Introduction to Stacks and Moduli

Lecture notes for Math 582C, *working draft*

University of Washington, Winter 2021

Jarod Alper

jarod@uw.edu

January 20, 2021

Abstract

These notes provide the foundations of moduli theory in algebraic geometry using the language of algebraic stacks with the goal of providing a self-contained proof of the following theorem:

Theorem A. *The moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus $g \geq 2$ is a smooth, proper and irreducible Deligne–Mumford stack of dimension $3g-3$ which admits a projective coarse moduli space.*[1]

Along the way we develop the foundations of algebraic spaces and stacks, and we hope to convey that this provides a convenient language to establish geometric properties of moduli spaces. Introducing these foundations requires developing several themes at the same time including:

- using the functorial and groupoid perspective in algebraic geometry: we will introduce the new algebro-geometric structures of algebraic spaces and stacks;
- replacing the Zariski topology on a scheme with the étale topology: we will generalize the concept of a topological space to Grothendieck topologies and systematically using descent theory for étale morphisms; and
- relying on several advanced topics not seen in a first algebraic geometry course: properties of flat, étale and smooth morphisms of schemes, algebraic groups and their actions, deformation theory, Artin approximation, existence of Hilbert schemes, and some deep results in birational geometry of surfaces.

Choosing a linear order in presenting the foundations is no easy task. We attempt to mitigate this challenge by relegating much of the background to appendices. We keep the main body of the notes always focused entirely on developing moduli theory with the above goal in mind.

[1] In a future course, I hope to establish an analogous result for the moduli of vector bundles: The moduli space $\mathcal{M}^{\mathrm{ss}}_{C,r,d}$ of semistable vector bundles of rank r and degree d over a smooth, connected and projective curve C of genus g is a smooth, universally closed and irreducible algebraic stack of dimension $r^2(g-1)$ which admits a projective good moduli space.

Contents

Chapter 0

Introduction and motivation

A *moduli space* is a space M (e.g. topological space, complex manifold or algebraic variety) where there is a natural one-to-one correspondence between points of M and isomorphism classes of certain types of algebro-geometric objects (e.g. smooth curves or vector bundles on a fixed curve). While any space M is the moduli space parameterizing points of M, it is much more interesting when alternative descriptions can be provided. For instance, projective space $\mathbb{P}^1$ can be described as the set of points in $\mathbb{P}^1$ (not so interesting) or as the set of lines in the plane passing through the origin (more interesting).

Moduli spaces arise as an attempt to answer one of the most fundamental problems in mathematics, namely the classification problem. In algebraic geometry, we may wish to classify all projective varieties, all vector bundles on a fixed variety or any number of other structures. The moduli space itself is the solution to the classification problem.

Depending on what objects are being parameterized, the moduli space could be discrete or continuous, or a combination of the two. For instance, the moduli space parameterizing line bundles on $\mathbb{P}^1$ is the discrete set $\mathbb{Z}$: every line bundle on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}(n)$ for a unique integer $n \in \mathbb{Z}$. On the other hand, the moduli space parameterizing quadric plane curve $C \subset \mathbb{P}^2$ is the connected space $\mathbb{P}^5$: a plane curve defined by $a_0x^2 + a_1xy + a_2xz + a_3y^2 + a_4yz + a_5z^2$ is uniquely determined by the point $[a_0, \dots, a_5] \in \mathbb{P}^5$, and as a plane curve varies continuously (i.e. by varying the coefficients a_i), the corresponding point in $\mathbb{P}^5$ does too.

The moduli space parameterizing smooth projective abstract curves has both a discrete and continuous component. While the genus of a smooth curve is a discrete invariant, smooth curves of a fixed genus vary continuously. For instance, varying the coefficients of a homogeneous degree d polynomial in x, y, z describes a continuous family of mostly non-isomorphic curves of genus $(d-1)(d-2)/2$. After fixing the genus g, the moduli space M_g parameterizing genus g curves is a connected (even irreducible) variety of dimension $3g-3$, a deep fact providing the underlying motivation of these notes. Similarly, the moduli space of vector bundles on a fixed curve has a discrete component corresponding to the rank r and degree d of the vector bundle, and it turns out that after fixing these invariants, the moduli space is also irreducible.

An inspiring feature of moduli spaces and one reason they garner so much attention is that their properties inform us about the properties of the objects themselves that are being classified. For instance, knowing that M_g is unirational (i.e. there is a dominant rational map $\mathbb{P}^N \dashrightarrow M_g$) for a given genus g tells us that a general genus g curve can be written down explicitly in a similar way to how a general genus 3 curve can be expressed as the solution set to a plane quartic whose coefficients are general complex numbers.

Before we can get started discussing the geometry of moduli spaces such as M_g, we need to ask: why do they even exist? We develop the foundations of moduli theory with this single question in mind. Our goal is to establish the truly spectacular result that there is a projective variety whose points are in natural one-to-one correspondence with isomorphism classes of curves (or vector bundles on a fixed curve). In this chapter, we motivate our approach for constructing projective moduli spaces through the language of algebraic stacks.

0.1 Moduli sets

A *moduli set* is a set where elements correspond to isomorphism classes of certain types of algebraic, geometric or topological objects. To be more explicit, defining a moduli set entails specifying two things:

1. a class of certain types of objects, and
2. an equivalence relation on objects.

The word 'moduli' indicates that we are viewing an element of the set of as an equivalence class of certain objects. In the same vein, we will discuss *moduli groupoids*, *moduli varieties/schemes* and *moduli stacks* in the forthcoming sections. Meanwhile, the word 'object' here is intentionally vague as the possibilities are quite broad: one may wish to discuss the moduli of really any type of mathematical structure, e.g. complex structures on a fixed space, flat connections, quiver representations, solutions to PDEs, or instantons. In these notes, we will entirely focus our study on moduli problems appearing in algebraic geometry although many of the ideas we present extend similarly to other branches of mathematics.

The two central examples in these notes are the moduli of curves and the moduli of vector bundles on a fixed curve—two of the most famous and studied moduli spaces in algebraic geometry. While there are simpler examples such as projective space and the Grassmanian that we will study first, the moduli spaces of curves and vector bundles are both complicated enough to reveal many general phenomena of moduli and simple enough that we can provide a self-contained exposition. Certainly, before you hope to study moduli of higher dimensional varieties or moduli of complexes on a surface, you better have mastered these examples.

0.1.1 Moduli of curves

Here's our first attempt at defining M_g:

Example 0.1.1 (Moduli set of smooth curves)**.** The *moduli set of smooth curves*, denoted as M_g, is defined as followed: the objects are smooth, connected and projective curves of genus g over $\mathbb{C}$ and the equivalence relation is given by isomorphism.

There are alternative descriptions. We could take the objects to be complex structures on a fixed oriented compact surface Σ of genus g and the equivalence relation to be biholomorphism. Or we could take the objects to be pairs (X, ϕ)

where X is a hyberbolic surface and $\phi\colon \Sigma \to X$ is a diffeomorphism (the set of such pairs is the Teichmüller space) and the equivalence relation is isotopy (induced from the action of the mapping class group of Σ).

Each description hints at different additional structures that M_g should inherit.

There are many related examples parameterizing curves with additional structures as well as different choices for the equivalences relations.

Example 0.1.2 (Moduli set of plane curves)**.** The objects here are degree d plane curves $C \subset \mathbb{P}^2$ but there are several choices for how we could define two plane curves C and C' to be equivalent:

(1) C and C' are equal as subschemes;

(2) C and C' are projectively equivalent (i.e. there is an automorphism of $\mathbb{P}^2$ taking C to C'); or

(3) C and C' are abstractly isomorphic.

The three equivalence relations define three different moduli sets. The moduli set (1) is naturally bijective to the projectivization $\mathbb{P}(\operatorname{Sym}^d \mathbb{C}^3)$ of the space of degree d homogeneous polynomials in x, y, z while the moduli set (2) is naturally bijective to the quotient set $\mathbb{P}(\operatorname{Sym}^d \mathbb{C}^3)/\operatorname{Aut}(\mathbb{P}^2)$. The moduli set (3) is the subset of the moduli set of (possibly singular) abstract curves which admit planar embeddings.

Example 0.1.3 (Moduli set of curves with level n structure)**.** The objects are smooth, connected and projective curves C of genus g over $\mathbb{C}$ together with a basis $(\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g)$ of $H_1(C, \mathbb{Z}/n\mathbb{Z})$ such that the intersection pairing is symplectic. We say that $(C, \alpha_i, \beta_i) \sim (C', \alpha'_i, \beta'_i)$ if there is an isomorphism $C \to C'$ taking α_i and β_i to α'_i and β'_i.

A rational function f/g on a curve C defines a map $C \to \mathbb{P}^1$ given by $x \mapsto [f(x), g(x)]$. Visualizing a curve as a cover of $\mathbb{P}^1$ is extremely instructive providing a handle to its geometry. Likewise it is instructive to consider the moduli of such covers.

Example 0.1.4 (Moduli of branched covers)**.** We define the *Hurwitz moduli set* $\operatorname{Hur}_{d,g}$ where an object is a smooth, connected and projective curve of genus g together with a finite morphisms $f\colon C \to \mathbb{P}^1$ of degree d, and we declare $(C \xrightarrow{f} \mathbb{P}^1) \sim (C' \xrightarrow{f'} \mathbb{P}^1)$ if there is an isomorphism $\alpha\colon C \to C'$ over $\mathbb{P}^1$ (i.e. $f' = f \circ \alpha$). By Riemann–Hurwitz, any such map $C \to \mathbb{P}^1$ has $2d + 2g - 2$ branch points. Conversely, given a general collection of $2d + 2g - 2$ points of $\mathbb{P}^1$, there exists a genus g curve C and a map $C \to \mathbb{P}^1$ branched over precisely these points. In fact there are only finitely many such covers $C \to \mathbb{P}^1$ as any cover is uniquely determined by the ramification type over the branched points and the finite number of permutations specifying how the unramified covering over the complement of the branched locus is obtained by gluing trivial coverings. In other words, the map $\operatorname{Hur}_{d,g} \to \operatorname{Sym}^{2d+2g-2} \mathbb{P}^1$, assigning a cover to its branched points, has dense image and finite fibers.

Likewise, for a fixed curve C, we could consider the moduli set $\operatorname{Hur}_{d,C}$ parameterizing degree d covers $C \to \mathbb{P}^1$ where the equivalence relation is equality. There is a map $\operatorname{Hur}_{d,g} \to M_g$ defined by $(C \to \mathbb{P}^1) \mapsto C$, and the fiber over a curve C is precisely $\operatorname{Hur}_{d,C}$. Equivalently, $\operatorname{Hur}_{d,C}$ can be described as parameterizing line bundles L on C together with linearly independent sections s_1, s_2 where $(L, s_1, s_2) \sim (L', s'_1, s'_2)$ if there exists an isomorphism $\alpha\colon L \to L'$ such that $s'_i = \alpha(s_i)$.

Application: number of moduli of M_g

Even before we attempt to give M_g the structure of a variety so that in particular its dimension makes sense, for $g \geq 2$ we can use a parameter count to determine the *number of moduli of* M_g or in modern terminology the *dimension of the local deformation spaces*. Historically Riemann computed the number of moduli in the mid 19th century (in fact using several different methods) well before it was known that M_g is a variety. Following [Rie57], the main idea is to compute the number of moduli of $\mathrm{Hur}_{d,g}$ in two different ways using the diagram

$$\begin{tikzcd} & \mathrm{Hur}_{d,C} \arrow[r, hook] \arrow[dl] & \mathrm{Hur}_{d,g} \arrow[dl] \arrow[drr, "\text{finite fibers}"] & & \\ \{C\} \arrow[r, hook] & M_g & & & \mathrm{Sym}^{2d+2g-2}\mathbb{P}^1 \end{tikzcd} \tag{0.1.1}$$

We first compute the number of moduli of $\mathrm{Hur}_{d,C}$ and we might as well assume that d is sufficiently large (or explicitly $d > 2g$). For a fixed curve C, a degree d map $f\colon C \to \mathbb{P}^1$ is determined by an effective divisor $D := f^{-1}(0) = \sum_i p_i \in \mathrm{Sym}^d C$ and a section $t \in H^0(C, \mathcal{O}(D))$ (so that $f(p) = [s(p), t(p)]$ where $s \in \Gamma(C, \mathcal{O}(D))$ defines D). Using that $H^1(C, \mathcal{O}(D)) = H^0(C, \mathcal{O}(K_C - D)) = 0$, Riemann–Roch implies that $h^0(\mathcal{O}(D)) = d - g + 1$. Thus the number of moduli of $\mathrm{Hur}_{d,C}$ is the sum of the number of parameters determining D and the section t

$$\#\text{ of moduli of } \mathrm{Hur}_{d,C} = d + (d - g + 1) = 2d - g + 1.$$

Using (0.1.1), we compute that

$$\begin{aligned} \#\text{ of moduli of } M_g &= \#\text{ of moduli of } \mathrm{Hur}_{d,g} - \#\text{ of moduli of } \mathrm{Hur}_{d,C} \\ &= \#\text{ of moduli of } \mathrm{Sym}^{2d+2g-2}\mathbb{P}^1 - \#\text{ of moduli of } \mathrm{Hur}_{d,C} \\ &= (2d + 2g - 2) - (2d - g + 1) \\ &= 3g - 3. \end{aligned}$$

One goal of these notes is to put this calculation on a more solid footing. The interested reader may wish to consult [GH78, pg. 255-257] or [Mir95, pg. 211-215] for further discussion on the number of moduli of M_g, or [AJP16] for a historical background of Riemann's computations.

0.1.2 Moduli of vector bundles

The moduli of vector bundles on a fixed curve provides our second primary example of a moduli set:

Example 0.1.5 (Moduli set of vector bundles on a curve)**.** Let C be a fixed smooth, connected and projective curve over $\mathbb{C}$, and fix integers $r \geq 0$ and d. The objects of interest are vector bundles E (i.e. locally free $\mathcal{O}_C$-modules of finite rank) of rank r and degree d, and the equivalence relation is isomorphism.

There are alternative descriptions. If V is a fixed C^∞-vector bundle V on C, we can take the objects to be connections on V and the equivalence relation to be gauge equivalence. Or we can take the objects to be representations $\pi_1(C) \to \mathrm{GL}_n(\mathbb{C})$ of the fundamental group $\pi_1(C)$ and declare two representations to be equivalent if they have the same dimension n and are conjugate under an element of $\mathrm{GL}_n(\mathbb{C})$. This last description uses the observation that a vector bundle induces a monodromy representation of $\pi_1(C)$ and conversely that a representation V of $\pi_1(C)$ induces a vector bundle $(\widetilde{C} \times V)/\pi_1(C)$ on C, where $\widetilde{C}$ denotes the universal cover of C.

Specializing to the rank one case is a model for the general case: the moduli set $\mathrm{Pic}^d(C)$ of line bundles on C of degree d is identified (non-canonically) with the abelian variety $H^1(C, \mathcal{O}_C)/H^1(C, \mathbb{Z})$ by means of the cohomology of the exponential exact sequence

$$\begin{array}{ccccccccc} H^1(C,\mathbb{Z}) & \longrightarrow & H^1(C,\mathcal{O}_C) & \longrightarrow & \mathrm{Pic}(C) & \longrightarrow & H^2(C,\mathbb{Z}) & \longrightarrow & 0 \\ & & & & & L \mapsto \deg(L) & & & \end{array}$$

There is a group structure on $\mathrm{Pic}^0(C)$ corresponding to the tensor product of line bundles.

Example 0.1.6 (Moduli of vector bundles on $\mathbb{P}^1$)**.** Since all vector bundles on $\mathbb{A}^1$ are trivial, a vector bundle of rank n on $\mathbb{P}^1$ is described by an element of $\mathrm{GL}_n(k[x]_x)$ specifying how trivial vector bundles on $\{x \neq 0\}$ and $\{y \neq 0\}$ are glued. We can thus describe this moduli set by taking the objects to be elements of $\mathrm{GL}_n(k[x]_x)$ where two elements g and g' are declared equivalent if there exists $\alpha \in \mathrm{GL}_n(k[x])$ and $\beta \in \mathrm{GL}_n(k[1/x])$ (i.e. automorphisms of the trivial vector bundles on $\{x \neq 0\}$ and $\{y \neq 0\}$) such that $g' = \alpha g \beta$.

The Birkhoff–Grothendieck theorem asserts that any vector bundle E on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$ for unique integers $a_1 \leq \cdots \leq a_r$.[1] This implies that the moduli set of degree d vector bundles of rank r on $\mathbb{P}^1$ is bijective to the set of increasing tuples $(a_1, \ldots, a_r) \in \mathbb{Z}^r$ of integers with $\sum_i a_i = d$. One would be mistaken though to think that the moduli space of vector bundles on $\mathbb{P}^1$ with fixed rank and degree is discrete. For instance, if $d = 0$ and $r = 2$, the group of extensions

$$\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(-1)) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}$$

is one-dimensional and the universal extension (see Example 0.4.21) is a vector bundle $\mathcal{E}$ on $\mathbb{P}^1 \times \mathbb{A}^1$ such that $\mathcal{E}|_{\mathbb{P}^1 \times \{t\}}$ is the non-trivial extension $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ for $t \neq 0$ and the trivial extension $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ for $t = 0$. This shows that $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ should be in the same connected component of the moduli space.

0.1.3 Wait—why are we just defining sets?

It is indeed a bit silly to define these moduli spaces as sets. After all, any two complex projective varieties are bijective so we should be demanding a lot more structure than a variety whose points are in bijective correspondence with isomorphism classes. However, spelling out what properties we desire of the moduli space is by no means easy. What we would really like is a quasi-projective variety

[1] Birkhoff proved this in 1909 using linear algebra by explicitly showing that an element $\mathrm{GL}_n(k[x]_x)$ can be multiplied on the left and right by elements of $\mathrm{GL}_n(k[x])$ and $\mathrm{GL}_n(k[1/x])$ to be a diagonal matrix $\mathrm{diag}(x^{a_1}, \ldots, x^{a_r})$ [Bir09] while Grothendieck proved this in 1957 via induction and cohomology by exhibiting a line subbundle $\mathcal{O}(a) \subset E$ such that the corresponding short exact sequence splits [Gro57].

M_g with a universal family $U_g \to M_g$ such that the fiber of a point $[C] \in M_g$ is precisely that curve. This is where the difficulty lies—automorphisms of curves obstruct the existence of such a family—and this is the main reason we want to expand our notion of a geometric space from schemes to algebraic stacks. Algebraic stacks provide a nice approach ensuring the existence of a universal family but it is by no means the only approach.

Historically, it was not clear what structure M_g should have. Riemann introduced the word 'Mannigfaltigkeiten' (or 'manifoldness') but did not specify what this means–complex manifolds were only introduced in the 1940s following Teichmüller, Chern and Weil. The first claim that M_g exists as an algebraic variety was perhaps due to Weil in [Wei58]: "As for M_g there is virtually no doubt that it can be provided with the structure of an algebraic variety." Grothendieck, aware that the functor of smooth families of curves was not representable, studied the functor of smooth families of curves with level structure $r \geq 3$ [Gro61]. While he could show representability, he struggled to show quasi-projectivity. It was only later that Mumford proved that M_g is a quasi-projective variety, an accomplishment for which he was awarded the Field Medal in 1974, by introducing and then applying Geometric Invariant Theory (GIT) to construct M_g as a quotient [GIT]. For further historical background, we recommend [JP13], [AJP16] and [Kol18].

In these notes, we take a similar approach to Mumford's original construction and integrate later influential results due to Deligne, Kollár, Mumford and others such as the seminal paper [DM69] which simultaneously introduced stable curves and stacks with the application of irreducibility of M_g in any characteristic. In this chapter, we motivate our approach by gradually building in additional structure: first as a groupoid (Section 0.3), then as a presheaf (i.e. contravariant functor) (Section 0.4), then as a stack (Section 0.7) and then ultimately as a projective variety (Section 0.9).

One of the challenges of learning moduli stacks is that it requires simultaneously extending the theory of schemes in several orthogonal directions including:

(1) the functorial approach: thinking of a scheme X not as topological space with a sheaf of rings but rather in terms of the functor $\mathrm{Sch} \to \mathrm{Sets}$ defined by $T \mapsto \mathrm{Mor}(T, X)$. For moduli problems, this means specifying not just objects but families of objects; and

(2) the groupoid approach: rather than specifying just the points we also specify their symmetries. For moduli problems, this means specifying not just the objects but their automorphism groups.

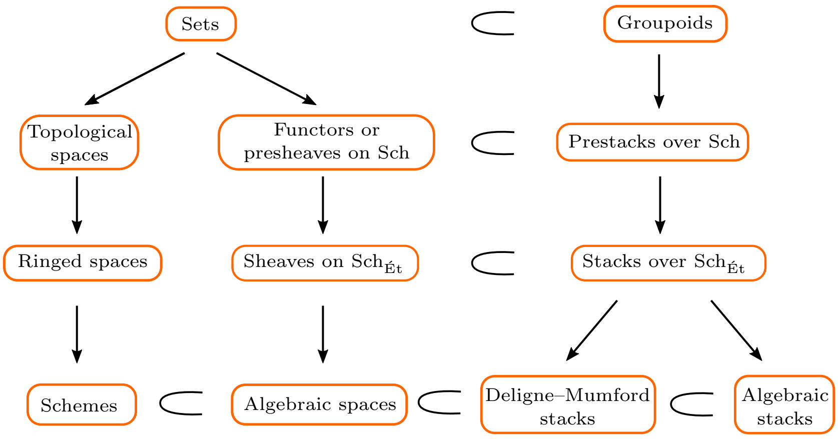

Figure 1: Schematic diagram featuring algebro-geometric enrichments of sets and groupoids where arrows indicate additional geometric conditions.

0.2 Toy example: moduli of triangles

Before we dive deeper into the moduli of curves or vector bundles, we will study the simple yet surprisingly fruitful example of the moduli of triangles which is easy both to visualize and construct. In fact, we present several variants of the moduli of triangles that highlight various concepts in moduli theory. The moduli spaces of labelled triangles and labelled triangles up to similarity have natural functorial descriptions and universal families while the moduli space of *unlabelled* triangles does not admit a universal family due to the presence of symmetries—in exploring this example, we are led to the concept of a moduli groupoid and ultimately to moduli stacks. Michael Artin is attributed to remarking that you can understand most concepts in moduli through the moduli space of triangles.

0.2.1 Labelled triangles

A *labelled triangle* is a triangle in $\mathbb{R}^2$ where the vertices are labelled with '1', '2' and '3', and the distances of the edges are denoted as a, b, and c. We require that triangles have non-zero area or equivalently that their vertices are not colinear.

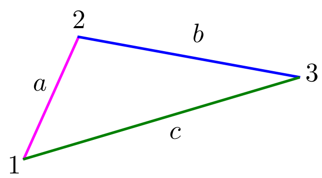

Figure 2: To keep track of the labelling, we color the edges as above.

We define the *moduli set of labelled triangles* M as the set of labelled triangles where two triangles are said to be equivalent if they are the same triangle in $\mathbb{R}^2$ with the same vertices and same labeling. By writing (x_1, y_1), (x_2, y_2) and (x_3, y_3)

as the coordinates of the labelled vertices, we obtain a bijection

$$M \cong \left\{(x_1, y_1, x_2, y_2, x_3, y_3) \;\middle|\; \det \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \neq 0\right\} \subset \mathbb{R}^6 \tag{0.2.1}$$

with the open subset of $\mathbb{R}^6$ whose complement is the codimension 1 closed subset defined by the condition that the vectors $(x_2, y_2) - (x_1, y_1)$ and $(x_3, y_3) - (x_1, y_1)$ are linearly dependent.

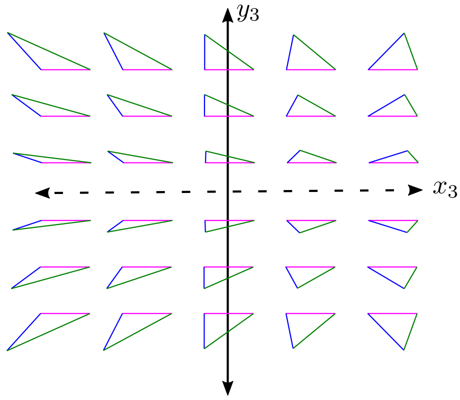

Figure 3: Picture of the slice of the moduli space M where $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (1, 0)$. Triangles are described by their third vertex (x_3, y_3) with $y_3 \neq 0$. We've drawn representative triangles for a handful of points in the x_3y_3- plane.

0.2.2 Labelled triangles up to similarity

We define the *moduli set of labelled triangles up to similarity*, denoted by M^{lab}, by taking the same class of objects as in the previous example—labelled triangles—but changing the equivalence relation to label-preserving similarity.

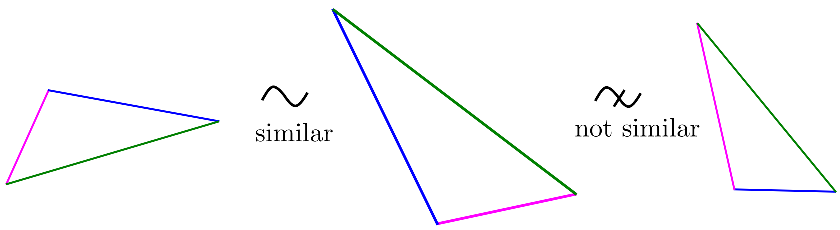

Figure 4: The two triangles on the left are similar, but the third is not.

Every labelled triangle is similar to a unique labelled triangle with perimeter $a + b + c = 2$. We have the description

$$M^{\text{lab}} = \left\{ (a, b, c) \;\middle|\; \begin{array}{c} a + b + c = 2 \\ 0 < a < b + c \\ 0 < b < a + c \\ 0 < c < a + b \end{array} \right\}. \tag{0.2.2}$$

By setting $c = 2 - a - b$, we may visualize M^{lab} as the analytic open subset of $\mathbb{R}^2$ defined by pairs (a, b) satisfying $0 < a, b < 1$ and $a + b > 1$.

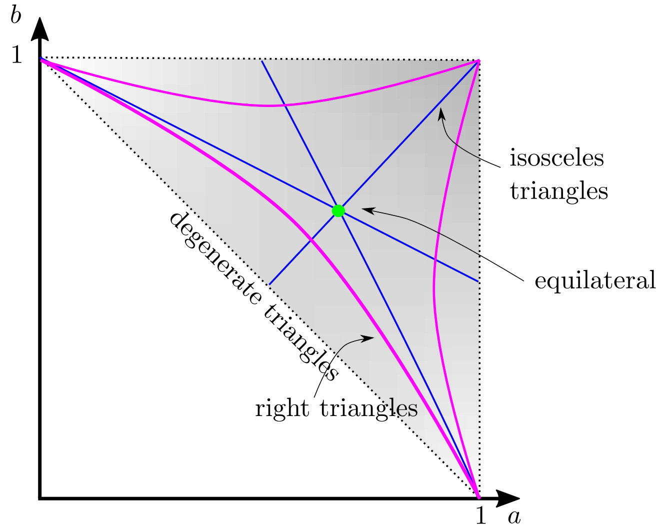

Figure 5: M^{lab} is the shaded area above. The pink lines represent the right triangles defined by $a^2+b^2=c^2$, $a^2+c^2=b^2$ and $b^2+c^2=a^2$, the blue lines represent isosceles triangles defined by $a=b$, $b=c$ and $a=c$, and the green point is the unique equilateral triangle defined by $a=b=c$.

0.2.3 Unlabelled triangles up to similarity

We now turn to the moduli of unlabelled triangles up to similarity, which reveals a new feature not seen in to the two above examples: symmetry!

We define the *moduli set of unlabelled triangles up to similarity*, denoted by M^{unl}, where the objects are unlabelled triangles in $\mathbb{R}^2$ and the equivalence relation is symmetry. We can describe a unlabelled triangle uniquely by the ordered tuple (a,b,c) of increasing side lengths as follows:

$$M^{\mathrm{unl}} = \left\{ (a,b,c) \;\middle|\; \begin{array}{l} 0<a\le b\le c<a+b \\ a+b+c=2 \end{array} \right\}. \tag{0.2.3}$$

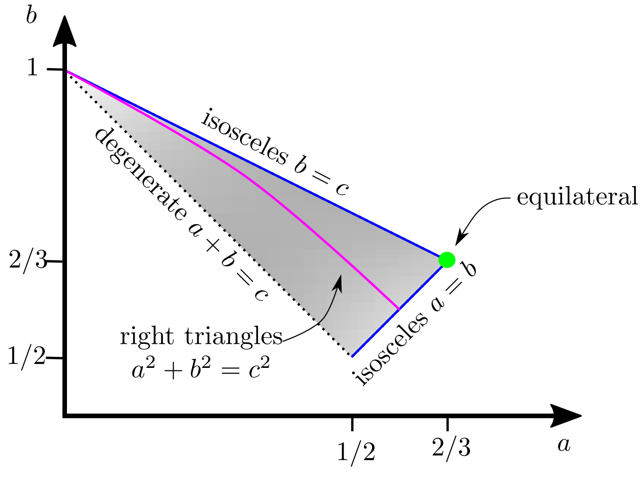

Figure 6: Picture of M^{unl} where $c=2-a-b$.

The isosceles triangles with $a = b$ or $b = c$ and the equilateral triangle with $a = b = c$ have symmetry groups of $\mathbb{Z}/2$ and S_3, respectively. This is unfortunately not encoded into our description M^{unl} above. However, we can identify M^{unl} as the quotient M^{lab}/S_3 of the moduli set of labelled triangles up to similarity modulo the natural action of S_3 on the labellings. Under this action, the stabilizers of isosceles and equilateral triangles are precisely their symmetry groups $\mathbb{Z}/2$ and S_3. The action of S_3 on the complement of the set of isosceles and equilateral triangles is free.

0.3 Moduli groupoids

We now change our perspective: rather than specifying *when* two objects are identified, we specify *how*! One of the most desirable properties of a moduli space is the existence of a universal family (see §0.4.5) and the presence of automorphisms obstructs its existence (see §0.4.6). Encoding automorphisms into our descriptions will allow us to get around this problem. A convenient mathematical structure to encode this information is a *groupoid*.

Definition 0.3.1. A *groupoid* is a category $\mathcal{C}$ where every morphism is an isomorphism.

0.3.1 Specifying a moduli groupoid

A *moduli groupoid* is described by

1. a class of certain algebraic, geometric or topological objects; and
2. a set of equivalences between two objects.

where (1) describes the objects and (2) the morphisms of a groupoid. In particular, the moduli groupoid encodes $\mathrm{Aut}(E)$ for every object E.

We say that two groupoids $\mathcal{C}_1$ and $\mathcal{C}_2$ are *equivalent* if there is an equivalence of categories (i.e. a fully faithful and essentially surjective functor) $\mathcal{C}_1 \to \mathcal{C}_2$. Moreover, we say that a groupoid $\mathcal{C}$ *is equivalent to a set* Σ if there is an equivalence of categories $\mathcal{C} \to \mathcal{C}_\Sigma$ (where $\mathcal{C}_\Sigma$ is defined in Example 0.3.2).

0.3.2 Examples

We will return to our two main examples—curves and vector bundles—in a moment but it will be useful first to consider a number of simpler examples.

Example 0.3.2. If Σ is a set, the category $\mathcal{C}_\Sigma$, whose objects are elements of Σ and whose morphisms consist of only the identity morphism, is a groupoid.

Example 0.3.3. If G is a group, the *classifying groupoid* BG of G, defined as the category with one object $\star$ such that $\mathrm{Mor}(\star, \star) = G$, is a groupoid.

Example 0.3.4. The category FB of finite sets where morphisms are bijections is a groupoid. Observe that the isomorphism classes of FB are in bijection with $\mathbb{N}$ but the groupoid FB retains the information of the permutation groups S_n.

Example 0.3.5 (Projective space)**.** Projective space can be defined as a moduli groupoid where the objects are lines $L \subset \mathbb{A}^{n+1}$ through the origin and whose morphisms consist of only the identity, or alternatively where the objects are non-zero linear maps $x = (x_0, \ldots, x_n)\colon \mathbb{C} \to \mathbb{C}^{n+1}$ such that there is a unique morphism $x \to x'$ if $\mathrm{im}(x) = \mathrm{im}(x') \subset \mathbb{C}^{n+1}$ (i.e. there exists a $\lambda \in \mathbb{C}^*$ such that $x' = \lambda x$) and no morphisms otherwise.

0.3.3 Moduli groupoid of orbits

Example 0.3.6 (Moduli groupoid of orbits)**.** Given an action of a group G on a set X, we define the *moduli groupoid of orbits* $[X/G]$[2] by taking the objects to be all elements $x \in X$ and by declaring $\mathrm{Mor}(x, x') = \{g \in G \mid x' = gx\}$.

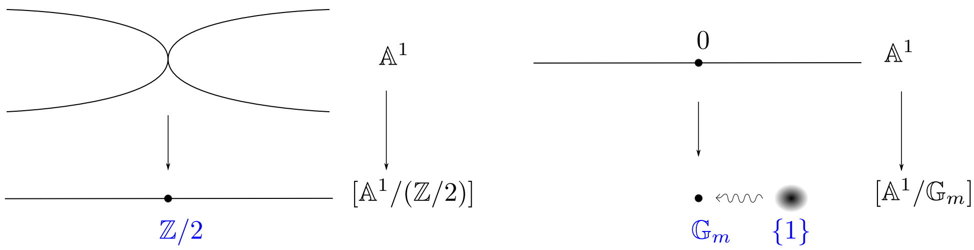

Figure 7: Pictures of the scaling actions of $\mathbb{Z}/2 = \{\pm 1\}$ and $\mathbb{G}_m$ on $\mathbb{A}^1$ over $\mathbb{C}$ with the automorphism groups listed in blue. Note that $[\mathbb{A}^1/\mathbb{G}_m]$ has two isomorphism classes of objects—0 and 1—corresponding to the two orbits—0 and $\mathbb{A}^1 \setminus 0$—such that $0 \in \overline{\{1\}}$ if the set $\mathbb{A}^1/\mathbb{G}_m$ is endowed with the quotient topology.

Exercise 0.3.7.

(1) Show that the moduli groupoid of orbits $[X/G]$ in Example 0.3.6 is equivalent to a set if and only if the action of G on X is free.

(2) Show that a groupoid $\mathcal{C}$ is equivalent to a set if and only if $\mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is fully faithful.

Example 0.3.8. Consider the category $\mathcal{C}$ with two objects x_1 and x_2 such that $\mathrm{Mor}(x_i, x_j) = \{\pm 1\}$ for $i, j = 1, 2$ where composition of morphisms is given by multiplication. Then $\mathcal{C}$ is equivalent $B\mathbb{Z}/2$.

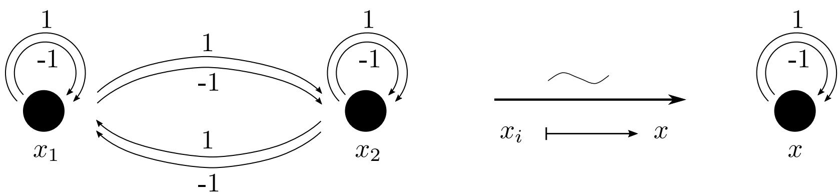

Figure 8: An equivalence of groupoids

Exercise 0.3.9. In Example 0.3.8, show that there is an equivalence of categories inducing a bijection on objects between $\mathcal{C}$ and either $[(\mathbb{Z}/2)/(\mathbb{Z}/4)]$ or $[(\mathbb{Z}/2)/(\mathbb{Z}/2 \times \mathbb{Z}/2)]$ where the action is given by the surjections $\mathbb{Z}/4 \to \mathbb{Z}/2$ or $\mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{Z}/2$.

[2] We use brackets to distinguish the groupoid quotient $[X/G]$ from the set quotient X/G. Later when G and X are enriched with more structure (e.g. an algebraic group acting on a variety), then $[X/G]$ will be correspondingly enriched (e.g. as an algebraic stack).

Example 0.3.10 (Projective space as a quotient)**.** The moduli groupoid of projective space (Example 0.3.5) can also be described as the moduli groupoid of orbits $[(\mathbb{A}^{n+1} \setminus 0)/\mathbb{G}_m]$.

We can also consider the quotient groupoid $[\mathbb{A}^{n+1}/\mathbb{G}_m]$, which is equivalent to the groupoid whose objects are (possibly zero) linear maps $x = (x_0, \ldots, x_n) \colon \mathbb{C} \to \mathbb{C}^{n+1}$ such that $\mathrm{Mor}(x, x') = \{t \in \mathbb{C}^* \mid x'_i = tx_i \text{ for all } i\}$. We can thus view $\mathbb{P}^n$ as a subgroupoid of $[\mathbb{A}^{n+1}/\mathbb{G}_m]$.

Exercise 0.3.11. If a group G acts on a set X and $x \in X$ is any point, there exists a fully faithful functor $BG_x \to [X/G]$. If the action is transitive, show that it is an equivalence.

A morphisms of groupoids $\mathcal{C}_1 \to \mathcal{C}_2$ is simply a functor, and we define the category $\mathrm{MOR}(\mathcal{C}_1, \mathcal{C}_2)$ whose objects are functors and whose morphisms are natural transformations.

Exercise 0.3.12. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are groupoids, show that $\mathrm{MOR}(\mathcal{C}_1, \mathcal{C}_2)$ is a groupoid.

Exercise 0.3.13. If H and G are groups, show that there is an equivalence

$$\mathrm{MOR}(BH, BG) = \coprod_{\phi \in \mathrm{Conj}(H,G)} BN_G(\operatorname{im} \phi)$$

where $\mathrm{Conj}(H, G)$ denotes a set of representatives of homomorphisms $H \to G$ up to conjugation by G, and $N_G(\operatorname{im} \phi)$ denotes the normalizer of $\operatorname{im} \phi$ in G.

Exercise 0.3.14. Provide an example of group actions of H and G on sets X and Y and a map $[X/H] \to [Y/G]$ of groupoids that *does not* arise from a group homomorphism $\phi \colon H \to G$ and a ϕ-equivariant map $X \to Y$.

0.3.4 Moduli groupoids of curves and vector bundles

We return to the two main examples in these notes.

Example 0.3.15 (Moduli groupoid of smooth curves)**.** In this case, the objects are smooth, connected and projective curves of genus g over $\mathbb{C}$ and for two curves C, C', the set of morphisms is defined as the set of isomorphisms

$$\mathrm{Mor}(C, C') = \{\text{isomorphisms } \alpha \colon C \xrightarrow{\sim} C'\}.$$

Example 0.3.16 (Moduli groupoid of vector bundles on a curve)**.** Let C be a fixed smooth, connected and projective curve over $\mathbb{C}$, and fix integers $r \geq 0$ and d. The objects are vector bundles E of rank r and degree d, and the morphisms are isomorphisms of vector bundles.

0.3.5 Moduli groupoid of unlabelled triangles up to similarity

We now revisit Section 0.2.3 of the moduli set M^{unl} of unlabelled triangles up to similarity. We will show later that this moduli set does not admit a natural functorial descriptions nor universal family due to presence of symmetries

(Example 0.4.37). Since these are such desirable properties, we will pursue a work around where we encode the symmetries into the definition.

