Introduction to statistics: Linear mixed models

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Summary

1. We know how to do simple t-tests.
2. We know how to fit simple linear models.
3. We saw that the paired t-test is identical to the varying intercepts linear mixed model.

Now we are ready to look at linear mixed models in detail.

Linear models

Returning to our SR/OR relative clause data from English (Grodner and Gibson, Expt 1). First we load the data as usual (not shown).

```
gge1crit<-read.table("data/grodnergibson05data.txt",
                     header=TRUE)

gge1crit$so<-ifelse(gge1crit$condition=="objgap",1,-1)

dat<- gge1crit
dat$logrt<-log(dat$rawRT)

bysubj<-aggregate(logrt~subject+condition,
                  mean,data=dat)
```

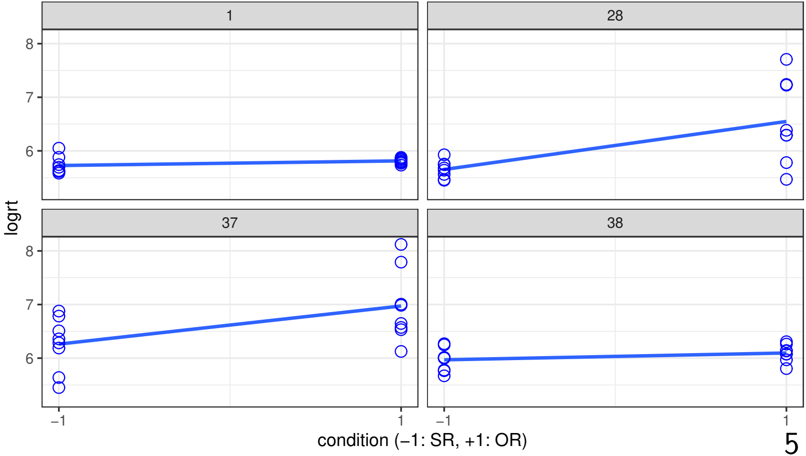
Linear models

The simple linear model (incorrect for these data):

```
summary(m0<-lm(logrt~so,dat))$coefficients

##             Estimate Std. Error  t value  Pr(>|t|)
## (Intercept) 5.883056   0.019052 308.7841 0.0000000
## so          0.062017   0.019052   3.2551 0.0011907
```

Linear models

We can visualize the different responses of subjects (four subjects shown):



Linear models

Given these differences between subjects, you could fit a separate linear model for each subject, collect together the intercepts and slopes for each subject, and then check if the intercepts and slopes are significantly different from zero.

We will fit the model using log reading times because we want to make sure we satisfy model assumptions (e.g., normality of residuals).

Linear models

There is a function in the package `lme4` that computes separate linear models for each subject: `lmList`.

```
library(lme4)

## Loading required package:  Matrix

lmlist.fm1<-lmList(logrt~so|subject,dat)
```

Linear models

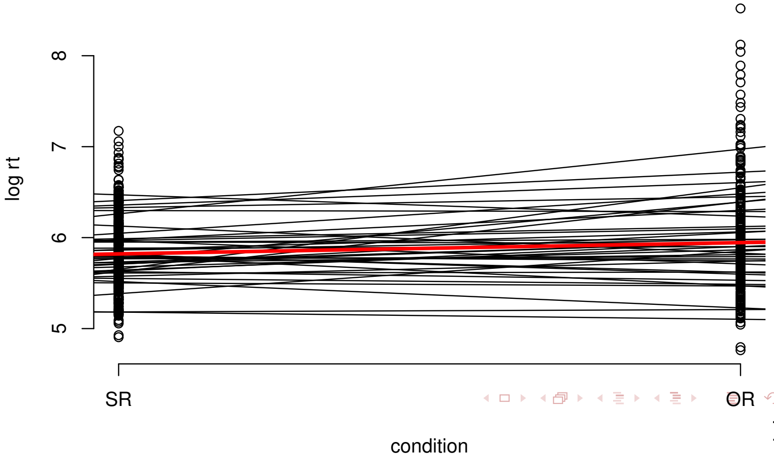
Intercept and slope estimates for three subjects:

```
lmlist.fm1$`1`$coefficients

## (Intercept)          so
##    5.769617    0.043515

lmlist.fm1$`28`$coefficients

## (Intercept)          so
##     6.10021     0.44814

lmlist.fm1$`37`$coefficients

## (Intercept)          so
##     6.61699     0.35537
```

