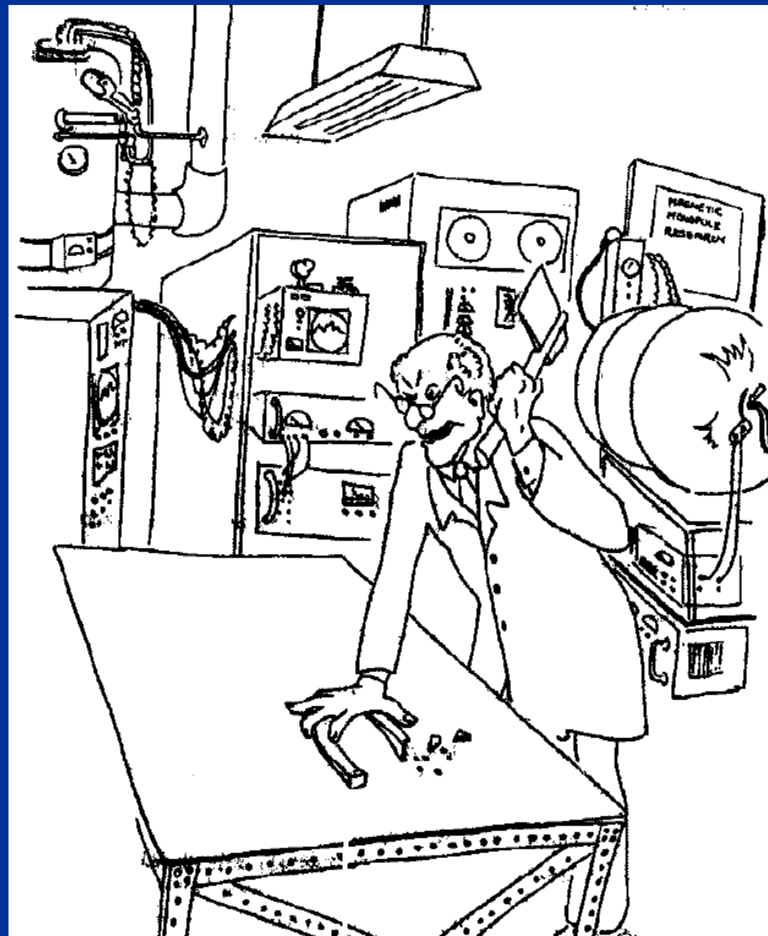
Predictions of the Big Bang scenario (adiabatic expansion)
1 monopole per 10^4 nucleons!



Inflation scenario: The potential is suffuciently flat at Φ=0, the phase transition occurs at **$T_c \sim 10^9$ GeV**
- only a few monopoles may survive the inflation!

Experimental search for monopoles

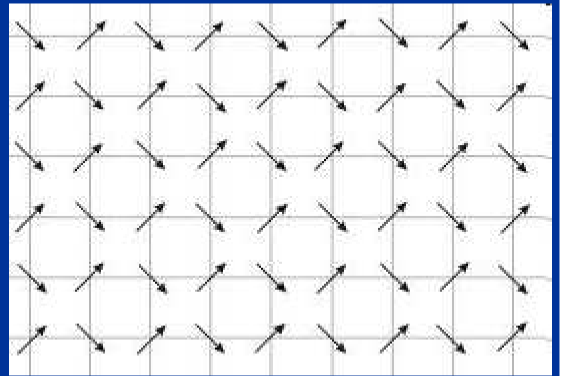
- **Accelerator search (Fermilab, CERN, DESY…)**
- **Superconducting coils**
- **Indirect limits (Parker‘s bound, neutron stars…)**
- **Search for monopole catalysis (IceCube, Berkeley, Stanford, IBM…)**
- **Monopoles in cosmic rays (Scintillators and ionization detectors)**



No monopole detected yet!