We define the *moduli groupoid of unlabelled triangles up to similarity*, denoted by $\mathcal{M}^{\mathrm{unl}}$ (note the calligraphic font), where the objects are unlabelled triangles in $\mathbb{R}^2$ and where for triangles $T_1, T_2 \subset \mathbb{R}^2$, the set $\mathrm{Mor}(T_1, T_2)$ consists of the symmetries σ (corresponding to the permutations of the vertices) such that T_1 is similar to $\sigma(T_2)$. For example, an isosceles triangle (resp. equilateral triangle) has automorphism group $\mathbb{Z}/2$ (resp. S_3).

We can draw essentially the same picture as Figure 6 except we mark the automorphisms of triangles.

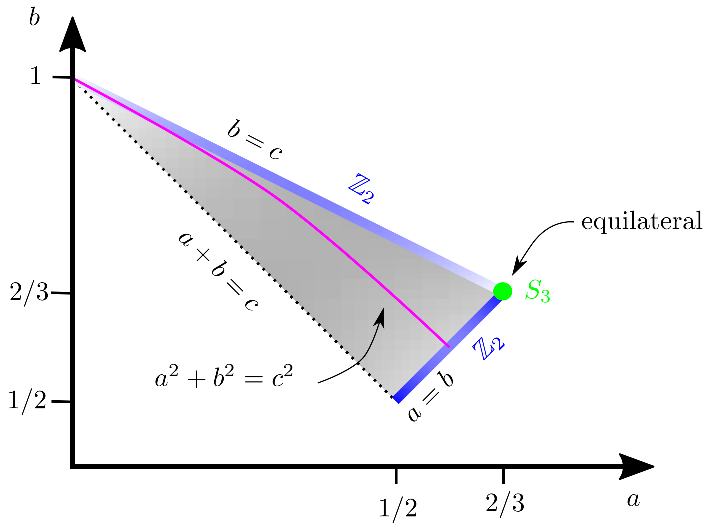

Figure 9: Picture of the moduli groupoid $\mathcal{M}^{\mathrm{unl}}$ with non-trivial automorphism groups labelled.

There is a functor

$$\mathcal{M}^{\mathrm{unl}} \to M^{\mathrm{unl}}$$

which is the identity on objects and collapses all morphisms to the identity. This could be called a *coarse moduli set* where by forgetting some information (i.e. the symmetry groups of isosceles and equilateral triangles), we can study the moduli problem as a more familiar object (i.e. a set rather than groupoid).

Exercise 0.3.17. Recall that the moduli set M^{lab} of labelled triangles up to similarity has the description as the set of tuples (a, b, c) such that $a + b + c = 2$, $0 < a < b + c$, $0 < b < a + c$, and $0 < c < a + b$ (see from (0.2.1)) Show that there is a natural action of S_3 on the moduli set M^{lab} of unlabelled triangles up to similarity and that the functor obtained by forgetting the labelling

$$[M^{\mathrm{lab}}/S_3] \to \mathcal{M}^{\mathrm{unl}}$$

is an equivalence of categories.

Exercise 0.3.18. Define a moduli groupoid of *oriented triangles* and investigate its relation to the moduli sets and groupoids of triangles we've defined above.

0.4 Moduli functors

We now undertake the challenging task of motivating moduli functors, which will be our approach for endowing moduli sets with the enriched structure of a topological space or scheme. This will require a leap in abstraction that is not at all the most intuitive, especially if you are seeing for the first time. The idea due to Grothendieck is to study a scheme X by studying all maps to it!

It may seem that this leap made life more difficult for us: rather than just specifying the points of a moduli space, we need to define all maps to the moduli space. In fact, it is easier than you may expect. Let's take M_g as an example. If S is a scheme and $f\colon S \to M_g$ is a map of sets, then for every point $s \in S$, the image $f(s) \in M_g$ corresponds to an isomorphism class of a curve C_s. But we don't want to consider arbitrary maps of sets. If M_g is enriched as a topological space (resp. scheme), then a continuous (resp. algebraic) map $f\colon S \to M_g$ should mean that the curves C_s are varying continuously (resp. algebraically). A nice way of packaging this is via *families of curves*, i.e. smooth and proper morphisms $\mathcal{C} \to S$ such that every fiber $\mathcal{C}_s$ is a curve.

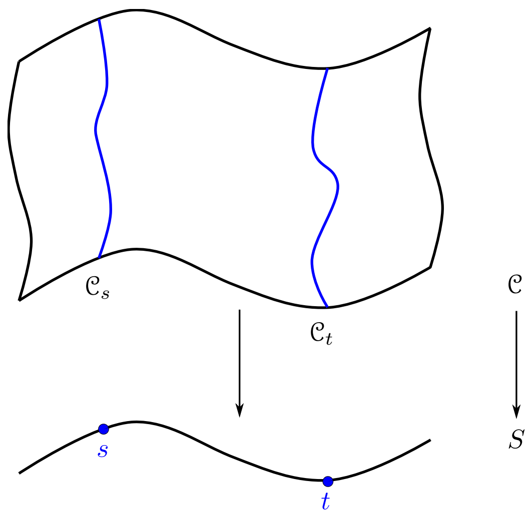

Figure 10: A family of curves over a curve S.

This suggests we define M_g as a functor Sch $\to$ Sets assigning a scheme S to the set of families of curves over S.

0.4.1 Yoneda's lemma

The fact that schemes are determined by maps into it follows from a completely formal argument that holds in any category. If X is an object of a category $\mathcal{C}$, the contravariant functor

$$h_X\colon \mathcal{C} \to \text{Sets}, \qquad S \mapsto \text{Mor}(S, X)$$

recovers the object X itself: this is the content of Yoneda's lemma:

Lemma 0.4.1 (Yoneda's lemma)**.** *Let $\mathcal{C}$ be a category and X be an object. For any contravariant functor $G\colon \mathcal{C} \to \mathrm{Sets}$, the map*

$$\mathrm{Mor}(h_X, G) \to G(X), \qquad \alpha \mapsto \alpha_X(\mathrm{id}_X)$$

is bijective and functorial with respect to both X and G.

Remark 0.4.2. The set $\mathrm{Mor}(h_X, G)$ consists of morphisms or natural transformations $h_X \to G$, and α_X denotes the map $h_X(X) = \mathrm{Mor}(X, X) \to G(X)$.

⚠ **Warning 0.4.3.** We will consistently abuse notation by conflating an element $g \in G(X)$ and the corresponding morphism $h_X \to G$, which we will often write simply as $X \to G$.

Exercise 0.4.4.

1. Spell out precisely what 'functorial with respect to both X and G' means.
2. Prove Yoneda's lemma.

Remark 0.4.5. It is instructive to imagine constructive proofs of Yoneda's lemma. Here we try to explicitly recover a *variety* X over $\mathbb{C}$ from its functor $h_X\colon \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}$. Clearly, we can recover the closed points of X by simply evaluating $h_X(\operatorname{Spec}\mathbb{C})$. To get all points, we need to allow points whose residue fields are extensions of $\mathbb{C}$. The underlying set of X is

$$\Sigma_X := \bigsqcup_{\mathbb{C}\subset k} h_X(\operatorname{Spec} k)/\sim$$

where we say $x \in h_X(k)$ and $x' \in h_X(k')$ are equivalent if there is a further field extension $\mathbb{C} \subset k''$ containing both k and k' such that the images of x and x' in $h_X(k'')$ are equal under the natural maps $h_X(k) \to h_X(k'')$ and $h_X(k') \to h_X(k'')$. Later, we will follow the same approach when *defining* points of algebraic spaces and stacks (see **??**).

How can we recover the topological space? Here's a tautological way: we say a subset $A \subset \Sigma_X$ is open if there is an open immersion $U \hookrightarrow X$ with image A. Here's a better approach: we say a subset $A \subset \Sigma_X$ is open if for every map $f\colon S \to X$ of schemes, the subset $f^{-1}(A) \subset S$ is open.

What about recovering the sheaf of rings $\mathcal{O}_X$? For an open subset $U \subset \Sigma_X$, we define the functions on U as continuous maps $U \to \mathbb{A}^1$ such that for every morphism $f\colon S \to X$ of schemes, the composition (as a continuous map) $f^{-1}(U) \to U \to \mathbb{A}^1$ is an algebraic function (i.e. corresponds to an element $\Gamma(S, f^{-1}(U))$).

Exercise 0.4.6.

(a) Can the above argument be extended if X is non-reduced?

(b) Is it possible to explicitly recover a scheme X from its *covariant* functor $\mathrm{Sch} \to \mathrm{Sets}, S \mapsto \mathrm{Mor}(X, S)$?

0.4.2 Specifying a moduli functor

Defining a moduli functor requires specifying:

(1) families of objects;

(2) when two families of objects are isomorphic; and

(3) and how families pull back under morphisms.

In defining a moduli functor $F\colon \text{Sch} \to \text{Sets}$, then (1) and (2) specify $F(S)$ for a scheme S and (3) specifies the pull back $F(S) \to F(S')$ for maps $S' \to S$.

Example 0.4.7 (Moduli functor of smooth curves)**.** A *family of smooth curves* (of genus g) is a smooth, proper morphism $\mathcal{C} \to S$ of schemes such that for every $s \in S$, the fiber $\mathcal{C}_s$ is a connected curve (of genus g). The *moduli functor of smooth curves of genus g* is

$$F_{M_g}\colon \text{Sch} \to \text{Sets}, \quad S \mapsto \{\text{families of smooth curves } \mathcal{C} \to S \text{ of genus } g\} / \sim,$$

where two families $\mathcal{C} \to S$ and $\mathcal{C}' \to S$ are equivalent if there is a S-isomorphism $\mathcal{C} \to \mathcal{C}'$. If $S' \to S$ is a map of schemes and $\mathcal{C} \to S$ is a family of curves, the pull back is defined as the family $\mathcal{C} \times_S S' \to S'$.

Example 0.4.8 (Moduli functor of vector bundles on a curve)**.** Let C be a fixed smooth, connected and projective curve over $\mathbb{C}$, and fix integers $r \geq 0$ and d. A *family of vector bundles* (of rank r and degree d) over a scheme S is a vector bundle $\mathcal{E}$ on $C \times S$ (such that for all $s \in S$, the restriction $\mathcal{E}_s := \mathcal{E}|_{C \times \operatorname{Spec} \kappa(s)}$ has rank r and degree d on $C_{\kappa(s)}$). The *moduli functor of vector bundles on C of rank r and degree d* is

$$\text{Sch} \to \text{Sets} \qquad S \mapsto \left\{ \begin{array}{l} \text{families of vector bundles } \mathcal{E} \text{ on } C \times S \\ \text{of rank } r \text{ and degree } d \end{array} \right\} / \sim,$$

where equivalence $\sim$ is given by isomorphism. If $S' \to S$ is a map of schemes and E is a vector bundle on $C \times S$, the pull back is defined as the vector bundle $(\operatorname{id} \times f)^* \mathcal{E}$ on $C \times S'$

Example 0.4.9 (Moduli functor of orbits)**.** Revisiting Example 0.3.6, consider an algebraic group G acting on a scheme X. For every scheme S, the abstract group $G(S)$ acts on the set $X(S)$ (in fact, giving such actions functorial in S uniquely specifies the group action). We can consider the functor

$$\text{Sch} \to \text{Sets} \qquad S \mapsto X(S)/G(S).$$

Elements of the quotient set $X(S)/G(S)$ is our first candidate for a notion of a family of orbits, which we will modify later.

To gain intuition of any moduli functor $F\colon \text{Sch} \to \text{Sets}$, it is always useful to plug in special test schemes. For instance, plugging in a field K should give the K-points of the moduli problem, plugging in $\mathbb{C}[\epsilon]$ should give pairs of $\mathbb{C}$-points together with tangent vectors, and plugging in a curve (e.g. a DVR) gives families of objects over the curve.

In some cases, even though you may know exactly what objects you want to parameterize, it is not always clear how to define families of objects. In fact, there may be several candidates for families corresponding to different scheme structures on the same topological space. This is the case for instance for the moduli of higher dimensional varieties.

0.4.3 Representable functors

Definition 0.4.10. We say that a functor $F\colon \text{Sch} \to \text{Sets}$ is *representable by a scheme* if there exists a scheme X and an isomorphism of functors $F \xrightarrow{\sim} h_X$.

We would like to know when a given a moduli functor F is representable by a scheme. Unfortunately, each of the functors considered in Examples 0.4.7 to 0.4.9 is *not* representable; see Section 0.4.6. We begin though by considering a few simpler moduli functors which are in fact representable.

Theorem 0.4.11 (Projective space as a functor)**.** *[Har77, Thm. II.7.1] There is a functorial bijection*

$$\operatorname{Mor}(S, \mathbb{P}^n_{\mathbb{Z}}) \cong \left\{ \big(L, (s_0, \ldots, s_n)\big) \;\middle|\; \begin{array}{l} L \textit{ is a line bundle on } S \textit{ globally generated} \\ \textit{by sections } s_0, \ldots, s_n \in \Gamma(S, L) \end{array} \right\} / \sim,$$

where $(L, (s_i)) \sim (L', (s'_i))$ *if there exists* $t \in \Gamma(S, \mathcal{O}_S)^*$ *such that* $s'_i = ts_i$ *for all* i*.*

In other words, the theorem states the functor defined on the right is representable by the scheme $\mathbb{P}^n_{\mathbb{Z}}$. The condition that the sections s_i are globally generated translates to the condition that for every $x \in S$, at least one section $s_i(x) \in L \otimes \kappa(t)$ is non-zero, or equivalently to the surjectivity of $(s_0, \ldots, s_n)\colon \mathcal{O}_S^{n+1} \to L$. This perspective of viewing projective space as parameterizing rank 1 quotients of the trivial bundle will be generalized when we study the Grassmanian in Section 0.5 and even further generalized when we study the Hilbert and Quot schemes. For now, we mention the following mild generalization:

Definition 0.4.12. If S is a scheme and E is a vector bundle on S, we define the contravariant functor

$$\begin{aligned} \mathbb{P}(E)\colon \operatorname{Sch}/S &\to \operatorname{Sets} \\ (T \xrightarrow{f} S) &\mapsto \{\text{quotients } f^*E \overset{q}{\twoheadrightarrow} L \text{ where } L \text{ is a line bundle on } T\}/\sim \end{aligned}$$

where $[f^*E \overset{q}{\twoheadrightarrow} L] \sim [f^*E \overset{q'}{\twoheadrightarrow} L']$ if there is an isomorphism $\alpha\colon L \to L'$ with $q' = \alpha' \circ q'$.

Observe that there is an isomorphism $\mathbb{P}^n_{\mathbb{Z}} \cong \mathbb{P}(\mathcal{O}^{n+1}_{\operatorname{Spec}\mathbb{Z}})$ of functors.

Exercise 0.4.13. Show that $\mathbb{P}(E)$ is representable by the usual projectivization of a vector bundle.

Exercise 0.4.14. Provide functorial descriptions of:

(a) $\mathbb{A}^n \setminus 0$; and

(b) the blowup $\operatorname{Bl}_p \mathbb{P}^n$ of $\mathbb{P}^n$ at a point.

Exercise 0.4.15. Let X be a scheme, and let E and G be $\mathcal{O}_X$-modules. The group $\operatorname{Ext}^1(G, E)$ classifies extensions $0 \to E \to F \to G \to 0$ of $\mathcal{O}_X$-modules where two extensions are identified if there is an isomorphism of short exact sequences inducing the identity map on E and G [Har77, Exer. III.6.1].

Show that the affine scheme $\underline{\operatorname{Ext}}^1_{\mathcal{O}_X}(G, E) := \operatorname{Spec}\operatorname{Sym}\operatorname{Ext}^1(G, F)^\vee$ represents the functor

$$\operatorname{Sch} \to \operatorname{Sets}, \quad T \mapsto \operatorname{Ext}^1_{\mathcal{O}_{X\times T}}(p_1^*G, p_1^*E).$$

0.4.4 Working with functors

We can form a category $\operatorname{Fun}(\operatorname{Sch}, \operatorname{Sets})$ whose objects are contravariant functors $F\colon \operatorname{Sch} \to \operatorname{Sets}$ and whose morphisms are natural transformations. This category has fiber products: given a morphism $F \xrightarrow{\alpha} G$ and $G' \xrightarrow{\beta} G$, we define

$$\begin{aligned} F \times_G G'\colon \operatorname{Sch} &\to \operatorname{Sets} \\ S &\mapsto \{(a, b) \in F(S) \times G'(S) \mid \alpha_S(a) = \beta_S(b)\} \end{aligned}$$

Exercise 0.4.16. Show that that $F \times_G G'$ satisfies the universal property for fiber products in $\mathrm{Fun}(\mathrm{Sch}, \mathrm{Sets})$.

Definition 0.4.17.

(1) We say that a morphism $F \to G$ of contravariant functors is *representable by schemes* if for any map $S \to G$ from a scheme S, the fiber product $F \times_G S$ is representable by a scheme.

(2) We say that a morphism $F \to G$ is an *open immersion* or that a subfunctor $F \subset G$ is *open* if for any morphism $S \to G$ from a scheme S, $F \times_G S$ is representable by an open subscheme of S.

(3) We say that a set of open subfunctors $\{F_i\}$ is a *Zariski-open cover* of F if for any morphism $S \to F$ from a scheme S, $\{F_i \times_F S\}$ is a Zariski-open cover of S.

Each of these conditions can be checked on *affine* schemes

By appealing to Yoneda's lemma (Lemma 0.4.1), one can define a scheme as a functor $F\colon \mathrm{Sch} \to \mathrm{Sets}$ such that there exists a Zariski-open cover $\{F_i\}$ where each F_i is representable by an affine scheme. Furthermore, this perspective also gives us a recipe for checking that a given functor F is representable by a scheme: simply find a Zariski-open cover $\{F_i\}$ where each F_i is representable.

Exercise 0.4.18. Show that a scheme can be equivalently defined as a contravariant functor $F\colon \mathrm{AffSch} \to \mathrm{Sets}$ on the category of affine schemes (or covariant functor on the category of rings) such that there is Zariski-open cover $\{F_i\}$ where each F_i is representable by an affine scheme.

Replacing Zariski-opens with étale-opens (see Section 0.6) leads to the definition of an algebraic space (Definition 2.1.2).

0.4.5 Universal families

Definition 0.4.19. Let $F\colon \mathrm{Sch} \to \mathrm{Sets}$ be a moduli functor representable by a scheme X via an isomorphism $\alpha\colon F \xrightarrow{\sim} h_X$ of functors. The *universal family* of F is the object $U \in F(X)$ corresponding under α to the identity morphism $\mathrm{id}_X \in h_X(X) = \mathrm{Mor}(X, X)$.

Suspend your skepticism for a moment and suppose that there actually exists a scheme M_g representing the moduli functor of smooth curves of genus g (Example 0.4.7). Then corresponding to the identity map $M_g \to M_g$ is a family of genus g curves $U_g \to M_g$ satisfying the following universal property: for any smooth family of curves $\mathcal{C} \to S$ over a scheme S, there is a unique map $S \to M_g$ and cartesian diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U_g \arrow[d] \\ S \arrow[r] & M_g. \end{tikzcd}$$

The map $S \to M_g$ sends a point $s \in S$ to the curve $[\mathcal{C}_s] \in M_g$.

Example 0.4.20. The universal family of the moduli functor of projective space (Theorem 0.4.11) is the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^n$ together with the sections $x_0, \dots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$.

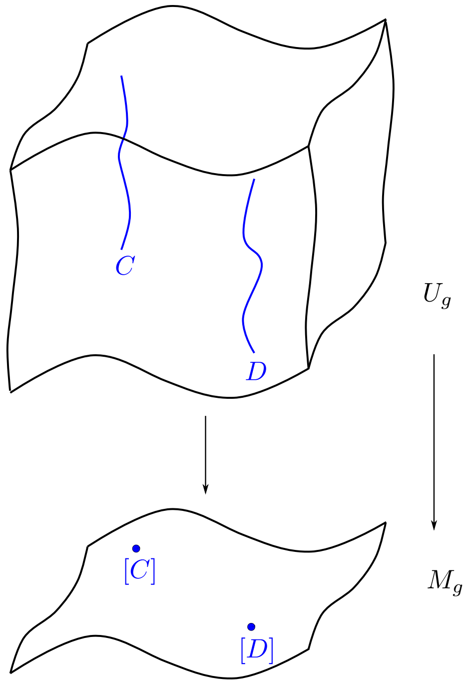

Figure 11: Visualization of a (non-existent) universal family over M_g.

Example 0.4.21 (Universal extensions)**.** If X is a scheme with vector bundles E and G, the universal family for the moduli functor $\underline{\mathrm{Ext}}^1_{\mathcal{O}_X}(G, F)$ of extensions of Exercise 0.4.15 is the extension $0 \to p_1^* G \to \mathcal{F} \to p_2^* E \to 0$ of vector bundle on $X \times \underline{\mathrm{Ext}}^1_{\mathcal{O}_X}(G, E)$. The restriction of this extension to $X \times \{t\}$ is the extension corresponding to $t \in \mathrm{Ext}^1(G, E)$.

Example 0.4.22 (Classifying spaces in algebraic topology)**.** Let G be a topological group and $\mathrm{Top}^{\mathrm{para}}$ be the category of paracompact topological spaces where morphisms are defined up to homotopy. It is a theorem in algebraic topology that the functor

$$\mathrm{Top}^{\mathrm{para}} \to \mathrm{Sets}, \qquad S \mapsto \{\text{principal } G\text{-bundles } P \to S\}/\sim,$$

where $\sim$ denotes isomorphism, is represented by a topological space, which we denote by BG and call the *classifying space*. The universal family is usually denoted by $EG \to BG$.

For example, the classifying space $B\mathbb{C}^*$ is the infinite-dimensional manifold $\mathbb{CP}^\infty$; in algebraic geometry however the classifying stack $B\mathbb{G}_{m,\mathbb{C}}$ is an algebraic stack of dimension -1.

0.4.6 Non-representability of some moduli functors

Suppose $F\colon \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}$ is a moduli functor parameterizing isomorphism classes of objects, and let's suppose that there is an object E over $\operatorname{Spec}\mathbb{C}$ with a non-trivial automorphism α. This can obstruct the representability of F as the automorphism α can sometimes be used to construct non-trivial families: namely, if $S = S_1 \cup S_2$ is an open cover of a scheme S, we can glue the trivial families $E \times S_1$ and $E \times S_2$ using α to obtain a family $\mathcal{E}$ over S which might be non-trivial.

Proposition 0.4.23. *Let $F\colon \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}$ be a moduli functor parameterizing isomorphism classes of objects. Suppose there is a family of objects $\mathcal{E} \in F(S)$ over a variety S. For a point $s \in S(\mathbb{C})$, denote by $\mathcal{E}_s$ the pull back of $\mathcal{E}$ along $s\colon \operatorname{Spec}\mathbb{C} \to S$. If*

(a) the fibers $\mathcal{E}_s$ are isomorphic for $s \in S(\mathbb{C})$; and

(b) the family $\mathcal{E}$ is non-trivial, i.e. is not equal to the pull back of an object $E \in F(\mathbb{C})$ along the structure map $S \to \operatorname{Spec}\mathbb{C}$,

then F is not representable.

Proof. Suppose by way of contradiction that F is represented by a scheme X. By condition (a), the restriction $E := \mathcal{E}_s$ is independent of $s \in S(\mathbb{C})$ and defines a unique point $x \in X(\mathbb{C})$. As S is reduced, the map $S \to X$ factors as $S \to \operatorname{Spec}\mathbb{C} \xrightarrow{x} X$. Thus both the family $\mathcal{E}$ and the trivial family correspond to the same constant map $S \to \operatorname{Spec}\mathbb{C} \xrightarrow{x} X$, contradicting condition (b). □

Example 0.4.24 (Moduli of vector bundles over a point)**.** Consider the moduli functor $F\colon \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}$ assigning a scheme S to the set of isomorphism classes of vector bundles over S. Note that $F(\operatorname{Spec}\mathbb{C}) = \bigsqcup_{r \geq 0}\{\mathcal{O}^r_{\operatorname{Spec}\mathbb{C}}\}$. Since we know there exist non-trivial vector bundles (of any positive rank), we see that F cannot be representable by a scheme.

Exercise 0.4.25. Show that the moduli functor of vector bundles over a curve C is not representable.

Example 0.4.26 (Moduli of elliptic curves)**.** An *elliptic curve* over a field K is a pair (E, P) where E is a smooth, geometrically connected (i.e. $E_{\overline{K}}$ is connected), and projective curve E of genus 1 and $p \in E(K)$. A *family of elliptic curves* over a scheme S is a pair $(\mathcal{E} \to S, \sigma)$ where $\mathcal{E} \to S$ is smooth proper morphism with a section $\sigma\colon S \to \mathcal{E}$ such that for every $s \in S$, the fiber $(\mathcal{E}_s, \sigma(t))$ is an elliptic curve over the residue field $\kappa(s)$. The *moduli functor of elliptic curves* is

$$\begin{aligned} F_{M_{1,1}}\colon \mathrm{Sch} &\to \mathrm{Sets} \\ S &\mapsto \{\text{families } (\mathcal{E} \to S, \sigma) \text{ of elliptic curves }\}/\sim, \end{aligned}$$

where $(\mathcal{E} \to S, \sigma) \sim (\mathcal{E}' \to S, \sigma')$ if there is a S-isomorphism $\alpha\colon \mathcal{E} \to \mathcal{E}'$ compatible with the sections (i.e. $\sigma' = \alpha \circ \sigma$).

Exercise 0.4.27. Consider the family of elliptic curves defined over $\mathbb{A}^1 \setminus 0$ (with coordinate t) by

$$\begin{tikzcd} \mathcal{E} := V(y^2z - x^3 + tz^3) \arrow[r, hook] \arrow[d] & (\mathbb{A}^1 \setminus 0) \times \mathbb{P}^2 \\ \mathbb{A}^1 \setminus 0 & \end{tikzcd}$$

with section $\sigma\colon \mathbb{A}^1 \setminus 0 \to \mathcal{E}$ given by $t \mapsto [0, 1, 0]$. Show that $(\mathcal{E} \to \mathbb{A}^1 \setminus 0, \sigma)$ satisfies (a) and (b) in Proposition 0.4.23.

Example 0.4.28 (Moduli functor of smooth curves)**.** Let C be a curve with a non-trivial automorphism $\alpha \in \mathrm{Aut}(C)$ and let N be a the nodal cubic curve which we can think of as $\mathbb{P}^1$ after glueing 0 and ∞. We can construct a family $\mathcal{C} \to N$ by taking the trivial family $\pi\colon C \times \mathbb{P}^1 \to \mathbb{P}^1$ and gluing the fiber $\pi^{-1}(0)$ with $\pi^{-1}(\infty)$ via the automorphism α.

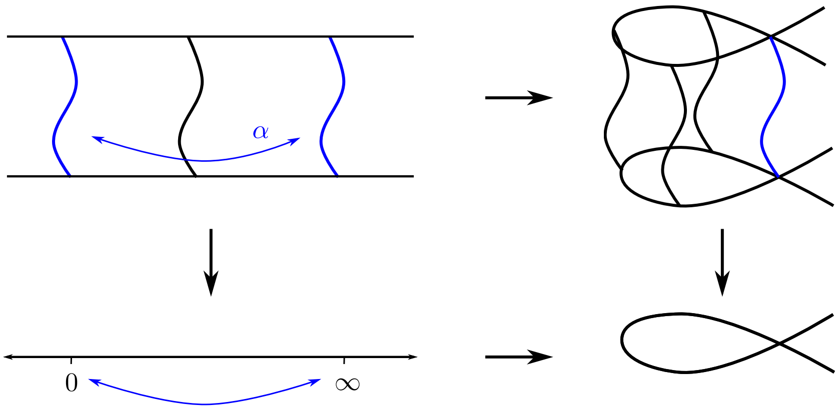

Figure 12: Family of curves over the nodal cubic obtaining by gluing the fibers over 0 and ∞ of the trivial family over $\mathbb{P}^1$ via α. (It would be more illustrative to draw a Mobius band as the family of curves over the nodal cubic.)

To show that the moduli functor of curves is not representable, it suffices to show that $\mathcal{C} \to N$ is non-trivial.

Exercise 0.4.29. Show that $\mathcal{C} \to N$ is a non-trivial family.

0.4.7 Schemes are sheaves

If $F\colon \mathrm{Sch} \to \mathrm{Sets}$ is representable by a scheme X (i.e. $F = \mathrm{Mor}(-, X)$), then F is necessarily a *sheaf in the big Zariski topology*, that is, for any scheme S, the presheaf on the Zariski topology of S defined by assigning to an open subset $U \subset S$ the set $F(U)$ is a sheaf on the Zariski topology of S. This is simply stating that morphisms into the fixed scheme X glue uniquely.

This therefore gives a potential obstruction to the representability of a given moduli functor F: if F is not a sheaf in the big Zariski topology, then F can not be representable.

Example 0.4.30. Consider the functor

$$F\colon \mathrm{Sch} \to \mathrm{Sets}, \quad S \mapsto \{\text{quotients } q\colon \mathcal{O}_S^n \twoheadrightarrow \mathcal{O}_S^k\}/\sim$$

where quotients q and q' are identified if there exists an automorphism Ψ of $\mathcal{O}_S^k$ such that $q' = \Psi \circ q$ or equivalently if $\ker(q) = \ker(q')$.

If F were representable by a scheme, then since morphisms glue in the Zariski topology, sections of F should also glue. But it easy to see that this fails: specializing to $k = 1$ and $S = \mathbb{P}^1$ (with coordinates x and y), consider the cover $S_1 = \{y \neq 0\} = \operatorname{Spec} \mathbb{C}[\frac{x}{y}]$ and $S_2 = \{x \neq 0\} = \operatorname{Spec} \mathbb{C}[\frac{y}{x}]$. The quotients

$$[(1, \frac{x}{y}, 0, \cdots, 0)\colon \mathcal{O}_{S_1}^{\oplus n} \to \mathcal{O}_{S_1}] \in F(S_1) \quad \text{and} \quad [(\frac{y}{x}, 1, 0, \cdots, 0)\colon \mathcal{O}_{S_2}^{\oplus n} \to \mathcal{O}_{S_2}] \in F(S_2)$$

become equivalent in $F(S_1 \cap S_2)$ under the automorphism $\Psi = \frac{y}{x}$ of $\mathcal{O}_{S_1 \cap S_2}$ and *do not* glue to a section of $F(\mathbb{P}^1)$. Of course, the issue is that the structure sheaves on S_1 and S_2 glue to $\mathcal{O}_{\mathbb{P}^1}(1)$—not $\mathcal{O}_{\mathbb{P}^1}$—under Ψ.

The above functor can be modified to define the Grassmanian functor (Definition 0.5.1) where instead of parameterizing *free* rank k quotients of $\mathcal{O}_S^n$, we parameterize *locally free* quotients.

Example 0.4.31. In Example 0.4.9, we introduced the functor $S \mapsto X(S)/G(S)$ associated to an action of an algebraic group G on a scheme X. Even in simple examples of free actions, this functor is not a sheaf; see Exercise 0.4.32

Exercise 0.4.32. Consider $\mathbb{G}_m$ acting on $\mathbb{A}^{n+1} \setminus 0$ with the usual scaling action. Show that the functor $S \mapsto (\mathbb{A}^{n+1} \setminus 0)(S)/\mathbb{G}_m(S)$ is not a sheaf.

Remark 0.4.33. The obstruction of representability due to non-sheafiness is intimately related to the existence of automorphisms. Indeed, the presence of a non-trivial automorphism often implies that a given moduli functor is not a sheaf.

Consider the moduli functor F_{M_g} of smooth curves from Example 0.4.7. Let $\{S_i\}$ be a Zariski-open covering of a scheme S. Suppose we have families of smooth curves $\mathcal{C}_i \to S_i$ and isomorphisms $\alpha_{ij} : \mathcal{C}_i|_{S_{ij}} \xrightarrow{\sim} \mathcal{C}_j|_{S_{ij}}$ on the intersection $S_{ij} := S_i \cap S_j$. The requirement that F_{M_g} be a sheaf (when restricted to the Zariski topology on S) implies that the families $\mathcal{C}_i \to S_i$ glue uniquely to a family of curves $\mathcal{C} \to S$. However, we have not required the isomorphisms α_i to be compatible on the triple intersection (i.e. $\alpha_{ij}|_{S_{ijk}} \circ \alpha_{jk}|_{S_{ijk}} = \alpha_{ik}|_{S_{ijk}}$) as is usual with gluing of schemes ([Har77, Exercise II.2.12]). For this reason, F_{M_g} fails to be a sheaf.

Exercise 0.4.34. Show that the moduli functors of smooth curves and elliptic curves are not sheaves by explicitly exhibiting a scheme S, an open cover $\{S_i\}$ and families of curves over S_i that do not glue to a family over S.

0.4.8 Moduli functors of triangles

We will now attempt to define moduli functors of labelled and unlabelled triangles. Since we are primarily interested in constructing these moduli spaces as topological spaces, we will consider the category Top of topological spaces and consider representability as a topological space.

Example 0.4.35 (Labelled triangles)**.** If S is a topological space, then we define a *family of labelled triangles over* S as a tuple $(\mathcal{T}, \sigma_1, \sigma_2, \sigma_3)$ where $\mathcal{T} \subset S \times \mathbb{R}^2$ is a closed subset and $\sigma_i : S \to \mathcal{T}$ are continuous sections for $i = 1, 2, 3$ of the projection $\mathcal{T} \to S$ such that for every $s \in S$, the subset $\mathcal{T}_s \subset \mathbb{R}^2$ is a labelled triangle with vertices $\sigma_1(s)$, $\sigma_2(s)$, and $\sigma_3(s)$.

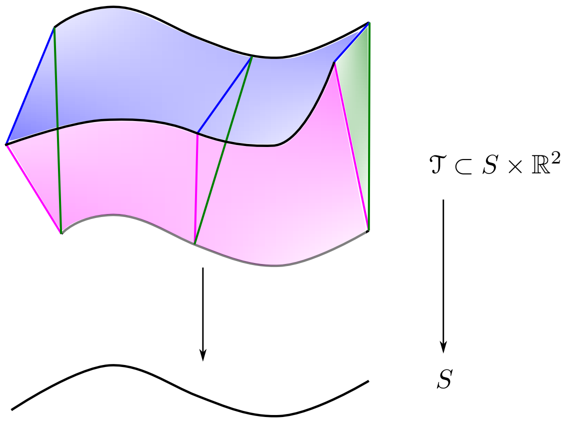

Figure 13: A family of labelled triangles over a curve.

Likewise, we define the *moduli functor of labelled triangles* as

$$F_M\colon \text{Top} \to \text{Sets}, \quad S \mapsto \{\text{families } (\mathcal{T}, \sigma_1, \sigma_2, \sigma_3) \text{ of labelled triangles}\}$$

We claim this functor is represented by the topological space of full rank 2×3 matrices

$$M := \left\{(x_1, y_1, x_2, y_2, x_3, y_3) \mid \det \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \neq 0\right\} \subset \mathbb{R}^6.$$

There is a bijection of the set $F_M(\text{pt})$ of labelled triangles and M given by taking the coordinates of the vertices. It is easy to see that this bijection can be promoted to an equivalence of functors $F_M \xrightarrow{\sim} h_M$, i.e. to a functorial bijection

$$F_M(S) \xrightarrow{\sim} \text{Mor}(S, M)$$

for each $S \in \text{Top}$, which assigns a family $(\mathcal{T}, \sigma_i)$ of labelled triangles to the map $S \to M$ where $s \mapsto (\sigma_1(s), \sigma_2(s), \sigma_3(s)) \in \mathcal{T}$.

Since F_M is representable by the topological space M, we have a universal family $\mathcal{T}_{\text{univ}} \subset M \times \mathbb{R}^2$ with $\sigma_1, \sigma_2, \sigma_3\colon M \to \mathcal{T}_{\text{univ}}$. This universal family can be visualized over the locus $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (1, 0)$ by taking Figure 3 and drawing the triangles *above* each point rather than *at* each point.

Example 0.4.36 (Labelled triangles up to similarity)**.** We say two families $(\mathcal{T}, (\sigma_i))$ and $(\mathcal{T}', (\sigma_i'))$ of labelled triangles over $S \in \text{Top}$ are *similar* if for each $s \in S$, the labelled triangles $\mathcal{T}_s$ and $\mathcal{T}'_s$ are similar. We define the functor

$$F_{M^{\text{lab}}}\colon \text{Top} \to \text{Sets}, \quad S \mapsto \{\text{families } \mathcal{T} \subset S \times \mathbb{R}^2 \text{ of labelled triangles}\}/\sim$$

where $\sim$ denotes similarity. Recall from (0.2.2) that the assignment of a triangle to its side lengths yields a bijection between $F_{M^{\text{lab}}}$ and

$$M^{\text{lab}} = \left\{ (a, b, c) \,\middle|\, \begin{array}{l} a + b + c = 2 \\ 0 < a < b + c \\ 0 < b < a + c \\ 0 < c < a + b \end{array} \right\};$$

As in the previous example, this extends to an isomorphism of functors $F_{M^{\text{lab}}} \to \text{Mor}(-, M^{\text{lab}})$, showing that the topological space M^{lab} represents the functor $F_{M^{\text{lab}}}$.

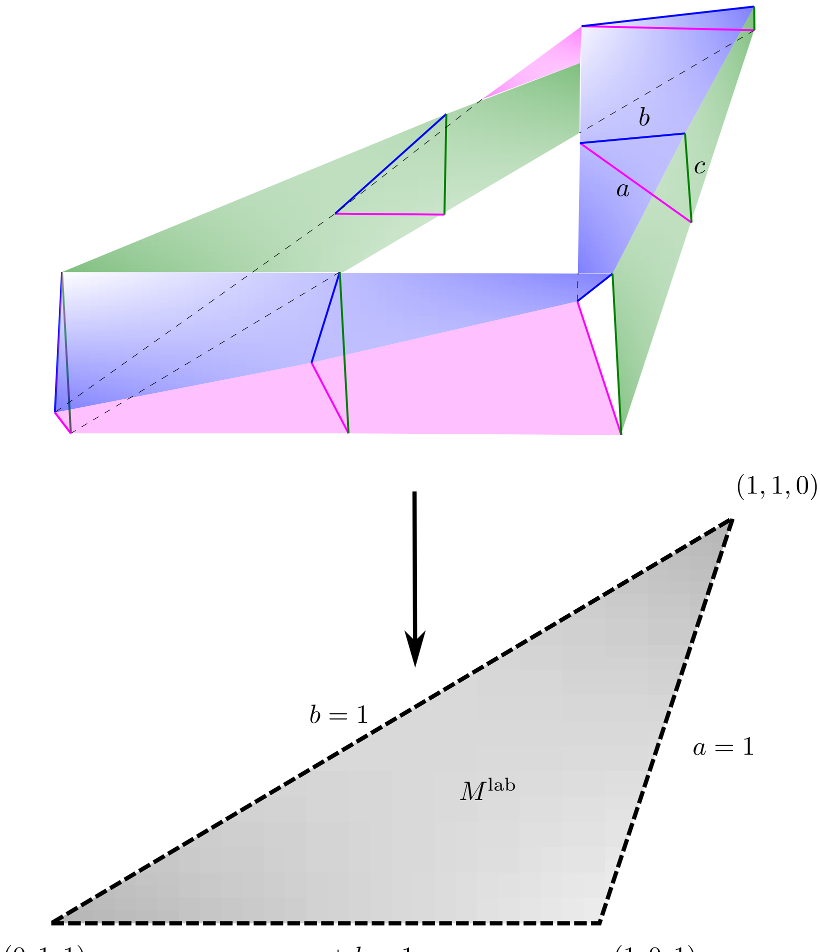

Figure 14: The universal family $U^{\mathrm{lab}} \to M^{\mathrm{lab}}$ of labelled triangles up to similarity.