Linear models

One can plot the individual lines for each subject, as well as the linear model m0's line (this shows how each subject deviates in intercept and slope from the model m0's intercept and slopes).

Linear models



Linear models

To find out if there is an effect of RC type, you can simply check whether the slopes of the individual subjects' fitted lines taken together are significantly different from zero.

Linear models

```
t.test(coef(lmlist.fm1)[2])

##
##  One Sample t-test
##
## data:  coef(lmlist.fm1)[2]
## t = 2.81, df = 41, p-value = 0.0076
## alternative hypothesis: true mean is not equal to 0
## 95 percent confidence interval:
##  0.017449 0.106585
## sample estimates:
## mean of x
##  0.062017
```

Linear models

The above test is exactly the same as the paired t-test and the varying intercepts linear mixed model **on aggregated data**:

```
t.test(logrt~condition,bysubj,paired=TRUE)$statistic

##      t
## 2.8102

## also compare with linear mixed model:
summary(lmer(logrt~condition+(1|subject),
             bysubj))$coefficients[2,]

##   Estimate Std. Error    t value
##  -0.124033   0.044137  -2.810207
```

Linear models

- The above lmList model we fit is called **repeated measures regression**. We now look at how to model unaggregated data using the linear mixed model.
- This model is now only of historical interest, and useful only for understanding the linear mixed model, which is the modern standard approach.

Linear mixed models

- The **linear mixed model** does something related to the above by-subject fits, but with some crucial twists, as we see below.
- In the model shown in the next slide, the statement
  (1|subject)
  adjusts the grand mean estimates of the intercept by a term (a number) for each subject.

Linear mixed models

Notice that we did not aggregate the data here.

```
m0.lmer<-lmer(logrt~so+(1|subject),dat)
```

Abbreviated output:

```
Random effects:
 Groups   Name        Variance Std.Dev.
 subject  (Intercept) 0.09983  0.3160
 Residual             0.14618  0.3823
Number of obs: 672, groups:  subject, 42

Fixed effects:
            Estimate Std. Error t value
(Intercept)  5.88306    0.05094 115.497
so           0.06202    0.01475   4.205
```

Linear mixed models

One thing to notice is that the coefficients (intercept and slope) of the fixed effects of the above model are identical to those in the linear model m0 above.

The varying intercepts for each subject can be viewed by typing:

```
ranef(m0.lmer)$subject[,1][1:10]

##  [1] -0.1039283  0.0771948 -0.2306209  0.2341978  0.0088
##  [7] -0.2055713 -0.1553708  0.0759436 -0.3643671
```

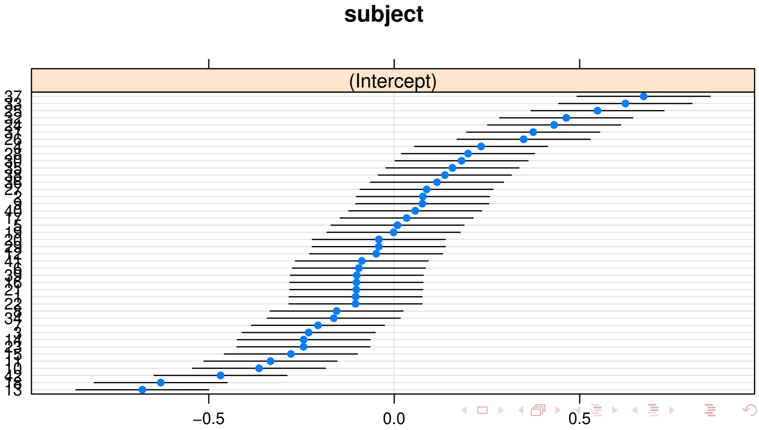
Visualizing random effects

Here is another way to summarize the adjustments to the grand mean intercept by subject. The error bars represent 95% confidence intervals.

```
library(lattice)
print(dotplot(ranef(m0.lmer,condVar=TRUE)))
```