(©Picture by courtesy G.Giacomelli)

Fake monopoles

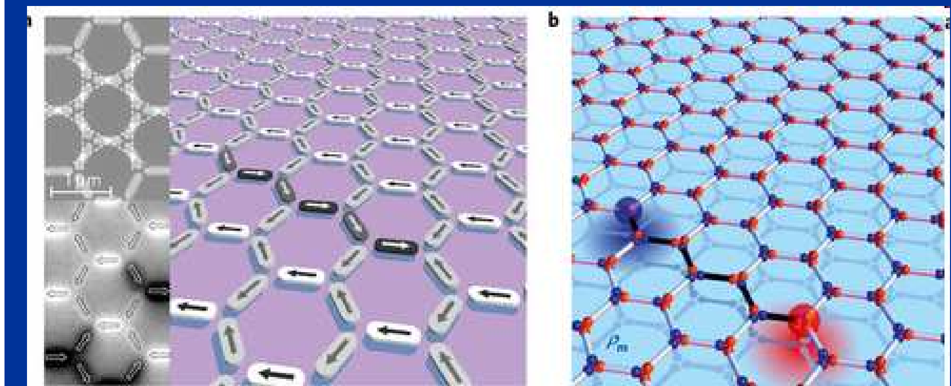


- ***„Monopoles“ in spin-ice crystal structures***

(Castelnovo, C., R. Moessner, and S. L. Sondhi, *Magnetic monopoles in spin ice,* Nature, Vol. 451, 42-45, 2008;
D.J.P. Morris et al, Dirac Strings and Magnetic Monopoles in the Spin Ice; **Science, Vol. 326, 411-414, 2009)**

$$H = J\sum_{ij} S_i S_J + \sigma \sum_{ij} \frac{3(\hat{e}_i\cdot\hat{r}_{ij})(\hat{e}_j\cdot\hat{r}_{ij}) - (e_i\cdot e_j)}{r_{ij}^3}$$

A sum of nearest-neighbor Ising model term and long range dipolar interactions



(Mengotti et al, Nature Physics , 7 (2011) 68)

Where is the cheat?

Each dipole is replaced by a pair of equal and opposite magnetic charges

Fake monopoles

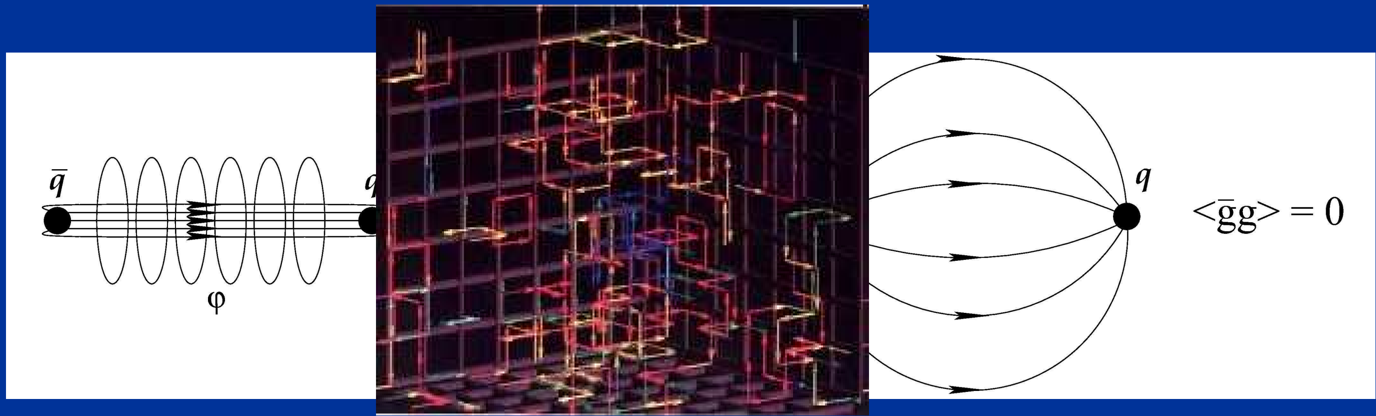
- ***„Monopoles“ and low energy QCD***

Λ_{χ} ~ 1 *GeV* **Λ_{QCD} ~ 180 *MeV***

Perturbative QCD
(Quarks & gluons)

Low energy effective theory
(Pions and quarks)

Hadrons



QCD confinement as dual Meissner effect: monopole condencation as a reason of formation of the chromoelectric flux tube and QCD is taking a form of the dual Ginzburg-Landau model ***(S.Mandelstam, G `t Hooft et al (1970s))***

Where is the cheat?

There is no monopoles in QCD!

