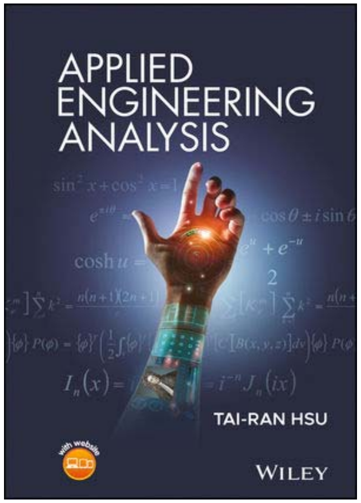
Applied Engineering Analysis
- slides for class teaching*

Chapter 6

Introduction to the Laplace Transform and Applications

* Based on the book of "Applied Engineering Analysis", by Tai-Ran Hsu, published by John Wiley & Sons, 2018 (ISBN 9781119071204)



(Chapter 6 Laplace transform)

Chapter Learning Objectives

- Learn the application of Laplace transform in engineering analysis.
- Learn the required conditions for transforming variable or variables in functions by the Laplace transform.
- Learn the use of available Laplace transform tables for transformation of functions and the inverse transformation.
- Learn to use partial fractions and convolution methods in inverse Laplace transforms.
- Learn the Laplace transform for ordinary derivatives and partial derivatives of different orders.
- Learn how to use Laplace transform methods to solve ordinary and partial differential equations.
- Learn the use of special functions in solving indeterminate beam bending problems using Laplace transform methods.

6.1 Introduction

Laplace, Pierre-Simon (1749-1829)
- a French mathematician, astronomer and statistician.

Major accomplishments in mathematics:

- Laplace equation for electrical and mechanical potentials

$$\frac{\partial^2 P(x,y)}{\partial x^2} + \frac{\partial^2 P(x,y)}{\partial y^2} = 0$$

where P(x,y) = Temperature for thermal potential or electric charge in electrostatics

- Laplacian differential operator:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

, and

- Laplace transform:

$$L_x[F(x)] = \int_0^\infty e^{-sx} F(x) dx$$

or

$$L_t[f(t)] = \int_0^\infty e^{-st} f(t) dt$$

where F(x) is a function of variable x and f(t) is a function of variable t, and s = Laplace transform parameter

Laplace Transform in Engineering Analysis

- Laplace transform is a mathematical operation that is used to "transform" a *variable (such as x, or y, or z in space, or at time t)* to a *parameter (s) – a "constant" under certain conditions. It transforms ONE variable at a time*.
  Mathematically, it can be expressed as:

$$L_t\left[f(t)\right] = \int_0^{\infty} e^{-st}\, f(t)dt = F(s) \qquad (6.1)$$

  where F(s) = expression of Laplace transform of function f(**t**) involving the parameter s

- In a layman's term, Laplace transform is used to "transform" a variable in a function into a parameter - *a parameter is a "constant" under certain conditions*

- So, after the Laplace transformation that variable is no longer a variable anymore, but it should be treated as a "parameter", i.e a "constant under specific conditions"

- This "specific condition" for the Laplace transform is:
  - *Laplace transform can only be used to transform variables that cover a range from "zero (**0**)" to infinity, (∞), for instance: $0 \le t < \infty$ if t is the variable to be transformed*
- ***Any variable that does not vary within this range cannot be transformed using Laplace transform***

- Because time variable t is the most common variable that varies from (0 to ∞), functions with variable t are commonly transformed by Laplace transform
- Lapalce transform is a valuable "tool" in solving:
  - Differential equations for example: electronic circuit equations, and
  - In "feedback control" for example, in stability and control of aircraft systems

6.2 Mathematical Operator of Laplace Transform

The Laplace transform of a function f(t) is designated as L[f(t)], with the variable t covers a spectrum of (0,∞).

The mathematical expression of the Laplace transform of this function with $0 \le t < \infty$ has the form:

$$L[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt = F(s) \tag{6.1}$$

where s is the parameter of the Laplace transform, and F(s) is the expression of the Laplace transform of function f(t) with $0 \le t < \infty$.

The "inverse Laplace transform" operates in a reverse way; That is to invert the transformed expression of F(s) in Equation (6.1) to its original function f(t). Mathematically, it has the form:

$$L^{-1}[F(s)] = f(t) \tag{6.2}$$

The above definition of Laplace transform as expressed in Equation (6.1) provides us with the "specific condition" for treating the Laplace transform parameter s as a constant is that the variable in the function to be transformed must SATISFY the condition that

$$0 \le (\text{variable } t) < \infty$$

Examples 6.1 (p.172)

Express the Laplace transforms of the following simple functions:

(1) For f(t) = t^2 with $0 \le t < \infty$:

$$L[f(t)] = \int_0^\infty e^{-st}\left(t^2\right)dt = e^{-st}\left[-\frac{2t^2}{2s} - \frac{2t}{s^2} - \frac{2}{s^3}\right]\Big|_0^\infty = \frac{2}{s^3} = F(s) \qquad \text{(a)}$$

(2) For f(t) = e^{at} with a = constant and $0 \le t < \infty$:

$$L[f(t)] = \int_0^\infty e^{-st}\left(e^{at}\right)dt = \int_0^\infty e^{(-s+a)t}dt = \frac{1}{-s+a}e^{(-s+a)}\Big|_0^\infty = \frac{1}{s-a} \qquad \text{(b)}$$

(3) For f(t) = Cosωt with ω = constant and $0 \le t < \infty$:

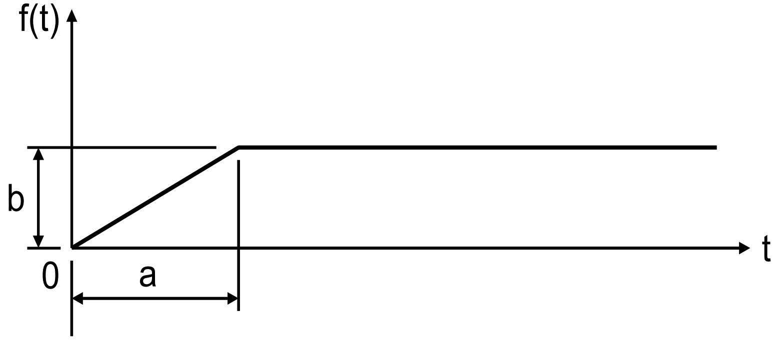
$$L[Cos\,\omega t] = \int_0^\infty e^{-st}\left(Cos\,\omega t\right)dt = \frac{e^{-st}}{(-s)^2+\omega^2}\left(-s\,Cos\,\omega t + \omega\,Sin\,\omega t\right)\Big|_0^\infty = \frac{s}{s^2+\omega^2} \qquad \text{(c)}$$

Appendix 1 of the book provides a Table of Laplace transforms of simple functions (p.463)

For example, L[f(t)] of a polynomial t^2 in Equation (a) is Case 3 with n = 3 in the Table,
exponential function e^{at} in Equation (b) is Case 7, and
trigonometric function Cosωt in Equation (c) is Case 18

Example 6.2 (p.172)

Perform the Laplace transform on the ramp function illustrated below:



Solution:

We may express the ramp function in the above figure as:

$$f(t) = \frac{b}{a}t \qquad 0 \le t < a$$
$$= b \qquad a < t < \infty \qquad \text{(a)}$$