Example 0.4.37 (Unlabelled triangles up to similarity)**.** In Examples 0.4.35 and 0.4.36, we considered the moduli functor of *labelled* triangles up to isomorphism and similarity, respectively. We now consider the unlabelled version.

If S is a topological space, a *family of triangles* is a closed subset $\mathcal{T} \subset S \times \mathbb{R}^2$ such that for all $s \in S$, the fiber $\mathcal{T}_s \subset \mathbb{R}^2$ is a triangle. We say two families $\mathcal{T}, \mathcal{T}'$ over S are similar if the fibers $\mathcal{T}_s$ and $\mathcal{T}'_s$ are similar for all $s \in S$.

We define the functor

$$F\colon \text{Top} \to \text{Sets}, \quad S \mapsto \{\text{families } \mathcal{T} \subset S \times \mathbb{R}^2 \text{ of triangles}\}/\sim$$

where $\sim$ denotes similarity.

This functor is *not* representable as there are non-trivial families of triangles $\mathcal{T}$ such that all fibers are similar triangles (Proposition 0.4.23). For instance, we construct a non-trivial family of triangles over S^1 by gluing two trivial families via a symmetry of an equilateral triangle.

Figure 15: A trivial (left) and non-trivial (right) family of equilateral triangles. Image taken from a video produced by Jonathan Wise: see `http://math.colorado.edu/~jonathan.wise/visual/moduli/index.html`.

0.5 Illustrating example: Grassmanian

As an illustration of the utility of the functorial approach, we introduce the Grassmanian functor $\text{Gr}(k, n)$ over $\mathbb{Z}$ (Definition 0.5.1) and show that it is representable by a projective scheme (Proposition 0.5.7). Since the Grassmanian parameterizes subspaces V of a fixed vector space, this moduli problem does not have non-trivial symmetries, i.e. automorphisms, and thus we do not need the language of groupoids or stacks. This also provides a warmup to the representability and projectivity of Hilbert and Quot schemes (Chapter D).

0.5.1 Functorial definition

The points of the Grassmanian $\text{Gr}(k, n)$ are k-dimensional *quotients* of n-dimensional space.[3] But what are families of k-dimensional quotients over a scheme S? As motivated by Example 0.4.30, they should be locally free quotients of $\mathcal{O}_S^n$:

Definition 0.5.1. The *Grassmanian functor* is

$$\begin{aligned} \text{Gr}(k, n)\colon \ &\text{Sch} \to \text{Sets} \\ &S \mapsto \left\{ [\mathcal{O}_S^n \twoheadrightarrow Q] \;\middle|\; Q \text{ is a vector bundle of rank } k \right\} / \sim \end{aligned}$$

[3] Alternatively, the points could be considered as k-dimensional *subspaces* but in these notes, we will follow Grothendieck's convention of quotients.

where $[\mathcal{O}_S^n \xrightarrow{q} Q] \sim [\mathcal{O}_S^n \xrightarrow{q'} Q']$ if there exists an isomorphism $\Psi\colon Q \xrightarrow{\sim} Q'$ such that

$$\begin{tikzcd} \mathcal{O}_S^n \arrow[r, "q"] \arrow[dr, "q'"'] & Q \arrow[d, "\Psi"] \\ & Q' \end{tikzcd}$$

commutes (i.e. $q' = \Psi \circ q$) or equivalently $\ker(q) = \ker(q')$.

Pullbacks are defined in the obvious manner. Observe that if $k = 1$, then $\mathrm{Gr}(1, n) \cong \mathbb{P}^{n-1}$.

0.5.2 Representability by a scheme

In this subsection, we show that $\mathrm{Gr}(k, n)$ is representable by a scheme (Proposition 0.5.4). Our strategy will be to find a Zariski-open cover of $\mathrm{Gr}(k, n)$ by representable functors; see Definition 0.4.17. Given a subset $I \subset \{1, \ldots, n\}$ of size k, let $\mathrm{Gr}(k, n)_I \subset \mathrm{Gr}(k, n)$ be the subfunctor where for a scheme S, $\mathrm{Gr}(k, n)_I(S)$ is the subset of $\mathrm{Gr}(k, n)(S)$ consisting of surjections $\mathcal{O}_S^n \xrightarrow{q} Q$ such that the composition

$$\mathcal{O}_S^I \xrightarrow{e_I} \mathcal{O}_S^n \xrightarrow{q} Q$$

is an isomorphism, where e_I is the canonical inclusion. When there is no possible ambiguity, we set $\mathrm{Gr}_I := \mathrm{Gr}(k, n)_I$.

Lemma 0.5.2. *For each $I \subset \{1, \ldots, n\}$ of size k, the functor Gr_I is representable by affine space $\mathbb{A}_{\mathbb{Z}}^{k \times (n-k)}$*

Proof. We may assume that $I = \{1, \ldots, k\}$. We define a map of functors $\phi\colon \mathbb{A}^{k \times (n-k)} \to \mathrm{Gr}_I$ where over a scheme S, a $k \times (n-k)$ matrix $f = \{f_{i,j}\}_{\substack{1 \le i \le n \\ 1 \le j \le k}}$ of global functions on S is mapped to the quotient

$$\left(\begin{array}{cccc|ccc} 1 & & & & f_{1,1} & \cdots & f_{1,n-k} \\ & 1 & & & f_{2,1} & \cdots & f_{2,n-k} \\ & & \ddots & & \vdots & & \\ & & & 1 & f_{k,1} & \cdots & f_{k,n-k} \end{array}\right) : \mathcal{O}_S^n \to \mathcal{O}_S^k. \tag{0.5.1}$$

The injectivity of $\phi(S)\colon \mathbb{A}^{k \times (n-k)}(S) \to \mathrm{Gr}_I(S)$ is clear. To see surjectivity, let $[\mathcal{O}_S^n \xrightarrow{q} Q] \in \mathrm{Gr}_I(S)$ where by definition $\mathcal{O}_S^I \xrightarrow{e_I} \mathcal{O}_S^n \xrightarrow{q} Q$ is an isomorphism. The tautological commutative diagram

$$\begin{tikzcd} \mathcal{O}_S^n \arrow[r, "q"] \arrow[dr, "(q \circ e_I)^{-1} \circ q"'] & Q \arrow[d, "(q \circ e_I)^{-1}"] \\ & \mathcal{O}_S^I \end{tikzcd}$$

shows that $[\mathcal{O}_S^n \xrightarrow{q} Q] = [\mathcal{O}_S^n \xrightarrow{(q \circ e_I)^{-1} \circ q} \mathcal{O}_S^I] \in \mathrm{Gr}(k, n)(S)$. Since the composition $\mathcal{O}_S^I \xrightarrow{e_I} \mathcal{O}_S^n \xrightarrow{(q \circ e_I)^{-1}} \mathcal{O}_S^I$ is the identity, the $k \times n$ matrix corresponding to $(q \circ e_I)^{-1} \circ q$ has the same form as (0.5.1) for functions $f_{i,j} \in \Gamma(S, \mathcal{O}_S)$ and therefore $\phi(S)(\{f_{i,j}\}) = [\mathcal{O}_S^n \xrightarrow{q} Q] \in \mathrm{Gr}(k, n)(S)$. $\square$

Lemma 0.5.3. *$\{\mathrm{Gr}_I\}$ is an open cover of $\mathrm{Gr}(k,n)$ where I ranges over all subsets of size k.*

Proof. For a fixed subset I, we first show that $\mathrm{Gr}_I \subset \mathrm{Gr}(k,n)$ is an open subfunctor. To this end, we consider a scheme S and a morphism $S \to \mathrm{Gr}(k,n)$ corresponding to a quotient $q\colon \mathcal{O}_S^n \to Q$. Let C denote the cokernel of the composition $q \circ e_I \colon \mathcal{O}_S^I \to Q$. Notice that if $C = 0$, then q is an isomorphism. The fiber product

$$\begin{tikzcd} F_I \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \arrow[d, "{[\mathcal{O}_S^n \overset{q}{\twoheadrightarrow} Q]}"] \\ \mathrm{Gr}_I \arrow[r] & \mathrm{Gr}(k,n) \end{tikzcd}$$

of functors is representable by the open subscheme $U = S \setminus \mathrm{Supp}(C)$ (the reader is encouraged to verify this claim).

To check the surjectivity of $\bigsqcup_I F_I \to S$, let $s \in S$ be a point. Since $\kappa(s)^n \overset{q\otimes\kappa(s)}{\twoheadrightarrow} Q \otimes \kappa(s)$ is a surjection of vector spaces, there is a non-zero $k \times k$ minor, given by a subset I, of the $k \times n$ matrix $q \otimes \kappa(s)$. This implies that $[\kappa(s)^n \overset{q\otimes\kappa(s)}{\twoheadrightarrow} Q \otimes \kappa(s)] \in F_I(\kappa(s))$. $\square$

Lemmas 0.5.2 and 0.5.3 together imply:

Proposition 0.5.4. *The functor $\mathrm{Gr}(k,n)$ is representable by a scheme.* $\square$

⚠ **Warning 0.5.5.** We will abuse notation by denoting both the functor and the scheme as $\mathrm{Gr}(k,n)$.

Exercise 0.5.6. Use the valuative criterion of properness to show that $\mathrm{Gr}(k,n) \to \operatorname{Spec} \mathbb{Z}$ is proper.

0.5.3 Projectivity of the Grassmanian

We show that the Grassmanian scheme $\mathrm{Gr}(k,n)$ is projective (Proposition 0.5.7) by explicitly providing a projective embedding using the functorial approach. The *Plücker embedding* is the map of functors

$$P\colon\ \mathrm{Gr}(k,n) \to \mathbb{P}\Big(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}}\Big)$$

defined over a scheme S by mapping a rank k quotient $\mathcal{O}_S^n \overset{q}{\twoheadrightarrow} Q$ to the corresponding rank 1 quotient $\bigwedge^k \mathcal{O}_S^n \to \bigwedge^k Q$. As both sides are representable by schemes, the morphism P corresponds to a morphism of schemes via Yoneda's lemma.

Proposition 0.5.7. *The morphism $P\colon \mathrm{Gr}(k,n) \to \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})$ of schemes is a closed immersion. In particular, $\mathrm{Gr}(k,n)$ is a projective scheme.*

Proof. Let $I \subset \{1, \dots, n\}$ be a subset which corresponds to a coordinate x_I on $\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})$. Let $\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})_I$ be the open locus where $x_I \neq 0$. Viewing

$\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}}) \cong \operatorname{Gr}(1, \binom{n}{k})$, then $\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})_I \cong \operatorname{Gr}(1, \binom{n}{k})_{\{I\}}$ (viewing $\{I\}$ as the corresponding subset of $\{1, \dots, \binom{n}{k}\}$ of size 1). Since

$$\begin{tikzcd} \operatorname{Gr}(k,n)_I \arrow[r, "P_I"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})_I \arrow[d] \\ \operatorname{Gr}(k,n) \arrow[r, "P"] & \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}}) \end{tikzcd}$$

is a cartesian diagram of functors, it suffices to show that P_I is a closed immersion. Under the isomorphisms of Lemma 0.5.2, P_I corresponds to the map

$$\mathbb{A}_{\mathbb{Z}}^{k\times(n-k)} \to \mathbb{A}_{\mathbb{Z}}^{\binom{n}{k}-1}$$

assigning a $k \times (n-k)$ matrix $A = \{a_{i,j}\}$ to the element of $\mathbb{A}_{\mathbb{Z}}^{\binom{n}{k}-1}$ whose Jth coordinate, where $J \subset \{1, \dots, n\}$ is a subset of length k distinct from I, is the $\{1, \dots, k\} \times J$ minor of the $k \times n$ block matrix

$$\left(\begin{array}{cccc|ccc} 1 & & & & a_{1,1} & \cdots & a_{1,n-k} \\ & 1 & & & a_{2,1} & \cdots & a_{2,n-k} \\ & & \ddots & & \vdots & & \\ & & & 1 & a_{k,1} & \cdots & a_{k,n-k} \end{array}\right)$$

(of the same form as (0.5.1)). The coordinate $x_{i,j}$ on $\mathbb{A}_{\mathbb{Z}}^{k\times(n-k)}$ is the pull back of the coordinate corresponding to the subset $\{1, \cdots, \widehat{i}, \cdots, k, k+j\}$ (see Figure 16). This shows that the corresponding ring map is surjective thereby establishing that P_I is a closed immersion.

$$x_{i,j} = \det \left(\begin{array}{ccccc|ccccc} 1 & & & & & x_{1,1} & \cdots & x_{1,j} & \cdots & x_{1,n-k} \\ & \ddots & & & & & & \vdots & & \vdots \\ & & 1 & & & x_{i,1} & \cdots & x_{i,j} & \cdots & x_{i,n-k} \\ & & & \ddots & & & & \vdots & & \vdots \\ & & & & 1 & x_{k,1} & \cdots & x_{k,j} & \cdots & x_{k,n-k} \end{array}\right)$$

Figure 16: The minor obtained by removing the ith column and all columns $k+1, \dots, n$ other than $k+j$ is precisely $a_{i,j}$.

$\square$

Exercise 0.5.8. For a field K, let $\operatorname{Gr}(k,n)_K$ be the K-scheme $\operatorname{Gr}(k,n) \times_{\mathbb{Z}} K$, and $p = [K^n \overset{q}{\twoheadrightarrow} Q]$ be a quotient with kernel $K = \ker(q)$. Show that there is a natural bijection of the tangent space

$$T_p \operatorname{Gr}(k,n)_K \xrightarrow{\sim} \operatorname{Hom}(K, Q).$$

with the vector space of K-linear maps $K \to Q$.

Exercise 0.5.9.

(1) Show that the functor $P\colon \mathrm{Gr}(k,n) \to \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})$ is injective on points and tangent spaces.

Hint: You may want to use the identification of the tangent space of $\mathrm{Gr}(k,n)$ *from Exercise 0.5.8. Alternatively you can also show it is a monomorphism.*

(2) Use Exercise 0.5.6, part (1) above and a criterion for a closed immersion (c.f.[Har77, Prop. II.7.3]) to provide an alternative proof that $\mathrm{Gr}(k,n)_K$ is projective.

0.6 Motivation: why the étale topology?

Why is the Zariski topology not sufficient for our purposes? The short answer is that there are not enough Zariski-open subsets and that étale morphisms are an algebro-geometric replacement of analytic open subsets.

0.6.1 What is an étale morphism anyway?

I'm always baffled when a student is intimidated by étale morphisms, especially when the student has already mastered the conceptually more difficult notions of say properness and flatness. One reason may be due to the fact that the definition is buried in [Har77, Exercises III.10.3-6] and its importance is not highlighted there.

The geometric picture of étaleness that you should have in your head is a covering space. The precise definition of an étale morphism is of course more algebraic, and there are in fact many equivalent formulations. This is possibly another point of intimidation for students as it is not at all obvious why the different notions are equivalent, and indeed some of the proofs are quite involved. Nevertheless, if you can take the equivalences on faith, it requires very little effort to not only internalize the concept, but to master its use.

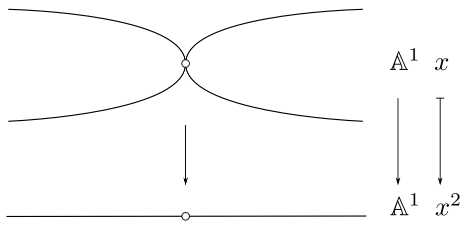

Figure 17: Picture of an étale double cover of $\mathbb{A}^1 \setminus 0$

For a morphism $f\colon X \to Y$ of schemes of finite type over $\mathbb{C}$, the following are equivalent characterizations of étaleness:

- f is smooth of relative dimension 0 (i.e. f is flat and all fibers are smooth of dimension 0);
- f is flat and unramified (i.e. for all $y \in Y(\mathbb{C})$, the scheme-theoretic fiber X_y is isomorphic to a disjoint union $\bigsqcup_i \operatorname{Spec}\mathbb{C}$ of points);
- f is flat and $\Omega_{X/Y} = 0$;

- for all $x \in X(\mathbb{C})$, the induced map $\widehat{\mathcal{O}}_{Y,f(x)} \to \widehat{\mathcal{O}}_{X,x}$ on completions is an isomorphism; and
- (assuming in addition that X and Y are smooth) for all $x \in X(\mathbb{C})$, the induced map $T_{X,x} \to T_{Y,f(x)}$ on tangent spaces is an isomorphism.

We say that f is *étale at* $x \in X$ if there is an open neighborhood U of x such that $f|_U$ is étale.

Exercise 0.6.1. Show that $f\colon \mathbb{A}^1 \to \mathbb{A}^1, x \mapsto x^2$ is étale over $\mathbb{A}^1 \setminus 0$ but is not étale at the origin.

Try to show this for as many of the above definitions as you can.

Étale and smooth morphisms are discussed in much greater detail and generality in Section A.3.

0.6.2 What can you see in the étale topology?

Working with the étale topology is like putting on a better pair of glasses allowing you to see what you couldn't before. Or perhaps more accurately, it is like getting magnifying lenses for your algebraic geometry glasses allowing you to visualize what you already could using your differential geometry glasses.

Example 0.6.2 (Irreducibility of the node)**.** Consider the plane nodal cubic C defined by $y^2 = x^2(x-1)$ in the plane. While there is an analytic open neighborhood of the node $p = (0,0)$ which is reducible, there is no such Zariski-open neighborhood. However, taking a 'square root' of $x-1$ yields a reducible étale neighborhood. More specifically, define $C' = \operatorname{Spec} k[x,y,t]_t/(y^2-x^3+x^2, t^2-x+1)$ and consider

$$C' \to C, \qquad (x,y,t) \mapsto (x,y)$$

Since $y^2 - x^3 + x^2 = (y-xt)(y+xt)$, we see that C' is reducible.

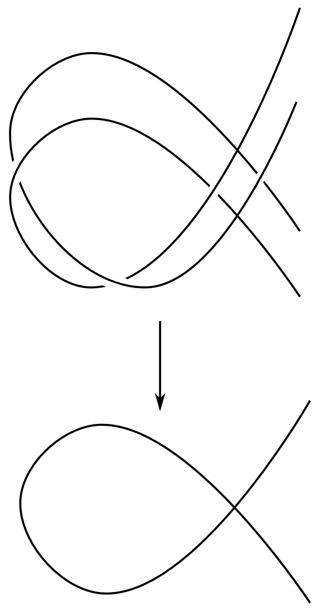

Figure 18: After an étale cover, the nodal cubic becomes reducible.

Example 0.6.3 (Étale cohomology)**.** Sheaf cohomology for the Zarisk-topology can be extended to the étale topology leading to the extremely robust theory of *étale cohomology*. As an example, consider a smooth projective curve C over $\mathbb{C}$ (or equivalently a Riemann surface of genus g), then the étale cohomology $H^1(C_{\text{ét}}, \mathbb{Z}/n)$ of the finite constant sheaf is isomorphic to $(\mathbb{Z}/n)^{2g}$ just like the ordinary cohomology groups, while the sheaf cohomology $H^1(C, \mathbb{Z}/n)$ in the Zariski-topology is 0.

Finally, we would be remiss without mentioning the spectacular application of étale cohomology to prove the Weil conjectures.

Example 0.6.4 (Étale fundamental group)**.** Have you ever thought that there is a similarity between the bijection in Galois theory between intermediate field extensions and subgroups of the Galois group, and the bijection in algebraic topology between covering spaces and subgroups of the fundamental group? Well, you're in good company—Grothendieck also considered this and developed a beautiful theory of the *étale fundamental group* which packages Galois groups and fundamental groups in the same framework.

We only point out here that this connection between étale morphisms and Galois theory is perhaps not so surprising given that a finite field extension L/K is étale (i.e. $\operatorname{Spec} L \to \operatorname{Spec} K$ is étale) if and only if L/K is separable. While we only defined étaleness above for $\mathbb{C}$-varieties, the general notion is not much more complicated; see Étale Equivalences A.3.2.

For the reader interested in reading more about étale cohomology or the étale fundamental group, we recommend [Mil80].

Example 0.6.5 (Quotients by free actions of finite groups)**.** If G is a finite group acting freely on a projective variety X, then there exists a quotient X/G as a projective variety. The essential reason for this is that any G-orbit (or in fact any finite set of points) is contained in an affine variety U, which is the complement of some hypersurface. Then the intersection $V = \bigcap_g gU$ of the G-translates is a G-invariant affine open containing Gx. One can then show that $V/G = \operatorname{Spec} \Gamma(V, \mathcal{O}_V)^G$ and that these local quotients glue to form X/G.

However, if X is not projective, the quotient does not necessarily exist as a scheme. As with most phenomenon for smooth proper varieties that are not-projective, a counterexample is provided by Hironaka's examples of smooth, proper 3-folds; [Har77, App. B, Ex. 3.4.1]. One can construct an example which has a free action by $G = \mathbb{Z}/2$ such that there is an orbit Gx not contained in any G-invariant affine open. This shows that X/G cannot exist as a scheme; indeed, if it did, then the image of x under the finite morphism $X \to X/G$ would be contained in some affine and its inverse would be an affine open containing Gx. See [Knu71, Ex. 1.3] or [Ols16, Ex. 5.3.2] for details.

Nevertheless, for any free action of a finite group G on a scheme X, there does exist a G-invariant étale morphism $U \to X$ from an affine scheme, and the quotients U/G can be glued in the *étale topology* to construct X/G as an *algebraic space*. The upshot is that we can always take quotients of free actions by finite groups, a very desirable feature given the ubiquity of group actions in algebraic geometry; this however comes at the cost of enlarging our category from schemes to algebraic spaces.

Example 0.6.6 (Artin approximation)**.** *Artin approximation* is a powerful and extremely deep result, due to Michael Artin, which implies that most properties which hold for the completion $\widehat{\mathcal{O}}_{X,x}$ of the local ring is also true in an étale neighborhood of x. More precisely, let $F\colon \mathrm{Sch}/X \to \mathrm{Sets}$ be a functor locally of finite presentation (i.e. satisfying the functorial property of Proposition A.1.2), $\widehat{a} \in F(\widehat{\mathcal{O}}_{X,x})$ and N a positive integer. Under the weak hypothesis of excellency on X (which holds if X is locally of finite type over $\mathbb{Z}$ or a field), Artin approximation states that there exists an étale neighborhood $(X', x') \to (X, x)$ with $\kappa(x') = \kappa(x)$ and an element $a' \in F(X')$ agreeing with a on the Nth order neighborhood of x.

For example, in Example 0.6.2, it's not hard to use properties of power series rings to establish that $\widehat{\mathcal{O}}_{C,p} \cong \mathbb{C}[[x,y]]/(y^2 - x^2)$ (e.g. take a power series expansion of $\sqrt{x-1}$), which is reducible. If we consider the functor

$$F\colon \mathrm{Sch}/C \to \mathrm{Sets}, \qquad (C' \xrightarrow{\pi} C) \mapsto \{\text{decompositions } C' = C'_1 \cup C'_2\}$$

then applying Artin approximation yields an étale cover $C' \to C$ with C' reducible. Of course, we already knew this from an explicit construction in Example 0.6.2, but hopefully this example shows the potential power of Artin approximation.

0.6.3 Working with the étale topology: descent theory

Another reason why the étale topology is so useful is that many properties of schemes and their morphisms can be checked on étale covers. For instance, you already know that to check if a scheme X is noetherian, finite type over $\mathbb{C}$, reduced or smooth, it suffices to find a Zariski-open cover $\{U_i\}$ such that the property holds for each U_i. *Descent theory* implies the same with respect to a collection $\{U_i \to U\}$ of étale morphisms such that $\coprod_i U_i \to U$ is surjective: X has the property if and only if each U_i does. Descent theory is developed in Chapter B and is used to prove just about everything concerning algebraic spaces and stacks.

0.7 Moduli stacks: moduli with automorphisms

The failure of the representability of the moduli functors of curves and vector bundles is a motivating factor for introducing moduli stacks, which encode the automorphisms groups as part of the data. We will synthesize the approaches from Section 0.3 on moduli groupoids and Section 0.4 on moduli functors.

0.7.1 Specifying a moduli stack

To define a moduli stack, we need to specify

1. families of objects;
2. how two families of objects are isomorphic; and
3. how families pull back under morphisms.

Notice the difference from specifying a moduli functor (Section 0.4.2) is that rather than specifying *when* two families are isomorphic, we specify *how*.

To specify a moduli stack in the algebro-geometric setting, we need to specify for each scheme T a *groupoid* Fam_T of families of objects over T. As a natural generalization of functors to sets, we could consider assignments

$$F\colon \mathrm{Sch} \to \mathrm{Groupoids}, \quad T \mapsto \mathrm{Fam}_T\,.$$

This presents the technical difficulty of considering functors between the category of schemes and the 'category' of groupoids. Morphisms of groupoids are functors but there are also morphisms of functors (i.e. natural transformations) which we call *2-morphisms*. This leads to a '2-category' of groupoids.

What is actually involved in defining such an assignment F? In addition to defining the groupoids Fam_T over each scheme T, we need pullback functors $f^*\colon \mathrm{Fam}_T \to \mathrm{Fam}_S$ for each morphism $f\colon S \to T$. But what should be the compatibility for a composition $S \xrightarrow{f} T \xrightarrow{g} U$ of schemes? Well, there should be an isomorphism of functors (i.e. a 2-morphism) $\mu_{f,g}\colon (f^* \circ g^*) \xrightarrow{\sim} (g \circ f)^*$. Should the isomorphisms $\mu_{f,g}$ satisfy a compatibility condition under triples $S \xrightarrow{f} T \xrightarrow{g} U \xrightarrow{h} V$? Yes, but we won't spell it out here (although we encourage the reader to work it out). Altogether this leads to the concept of a *pseudo-functor* (see [SP, Tag 003N]). We will take another approach however in specifying prestacks that avoids specifying such compatibility data.

0.7.2 Motivating the definition of a prestack

Instead of trying to define an assignment $T \mapsto \mathrm{Fam}_T$, we will build one massive category $\mathcal{X}$ encoding all of the groupoids Fam_T which will live over the category Sch of schemes. Loosely speaking, the objects of $\mathcal{X}$ will be a family a of objects over a scheme S, i.e. $a \in \mathrm{Fam}_S$. If $a \in \mathrm{Fam}_S$ and $b \in \mathrm{Fam}_T$, a morphism $a \to b$ in $\mathcal{X}$ will be a morphism $f\colon S \to T$ together with an isomorphism $a \xrightarrow{\sim} f^*b$.

A *prestack* over Sch is a category $\mathcal{X}$ together with functor $p\colon \mathcal{X} \to \mathrm{Sch}$, which we visualize as

$$\begin{tikzcd} \mathcal{X} \arrow[d, "p"] \\ \mathrm{Sch} \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d, mapsto] & b \arrow[d, mapsto] \\ S \arrow[r, "f"] & T \end{tikzcd}$$

where the lower case letters a, b are objects in $\mathcal{X}$ and the upper case letters S, T are objects in Sch. We say that a is over S and $\alpha\colon a \to b$ is over $f\colon S \to T$. Moreover, we need to require certain natural axioms to hold for $\mathcal{X} \xrightarrow{p} \mathrm{Sch}$. This will be given in full later but vaguely we need to require the existence and uniqueness of pullbacks: given a map $S \to T$ and object $b \in \mathcal{X}$ over T, there should exist an arrow $a \xrightarrow{\alpha} b$ over f satisfying a suitable universal property. See Definition 1.3.1 for a precise definition.

Given a scheme S, the *fiber category* $\mathcal{X}(S)$ is the category of objects over S whose morphisms are over id_S. If $\mathcal{X}$ is built from the groupoids Fam_S as above, then the fiber category $\mathcal{X}(S) = \mathrm{Fam}_S$.

Example 0.7.1 (Viewing a moduli functor as a moduli prestack)**.** A moduli functor $F\colon \mathrm{Sch} \to \mathrm{Sets}$ can be encoded as a moduli prestack as follows: we define the category $\mathcal{X}_F$ of pairs (S, a) where S is a scheme and $a \in F(S)$. A map $(S', a) \to (S, a)$ is a map $f\colon S' \to S$ such that $a' = f^*a$, where f^* is convenient shorthand for $F(f)\colon F(S) \to F(S')$. Observe that the fiber categories $\mathcal{X}_F(S)$ are equivalent (even equal) to the set $F(S)$.

Example 0.7.2 (Moduli prestack of smooth curves)**.** We define the *moduli prestack of smooth curves* as the category $\mathcal{M}_g$ of families of smooth curves $\mathcal{C} \to S$ together with the functor $p\colon \mathcal{M}_g \to \mathrm{Sch}$ where $(\mathcal{C} \to S) \mapsto S$. A map $(\mathcal{C}' \to S') \to$

$(\mathcal{C} \to S)$ is the data of maps $\alpha\colon \mathcal{C}' \to \mathcal{C}$ and $f\colon S' \to S$ such that the diagram

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

is cartesian.

Example 0.7.3 (Moduli prestack of vector bundles)**.** Let C be a fixed smooth, connected and projective curve over $\mathbb{C}$, and fix integers $r \geq 0$ and d. We define the *moduli prestack of vector bundles on* C as the category $\mathcal{M}_{C,r,d}$ of pairs (E, S) where S is a scheme and E is a vector bundle on $C_S = C \times_{\mathbb{C}} S$ together with the functor $p\colon \mathcal{M}_{C,r,d} \to \mathrm{Sch}/\mathbb{C}, (E, S) \mapsto S$. A map $(E', S') \to (E, S)$ consists of a map of schemes $f\colon S' \to S$ together with a map $E \to (\mathrm{id} \times f)_* E'$ of $\mathcal{O}_{C_S}$-modules whose adjoint is an isomorphism (i.e. for any choice of pull back $(\mathrm{id} \times f)^* E$, the adjoint map $(\mathrm{id} \times f)^* E \to E'$ is an isomorphism). Note that a map $(E', S) \to (E, S)$ over the identity map id_S consists simply of an isomorphism $E' \to E$.

Remark 0.7.4. We have formulated morphisms using the adjoint because the pull back is only defined up to isomorphism while the pushforward is canonical. If we were to instead parameterize the total spaces of vector bundles (i.e. $\mathbb{A}(E)$ rather than E), then a morphism $(V', S') \to (V, S)$ would consist of morphisms $\alpha\colon V' \to V$ and $f\colon S' \to S$ such that $V' \to V \times_{C_S} C_{S'}$ is an isomorphism of vector bundles.

0.7.3 Motivating the definition of a stack

A stack is to a prestack as a sheaf is to a presheaf. The concept could not be more intuitive: we require that objects and morphisms glue uniquely.

Example 0.7.5 (Moduli stack of sheaves over a point)**.** Define the category $\mathcal{X}$ over Sch of pairs (E, S) where E is a sheaf of abelian groups on a scheme S, and the functor $p\colon \mathcal{X} \to \mathrm{Sch}$ given by $(E, S) \mapsto S$. A map $(E', S') \to (E, S)$ in $\mathcal{X}$ is a map of schemes $f\colon S' \to S$ together with a map $E \to f_* E'$ of $\mathcal{O}_{S'}$-modules whose adjoint is an isomorphism.

You already know that morphisms of sheaves glue [Har77, Exercise II.1.15]: let E and F be sheaves on schemes S and T, and let $f\colon S \to T$ be a map. If $\{S_i\}$ is a Zariski-open cover of S, then giving a morphism $\alpha\colon (E, S) \to (F, T)$ is the same data as giving morphisms $\alpha_i\colon (E|_{S_i}, S_i) \to (F, T)$ such that $\alpha_i|_{S_{ij}} = \alpha_j|_{S_{ij}}$.

You also know how sheaves themselves glue [Har77, Exercise II.1.22]—it is more complicated than gluing morphisms since sheaves have automorphisms and given two sheaves, we prefer to say that they are isomorphic rather than equal. If $\{S_i\}$ is a Zariski-open cover of a scheme S, then giving a sheaf E on S is equivalent to giving a sheaf E_i on S_i and isomorphisms $\phi_{ij}\colon E_i|_{S_{ij}} \to E_j|_{S_{ij}}$ such that $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on the triple intersection S_{ijk}.

In an identical way, we could have considered the moduli stack of $\mathcal{O}$-modules, quasi-coherent sheaves or vector bundles.

The definition of a stack simply axiomitizes these two natural gluing concepts; it is postponed until Definition 1.4.1.

Exercise 0.7.6. Convince yourself that Examples 0.7.2 and 0.7.3 satisfy the same gluing axioms. (See also Propositions 1.4.6 and 1.4.9.)

0.7.4 Motivating the definition of an algebraic stack

There are functors $F\colon \mathrm{Sch} \to \mathrm{Sets}$ that are sheaves when restricted to the Zariski topology on any scheme T but that are not necessarily representable by schemes; see for instance **????**. In a similar way, there are prestacks $\mathcal{X}$ that are stacks but that are not sufficiently algebro-geometric. If we wish to bring our algebraic geometry toolkit (e.g. coherent sheaves, commutative algebra, cohomology, ...) to study stacks in a similar way that we study schemes, we must impose an algebraicity condition.

The condition we impose on a stack to be algebraic is very natural. Recall that a functor $F\colon \mathrm{Sch} \to \mathrm{Sets}$ is representable by a scheme if and only if there is a Zariski-open cover $\{U_i \subset F\}$ such that U_i is an affine scheme. Similarly, we will say that a stack $\mathcal{X} \to \mathrm{Sch}$ is *algebraic* if

- there is a smooth cover $\{U_i \to \mathcal{X}\}$ where each U_i is an affine scheme.

To make this precise, we need to define what it means for $\{U_i \to \mathcal{X}\}$ to be a smooth cover. Just like in the definition of Zariski-open cover (Definition 0.4.17(3)), we require that for every morphism $T \to \mathcal{X}$ from a scheme T, the fiber product (fiber products of prestacks will be formally introduced in §1.3.5) $U_i \times_{\mathcal{X}} T$ is representable (by an algebraic space) such that $\coprod_i U_i \times_{\mathcal{X}} T \to T$ is a smooth and surjective morphism. See Definition 2.1.5 for the precise definition of an algebraic stack.

Constructing a smooth cover of a given moduli stack is a geometric problem inherent to the moduli problem. It can often be solved by ridigifying the moduli problem by parameterizing additional information. This concept is best absorbed in examples.

Example 0.7.7 (Moduli stack of elliptic curves)**.** An elliptic curve (E, p) over $\mathbb{C}$ is embedded into $\mathbb{P}^2$ via $\mathcal{O}_E(3p)$ such that E is defined by a Weierstrass equation $y^2z = x(x-z)(x-\lambda z)$ for some $\lambda \neq 0, 1$ [Har77, Prop. 4.6]. Let $U = \mathbb{A}^1 \setminus \{0, 1\}$ with coordinate λ. The family $\mathcal{E} \subset U \times \mathbb{P}^2$ of elliptic curves defined by the Weierstrass equation gives a smooth (even étale) cover $U \to \mathcal{M}_{1,1}$.

Example 0.7.8 (Moduli stack of smooth curves)**.** For any smooth, connected and projective curve C of genus $g \geq 2$, the third tensor power $\omega_C^{\otimes 3}$ is very ample and gives an embedding $C \hookrightarrow \mathbb{P}(H^0(C, \omega_c^{\otimes 3})) \cong \mathbb{P}^{5g-6}$. There is a Hilbert scheme H parameterizing closed subschemes of $\mathbb{P}^{5g-6}$ with the same Hilbert polynomial as $C \subset \mathbb{P}^{5g-6}$, and there is a locally closed subscheme $H' \subset H$ parameterizing *smooth* subschemes such that $\omega_C^{\otimes 3} \cong \mathcal{O}_C(1)$. The universal subscheme over H' yields a smooth cover $H' \to \mathcal{M}_g$.

Example 0.7.9 (Moduli stack of vector bundles)**.** For any vector bundle E of rank r and degree d on a smooth, connected and projective curve C, the twist $E(m)$ is globally generated for sufficiently large m. Taking $N = h^0(C, E(m))$, we can view E as a quotient $\mathcal{O}_C(-m)^N \twoheadrightarrow E$. There is a Quot scheme Q_m parameterizing quotients $\mathcal{O}_C(-m)^N \overset{\pi}{\twoheadrightarrow} F$ with the same Hilbert polynomial as E and a locally closed subscheme $Q'_m \subset Q$ parameterizing quotients where E is a vector bundle and such that the induced map $H^0(\pi \otimes \mathcal{O}_C(m))\colon \mathbb{C}^N \to H^0(C, E(m))$ is an isomorphism. The universal quotient over Q'_m defines a smooth map $Q'_m \to \mathcal{M}_{C,r,d}$ and the collection $\{Q'_m \to \mathcal{M}_{C,r,d}\}$ over $m \gg 0$ defines a smooth cover.

0.7.5 Deligne–Mumford stacks and algebraic spaces

A *Deligne–Mumford stack* can be defined in two equivalent ways:

- a stack $\mathcal{X}$ such that there exists an étale (rather than smooth) cover $\{U_i \to \mathcal{X}\}$ by schemes; or
- an algebraic stack such that all automorphisms groups of field-valued points are étale, i.e. discrete (e.g. finite) and reduced.

The moduli stacks $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ are Deligne–Mumford for $g \geq 2$, but $\mathcal{M}_{C,r,d}$ is not. Similarly, an *algebraic space* can be defined in two equivalent ways:

- a sheaf (i.e. a contravariant functor $F\colon \mathrm{Sch} \to \mathrm{Sets}$ that is a sheaf in the big étale topology) such that there exists an étale cover $\{U_i \to F\}$ by schemes; or
- an algebraic stack such that all automorphisms groups of field-valued points are trivial.

In other words, an algebraic space is an algebraic stack without any stackiness.

Table 1: Schemes, algebraic spaces, Deligne–Mumford stacks, and algebraic stacks are obtained by gluing affine schemes in certain topologies

Algebro-geometric space	Type of object	Obtained by gluing
Schemes	sheaf	affine schemes in the Zariski topology
Algebraic spaces	sheaf	affine schemes in the étale topology
Deligne–Mumford stacks	stack	affine schemes in the étale topology
Algebraic stacks	stack	affine schemes in the smooth topology

Example 0.7.10 (Quotients by finite groups)**.** Quotients by free actions of finite groups exist as algebraic spaces! See Corollary 2.1.9.

0.8 Moduli stacks and quotients

One of the most important examples of a stack is a quotient stack $[X/G]$ arising from an action of a smooth algebraic group G on a scheme X. The geometry of $[X/G]$ couldn't be simpler: it's the G-equivariant geometry of X.