Visualizing random effects

```
## $subject
```



Linear mixed models

The model m0.lmer above prints out the following type of linear model. i indexes subject, and j indexes items.
Once we know the subject id and the item id, we know which subject saw which condition:

```
subset(dat,subject==1 & item == 1)

##   subject item condition rawRT so  logrt
## 6       1    1    objgap   320  1 5.7683
```

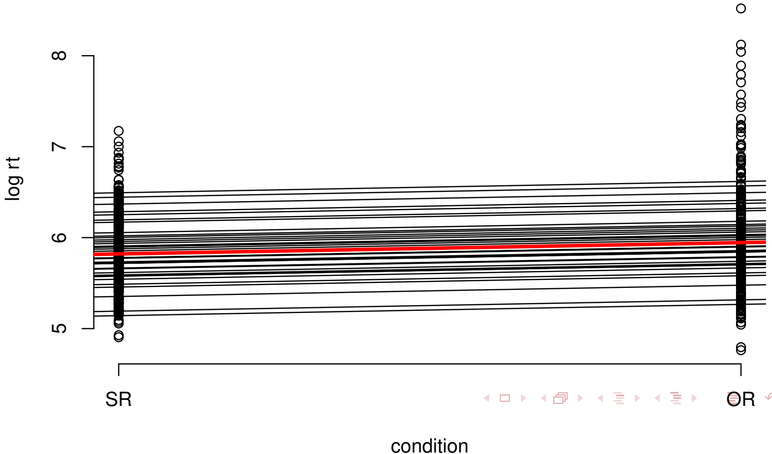
$$y_{ij} = \beta_0 + u_{0i} + \beta_1 \times so_{ij} + \epsilon_{ij} \tag{1}$$

The **only** new thing here is the by-subject adjustment to the intercept.

Linear mixed models

- Note that these by-subject adjustments to the intercept u_{0i} are assumed by lmer to come from a normal distribution centered around 0:
  $u_{0i} \sim Normal(0, \sigma_{u0})$
- The ordinary linear model m0 has one intercept β_0 for all subjects, whereas the linear mixed model with varying intercepts m0.lmer has a different intercept $(\beta_0 + u_{0i})$ for each subject i.
- We can visualize the adjustments for each subject to the intercepts as shown below.

Linear mixed models



Formal statement of varying intercepts linear mixed model

i indexes subjects, j items.

$$y_{ij} = \beta_0 + u_{0i} + (\beta_1) \times so_{ij} + \epsilon_{ij} \tag{2}$$

Variance components:

- $u_0 \sim Normal(0, \sigma_{u0})$
- $\epsilon \sim Normal(0, \sigma)$

Linear mixed models

Note that, unlike the figure associated with the lmlist.fm1 model above, which also involves fitting separate models for each subject, the model m0.lmer assumes **different intercepts** for each subject **but the same slope**.
We can have lmer fit different intercepts AND slopes for each subject.

Linear mixed models

Varying intercepts and slopes by subject

We assume now that each subject's slope is also adjusted:

$$y_{ij} = \beta_0 + u_{0i} + (\beta_1 + u_{1i}) \times so_{ij} + \epsilon_{ij} \tag{3}$$