We may perform the Laplace transform of the function expressed in Equation (a) by using the integral in Equation (6.1) as follows:

$$L[f(t)] = \int_0^\infty f(t)\,e^{-st}\,dt = F(s) = \int_0^a \frac{b}{a}t\,e^{-st}\,dt + \int_a^\infty b\,e^{-st}\,dt \qquad \text{(b)}$$

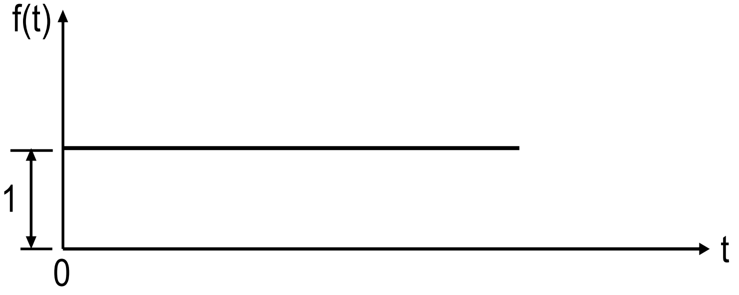
The Laplace transform of this ramp function is thus obtained after integrating the above expression:

$$F(s) = \frac{b}{a}\frac{e^{-st}}{(-s)^2}(-st-1)\Bigg|_0^a + \frac{b}{(-s)}e^{-st}\Bigg|_a^\infty = -\frac{b}{as^2}(as+1)e^{-as} + \frac{b}{as^2} + \frac{b}{s}e^{-as}$$

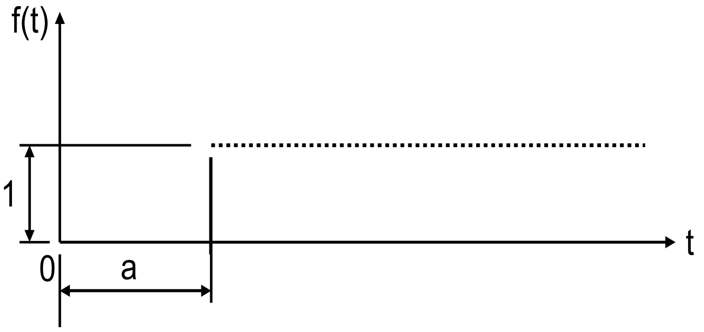
Example 6.3 (p.173)

Perform the Laplace transforms on (a) step function $u_0(t)$, and (b) $u_a(t)$ in the following two figures:

Step function $u_0(t)$:



Step function $u_a(t)$:



Solution:

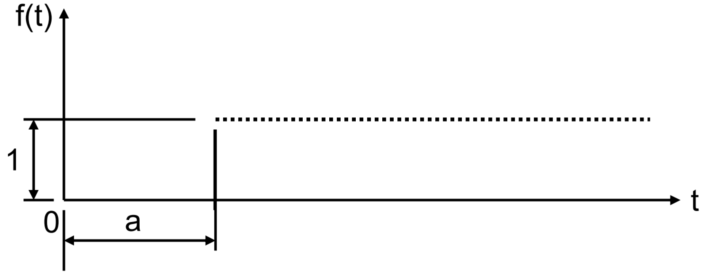
(A) Laplace transform of function $u_0(t)$:

We learned from Chapter 2 that both ramp and step functions provide math expressions for physical phenomena that begin to exist at t=a in the function illustrated in Example 6.2, and t=0 for the step function $u_0(t)$ at t= 0, and $u_a(t)$ at t = a in the above figures. Laplace transforms for both these step functions in this example may be obtained as:

We have the Laplace transform of this function using in integral in Equation (6.1) or as included in Case 1 in Appendix 1 to be:

$$L[u_0(t)] = \int_0^\infty (1)\, e^{-st}\, dt = -\frac{1}{s} e^{-st}\Big|_0^\infty = \frac{1}{s}$$

(B) Laplace transform of function $u_a(t)$:



The mathematical expression of function f(t) in this case is available in Equation (2.36a) (p.58) with α = 1, that is:

$$f(t) = u_a(t) = \begin{cases} 0 & 0 \le t < a \\ 1 & a < t < \infty \end{cases}$$

The corresponding Laplace transform is:

$$L[u_a(t)] = \int_0^a (0)e^{-st}\,dt + \int_a^\infty (1)e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_a^\infty = \frac{1}{s}e^{-as}$$

We will find that the result of the step function $u_a(t)$ as shown above is identical to that shown in Case 15 in the Laplace Transform Table in Appendix 1.

6.3 **Properties of Laplace Transform** (p.174)

Laplace transform of functions by integration:

$$L_t[f(t)] = \int_0^\infty e^{-st}\, f(t)dt = F(s) \qquad (6.1)$$

is not always easy to determine.

Laplace transform (LT) Table in Appendix 1 is useful, but does not always have the required answer for the specific functions. Following properties are selected for the LT of some functions:

6.3.1. **Linear operators:**

L[a f(t) + b g(t)] = a L[f(t)] + b L[g(t)] (6.5)

where a, b = constant coefficients

Example 6.4**:**

Find Laplace transform of function: $f(t) = 4t^2 – 3\text{Cos } t + 5e^{-t}$ with $0 \le t < \infty$:

Solution:

By using the linear operator, we may break up the transform of f(t) into three individual transformations:

$L(4t^2 – 3\text{Cos } t + 5e^{-t}) = 4L[t^2] – 3L[\text{Cos } t] + 5L[e^{-t}] = F(s)$

Case 3 with n = 3 Case 18 with ω=1 Case 7 with a = -1 from the LT Table

Hence

$$F(s) = \frac{8}{s^3} - \frac{3s}{s^2+1} + \frac{5}{s+1}$$

Properties of Laplace Transform – Cont'd

6.3.2. **Shifting property** (p.175):

If the Laplace transform of a function f(t) is **L[f(t)] = F(s)** by integration, or from the Laplace Transform (LT) Table, the Laplace transform of **G(t) = e^{at}f(t)** can be obtained by the following relationship:

$$L[G(t)] = L[e^{at}f(t)] = F(s-a) \qquad (6.6)$$

where a in the above formulation is the shifting factor, i.e. the parameter s in the transformed function f(t) that has been shifted by (s-a)

Example 6.5**:**

Perform the Laplace transform on function: F(t) = e^{2t} Sin(at), where a = constant

Solution:

We may either use the Laplace integral transform in Equation (6.1) to get the solution, or we could get the solution available the LT Table in Appendix 1 with the shifting property for the solution. We will use the latter method in this example, with:

$$L[f(t)] = L[Sin\,at] = \frac{a}{s^2+a^2}$$ (Case 17 in Appendix 1),

The Laplace transform of F(t) = e^{2t}sin(at) can thus be obtained by using the shift amount of 2 in Equation (6.6), or in the form:

$$L[F(t)] = L[e^{2t}\ Sin\,at] = \frac{a}{(s-2)^2+a^2}$$

6.3.3. **Change of scale property** (p.175):

If we know L[f(t)] = F(s) either from the LT Table, or by integral in Equation (6.1), we may find the Laplace transform of function f(at) by the following expression:

$$L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right) \qquad (6.7)$$

where a = scale factor for the change

Example 6.6**:**

Perform the Laplace transform of function F(t) = sin3t.