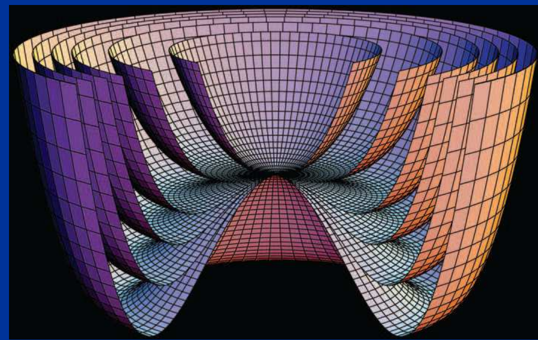
Here we are: Yang-Mills-Higgs Theory

$$S = \frac{1}{2}\int d^4x \{ F_{\mu\nu}F^{\mu\nu} + (D_\mu\Phi)(D^\mu\Phi) - V(\Phi)\}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu]$$

$$D_\mu\Phi = \partial_\mu\Phi + ie[A_\mu, \Phi]$$

$$V(\Phi) = \lambda\,(\Phi^2 - a^2)^2$$

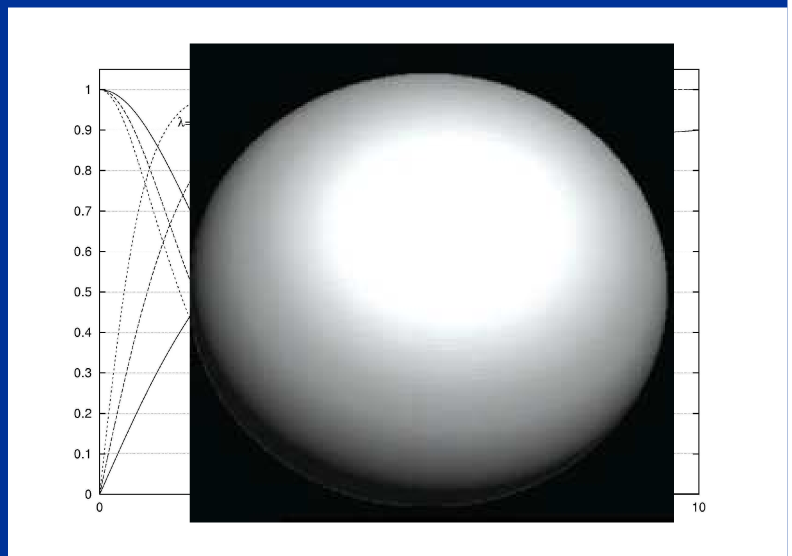


´t Hooft-Polyakov static spherically symmetric solution

$$\phi^a = \frac{r^a}{er^2}H(ear)$$

$$A_n^a = \varepsilon_{amn}\frac{r^m}{er^2}(1 - K(ear))$$

Monopole core: $R_c \sim m_V^{-1}$



BPS monopoles

Bogomolny equations: λ =0, $B_k = D_k \Phi$

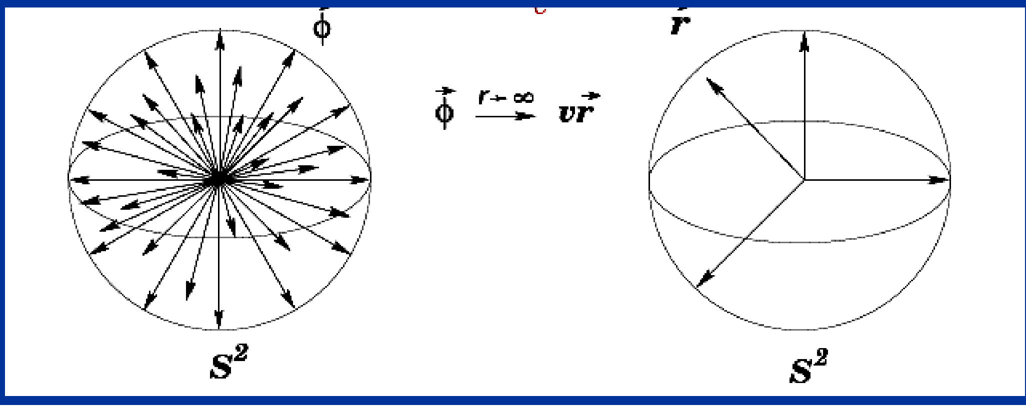
BPS monopole mass: $M = 4\pi\eta / e$ *Homotopy group* $\pi_2(S^2)$