Similar to how toric varieties provide concrete examples of schemes, quotient stacks provide both concrete examples useful to gain geometric intuition of general algebraic stacks and a fertile testing ground for conjectural results. On the other hand, it turns out that many algebraic stacks are quotient stacks (or at least locally quotient stacks) and therefore any (local) property that holds for quotient stacks also holds for many algebraic stacks.

0.8.1 Motivating the definition of the quotient stack

The quotient functor $\mathrm{Sch} \to \mathrm{Sets}$ defined by $S \mapsto X(S)/G(S)$ is not a sheaf even when the action is free (see Example 0.4.31). We therefore first need to consider a better notion for a family of orbits.

For simplicity, let's assume that G and X are defined over $\mathbb{C}$. For $x \in X(\mathbb{C})$, there is a G-equivariant map $\sigma_x \colon G \to X$ defined by $g \mapsto g \cdot x$. Note that two points x, x' are in the same G-orbit (say $x = hx'$), if and only if there is a G-equivariant morphism $\varphi \colon G \to G$ (say by $g \mapsto gh$) such that $\sigma_x = \sigma_{x'} \circ \varphi$.

We can try the same thing for a T-point $T \xrightarrow{f} X$ by considering

$$\begin{tikzcd} G \times T \arrow[r, "f"] \arrow[d, "p_2"] & X, & (g,t) \arrow[r, mapsto] & g \cdot f(t) \\ T & & & \end{tikzcd}$$

and noting that $f \colon G \times T \to X$ is a G-equivariant map. If we define a prestack consisting of such families, it fails to be a stack as objects don't glue: given a Zariski-cover $\{T_i\}$ of T, maps $T_i \xrightarrow{f_i} X$ and isomorphisms of the restrictions to T_{ij}, the trivial bundles $G \times T_i \to T_i$ will glue to a G-torsor $P \to T$ but it will *not* necessarily be trivial (i.e. $P \cong G \times T$). It is clear then how to correct this using the language of G-torsors (see Section C.3):

Definition 0.8.1 (Quotient stack)**.** We define $[X/G]$ as the category over Sch whose objects over a scheme S are diagrams

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] & X \\ S & \end{tikzcd}$$

where $P \to S$ is a G-torsor and $f \colon P \to X$ is a G-equivariant morphism. A morphism $(P' \to S', P' \xrightarrow{f'} X) \to (P \to S, P \xrightarrow{f} X)$ consists a maps $g \colon S' \to S$ and $\varphi \colon P' \to P$ of schemes such that the diagram

$$\begin{tikzcd} P' \arrow[r, "\varphi"] \arrow[d] \arrow[rr, bend left, "f'"] \arrow[dr, phantom, "\square"] & P \arrow[r, "f"] \arrow[d] & X \\ S' \arrow[r, "g"] & S & \end{tikzcd}$$

commutes with the left square cartesian.

There is an object of $[X/G]$ over X given by the diagram

$$\begin{tikzcd} G \times X \arrow[r, "\sigma"] \arrow[d, "p_2"] & X \\ X, & \end{tikzcd}$$

where σ denotes the action map. This corresponds to a map $X \to [X/G]$ via a 2-categorical version of Yoneda's lemma.

The map $X \to [X/G]$ is a G-torsor even if the action of G on X is not free. We state that again: **the map $X \to [X/G]$ is a G-torsor even if the action of G on X is not free**. Pause for a moment to appreciate how remarkable that is!

In particular, the map $X \to [X/G]$ is smooth and it follows that $[X/G]$ is algebraic. At the expense of enlarging our category from schemes to algebraic stacks, we are able to (tautologically) construct the quotient $[X/G]$ as a 'geometric space' with desirable geometric properties.

Example 0.8.2. Specializing to the case that $X = \operatorname{Spec} \mathbb{C}$ is a point, we define the *classifying stack of* G as the category $BG := [\operatorname{Spec} \mathbb{C}/G]$ of G-torsors $P \to S$. The projection $\operatorname{Spec} \mathbb{C} \to BG$ is not only a G-torsor; it is the *universal G-torsor*. Given any other G-torsor $P \to S$, there is a unique map $S \to BG$ and a cartesian diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} \mathbb{C} \arrow[d] \\ S \arrow[r] & BG. \end{tikzcd}$$

Exercise 0.8.3. What is the universal family over the quotient stack $[X/G]$?

0.8.2 Moduli as quotient stacks

Moduli stacks can often be described as quotient stacks, and these descriptions can be leveraged to establish properties of the moduli stack.

Example 0.8.4 (Moduli stack of smooth curves)**.** In Example 0.7.8, the embedding of a smooth curve C via $C \overset{|\omega_C^{\otimes 3}|}{\hookrightarrow} \mathbb{P}^{5g-6}$ depends on a choice of basis $H^0(C, \omega_C^{\otimes 3}) \cong \mathbb{C}^{5g-5}$ and therefore is only unique up to a projective automorphism, i.e. an element of $\mathrm{PGL}_{5g-5} = \operatorname{Aut}(\mathbb{P}^{5g-6})$. The action of the algebraic group PGL_{5g-5} on the scheme H', parameterizing *smooth* subschemes such that $\omega_C \cong \mathcal{O}_C(3)$, yields an identification $\mathcal{M}_g \cong [H'/\mathrm{PGL}_{5g-6}]$. See Theorem 2.1.11.

Example 0.8.5 (Moduli stack of vector bundles)**.** In Example 0.7.9, the presentation of a vector bundle E as a quotient $\mathcal{O}_C(-m)^N \twoheadrightarrow E$ depends on a choice of basis $H^0(C, E(m)) \cong \mathbb{C}^N$. The algebraic group PGL_{N-1} acts on the scheme Q'_m, parameterizing vector bundle quotients of $\mathcal{O}_C(-m)^N$ such that $\mathbb{C}^N \xrightarrow{\sim} H^0(C, E(m))$, yields an identification $\mathcal{M}_{C,r,d} \cong \bigcup_{m \gg 0} [Q'_m/\mathrm{PGL}_{N-1}]$. See Theorem 2.1.15.

0.8.3 Geometry of $[X/G]$

While the definition of the quotient stack $[X/G]$ may appear abstract, its geometry is very familiar. The table below provides a dictionary between the geometry of a quotient stack $[X/G]$ and the G-equivariant geometry of X. The stack-theoretic concepts on the left-hand side will be introduced later. For simplicity we work over $\mathbb{C}$.

Table 2: Dictionary

Geometry of $[X/G]$	G-equivariant geometry of X
$\mathbb{C}$-point $x \in [X/G]$	orbit Gx
automorphism group $\mathrm{Aut}(x)$	stabilizer G_x
function $f \in \Gamma([X/G], \mathcal{O}_{[X/G]})$	G-equivariant function $f \in \Gamma(X, \mathcal{O}_X)^G$
map $[X/G] \to Y$ to a scheme Y	G-equivariant map $X \to Y$
line bundle	G-equivariant line bundle (or linearization)
quasi-coherent sheaf	G-equivariant quasi-coherent sheaf
tangent space $T_{[X/G],x}$	normal space $T_{X,x}/T_{Gx,x}$ to the orbit
coarse moduli space $[X/G] \to Y$	geometric quotient $X \to Y$
good moduli space $[X/G] \to Y$	good GIT quotient $X \to Y$

0.9 Constructing moduli spaces as projective varieties

One of the primary reasons for introducing algebraic stacks to begin with is to ensure that a given moduli problem $\mathcal{M}$ is in fact represented by a bona fide algebro-geometric space equipped with a universal family. Many geometric questions can be answered (and arguably should be answered) by studying the moduli stack $\mathcal{M}$ itself. However, even in the presence of automorphisms, there still may exist a scheme—even a projective variety— that closely approximates the moduli problem. If we are willing to sacrifice some desirable properties (e.g. a universal family), we can sometimes construct a more familiar algebro-geometric space—namely a projective variety—where we have the much larger toolkit of projective geometry (e.g., Hodge theory, birational geometry, intersection theory, ...) at our disposal.

In this section, we present a general strategy for a constructing a moduli space specifically as a *projective variety*.

0.9.1 Boundedness

The first potential problem is that our moduli problem may simply have too many objects so that there is no hope of representing it by a *finite type* or *quasi-compact* scheme. We say that a moduli functor or stack $\mathcal{M}$ over $\mathbb{C}$ is *bounded* if there exists a scheme X of finite type over $\mathbb{C}$ and a family of objects $\mathcal{E}$ over X such that every object E of $\mathcal{M}$ is isomorphic to a fiber $E \cong \mathcal{E}_x$ for some (not necessarily unique) $x \in X(\mathbb{C})$.

Example 0.9.1. Let Vect be the algebraic stack over $\mathbb{C}$ where objects over a scheme S consist of vector bundles. Since we have not specified the rank, $\mathrm{Vect}_{\mathbb{C}}$ is not bounded. In fact, if we let $\mathrm{Vect}_r \subset \mathrm{Vect}$ be the substack parameterizing vector bundles of rank r, then $\mathrm{Vect} = \bigsqcup_{r\geq 0} \mathrm{Vect}_r$ While Vect is locally of finite type over $\mathbb{C}$, it is not of finite type (or equivalently quasi-compact).

Exercise 0.9.2. Show that Vect_r is isomorphic to the classifying stack $B\,\mathrm{GL}_r$ (Example 0.8.2).

Example 0.9.3. Let $\mathcal{V}$ be the stack of *all* vector bundles over a smooth, connected and projective curve C. The stack $\mathcal{V}$ is clearly not bounded since we haven't specified the rank and degree. But even the substack $\mathcal{M}_{C,r,d}$ of vector bundles with prescribed rank and degree is not bounded! For example, on $\mathbb{P}^1$, there are vector bundles $\mathcal{O}(-d) \oplus \mathcal{O}(d)$ of rank 2 and degree 0 for every $d \in \mathbb{Z}$, and not all of them can arise as the fibers of a single vector bundle on a finite type $\mathbb{C}$-scheme.

Exercise 0.9.4. Prove that $\mathcal{M}_{C,r,d}$ is not bounded for any curve C.

Although $\mathcal{M}_{C,r,d}$ is not bounded, we will study the substack $\mathcal{M}^{\mathrm{ss}}_{C,r,d}$ of *semistable* vector bundles which is bounded. Semistable vector bundles admit a number of remarkable properties with boundedness being one of the most important.

0.9.2 Compactness

Projective varieties are compact so if we are going to have any hope to construct a projective moduli space, the moduli stack better be compact as well. However, many moduli stacks such as $\mathcal{M}_g$ are not compact as they don't have *enough* objects. This is in contrast to the issue of non-boundedness where there may be *too many* objects.

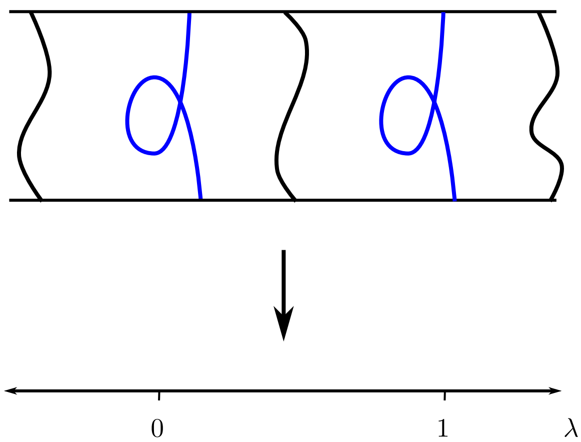

Figure 19: The family of elliptic curves $y^2z = x(x-z)(x-\lambda z)$ degenerates to the nodal cubic over $\lambda = 0, 1$.

The scheme-theoretic notion for compactness is *properness*—universally closed, separated and of finite type. There is a conceptual criterion to test properness called the *valuative criterion* which loosely speaking requires one-dimensional limits to exist. The usefulness of the valuative criterion is arguably best witnessed through studying moduli problems.

More precisely, a moduli stack $\mathcal{M}$ of finite type over $\mathbb{C}$ is *proper* (resp. *universally closed*, *separated*) if for every DVR R with fraction field K and for any diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \mathcal{M} \\ \operatorname{Spec} R, \arrow[ur, dashed] & \end{tikzcd} \tag{0.9.1}$$

after possibly allowing for an extension of R, there exists a unique extension (resp. there exists an extension, resp. there exists at most one extension) of the above diagram.[4] Since $\mathcal{M}$ is a moduli stack, a map $\operatorname{Spec} K \to \mathcal{M}$ corresponds to an object $E^\times$ over $\operatorname{Spec} K$ and a dotted arrow corresponds to a family of objects E over $\operatorname{Spec} R$ and an isomorphism $E|_{\operatorname{Spec} K} \cong E^\times$. In other words, properness of $\mathcal{M}$ means that every object E^* over the punctured disk $\operatorname{Spec} K$ extends uniquely (after possibly allowing for an extension of R) to a family E of objects over the entire disk $\operatorname{Spec} R$.

Example 0.9.5. The moduli stack $\mathcal{M}_g$ of smooth curves is not proper as exhibited in Figure 19. The pioneering insight of Deligne and Mumford is that there is a *moduli-theoretic compactification*! Namely, there is an algebraic stack $\overline{\mathcal{M}}_g$ parameterizing *Deligne–Mumford stable curves*, i.e. proper curves C with at worst nodal singularities such that any smooth rational subcurve $\mathbb{P}^1 \subset C$ intersects the rest of the curve along at least three points. The stack $\overline{\mathcal{M}}_g$ is a proper algebraic stack (due to the stable reduction theorem for curves) and contains $\mathcal{M}_g$ as an open substack.

Example 0.9.6. Let $\mathcal{M}^{\mathrm{ss}}_{C,r,d}$ be the moduli stack parameterizing semistable vector bundles over a curve of prescribed rank and degree. We will later show that $\mathcal{M}^{\mathrm{ss}}_{C,r,d}$ is an algebraic stack of finite type over $\mathbb{C}$. Langton's semistable reduction theorem states that $\mathcal{M}^{\mathrm{ss}}_{C,r,d}$ is universally closed, i.e. satisfies the existence part of the above valuative criterion.

However $\mathcal{M}^{\mathrm{ss}}_{C,r,d}$ is not separated as there may exist several non-isomorphic extensions of a vector bundle on C_K to C_R. Indeed, let E be vector bundle and consider the trivial family E_K on C_K. This extends to trivial family E_R over C_R but the data of an extension

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "{[E_K]}"] \arrow[d] & \mathcal{M}^{\mathrm{ss}}_{C,r,d} \\ \operatorname{Spec} R, \arrow[ur, dashed, "{[E_R]}"'] & \end{tikzcd}$$

also consists of an isomorphism $E_R|_{C_K} = E_K \xrightarrow{\sim} E_K$ or equivalently a K-point of $\operatorname{Aut}(E)$. There are many such isomorphisms and some don't extend to R-points. The automorphism group of a vector bundle is a positive dimensional (affine) algebraic group containing a copy of $\mathbb{G}_m$ corresponding to scaling. For instance, if $\pi \in K$ is a uniformizing parameter, the automorphism $1/\pi \in \mathbb{G}_m(K)$ does not extend to $\mathbb{G}_m(R)$ so (E_R, id) and $(E_R, 1/\pi)$ give non-isomorphic extensions of E_K. In a similar way, any moduli stack which has an object with a positive dimensional affine automorphism group is not separated.

[4] The valuative criterion can be equivalently formulated by replacing the local curve $\operatorname{Spec} R$ with a smooth curve C and $\operatorname{Spec} K$ with a puncture curve $C \setminus p$.

0.9.3 Enlarging a moduli stack

It is often useful to consider enlargements $\mathcal{X} \subset \mathcal{M}$ of a given moduli stack $\mathcal{X}$ by parameterizing a larger collection of objects. For instance, rather than just considering smooth or Deligne–Mumford stable curve, you could consider all curves, or rather than considering semistable vector bundles, you could consider all vector bundles or even all coherent sheaves.

Let's call an object of $\mathcal{M}$ *semistable* if is isomorphic to an object of $\mathcal{X}$; in this way, we can view $\mathcal{X} = \mathcal{M}^{\mathrm{ss}} \subset \mathcal{M}$ as the substack of semistable objects. Often it is easier to show properties (e.g. algebraicity) for $\mathcal{M}$ and then infer the corresponding property for $\mathcal{M}^{\mathrm{ss}}$.

0.9.4 The six steps toward projective moduli

In the setting of a moduli stack $\mathcal{M}^{\mathrm{ss}}$ of semistable objects and an enlargement $\mathcal{M}^{\mathrm{ss}} \subset \mathcal{M}$, we outline the steps to construct a projective moduli scheme M^{ss} approximating $\mathcal{M}^{\mathrm{ss}}$.[5]

Step 1 (Algebraicity): $\mathcal{M}$ is an algebraic stack locally of finite type over $\mathbb{C}$.

This requires first defining $\mathcal{M}$ by specifying both (1) families of objects over an arbitrary $\mathbb{C}$-scheme S, (2) how two families are isomorphic, and (3) how families pull back; see Section 0.7.1. One must then check that $\mathcal{M}$ is a stack.

To check that $\mathcal{M}$ is an algebraic stack locally of finite type over $\mathbb{C}$ entails finding a smooth cover of $\{U_i \to \mathcal{M}\}$ by affine schemes (see Section 0.7.4) where each U_i is of finite type over $\mathbb{C}$.

An alternative approach is to verify 'Artin's criteria' for algebraicity which essentially amounts to verifying local properties of the moduli problem and in particular requires an understanding of the deformation and obstruction theory.

Step 2 (Openness of semistability): semistability is an open condition, i.e. $\mathcal{M}^{\mathrm{ss}} \subset \mathcal{M}$ is an open substack.

If E is an object of $\mathcal{M}$ over T, one must show that the locus of points $t \in T$ such that the restriction E_t is semistable is an open subset of T. Indeed, just like in the definition of an open subfunctor, a substack $\mathcal{M}^{\mathrm{ss}} \subset \mathcal{M}$ is open if and only if for all maps $T \to \mathcal{M}$, the fiber product $\mathcal{M}_{\mathrm{ss}} \times_{\mathcal{M}} T$ is an open subscheme of T. This ensures in particular that $\mathcal{M}^{\mathrm{ss}}$ is also an algebraic stack locally of finite type.

Step 3 (Boundedness of semistability): semistability is bounded, i.e. $\mathcal{M}^{\mathrm{ss}}$ is of finite type over $\mathbb{C}$.

One must verify the existence of a scheme T of finite type over $\mathbb{C}$ and a family $\mathcal{E}$ of objects over T such that every semistable object $E \in \mathcal{M}^{\mathrm{ss}}(\mathbb{C})$ appears as a fiber of $\mathcal{E}$; see Section 0.9.1. In other words, one must exhibit a surjective map $U \to \mathcal{M}$ from a scheme U of finite type. It is worth noting that since we already know $\mathcal{M}$ is locally of finite type, the finite typeness of $\mathcal{M}$ is equivalent to quasi-compactness; *boundedness* is casual term often used to refer to this property.

[5]The calligraphic font $\mathcal{M}^{\mathrm{ss}}$ denotes an algebraic stack while the Roman font M^{ss} denotes an algebraic space. This notation will be continued throughout the notes.

Step 4 (Existence of coarse/good moduli space): there exists either a coarse or good moduli space $\mathcal{M}^{ss} \to M^{ss}$ where M^{ss} is a separated algebraic space.

The algebraic space M^{ss} can be viewed as the best possible approximation of $\mathcal{M}^{ss}$ which is an algebraic space. If automorphisms are finite and $\mathcal{M}^{ss}$ is a proper Deligne–Mumford stack, the Keel–Mori theorem ensures that there exists a *coarse moduli space* $\pi : \mathcal{M}^{ss} \to M^{ss}$ with M^{ss} proper; this means that (1) π is universal for maps to algebraic spaces and (2) π induces a bijection between the isomorphism classes of $\mathbb{C}$-points of $\mathcal{M}^{ss}$ and the $\mathbb{C}$-points of M^{ss}.

In the case of infinite automorphisms, we often cannot expect the existence of a coarse moduli space (as defined above) and we therefore relax the notion to a *good moduli space* $\pi : \mathcal{M}^{ss} \to M^{ss}$ which may identify non-isomorphic objects. In fact, it identifies precisely the $\mathbb{C}$-points whose closures in $\mathcal{M}^{ss}$ intersect in an analogous way to the orbit closure equivalence relation in GIT. A good moduli space is also universal for maps to algebraic spaces even if this property is not obvious from the definitions. We will use an analogue of the Keel–Mori theorem which ensures the existence of a proper good moduli space as long as $\mathcal{M}^{ss}$ can be verified to be both 'S-complete' and 'Θ-reductive'.

Step 5 (Semistable reduction): $\mathcal{M}^{ss}$ is universally closed, i.e. satisfies the existence part of the valuative criterion for properness.

This requires checking that any family of objects $E^{\times}$ over a punctured DVR or smooth curve $C^{\times} = C \setminus p$ has at least one extension to a family of objects over C after possibly taking an extension of C; see Section 0.9.2. For moduli problems with finite automorphisms, the uniqueness of the extension can usually be verified, which implies the properness of $\mathcal{M}$. For moduli problems with infinite affine automorphism groups, the extension is never unique. While $\mathcal{M}$ is therefore not separated, you can often still verify a condition called 'S-completeness', which enjoys properties analogous to separatedness. This property is often referred to as *stable or semistable reduction.*

As a consequence, we conclude that M^{ss} is a proper algebraic space.

Step 6 (Projectivity): a tautological line bundle on $\mathcal{M}^{ss}$ descends to an ample line bundle on M^{ss}.

This is often the most challenging step in this process. It requires a solid understanding of the geometry of the moduli problem and often relies on techniques in higher dimensional geometry.

0.9.5 An alternative approach using Geometric Invariant Theory

The approach outlined above is by no means the only way to construct moduli spaces. One alternative approach is Mumford's Geometric Invariant Theory, which has been wildly successful in both constructing and studying moduli spaces. The main idea is to rigidify the moduli stack $\mathcal{M}^{ss}$ (e.g. $\overline{\mathcal{M}}_g$) by parameterizing additional data (e.g. a stable curve C *and* an embedding $C \overset{|\omega_C^{\otimes 3}|}{\hookrightarrow} \mathbb{P}^N$) in such way that it represented by a projective scheme X and such that the different choices of additional data correspond to different orbits for the action of an algebraic group G acting on X. This provides an identification of the moduli stack $\mathcal{M}^{\mathrm{ss}}$ as an open substack of the quotient stack $[X/G]$. Given a *choice* of equivariant embedding $X \hookrightarrow \mathbb{P}^n$, GIT constructs the quotient as the projective variety

$$X/\!\!/G := \operatorname{Proj} \bigoplus_{d \geq 0} \Gamma(X, \mathcal{O}(d))^G$$

The rational map $X \dashrightarrow X/\!\!/G$ is defined on an open subscheme X^{ss}, which we call the *GIT semistable locus*. To make this procedure work (and this is the hard part!), one must show that an element $x \in X$ is GIT semistable if and only if the corresponding object of $[X/G]$ is semistable (i.e. is in $\mathcal{M}^{\mathrm{ss}}$).

One of the striking features of GIT is that it handles all six steps at once and in particular constructs the moduli space as a projective variety. Moreover, if we do not know a priori how to compactify a moduli problem, GIT can sometimes tell you how.

Example 0.9.7 (Deligne–Mumford stable curves)**.** Using the quotient presentation $\overline{\mathcal{M}}_g = [H'/\operatorname{PGL}_{5g-6}]$ of Example 0.8.4, the closure $\overline{H}'$ of H' in the Hilbert scheme inherits an action of $\operatorname{PGL}_{5g-6}$ and one must show than an element in H' is GIT semistable if and only if the corresponding curve is Deligne–Mumford stable.

Example 0.9.8 (Semistable vector bundles)**.** Using the quotient presentation $\mathcal{M}^{\mathrm{ss}}_{C,r,d} = [Q'_m/\operatorname{PGL}_{N-1}]$ of Example 0.8.5, the closure $\overline{Q}'_m$ has a PGL_{N-1}-action and one must show that an element in $\overline{Q}'_m$ is GIT semistable if and only if the corresponding quotient is semistable.

0.9.6 Trichotomy of moduli spaces

Table 3: The trichotomy of moduli

	No Auts	Finite Auts	Infinite Auts
Type of space	Algebraic variety / space	Deligne–Mumford stack	algebraic stack
Defining property	Zariski/étale-locally an affine scheme	étale-locally an affine scheme	smooth-locally an affine scheme
Examples	$\mathbb{P}^n$, $\operatorname{Gr}(k,n)$, Hilb, Quot	$\mathcal{M}_g$	$\mathcal{M}_{C,r,d}$
Quotient stacks $[X/G]$	action is free	finite stabilizers	any action
Existence of moduli varieties / spaces	already an algebraic variety/space	coarse moduli space	good moduli space

Notes

For a more detailed exposition of the moduli stack of triangles, we recommend Behrend's notes [Beh14].

Chapter 1

Sites, sheaves and stacks

1.1 Grothendieck topologies and sites

We would like to consider a topology on a scheme where étale morphisms are the open sets. This doesn't make sense using the conventional notion of a topological space so we simply adapt our definitions.

Definition 1.1.1. A *Grothendieck topology* on a category $\mathcal{S}$ consists of the following data: for each object $X \in \mathcal{S}$, there is a set $\mathrm{Cov}(X)$ consisting of *coverings* of X, i.e. collections of morphisms $\{X_i \to X\}_{i \in I}$ in $\mathcal{S}$. We require that:

(1) (identity) If $X' \to X$ is an isomorphism, then $(X' \to X) \in \mathrm{Cov}(X)$.

(2) (restriction) If $\{X_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$ and $Y \to X$ is any morphism, then the fiber products $X_i \times_X Y$ exist in $\mathcal{S}$ and the collection $\{X_i \times_X Y \to Y\}_{i \in I} \in \mathrm{Cov}(Y)$.

(3) (composition) If $\{X_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$ and $\{X_{ij} \to X_i\}_{j \in J_i} \in \mathrm{Cov}(X_i)$ for each $i \in I$, then $\{X_{ij} \to X_i \to X\}_{i \in I, j \in J_i} \in \mathrm{Cov}(X)$.

A *site* is a category $\mathcal{S}$ with a Grothendieck topology.

Example 1.1.2 (Topological spaces)**.** If X is a topological space, let $\mathrm{Op}(X)$ denote the category of open sets $U \subset X$ where there is a unique morphism $U \to V$ if $U \subset V$ and no other morphisms. We say that a covering of U (i.e. an element of $\mathrm{Cov}(U)$) is a collection of open immersions $\{U_i \to U\}_{i \in I}$ such that $U = \bigcup_{i \in I} U_i$. This defines a Grothendieck topology on $\mathrm{Op}(X)$.

In particular, if X is a scheme, the Zariski-topology on X yields a site, which we refer to as the *small Zariski site on* X.

Example 1.1.3 (Small étale site)**.** If X is a scheme, the *small étale site on* X is the category $X_{\text{ét}}$ of étale morphisms $U \to X$ such that a morphism $(U \to X) \to (V \to X)$ is simply an X-morphism $U \to V$ (which is necessarily étale). In other words, $X_{\text{ét}}$ is the full subcategory of Sch/X consisting of schemes étale over X. A covering of an object $(U \to X) \in X_{\text{ét}}$ is a collection of étale morphisms $\{U_i \to U\}$ such that $\coprod_i U_i \to U$ is surjective.

Example 1.1.4 (Big Zariski and étale sites)**.** The *big Zariski site* (resp. *big étale site*) is the category Sch where a covering of a scheme U is a collection of open immersions (resp. étale morphisms) $\{U_i \to U\}$ in Sch such that $\coprod_i U_i \to U$ is surjective. We denote these sites as $\mathrm{Sch}_{\mathrm{Zar}}$ and $\mathrm{Sch}_{\text{Ét}}$.

Example 1.1.5 (Localized categories and sites)**.** If $\mathcal{S}$ is a category and $S \in \mathcal{S}$, define the category $\mathcal{S}/S$ whose objects are maps $T \to S$ in $\mathcal{S}$. A morphism $(T' \to S) \to (T \to S)$ is a map $T' \to T$ over S. If $\mathcal{S}$ is a site, $\mathcal{S}/S$ is also a site where a covering of $T \to S$ in $\mathcal{S}/S$ is a covering $\{T_i \to T\}$ in $\mathcal{S}$.

Applying this construction for a scheme S yields the big Zariski and étale sites $(\mathrm{Sch}\,/S)_{\mathrm{Zar}}$ and $(\mathrm{Sch}\,/S)_{\text{Ét}}$ over a scheme S.

Replacing étale morphisms with other properties of morphisms yields other sites.

1.2 Presheaves and sheaves

Recall that if X is a topological space, a presheaf of sets on X is simply a contravariant functor $F\colon \mathrm{Op}(X) \to \mathrm{Sets}$ on the category $\mathrm{Op}(X)$ of open sets. The sheaf axiom translates succinctly into the condition that for each covering $U = \bigcup_i U_i$, the sequence

$$F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

is exact (i.e. is an equalizer diagram), where the two maps $F(U_i) \rightrightarrows F(U_i \cap U_j)$ are induced by the two inclusions $U_i \cap U_j \subset U_i$ and $U_i \cap U_j \subset U_j$. Also note that the intersections $U_i \cap U_j$ can also be viewed as fiber products $U_i \times_X U_j$.

1.2.1 Definitions

Definition 1.2.1. A *presheaf* on a category $\mathcal{S}$ is a contravariant functor $\mathcal{S} \to \mathrm{Sets}$.

Remark 1.2.2. If $F\colon \mathcal{S} \to \mathrm{Sets}$ is a presheaf and $S \xrightarrow{f} T$ is a map in $\mathcal{S}$, then the pullback $F(f)(b)$ of an element $b \in F(T)$ is sometimes denoted as f^*b or $b|_S$.

Definition 1.2.3. A *sheaf* on a site $\mathcal{S}$ is a presheaf $F\colon \mathcal{S} \to \mathrm{Sets}$ such that for every object S and covering $\{S_i \to S\} \in \mathrm{Cov}(S)$, the sequence

$$F(S) \to \prod_i F(S_i) \rightrightarrows \prod_{i,j} F(S_i \times_S S_j) \tag{1.2.1}$$

is exact, where the two maps $F(S_i) \rightrightarrows F(S_i \times_S S_j)$ are induced by the two maps $S_i \times_S S_j \to S_i$ and $S_i \times_S S_j \to S_i$.

Remark 1.2.4. The exactness of (1.2.1) means that it is an equalizer diagram: $F(S)$ is precisely the subset of $\prod_{i,j} F(S_i \times_S S_j)$ consisting of elements whose images under the two maps $F(S_i) \rightrightarrows F(S_i \times_S S_j)$ are equal.

Example 1.2.5 (Schemes are sheaves)**.** If X is a scheme, then $\mathrm{Mor}(-, X)\colon \mathrm{Sch} \to \mathrm{Sets}$ is a sheaf on $\mathrm{Sch}_{\text{Ét}}$ since morphisms glue uniquely in the étale topology. Indeed, Proposition B.2.1 implies that the sheaf axiom holds for a cover given by a single morphism $S' \to S$ which is étale and surjective. The sheaf axiom for an an étale covering $\{S_i \to S\}$ can be easily reduced to this case (see **??**).

Similarly, if $Y \to X$ is a morphism of schemes, then $\mathrm{Mor}_X(-, Y)\colon \mathrm{Sch}\,/X \to \mathrm{Sets}$ is a sheaf on $(\mathrm{Sch}\,/X)_{\text{Ét}}$. We will abuse notation by using X and $X \to Y$ to denote the sheaves $\mathrm{Mor}(-, X)$ and $\mathrm{Mor}_X(-, Y)$.

Exercise 1.2.6. Let F be a presheaf on Sch.

(1) Show that F is a sheaf on $\mathrm{Sch}_{\text{Ét}}$ if and only if for every étale surjective morphism $S' \to S$ of schemes, the sequence $F(S) \to F(S') \rightrightarrows S' \times_S S'$ is exact.

(2) Show that F is a sheaf on $\mathrm{Sch}_{\text{Ét}}$ if and only if

- F is a sheaf in the big Zariski topology $\mathrm{Sch}_{\mathrm{Zar}}$; and
- or every étale surjective morphism $S' \to S$ of *affine* schemes, the sequence $F(S) \to F(S') \rightrightarrows F(S' \times_S S')$ is exact.

Exercise 1.2.7. If $X \to Y$ is a surjective smooth morphism of schemes, show that $X \to Y$ is an epimorphism of sheaves on $\mathrm{Sch}_{\text{Ét}}$.

1.2.2 Morphisms and fiber products

A *morphism* of presheaves or sheaves is by definition a natural transformation. By Yoneda's lemma (Lemma 0.4.1), if X is a scheme and F is a presheaf on Sch, a morphism $\alpha\colon X \to F$ (which we interpret as a morphism of presheaves $\mathrm{Mor}(-, X) \to F$) corresponds to an element in $F(X)$, which by abuse of notation we also denote by α.

Given morphisms $F \xrightarrow{\alpha} G$ and $G' \xrightarrow{\beta} G$ of presheaves on a category $\mathcal{S}$, consider the presheaf

$$\begin{aligned} &\mathcal{S} \to \mathrm{Sets} \\ &S \mapsto F(S) \times_{G(S)} G'(S) = \{(a, b) \in F(S) \times G'(S) \mid \alpha_S(a) = \beta_S(b)\}\,. \end{aligned} \tag{1.2.2}$$

Exercise 1.2.8.

(1) Show that that (1.2.2) is a fiber product $F \times_G G'$ in $\mathrm{Pre}(\mathcal{S})$. (This is a generalization of Exercise 0.4.16 but the same proof should work.)

(2) Show that if F, G and G' are sheaves on a site $\mathcal{S}$, then so is $F \times_G G'$. In particular, (1.2.2) is also a fiber product $F \times_G G'$ in $\mathrm{Sh}(\mathcal{S})$.

1.2.3 Sheafification

Theorem 1.2.9 (Sheafification)**.** *Let $\mathcal{S}$ be a site. The forgetful functor* $\mathrm{Sh}(\mathcal{S}) \to \mathrm{Pre}(\mathcal{S})$ *admits a left adjoint $F \mapsto F^{\mathrm{sh}}$, called the sheafification.*

Proof. A presheaf F on $\mathcal{S}$ is called *separated* if for every covering $\{S_i \to S\}$ of an object S, the map $F(S) \to \prod_i F(S_i)$ is injective (i.e. if sections glue, they glue uniquely). Let $\mathrm{Pre}^{\mathrm{sep}}(\mathcal{S})$ be the full subcategory of $\mathrm{Pre}(\mathcal{S})$ consisting of separated presheaves. We will construct left adjoints

$$\begin{tikzcd} \mathrm{Sh}(\mathcal{S}) \arrow[r, hook] & \mathrm{Pre}^{\mathrm{sep}}(\mathcal{S}) \arrow[l, bend right, "\mathrm{sh}_2"'] \arrow[r, hook] & \mathrm{Pre}(\mathcal{S}). \arrow[l, bend right, "\mathrm{sh}_1"'] \end{tikzcd}$$

For $F \in \mathrm{Pre}(\mathcal{S})$, we define $\mathrm{sh}_1(F)$ by $S \mapsto F(S)/\sim$ where $a \sim b$ if there exists a covering $\{S_i \to S\}$ such that $a|_{S_i} = b|_{S_i}$ for all i.

For $F \in \mathrm{Pre}^{\mathrm{sep}}(\mathcal{S})$, we define $\mathrm{sh}_2(F)$ by

$$S \mapsto \left\{ (\{S_i \to S\}, \{a_i\}) \,\middle|\, \begin{array}{l} \text{where } \{S_i \to S\} \in \mathrm{Cov}(S) \text{ and } a_i \in F(S_i) \\ \text{such that } a_i|_{S_{ij}} = a_j|_{S_{ij}} \text{ for all } i, j \end{array} \right\} / \sim$$

where $(\{S_i \to S\}, \{a_i\}) \sim (\{S'_j \to S\}, \{a'_j\})$ if $a_i|_{S_i \times_S S'_j} = a'_j|_{S_i \times_S S'_j}$ for all i, j. The details are left to the reader. $\square$

Remark 1.2.10 (Topos)**.** A *topos* is a category equivalent to the category of sheaves on a site. Two different sites may have equivalent categories of sheaves, and the topos can be viewed as a more fundamental invariant. While topoi are undoubtedly useful in moduli theory, they will not play a role in these notes.

1.3 Prestacks

In Section 0.7.1, we motivated the concept of a prestack on a category $\mathcal{S}$ as a generalization of a presheaf $\mathcal{S} \to \mathrm{Sets}$. By trying to keep track of automorphisms, we were naively led to consider a 'functor' $F \colon \mathcal{S} \to \mathrm{Groupoids}$ but decided instead to package this data into one large category $\mathcal{X}$ over $\mathcal{S}$ parameterizing pairs (a, S) where $S \in \mathcal{S}$ and $a \in F(S)$.

1.3.1 Definition of a prestack

Let $\mathcal{S}$ be a category and $p \colon \mathcal{X} \to \mathcal{S}$ be a functor of categories. We visualize this data as

$$\begin{tikzcd} \mathcal{X} \arrow[d, "p"] \\ \mathcal{S} \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d, mapsto] & b \arrow[d, mapsto] \\ S \arrow[r, "f"] & T \end{tikzcd}$$

where the lower case letters a, b are objects of $\mathcal{X}$ and the upper case letters S, T are objects of $\mathcal{S}$. We say that a is over S and $\alpha \colon a \to b$ is over $f \colon S \to T$.

Definition 1.3.1. A functor $p \colon \mathcal{X} \to \mathcal{S}$ is a *prestack over a category* $\mathcal{S}$ if

(1) (pullbacks exist) for any diagram

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[d, mapsto, dashed] & b \arrow[d, mapsto] \\ S \arrow[r] & T \end{tikzcd}$$

of solid arrows, there exist a morphism $a \to b$ over $S \to T$; and

(2) (universal property for pullbacks) for any diagram

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[rr, bend left] \arrow[d, mapsto] & b \arrow[r] \arrow[d, mapsto] & c \arrow[d, mapsto] \\ R \arrow[r] & S \arrow[r] & T \end{tikzcd}$$

of solid arrows, there exists a unique arrow $a \to b$ over $R \to S$ filling in the diagram.

Warning 1.3.2. When defining and discussing prestacks, we often simply write $\mathcal{X}$ instead of $\mathcal{X} \to \mathcal{S}$. In most examples it is clear what the functor $\mathcal{X} \to \mathcal{S}$ is. When necessary, we denote the projection by $p_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{S}$.

Moreover, when defining a prestack $\mathcal{X}$, we often only define the objects and morphisms in $\mathcal{X}$, and we leave the definition of the composition law to the reader.

Remark 1.3.3. Axiom (2) above implies that the pullback in Axiom (1) is unique up to unique isomorphism. We often write f^*b or simply $b|_S$ to indicate a *choice* of a pullback.

Definition 1.3.4. If $\mathcal{X}$ is a prestack over $\mathcal{S}$, the *fiber category* $\mathcal{X}(S)$ over $S \in \mathcal{S}$ is the category of objects in $\mathcal{X}$ over S with morphisms over id_S.

Exercise 1.3.5. Show that the fiber category $\mathcal{X}(S)$ is a groupoid.