Since we know the Laplace transform of f(t) = sint from the LT Table in Appendix 1 as:

$$L[f(t)] = L[Sint] = \frac{1}{s^2+1} = F(s)$$

We may find the Laplace transform of F(t) using the “Change scale property” with scale factor a=3 to take a form:

$$L[Sin3t] = \frac{1}{3}\frac{1}{\left(\frac{s}{3}\right)^2+1} = \frac{3}{s^2+9}$$

6.4 **Inverse Laplace Transform** (p.176)

We have defined the Laplace transform of a function f(t) to be:

Laplace transform

From here → to there

$$L_t\left[f(t)\right] = \int_0^{\infty} e^{-st} f(t)dt = F(s) \qquad (6.1)$$

there are times we need to do the following:

$$L_t\left[f(t)\right] = \int_0^{\infty} e^{-st} f(t)dt = F(s)$$

to there ← From here

Inverse Laplace transform

There are **4** available ways to inverse Laplace transforms to engineers:

- Use LT Table by looking at F(s) in right column for corresponding f(t) in middle column - the chance of success is not very good.
- Use partial fraction method for F(s) = rational function (i.e. fraction functions involving polynomials), and
- The convolution theorem involving integrations.
- Use the Bromwich contour integrations around residues in the approximate form of F(s) using complex variable theories. This method will not be presented in this class because it is beyond the scope of this course.

6.4.2 The Partial Fraction Method for Inverse Laplace Transform (p. 176)

- The expression of F(s) to be inversed in Laplace transform is expressed in the following partial fractions:

$$F(s) = \frac{P(s)}{Q(s)}$$

where polynomial P(s) is at least one order less than the order of polynomial Q(s)

- "Break" up the above rational function into summation of "simple fractions":

$$F(s) = \frac{P(s)}{Q(s)} = \frac{A_1}{s-a_1} + \frac{A_2}{s-a_2} + \ldots\ldots\ldots\ldots\ldots + \frac{A_n}{s-a_n} \qquad (6.8)$$

where A_1; A_2,……..A_n, and a_1, a_2, ……….a_n are constants to be determined by comparing coefficients of terms on both sides of the equality:

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-a_1} + \frac{A_2}{s-a_2} + \ldots\ldots\ldots\ldots\ldots + \frac{A_n}{s-a_n}$$

- The inverse Laplace transform of F(s) = P(s)/Q(s) becomes:

$$L^{-1}\left[\frac{P(s)}{Q(s)}\right] = L^{-1}\left(\frac{A_1}{s-a_1}\right) + L^{-1}\left(\frac{A_2}{s-a_2}\right) + \ldots\ldots\ldots\ldots + L^{-1}\left(\frac{A_n}{s-a_n}\right)$$

↑ ↑

a fraction = a sum of partial fractions

Example 6.7 (p.177)**:**

Perform the inverse Laplace transform of the following expression:

$$F(s) = \frac{P(s)}{Q(s)} = \frac{3s+7}{s^2-2s-3}$$

Solution:

We may express F(s) in the following partial fraction form:

$$F(s) = \frac{3s+7}{s^2-2s-3} = \frac{3s+7}{(s-3)(s+1)} = \frac{A}{s-3}+\frac{B}{s+1}=\frac{A(s+1)+B(s-3)}{(s-3)(s+1)}$$

where A and B are constant coefficients

After expanding the above rational function and equating the terms in the numerators of both sides of the above equality:

3s + 7 = A(s+1) + B(s-3) = (A + B)s + (A – 3B)

We may solve for A and B from the above simultaneous equations:

A + B = 3 and A – 3B = 7 resulting in: A = 4 and B = -1

We will thus have: $\frac{3s+7}{s^2-2s-3} = \frac{4}{s-3}-\frac{1}{s+1}$

The required Laplace transform is:

$$L^{-1}\left[\frac{3s+7}{(s-3)(s+1)}\right] = 4L^{-1}\left(\frac{1}{s-3}\right)-L^{-1}\left(\frac{1}{s+1}\right) = 4e^{3t}-e^{-t}$$

Example 6.8:

Perform the following inverse Laplace transform:

$$L^{-1}[F(s)] = L^{-1}\left[\frac{P(s)}{Q(s)}\right] = L^{-1}\left[\frac{3s+1}{s^3-s^2+s-1}\right]$$

Solution:

We may break up F(s) in the above expression in the following form:

$$\frac{P(s)}{Q(s)} = \frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

The polynomial in numerator is always one order less than that in the denominator

By following the same procedure, we determine the coefficients A = 2, B = -2 and C = 1, or:

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{2}{s-1} - \frac{2s}{s^2+1} + \frac{1}{s^2+1}$$

We will thus have the inversed Laplace transform, and thus the original function f(t) to be:

$$f(t) = L^{-1}[F(s)] = L^{-1}\left[\frac{3s+1}{s^3-s^2+s-1}\right] = L^{-1}\left(\frac{2}{s-1}\right) - L^{-1}\left(\frac{2s}{s^2+1}\right) + L^{-1}\left(\frac{1}{s^2+1}\right) = 2e^t - 2Cos\,t + Sin\,t$$

6.4.3 Inverse Laplace Transform by Convolution Theorem (p.178)

- This method involves the use of integration of the expressions involving LT parameter s - F(s)
- There is no restriction on the form of the expression F(s): – they can be rational functions of polynomial, trigonometric functions or exponential functions
- The convolution theorem works in the following ways for inverse Laplace transforms:

If we know the following:

$L^{-1}[F(s)] = f(t)$ and $L^{-1}[G(s)] = g(t)$, with: $F(s) = \int_0^\infty e^{-st} f(t)\,dt$, and $G(s) = \int_0^\infty e^{-st} g(t)\,dt$

from the LT Table in Appendix 1 or using integration in Equation (6.1), then the desired inverse Laplace transform of Q(s) = F(s) or G(s) can be obtained by:
the following integrals:

Either by: $$L^{-1}\left[Q(s)\right] = L^{-1}\left[F(s)G(s)\right] = \int_0^t f(\tau)\,g(t-\tau)\,d\tau \qquad (6.9a)$$

Or by: $$L^{-1}\left[Q(s)\right] = L^{-1}\left[F(s)G(s)\right] = \int_0^t f(t-\tau)\,g(\tau)\,d\tau \qquad (6.9b)$$

NOTE: the variable τ in Equations (6.9)is the dummy integration variable, which means the variable t in the integrals is treated as "constant,“ and it may be "factored out" from the integrals.

Example 6.9 (p. 178)**:**

Find the inverse of a Laplace transformed function with: $Q(s) = \frac{s}{\left(s^2 + a^2\right)^2}$

Solution:

We may express F(s)=Q(s) in the following expression:

$$Q(s) = \frac{s}{(s^2+a^2)^2} = \frac{s}{s^2+a^2}\frac{1}{s^2+a^2} = F(s)G(s) \qquad \text{(a)}$$

It is our choice to select F(s) and G(s) in the above expression for the integrals in Equation (6.9a) or (6.9b).