- **Long-range scalar field** $\Phi \sim 1/r$
- **No net interaction between the BPS monopoles**
- **Analytical solution of the BPS equations:**

$$K = \frac{\xi}{\sinh \xi}; \quad H = \xi \coth \xi - 1$$

Sir M. Atiyah, R. Ward (1977),
P. Forgacs *et al* (1981),
W. Nahm (1982),
P. Sutcliffe (1996) *and other*

Magnetic charge of a monopole is
a topological number $\Phi: \; S^2 \to S^2$



Self-dual monopoles vs non-self dual monopoles

BPS monopoles:

- In the limit $\eta=0$ the energy becomes:

$$E = \mathrm{Tr} \int d^3x \left\{ \frac{1}{4}(\varepsilon_{ijk}F_{ij} \pm D_i\Phi)^2 \mp \frac{1}{2}\varepsilon_{ijk}F_{ij}D_k\Phi \right\}$$

- The first order Bogomol'nyi equations $B_k = \pm D_k\Phi$ yield absolute minimum:

$$M = 4\pi g$$

- No net interaction between the BPS monopoles: the electromagnetic repulsion is compensated by the long-range scalar interaction.

Non self-dual monopoles:

- They are solutions of the second order Yang-Mills equations: $\partial_\mu F_{\mu\nu} = 0$
- $E > M_{BPS}$ even if g=0 (deformations of the topologically trivial sector)
- The constituents are non BPS monopoles and/or vortices in a static equilibrium; separation is relatively small, there are no long-range forces

Self-dual monopoles vs non-self dual monopoles

BPS monopoles:

- Integrability of the BPS equations: there is a correspondence to the reduced self-duality equations of the Yang-Mills theory:

$$D_k\Phi^a \rightleftharpoons D_k A_0^a \equiv F_{0k}^a \qquad B_k^a \rightleftharpoons \frac{1}{2}\varepsilon_{kmn}F_{mn} \equiv \widetilde{F}_{0k}^a \equiv F_{0k}^a$$

- Properties of the BPS monopole are completely defined by the Higgs field:

$$D_k\Phi^a D_k\Phi^a = \frac{1}{2}\partial_k\partial_k|\Phi|^2 \qquad E = \frac{1}{2}\int d^3x\partial_k\partial_k|\Phi|^2 = 4\pi g$$

- An infinite chain of YM instantons along Euclidean time axis

$$A_k^a = \varepsilon_{akn}\partial_n \ln\rho + \delta_{ak}\partial_0 \ln\rho; \qquad A_0^a = -\partial_a \ln\rho \quad \text{where} \quad \rho = \sum_{n=-\infty}^{\infty}\frac{1}{r^2+(\tau-2\pi n)^2}$$

is equal to the BPS monopole $(z = r + i\tau)$:

$$\rho = \frac{1}{2r}\left\{\sum_{n=-\infty}^{n=\infty}\frac{1}{z-i\omega_n} + \sum_{n=-\infty}^{n=\infty}\frac{1}{z^*+i\omega_n}\right\} = \frac{1}{2r}\left\{\coth\frac{z}{2} + \coth\frac{z^*}{2}\right\} = \frac{1}{2r}\frac{\sinh r}{\cosh r - \cos\tau}.$$

Rational map monopoles

There is a transformation of a monopole into a rational map from the Riemanian sphere to inself: *R: $S^2 \mapsto S^2$* (P. Sutcliffe, N.Manton et al)

$$R(z) = \frac{a(z)}{b(z)} = \frac{a_1 z^{n-1} + \cdots + a_n}{z^n + b_1 z^{n-1} + \cdots + b_n}, \qquad z = x_1 + i x_2$$

Construction of the rational maps monopoles:

- **Represent BPS equation in spherical coordinates** $r, z, \bar{z}$
- **Impose a complex gauge** $\Phi = -iA_r = \frac{i}{2} U^{-1} \partial_r U, \quad A_z = U^{-1} \partial_z U, \quad A_{\bar{z}} = 0$
- **Construct the monopoles using**