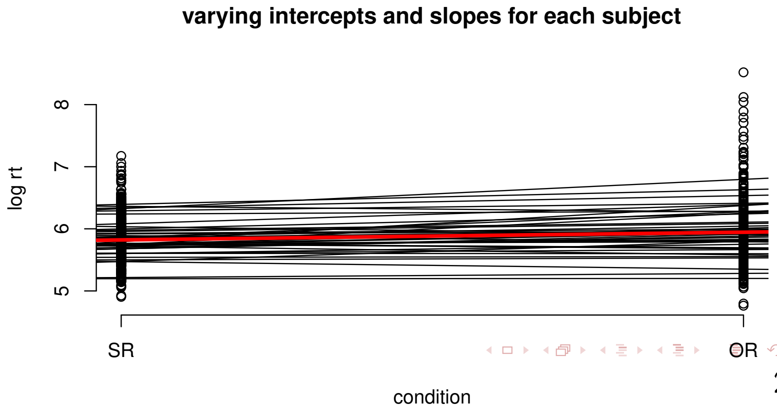
That is, we additionally assume that $u_{1i} \sim Normal(0, \sigma_{u1})$.

```
m1.lmer<-lmer(logrt~so+(1+so||subject),dat)
```

```
Random effects:
 Groups    Name        Variance Std.Dev.
 subject   (Intercept) 0.1006   0.317
 subject.1 so          0.0121   0.110
 Residual              0.1336   0.365
Number of obs: 672, groups:  subject, 42

Fixed effects:
```

Linear mixed models

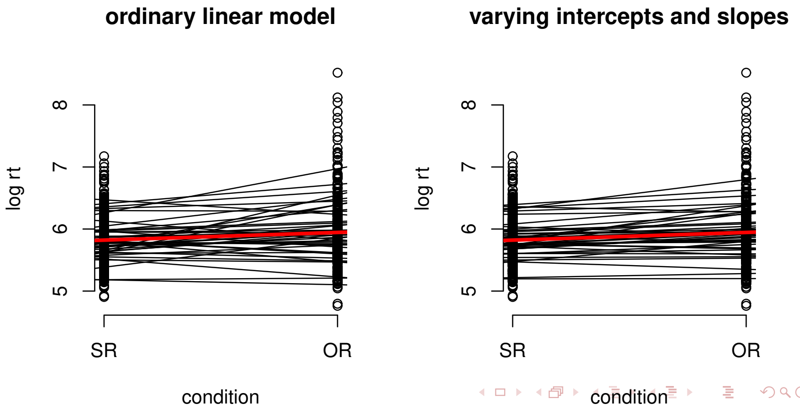
These fits for each subject are visualized below (the red line shows the model with a single intercept and slope, i.e., our old model m0):



Linear mixed models

Comparing lmList model with varying intercepts model

Compare this model with the lmlist.fm1 model we fitted earlier:

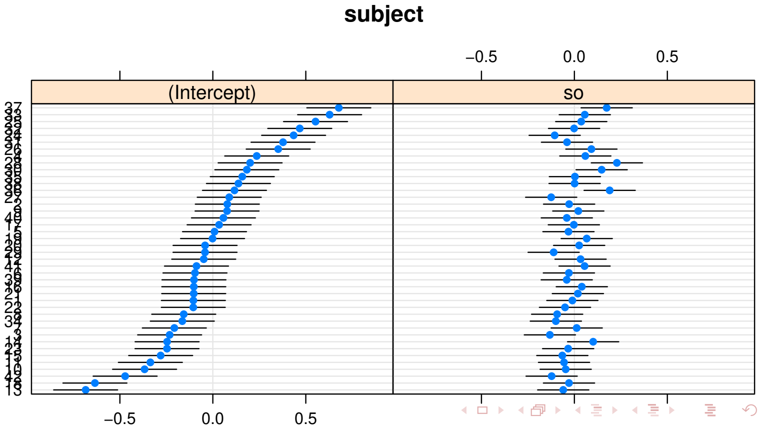


Visualizing random effects

```
print(dotplot(ranef(m1.lmer,condVar=TRUE)))
```

Visualizing random effects

```
## $subject
```



Formal statement of varying intercepts and varying slopes linear mixed model

i indexes subjects, j items.