Let us choose: $F(s) = \frac{s}{s^2 + a^2} \quad and \quad G(s) = \frac{1}{s^2 + a^2}$

From the LT Table in Appendix 1, we get the following:

$$L^{-1}[F(s)] = Cos(at) = f(t) \quad and \quad L^{-1}[G(s)] = \frac{Sin(at)}{a} = g(t)$$

The inverse of Q(s) = F(s)G(s) is obtained by Equation (6.9a) as:

$$L^{-1}\left[\frac{s}{\left(s^2 + a^2\right)^2}\right] = \int_0^t Cos\,a\tau \,\frac{Sin\,a(t-\tau)}{a} d\tau = \frac{t\,Sin\,at}{2a}$$

One will get the same result by using another convolution integral in Equation (6.9b), or using partial fraction method in Equation (6.8)

Example 6.11 (p.179)**:**

Use convolution theorem to find the inverse Laplace transform: $Q(s) = \dfrac{1}{(s+1)(s^2+4)}$

Solution:

We may express Q(s) in the following form:

$$Q(s) = \frac{1}{(s+1)(s^2+4)} = \frac{1}{s+1} \cdot \frac{1}{s^2+4} \qquad \text{(a)}$$

We choose F(s) and G(s) as:

$F(s) = \dfrac{1}{s+1}$ or $e^{-t} = f(t)$ and $G(s) = \dfrac{1}{s^2+4}$ or $\dfrac{1}{2} Sin\, 2t = g(t)$

Let us use Equation (6.9b) for the inverse of Q(s) in Equation (a):

$$q(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau = \int_0^t e^{-(t-\tau)} \left(\frac{1}{2} Sin\, 2\tau \right) d\tau = \frac{1}{2} e^{-t} \int_0^t e^{\tau}\, Sin\, 2\tau \; d\tau$$

After the integration , we get the inverse of Laplace transform Q(s) to be:

$$q(t) = \frac{1}{2} e^{-t} \left[\frac{e^{\tau}\left(Sin\, 2\tau - 2\, Cos\, 2\tau \right)}{1+2^2} \right]\Bigg|_0^t = \frac{1}{10} Sin\, 2t - \frac{1}{5} Cos\, 2t + \frac{1}{5} e^{-t}$$

6.5 **Laplace Transform of Derivatives** (p.180)

- We have learned the Laplace transform of function f(t) by:

$$L_t\left[f(t)\right] = \int_0^\infty e^{-st} f(t)dt = F(s) \qquad (6.1)$$

We realize the derivative of function f(t): $f'(t) = \frac{df(t)}{dt}$ is also a FUNCTION

So, there should be a possible way to perform the Laplace transform of the derivatives as functions, as long as its variable varies from zero to infinity.

- Laplace transform of derivatives is necessary steps in solving DEs using Laplace transform
- By following the mathematical expression for Laplace transform of functions in Equation (6.1), we may expression Laplace transform of derivative f'(t) in the following form:

$$L\left[f'(t)\right] = \int_0^\infty e^{-st} f'(t)\,dt = \int_0^\infty e^{-st}\left[\frac{df(t)}{dt}\right]dt \qquad (6.10)$$

We will use "integration by parts" technique
For this integral by letting: $\Rightarrow$

$$\int_0^\infty udv = uv\Big|_0^\infty - \int_0^\infty vdu$$

We will further assign:

$$u = e^{-st} \qquad and \qquad dv = \left[\frac{df(t)}{dt}\right]$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$du = -se^{-st}\,dt \qquad\qquad v = f(t)$$

By substituting the above ‘u’, “du”, “dv” and “v” into the following relationship:

$$L\left[f'(t)\right] = \int_0^\infty e^{-st}\, f'(t)\, dt = \int_0^\infty e^{-st}\left[\frac{df(t)}{dt}\right] dt$$

$$\int_0^\infty u dv = uv\Big|_0^\infty - \int_0^\infty v du$$

We will have:

$$L\left[f'(t)\right] = \int_0^\infty e^{-st}\, f'(t)\, dt = \int_0^\infty e^{-st}\left[\frac{df(t)}{dt}\right] dt = e^{-st} f(t)\Big|_0^\infty - \int_0^\infty f(t)\left(-se^{-st}\right) dt \qquad (6.11)$$

, leading to:

$$L\left[f'(t)\right] = \int_0^\infty e^{-st}\, f'(t)\, dt = e^{-st} f(t)\Big|_0^\infty - \int_0^\infty f(t)\left(-se^{-st}\right) dt = -f(0) + s\int_0^\infty e^{-st} f(t) dt = -f(0) + sL\left[f(t)\right]$$

or in a simplified form:

$$L[f'(t)] = s\, L[f(t)] - f(0) \qquad (6.12)$$

- Likewise, we may find the Laplace transform of second order derivative of function f(t) to be:

$$L[f''(t)] = s^2\, L[f(t)] - sf(0) - f'(0) \qquad (6.13)$$

- A recurrence relation for Laplace transform of higher order (n) derivatives of function f(t) may be expressed as:

$$L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \ldots\ldots. f^{n-1}(0) \qquad (6.14)$$

Example 6.12 (p.181)**:**

Find the Laplace transform of the second order derivative of function: f(t) = t Sint

The second order derivative of f(t) meaning n = 2 in Equation (6.14), or as in Equation (6.13):

$$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0) \tag{6.13}$$

We thus have:

$$L\left[\frac{d^2 f(t)}{dt^2}\right] = s^2 L[f(t)] - sf(0) - f'(0)$$

Since $$f'(t) = \frac{df(t)}{dt} = \frac{d(t\,Sin\,t)}{dt} = t\,Cos\,t + Sin\,t$$

We thus have:

$$L[f''(t)] = s^2 L[f(t)] - s\,f(0) - f'(0) = s^2 L[t\,Sin\,t] - s(t\,Sin\,t)\big|_{t=0} - (t\,Cos\,t + Sin\,t)\big|_{t=0}$$
$$= s^2\,L[t\,Sin\,t]$$

6.5.2 Laplace Transform of Partial Derivatives (p.181)

In Section 2.2.5 (p.36), we learned that partial derivatives involving more than one independent variable in the function and they appear frequently in engineering analyses. It is necessary for engineers to learn how to perform Laplace transform of these functions and their derivatives.

Functions involving more than one independent variable in forms such as: f(x,t), f(x,y,t) or f(x,y,z,t), in which x, y, z and t are independent variables - with (x,y,z) represent variables in space and t for the time.

Laplace transform can be performed on partial derivatives, BUT with one variable at a time ONLY, as long as the variable to be transformed satisfy the condition of:

$0 \leq (\text{variable}) < \infty$

Let us elaborate the above statement by an example for a function f(x,t) with a space variable x and another independent variable time t.
The rate of change of the values of this function f(x,t) depends on both values of these two independent variables –x and t, or mathematically to be expressed as:

$\frac{\partial f(x,t)}{\partial x}$ for the rate of change of function f(x,t) with respect to variable x, and

$\frac{\partial f(x,t)}{\partial t}$ for the rate of change of function f(x,t) with respect to the other variable t,

$\frac{\partial^2 f(x,t)}{\partial x^2}$ and $\frac{\partial^2 f(x,t)}{\partial t^2}$ are the corresponding second order partial derivatives of the function f(x,t)

6.5.2 Laplace Transform of Partial Derivatives – Cont'd

Let us designate the Laplace transform of a function with two independent variables x and t by:

$$L_x[f(x,t)] = \int_0^\infty e^{-sx} f(x,t)dx = F^*(s,t) \quad \text{with } 0 < x < \infty \tag{6.15}$$

to be the Laplace transform of function f(x,t) with respect to variable x, and

$$L_t[f(x,t)] = \int_0^\infty e^{-st} f(x,t)dt = F^*(x,s) \quad \text{with } 0 < t < \infty \tag{6.16}$$

to be the Laplace transform of function f(x,t) with respect to the variable t.