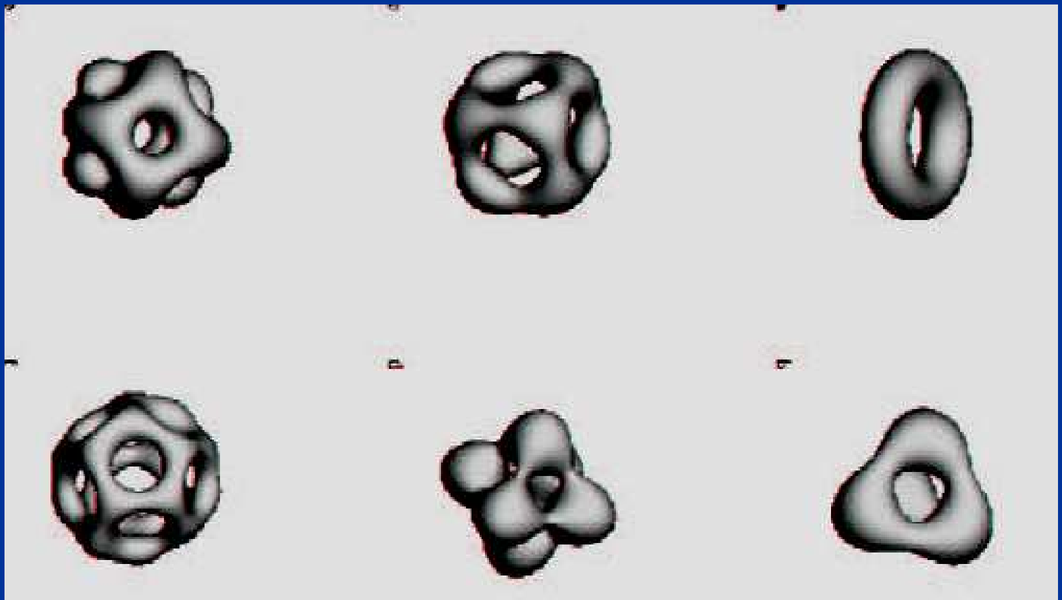
$$U \sim \exp\left\{ \frac{2r}{1+|R|^2} \begin{pmatrix} |R|^2 - 1 & 2\bar{R} \\ 2R & 1 - |R|^2 \end{pmatrix} \right\}$$

R = 1/z: **one spherically symmetric monopole centered at the origin;**

$$R(z) = \frac{a_1 z + a_2}{z^2 + b_1 z + b_2}$$ **: two monopoles**

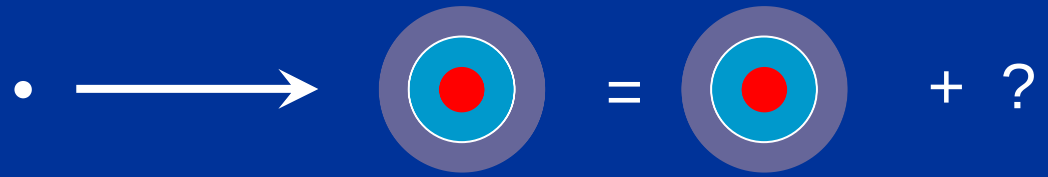
$$R(z) = \frac{i\sqrt{3}z^2 - 1}{z(z^2 - i\sqrt{3})}$$ **: Tetrahedral monopoles (degree 3 map)**

$$R(z) = \frac{z^4 + 2i\sqrt{3}z^2 + 1}{z^4 - 2iz^2\sqrt{3} + 1}$$ **: Octahedral monopoles (degree 4 map)**



Monopole catalysis of proton decay

Naive question: ***What happened when a fermion collides with a monopole?***



There are zero-energy solutions of the Dirac equation for a massless fermion coupled to a monopole;

$$\gamma^\mu(\partial_\mu + eA_\mu)\psi = 0; \qquad \vec{J} = \vec{L} + \vec{T} + \vec{S}; \qquad \vec{L} = 0, \qquad \vec{T} + \vec{S} = 0$$

The ground state of a monopole becomes two-fold degenerated:

(i) $|\Omega\rangle$ **(no fermions;** $Q_F = -1/2$**)** **(ii)** $a^\dagger|\Omega\rangle$ **(zero mode);** $Q_F = 1/2$**)**