⚠ **Warning 1.3.6.** Our terminology is not standard. Prestacks are usually referred to as *categories fibered in groupoids*. In the literature (c.f. [Vis05], [Ols16]) a prestack is sometimes defined as a category fibered in groupoids together with Axiom (1) of a stack (Definition 1.4.1).

It is also standard to call a morphism $b \to c$ in $\mathcal{X}$ *cartesian* if it satisfies the universal property in (2) and $p\colon \mathcal{X} \to S$ a *fibered category* if for any diagram as in (1), there exists a cartesian morphism $a \to b$ over $S \to T$. With this terminology, a prestack (as we've defined it) is a fibered category where every arrow is cartesian or equivalently where every fiber category $\mathcal{X}(S)$ is a groupoid.

1.3.2 Examples

Example 1.3.7 (Presheaves are prestacks)**.** If $F\colon \mathcal{S} \to \mathrm{Sets}$ is a presheaf, we can construct a prestack $\mathcal{X}_F$ as the category of pairs (a, S) where $S \in \mathcal{S}$ and $a \in F(S)$. A map $(a', S') \to (a, S)$ is a map $f\colon S' \to S$ such that $a' = f^*a$, where f^* is convenient shorthand for $F(f)\colon F(S) \to F(S')$. Observe that the fiber categories $\mathcal{X}_F(S)$ are equivalent (even equal) to the set $F(S)$. We will often abuse notation by conflating F and $\mathcal{X}_F$.

Example 1.3.8 (Schemes are prestacks)**.** For a scheme X, applying the previous example to the functor $\mathrm{Mor}(-, X)\colon \mathrm{Sch} \to \mathrm{Sets}$ yields a prestack $\mathcal{X}_X$. This allows us to view a scheme X as a prestack and we will often abuse notation by referring to $\mathcal{X}_X$ as X.

Example 1.3.9 (Prestack of smooth curves)**.** We define the prestack $\mathcal{M}_g$ over Sch as the category of families of smooth curves $\mathcal{C} \to S$ of genus g, i.e. smooth and proper morphisms $\mathcal{C} \to S$ (of finite presentation) of schemes such that every geometric fiber is a connected curve of genus g. A map $(\mathcal{C}' \to S') \to (\mathcal{C} \to S)$ is the data of maps $\alpha\colon \mathcal{C}' \to \mathcal{C}$ and $f\colon S' \to S$ such that the diagram

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

is cartesian. Note that the fiber category $\mathcal{M}_g(\mathbb{C})$ over $\operatorname{Spec}\mathbb{C}$ is the groupoid of smooth connected projective complex curves C of genus g such that $\mathrm{Mor}_{\mathcal{M}_g(\mathbb{C})}(C, C') = \mathrm{Isom}_{\mathrm{Sch}/\mathbb{C}}(C, C')$.

Example 1.3.10 (Prestack of vector bundles)**.** Let C be a fixed smooth connected projective curve over $\mathbb{C}$, and fix integers $r \geq 0$ and d. We define the prestack $\mathcal{M}_{C,r,d}$ over $\mathrm{Sch}/\mathbb{C}$ where objects are pairs (E, S) where S is a scheme over $\mathbb{C}$ and E is a vector bundle on $C_S = C \times_{\mathbb{C}} S$. A map $(E', S') \to (E, S)$ consists of a map of schemes $f\colon S' \to S$ together with a map $E \to (\mathrm{id} \times f)_* E'$ of $\mathcal{O}_{C_S}$-modules whose adjoint is an isomorphism (i.e. for any *choice* of pullback $(\mathrm{id} \times f)^* E$, the adjoint map $(\mathrm{id} \times f)^* E \to E'$ is an isomorphism).

Exercise 1.3.11. Verify that $\mathcal{M}_g$ and $\mathcal{M}_{C,r,d}$ are prestacks.

Definition 1.3.12 (Quotient and classifying prestacks)**.** Let $G \to S$ be a group scheme acting on a scheme $X \to S$ via $\sigma\colon G \times_S X \to X$. We define the *quotient prestack* $[X/G]^{\mathrm{pre}}$ as the category over Sch/S where the fiber category over an S-scheme T is quotient groupoid $[X(T)/G(T)]$ of the (abstract) group $G(T)$ acting on the set $X(T)$; see Example 0.3.6. A morphism $(T' \to X) \to (T \to X)$ over $T' \to T$ is an element $\gamma \in G(T')$ such that $(T' \to X) = \gamma \cdot (T' \to T \to X) \in X(T')$

We now define the prestack $[X/G]$ (which we will call the *quotient stack*) as the category over Sch/S whose objects over an S-scheme T are diagrams

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] & X \\ T & \end{tikzcd}$$

where $P \to T$ is a G-torsor (see **??**) and $f\colon P \to X$ is a G-equivariant morphism. A morphism $(P' \to T', P' \xrightarrow{f'} X) \to (P \to T, P \xrightarrow{f} X)$ consists a maps $g\colon T' \to T$ and $\varphi\colon P' \to P$ of schemes such that the diagram

$$\begin{tikzcd} P' \arrow[r, "\varphi"] \arrow[d] \arrow[rr, bend left, "f'"] \arrow[dr, phantom, "\Box"] & P \arrow[r, "f"] \arrow[d] & X \\ T' \arrow[r, "g"] & T & \end{tikzcd}$$

commutes with the left square cartesian. See Section 0.8.1 for motivation of the above definition.

We define the *classifying prestack* as $\mathbf{B}_S G = [S/G]$ arising as the special case when $X = S$. When S is understood, we simply write $\mathbf{B}G$.

Exercise 1.3.13. Verify that $[X/G]^{\mathrm{pre}}$ and $[X/G]$ are prestacks over Sch/S.

1.3.3 Morphisms of prestacks

Definition 1.3.14.

(1) A *morphism of prestacks* $f\colon \mathcal{X} \to \mathcal{Y}$ is a functor $f\colon \mathcal{X} \to \mathcal{Y}$ such that the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr, "p_{\mathcal{X}}"'] & & \mathcal{Y} \arrow[dl, "p_{\mathcal{Y}}"] \\ & \mathcal{S} & \end{tikzcd}$$

strictly commutes, i.e. for every object $a \in \mathrm{Ob}(\mathcal{X})$, the schemes $p_{\mathcal{X}}(a) = p_{\mathcal{Y}}(f(a))$ are *equal*.

(2) If $f, g\colon \mathcal{X} \to \mathcal{Y}$ are morphisms of prestacks, a *2-morphism* (or *2-isomorphism*) $\alpha\colon f \to g$ is a natural transformation $\alpha\colon f \to g$ such that for every object $a \in \mathcal{X}$, the morphism $\alpha_a\colon f(a) \to g(a)$ in $\mathcal{Y}$ (which is an isomorphism) is over the identity in $\mathcal{S}$. We often describe the 2-morphism α schematically as

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & \mathcal{Y}. \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] \end{tikzcd}$$

(3) We define the category $\mathrm{MOR}(\mathcal{X}, \mathcal{Y})$ whose objects are morphisms of prestacks and whose morphisms are 2-morphisms.

(4) We say that a diagram

$$\begin{tikzcd} \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \arrow[r, "f'"] \arrow[d, "g'"] \arrow[dr, phantom, "\swarrow\alpha"] & \mathcal{Y}' \arrow[d, "g"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

together with a 2-isomorphism $\alpha\colon g \circ f' \xrightarrow{\sim} f \circ g'$ is *2-commutative.*

(5) A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of prestacks is an *isomorphism* if there exists a morphism $g\colon \mathcal{Y} \to \mathcal{X}$ and 2-isomorphisms $g \circ f \xrightarrow{\sim} \mathrm{id}_{\mathcal{X}}$ and $f \circ g \xrightarrow{\sim} \mathrm{id}_{\mathcal{Y}}$.

Exercise 1.3.15. Show that any 2-morphism is an isomorphism of functors, or in other words that $\mathrm{MOR}(\mathcal{X}, \mathcal{Y})$ is a groupoid.

Exercise 1.3.16. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of prestacks over a category $\mathcal{S}$.

(a) Show that f is fully faithful if and only if $f_S\colon \mathcal{X}(S) \to \mathcal{Y}(S)$ is fully faithful for every $S \in \mathcal{S}$.

(b) Show that f is an isomorphism if and only if $f_S\colon \mathcal{X}(S) \to \mathcal{Y}(S)$ is an equivalence of categories for every $S \in \mathcal{S}$.

A prestack $\mathcal{X}$ is *equivalent to a presheaf* if there is a presheaf F and an isomorphism between $\mathcal{X}$ and the stack $\mathcal{X}_F$ corresponding to F (see Example 1.3.7).

Exercise 1.3.17. Let $G \to S$ be a group scheme acting on a scheme $X \to S$ via $\sigma\colon G \times_S X \to X$. Show that the prestacks $[X/G]^{\mathrm{pre}}$ and $[X/G]$ are equivalent to presheaves if and only if the action is free (i.e. $(\sigma, p_2)\colon G \times_S X \to X \times_S X$ is a monomorphism).

1.3.4 The 2-Yoneda lemma

Recall that Yoneda's lemma (Lemma 0.4.1) implies that for a presheaf $F\colon \mathcal{S} \to \mathrm{Sets}$ on a category $\mathcal{S}$ and an object $X \in \mathcal{S}$, there is a bijection $\mathrm{Mor}(S, F) \xrightarrow{\sim} F(S)$, where we view S as a presheaf via $\mathrm{Mor}(-, S)$. We will need an analogue of Yoneda's lemma for prestacks. First we recall that an object $S \in \mathcal{S}$ defines a prestack over $\mathcal{S}$, which we also denote by S, whose objects over $T \in \mathcal{S}$ are morphisms $T \to S$ and a morphism $(T \to S) \to (T' \to S)$ is an S-morphism $T \to T'$.

Lemma 1.3.18 (The 2-Yoneda Lemma)**.** *Let $\mathcal{X}$ be a prestack over a category $\mathcal{S}$ and $S \in \mathcal{S}$. The functor*

$$\mathrm{MOR}(S, \mathcal{X}) \to \mathcal{X}(S), \qquad f \mapsto f_S(\mathrm{id}_S)$$

is an equivalence of categories.

Proof. We will construct a quasi-inverse $\Psi\colon \mathcal{X}(S) \to \mathrm{MOR}(S, \mathcal{X})$ as follows.

On objects: For $a \in \mathcal{X}(S)$, we define $\Psi(a)\colon S \to \mathcal{X}$ as the morphism of prestacks sending an object $(T \xrightarrow{f} S)$ (of the prestack corresponding to S) over T to a *choice* of pullback $f^*a \in \mathcal{X}(T)$ and a morphism $(T' \xrightarrow{f'} S) \to (T \xrightarrow{f} S)$ given by an S-morphism $g\colon T' \to T$ to the morphism $f'^*a \to f^*a$ uniquely filling in the diagram

$$\begin{tikzcd} f'^*a \arrow[r, dashed] \arrow[d] \arrow[rr, bend left] & f^*a \arrow[r] \arrow[d, mapsto] & a \arrow[d, mapsto] \\ T' \arrow[r, "g"] & T \arrow[r, "f"] & S, \end{tikzcd}$$

using Axiom (2) of a prestack.

On morphisms: If $\alpha\colon a' \to a$ is a morphism in $\mathcal{X}(S)$, then $\Psi(\alpha)\colon \Psi(a') \to \Psi(a)$ is defined as the morphism of functors which maps a morphism $T \xrightarrow{f} S$ (i.e. an object in S over T) to the unique morphism $f^*a' \to f^*a$ filling in the diagram

$$\begin{tikzcd} f^*a' \arrow[r, dashed] \arrow[d] & f^*a \arrow[d] \\ a' \arrow[r, "\alpha"] & a \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} T \arrow[d, "f"] \\ S \end{tikzcd}$$

using again Axiom (2) of a prestack.

We leave the verification that Ψ is a quasi-inverse to the reader. □

We will use the 2-Yoneda lemma, often without mention, throughout these notes in passing between morphisms $S \to \mathcal{X}$ and objects of $\mathcal{X}$ over S.

Example 1.3.19 (Quotient stack presentations)**.** Consider the prestack $[X/G]$ in Definition 1.3.12 arising from a group action $\sigma\colon G \times_S X \to X$. The object of $[X/G]$ over X given by the diagram

$$\begin{tikzcd} G \times_S X \arrow[r, "\sigma"] \arrow[d, "p_2"] & X \\ X & \end{tikzcd}$$

corresponds via the 2-Yoneda lemma (Lemma 1.3.18) to a morphism $X \to [X/G]$.

Exercise 1.3.20.

(1) Show that there is a morphism $p\colon X \to [X/G]^{\mathrm{pre}}$ and a 2-commutative diagram

$$\begin{tikzcd} G \times_S X \arrow[r, "\sigma"] \arrow[d, "p_2"] \arrow[dr, phantom, "\swArrow_{\alpha}"] & X \arrow[d, "p"] \\ X \arrow[r, "p"] & {[X/G]^{\mathrm{pre}}} \end{tikzcd}$$

(2) Show that $X \to [X/G]^{\mathrm{pre}}$ is a categorical quotient among prestacks, i.e. for

any 2-commutative diagram

$$
\begin{tikzcd}
G \times_S X \arrow[r, "\sigma"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\not\Downarrow_\alpha"] & X \arrow[d, "p"] \arrow[ddr, bend left, "\varphi"] & \\
X \arrow[r, "p"] \arrow[drr, bend right, "\varphi"'] & {[X/G]^{\mathrm{pre}}} \arrow[dr, phantom, "\not\Downarrow_\tau"] & \\
& & \mathcal{Z}
\end{tikzcd}
$$

of prestacks, there exists a morphism $\chi\colon [X/G]^{\mathrm{pre}} \to \mathcal{Z}$ and a 2-isomorphism $\beta\colon \varphi \xrightarrow{\sim} \chi \circ p$ which is compatible with α and τ (i.e. the two natural transformations $\varphi \circ \sigma \xrightarrow{\beta\circ\sigma} \chi \circ p \circ \sigma \xrightarrow{\chi\circ\alpha} \chi \circ p \circ p_2$ and $\varphi \circ \sigma \xrightarrow{\tau} \varphi \circ p_2 \xrightarrow{\beta\circ p_2} \chi \circ p \circ p_2$ agree.

1.3.5 Fiber products

We discuss fiber products for prestacks and in particular prove their existence. Recall that for morphisms $X \to Y$ and $Y' \to Y$ of presheaves on a category $\mathcal{S}$, the fiber product can be constructed as the presheaf mapping an object $S \in \mathcal{S}$ to the fiber product $X(S) \times_{Y(S)} Y'(S)$ of sets. Essentially the same construction works for morphisms $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y}' \to \mathcal{Y}$ of prestacks but since we are dealing with groupoids rather than sets, the fiber category over an object $S \in \mathcal{S}$ should be the fiber product $\mathcal{X}(S) \times_{\mathcal{Y}(S)} \mathcal{Y}'(S)$ of groupoids.

The reader may first want to work on Exercises 1.3.24 and 1.3.25 on fiber products of groupoids as they not only provide a warmup to fiber products of prestacks but motivate its construction.

Construction 1.3.21. Let $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{Y}' \to \mathcal{Y}$ be morphisms of prestacks over a category $\mathcal{S}$. Define the prestack $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ over $\mathcal{S}$ as the category of triples (x, y', γ) where $x \in \mathcal{X}$ and $y' \in \mathcal{Y}'$ are objects over the *same* object $S := p_{\mathcal{X}}(x) = p_{\mathcal{Y}'}(y') \in \mathcal{S}$, and $\gamma\colon f(x) \xrightarrow{\sim} g(y')$ is an isomorphism in $\mathcal{Y}(S)$. A morphism $(x_1, y_1', \gamma_1) \to (x_2, y_2', \gamma_2)$ consists of a triple (f, χ, γ') where $f\colon p_{\mathcal{X}}(x_1) = p_{\mathcal{Y}'}(y_1') \to p_{\mathcal{Y}'}(y_2') = p_{\mathcal{X}}(x_2)$ is a morphism in $\mathcal{S}$, and $\chi\colon x_1 \xrightarrow{\sim} x_2$ and $\gamma'\colon y_1' \xrightarrow{\sim} y_2'$ are morphisms in $\mathcal{X}$ and $\mathcal{Y}'$ over f such that

$$
\begin{tikzcd}
f(x_1) \arrow[r, "f(\chi)"] \arrow[d, "\gamma_1"] & f(x_2) \arrow[d, "\gamma_2"] \\
g(y_1') \arrow[r, "g(\gamma')"] & g(y_2')
\end{tikzcd}
$$

commutes.

Let $p_1\colon \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{X}$ and $p_2\colon \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{X}$ denote the projections $(x, y', \gamma) \mapsto x$ and $(x, y', \gamma) \mapsto y'$. There is a 2-isomorphism $\alpha\colon f \circ p_1 \xrightarrow{\sim} g \circ p_2$ defined on an object $(x, y', \gamma) \in \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ by $\alpha_{(x,y',\gamma)}\colon f(x) \xrightarrow{\gamma} g(y')$. This yields a 2-commutative diagram

$$
\begin{tikzcd}
\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \arrow[r, "p_2"] \arrow[d, "p_1"] \arrow[dr, phantom, "\alpha\not\Nearrow"] & \mathcal{Y}' \arrow[d, "g"] \\
\mathcal{X} \arrow[r, "f"] & \mathcal{Y}
\end{tikzcd}
\tag{1.3.1}
$$

Theorem 1.3.22. *The prestack $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ together with the morphisms p_1 and p_2 and the 2-isomorphism α as in (1.3.1) satisfy the following universal property: for any 2-commutative diagram*

$$\begin{tikzcd}
& & \mathcal{Y}' \arrow[dr, "g"] & \\
\mathcal{T} \arrow[urr, bend left, "q_2"] \arrow[drr, bend right, "q_1"'] \arrow[r, phantom, "\tau\Uparrow"] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \arrow[ur, "p_2"] \arrow[dr, "p_1"'] \arrow[rr, phantom, "\alpha\Uparrow"] & & \mathcal{Y} \\
& & \mathcal{X} \arrow[ur, "f"'] &
\end{tikzcd}$$

with 2-isomorphism $\tau\colon f \circ q_1 \xrightarrow{\sim} g \circ q_2$, there exist a morphism $h\colon \mathcal{T} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ and 2-isomorphisms $\beta\colon q_1 \to p_1 \circ h$ and $\gamma\colon q_2 \to p_2 \circ h$ yielding a 2-commutative diagram

$$\begin{tikzcd}
& & \mathcal{Y}' \arrow[dr, "g"] & \\
\mathcal{T} \arrow[urr, bend left, "q_2"] \arrow[drr, bend right, "q_1"'] \arrow[r, "h"] \arrow[urr, phantom, "\Downarrow\gamma"] \arrow[drr, phantom, "\Uparrow\beta"'] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \arrow[ur, "p_2"] \arrow[dr, "p_1"'] \arrow[rr, phantom, "\alpha\Uparrow"] & & \mathcal{Y} \\
& & \mathcal{X} \arrow[ur, "f"'] &
\end{tikzcd}$$

such that

$$\begin{tikzcd}
f \circ q_1 \arrow[r, "f(\beta)"] \arrow[d, "\tau"] & f \circ p_1 \circ h \arrow[d, "\alpha \circ h"] \\
g \circ q_2 \arrow[r, "g(\gamma)"] & g \circ p_2 \circ h
\end{tikzcd}$$

commutes. The data (h, β, γ) is unique up to unique isomorphism.

Proof. We define $h\colon \mathcal{T} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ on objects by $t \mapsto \big(q_1(t), q_2(t), f(q_1(t)) \xrightarrow{\tau_t} g(q_2(t))\big)$ and on morphisms as $(t \xrightarrow{\Psi} t') \mapsto (p_{\mathcal{T}}(\Psi), q_1(\Psi), q_2(\Psi))$. There are *equalities* of functors $q_1 = p_1 \circ h$ and $q_2 = p_2 \circ h$ so we define β and γ as the identity natural transformation. The remaining details are left to the reader. $\square$

Definition 1.3.23. We say that a 2-commutative diagram

$$\begin{tikzcd}
\mathcal{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow_{\alpha}"] & \mathcal{Y}' \arrow[d] \\
\mathcal{X} \arrow[r] & \mathcal{Y}
\end{tikzcd}$$

is *cartesian* if it satisfies the universal property of Theorem 1.3.22.

1.3.6 Examples of fiber products

Exercise 1.3.24.

(1) If $\mathcal{C} \xrightarrow{f} \mathcal{D}$ and $\mathcal{D}' \xrightarrow{g} \mathcal{D}$ are morphisms of groupoids, define the groupoid $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}'$ whose objects are triples (c, d', δ) where $c \in \mathcal{C}$ and $d' \in \mathcal{D}'$ are objects, and $\delta \colon f(c) \xrightarrow{\sim} g(d')$ is an isomorphism in $\mathcal{D}$. A morphism $(c_1, d_1', \delta_1) \to (c_2, d_2', \delta_2)$ is the data of morphisms $\gamma \colon c_1 \xrightarrow{\sim} c_2$ and $\delta' \colon d_1' \xrightarrow{\sim} d_2'$ such that

$$\begin{tikzcd}
f(c_1) \arrow[r, "f(\gamma)"] \arrow[d, "\delta_1"] & f(c_2) \arrow[d, "\delta_2"] \\
g(d_1') \arrow[r, "g(\delta')"] & g(d_2')
\end{tikzcd}$$

commutes. Formulate a university property for fiber products of groupoids and show that $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}'$ satisfies it.

(2) If $f \colon \mathcal{X} \to \mathcal{Y}$ and $g \colon \mathcal{Y}' \to \mathcal{Y}$ are morphisms of prestacks over a category $\mathcal{S}$, show that for every $S \in \mathcal{S}$, the fiber category $(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}')(S)$ is a fiber product $\mathcal{X}(S) \times_{\mathcal{Y}(S)} \mathcal{Y}'(S)$ of groupoids.

Exercise 1.3.25. Let G be a group acting on a set X via $\sigma \colon G \times X \to X$. Let $[X/G]$ denote the quotient groupoid (Exercise 0.3.7) with projection $p \colon X \to [X/G]$.

(1) Show that there are cartesian diagrams

$$\begin{tikzcd}
G \times X \arrow[r, "\sigma"] \arrow[d, "p_2"] \arrow[dr, phantom, "\not\Downarrow"] & X \arrow[d, "p"] \\
X \arrow[r, "p"] & {[X/G]}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
G \times X \arrow[r, "{(\sigma, p_2)}"] \arrow[d] \arrow[dr, phantom, "\not\Downarrow"] & X \times X \arrow[d, "p \times p"] \\
{[X/G]} \arrow[r, "\Delta"] & {[X/G] \times [X/G].}
\end{tikzcd}$$

(2) Show that if $P \to T$ is any G-torsor and $P \to X$ is a G-equivariant map, there is a morphism $T \to [X/G]$, unique up to unique isomorphism, and a cartesian diagram

$$\begin{tikzcd}
P \arrow[r] \arrow[d] \arrow[dr, phantom, "\not\Downarrow"] & X \arrow[d] \\
T \arrow[r] & {[X/G].}
\end{tikzcd}$$

(If $G \to S$ is a smooth affine group scheme, we will later see that $[X/G]$ is an algebraic stack and that $X \to [X/G]$ is G-torsor (Theorem 2.1.8). Therefore the G-torsor $X \to [X/G]$ and the identity map $X \to X$ is the universal family over $[X/G]$ (corresponding to the identiy map $[X/G] \to [X/G]$).

(3) Assume in addition that $G \to S$ is a smooth group scheme. If $T \to [X/G]$ is any morphism from a scheme T, show that there is an étale cover $T' \to T$ and a commutative diagram

$$\begin{tikzcd}
T' \arrow[r] \arrow[d] \arrow[dr, phantom, "\not\Downarrow"] & X \arrow[d] \\
T \arrow[r] & {[X/G].}
\end{tikzcd}$$

Exercise 1.3.26.

(1) If $x \in X$, show that there is a morphism $\mathbf{B}G_x \to [X/G]$ of groupoids and a cartesian diagram

$$\begin{tikzcd} Gx \arrow[r] \arrow[d] \arrow[dr, phantom, "\swArrow"] & X \arrow[d, "p"] \\ \mathbf{B}G_x \arrow[r] & {[X/G].} \end{tikzcd}$$

(2) Let $\phi \colon H \to G$ be a homomorphism of groups. Show that there is an induced morphism $\mathbf{B}H \to \mathbf{B}G$ of groupoids and that $\mathbf{B}H \times_{\mathbf{B}G} \mathrm{pt} \cong [G/H]$. If $G' \to G$ is a homomorphism of groups, can you describe $\mathbf{B}H \times_{\mathbf{B}G} \mathbf{B}G'$?

Exercise 1.3.27 (Magic Square)**.** Let $\mathcal{X}$ be a prestack. Show that for any morphism $a \colon S \to \mathcal{X}$ and $b \colon T \to \mathcal{X}$, there is a cartesian diagram

$$\begin{tikzcd} S \times_{\mathcal{X}} T \arrow[r] \arrow[d] \arrow[dr, phantom, "\swArrow"] & S \times T \arrow[d, "a \times b"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X}. \end{tikzcd}$$

Exercise 1.3.28 (Isom presheaf)**.**

(1) Let $\mathcal{X}$ be a prestack over a category $\mathcal{S}$ and let a and b be objects over $S \in \mathcal{S}$. Recall that $\mathcal{S}/S$ denotes the localized category whose objects are morphisms $T \to S$ in $\mathcal{S}$ and whose morphisms are S-morphisms. Show that

$$\begin{aligned} \underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a, b) \colon \mathcal{S}/S &\to \mathrm{Sets} \\ (T \xrightarrow{f} S) &\mapsto \mathrm{Mor}_{\mathcal{X}(T)}(f^*a, f^*b), \end{aligned}$$

where f^*a and f^*b are choices of a pullback, defines a presheaf on $\mathcal{S}/S$.

(2) Show that there is a cartesian diagram

$$\begin{tikzcd} \underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a, b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\swArrow"] & S \arrow[d, "{(a,b)}"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X}. \end{tikzcd}$$

(3) Show that the presheaf $\underline{\mathrm{Aut}}_{\mathcal{X}(T)}(a) = \underline{\mathrm{Isom}}_{\mathcal{X}(T)}(a, a)$ is naturally a presheaf in groups.

Exercise 1.3.29. If $n \geq 2$, show that $[\mathbb{A}^n/\mathbb{G}_m^n] \cong \underbrace{[\mathbb{A}^1/\mathbb{G}_m] \times \cdots \times [\mathbb{A}^1/\mathbb{G}_m]}_{n \text{ times}}$.

Exercise 1.3.30.

(1) Show that if $H \to G$ is a morphism of group schemes over a scheme S, there is an induced morphism of prestacks $\mathbf{B}H \to \mathbf{B}G$ over Sch/S.

(2) Show that $\mathbf{B}H \times_{\mathbf{B}G} S \cong [G/H]$.

1.4 Stacks

In this subsection, we will define a stack over a site $\mathcal{S}$ as a prestack $\mathcal{X}$ such that objects and morphisms glue uniquely in the Grothendieck topology of $\mathcal{S}$

(Definition 1.4.1). Verifying a given prestack is a stack reduces to a *descent* condition on objects and morphisms with respect to the covers of $\mathcal{S}$. The theory of descent is discussed in Section B.1 and is essential for verifying the stack axioms.

For a motivating example, consider the prestack of sheaves (Example 0.7.5) over the big Zariski site $(\mathrm{Sch})_{\mathrm{Zar}}$ whose objects over a scheme S are sheaves of abelian groups. Since sheaves and morphisms of sheaves glue in the Zariski-topology, this is a stack. It is also a stack in the big étale site $(\mathrm{Sch})_{\text{Ét}}$ and this requires the analogous gluing results in the étale topology (Propositions B.1.3 and B.1.5).

1.4.1 Definition of a stack

Definition 1.4.1. A prestack $\mathcal{X}$ over a site $\mathcal{S}$ is a *stack* if the following conditions hold for all coverings $\{S_i \to S\}$ of an object $S \in \mathcal{S}$:

(1) (morphisms glue) For objects a and b in $\mathcal{X}$ over S and morphisms $\phi_i\colon a|_{S_i} \to b$ such that $\phi_i|_{S_{ij}} = \phi_j|_{S_{ij}}$ as displayed in the diagram

$$\begin{tikzcd} & a|_{S_i} \arrow[dr] \arrow[drr, bend left, "\phi_i"] & & \\ a|_{S_{ij}} \arrow[ur] \arrow[dr] & & a \arrow[r, dashed] & b \\ & a|_{S_j} \arrow[ur] \arrow[urr, bend right, "\phi_j"'] & & \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} & S_i \arrow[dr] & \\ S_{ij} \arrow[ur] \arrow[dr] & & S, \\ & S_j \arrow[ur] & \end{tikzcd}$$

there exists a unique morphism $\phi\colon a \to b$ with $\phi|_{S_i} = \phi_i$.

(2) (objects glue) For objects a_i over S_i and isomorphisms $\alpha_{ij}\colon a_i|_{S_{ij}} \to a_j|_{S_{ij}}$, as displayed in the diagram

$$\begin{tikzcd} & & a_i \arrow[dr, dashed] & \\ a_i|_{S_{ij}} \arrow[r, "\alpha_{ij}"] & a_j|_{S_{ij}} \arrow[ur] \arrow[dr] & & a \\ & & a_j \arrow[ur, dashed] & \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} & S_i \arrow[dr] & \\ S_{ij} \arrow[ur] \arrow[dr] & & S \\ & S_j \arrow[ur] & \end{tikzcd}$$

satisfying the cocycle condition $\alpha_{ij}|_{S_{ijk}} \circ \alpha_{jk}|_{S_{ijk}} = \alpha_{ik}|_{S_{ijk}}$ on S_{ijk}, then there exists an object a over S and isomorphisms $\phi_i\colon a|_{S_i} \to a_i$ such that $\alpha_{ij} \circ \phi_i|_{S_{ij}} = \phi_j|_{S_{ij}}$ on S_{ij}.

Remark 1.4.2. There is an alternative description of the stack axioms analogous to the sheaf axiom of a presheaf $F\colon \mathcal{S} \to \mathrm{Sets}$, i.e. that $F(S) \to \prod_i F(S_i) \rightrightarrows \prod_{i,j} F(S_i \times_S S_j)$ is exact for coverings $\{S_i \to S\}$. Namely, we add an additional layer to the diagram corresponding to triple intersections and the stack axiom translates to the 'exactness' of

$$\mathcal{X}(S) \longrightarrow \prod_i \mathcal{X}(S_i) \rightrightarrows \prod_{i,j} \mathcal{X}(S_i \times_S S_j) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \prod_{i,j,k} \mathcal{X}(S_i \times_S S_j \times_S S_k).$$

Exercise 1.4.3. Show that Axiom (1) is equivalent to the condition that for all objects a and b of $\mathcal{X}$ over $S \in \mathcal{S}$, the Isom presheaf $\underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a, b)$ (see Exercise 1.3.28) is a sheaf on $\mathcal{S}/S$.

A *morphism* of stacks is a morphism of prestacks.

Exercise 1.4.4 (Fiber product of stacks)**.** Show that if $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y}' \to \mathcal{Y}$ are morphisms of stacks over a site $\mathcal{S}$, then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ is also a stack over $\mathcal{S}$.

1.4.2 Examples of stacks

Example 1.4.5 (Sheaves and schemes are stacks)**.** Recall that if F is a presheaf on a site $\mathcal{S}$, we can construct a prestack $\mathcal{X}_F$ over $\mathcal{S}$ as the category of pairs (a, S) where $S \in \mathcal{S}$ and $a \in F(S)$ (see Example 1.3.7). If F is a sheaf, then $\mathcal{X}_F$ is a stack. We often abuse notation by writing F also as the stack $\mathcal{X}_F$.

Since schemes are sheaves on $\mathrm{Sch}_{\text{Ét}}$ (Example 1.2.5), a scheme X defines a stack over $\mathrm{Sch}_{\text{Ét}}$ (where objects over a scheme S are morphisms $S \to X$), which we also denote as X.

Let $\mathcal{M}_g$ denote the prestack of families of smooth curves $\mathcal{C} \to S$ of genus g; see Example 1.3.9.

Proposition 1.4.6 (Moduli stack of smooth curves)**.** *If $g \geq 2$, then $\mathcal{M}_g$ is a stack over* $\mathrm{Sch}_{\text{Ét}}$.

Proposition 1.4.7 (Properties of Families of Smooth Curves)**.** *Let $\mathcal{C} \to S$ be a family of smooth curves of genus $g \geq 2$. Then for $k \geq 3$, $\Omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample and $\pi_*(\Omega_{\mathcal{C}/S}^{\otimes k})$ is a vector bundle of rank $(2k-1)(g-1)$.*

Proof. Axiom (1) translates to: for families of smooth curves $\mathcal{C} \to S$ and $\mathcal{D} \to S$ of genus g and commutative diagrams

$$\begin{tikzcd}
\mathcal{C}_{S_{ij}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C}_{S_i} \arrow[r] \arrow[d] \arrow[rr, bend left, "f_i"] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[r, dashed, "f"] \arrow[d] & \mathcal{D} \arrow[dl] \\
S_{ij} \arrow[r] & S_i \arrow[r] & S &
\end{tikzcd}$$

of solid arrows for all i, j (i.e. morphisms $f_i \colon \mathcal{C}_{S_i} \to \mathcal{D}$ such that $f_i|_{\mathcal{C}_{S_{ij}}} = f_j|_{\mathcal{C}_{S_{ij}}}$), there exists a unique morphism filling in the diagram (i.e. $f_i = f|_{\mathcal{C}_{S_i}}$). The existence and uniqueness of f follows from étale descent for morphisms (Proposition B.2.1). The fact that f is an isomorphism also follows from étale descent (Proposition B.4.1).

Axiom (2) is more difficult: we must show that given diagrams

$$\begin{tikzcd}
\mathcal{C}_i|_{S_{ij}} \arrow[r, "\alpha_{ij}"] \arrow[dr] \arrow[rr, bend left] & \mathcal{C}_j|_{S_{ij}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C}_j \arrow[r, dashed] \arrow[d, "\pi_j"] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d, dashed] \\
& S_{ij} \arrow[r] & S_j \arrow[r] & S
\end{tikzcd}$$

for all i, j where $\pi_i \colon \mathcal{C}_i \to S_i$ are families of smooth curves of genus g and $\alpha_{ij} \colon \mathcal{C}_i|_{S_{ij}} \to \mathcal{C}_j|_{S_{ij}}$ are isomorphisms satisfying the cocycle condition $\alpha_{ij} \circ \alpha_{jk} = \alpha_{ik}$, there is family of smooth curves $\mathcal{C} \to S$ and isomorphisms $\phi_i \colon \mathcal{C}|_{S_i} \to \mathcal{C}_i$ such that $\alpha_{ij} \circ \phi_i|_{\mathcal{C}_{S_{ij}}} = \phi_j|_{\mathcal{C}_{S_{ij}}}$.

We will use the following property of families of smooth curves: for a family of smooth curves $\pi\colon \mathcal{C} \to S$, $\omega_{\mathcal{C}/S}^{\otimes 3}$ is relatively very ample on S (as $g > 2$) and $F := \pi_* \omega_{\mathcal{C}/S}^{\otimes 3}$ is a vector bundle of rank $5(g-1)$. In particular, $\omega_{\mathcal{C}/S}^{\otimes 3}$ yields a closed immersion $\mathcal{C} \hookrightarrow \mathbb{P}(F)$ over S.

Therefore, if we set $E_i = (\pi_i)_*(\omega_{\mathcal{C}_i/S_i})$, there is a closed immersion $\mathcal{C}_i \hookrightarrow \mathbb{P}(E_i)$ over S_i. The isomorphisms α_{ij} induce isomorphisms $\beta_{ij}\colon E_i|_{S_{ij}} \to E_j|_{S_{ij}}$ satisfying the cocycle condition $\beta_{ij} \circ \beta_{jk} = \beta_{ik}$ on S_{ijk}. Descent for quasi-coherent sheaves (Proposition B.1.5) implies there is a quasi-coherent sheaf E on S and isomorphisms $\Psi_i\colon E|_{S_{ij}} \to E_i$ such that $\beta_{ij} \circ \Psi_i|_{S_{ij}} = \Psi_j|_{S_{ij}}$. It follows again from descent that E is in fact a vector bundle (Proposition B.4.4). Pictorially, we have

$$\begin{tikzcd}
& \mathbb{P}(E_{ij}) \arrow[r] & \mathbb{P}(E_i) \arrow[r] & \mathbb{P}(E) \\
\mathcal{C}_i|_{S_{ij}} \arrow[ur, hook] \arrow[r] \arrow[d] & \mathcal{C}_i \arrow[ur, hook] \arrow[r, dashed] \arrow[d] & \mathcal{C} \arrow[ur, hook, dashed] \arrow[d, dashed] \\
S_{ij} \arrow[r] & S_i \arrow[r] & S.
\end{tikzcd}$$

Since the preimages of $\mathcal{C}_i \subset \mathbb{P}(E_i)$ and $\mathcal{C}_j \subset \mathbb{P}(E_j)$ in $\mathbb{P}(E_{ij})$ are equal, it follows from descent for closed subschemes (Proposition B.3.1) that there exists $\mathcal{C} \to S$ and isomorphisms ϕ_i such that $\alpha_{ij} \circ \phi_i|_{\mathcal{C}_{S_{ij}}} \to \phi_j|_{\mathcal{C}_{S_{ij}}}$. Since smoothness and properness are étale-local property on the target (Proposition B.4.1), $\mathcal{C} \to S$ is smooth and proper. The geometric fibers of $\mathcal{C} \to S$ are connected genus g curves since the geometric fibers of $\mathcal{C}_i \to S_i$ are. □

Exercise 1.4.8.

(1) Show that the prestack $\mathcal{M}_0$ is a stack on $\mathrm{Sch}_{\text{Ét}}$ isomorphic to $B\,\mathrm{PGL}_2$.

(2) Show that the moduli stack $\mathcal{M}_{1,1}$, whose objects are families of elliptic curves (see Example 0.4.26) is a stack on $\mathrm{Sch}_{\text{Ét}}$.

(3) Can you show that $\mathcal{M}_1$ is a stack on $\mathrm{Sch}_{\text{Ét}}$?

Let C be a smooth connected projective curve over $\mathbb{C}$, and fix integers $r \geq 0$ and d. Recall from Example 1.3.10 that $\mathcal{M}_{C,r,d}$ denotes the prestack over $\mathrm{Sch}/\mathbb{C}$ consisting of pairs (E, S) where S is a scheme over $\mathbb{C}$ and E is a vector bundle on C_S.

Proposition 1.4.9 (Moduli stack of vector bundles over a curve)**.** *For all integers r, d with $r \geq 0$, $\mathcal{M}_{C,r,d}$ is a stack over $(\mathrm{Sch}/\mathbb{C})_{\text{Ét}}$.*

Proof. The prestack $\mathcal{M}_{C,r,d}$ is a stack: Axioms (1) and (2) are precisely descent for morphisms of quasi-coherent sheaves (Propositions B.1.3 and B.1.5) coupled with the fact that the property of a quasi-coherent sheaf being a vector bundle is étale-local (Proposition B.4.4). □

Let $G \to S$ be a smooth affine group scheme acting on a scheme $X \to S$. Let $[X/G]$ be the prestack defined in Definition 1.3.12 whose objects over a scheme S are G-torsors $P \to S$ together with G-equivariant maps $P \to X$. The following proposition justifies calling $[X/G]$ the *quotient stack*.