$$y_{ij} = \beta_0 + u_{0i} + (\beta_1 + u_{1i}) \times so_{ij} + \epsilon_{ij} \tag{4}$$

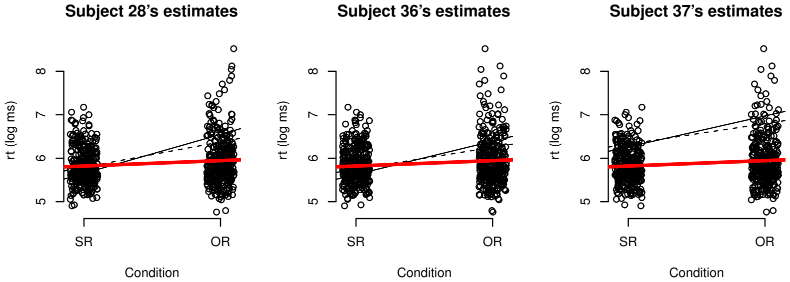
Variance components:

- $u_0 \sim Normal(0, \sigma_{u0})$
- $u_1 \sim Normal(0, \sigma_{u1})$
- $\epsilon \sim Normal(0, \sigma)$

Shrinkage in linear mixed models

- The estimate of the effect by participant is smaller than when we fit a separate linear model to the subject's data.
- This is called shrinkage in linear mixed models: the individual level estimates are shunk towards the mean slope.
- The less data we have from a given subject, the more the shrinkage.

Shrinkage in linear mixed models



Shrinkage in linear mixed models

The effect of missing data on estimation in LMMs

Let's randomly delete some data from one subject:

```
set.seed(4321)
## choose some data randomly to remove:
rand<-rbinom(1,n=16,prob=0.5)
```

Shrinkage in linear mixed models

The effect of missing data on estimation in LMMs

```
dat[which(dat$subject==37),]$rawRT

##  [1]  770  536  686  578  457  487 2419  884 3365  233
## [15] 1081  971

dat$deletedRT<-dat$rawRT
dat[which(dat$subject==37),]$deletedRT<-
  ifelse(rand,NA,
         dat[which(dat$subject==37),]$rawRT)
```

Shrinkage in linear mixed models

The effect of missing data on estimation in LMMs

Now subject 37's estimates are going to be pretty wild:

```
subset(dat,subject==37)$deletedRT

##  [1]  770   NA  686  578   NA   NA   NA   NA 3365  233
## [15]   NA  971
```

Shrinkage in linear mixed models

The effect of missing data on estimation in LMMs

```
## original no pooling estimate:
lmList.fm1_old<-lmList(log(rawRT)~so|subject,dat)
coefs_old<-coef(lmList.fm1_old)
intercepts_old<-coefs_old[1]
colnames(intercepts_old)<-"intercept"
slopes_old<-coefs_old[2]
## subject 37's original estimates:
intercepts_old$intercept[37]

## [1] 6.617

slopes_old$so[37]

## [1] 0.35537
```