The subscripts attached to the Laplace transform operator (L) in Equations (6.15) and (6.16) denotes the variable to be transformed by the Laplace transform.

We were reminded again and again that function this is qualified for Laplace transform must satisfy the condition that the variable to be transformed must satisfy the condition:

$0 \leq$ (variable) $< \infty$.

Let us assume that of the two independent variable x and t involved in the function f(x,t), but only the variable t satisfies this condition $0 \leq t < \infty$**,** (the other variable x does not satisfy this condition). We can thus only have Laplace transform on the variable t in this case. Consequently, we will have the following Laplace transform of the function f(x,t):

$$L_t\left[\frac{\partial f(x,t)}{\partial x}\right] = \int_0^\infty e^{-st}\left[\frac{\partial f(x,t)}{\partial x}\right]dt = \frac{\partial}{\partial x}\int_0^\infty e^{-st} f(x,t)dt = \frac{\partial F^*(x,s)}{\partial x} \tag{6.17}$$

6.5.2 Laplace Transform of Partial Derivatives – Cont'd

The expression of Laplace transform of the partial derivative of function f(x,t):

$$L_t\left[\frac{\partial f(x,t)}{\partial t}\right] = \int_0^\infty e^{-st}\left[\frac{\partial f(x,t)}{\partial t}\right]dt$$

is not straightforward as one would imagine. The following steps need to be taken to get the appropriate expression for this partial derivative. Let us follow what we did in the case for functions with single variable in Section 6.5.1, with the following integration:

$\mathrm{I} = \int_0^\infty e^{-st}\left[\frac{\partial f(x,t)}{\partial t}\right]dt = \int_0^\infty u\,dv = uv\Big|_0^\infty - \int_0^\infty v\,du$ in which u and v are parts of the integral (I).

If we let u = e^{-st} leading to du = -se^{-st}dt, and $dv = \frac{\partial f(x,t)}{\partial t}dt$ leading to v = f(x,t).

Resulting in: $$L_t\left[\frac{\partial f(x,t)}{\partial t}\right] = I = e^{-st}\, f(x,t)\Big|_{t=0}^{\infty} + \int_0^\infty s\,e^{-st}\, f(x,t)dt$$

$$= -f(x,0) + s\int_0^\infty e^{-st} f(x,t)dt = -f(x,0) + sF^*(x,s) \qquad (6.19)$$

We may thus express the Laplace transform of partial derivatives in the following expression for the Laplace transform of the partial derivative as:

$$L_t\left[\frac{\partial f(x,t)}{\partial t}\right] = sF^*(x,s) - f(x,0) \qquad (6.20)$$

Likewise, we may show that the Laplace transform of second order partial derivatives as follows:

$$L_t\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = s^2F^*(x,s) - sF^*(x,s) - \frac{\partial f(x,t)}{\partial t}\Big|_{t=0} \text{ and } L_x\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] = \frac{\partial^2 F^*(x,s)}{\partial x^2} \qquad (6.21a,b)$$

Example 6.13 (p.183)

If the Laplace transform of the function θ(x,t) = xe^{-t} is defined as:

$$L_t[\theta(x,t)] = \theta^*(x,s) = \int_0^\infty e^{-st}\theta(x,t)dt \qquad \text{(a)}$$

and with: 0< t <∞

Determine: $(a)\ L_t\left[\frac{\partial\theta(x,t)}{\partial x}\right]\quad (b)\ L_t\left[\frac{\partial^2\theta(x,t)}{\partial x^2}\right]\quad (c)\ L_t\left[\frac{\partial\theta(x,t)}{\partial t}\right]\quad (d)\ L_t\left[\frac{\partial^2\theta(x,t)}{\partial t^2}\right]$

Solution:

We first establish the Laplace transform of the multi-variable function θ(x,t) = xe^{-t} by using Equation (a) and obtain:

$$L_t[\theta(x,t)] = x L_t[e^{-t}] = \frac{x}{s+1} = \theta^*(x,s)$$

We will then proceed to determine the four required Laplace transforms of the derivatives of function θ(x,t) as required by using Equations (6.21 a,b).

$$(a)\ L_t\left[\frac{\partial\theta(x,t)}{\partial x}\right] = \frac{\partial\theta^*(x,s)}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{s+1}\right) = \frac{1}{s+1}$$

$$(b)\ L_t\left[\frac{\partial^2\theta(x,t)}{\partial x^2}\right] = \frac{\partial^2\theta^*(x,s)}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{1}{s+1}\right) = 0$$

$$(c)\ L_t\left[\frac{\partial\theta(x,t)}{\partial t}\right] = s\theta^*(x,s) - \theta(x,0) = \frac{sx}{s+1} - x = -\frac{x}{s+1}$$

$$(d)\ L_t\left[\frac{\partial^2\theta(x,t)}{\partial t^2}\right] = s^2\,\theta^*(x,t) - s\theta(x,0) - \left.\frac{\partial\theta(x,t)}{\partial t}\right|_{t=0} = \frac{s^2x}{s+1} - sx - (-x)$$

$$= \frac{s^2x}{s+1} - sx + x = \frac{x}{s+1}$$

6.6 Solution of Differential Equations Using Laplace Transforms (p.184)

- One popular application of Laplace transform is solving differential equations
- However, such application **MUST** satisfy the following two conditions:

(1) The variable(s) in the function for the solution, e.g., x, y, z, t must cover the range of (0, ∞).
It means that the solution, e.g., u(x) or u(t) MUST also be VALID for the range of (0, ∞), and

(2) ALL appropriate conditions for the differential equation MUST be available with the problem

- The solution procedure is presented below:

(1) Apply Laplace transform on EVERY term in the differential equation (DE)
(2) The Laplace transform of derivatives results in given conditions, such as f(0), f'(0), f"(0), etc. as shown in Equation (6.14)
(3) After apply the given values of the given conditions as required in Step (2), we will get an ALGEBRAIC equation for F(s) as defined in Equation (6.1):

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt = F(s) \tag{6.1}$$

(4) We thus can obtain an expression for F(s) from Step (3)
(5) The solution of the DE is the inverse of the Laplace transformed F(s), i.e.,:

$$y(t) = L^{-1}[F(s)]$$

Example 6.14 (p.184):

Solve the following DE with given conditions:

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 5y(t) = e^{-t}\,Sint \qquad 0 \le t \le \infty \qquad \text{(a)}$$

given conditions: y(0) = 0 and y'(0) =1 (b)

Solution:

(1) Apply Laplace transform to EVERY term in the DE in Equation (a):

$$L\left[\frac{d^2y(t)}{dt^2}\right] + 2L\left[\frac{dy(t)}{dt}\right] + 5L[y(t)] = L[e^{-t}\,Sint] \qquad \text{(c)}$$

where $L[y(t)] = \int_0^\infty y(t)\,e^{-st}\,dt = Y(s)$

Equations (6.12) and (6.13) for the Laplace transforms of the first and second order derivatives in Equation (c) will result in:

$$[s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 5Y(s) = \frac{1}{(s+1)^2+1} = \frac{1}{s^2+2s+2} \qquad \text{(d)}$$