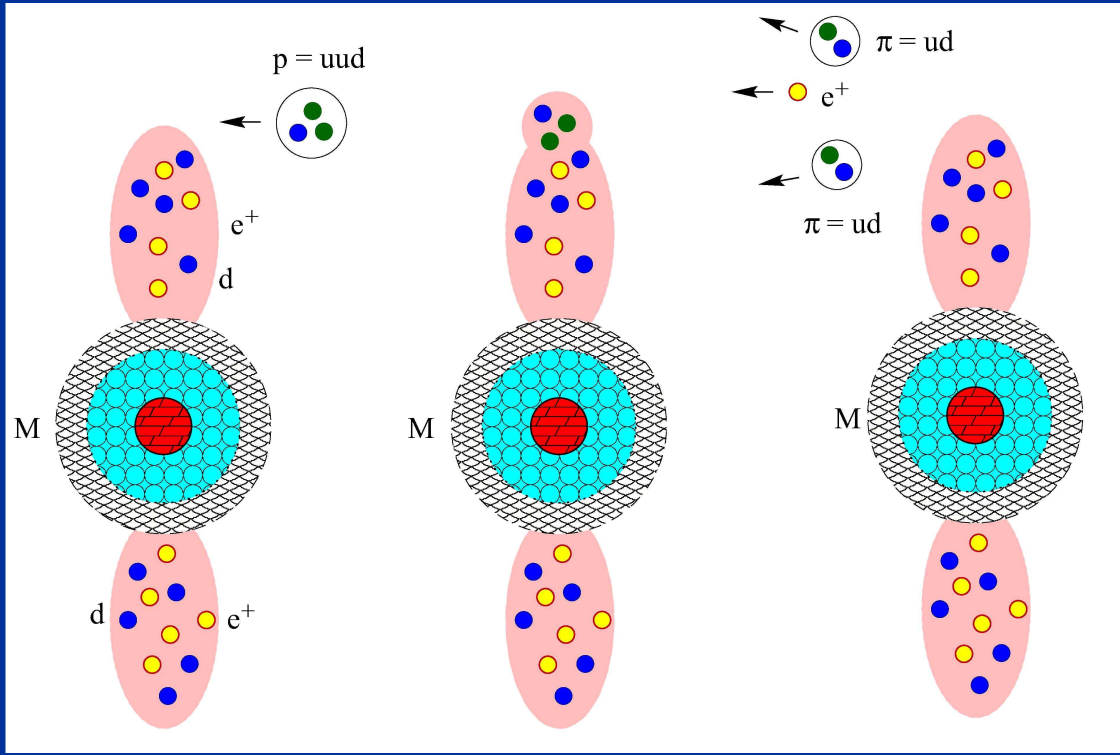
Spin-flip? Charge conjugation? Chirality?

There are non-suppressed fermionic condensates on the monopole background:

$$\langle (e^+e^- - d^3\bar{d}^3)(\bar{u}^1u^1 - \bar{u}^2u^2)\rangle \sim r^{-6}$$