Proposition 1.4.10 (Quotient stack)**.** *The prestack $[X/G]$ is a stack.*

Proof. Axiom (1) follows from descent for morphisms of schemes (Proposition B.2.1). For Axiom (2), if $\{T_i \to T\}$ is an étale covering and $(\mathcal{P}_i \to T_i, \mathcal{P}_i \to X)$ are objects over T_i with isomorphisms on the restrictions satisfying the cocycle condition, then the existence of a G-torsor $\mathcal{P} \to T$ follows from descent for G-torsors (Proposition C.3.11) and the existence of $\mathcal{P} \to X$ follows from descent for morphisms of schemes (Proposition B.2.1). □

1.4.3 Stackification

To any presheaf F on a site $\mathcal{S}$, there is a sheafification $F \to F^{\mathrm{sh}}$ which is a left adjoint to the inclusion, i.e. $\mathrm{Mor}(F^{\mathrm{sh}}, G) \to \mathrm{Mor}(F, G)$ is bijective for any sheaf G on $\mathcal{S}$ (Theorem 1.2.9). Similarly, there is a stackification $\mathcal{X} \to \mathcal{X}^{\mathrm{st}}$ of any prestack $\mathcal{X}$ over $\mathcal{S}$.

Theorem 1.4.11 (Stackification)**.** *If $\mathcal{X}$ is a prestack over a site $\mathcal{S}$, there exists a stack $\mathcal{X}^{\mathrm{st}}$, which we call the* stackification*, and a morphism $\mathcal{X} \to \mathcal{X}^{\mathrm{st}}$ of prestacks such that for any stack $\mathcal{Y}$ over $\mathcal{S}$, the induced functor*

$$\mathrm{MOR}(\mathcal{X}^{\mathrm{st}}, \mathcal{Y}) \to \mathrm{MOR}(\mathcal{X}, \mathcal{Y}) \tag{1.4.1}$$

is an equivalence of categories.

Proof. As in the construction of the sheafification (see the proof of Theorem 1.2.9), we construct the stackification in stages. Most details are left to the reader.

First, given a prestack $\mathcal{X}$, we can construct a prestack $\mathcal{X}^{\mathrm{st}_1}$ satisfying Axiom (1) and a morphism $\mathcal{X} \to \mathcal{X}^{\mathrm{st}_1}$ of prestacks such that

$$\mathrm{MOR}(\mathcal{X}^{\mathrm{st}_1}, \mathcal{Y}) \to \mathrm{MOR}(\mathcal{X}, \mathcal{Y})$$

is an equivalence for all prestacks $\mathcal{Y}$ satisfying Axiom (1). Specifically, the objects of $\mathcal{X}^{\mathrm{st}_1}$ are the same as $\mathcal{X}$, and for objects $a, b \in \mathcal{X}$ over $S, T \in \mathcal{S}$, the set of morphisms $a \to b$ in $\mathcal{X}^{\mathrm{st}_1}$ over a given morphism $f\colon S \to T$ is the global sections $\Gamma(S, \underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a, f^*b)^{\mathrm{sh}})$ of the sheafification of the Isom presheaf (Exercise 1.3.28).

Second, given a prestack $\mathcal{X}$ satisfying Axiom (1), we construct a stack $\mathcal{X}$ and a morphism $\mathcal{X} \to \mathcal{X}^{\mathrm{st}}$ of prestacks such that (1.4.1) is an equivalence for all stacks $\mathcal{Y}$. An object of $\mathcal{X}^{\mathrm{st}}$ over $S \in \mathcal{S}$ is given by a triple consisting of a covering $\{S_i \to S\}$, objects a_i of $\mathcal{X}$ over S_i, and isomorphisms $\alpha_{ij}\colon a_i|_{S_{ij}} \to a_j|_{S_{ij}}$ satisfying the cocycle condition $\alpha_{ij}|_{S_{ijk}} \circ \alpha_{jk}|_{S_{ijk}} = \alpha_{ik}|_{S_{ijk}}$ on S_{ijk}. Morphisms

$$\big(\{S_i \to S\}, \{a_i\}, \{\alpha_{ij}\}\big) \to \big(\{T_\mu \to T\}, \{b_\mu\}, \{\beta_{\mu\nu}\}\big)$$

in $\mathcal{X}^{\mathrm{st}}$ over $S \to T$ are defined as follows: first consider the induced cover $\{S_i \times_S T_\mu \to S\}_{i,\mu}$ and choose pullbacks $a_i|_{S_i \times_S T_\mu}$ and $b_\mu|_{S_i \times_S T_\mu}$. A morphism is then the data of maps $\Psi_{i\mu}\colon a_i|_{S_i \times_S T_\mu} \to b_\mu|_{S_i \times_S T_\mu}$ for all i, μ which are compatible with α_{ij} and $\beta_{\mu\nu}$ (i.e. $\Psi_{j\nu} \circ \alpha_{ij} = \beta_{\mu\nu} \circ \Psi_{i\mu}$ on $S_{ij} \times_T T_{\mu\nu}$). □

Exercise 1.4.12. Show that stackification commutes with fiber products: if $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Z} \to \mathcal{Y}$ are morphisms of prestacks, then $(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})^{\mathrm{st}} \cong \mathcal{X}^{\mathrm{st}} \times_{\mathcal{Y}^{\mathrm{st}}} \mathcal{Z}^{\mathrm{st}}$.

Exercise 1.4.13. Recall the prestacks $[X/G]^{\mathrm{pre}}$ and $[X/G]$ from Definition 1.3.12.

(1) Show that $[X/G]^{\mathrm{pre}}$ satisfies Axiom (1) of a stack.

(2) Show that the $[X/G]$ is isomorphic to the stackification of $[X/G]^{\mathrm{pre}}$ and that $[X/G]^{\mathrm{pre}} \to [X/G]$ is fully faithful.

Exercise 1.4.14. Extending Exercise 1.3.20, show that $X \to [X/G]$ is is a categorical quotient among stacks.

Notes

Grothendieck topologies and stacks were introduced in [SGA4] and our exposition closely follows [Art62], [Vis05], and [Ols16, Ch. 2].

Chapter 2

Algebraic spaces and stacks

2.1 Definitions of algebraic spaces and stacks

We present a streamlined approach to defining algebraic spaces (Definition 2.1.2), Deligne–Mumford stacks (Definition 2.1.4) and algebraic stacks (Definition 2.1.5), and we verify the algebraicity of quotient stacks (Theorem 2.1.8), the moduli stack of curves (Theorem 2.1.11) and the moduli stack of vector bundles (Theorem 2.1.15).

2.1.1 Algebraic spaces

Definition 2.1.1 (Morphisms representable by schemes)**.** A morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks (or presheaves) over Sch is *representable by schemes* if for every morphism $V \to \mathcal{Y}$ from a scheme, the fiber product $\mathcal{X} \times_{\mathcal{Y}} V$ is a scheme.

If $\mathcal{P}$ is a property of morphisms of schemes (e.g. surjective or étale), a morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks representable by schemes *has property* $\mathcal{P}$ if for every morphism $V \to \mathcal{Y}$ from a scheme, the morphism $\mathcal{X} \times_{\mathcal{Y}} V \to V$ of schemes has property $\mathcal{P}$.

Definition 2.1.2. An *algebraic space* is a sheaf X on $\mathrm{Sch}_{\text{Ét}}$ such that there exist a scheme U and a surjective étale morphism $U \to X$ representable by schemes.

The morphism $U \to X$ is called an *étale presentation.* Morphisms of algebraic spaces are by definition morphisms of sheaves. Any scheme is an algebraic space.

2.1.2 Deligne–Mumford stacks

Definition 2.1.3 (Representable morphisms)**.** A morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks (or presheaves) over Sch is *representable* if for every morphism $V \to \mathcal{Y}$ from a scheme V, the fiber product $\mathcal{X} \times_{\mathcal{Y}} V$ is an algebraic space.

If $\mathcal{P}$ is a property of morphisms of schemes which is étale-local on the source (e.g., surjective, étale, or smooth), we say that a representable morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks *has property* $\mathcal{P}$ if for every morphism $V \to \mathcal{Y}$ from a scheme and étale presentation $U \to \mathcal{X} \times_{\mathcal{Y}} V$ by a scheme, the composition $U \to \mathcal{X} \times_{\mathcal{Y}} V \to V$ has property $\mathcal{P}$.

Definition 2.1.4. A *Deligne–Mumford stack* is a stack $\mathcal{X}$ over $\mathrm{Sch}_{\text{Ét}}$ such that there exist a scheme U and a surjective, étale and representable morphism $U \to \mathcal{X}$.

The morphism $U \to \mathcal{X}$ is called an *étale presentation.* Morphisms of Deligne–Mumford stacks are by definition morphisms of stacks. Any algebraic space is a Deligne–Mumford stack via Example 1.3.7.

2.1.3 Algebraic stacks

Definition 2.1.5. An *algebraic stack* is a stack $\mathcal{X}$ over $\mathrm{Sch}_{\text{Ét}}$ such that there exist a scheme U and a surjective, smooth and representable morphism $U \to \mathcal{X}$.

The morphism $U \to \mathcal{X}$ is called a *smooth presentation.* For any smooth-local property $\mathcal{P}$ of schemes, we can say that $\mathcal{X}$ has $\mathcal{P}$ if U does. Morphisms of algebraic stacks are by definition morphisms of prestacks. Any scheme, algebraic space or Deligne–Mumford stack is also an algebraic stack.

⚠ **Warning 2.1.6.** The definitions above are not standard as most authors also add a representability condition on the diagonal. They are nevertheless equivalent to the standard definitions: we show in **??** that the diagonal of an algebraic space is representable by schemes and that the diagonal of an algebraic stack is representable.

Exercise 2.1.7 (Fiber products)**.** Show that fiber products exist for algebraic spaces, Deligne–Mumford stacks and algebraic stacks

2.1.4 Algebraicity of quotient stacks

We will now show that if G is a smooth affine group scheme acting on an algebraic space U over a base T, the quotient stack $[U/G]$ is algebraic and $U \to [U/G]$ is a G-torsor (Theorem 2.1.8).

Since we want to allow for the case that U is not a scheme, we need to generalize a few definitions. An *action* of a smooth affine group scheme $G \to T$ on an algebraic space U over T is a morphism $\sigma\colon G \times_T U \to U$ satisfying the same axioms as in Definition C.1.7, and we define as in Definition 1.3.12 the quotient stack $[U/G]$ as the stackification of the prestack $[U/G]^{\mathrm{pre}}$, whose fiber category over an T-scheme S is the quotient groupoid $[U(S)/G(S)]$. Objects of $[U/G]$ over an T-scheme S are G-torsors $P \to S$ and G-equivariant morphisms $S \to U$. Since morphisms to algebraic spaces glue uniquely in the étale topology (by definition), the argument of Proposition 1.4.10 shows that $[U/G]$ is a stack. Using Definition 2.1.3, the morphism $U \to [U/G]$ is a *G-torsor* if for every morphism $S \to \mathcal{X}$ from a scheme S, the algebraic space $U \times_{\mathcal{X}} S$ with the induced G-action is a G-torsor over S.

Theorem 2.1.8 (Algebraicity of Quotient Stacks)**.** *If $G \to T$ is a smooth, affine group scheme acting on an algebraic space $U \to T$, the quotient stack $[U/G]$ is an algebraic stack over T such that $U \to [U/G]$ is a G-torsor and in particular surjective, smooth and affine.*

Proof. Set $\mathcal{X} = [U/G]$. We need to show that for any map $S \to \mathcal{X}$ from a scheme, the fiber product $U_S := U \times_{\mathcal{X}} S$ is a G-torsor over S. It follows from the definition of $[U/G]$ as the stackification of $[U/G]^{\mathrm{pre}}$ that there exists an étale cover $S' \to S$

of schemes and a commutative diagram

$$
\begin{tikzcd}
S' \arrow[r] \arrow[d] & U \arrow[d] \\
S \arrow[r] & \mathcal{X}.
\end{tikzcd}
$$

In the commutative cube

$$
\begin{tikzcd}
& U_{S'} \arrow[rr] \arrow[dl] \arrow[dd] & & S' \arrow[dl] \arrow[dd] \\
G \times U \arrow[rr, crossing over] \arrow[dd] & & U & \\
& U_S \arrow[rr] \arrow[dl] & & S \arrow[dl] \\
U \arrow[rr] & & \mathcal{X} \arrow[from=uu, crossing over] &
\end{tikzcd}
$$

the front, back, top and bottom squares are cartesian, and U_S is a sheaf. Since $G \times U \to U$ is a G-torsor, so is $U_{S'} \to S'$. By Effective Descent for G-torsors (Proposition C.3.11), $U_S \to S$ is a G-torsor. $\square$

Corollary 2.1.9. *If G is a finite group acting freely on an algebraic space U, then the quotient sheaf U/G is an algebraic space.*

Proof. Theorem 2.1.8 implies that U/G is an algebraic stack and that $U \to U/G$ is a G-torsor so in particular finite, étale, surjective and representable by schemes. Taking $U' \to U$ to be any étale presentation by a scheme, the composition $U' \to U \to U/G$ yields an étale presentation of U/G. $\square$

Remark 2.1.10. This resolves the troubling issue from Example 0.6.5 where we saw that the quotient of a finite group acting freely on a scheme need not exist as a scheme. In addition, it shows that the category of algebraic spaces is reasonably well-behaved as it is closed under taking quotients by free actions of finite groups.

2.1.5 Algebraicity of $\mathcal{M}_g$

We now show that $\mathcal{M}_g$ is an algebraic stack. The main idea is quite simple: every smooth connected projective curve C is tri-canonically embedded $C \overset{|\omega_C^{\otimes 3}|}{\hookrightarrow} \mathbb{P}^{5g-6}$ and the locally closed subscheme $H' \subset \mathrm{Hilb}_P(\mathbb{P}^{5g-6})$ parameterizing smooth families of tri-canonically embedded curves provides a smooth presentation $H' \to \mathcal{M}_g$

Theorem 2.1.11 (Algebraicity of the stack of smooth curves)**.** *If $g \geq 2$, then $\mathcal{M}_g$ is an algebraic stack over* $\operatorname{Spec} \mathbb{Z}$*.*

Proof. As in the proof that $\mathcal{M}_g$ is a stack (Proposition 1.4.6), we will use Properties of Families of Smooth Curves (Proposition 1.4.7) which implies that for a family of smooth curves $\pi \colon \mathcal{C} \to S$, $\omega_{\mathcal{C}/S}^{\otimes 3}$ is relatively very ample on S (as $g > 2$) and $\pi_* \omega_{\mathcal{C}/S}^{\otimes 3}$ is a vector bundle of rank $5(g-1)$. In particular, $\omega_{\mathcal{C}/S}^{\otimes 3}$ yields a closed immersion $\mathcal{C} \hookrightarrow \mathbb{P}(\pi_* \omega_{\mathcal{C}/S}^{\otimes 3})$ over S. By Riemann–Roch, the Hilbert polynomial of any fiber $\mathcal{C}_s \hookrightarrow \mathbb{P}^{5g-6}_{\kappa(s)}$ is given by

$$P(n) := \chi(\mathcal{O}_{\mathcal{C}_s}(n)) = \deg(\omega_{\mathcal{C}_s}^{\otimes 3n}) + 1 - g = (6n-1)(g-1).$$

Let

$$H := \mathrm{Hilb}_P(\mathbb{P}^{5g-6}_{\mathbb{Z}})$$

by the Hilbert scheme parameterizing closed subschemes of $\mathbb{P}^{5g-6}$ with Hilbert polynomial P (Theorem D.0.1). Let $\mathcal{C} \hookrightarrow \mathbb{P}^{5g-6} \times H$ be the universal closed subscheme and let $\pi \colon \mathcal{C} \to H$. We claim that there is a unique locally closed subscheme $H' \subset H$ consisting of points $h \in H$ satisfying

(a) $\mathcal{C}_h \to \operatorname{Spec} \kappa(h)$ is smooth and geometrically connected; and
(b) $\mathcal{C}_h \hookrightarrow \mathbb{P}^{5g-6}_{\kappa(h)}$ is embedded by the complete linear series $\omega_{\mathcal{C}_h/\kappa(h)}^{\otimes 3}$.
(c) denoting $\mathcal{C}' = \mathcal{C}|_{H'} \to H'$, the coherent sheaves $\omega_{\mathcal{C}'/H'}^{\otimes 3}$ and $\mathcal{O}_{\mathcal{C}'}(1)$ differ by a pullback of a line bundle from H'.

Since the condition that a fiber of a proper morphism (of finite presentation) is smooth is an open condition on the target (Corollary A.3.8), the condition that $\mathcal{C}_h$ is smooth is open. Consider the Stein factorization [Har77, Cor. 11.5] $\mathcal{C} \to \widetilde{H} = \operatorname{Spec}_H \pi_* \mathcal{O}_{\mathcal{C}} \to H$ where $\mathcal{C} \to \widetilde{H}$ has geometrically connected fibers and $\widetilde{H} \to H$ is finite. Since the kernel and cokernel of $\mathcal{O}_H \to \pi_* \mathcal{O}_{\mathcal{C}}$ have closed support (as they are coherent), $\widetilde{H} \to H$ is an isomorphism over an open subscheme of H, which is precisely where the fibers of $\mathcal{C} \to H$ are geometrically connected. In summary, the set of $h \in H$ satisfying (a) is an open subscheme of H, which we will denote by H_1.

The relative canonical sheaf $\omega_{\mathcal{C}_1/H_1}$ of the family $\mathcal{C}_1 := \mathcal{C}|_{H_1}$ is a line bundle. As a consequence Theorem 2.1.12, there exists a locally closed subscheme $H_2 \hookrightarrow H_1$ such that a morphism $T \to H_1$ factor through H_2 if and only if $\omega_{\mathcal{C}_1/H_1}|_{\mathcal{C}_T}$ and $\mathcal{O}_{\mathcal{C}}(1)|_{\mathcal{C}_T}$ differ by the pullback of a line bundle on T. In particular, (c) holds and for every $h \in H_2$, there is an isomorphism $\omega_{\mathcal{C}_h/\kappa(h)}^{\otimes 3} \cong \mathcal{O}_{\mathcal{C}_h}(1)$. To arrange (b), consider the restriction of the universal curve $\pi_2 \colon \mathcal{C}_2 \to H_2$. There is a canonical map $\alpha \colon H^0(\mathbb{P}^{5g-6}, \mathcal{O}(1)) \otimes \mathcal{O}_{H_2} \to (\pi_2)_* \omega_{\mathcal{C}_2/H_2}$ of vector bundles of rank $5g-5$ on H_2 whose fiber over a point $h \in H_2$ is the map $\alpha_h \colon H^0(\mathbb{P}^{5g-6}_{\kappa(h)}, \mathcal{O}(1)) \to H^0(\mathcal{C}_h, \omega_{\mathcal{C}_h/\kappa(h)}^{\otimes 3})$. The closed locus defined by the support of $\operatorname{coker}(\alpha)$ is precisely the locus where α_h is not an isomorphism (as the vector bundles have the same rank). The closed subscheme $H' = H_2 \setminus \operatorname{Supp}(\operatorname{coker}(\alpha))$ satisfies (a)-(c).

The group scheme $\mathrm{PGL}_{5g-5} = \underline{\mathrm{Aut}}(\mathbb{P}^{5g-6}_{\mathbb{Z}})$ over $\mathbb{Z}$ acts naturally on H: if $g \in \mathrm{Aut}(\mathbb{P}^{5g-6}_S)$ and $[\mathcal{D} \subset \mathbb{P}^{5g-6}_S] \in H(S)$, then $g \cdot [\mathcal{D} \subset \mathbb{P}^{5g-6}_S] = [g(\mathcal{D}) \subset \mathbb{P}^{5g-6}_S]$. The closed subscheme $H' \subset H$ is PGL_{5g-5}-invariant and we claim that $\mathcal{M}_g \cong [H'/\mathrm{PGL}_{5g-5}]$. This establishes the theorem since $[H'/\mathrm{PGL}_{5g-5}]$ is algebraic (Theorem 2.1.8).

Consider the morphism $H' \to \mathcal{M}_g$ which forgets the embedding, i.e. assigns a closed subscheme $\mathcal{C} \subset \mathbb{P}^{5g-6}_S$ to the family $\mathcal{C} \to S$. This morphism descends to a morphism $\Psi^{\mathrm{pre}} \colon [H'/\mathrm{PGL}_{5g-5}]^{\mathrm{pre}} \to \mathcal{M}_g$ of prestacks. The map Ψ^{pre} is fully faithful since for a family $\mathcal{C} \subset \mathbb{P}^{5g-6}_S$ of closed subschemes in H', any automorphism of $\mathcal{C} \to S$ induces an automorphism of $\omega_{\mathcal{C}/S}^{\otimes 3}$ and therefore an automorphism of $\mathbb{P}^{5g-6}_S$ preserving $\mathcal{C}$.

Since $\mathcal{M}_g$ is a stack (Theorem 2.1.8), the universal property of stackification yields a morphism $\Psi\colon [H'/\operatorname{PGL}_{5g-5}] \to \mathcal{M}_g$. Since $[H'/\operatorname{PGL}_{5g-5}]^{\mathrm{pre}} \to [H'/\operatorname{PGL}_{5g-5}]$ is fully faithful (Exercise 1.4.13), so is Ψ. It remains to check that Ψ is essentially surjective. For this, it suffices to check that if $\pi\colon \mathcal{C} \to S$ is a family of smooth curves, then there exists an étale cover $\{S_i \to S\}$ such that each $\mathcal{C}|_{S_i}$ is in the image of $H' \to \mathcal{M}_g$. Since $\pi_*\omega_{\mathcal{C}/S}$ is locally free of rank $5g-5$ and there is a closed immersion $\mathcal{C} \hookrightarrow \mathbb{P}(\pi_*\omega_{\mathcal{C}/S}^{\otimes 3})$ over S, we may simply take $\{S_i\}$ to be any Zariski-open cover (and thus étale cover) where $\pi_*\omega_{\mathcal{C}/S}^{\otimes 3}$ is free. $\square$

The above proof used the following fact asserting under certain hypotheses, for a morphism $X \to S$ and a line bundle L on X, the locus in S consisting of points $s \in S$ such that $L|_{X_s}$ is trivial is closed. See [SP, Tag 0BEZ,Tag 0BF0] (and [Mum70, Cor. II.5.6, Thm. III.10] for the case when X is a product over S) .

Theorem 2.1.12. *Let $f\colon X \to S$ be a flat, proper morphism of finite presentation with geometrically integral fibers. Let L be a line bundle on X. Assume that for any morphism $T \to S$, the base change $f_T\colon X_T \to T$ satisfies $\mathcal{O}_T \xrightarrow{\sim} (f_T)_*\mathcal{O}_{X_T}$. Let L be a line bundle on X. Then there exists a closed subscheme $Z \hookrightarrow S$ of finite presentation such that a morphism $T \to S$ factors through Z if and only if $L|_{X_T}$ is the pullback of a line bundle on T.*

Exercise 2.1.13. Let $f\colon X \to S$ be a morphism as in Theorem 2.1.12. Define the *Picard functor of $f\colon X \to S$* as

$$\operatorname{Pic}_{X/S}\colon \operatorname{Sch}/S \to \operatorname{Sets}, \quad T \mapsto \operatorname{Pic}(X_T)/f_T^*\operatorname{Pic}(T).$$

Show that the above theorem is equivalent to the diagonal morphism $\operatorname{Pic}_{X/S} \to \operatorname{Pic}_{X/S} \times_S \operatorname{Pic}_{X/S}$ of presheaves over Sch/S being representable by closed immersions, i.e. $\operatorname{Pic}_{X/S}$ is separated over S.

Exercise 2.1.14. Show that $\mathcal{M}_{1,1}$ is an algebraic stack.

2.1.6 Algebraicity of $\mathcal{M}_{C,r,d}$

We now show that the stack of vector bundles over a fixed curve is algebraic.

Theorem 2.1.15 (Algebraicity of the stack of vector bundles)**.** *Let C be a smooth, projective and connected curve over a field k, and let r and d be integers with $r \geq 0$. The stack $\mathcal{M}_{C,r,d}$ is an algebraic stack over* $\operatorname{Spec} k$.

Proof. For any vector bundle E on C of rank r and degree d, by Serre vanishing $E(m)$ is globally generated and $H^1(C, E(m)) = 0$ for $m \gg 0$. In particular,

$$\Gamma(C, E(m)) \otimes \mathcal{O}_C \twoheadrightarrow E(m)$$

is surjective which by construction induces an isomorphism on global sections. By Riemann–Roch, the Hilbert polynomial of E is

$$P(n) = \chi(E(n)) = \deg(E(n)) + \operatorname{rk}(E(n))(1-g) = d + rn + r(1-g).$$

For any scheme S, we have the diagram

$$\begin{tikzcd} & C \times S \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ C & & S. \end{tikzcd}$$

For each integer m, consider the substack $\mathcal{M}^m_{C,r,d}$ parameterizing families $\mathcal{E}$ of vector bundles on $C \times S$ over S such that $p_1^* p_{2,*}\mathcal{E}(m) \to \mathcal{E}(m)$ is surjective and $R^1 p_{2,*}\mathcal{E}(m) = 0$. It follows from Cohomology and Base Change [Har77, Thm III.12.11] that $\mathcal{M}^m_{C,r,d} \subset \mathcal{M}_{C,r,d}$ is an open substack.

For each m, let $N_m = P(m)$ and consider the Quot scheme

$$Q_m := \mathrm{Quot}^P(C, \mathcal{O}_C(-m)^{N_m})$$

parameterizing quotients $\mathcal{O}_C(-m)^{N_m} \twoheadrightarrow F$ with Hilbert polynomial P (Theorem D.0.2). Let $\mathcal{O}_{C\times Q_m}(-m)^{N_m} \to \mathcal{E}_m$ be the universal quotient on $C \times Q_m$ and consider the induced map

$$\Psi \colon \mathcal{O}_{Q_m}^{N_m} \xrightarrow{\sim} p_{2,*}\mathcal{O}_{C\times Q_m}^{N_m} \to p_{2,*}(\mathcal{E}_m(m))$$

The cokernel of Ψ has closed support in Q_m and its complement $Q'_m \subset Q_m$ is precisely the locus over which Ψ is an isomorphism.

The Quot scheme Q_m inherits a natural action from GL such that Q'_m is invariant. The morphism $Q'_m \to \mathcal{M}^m_{C,r,d}$, defined by $[\mathcal{O}_C(-m)^{N_m} \twoheadrightarrow F] \mapsto F$, factors to a yield a morphism $\Psi^{\mathrm{pre}} \colon [Q'_m/\,\mathrm{GL}_{N_m}]^{\mathrm{pre}} \to \mathcal{M}^m_{C,r,d}$ of prestacks. The map Ψ^{pre} is fully faithful since any automorphism of a family $\mathcal{F} \in \mathcal{M}^m_{C,r,d}(S)$ of vector bundles on $C \times S$ induces an automorphism of $p_{2,*}\mathcal{F}(m) = \mathcal{O}_S^{N_m}$ which is an element of $\mathrm{GL}_{N_m}(S)$, and this element acts on $\mathcal{O}_C(-m)^{N_m}$ preserving the quotient F.

Since $\mathcal{M}_{C,r,d}$ is a stack (Proposition 1.4.9), there is an induced morphism $\Psi \colon [Q'_m/\,\mathrm{GL}_{N_m}] \to \mathcal{M}^m_{C,r,d}$ of stacks which is also fully faithful (Exercise 1.4.13) and by construction essentially surjective. We conclude that

$$\mathcal{M}_{C,r,d} = \bigcup_m \,[Q'_m/\,\mathrm{GL}_{N_m}]$$

and the result follows from the algebraicity of quotient stacks (Theorem 2.1.8. $\square$

Remark 2.1.16. Note that while $\mathcal{M}_{C,r,d}$ itself is not quasi-compact (**??**), the proof establishes that any quasi-compact open substack of $\mathcal{M}_{C,r,d}$ is a quotient stack.

2.1.7 Survey of important results

We will develop the foundations of algebraic spaces and stacks in the forthcoming chapters but it is worth first highlighting some of the most important results.

The importance of the diagonal

When overhearing others discussing algebraic stacks, you may have wondered what's all the fuss about the diagonal? Well, I'll tell you—the diagonal encodes the stackiness!

First and foremost, the diagonal $\mathcal{X} \to \mathcal{X}\times\mathcal{X}$ of an algebraic stack is representable and the diagonal $X \to X \times X$ of an algebraic space is representable by schemes. Many authors in fact include this condition in the definition of algebraicity.

Recall that if $\mathcal{X}$ is a prestack over Sch and x, y are objects over a scheme T, then there is a cartesian diagram

$$\begin{tikzcd} \underline{\mathrm{Isom}}_{\mathcal{X}(T)}(x,y) \arrow[r] \arrow[d] & T \arrow[d, "{(x,y)}"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X}; \end{tikzcd}$$

see Exercise 1.3.28. Axiom (1) of a stack is the condition that $\underline{\mathrm{Isom}}_{\mathcal{X}(T)}(x,y)$ is a sheaf on $(\mathrm{Sch}/T)_{\text{Ét}}$ and Representability of the Diagonal (**??**) shows that $\underline{\mathrm{Isom}}_{\mathcal{X}(T)}(x,y)$ is an algebraic space. Moreover, $\underline{\mathrm{Aut}}_{\mathcal{X}(T)}(x) = \underline{\mathrm{Isom}}_{\mathcal{X}(T)}(x,x)$ is naturally a sheaf in groups and thus a group algebraic space over T. Taking T to be the spectrum of a field K, we define the *stabilizer of* $x\colon \operatorname{Spec} K \to \mathcal{X}$ as

$$G_x := \underline{\mathrm{Aut}}_{\mathcal{X}(K)}(a).$$

For schemes (resp. separated schemes), the diagonal is an immersion (resp. closed immersion). For algebraic stacks, the diagonal is not necessarily a monomorphism as the fiber over $(x,x)\colon \operatorname{Spec} K \to \mathcal{X} \times \mathcal{X}$, or in other words the stabilizer G_x, may be non-trivial. Properties of the diagonal in fact characterize algebraic spaces and Deligne–Mumford stacks: an algebraic stack is an algebraic space (resp. Deligne–Mumford stack) if and only if $\mathcal{X} \to \mathcal{X} \to \mathcal{X}$ is a monomorphism (resp. unramified)—see **??**. Properties of the stabilizer also provide characterizations as in the table below:

Table 2.1: Characterization of algebraic spaces and Deligne–Mumford stacks

Type of space	Property of the diagonal	Property of stabilizers
algebraic space	monomorphism	trivial
Deligne–Mumford stack	unramified	discrete and reduced groups
algebraic stack	arbitrary	arbitrary

As a consequence of these characterizations, we will generalize Corollary 2.1.9: the quotient of a free action of a smooth algebraic group on an algebraic space exists as an algebraic space. We will also be able to establish that $\mathcal{M}_g$ is Deligne–Mumford rather than just algebraic (Theorem 2.1.11).

We now summarize additional important properties of algebraic spaces, Deligne–Mumford stacks and algebraic stacks. The reader may also wish to consult Table 3 for a brief recap of the trichotomy of moduli spaces.

Properties of algebraic spaces

- If $R \rightrightarrows X$ is an étale equivalence relation of schemes, the quotient sheaf X/R is an algebraic space.
- If X is a quasi-separated algebraic space, there exists a dense open subspace $U \subset X$ which is a scheme.

- If $X \to Y$ is a separated and quasi-finite morphism of noetherian algebraic spaces, then there exists a factorization $X \hookrightarrow \widetilde{X} \to Y$ where $X \hookrightarrow \widetilde{X}$ is an open immersion and $\widetilde{X} \to Y$ is finite (Zariski's Main Theorem). In particular, $X \to Y$ is quasi-affine.

Properties of Deligne–Mumford stacks

- If $R \rightrightarrows X$ is an étale groupoid of scheme, the quotient stack $[X/R]$ is a Deligne–Mumford stack.
- If $\mathcal{X}$ is a Deligne–Mumford stack (e.g. algebraic space), there exists a scheme U and a finite morphism $U \to \mathcal{X}$.
- If $\mathcal{X}$ is a Deligne–Mumford stack and $x \in \mathcal{X}(k)$ is any field-valued point, there exists an étale neighborhood $[\mathrm{Spec}(A)/G] \to \mathcal{X}$ of x where G is a finite group, which can be arranged to be the stabilizer of x (Local Structure of Deligne–Mumford Stacks).
- If $\mathcal{X}$ is a separated Deligne–Mumford stack, there exists a coarse moduli space $\mathcal{X} \to X$ where X is a separated algebraic space (Keel-Mori theorem).

Properties of algebraic stacks

- If $R \rightrightarrows X$ is a smooth groupoid of scheme, the quotient stack $[X/R]$ is an algebraic stack.
- If $\mathcal{X}$ is an algebraic stack of finite type over an algebraically closed field k with affine diagonal, any point $x \in \mathcal{X}(k)$ with linearly reductive stabilizer has an affine étale neighborhood $[\mathrm{Spec}(A)/G_x] \to \mathcal{X}$ of x where G is a finite group (Local Structure of Algebraic Stacks).
- Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field k of characteristic 0 with affine diagonal. If $\mathcal{X}$ is S-complete and Θ-reductive, there exists a good moduli space $\mathcal{X} \to X$ where X is a separated algebraic space of finite type over k.

Notes

Deligne–Mumford and algebraic stacks were first introduced in [DM69] and [Art74]—and in both cases referred to as *algebraic stacks*—with conventions slightlly different than ours. Namely, [DM69, Def. 4.6] assumed in addition to the existence of an étale presentation that the diagonal is representable by schemes (which is automatic if the diagonal is separated and quasi-compact). On the other hand, [Art74, Def. 5.1] assumed in addition to the existence of a smooth presentation that the stack is locally of finite type over an excellent Dedekind domain. We will not use the term *Artin stack* which is often used to refer to algebraic stacks that satisfy Artin's axioms (e.g. algebraic stacks locally of finite type over an excellent scheme with quasi-compact and separated diagonal) as *Artin stacks*.

We follow the conventions of [Ols16] and [SP] (with the exception that we work over the site $\mathrm{Sch}_{\text{Ét}}$ while [SP] works over $\mathrm{Sch}_{\mathrm{fppf}}$).

Appendix A

Properties of morphisms

In this appendix, we recall definitions and summarize properties for certain types morphisms of schemes—locally of finite presentation, flat, smooth, étale, and unramified.

We pay particular attention to properties that can be described *functorially*, i.e. properties of schemes and their morphisms that can be characterized in terms of their functors. The following properties of morphisms can be characterized functorially:

- separated, universally closed and proper;
- locally of finite presentation; and
- smooth, étale and unramified.

Such descriptions are particularly advantageous for us since we systematically study moduli problems via functors and stacks. For example, the valuative criterion for properness for $\overline{\mathcal{M}}_g$ amounts to checking that every family of curves over a punctured curve (i.e. over the generic point of a DVR) can be extended uniquely (after possibly a finite extension of the curve) to the entire curve (i.e. DVR). Similarly, the smoothness of $\overline{\mathcal{M}}_g$ can be shown by using the functorial formal lifting criterion for smoothness.

A.1 Morphisms locally of finite presentation

A morphism of schemes $f\colon X \to Y$ is *locally of finite type* (resp. *locally of finite presentation*) if for all affine open $\operatorname{Spec} B \subset Y$ and $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$, there is surjection $A[x_1, \ldots, x_n] \to B$ of A-algebras (resp. a surjection $\phi\colon A[x_1, \ldots, x_n] \to B$ such that the ideal $\ker(\phi) \subset A[x_1, \ldots, x_n]$) is finitely generated). If in addition f is quasi-compact (resp. quasi-compact and quasi-separated), we say that f is *of finite type* (resp. *of finite presentation*).

Remark A.1.1. When Y is locally noetherian, these two notions coincide. However, in the non-noetherian setting even closed immersions may not be locally of finite presentation; e.g. $\operatorname{Spec} \mathbb{C} \hookrightarrow \operatorname{Spec} \mathbb{C}[x_1, x_2, \ldots]$. Since functors and stacks are defined in these notes on the entire category of schemes, it is often necessary to work with non-noetherian schemes. In particular, when defining a moduli functor or stack, we need to specify what families of objects are over possibly non-noetherian schemes. Morphisms of finite presentation are better behaved than morphisms of finite type and so we often use the former condition. For example, when defining a family of smooth curves $\pi\colon \mathcal{C} \to S$, we require not only that π is proper and smooth, but also of finite presentation.

The following is a very useful functorial criterion for a morphism to be locally of finite presentation. First recall that an *inverse system* (or *projective system*) in a category $\mathcal{C}$ is a partially ordered set $(I, \geq)$ which is filtered (i.e. for every $i, j \in I$ there exists $k \in I$ such that $k \geq i$ and $k \geq j$) together with a functor $I \to \mathcal{C}$.

Proposition A.1.2. *A morphism $f\colon X \to Y$ of schemes is locally of finite presentation if and only if for every inverse system $\{\operatorname{Spec} A_\lambda\}_{\lambda \in I}$ of schemes over Y, the natural map*

$$\varinjlim_{\lambda} \operatorname{Mor}_Y(\operatorname{Spec} A_\lambda, X) \to \operatorname{Mor}_Y(\operatorname{Spec}(\varinjlim_{\lambda} A_\lambda), X) \tag{A.1.1}$$

is bijective.

We won't include a proof here but we will mention a conceptual reason for why you might expect this to be true: any ring A (e.g. $\mathbb{C}[x_1, x_2, \ldots]$) is the union (or colimit) of its finitely generated subalgebras A_λ. The requirement that any map $\operatorname{Spec} A \to X$ factors through $\operatorname{Spec} A_\lambda \to X$ for some λ can be viewed as the condition that specifying $\operatorname{Spec} A \to X$ over Y depends on only a finite amount of data and therefore can is a type of finiteness condition on X over Y. We encourage the reader to convince themselves the above proposition holds in the case of a morphism of affine schemes.

Remark A.1.3. As we desire to define and study moduli stacks $\mathcal{X}$ that are of finite type over a field k, the following analogous condition to (A.1.1) better hold: for all inverse system $\{\operatorname{Spec} A_\lambda\}_{\lambda \in I}$ of k-schemes, the natural functor

$$\varinjlim_{\lambda} \operatorname{MOR}_k(\operatorname{Spec} A_\lambda, \mathcal{X}) \to \operatorname{MOR}_k(\operatorname{Spec}(\varinjlim_{\lambda} A_\lambda), \mathcal{X})$$

is an equivalence. It turns out for many moduli stacks, this condition can be checked directly even before knowing algebraicity. In fact, this locally of finite presentation condition (often also referred to as *limit-preserving*) is the first axiom in Artin's criteria for algebraicity.