(2) Apply the given conditions in Equation (b) in Equation (d):

$$[s^2Y(s) - s\overset{=\,0}{y(0)} - \overset{=\,1}{y'(0)}] + 2[sY(s) - \overset{=\,0}{y(0)}] + 5Y(s) = \frac{1}{(s+1)^2+1} = \frac{1}{s^2+2s+2}$$

(3) We can obtain the expression:

$$Y(s) = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} \qquad \text{(e)}$$

(4) The solution of the DE in Equation (a) will be obtained by the inverse Laplace transform of Y(s) in Equation (e), i.e. $y(t) = L^{-1}[Y(s)]$, or:

$$y(t) = L^{-1}[Y(s)] = L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right]$$

(5) The inverse Laplace transform of Y(s) in Equation (e) is obtained by using either "partial fraction method" or "convolution theorem." The expression of Y(s) can be shown in the following form by "partial fractions:"

$$Y(s) = \frac{\frac{1}{3}}{s^2+2s+2} + \frac{\frac{2}{3}}{s^2+2s+5} = \frac{1}{3}\cdot\frac{1}{s^2+2s+2} + \frac{2}{3}\cdot\frac{1}{s^2+2s+5}$$

The inversion of Y(s) in the above form is:

$$y(t) = L^{-1}[Y(s)] = \frac{1}{3}L^{-1}\left[\frac{1}{s^2+2s+2}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{s^2+2s+5}\right] = \frac{1}{3}L^{-1}\left[\frac{1}{(s+1)^2+1}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{(s+1)^2+4}\right]$$

Leading to the solution of the DE in Equation (a) to be:

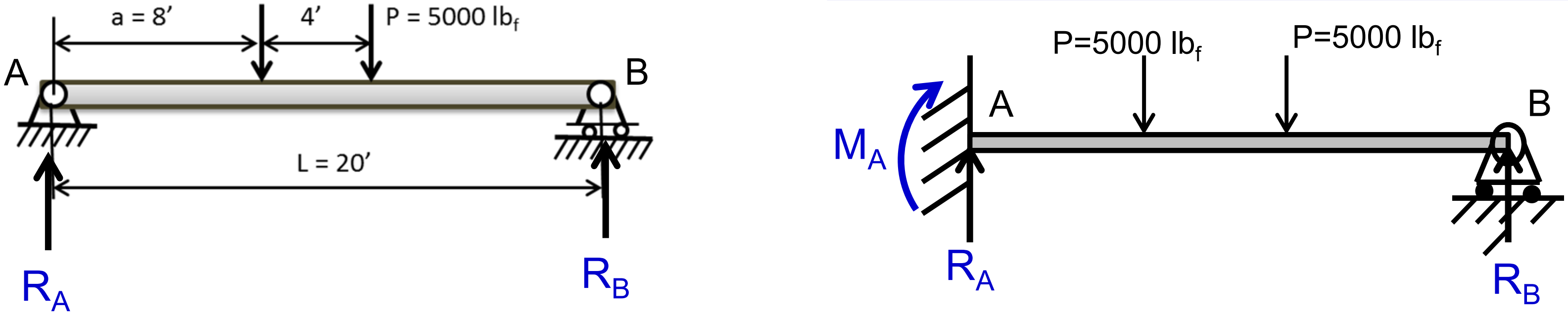
$$y(t) = L^{-1}[Y(s)] = \frac{1}{3}e^{-t}\, Sin\, t + \frac{2}{3}\,\frac{1}{2}\,e^{-t}\, Sin\, 2t = \frac{1}{3}e^{-t}\,(Sin\, t + Sin\, 2t)$$

6.6 **Solution of Differential Equations Using Laplace Transforms-**Cont'd

6.6.2 Differential equations for the bending of beams (p.186):

We offer this method for solving problems of indeterminate beam bending (the cases in which the number of unknowns in the mechanic analysis of beams subject to bending loads exceed the total number of end conditions that engineers can derive from the static equilibrium conditions).

Laplace transform method is found to be effective in solving problems of beam bending of this kind. The following figure illustrates the different situations of statically determinate and statically indeterminate beams; we notice the loaded beam in the left involves 2 unknowns of reactions at supports A and B which can be determined by the two available end conditions from free-body force diagram. The similar situation is illustrated in the right of the diagram , in which we realize that it has three (3) unknowns involving two reactions R_A and R_B at the supports A and B and the bending moment M_A in Support A. These three unknowns cannot be solved with the available two (2) conditions by the Supports A and B. This situation is termed as "indeterminate." Solutions to indeterminate beam bending are usually tedious and cumbersome. We will demonstrate the use of Laplace transform technique involving special functions to be an effective alternative solution method for the solution of bending of indeterminate beams subjected to loading to parts of the beams.



Statically determinate beam with 2 unknowns:

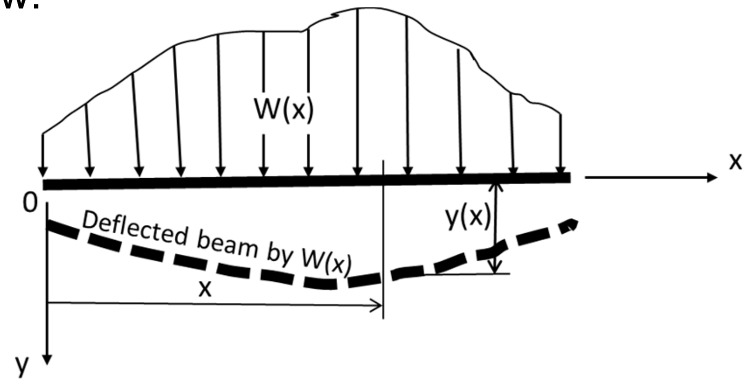
Statically indeterminate beam with 3 unknowns:

Figure 1.10 Static bending of beams

6.6 Solution of Differential Equations Using Laplace Transforms-Cont'd

6.6.2 Differential equations for the bending of beams-Cont'd (p.186):

We may derive a differential equation from the theory of elasticity to determine the deflection of a beam y(x) induced by a distributed bending load W(x) per unit length as illustrated in the figure below:



The induced deflection in the beam y(x) at location x can be obtained by solving the Euler-Bernoulli equation in the following differential equation:

$$\frac{d^4 y(x)}{dx^4} = \frac{W(x)}{EI} \tag{6.22}$$

where E is the Young's modulus of the beam material and I is the section moment of inertia of the beam cross-section.

The product of EI is referred to as the *flexural rigidity* of the beam structure.

6.6 Solution of Differential Equations Using Laplace Transforms-Cont'd

6.6.2 Differential equations for the bending of beams – cont'd:

Example 6.15 (p.187):

Use the Laplace transform method to find the induced deflection function y(x) of a cantilever beam by a uniform distributed load with intensity w_0 on half of the beam span as illustrated in Figure 6.5.