Rubakov-Callan effect

SU(5) model with massless fermions in s-wave

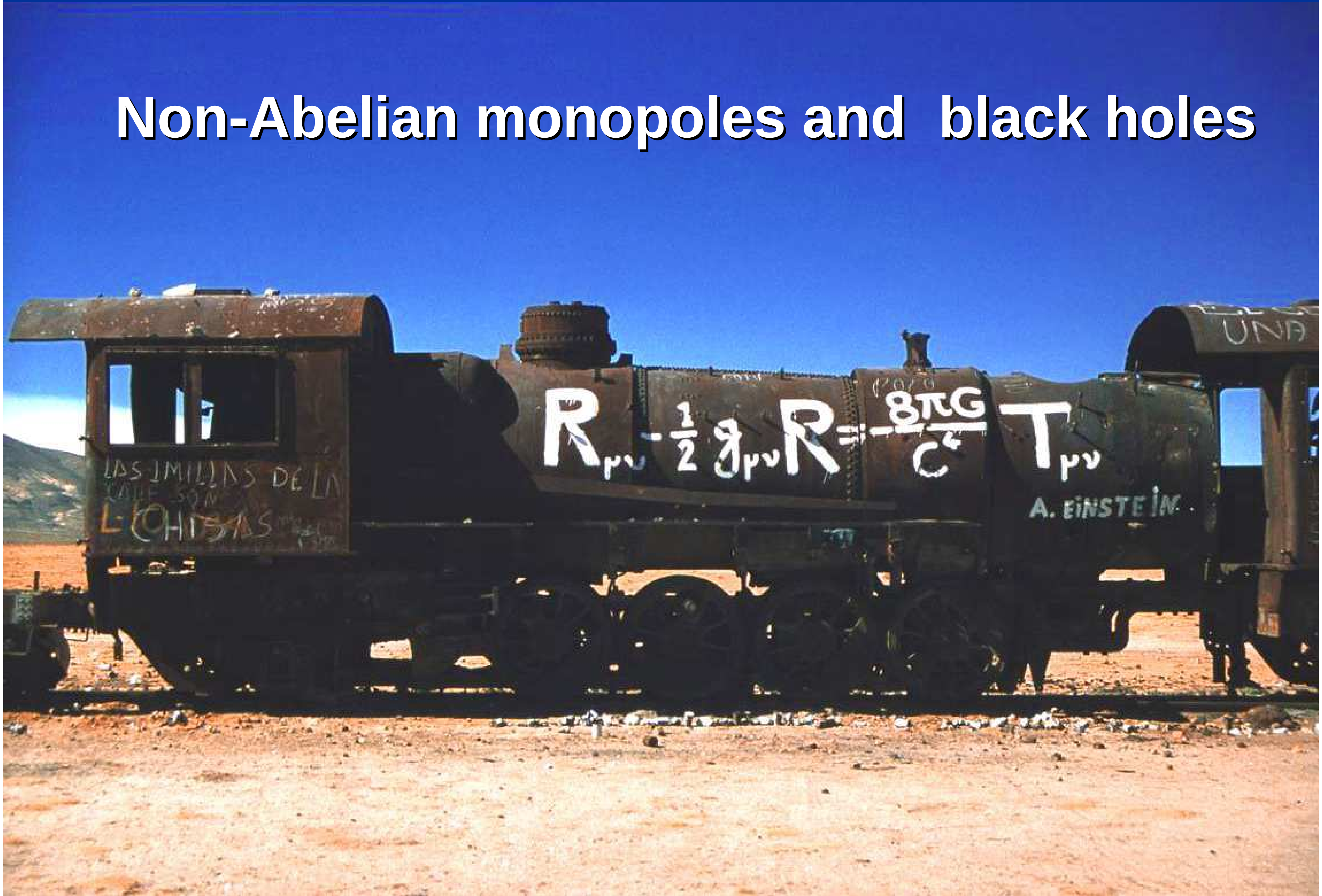


Monopole could catalyse baryon number violating processes like

P + monopole $\rightarrow$ e$^+$ + mesons + monopole

Caution: **the effect is model-dependent!**

Non-Abelian monopoles and black holes



Gravitating monopoles

$$\frac{1}{\sqrt{-g}} D_\mu(\sqrt{-g} F^{\mu\nu}) - \frac{1}{4} ie[\Phi, D^\nu \Phi] = 0 \;;$$

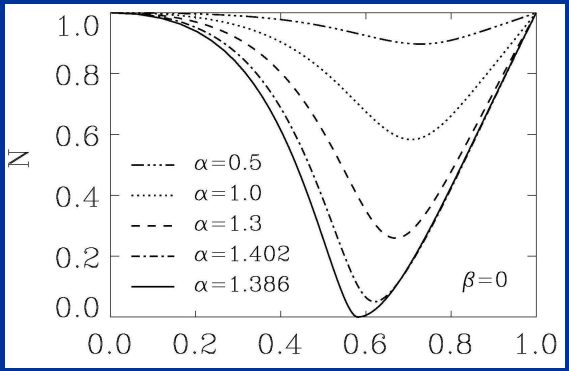
$$\frac{1}{\sqrt{-g}} D_\mu(\sqrt{-g} D^\mu \Phi) + \lambda(\Phi^2 - a^2)\Phi = 0 \;.$$

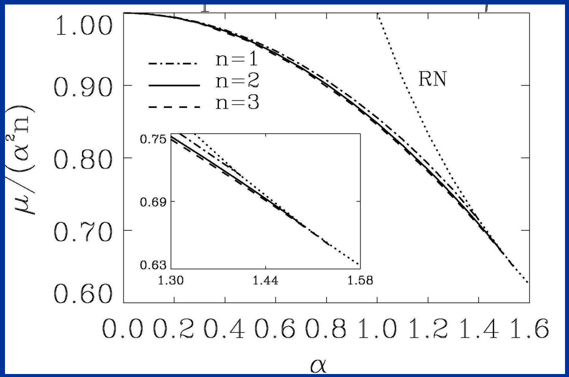
(Cho, Freund (1975), van Nieuwenhuizen et al (1976), Breitenlohner, Forgacs, Maison (1992), Lee, Nair, Weinberg (1992), Hartmann, Kleihaus Kunz, Shnir…)