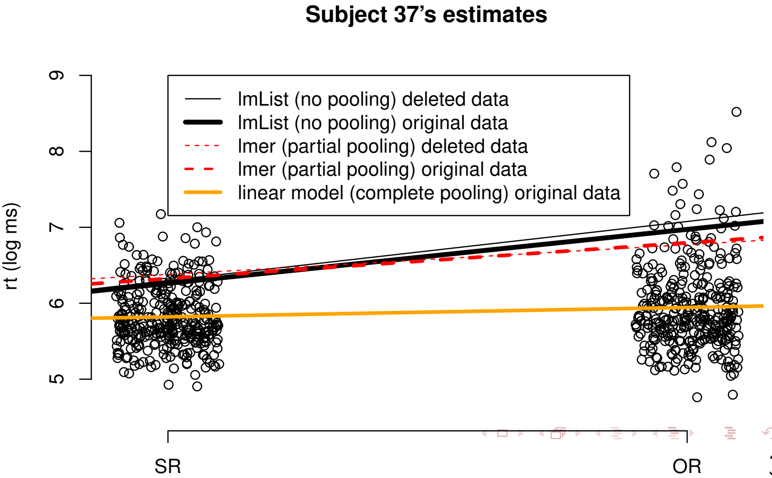
Shrinkage in linear mixed models

The effect of missing data on estimation in LMMs

```
## on deleted data:
lmList.fm1_deleted<-lmList(log(deletedRT)~so|subject,dat)
coefs<-coef(lmList.fm1_deleted)
intercepts<-coefs[1]
colnames(intercepts)<-"intercept"
slopes<-coefs[2]
## subject 37's new estimates on deleted data:
intercepts$intercept[37]

## [1] 6.6879

slopes$so[37]

## [1] 0.38843
```

Shrinkage in linear mixed models

The effect of missing data on estimation in LMMs



Shrinkage in linear mixed models

The effect of missing data on estimation in LMMs

- What we see here is that the estimates from the hierarchical model are barely affected by the missingness, but the estimates from the no-pooling model are heavily affected.
- This means that linear mixed models will give you more robust estimates (think Type M error!) compared to no pooling models.
- This is one reason why linear mixed models are such a big deal.

Crossed subjects and items in LMMs

Subjects and items are fully crossed:

```
head(xtabs(~subject+item,dat))

##          item
## subject 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
##       1 1 1 1 1 1 1 1 1 1  1  1  1  1  1  1  1
##       2 1 1 1 1 1 1 1 1 1  1  1  1  1  1  1  1
##       3 1 1 1 1 1 1 1 1 1  1  1  1  1  1  1  1
##       4 1 1 1 1 1 1 1 1 1  1  1  1  1  1  1  1
##       5 1 1 1 1 1 1 1 1 1  1  1  1  1  1  1  1
##       6 1 1 1 1 1 1 1 1 1  1  1  1  1  1  1  1
```

Linear mixed models

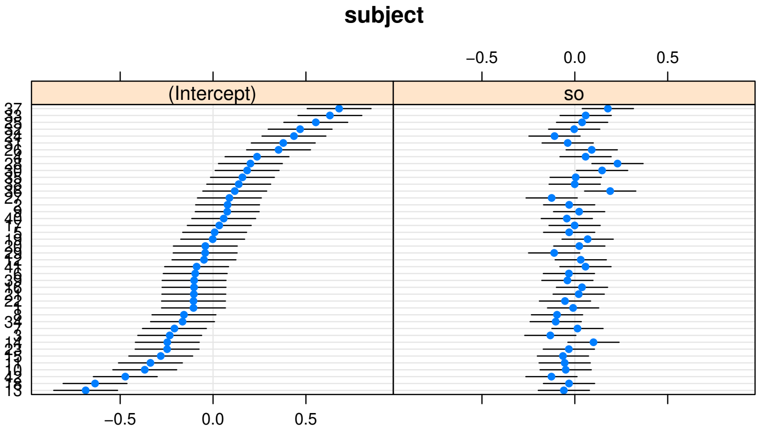
Linear mixed model with crossed subject and items random effects.

```
m2.lmer<-lmer(logrt~so+(1+so||subject)+(1+so||item),dat)
```