A.2 Flatness

You can't get very far in moduli theory without internalizing the concept of flatness. While its definition is seemingly abstract and algebraic, it is a magical geometric property of a morphism $X \to Y$ that ensures that fibers X_y 'vary nicely' as $y \in Y$ varies. This principle is nicely illustrated by the fact that a subscheme $X \subset \mathbb{P}^n_Y$ is flat over an integral scheme Y if and only if the function assigning a point y to the Hilbert polynomial of the fiber $X_y \subset \mathbb{P}^n_{\kappa(y)}$ is constant (Proposition A.2.5).

A.2.1 Definition and equivalences

A morphism $f\colon X \to Y$ of schemes is *flat* if for all affine opens $\operatorname{Spec} B \subset Y$ and $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$, the ring map $B \to A$ is flat, i.e. the functor

$$- \otimes_B A\colon \operatorname{Mod}(B) \to \operatorname{Mod}(A)$$

is exact. More generally, a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is *flat over* Y if for all affine opens as above, $\Gamma(\operatorname{Spec} A, \mathcal{F})$ is a flat B-module, i.e. the functor $- \otimes_B \Gamma(\operatorname{Spec} A, \mathcal{F})$ is exact.

Flat Equivalences A.2.1. Let $f\colon X \to Y$ be a morphism of schemes and $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent:

(1) $\mathcal{F}$ is flat over Y;

(2) There exists a Zariski-cover $\{\operatorname{Spec} B_i\}$ of Y and $\{\operatorname{Spec} A_{ij}\}$ of $f^{-1}(\operatorname{Spec} B_i)$ such that $\Gamma(\operatorname{Spec} A_{ij}, \mathcal{F})$ is flat as an B_i-module under the ring map $B_i \to A_{ij}$;

(3) For all $x \in X$, the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$ is flat as an $\mathcal{O}_{Y,y}$-module.

(4) The functor

$$\operatorname{QCoh}(Y) \to \operatorname{QCoh}(X), \qquad \mathcal{G} \mapsto f^*\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$$

is exact.

If $x \in X$, we say that a morphism $f\colon X \to Y$ of schemes is *flat at* x (resp. a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is *flat at* x) if there exists a Zariski-open neighborhood $U \subset X$ containing x such that $f|_U$ (resp. $\mathcal{F}|_U$) is flat over Y. This is equivalent to the flatness of $\mathcal{O}_{X,x}$ (resp. $\mathcal{F}_x$) as an $\mathcal{O}_{Y,y}$-module.

A.2.2 Useful geometric properties

Proposition A.2.2 (Flat Morphisms are Open)**.** *Let $f\colon X \to Y$ be a morphism of schemes. If f is flat and locally of finite presentation, then $f(U) \subset Y$ is open for every open $U \subset X$.*

The following simple corollary will be used to reduce certain properties of flat and locally of finite presentation morphisms to the affine case.

Corollary A.2.3. *If $f\colon X \to Y$ is a faithfully flat and locally of finite presentation morphism of schemes and $\{V_i\}$ is an affine open cover of Y, then there exist an open cover $\{U_{ij}\}_{j\in J}$ of $f^{-1}(V_i)$ for each i such that U_{ij} is quasi-compact and $f(U_{ij}) = V_i$.*

Proposition A.2.4 (Flatness Criterion over Smooth Curves)**.** *Let C be an integral and regular scheme of dimension 1 (e.g. the spectrum of a DVR or a smooth connected curve over a field) and $X \to C$ a quasi-compact and quasi-separated morphism of schemes. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is flat over C if and only if every associated point of $\mathcal{F}$ maps to the generic point of C.*

Recall that if $X \subset \mathbb{P}^n_K$ is a subscheme and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, the *Hilbert polynomial* of $\mathcal{F}$ is $P_{\mathcal{F}}(n) = \chi(X, \mathcal{F}(n)) \in \mathbb{Q}[n]$.

Proposition A.2.5 (Flatness vs the Hilbert Polynomial)**.** *Let Y be an integral scheme and $X \subset \mathbb{P}^n_Y$ a closed subscheme. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is flat over Y if and only if the function*

$$Y \to \mathbb{Q}[n], \qquad y \mapsto P_{\mathcal{F}|_{X_y}}$$

assigning a point $y \in Y$ to the Hilbert polynomial of the restriction $\mathcal{F}|_{X_y}$ to the fiber $X_y \subset \mathbb{P}^n_{\kappa(y)}$ is constant.

Proposition A.2.6 (Generic flatness)**.** *Let $f\colon X \to S$ be a finite type morphism of schemes and $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. If S is reduced, there exists an open dense subscheme $U \subset S$ such that $X_U \to U$ is flat and of presentation and such that $\mathcal{F}|_{X_U}$ is flat over U and of finite presentation as on $\mathcal{O}_{X_U}$-module.*

A.2.3 Faithful flatness

For a ring A, an A-module M is *faithfully flat* if for all non-zero map $\phi\colon N \to N'$ of A-modules, the induced map $\phi \otimes_A M\colon N \otimes_A M \to N' \otimes_A M$ is also non-zero.

Faithfully Flat Equivalences A.2.7. Let R be a ring and M be an A-module. The following are equivalent:

(1) M is faithfully flat;

(2) for any A-module N and non-zero element $n \in N$, the map $M \to N \otimes M$ given by $m \mapsto m \otimes n$ is non-zero;

(3) for any non-zero A-module N, we have $N \otimes_A M$ is non-zero;

(4) the functor $- \otimes_R M\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R)$ is faithfully exact, i.e. a sequence $N' \to N \to N''$ of A-modules is exact if and only if $N' \otimes_A M \to N \otimes_A M \to N'' \otimes_A M$ is exact; and

(5) M is flat and for all maximal ideals $\mathfrak{m} \subset A$, the quotient $M/\mathfrak{m}M$ is non-zero.

If in addition $M = B$ is an A-algebra, then the above are also equivalent to:

(6) $\operatorname{Spec} B \to \operatorname{Spec} A$ is flat and surjective.

A morphism $f\colon X \to Y$ of schemes is *faithfully flat* if f is flat and surjective. This is equivalent to the condition that $f^*\colon \operatorname{QCoh}(Y) \to \operatorname{QCoh}(X)$ is faithfully exact. It is also equivalent to the condition that a quasi-coherent $\mathcal{O}_Y$-module (resp. a morphism of quasi-coherent $\mathcal{O}_Y$-modules) is zero if and only if its pullback is.

A.3 Étale, smooth and unramified morphisms

A.3.1 Smooth morphisms

A morphism $f\colon X \to Y$ of schemes is *smooth* if f is locally of finite presentation and flat, and the geometric fiber $X_{\overline{\kappa(y)}} = X \times_Y \operatorname{Spec} \overline{\kappa(y)}$ of any point $y \in Y$ is regular.

Smooth Equivalences A.3.1. Let $f\colon X \to Y$ be morphism of schemes locally of finite presentation. The following are equivalent:

(1) f is smooth;

(2) f is formally smooth, i.e. for any surjection $A \to A_0$ of rings with nilpotent kernel and any commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

of solid arrows, there exists a dotted arrow filling in the diagram;

(This is often referred to as the *Formal Lifting Criterion for Smoothness.*)

(3) for every point $x \in X$, there exist affine open neighborhoods $\operatorname{Spec} B$ of $f(x)$ and $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$ of x and an A-algebra isomorphism

$$B \cong \big(A[x_1, \dots, x_n]/(f_1, \dots, f_r)\big)_g$$

for some $f_1, \dots, f_r, g \in A[x_1, \dots, x_n]$ with $r \leq n$ such that the determinant $\det(\frac{\delta f_j}{\delta x_i})_{1 \leq i,j \leq r} \in B$ of the Jacobi matrix, defined by the partial derivatives with respect *first* r x_i's, is a unit.

(This is often referred to as the *Jacobi Criterion for Smoothness.*)

If in addition X and Y are locally of finite type over an algebraically closed field K, then the above are equivalent to:

(4) for all $x \in X(K)$, there is an isomorphism $\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_{Y,y}[[x_1, \dots, x_r]]$ of $\widehat{\mathcal{O}}_{Y,y}$-algebras.

If $f \colon X \to Y$ is a smooth morphism of schemes, then $\Omega_{X/Y}$ is a locally free $\mathcal{O}_X$-module of finite rank. If Y is connected, the rank of $\Omega_{X/Y}$ is the dimension of any fiber.

A.3.2 Étale morphisms

A morphism $f \colon X \to Y$ of schemes is *étale* if f is smooth of relative dimension 0 (i.e. f is smooth and $\dim X_y = 0$ for all $y \in Y$).

Étale Equivalences A.3.2. Let $f \colon X \to Y$ be morphism of schemes locally of finite presentation. The following are equivalent:

(1) f is étale;

(2) f is smooth and $\Omega_{X/Y} = 0$;

(3) f is flat and for all $y \in Y$, the fiber X_y is isomorphic to a disjoint union $\bigsqcup_i \operatorname{Spec} K_i$ where each K_i is separable field extension of $\kappa(y)$; (This is exactly the condition that f is flat and unramified; see Section A.3.3.)

(4) f is formally étale, i.e. for any surjection $A \to A_0$ of rings with nilpotent kernel and any commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

of solid arrows, there exists a unique dotted arrow filling in the diagram;

(This is often referred to as the *Formal Lifting Criterion for Étaleness.*)

(5) for every point $x \in X$, there exist affine open neighborhoods $\operatorname{Spec} B$ of $f(x)$ and $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$ of x and an A-algebra isomorphism

$$B \cong \big(A[x_1, \dots, x_n]/(f_1, \dots, f_n)\big)_g$$

for some $f_1, \dots, f_n, g \in A[x_1, \dots, x_n]$ such that the determinant $\det(\frac{\delta f_j}{\delta x_i})_{1 \leq i,j \leq n} \in B$ is a unit.

(This is often referred to as the *Jacobi criterion for étaleness.*)

If in addition X and Y are locally of finite type over an algebraically closed field K, then the above are equivalent to:

(6) for all $x \in X(K)$, the induced map $\widehat{\mathcal{O}}_{Y,y} \to \widehat{\mathcal{O}}_{X,x}$ on completions is an isomorphism

If in addition X and Y are smooth over K, then the above are equivalent to:

(7) for all $x \in X(K)$, the induced map $T_{X,x} \to T_{Y,y}$ on tangent spaces is an isomorphism.

A.3.3 Unramified morphisms

A morphism $f \colon X \to Y$ of schemes is *unramified* if f is locally of finite type and every geometric fiber is discrete and reduced. Note that this second condition is equivalent to requiring that for all $y \in Y$, the fiber X_y is isomorphic to a disjoint union $\bigsqcup_i \operatorname{Spec} K_i$ where each K_i is separable field extension of $\kappa(y)$.

Warning A.3.3. We are following the conventions of [RG71] and [SP] rather than [EGA] as we only require that f is locally of finite type rather than locally of finite presentation.

Unramified Equivalences A.3.4. Let $f \colon X \to Y$ be morphism of schemes locally of finite type. The following are equivalent:

(1) f is unramified;

(2) $\Omega_{X/Y} = 0$;

(3) f is formally unramifed, i.e. for any surjection $A \to A_0$ of rings with nilpotent kernel and any commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

of solid arrows, there exists at most one dotted arrow filling in the diagram.

(This is often referred to as the *Formal Lifting Criterion for Unramifiedness.*)

If in addition X and Y are locally of finite type over an algebraically closed field K, then the above are equivalent to:

(4) for all $x \in X(K)$, the induced map $\widehat{\mathcal{O}}_{Y,y} \to \widehat{\mathcal{O}}_{X,x}$ on completions is surjective.

A.3.4 Further properties

The following proposition states that any smooth morphism $X \to Y$ is étale locally (on the source and target) of the form $\mathbb{A}^n_R \to \operatorname{Spec} R$ and in particular has sections étale locally on the target.

Proposition A.3.5. *Let $X \to Y$ be a morphism of schemes which is smooth at a point $x \in X$. There exists affine open subschemes* $\operatorname{Spec} A \subset X$ *and* $\operatorname{Spec} B \subset Y$ *with* $x \in \operatorname{Spec} A$*, and a commutative diagram*

$$\begin{tikzcd} X \arrow[d] & \operatorname{Spec} A \arrow[l, hook'] \arrow[r] \arrow[d] & \mathbb{A}^n_B \arrow[dl] \\ Y & \operatorname{Spec} B \arrow[l, hook'] & \end{tikzcd}$$

where $U \to \mathbb{A}^n_B$ is étale.

Proposition A.3.6 (Fiberwise criteria for étaleness/smoothness/unramifiedness)**.** *Consider a diagram*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

of schemes where $X \to S$ and $Y \to S$ are locally of finite presentation. Assume that $X \to S$ is flat in the étale/smooth case. Then $X \to Y$ is étale (resp. smooth, unramified) if and only if $X_s \to Y_s$ is for all $s \in S$.

Remark A.3.7. With the same hypotheses, let $x \in X$ be a point with image $s \in S$. Then $X \to Y$ is étale (resp. smooth, unramified) at $x \in X$ if and only if $X_s \to Y_s$ is at x.

Corollary A.3.8. *If $f\colon X \to Y$ is a proper morphism of finite presentation, then the set $y \in Y$ such that $X_y \to \operatorname{Spec} \kappa(y)$ is smooth defines an open subset.*

Proof. By Remark A.3.7, if $y \in Y$ is a point such that $X_y \to \operatorname{Spec} \kappa(y)$ is smooth, then $f\colon X \to Y$ is smooth in an open neighborhood of X_y. If $Z \subset X$ is the closed locus where $f\colon X \to Y$ is not smooth, then $f(Z) \subset Y$ is precisely the locus where the fibers of f are not smooth. Since f is proper, $f(Z)$ is closed. □

Proposition A.3.9. *Let $X \to Y$ be a smooth morphism of noetherian schemes. For any point $x \in X$ with image $y \in Y$,*

$$\dim_x(X) = \dim_y(Y) + dim_x(X_y).$$

Appendix B

Descent

It is hard to overstate the importance of descent in moduli theory. The central idea of descent is as simple as it is powerful. You already know that many properties of schemes and their morphisms can be checked on a Zariski-cover, and descent theory states that they can also be checked on étale covers or even faithfully flat covers. For example, if $Y' \to Y$ is étale and surjective, then a morphism $X \to Y$ is proper if and only if $X \times_Y Y' \to Y'$ is.

The applications of descent reach far beyond moduli theory. For instance, it can be used to reduce statements about schemes over a field k to the case when k is algebraically closed since $k \to \overline{k}$ is faithfully flat, or reduce statements over a local noetherian ring A to its completion $\widehat{A}$ since $A \to \widehat{A}$ is faithfully flat.

References: [BLR90, Ch.6], [Vis05], [Ols16, Ch. 4], [SP, Tag 0238], [EGA, §IV.2], and [SGA1, §VIII.7] (other descent results are scattered throughout EGA and SGA).

B.1 Descent for quasi-coherent sheaves

Descent theory rests on the following algebraic fact.

Proposition B.1.1. *If $\phi\colon A \to B$ is a faithfully flat ring map, then the sequence*

$$A \xrightarrow{\phi} B \underset{b \mapsto 1 \otimes b}{\overset{b \mapsto b \otimes 1}{\rightrightarrows}} B \otimes_A B$$

is exact. More generally, if M is an A-module, the sequence

$$M \xrightarrow{m \mapsto m \otimes 1} M \otimes_A B \underset{m \otimes b \mapsto m \otimes 1 \otimes b}{\overset{m \otimes b \mapsto m \otimes b \otimes 1}{\rightrightarrows}} M \otimes_A B \otimes_A B \tag{B.1.1}$$

is exact.

Remark B.1.2. By Faithfully Flat Equivalences A.2.7, $A \to B$ and $M \to M \otimes_A B$ are necessarily injective.

Proof. Since $A \to B$ is faithfully flat, the sequence (B.1.1) is exact if and only if the sequence

$$M \otimes_A B \xrightarrow{m \otimes b' \mapsto m \otimes 1 \otimes b'} M \otimes_A B \otimes_A B \underset{m \otimes b \otimes b' \mapsto m \otimes 1 \otimes b \otimes b'}{\overset{m \otimes b \otimes b' \mapsto m \otimes b \otimes 1 \otimes b'}{\rightrightarrows}} M \otimes_A B \otimes_A B \otimes_A B$$

is exact. The above sequence can be rewritten as

$$M \otimes_A B \xrightarrow{x \mapsto x \otimes 1} (M \otimes_A B) \otimes_B (B \otimes_A B) \underset{x\otimes y \mapsto x \otimes 1 \otimes y}{\overset{x \otimes y \mapsto x \otimes y \otimes 1}{\rightrightarrows}} (M \otimes_A B) \otimes_B (B \otimes_A B) \otimes_B (B \otimes_A B)$$

which is precisely sequence (B.1.1) applied to ring $B \to B \otimes_A B$ given by $b \mapsto 1 \otimes b$ and the B-module $M \otimes_A B$. Since this ring map has a section $B \otimes_A B \to B$ given by $b \otimes b' \mapsto bb'$, we can assume that in the statement $\phi\colon A \to B$ has a section $s\colon B \to A$ with $s \circ \phi = \mathrm{id}_A$. Let $x \in M \otimes_A B$ such that $x \otimes 1 = 1 \otimes x \in M \otimes_A B \otimes_A B$. Applying $\mathrm{id}_M \otimes \mathrm{id}_B \otimes s\colon M \otimes_A B \otimes_A B \to M \otimes_A B \otimes_A A \cong M \otimes_A B$ to the identity $x \otimes 1 = 1 \otimes x$ yields that $x = (\mathrm{id}_M \otimes s)(x) \in M$ where $\mathrm{id}_M \otimes s$ denotes the composition $M \otimes_A B \to M \otimes_A A \xrightarrow{\sim} M$. □

Proposition B.1.3. *Let $f\colon X \to Y$ be a faithfully flat morphism of schemes that is either quasi-compact or locally of finite presentation. Let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent $\mathcal{O}_Y$-modules. Let p_1, p_2 denote the two projections $X \times_Y X \to X$ and q denote the composition $X \times_Y X \xrightarrow{p_i} X \xrightarrow{f} Y$. Then the sequence*

$$\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{F}, f^*\mathcal{G}) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathrm{Hom}_{\mathcal{O}_{X \times_Y X}}(q^*\mathcal{F}, q^*\mathcal{G})$$

is exact.

Remark B.1.4. The special case that $\mathcal{F} = \mathcal{O}_Y$ implies that $0 \to \Gamma(Y, \mathcal{G}) \xrightarrow{f^*} \Gamma(X, f^*\mathcal{G}) \xrightarrow{p_1^* - p_2^*} \Gamma(X \times_Y X, q^*\mathcal{G})$ is exact. When X and Y are affine, this is precisely Proposition B.1.1.

Proof. This can be reduced to Proposition B.1.1 by first reducing to the case that Y is affine. If f is quasi-compact, we reduce to the case that X is affine by choosing a *finite* affine cover $\{U_i\}$ and replacing X with the affine scheme $\bigsqcup_i U_i$. If f is locally of finite presentation, we apply Corollary A.2.3 to reduce to the quasi-compact case. We leave the details to the reader. □

Proposition B.1.5. *Let $f\colon X \to Y$ be a faithfully flat morphism of schemes that is either quasi-compact or locally of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module and $\alpha\colon p_1^*\mathcal{F} \to p_2^*\mathcal{F}$ an isomorphism of $\mathcal{O}_{X \times_Y X}$-modules satisfying the cocycle condition $p_{12}^*\alpha \circ p_{23}^*\alpha = p_{13}^*\alpha$ on $X \times_Y X \times_X Y$. Then there exists a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{G}$ and an isomorphism $\phi\colon \mathcal{F} \to f^*\mathcal{G}$ such that $p_1^*\phi = p_2^*\phi \circ \alpha$ on $X \times_Y X$. The data $(\mathcal{F}, \phi)$ is unique up to unique isomorphism.*

Remark B.1.6. The following diagram may be useful to internalize the above statement:

$$p_{12}^*\alpha \circ p_{23}^*\alpha = p_{13}^*\alpha \qquad p_1^*\mathcal{F} \xrightarrow{\alpha} p_2^* \qquad \mathcal{F} \qquad \exists\mathcal{G}$$

$$\begin{tikzcd} X \times_Y X \times_Y X \arrow[r, shift left=2, "p_{12}"] \arrow[r, "p_{13}"] \arrow[r, shift right=2, "p_{23}"'] & X \times_Y X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Keep in mind the special case that $X = \bigsqcup_i Y_i$ where $\{Y_i\}$ is an open covering of Y in which case the above fiber products correspond to intersections.

The cocycle condition $p_{12}^*\alpha \circ p_{23}^*\alpha = p_{13}^*\alpha$ should be understood as the commutativity of

$$\begin{tikzcd}
p_{12}^*p_1^*\mathcal{F} \arrow[r, "p_{12}^*\alpha"] \arrow[d, equal] & p_{12}^*p_2^*\mathcal{F} \arrow[r, equal] & p_{23}^*p_1^*\mathcal{F} \arrow[d, "p_{23}^*\alpha"] \\
p_{13}^*p_1^*\mathcal{F} \arrow[r, "p_{13}^*\alpha"] & p_{13}^*p_2^*\mathcal{F} \arrow[r, equal] & p_{23}^*p_2^*\mathcal{F}
\end{tikzcd}$$

and the condition that $p_1^*\phi = p_2^*\phi \circ \alpha$ should be understood as the commutativity of

$$\begin{tikzcd}
p_1^*\mathcal{F} \arrow[r, "p_1^*\phi"] \arrow[d, "\alpha"] & p_1^*f^*\mathcal{G} \arrow[d, equal] \\
p_2^*\mathcal{F} \arrow[r, "p_2^*\phi"] & p_2^*f^*\mathcal{G}.
\end{tikzcd}$$

Remark B.1.7. Propositions B.1.3 and B.1.5 together can be reformulated as the statement that the category $\mathrm{QCoh}(Y)$ is equivalent to the *category of descent datum for* $X \to Y$, denoted by $\mathrm{QCoh}(X \to Y)$. Here the objects of $\mathrm{QCoh}(X \to Y)$ are pairs $(\mathcal{F}, \alpha)$ consisting of a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and an isomorphism $\alpha\colon p_1^*\mathcal{F} \to p_2^*\mathcal{F}$ satisfying the cocycle condition. A morphism $(\mathcal{F}', \alpha') \to (\mathcal{F}, \alpha)$ is a morphism $\beta\colon \mathcal{F}' \to \mathcal{F}$ such that

$$\begin{tikzcd}
p_1^*\mathcal{F}' \arrow[r, "\alpha'"] \arrow[d, "p_1^*\beta"] & p_2^*\mathcal{F}' \arrow[d, "p_2^*\beta"] \\
p_1^*\mathcal{F} \arrow[r, "\alpha"] & p_2^*\mathcal{F}
\end{tikzcd}$$

commutes.

B.2 Descent for morphisms

The following result implies that if Z is a scheme, the functor $\mathrm{Mor}(-, Z)\colon \mathrm{Sch} \to \mathrm{Sets}$ is a sheaf in the fppf topology.

Proposition B.2.1. *Let $f\colon X \to Y$ be a faithfully flat morphism of schemes that is either quasi-compact or locally of finite presentation. If $g\colon X \to Z$ is any morphism to a scheme such that $p_1 \circ g = p_2 \circ g$ on $X \times_Y X$, then there exists a unique morphism $h\colon Y \to Z$ filling in the commutative diagram*

$$\begin{tikzcd}
X \times_Y X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \arrow[r, "f"] \arrow[dr, "g"'] & Y \arrow[d, dashed, "h"] \\
& & Z
\end{tikzcd}$$

of solid arrows.

B.3 Descending schemes

Proposition B.3.1 (Effective Descent for Open and Closed Immersions)**.** *Let $f\colon X \to Y$ be a faithfully flat morphism of schemes that is either quasi-compact* *or locally of finite presentation. If $Z \subset X$ is a closed (resp. open) subscheme such that ${p_1}^{-1}(Z) = p_2^{-1}(Z)$ as closed (resp. open) subschemes of $X \times_Y X$, then there exists a closed (resp. open) subscheme $W \subset Y$ such that $Z = f^{-1}(W)$.*

To formulate effective descent for morphisms that are not monomorphisms, we need to specify an isomorphism of pullbacks satisfying a cocycle condition. We will use the following notation: if $f\colon X \to Y$ and $W \to Y$ are morphisms of schemes, we denote f^*W as the fiber product $X \times_Y W$.

Proposition B.3.2 (Effective Descent for Affine Immersions)**.** *Let $f\colon X \to Y$ be a faithfully flat morphism of schemes that is either quasi-compact or locally of finite presentation. If $Z \to X$ is an affine morphism and $\alpha\colon {p_1}^*(Z) \xrightarrow{\sim} p_2^*(Z)$ is an isomorphism over $X \times_Y X$ satisfying $p_{12}^*\alpha \circ p_{23}^*\alpha = p_{13}^*\alpha$, then there exists an affine morphism $W \to Y$ and an isomorphism $\phi\colon Z \to f^*(W)$ such that $p_1^*\phi = p_2^*\phi \circ \alpha$.*

Remark B.3.3. It is helpful to interpret the above statement using the diagram

$$
\begin{tikzcd}
p_{12}^*\alpha \circ p_{23}^*\alpha = p_{13}^*\alpha & p_1^*Z \arrow[r, "\alpha"] \arrow[d] & p_2^*Z \arrow[dl] & Z \arrow[r, dashed] \arrow[d] & W \arrow[d, dashed] \\
X \times_Y X \times_Y X \arrow[r, shift left=2, "p_{12}"] \arrow[r, "p_{13}"] \arrow[r, shift right=2, "p_{23}"'] & X \times_Y X \arrow[rr, shift left, "p_1"] \arrow[rr, shift right, "p_2"'] & & X \arrow[r, "f"] & Y.
\end{tikzcd}
$$

Proposition B.3.4 (Effective Descent for Quasi-affine Immersions)**.** *Let $f\colon X \to Y$ be a faithfully flat morphism of schemes that is either quasi-compact or locally of finite presentation. If $Z \to X$ is a quasi-affine morphism and $\alpha\colon {p_1}^*(Z) \xrightarrow{\sim} p_2^*(Z)$ is an isomorphism over $X \times_Y X$ satisfying $p_{12}^*\alpha \circ p_{23}^*\alpha = p_{13}^*\alpha$, then there exists an quasi-affine morphism $W \to Y$ and an isomorphism $\phi\colon Z \to f^*(W)$ such that $p_1^*\phi = p_2^*\phi \circ \alpha$.*

Proposition B.3.5 (Effective Descent for Separated and Locally Quasi-finite morphisms)**.** *Let $f\colon X \to Y$ be a faithfully flat morphism of schemes that is either quasi-compact or locally of finite presentation. If $Z \to X$ is a separated and locally quasi-finite morphism of schemes and $\alpha\colon {p_1}^*(Z) \xrightarrow{\sim} p_2^*(Z)$ is an isomorphism over $X \times_Y X$ satisfying $p_{12}^*\alpha \circ p_{23}^*\alpha = p_{13}^*\alpha$, then there exists an quasi-affine morphism $W \to Y$ and an isomorphism $\phi\colon Z \to f^*(W)$ such that $p_1^*\phi = p_2^*\phi \circ \alpha$.*

Corollary B.3.6. *Let $\mathcal{P}$ be one of the following properties of morphisms of schemes: open immersion, closed immersion, locally closed immersion, affine, quasi-affine or separated and locally quasi-finite. Let $f\colon X \to Y$ be a faithfully flat morphism of schemes that is either quasi-compact or locally of finite presentation. Let $Q \to Y$ be a map of presheaves and consider the fiber product*

$$
\begin{tikzcd}
Q_X \arrow[r] \arrow[d] & Q \arrow[d] \\
X \arrow[r, "f"] & Y.
\end{tikzcd}
$$

If Q_X is a scheme and $Q_X \to X$ has $\mathcal{P}$, then Q is a scheme and $Q \to Y$ has $\mathcal{P}$.

Proof. As Q_X is the pullback of Q, there is a canonical isomorphism $\alpha\colon p_1^*Q_X \to p_2^*Q_X$ satisfying the cocycle condition. By Propositions B.3.1, B.3.2, B.3.4 and B.3.5, there exists a quasi-affine morphism $W \to Y$ that pulls back to $Q_X \to X$. The reader to left to check that the natural map $Q \to W$ is an isomorphism. $\square$

B.4 Descending properties of schemes and their morphisms

B.4.1 Descending properties of morphisms

Proposition B.4.1 (Properties flat local on the target)**.** *Let $Y' \to Y$ be a faithfully flat morphism of schemes that is either quasi-compact or locally of finite presentation. Let $\mathcal{P}$ be one of the following properties of a morphism of schemes:*

(i) isomorphism;

(ii) surjective;

(iii) proper;

(iv) flat;

(v) smooth;

(vi) étale;

(vii) unramified.

Then $X \to Y$ has $\mathcal{P}$ if and only if $X \times_Y Y' \to Y'$ does.

Proposition B.4.2 (Properties smooth local on the source)**.** *Let $X' \to X$ be a smooth and surjective morphism of schemes. Let $\mathcal{P}$ be one of the following properties of a morphism of schemes:*

(i) surjective;

(ii) smooth;

Then $X \to Y$ has $\mathcal{P}$ if and only if $X' \to X \to Y$ does.

Proposition B.4.3 (Properties étale local on the source)**.** *Let $X' \to X$ be an étale and surjective morphism of schemes. Let $\mathcal{P}$ be one of the following properties of a morphism of schemes:*

(i) surjective;

(ii) étale;

(iii) smooth.

Then $X \to Y$ has $\mathcal{P}$ if and only if $X' \to X \to Y$ does.

MORE PROPERTIES TO BE ADDED

B.4.2 Descent for properties of quasi-coherent sheaves

Proposition B.4.4. *Let $f\colon X \to Y$ be a faithfully flat morphism of schemes that is either quasi-compact or locally of finite presentation. Let $\mathcal{P} \in \{$finite type, finite presentation, vector bundle$\}$ be a property of quasi-coherent sheaves. If $\mathcal{G}$ is a quasi-coherent $\mathcal{O}_Y$-module, then $\mathcal{G}$ has $\mathcal{P}$ if and only if $f^*\mathcal{G}$ does. If X and Y are noetherian, then the same holds for the property of coherence.*

Appendix C

Algebraic groups and actions

C.1 Algebraic groups

C.1.1 Group schemes

Definition C.1.1. A *group scheme over a scheme S* is a morphism $\pi\colon G \to S$ of schemes together with a multiplication morphism $\mu\colon G \times_S G \to G$, an inverse morphism $\iota\colon G \to G$ and an identity morphism $e\colon S \to G$ (with each morphism over S) such that the following diagrams commute:

$$\begin{tikzcd} G \times_S G \times_S G \arrow[r, "\mathrm{id}_G \times \mu"] \arrow[d, "\mu \times \mathrm{id}_G"'] & G \times_S G \arrow[d, "\mu"] \\ G \times_S G \arrow[r, "\mu"] & G \end{tikzcd}
\qquad
\begin{tikzcd} G \arrow[r, "(\mathrm{id}_G, \iota)"] \arrow[d, "(\iota, \mathrm{id}_G)"'] \arrow[dr, "e \circ \pi"] & G \times_S G \arrow[d, "\mu"] \\ G \times_S G \arrow[r, "\mu"] & G \end{tikzcd}
\qquad
\begin{tikzcd} G \arrow[r, "(e\circ\pi, \mathrm{id}_G)"] \arrow[d, "(\mathrm{id}_G, e\circ\pi)"'] \arrow[dr, "\mathrm{id}_G"] & G \times_S G \arrow[d, "\mu"] \\ G \times_S G \arrow[r, "\mu"] & G \end{tikzcd}$$

Associativity — Law of inverse — Law of identity

A morphism $\phi\colon H \to G$ of schemes over S is a *morphism of group schemes* if $\mu_G \circ (\phi \times \phi) = \phi \circ \mu_H$. A *closed subgroup* of G is a closed subscheme $H \subset G$ such that $H \to G \xrightarrow{\mu_G} G \times G$ factors through $H \times H$.

Remark C.1.2. If G and S are affine, then by reversing the arrows above gives $\Gamma(G, \mathcal{O}_G)$ the structure of a Hopf algebra of $\Gamma(S, \mathcal{O}_S)$.

Exercise C.1.3. Show that a group scheme over S is equivalently defined as a scheme G over S together with a factorization

$$\begin{tikzcd} \mathrm{Sch}/S \arrow[r] \arrow[dr, "\mathrm{Mor}_S(-,G)"'] & \mathrm{Gps} \arrow[d] \\ & \mathrm{Sets} \end{tikzcd}$$

where Gps $\to$ Sets is the forgetful functor.

(*We are not requiring that there exists a factorization; the factorization is part of the data. Indeed, the same scheme can have multiple structures as a group scheme, e.g.* $\mathbb{Z}/4$ *and* $\mathbb{Z}/2 \times \mathbb{Z}/2$ *over* $\mathbb{C}$.)

Example C.1.4. The following examples of group schemes are the most relevant for us. Let $S = \operatorname{Spec} R$ and V be a free R-module of finite rank:

1. The *multiplicative group scheme over* R is $\mathbb{G}_{m,R} = \operatorname{Spec} R[t]_t$ with comultiplication $\mu^*\colon R[t]_t \to R[t]_t \otimes_R R[t']_{t'}$ given by $t \mapsto tt'$.
2. The *additive group scheme over* R is $\mathbb{G}_{a,R} = \operatorname{Spec} R[t]$ with comultiplication $\mu^*\colon R[t] \to R[t] \otimes_R R[t']$ given by $t \mapsto t + t'$.
3. The *general linear group on* V is
$$\mathrm{GL}(V) = \operatorname{Spec}(\mathrm{Sym}^*(\mathrm{End}(V))_{\det})$$
with the comultiplication $\mu^*\colon \mathrm{Sym}^*(\mathrm{End}(V)) \to \mathrm{Sym}^*(\mathrm{End}(V)) \otimes_R \mathrm{Sym}^*(\mathrm{End}(V))$ which can be defined as following: choose a basis $v_1, \ldots, v_n$ of V and let $x_{ij}\colon V \to V$ where $v_i \mapsto v_j$ and $v_k \mapsto 0$ if $k \neq 0$, and then define $\mu^*(x_{ij}) = x_{i1}x'_{1j} + \cdots + x_{in}x'_{nj}$.
4. The *special linear group on* V is $\mathrm{SL}(V)$ is the closed subgroup of $\mathrm{GL}(V)$ defined by $\det = 1$.
5. The *projective linear group* PGL_n is the affine group scheme
$$\mathrm{Proj}(\mathrm{Sym}^*(\mathrm{End}(V)))_{\det}$$
with the comultiplication defined similarly to $\mathrm{GL}(V)$.

We write $\mathrm{GL}_{n,R} = \mathrm{GL}(R^n)$, $\mathrm{SL}_{n,R} = \mathrm{GL}(R^n)$ and $\mathrm{PGL}_{n,R} = \mathrm{PGL}(R^n)$. We often simply write $\mathbb{G}_m$, GL_n, SL_n and PGL_n when there is no possible confusion on what the base is.

Exercise C.1.5. (1) Provide functorial descriptions of each of the group schemes above.

(2) Show that any abstract group G can be given the structure of a group scheme $\bigsqcup_{g \in G} S$ over any base S. Provide both explicit and functorial descriptions.

Exercise C.1.6. Show that a group scheme $G \to S$ is trivial if and only if the fiber G_s is trivial for each $s \in S$.

C.1.2 Group actions

Definition C.1.7. An *action* of a group scheme $G \xrightarrow{\pi} S$ on a scheme $X \xrightarrow{p} S$ is a morphism $\sigma\colon G \times_S X \to X$ over S such that the following diagrams commute:

$$\begin{tikzcd} G \times_S G \times_S X \arrow[r, "\mathrm{id}_G \times \sigma"] \arrow[d, "\sigma \times \mathrm{id}_G"'] & G \times_S X \arrow[d, "\sigma"] \\ G \times_S X \arrow[r, "\sigma"] & G \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "e \circ p, \mathrm{id}_X"] \arrow[dr, "\mathrm{id}_X"] & G \times_S X \arrow[d, "\sigma"] \\ & X \end{tikzcd}$$

Compatibility — Law of identity

If $X \to S$ and $Y \to S$ are schemes with actions of $G \to S$, a morphism $f\colon X \to Y$ of schemes over S is G*-equivariant* if $\sigma_Y \circ (\mathrm{id} \times f) = f \circ \sigma_X$, and is G*-invariant* if G-equivariant and Y has the trivial G-action.

Exercise C.1.8. Show that giving a group action of $G \to S$ on $X \to S$ is the same as giving an action of the functor $\mathrm{Mor}_S(-, G)\colon \mathrm{Sch}/S \to \mathrm{Gps}$ on the functor $\mathrm{Mor}_S(-, X)\colon \mathrm{Sch}/S \to \mathrm{Sets}$.

(*This requires first spelling out what it means for a functor to groups to act on a functor to sets.*)

C.1.3 Representations

To define a representation, for simplicity we specialize to the case when $S = \operatorname{Spec} R$ and G are affine. The case that most interests us of course is when R is a field. A *representation* (or *comodule*) of a group scheme $G \to \operatorname{Spec} R$ is an R-module V together with a homomorphism $\widehat{\sigma}\colon V \to \Gamma(G, \mathcal{O}_G) \otimes_R V$ (often referred to as a *coaction*).

A representation V of G induces an action of G on $\mathbb{A}(V) = \operatorname{Spec} \operatorname{Sym}^* V$, which we refer to as a *linear action*. Morphisms of representations and subrepresentations are defined in the obvious way.

Exercise C.1.9. If $R = k$ is a field and V is a finite dimensional vector space, show that giving V the structure as a representation is the same as giving a homomorphism $G \to \mathrm{GL}(V)$ of group schemes.

A representation V of G is *irreducible* if for every subrepresentation $W \subset V$ is either 0 or V.

Example C.1.10 (Diagonaliable group schemes)**.** If A is a finitely generated abelian group, we let $R[A]$ be the free R-module generated by elements of A. The R-module $R[A]$ has the structure of an R-algebra with multiplication on generators induced from multiplication in A. The comultiplication $R[A] \to R[A] \otimes_R \to R[A']$ defined by $a \mapsto a \otimes a'$ defines a group scheme $D(A) = \operatorname{Spec} R[A]$ over $\operatorname{Spec} R$. A group scheme G over $\operatorname{Spec} R$ is *diagonalizable* if $G \cong D(A)$ for some A.

If $A = \mathbb{Z}^r$, then $D(A) = \mathbb{G}^r_{m,A}$ is the r-dimensional torus. If $A = \mathbb{Z}/n$, then $D(A) = \mu_n = \operatorname{Spec} k[t]/(t^n - 1)$. The classification of finitely generated abelian groups implies that any diagonalizable group scheme is a product of $\mathbb{G}^r_m \times \mu_{n_1} \times \cdots \mu_{n_k}$.

Exercise C.1.11. Describe $D(A)$ as a functor $\mathrm{Sch}/R \to \mathrm{Gps}$.

Each element $a \in A$ defines a one-dimensional representation $W_a = A$ of $D(A)$ defined by the coaction $W_a \to R[A] \otimes_R W_a$ defined by $1 \mapsto a \otimes 1$.