Dimensionless parameters of the model: $\alpha^2 = 4\pi^2\, G\eta^2,\; \beta^2 = e^2/\eta$

Monopole core $R_c \sim m_v^{-1} = (e\eta)^{-1}$ vs Schwarzschild radius $R_{Sch} = 2MG$;

$R_c \sim R_{Sch}$ as $\eta \sim M_{Pl} = G^{-1/2}$





Hairy black holes with axial symmetry are linked to monopoles

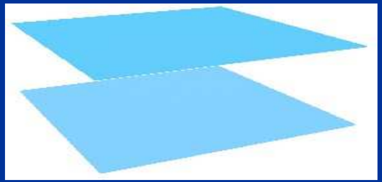
Global monopoles (monopole as big as Universe)

$$L = \frac{1}{2}(\partial_\mu \Phi^a)^2 - V(|\Phi|); \qquad SO(3) \rightarrow O(2)$$

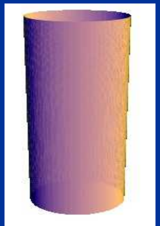
There is no vector (gauge) field - but gravity may be coupled to this system instead

Topological defects & extra dimensions:

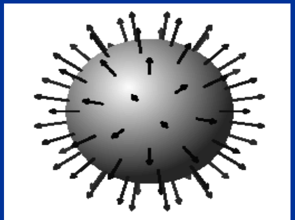
Our D=4 world: **the internal space of a topological defect living in a higher dimensional space-time**



d=5: **domain wall (kink in d=1 + D=4)**



d=6: **vortex (in d=2) + D=4**



hedgehog

d=7: **monopole (in d=3) + D=4**

d=8: **instanton (in d=4) + D=4**

Acknowledgments

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Other thanks to:

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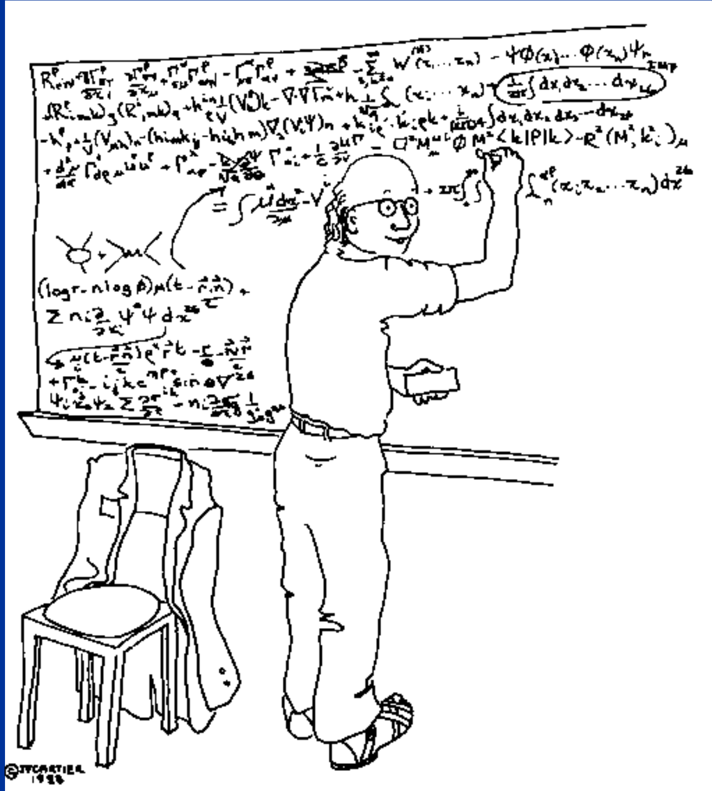
- P. Forgacs
- T. Ioannidou
- P.M.Sutcliffe
- W.Zakrzewski
- M.Volkov



"At this point we notice that this equation is beautifully simplified if we assume that space–time has 92 dimensions."

Work in progress!