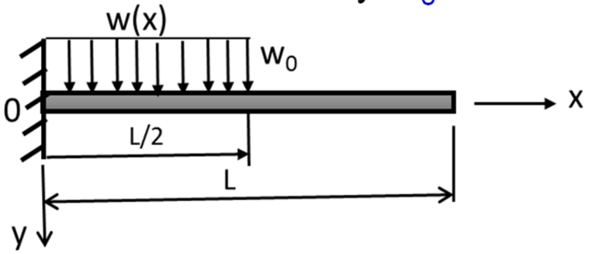


Figure 6.5 A Cantilever beam subjected to uniform distributed load

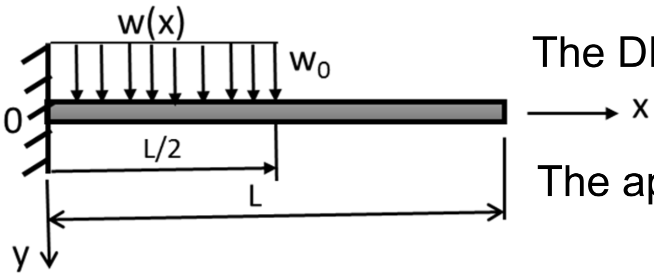
The beam subjected to bending as illustrated in Figure 6.5 would be difficult to solve for it's induced deflection and the bending stress by traditional simple beam theory because it is a statically indeterminate beam bending problem. Laplace transform method combined with a step function for the loading situation is a viable alternative method in solving this problem.

One critical issue, however, is that one must bear in mind that Laplace transform can only be used in the situation that the variable to be transformed must satisfy the condition of:

$0 \leq (\text{variable}) < \infty$

6.6 **Solution of Differential Equations Using Laplace Transforms-**Cont'd

6.6.2 Solving differential equations for the bending of beams using Laplace Transform– cont'd (p.187):



The DE: $$\frac{d^4y(x)}{dx^4} = \frac{W(x)}{EI} \qquad \text{(a1)}$$

The applied loading function W(x):

W(x) = W$_0$ for 0 ≤ x ≤ L/2
= 0 for L/2 ≤ x ≤ L

We thus have the differential equation for the solution with the form:

$$\frac{d^4y(x)}{dx^4} = \frac{W_0}{EI} \quad \text{with } 0 \le x \le L/2 \qquad \text{(a2)}$$

$$= 0 \quad \text{with } L/2 \le x \le L$$

with the following end (or boundary) conditions:

At x=0: $y(x)\big|_{x=0} = y(0) = 0$ (zero deflection) (b1)

$\left.\frac{dy(x)}{dx}\right|_{x=0} = y'(0) = 0$ (zero slope for the "built-in end support) (b2)

At =L: $\left.\frac{d^2y(x)}{dx^2}\right|_{x=L} = y''(L) = 0$ (zero bending moment at free-end) (b3)

$\left.\frac{d^3y(x)}{dx^3}\right|_{x=L} = y'''(L) = 0$ (zero shear force at free-end) (b4)

6.6 **Solution of Differential Equations Using Laplace Transforms-**Cont'd

6.6.2 Solving differential equations for the bending of beams using Laplace Transform– cont'd (p.188):

We notice that the variable x that is associated with the deflection function y(x) in the problem covers a finite range with $0 \le x \le L$. The legitimacy of using the Laplace transform method in solving the differential equation requires the expression of this function y(x) with its variable x covering the spectrum of $(0,\infty)$.

We thus need to derive the form of Equation (a1) from the variable range (0.L) to the domain (0,∞) in order to use the Laplace transform method for solving Equation (a1).

By using the definition of step functions in Section 2.4.2, we may convert the current loading function W(x) from the spectrum of (0,L) to $(0,\infty)$:

$$W(x) = W_0[u(x) - u(x-L/2)] \quad \text{with} \quad 0 \le x < \infty \tag{c}$$

where u(x) is the unit step function as defined in Section 2.4.2 (p.58), and with its shape illustrated in Figure 2.51 on p.60.

The equation for the deflection function y(x) in Equation (a1) will thus become:

$$\frac{d^4 y(x)}{dx^4} = \frac{W_0}{EI}\left[u(x) - u\left(x - \frac{L}{2}\right)\right] \qquad \text{with } 0 \le x < \infty, \tag{a3}$$

We may thus use the Laplace transform method to solve the deflection function y(x) in Equation (a3) togather with the end conditions in Equations (b1) to (b4). Let Y(s) to be the function y(x) after the Laplace transform with the following definition of Y(s):

$$L[f(x)] = \int_0^\infty e^{-sx} f(x)dx = Y(s) \tag{d}$$

6.6 **Solution of Differential Equations Using Laplace Transforms-**Cont'd

6.6.2 Solving differential equations for the bending of beams using Laplace Transform– cont'd:

Upon applying the Laplace transform in Equation (d) to the differential equation in (a3), we will get the following expression:

$$s^4Y(s)-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)=\frac{W_0}{RI}\left(\frac{1-e^{\frac{sL}{2}}}{s}\right) \qquad \text{(e)}$$

in which s = the Laplace transform parameter. NOTE: the LT on step functions is available in Laplace Transform table in Appendix 1, case No. 15.

Expression in Equation (e) will lead to the following expression of Y(s) to be:

$$Y(s)=\frac{c_1}{s^3}+\frac{c_2}{s^4}+\frac{W_0}{EIs^5}\left(1-e^{\frac{sL}{2}}\right) \qquad \text{(f)}$$

where c_1=y"(0) and c_2 = y'"(0) to be determined after we get the solution y(x).

The induced beam deflection function y(x) in Equations (a1), (a2) and (a3) may be obtained by inverting the expression Y(s) in Equation (f), resulting in the following:

$$y(x)=L^{-1}[Y(s)]=\frac{c_1x^2}{2!}+\frac{c_2x^3}{3!}+\frac{W_0}{EI}\frac{x^4}{4!}-\frac{W_0}{EI}\frac{(x-L/2)^4}{4!}u(x-L/2) \qquad \text{(g)}$$

The unit step function appears in the last part of Equation (g) will result in the following solution in the two separate portions in the following expressions:

$$y(x)=\frac{c_1x^2}{2}+\frac{c_2x^3}{6}+\frac{W_0x^4}{24EI} \qquad \text{for } 0 \le x \le L/2 \qquad \text{(h1)}$$

and $$y(x)=\frac{c_1x^2}{2}+\frac{c_2x^3}{6}+\frac{W_0x^4}{24EI}-\frac{W_0}{24EI}(x-L/2)^4 \qquad \text{for } x > L/2 \qquad \text{(h2)}$$

6.6 **Solution of Differential Equations Using Laplace Transforms-**Cont'd

6.6.2 Solving differential equations for the bending of beams using Laplace Transform– cont'd:

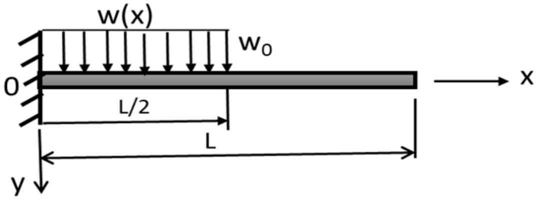
The two arbitrary constant coefficients c_1 and c_2 in the following solutions of the deflection of the bent beam, y(x), can be determine by the two unused end conditions in Equations (b3) and (b4):

$$y(x) = \frac{c_1 x^2}{2} + \frac{c_2 x^3}{6} + \frac{W_0 x^4}{24EI} \quad \text{for } 0 \le x \le L/2 \qquad \text{(h1)}$$

and $$y(x) = \frac{c_1 x^2}{2} + \frac{c_2 x^3}{6} + \frac{W_0 x^4}{24EI} - \frac{W_0}{24EI}(x - L/2)^4 \quad \text{for } x > L/2 \qquad \text{(h2)}$$