Proposition C.1.12. *Any free representation of a diagonalizable group scheme is a direct sum of one-dimensional representations.*

Proof. Let $G = D(A)$ and let $V = A^r$ be a free representation of G with coaction $\widehat{\sigma}\colon V \to R[A] \otimes_R V$. Then for each $a \in A$,

$$V_a := \{v \in V \mid \widehat{\sigma}(v) = a \otimes v\}$$

is isomorphic to $W_a^{\dim V_a}$ as G-representations. Then $V \cong \oplus_{a \in A} V_a$ as G-representations. The details are left to the reader. $\square$

If V is a representation of an affine group scheme G over $\operatorname{Spec} R$ with coaction $\widehat{\sigma}$, the *invariant subrepresentation* is defined as $V^G = \{v \in V \mid \widehat{\sigma}(v) = 1 \otimes v\}$. Observe that $V^G = V_0$ using the notation in the proof above.

C.2 Properties of algebraic groups

An *algebraic group over a field k* if a group scheme G of finite type over k. While we are not assuming that G is affine nor smooth. We are primarily interested in affine algebraic groups.

Algebraic Group Facts C.2.1. Let G be an affine algebraic group over a field k.

(1) Every representation V of G is a union of its finite dimensional subrepresentations.

(2) There exists a finite dimensional representation V and a closed immersion $G \hookrightarrow \mathrm{GL}(V)$ of group schemes.

(3) If G acts on an affine scheme X of finite type over k, there exist a finite dimensional representation V of G and a G-invariant closed immersion $X \hookrightarrow \mathbb{A}(V)$.

(4) If $\mathrm{char}(k) = 0$, then G is smooth.

C.3 Principal G-bundles

The following definition of a principal G-bundle is an algebraic formulation of the topological notion of a fiber bundle $P \to X$ with fiber G where G acts freely on P and $P \to X$ is G-invariant (i.e. equivariant with respect to the trivial action of G on X) with fibers isomorphic to G.

C.3.1 Definition and equivalences

Definition C.3.1. Let $G \to S$ be a flat group scheme locally of finite presentation. A *principal G-bundle over an S-scheme X* is flat morphism $P \to X$ locally of finite presentation with an action of G via $\sigma \colon G \times_S P \to P$ such that $P \to X$ is G-invariant and

$$(\sigma, p_2) \colon G \times_S P \to P \times_X P, \qquad (g, p) \mapsto (gp, p)$$

is an isomorphism.

A principal G-bundle is also often referred to as a *G-torsor* (see Definition C.3.12 and Exercise C.3.13).

Morphisms of principal G-bundles are G-equivariant morphisms.

Exercise C.3.2. Show that $P \to X$ is principal G-bundle over the S-scheme X if and only if $P \to X$ is a principal $G \times_S X$-bundle over the X-scheme X.

Exercise C.3.3. Show that a morphism of principal G-bundles is necessarily an isomorphism.

We call a principal G-bundle $P \to X$ *trivial* if there is a G-equivariant isomorphism $P \cong G \times X$ where G acts on $G \times X$ via multiplication on the first factor. The following proposition characterizes principal G-bundles as morphisms $P \to X$ which are locally trivial.

Proposition C.3.4. *Let $G \to S$ be a flat group scheme locally of finite presentation and $P \to X$ be a G-equivariant morphism of S-scheme where X has the trivial action. Then $P \to X$ is a principal G-bundle if and only if there exists a faithfully flat and locally of finite presentation morphism $X' \to X$, and an isomorphism $P \times_X X' \to G \times_S X'$ of principal G-bundles over X'. Moreover, if $G \to S$ is smooth, then $X' \to X$ can be arranged to be étale.*

Proof. The $\Rightarrow$ direction follows from the definition by taking $X' = P \to X$. For $\Leftarrow$, after base changing $G \to S$ by $X \to S$, we assume that G is defined over X (see Exercise C.3.2). Let $G_{X'}$ and $P_{X'}$ be the base changes of G and P along $X' \to X$. The base change of the action map $(\sigma, p_2): G \times_X P \to P \times_X P$ along $X' \to X$ is the action map $G_{X'} \times_{X'} P_{X'} \to P_{X'} \times_{X'} P_{X'}$ of $G_{X'}$ acting on $P_{X'}$ over X'. Since $P_{X'}$ is trivial, this latter action map is an isomorphism. Since the property of being an isomorphism descends along faithfully flat and locally of finite presentation morphisms (Proposition B.4.1), we conclude that $(\sigma, p_2): G \times_X P \to P \times_X P$ is an isomorphism.

The final statement follows from the fact that smooth morphisms have sections étale-locally (Proposition A.3.5). □

Exercise C.3.5. Let L/K be a finite Galois extension and $G = \mathrm{Gal}(L/K)$ be the finite group scheme over $\operatorname{Spec} K$. Show that $\operatorname{Spec} L \to \operatorname{Spec} K$ is a principal G-bundle.

Exercise C.3.6. If X is a scheme, show that there there is an equivalence of categories

$$\{\text{line bundles on } X\} \to \{\text{principal } \mathbb{G}_m\text{-bundle on } X\}$$
$$L \mapsto \mathbb{A}(L) \setminus 0$$

between the *groupoids* of line bundles on X and $\mathbb{G}_m$-torsors on X and (where the only morphisms allowed are isomorphisms). If L is a line bundle (i.e. invertible $\mathcal{O}_X$-module), then $\mathbb{A}(L)$ denotes the total space $\operatorname{Spec} \operatorname{Sym}^* L^\vee$ and 0 denotes the zero section $X \to \mathbb{A}(L)$.

Exercise C.3.7.

(1) Show that the standard projection $\mathbb{A}^{n+1} \setminus 0 \to \mathbb{P}^n$ is a principal $\mathbb{G}_m$-bundle.

(2) For each line bundle $\mathcal{O}(d)$ on $\mathbb{P}^n$, explicitly determine the corresponding principal $\mathbb{G}_m$-bundle. In particular, for which d does $\mathcal{O}(d)$ correspond to the principal $\mathbb{G}_m$-bundle of (1).

Exercise C.3.8. Let X be a scheme

(1) If E is a vector bundle on X of rank n, define the *frame bundle* is the functor

$$\mathrm{Frame}_X(E): \mathrm{Sch}/X \to \mathrm{Sets}, \quad (T \to X) \mapsto \{\text{trivializations } \alpha: f^*E \xrightarrow{\sim} \mathcal{O}_T^n\}.$$

Show that $\mathrm{Frame}_X(E)$ is representable by scheme and that $\mathrm{Frame}_X(E) \to X$ is a principal GL_n-bundle.

(2) If $P \to X$ is a principal GL_n-bundle, then define $P \times^{\mathrm{GL}_n} \mathbb{A}^n := (P \times \mathbb{A}^n)/\mathrm{GL}_n$ where GL_n acts diagonally via its given action on P and the standard action on $\mathbb{A}^n$. (The action is free and the quotient $(P \times \mathbb{A}^n)/\mathrm{GL}_n$ can be interpreted as the sheafification of the quotient presheaf $\mathrm{Sch}/X \to \mathrm{Sets}$ taking $T \mapsto (P \times \mathbb{A}^n)(T)/\mathrm{GL}_n(T)$ in the big Zariski (or big étale) topology or equivalently as the algebraic space quotient (???). Show that $(P \times \mathbb{A}^n)/\mathrm{GL}_n$ is representable by scheme and is the total space of a vector bundle over X.

(3) show that there there is an equivalence of categories

$$\{\text{vector bundles on } X\} \to \{\text{principal } \mathrm{GL}_n\text{-bundles on } X\}$$
$$E \mapsto \mathrm{Frame}_X(E)$$
$$\text{locally free sheaf associated to } (P \times \mathbb{A}^n)/\mathrm{GL}_n \hookleftarrow (P \to X)$$

between the *groupoids* of vector bundles on X and principal GL_n-bundles on X.

Exercise C.3.9. What is the GL_2-torsor on $\mathbb{P}^1 \times \mathbb{P}^1$ corresponding to $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$?

Exercise C.3.10. Let $G \to S$ be a smooth, affine group scheme. Let $P \to X$ and $Q \to X$ be principal G-bundles. Show that the functor

$$\begin{aligned}\underline{\mathrm{Isom}}_X(P,Q)\colon\ & \mathrm{Sch}/X \to \mathrm{Sets}\\ & (T \xrightarrow{f} X) \mapsto \mathrm{Isom}_{\text{principal } G\text{-bundles/T}}(f^*P, f^*Q)\end{aligned}$$

is representable by a scheme which is a principal G-bundle over X.

C.3.2 Descent for principal G-bundles

Proposition C.3.11 (Effective Descent for Principal G-bundles)**.** *Let $G \to S$ be a flat and affine group scheme of finite presentation Let $f\colon X \to Y$ be a faithfully flat morphism of schemes over S that is either quasi-compact or locally of finite presentation. If $P \to X$ is a principal G-bundle and $\alpha\colon {p_1}^*(P) \xrightarrow{\sim} p_2^*(P)$ is an isomorphism of principal G-bundles over $X \times_Y X$ satisfying $p_{12}^*\alpha \circ p_{23}^*\alpha = p_{13}^*\alpha$, then there exists a principal G-bundle $Q \to Y$ and an isomorphism $\phi\colon P \to f^*(Q)$ of principal G-bundles such that $p_1^*\phi = p_2^*\phi \circ \alpha$.*

C.3.3 G-torsors

A G-torsor is a categorical generalization of a principal G-bundle which makes sense with respect to any sheaf of groups on a site.

Definition C.3.12. Let $\mathcal{S}$ be a site and G a sheaf of groups on $\mathcal{S}$. A *G-torsor on $\mathcal{S}$* is a sheaf P of sets on $\mathcal{S}$ with a left action $\sigma\colon G \times P \to P$ of G such that

(a) For every object $X \in \mathcal{S}$, there exists a covering $\{X_i \to X\}$ such that $P(X_i) \neq 0$, and
(b) The action map $(\sigma, p_2)\colon G \times P \to P \times P$ is an isomorphism.

Exercise C.3.13. If $G \to S$ is a flat and affine group scheme of finite presentation, show that any G-torsor on the big étale topology $(\mathrm{Sch}/S)_{\text{Ét}}$ is representable by a principal G-bundle.

Appendix D

Hilbert and Quot schemes

In this section, we state that the Hilbert and Quot functors are representable by a projective scheme. Let $X \to S$ be a projective morphism of noetherian schemes and $\mathcal{O}_X(1)$ be a relatively ample line bundle on X. Let $P \in \mathbb{Q}[z]$ be a polynomial.

Theorem D.0.1. *The functor*

$$\begin{aligned} \operatorname{Hilb}^P(X/S) \colon & \operatorname{Sch}/S \to \operatorname{Sets} \\ & (T \to S) \mapsto \left\{ \begin{array}{l} \textit{subschemes } Z \subset X \times_S T \textit{ flat and finitely presented over } T \\ \textit{such that } Z_t \subset X \times_S \kappa(t) \textit{ has Hilbert polynomial } P \textit{ for all } t \in T \end{array} \right\} \end{aligned}$$

is represented by a scheme projective over S.

Theorem D.0.2. *If F is a coherent sheaf on X, the functor*

$$\begin{aligned} \operatorname{Quot}^P(F/X/S) \colon & \operatorname{Sch}/S \to \operatorname{Sets} \\ & (T \xrightarrow{f} S) \mapsto \left\{ \begin{array}{l} \textit{quotients } f^*F \to Q \textit{ of finite presentation such that} \\ Q_t \textit{ on } X \times_S \kappa(t) \textit{ has Hilbert polynomial } P \textit{ for all } t \in T \end{array} \right\} \end{aligned}$$

is represented by a scheme projective over S.

Remark D.0.3.

(1) Theorem D.0.1 is a special case of Theorem D.0.2 by taking $F = \mathcal{O}_X$.

(2) A morphism of noetherian schemes $X \to S$ is *projective* if there is a coherent sheaf E on S such that there is a closed immersion $X \hookrightarrow \mathbb{P}(E)$ over S [EGA, §II.5], [SP, Tag 01W8]. The definition of projectivity in [Har77, II.4] is stronger as it requires $X \hookrightarrow \mathbb{P}^n_S$. There is an intermediate notion of *strongly projective* morphisms requiring $X \hookrightarrow \mathbb{P}(E)$ where E is a vector bundle over S. In this case if $X \to S$ is strongly projective, one can show that $\operatorname{Hilb}^P(X/S) \to S$ and $\operatorname{Quot}^P(F/X/S) \to S$ are also strongly projective; [?].

(3) When T is noetherian, the conditions that Z be finitely presented and Q be of finite presentation in the definitions of $\operatorname{Hilb}^P(X/S)$ and $\operatorname{Quot}^P(F/X/S)$ are superfluous.

These theorems are the backbone of many results in moduli theory and in particular are essential for establishing properties about the moduli stacks $\overline{\mathcal{M}}_g$

of stable curves and $\mathcal{V}^{\mathrm{ss}}_{r,d}$ of vector bundles over a curve. While the reader could safely treat these results as black boxes (and we encourage some readers to do this), it is also worthwhile to dive into the details. The proof follows the same strategy as the construction of the Grassmanian (Proposition 0.5.7) but it involves several important new ingredients: Castelnuovo–Mumford regularity and flattening stratifications.

Appendix E

Artin approximation

In this section, we discuss the deep result of Artin Approximation (Theorem E.0.10) which can be vaguely expressed as the following principle:

> **Principle.** Algebraic properties that hold for the completion $\widehat{\mathcal{O}}_{S,s}$ of the local ring of a scheme S at a point s also hold in an étale neighborhood $(S', s') \to (S, s)$.

Artin approximation is related to another equally deep and powerful result known as Néron–Popescu Desingularization (Theorem E.0.4). Both Artin Approximation and Néron–Popescu are difficult theorems which we will not attempt to prove here. However, we will show at least how Artin Approximation easily follows from Néron–Popescu Desingularization.

E.0.1 Néron–Popescu Desingularization

Definition E.0.1. A ring homomorphism $A \to B$ of noetherian rings is called *geometrically regular* if $A \to B$ is flat and for every prime ideal $\mathfrak{p} \subset A$ and every finite field extension $k(\mathfrak{p}) \to k'$ (where $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}$), the fiber $B \otimes_A k'$ is regular.

Remark E.0.2. It is important to note that $A \to B$ is *not* assumed to be of finite type. In the case that $A \to B$ is a ring homomorphism (of noetherian rings) of finite type, then $A \to B$ is geometrically regular if and only if $A \to B$ is smooth (i.e. $\operatorname{Spec} B \to \operatorname{Spec} A$ is smooth).

Remark E.0.3. It can be shown that it is equivalent to require the fibers $B \otimes_A k'$ to be regular only for *inseparable* field extensions $k(\mathfrak{p}) \to k'$. In particular, in characteristic 0, $A \to B$ is geometrically regular if it is flat and for every prime ideal $\mathfrak{p} \subset A$, the fiber $B \otimes_A k(\mathfrak{p})$ is regular.

Theorem E.0.4 (Néron–Popescu Desingularization)**.** *Let $A \to B$ be a ring homomorphism of noetherian rings. Then $A \to B$ is geometrically regular if and only if $B = \varinjlim B_\lambda$ is a direct limit of smooth A-algebras.*

Remark E.0.5. This was result was proved by Néron in [Nér64] in the case that A and B are DVRs and in general by Popescu in [Pop85], [Pop86], [Pop90]. We recommend [Swa98] and [SP, Tag 07GC] for an exposition on this result.

Example E.0.6. If l is a field and l^s denotes its separable closure, then $l \to l^s$ is geometrically regular. Clearly, l^s is the direct limit of separable field extensions $l \to l'$ (i.e. étale and thus smooth l-algebras). If l is a perfect field, then any field extension $l \to l'$ is geometrically regular—but if $l \to l'$ is not algebraic, it is not possible to write l' is a direct limit of étale l-algebras. On the other hand, if l is a non-perfect field, then $l \to \bar{l}$ is not geometrically regular as the geometric fiber is non-reduced and thus not regular.

In order to apply Néron–Popescu Desingularization, we will need the following result, which we will also accept as a black box. The proof is substantially easier than Néron–Popescu's result but nevertheless requires some effort.

Theorem E.0.7. *If S is a scheme of finite type over a field k or $\mathbb{Z}$ and $s \in S$ is a point, then $\mathcal{O}_{S,s} \to \widehat{\mathcal{O}}_{S,s}$ is geometrically regular.*

Remark E.0.8. See [EGA, IV.7.4.4] or [SP, Tag 07PX] for a proof.

Remark E.0.9. A local ring A is called a *G-ring* if the homomorphism $A \to \widehat{A}$ is geometrically regular. We remark that one of the conditions for a scheme S to be *excellent* is that every local ring is a G-ring. Any scheme that is finite type over a field or $\mathbb{Z}$ is excellent.

E.0.2 Artin Approximation

Let S be a scheme and consider a contravariant functor

$$F\colon \operatorname{Sch}/S \to \operatorname{Sets}$$

where Sch/S denotes the category of schemes over S. An important example of a contravariant functor is the functor representing a scheme: if X is a scheme over S, then the *functor representing* X is:

$$h_X\colon \operatorname{Sch}/S \to \operatorname{Sets}, \qquad (T \to S) \mapsto \operatorname{Mor}_S(T, X). \tag{E.0.1}$$

We say that F is *locally of finite presentation* or *limit preserving* if for every direct limit $\varinjlim B_\lambda$ of $\mathcal{O}_S$-algebras B_λ (i.e. a direct limit of commutative rings B_λ together with morphisms $\operatorname{Spec} B_\lambda \to S$), the natural map

$$\varinjlim F(\operatorname{Spec} B_\lambda) \to F(\operatorname{Spec} \varinjlim B_\lambda)$$

is bijective. This should be viewed as a finiteness condition on the functor F. Indeed, a scheme X is locally of finite presentation over S if and only if its function $\operatorname{Mor}_S(-, X)$ is (Proposition A.1.2).

Theorem E.0.10 (Artin Approximation)**.** *Let S be an excellent scheme (e.g. a scheme of finite type over a field or $\mathbb{Z}$) and let*

$$F\colon \operatorname{Sch}/S \to \operatorname{Sets}$$

be a limit preserving contravariant functor. Let $s \in S$ be a point and $\widehat{\xi} \in F(\operatorname{Spec} \widehat{\mathcal{O}}_{S,s})$. For any integer $N \geq 0$, there exist a residually-trivial étale morphism

$$(S', s') \to (S, s) \qquad \text{and} \qquad \xi' \in F(S')$$

such that the restrictions of $\widehat{\xi}$ and ξ' to $\operatorname{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1})$ are equal.

Remark E.0.11. The following theorem was originally proven in [Art69, Cor. 2.2] in the case that S is of finite type over a field or an excellent dedekind domain. We also recommend [BLR90, §3.6] for an accessible account of the case of excellent and henselian DVRs.

Remark E.0.12. The condition that $(S', s') \to (S, s)$ is residually trivial means that the extension of residue fields $\kappa(s) \to \kappa(s')$ is an isomorphism. To make sense of the restriction ξ' to $\operatorname{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1})$, note that since $(S', s') \to (S, s)$ is a residually-trivial étale morphism, there are compatible identifications $\mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1} \cong \mathcal{O}_{S',s'}/\mathfrak{m}_{s'}^{N+1})$.

Remark E.0.13. It is not possible in general to find $\xi' \in F(S')$ restricting to $\widehat{\xi}$ or even such that the restrictions of ξ' and $\widehat{\xi}$ to $\operatorname{Spec} \mathcal{O}_{S,s}/\mathfrak{m}_s^{n+1}$ agree for all $n \geq 0$. For instance, F could be the functor $\operatorname{Mor}(-, \mathbb{A}^1)$ representing the affine line $\mathbb{A}^1$ and $\widehat{\xi} \in \widehat{\mathcal{O}}_{S,s}$ could be a non-algebraic power series.

E.0.3 Alternative formulation of Artin Approximation

Consider the functor $F \colon \operatorname{Sch}/S \to \operatorname{Sets}$ representing an affine scheme $X = \operatorname{Spec} A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ of finite type over an excellent affine scheme $S = \operatorname{Spec} A$. Restricted to the category of affine schemes over S (or equivalently A-algebras), the functor is:

$$\begin{aligned} F \colon \operatorname{AffSch}/S &\to \operatorname{Sets} \\ \operatorname{Spec} B &\mapsto \{a = (a_1, \ldots, a_n) \in B^{\oplus n} \mid f_i(a) = 0 \text{ for all } i\} \end{aligned}$$

Applying Artin Approximation to the functor F, we obtain:

Corollary E.0.14. *Let R be an excellent ring and A be a finitely generated R-algebra. Let $\mathfrak{m} \subset A$ be a maximal ideal. Let $f_1, \ldots, f_m \in A[x_1, \ldots, x_n]$ be polynomials. Let $\widehat{a} = (\widehat{a}_1, \ldots, \widehat{a}_n) \in \widehat{A}_{\mathfrak{m}}$ be a solution to the equations $f_1(x) = \cdots = f_m(x) = 0$. Then for every $N \geq 0$, there exist a residually-trivial étale ring homomorphism $(A, \mathfrak{m}) \to (A', \mathfrak{m}')$ and a solution $a' = (a'_1, \ldots, a'_n) \in A'^{\oplus n}$ to the equations $f_1(x) = \cdots = f_m(x) = 0$ such that $a' \cong \widehat{a} \mod \mathfrak{m}^{N+1}$.* □

Remark E.0.15. Although this corollary may seem weaker than Artin Approximation, it is not hard to see that it in fact directly implies Artin Approximation. Indeed, writing $S = \operatorname{Spec} A$, we may write $\widehat{\mathcal{O}}_{S,s}$ as a direct limit of finite type A-algebras and since F is limit preserving, we can find a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \arrow[d] \arrow[dr, "\widehat{\xi}"] & \\ \operatorname{Spec} A[x_1, \ldots, x_n]/(f_1, \ldots, f_m) \arrow[r, "\xi"'] & F. \end{tikzcd}$$

The vertical morphism corresponds to a solution $\widehat{a} = (\widehat{a}_1, \ldots, \widehat{a}_n) \in \widehat{\mathcal{O}}_{S,s}^{\oplus n}$ to the equations $f_1(x) = \cdots = f_m(x) = 0$. Applying Corollary E.0.14 yields the desired étale morphism $(\operatorname{Spec} A', s') \to (\operatorname{Spec} A, s)$ and a solution $a' = (a'_1, \ldots, a'_n) \in A'^{\oplus n}$

to the equations $f_1(x) = \cdots = f_m(x) = 0$ agreeing with $\widehat{a}$ up to order N (i.e. congruent modulo $\mathfrak{m}^{N+1}$). This induces a morphism

$$\xi' \colon \operatorname{Spec} A' \to \operatorname{Spec} A[x_1, \dots, x_n]/(f_1, \dots, f_m) \to F$$

which agrees with $\widehat{\xi} \colon \operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \to F$ to order N.

Alternatively, we can state Corollary E.0.14 using henselian rings. Recall that a local ring $(A, \mathfrak{m})$ is called *henselian* if the following analogue of the implicit function theorem holds: if $f_1, \dots, f_n \in A[x_1, \dots, x_n]$ and $\overline{a} = (\overline{a}_1, \dots, \overline{a}_n) \in (A/\mathfrak{m})^{\oplus n}$ is a solution to the equations $f_1(x) = \cdots = f_n(x) = 0$ modulo $\mathfrak{m}$ and $\det\left(\frac{\partial f_i}{\partial x_j}(\overline{a})\right)_{i,j=1,\dots,n} \neq 0$, then there exists a solution $a = (a_1, \dots, a_n) \in A^{\oplus n}$ to the equations $f_1(x) = \cdots = f_n(x) = 0$. Equivalently, if $(A, \mathfrak{m})$ is a local k-algebra with $A/\mathfrak{m} \cong k$, then $(A, \mathfrak{m})$ is henselian if every étale homomorphism $(A, \mathfrak{m}) \to (A', \mathfrak{m}')$ of local rings with $A/\mathfrak{m} \cong A'/\mathfrak{m}'$ is an isomorphism. Also, if S is a scheme and $s \in S$ is a point, one defines the *henselization* $\mathcal{O}^h_{S,s}$ *of* S *at* s to be

$$\mathcal{O}^h_{S,s} = \varinjlim_{(S',s') \to (S,s)} \Gamma(S', \mathcal{O}_{S'})$$

where the direct limit is over all étale morphisms $(S', s') \to (S, s)$. In other words, $\mathcal{O}^h_{S,s}$ is the local ring of S at s in the étale topology.

Corollary E.0.16. *Let $(A, \mathfrak{m})$ be an excellent local henselian ring (e.g. the henselization of the local ring of a scheme of finite type over a field or $\mathbb{Z}$). Let $f_1, \dots, f_m \in A[x_1, \dots, x_n]$. Suppose that $\widehat{a} = (\widehat{a}_1, \dots, \widehat{a}_n) \in \widehat{A}^{\oplus n}$ is a solution to the equations $f_1(x) = \cdots = f_m(x) = 0$. For any integer $N \geq 0$, there exists a solution $a = (a_1, \dots, a_n) \in A^{\oplus n}$ to the equations $f_1(x) = \cdots = f_m(x) = 0$ such that $\widehat{a} \cong a \mod \mathfrak{m}^{N+1}$.*

E.0.4 A first application of Artin Approximation

The next corollary states an important fact which you may have taken for granted: if two schemes are formally isomorphic at two points, then they are isomorphic in the étale topology.

Corollary E.0.17. *Let X_1, X_2 be schemes of finite type over an excellent scheme S. Suppose $x_1 \in X_1, x_2 \in X_2$ are points such that $\widehat{\mathcal{O}}_{X_1,x_1}$ and $\widehat{\mathcal{O}}_{X_2,x_2}$ are isomorphic as $\mathcal{O}_S$-algebras. Then there exists a common residually-trivial étale neighborhood*

$$\begin{tikzcd} & (X_3, x_3) \arrow[dl] \arrow[dr] & \\ (X_1, x_1) & & (X_2, x_2) . \end{tikzcd} \tag{E.0.2}$$

Proof. The functor

$$F \colon \mathrm{Sch}/X_1 \to \mathrm{Sets}, \quad (T \to X_1) \mapsto \operatorname{Mor}(T, X_2)$$

is limit preserving as it can be identified with the representable functor $\operatorname{Mor}_{X_1}(-, X_2 \times X_1)$ corresponding to the finite type morphism $X_2 \times X_1 \to X_1$. The isomorphism $\widehat{\mathcal{O}}_{X_1,x_1} \cong \widehat{\mathcal{O}}_{X_2,x_2}$ provides an element of $F(\operatorname{Spec} \widehat{\mathcal{O}}_{X_1,x_1})$. By applying Artin Approximation with $N = 1$, we obtain a diagram as in (E.0.2) with $X_3 \to X_1$ étale at x_3 with $\kappa(x_2) \xrightarrow{\sim} \kappa(x_3)$ and such that $\mathcal{O}_{X_2,x_2}/\mathfrak{m}_{x_2}^2 \to \mathcal{O}_{X_3,x_3}/\mathfrak{m}_{x_3}^2$ is an isomorphism. By Lemma E.0.18, $\widehat{\mathcal{O}}_{X_2,x_2} \to \widehat{\mathcal{O}}_{X_3,x_3}$ is surjective. But we also know that $\widehat{\mathcal{O}}_{X_3,x_3}$ is abstractly isomorphic to $\widehat{\mathcal{O}}_{X_2,x_2}$ and since any surjective endomorphism of a noetherian ring is an isomorphism, we conclude that $\widehat{\mathcal{O}}_{X_2,x_2} \to \widehat{\mathcal{O}}_{X_3,x_3}$ is an isomorphism and therefore that $(X_3, x_3) \to (X_2, x_2)$ is étale. $\square$

Lemma E.0.18. *Let $(A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ be a local homomorphism of noetherian complete local rings. If $A/\mathfrak{m}_A^2 \to B/\mathfrak{m}_B^2$ is surjective, so is $A \to B$.*

Proof. This follows from the following version of Nakayama's lemma for noetherian complete local rings $(A, \mathfrak{m})$: if M is a (not-necessarily finitely generated) A-module such that $\bigcap_k \mathfrak{m}^k M = 0$ and $m_1, \ldots, m_n \in F$ generate $M/\mathfrak{m}M$, then $m_1, \ldots, m_n$ also generate M (see [Eis95, Exercise 7.2]). $\square$

E.0.5 Néron–Pescue Desingularization $\implies$ Artin Approximation

By Theorem E.0.7, the morphism $\mathcal{O}_{S,s} \to \widehat{\mathcal{O}}_{S,s}$ is geometrically regular. By Néron–Popescu Desingularization (Theorem E.0.4), $\widehat{\mathcal{O}}_{S,s} = \varinjlim B_\lambda$ is a direct limit of smooth $\mathcal{O}_{S,s}$-algebras. Since F is limit preserving, there exist λ, a factorization $\mathcal{O}_{S,s} \to B_\lambda \to \widehat{\mathcal{O}}_{S,s}$ and an element $\xi_\lambda \in F(\operatorname{Spec} B_\lambda)$ whose restriction to $F(\operatorname{Spec} \widehat{\mathcal{O}}_{S,s})$ is $\widehat{\xi}$.

Let $B = B_\lambda$ and $\xi = \xi_\lambda$. Geometrically, we have a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \arrow[r, "g"] \arrow[rr, bend left, "\widehat{\xi}"] \arrow[dr] & \operatorname{Spec} B \arrow[r, "\xi"] \arrow[d] & F \\
& \operatorname{Spec} \mathcal{O}_{S,s} &
\end{tikzcd}$$

where $\operatorname{Spec} B \to \operatorname{Spec} \mathcal{O}_{S,s}$ is smooth. We claim that we can find a commutative diagram

$$\begin{tikzcd}
S' \arrow[r, hook] \arrow[dr] & \operatorname{Spec} B \arrow[d] \\
& \operatorname{Spec} \mathcal{O}_{S,s}
\end{tikzcd} \tag{E.0.3}$$

where $S' \hookrightarrow \operatorname{Spec} B$ is a closed immersion, $(S', s') \to (\operatorname{Spec} \mathcal{O}_{S,s}, s)$ is étale, and the composition $\operatorname{Spec} \mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1} \to S' \to \operatorname{Spec} B$ agrees with the restriction of $g\colon \operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \to \operatorname{Spec} B$.[1]

To see this, observe that the B-module of relative differentials $\Omega_{B/\mathcal{O}_{S,s}}$ is locally free. After shrinking $\operatorname{Spec} B$ around the image of the closed point under

[1] This is where the approximation occurs. It is not possible to find a morphism $S' \to \operatorname{Spec} B \to \operatorname{Spec} \mathcal{O}_{S,s}$ which is étale at a point s' over s such that the composition $\operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \to S' \to \operatorname{Spec} B$ is equal to g.

$\operatorname{Spec}\widehat{\mathcal{O}}_{S,s} \to \operatorname{Spec} B$, we may assume $\Omega_{B/\mathcal{O}_{S,s}}$ is free with basis $db_1, \dots, db_n$. This induces a homomorphism $\mathcal{O}_{S,s}[x_1, \dots, x_n] \to B$ defined by $x_i \mapsto b_i$ and provides a factorization

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r] \arrow[d] & \mathbb{A}^n_{\mathcal{O}_{S,s}} \arrow[dl] \\ \operatorname{Spec} \mathcal{O}_{S,s} & \end{tikzcd}$$

where $\operatorname{Spec} B \to \mathbb{A}^n_{\mathcal{O}_{S,s}}$ is étale. We may choose a lift of the composition

$$\mathcal{O}_{S,s}[x_1, \dots, x_n] \to B \to \widehat{\mathcal{O}}_{S,s} \to \mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1}$$

to a morphism $\mathcal{O}_{S,s}[x_1, \dots, x_n] \to \mathcal{O}_{S,s}$. This gives a section $s\colon \operatorname{Spec}\mathcal{O}_{S,s} \to \mathbb{A}^n_{\mathcal{O}_{S,s}}$ and we define S' as the fibered product

$$\begin{tikzcd} S' \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \operatorname{Spec} \mathcal{O}_{S,s} \arrow[d, hook, "s"] \\ \operatorname{Spec} B \arrow[r] & \mathbb{A}^n_{\mathcal{O}_{S,s}}. \end{tikzcd}$$

This gives the desired Diagram E.0.3. The composition $\xi'\colon S' \to \operatorname{Spec} B \xrightarrow{\xi} F$ is an element which agrees with $\widehat{\xi}$ up to order N.

By "standard direct limit" methods, we may "smear out" the étale morphism $(S', s') \to (\operatorname{Spec}\mathcal{O}_{S,s}, s)$ and the element $\xi'\colon S' \to F$ to find an étale morphism $(S'', s'') \to (S, s)$ and an element $\xi''\colon S'' \to F$ agreeing with $\widehat{\xi}$ up to order N. Since this may not be standard for everyone, we spell out the details. Let $\operatorname{Spec} A \subset S$ be an open affine containing s. We may write $S' = \operatorname{Spec} A'$ and $A' = \mathcal{O}_{S,s}[y_1, \dots, y_n]/(f_1', \dots, f_m')$. As $\mathcal{O}_{S,s} = \varinjlim_{g \notin \mathfrak{m}_s} A_g$, we can find an element $g \notin \mathfrak{m}_s$ and elements $f_1'', \dots, f_m'' \in A_g[y_1, \dots, y_n]$ restricting to $f_1', \dots, f_m'$. Let $S'' = \operatorname{Spec} A_g[y_1, \dots, y_n]/(f_1'', \dots, f_m'')$ and $s'' \in S''$ be the image of s' under $S' \to S''$. Then $S'' \to S$ is étale at s''. As $A' = \varinjlim_{h \notin \mathfrak{m}_s} A_{hg}[y_1, \dots, y_n]/(f_1', \dots, f_m')$ and F is limit preserving, we can, after replacing g with hg, find an element $\xi'' \in F(S'')$ restricting to ξ' and, in particular, agreeing with $\widehat{\xi}$ up to order N. Finally, we shrink S'' around s'' so that $S'' \to S$ is étale everywhere.

Bibliography

[AJP16] Norbert A'Campo, Lizhen Ji, and Athanase Papadopoulos, *On the early history of moduli and teichmller spaces*, 2016, arXiv:1602.07208.

[Art62] M. Artin, *Grothendieck topolgoies*, Notes from a seminar of M. Artin, Harvard U., 1962.

[Art69] ———, *Algebraic approximation of structures over complete local rings*, Inst. Hautes Études Sci. Publ. Math. (1969), no. 36, 23–58.

[Art74] ———, *Versal deformations and algebraic stacks*, Invent. Math. **27** (1974), 165–189.

[Beh14] K. Behrend, *Introduction to algebraic stacks*, Moduli spaces, London Math. Soc. Lecture Note Ser., vol. 411, Cambridge Univ. Press, Cambridge, 2014, pp. 1–131.

[Bir09] George D. Birkhoff, *Singular points of ordinary linear differential equations*, Trans. Amer. Math. Soc. **10** (1909), no. 4, 436–470.

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 21, Springer-Verlag, Berlin, 1990.

[DM69] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. (1969), no. 36, 75–109.

[EGA] A. Grothendieck, *Éléments de géométrie algébrique*, I.H.E.S. Publ. Math. **4, 8, 11, 17, 20, 24, 28, 32** (1960, 1961, 1961, 1963, 1964, 1965, 1966, 1967).

[Eis95] D. Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry.

[GH78] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Wiley-Interscience [John Wiley & Sons], New York, 1978, Pure and Applied Mathematics.

[GIT] David Mumford, *Geometric invariant theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Band 34, Springer-Verlag, Berlin-New York, 1965.

[Gro57] A. Grothendieck, *Sur la classification des fibrés holomorphes sur la sphère de Riemann*, Amer. J. Math. **79** (1957), 121–138.

[Gro61] Alexander Grothendieck, *Techniques de construction en géométrie analytique. x. construction de l'espace de teichmüller*, Séminaire Henri Cartan **13** (1960-1961), no. 2 (fr), talk:17.

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52.

[JP13] Lizhen Ji and Athanase Papadopoulos, *Historical development of Teichmüller theory*, Arch. Hist. Exact Sci. **67** (2013), no. 2, 119–147.

[Knu71] Donald Knutson, *Algebraic spaces*, Lecture Notes in Mathematics, Vol. 203, Springer-Verlag, Berlin-New York, 1971.

[Kol18] János Kollár, *Mumford's influence on the moduli theory of algebraic varieties*, 2018, arXiv:1809.10723.

[Mil80] James S. Milne, *Étale cohomology*, Princeton Mathematical Series, vol. 33, Princeton University Press, Princeton, N.J., 1980.

[Mir95] Rick Miranda, *Algebraic curves and Riemann surfaces*, Graduate Studies in Mathematics, vol. 5, American Mathematical Society, Providence, RI, 1995.

[Mum70] David Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, No. 5, Published for the Tata Institute of Fundamental Research, Bombay; Oxford University Press, London, 1970.

[Nér64] André Néron, *Modèles minimaux des variétés abéliennes sur les corps locaux et globaux*, Inst. Hautes Études Sci. Publ.Math. No. **21** (1964), 128.

[Ols16] Martin Olsson, *Algebraic spaces and stacks*, American Mathematical Society Colloquium Publications, vol. 62, American Mathematical Society, Providence, RI, 2016.

[Pop85] Dorin Popescu, *General Néron desingularization*, Nagoya Math. J. **100** (1985), 97–126.

[Pop86] D. Popescu, *General Néron desingularization and approximation*, Nagoya Math. J. **104** (1986), 85–115.

[Pop90] Dorin Popescu, *Letter to the editor: "General Néron desingularization and approximation"*, Nagoya Math. J. **118** (1990), 45–53.

[RG71] Michel Raynaud and Laurent Gruson, *Critères de platitude et de ivité. Techniques de "platification" d'un module*, Invent. Math. **13** (1971), 1–89.

[Rie57] B. Riemann, *Theorie der Abel'schen Functionen*, J. Reine Angew. Math. **54** (1857), 115–155.

[SGA1] A. Grothendieck and M. Raynaud, *Revêtements étales et groupe fondamental*, Séminaire de Géométrie Algébrique, I.H.E.S., 1963.

[SGA4] *Théorie des topos et cohomologie étale des schémas.*, Lecture Notes in Mathematics, Vol. 269, 270, 305, Springer-Verlag, Berlin, 1972, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[SP] The Stacks Project Authors, *Stacks Project*, `http://stacks.math.columbia.edu`, 2020.

[Swa98] Richard G. Swan, *Néron-Popescu desingularization*, Algebra and geometry (Taipei, 1995), Lect. Algebra Geom., vol. 2, Int. Press, Cambridge, MA, 1998, pp. 135–192.

[Vis05] Angelo Vistoli, *Grothendieck topologies, fibered categories and descent theory*, Fundamental algebraic geometry, Math. Surveys Monogr., vol. 123, Amer. Math. Soc., Providence, RI, 2005, pp. 1–104.

[Wei58] André Weil, *On the moduli of riemann surfaces*, To Emil Artin on his sixtieth birthday. Unpublished manuscript; Collected Papers, Vol. II (1958), 381–389.