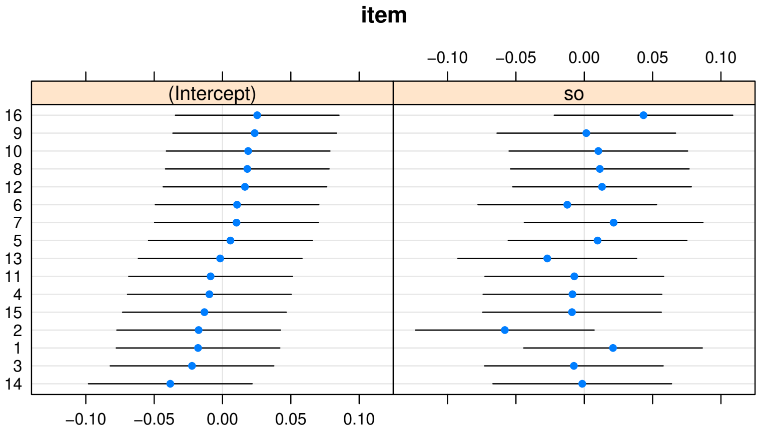
Linear mixed models

```
Random effects:
 Groups    Name        Variance Std.Dev.
 subject   (Intercept) 0.10090  0.3177
 subject.1 so          0.01224  0.1106
 item      (Intercept) 0.00127  0.0356
 item.1    so          0.00162  0.0402
 Residual              0.13063  0.3614
Number of obs: 672, groups:  subject, 42; item, 16

Fixed effects:
            Estimate Std. Error t value
(Intercept)   5.8831     0.0517  113.72
so            0.0620     0.0242    2.56
```

Visualizing random effects



Visualizing random effects



Linear mixed models

Linear mixed model with crossed subject and items random effects.

```
m3.lmer<-lmer(logrt~so+(1+so|subject)+(1+so|item),
              dat)

## boundary (singular) fit:  see ?isSingular
```

Linear mixed models

Linear mixed model with crossed subject and items random effects.

```
Random effects:
 Groups   Name        Variance Std.Dev. Corr
 subject  (Intercept) 0.10103  0.3178
          so          0.01228  0.1108   0.58
 item     (Intercept) 0.00172  0.0415
          so          0.00196  0.0443   1.00 <= degenerate
 Residual             0.12984  0.3603
Number of obs: 672, groups:  subject, 42; item, 16

Fixed effects:
            Estimate Std. Error t value
(Intercept)   5.8831     0.0520  113.09
so            0.0620     0.0247    2.51
```

Formal statement of varying intercepts and varying slopes linear mixed model with correlation

i indexes subjects, j items.

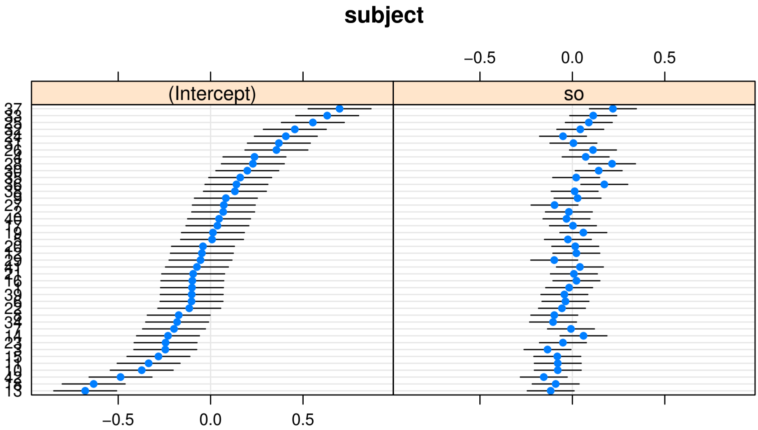
$$y_{ij} = \alpha + u_{0i} + w_{0j} + (\beta + u_{1i} + w_{1j}) * so_{ij} + \varepsilon_{ij} \tag{5}$$

where $\varepsilon_{ij} \sim Normal(0, \sigma)$ and

$$\Sigma_u = \begin{pmatrix} \sigma_{u0}^2 & \rho_u \sigma_{u0} \sigma_{u1} \\ \rho_u \sigma_{u0} \sigma_{u1} & \sigma_{u1}^2 \end{pmatrix} \quad \Sigma_w = \begin{pmatrix} \sigma_{w0}^2 & \rho_w \sigma_{w0} \sigma_{w1} \\ \rho_w \sigma_{w0} \sigma_{w1} & \sigma_{w1}^2 \end{pmatrix} \tag{6}$$

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma_u \right), \quad \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma_w \right) \tag{7}$$

Visualizing random effects



Visualizing random effects

These are degenerate estimates