Due to the fact that the given end conditions in Equations (b3) and (b4) are at x = L, which allows us to use the partial solution in Equation (h2) for: y"(L) =0 and y'"(L)=0 resulting in:

$$c_1 = \frac{W_0 L^2}{8EI} \quad and \quad c_2 = -\frac{W_0 L}{2EI}$$



We thus have the deflection of the beam y(x) in:

$$y(x) = \frac{W_0 L^2}{16EI}x^2 - \frac{W_0 L}{12EI}x^3 + \frac{W_0}{24EI}x^4 \quad \text{for } 0 \le x \le L/2 \qquad \text{(j1)}$$

$$y(x) = \frac{W_0 L^2}{16EI}x^2 - \frac{W_0 L}{12EI}x^3 + \frac{W_0}{24EI}x^4 - \frac{W_0}{24EI}\left(x - \frac{L}{2}\right)^4 \quad \text{for } x > L/2 \qquad \text{(j2)}$$

We will calculate the bending moment M(x) in the bent beam using the deflection function y(x) by:

$$M(x) = EI\frac{d^2y(x)}{dx^2}$$ in which y(x) is given in Equation (j1) or (j2)

The induced normal bending stress in the beam is: $\sigma_n(x) = \frac{M(x)c}{I}$

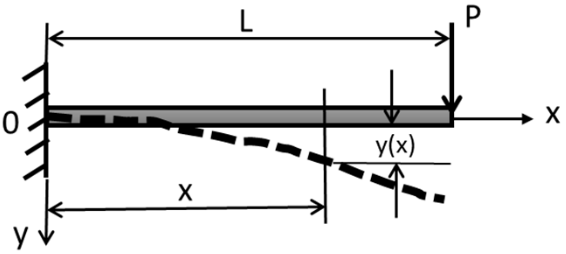
where c = the half depth of the beam cross-section and I = section moment of inertia of the beam cross-section

6.6 Solution of Differential Equations Using Laplace Transforms-Cont'd

6.6.2 Solving differential equations for the bending of beams using Laplace Transform– cont'd:

Example 6.16 (p. 189):

Use the Laplace transform method to solve Equation (6.22) and compute the induced deflection function y(x) of a cantilever beam induced by the application of a concentrate force P acting at its free end as illustrated in the figure: ⇨



Solution:

There are two issues involved in solving this problem:

(1) The loading function in Equation (6.22) is for distributed loads, **not** for concentrated force as in the current case. An equivalent distributed loading function W(x) to a concentrated force P needs to be derived, and

(2) The beam has a finite length of L meaning the variable x covers a spectrum (0,L) but NOT (0,∞) as required for using the Laplace transform method. A conversion of the range of coverage of the variable x to (0,∞) in the function y(x) from the current range of (0,L) is required.

We will first derive the equivalent concentrated force P for the current problem to the distribute load W(x) in the general beam bending situation in Figure 6.4 (p.186) by using the impulsive function described in Equation (2.43) (p. 62) by letting:

$$W(x) = P(x) = - P\,\delta(x-L) \qquad (6.23)$$

where δ(x-L) is an impulsive function defined in Section 2.4.2 (p. 58).

We notice from the above conversion leads to the loading function P(x) with x valid for the range (-∞,+∞) by the definition of the impulsive functions. This range however, includes the range (0,∞) and hence legitimizes us using the Laplace transform method.

6.6.2 Solving differential equations for the bending of beams using Laplace Transform– cont'd:

Example 6.16- Cont'd

We will thus have the differential equation for the solution of deflection function y(x) in Equation (6.22) expressed in the following equivalent form:

$$\frac{d^4 y(x)}{dx^4} = -\frac{P\,\delta(x-L)}{EI} \quad with \quad 0 < x < \infty \tag{a}$$

in which E and I are the Young's modulus of the beam material and section moment of inertia of the beam cross-section respectively.

The following end conditions need to be satisfied at the fixed end in Equation (b1) and those at the free-end in Equation (b2):

$y(x)\big|_{x=0} = 0$ (for zero deflection at the fixed end)

(b1)

$\left.\frac{dy(x)}{dx}\right|_{x=0} = y'(0) = 0$ (for zero slope of deflection curve at the fixed end)

$\left.\frac{d^2y(x)}{dx^2}\right|_{x=L} = y''(L) = 0$ (for zero bending moment at the free-end)

$\left.\frac{d^3y(x)}{dx^3}\right|_{x=L} = y'''(L) = 0$ (for zero shearing force at the free-end)

(b2)

6.6 Solution of Differential Equations Using Laplace Transforms-Cont’d

6.6.2 Solving differential equations for the bending of beams using Laplace Transform– cont’d:

Example 6.16- Cont’d

After applying the Laplace transform on each term in Equation (a), resulted in:

$$s^4Y(s)-s^3\cancelto{0}{y(0)}-s^2\cancelto{0}{y'(0)}-sy''(0)-y'''(0)=-\frac{P}{EI}e^{-Ls} \qquad \text{(c)}$$

where Y(s) is the expression of Laplace transform of function (or solution of Equation (a) with s being the Laplace transform parameter). One may get the Laplace transform of impulsive function in Equation (a) in Case 14 in the Laplace transform table in Appendix 1.

The specified conditions in equation (c) are given with y(0) = y’(0) = 0 as given in Equation (b1) to the above expression, leads to the following expression for Y(s):

$$Y(s)=\frac{c_1}{s^3}+\frac{c_2}{s^4}-\frac{P}{EI}\frac{e^{-Ls}}{s^4} \qquad \text{(d)}$$

We, however, do not have values for y”(0) and y”’(0) for the expression in Equation (c) but we may let y”(0) = c_1 and y”’(0) = c_2, in which c_1 and c_2 are constants to be determined later.

The solution of Equation (a) for the deflection function y(x) of the beam is obtained by inverting the Laplace transformed Y(s) in Equation (d) to yield:

$$y(x)=\frac{c_1x^2}{2!}+\frac{c_2x^3}{3!}-\frac{P}{EI}\frac{(x-L)^3}{3!}u(x-L) \qquad \text{(e)}$$

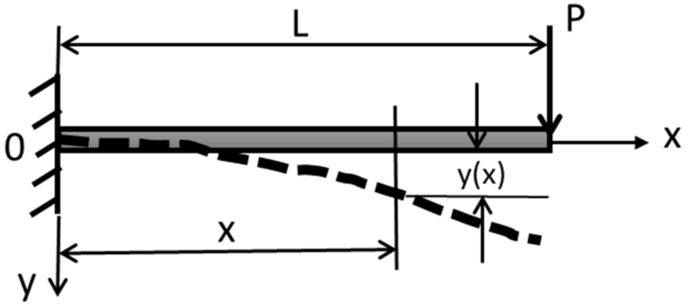
6.6 **Solution of Differential Equations Using Laplace Transforms-**Cont'd

6.6.2 Solving differential equations for the bending of beams using Laplace Transform– cont'd:

Example 6.16- Cont'd

We have obtained the solution of deflection of the cantilever beam subjected to a concentrated load P at the free-end of the beam to be:

$$y(x) = \frac{c_1 x^2}{2!} + \frac{c_2 x^3}{3!} - \frac{P}{EI}\frac{(x-L)^3}{3!}u(x-L)$$



By virtue of the definition of the step function u(x-L) appeared in Equation (e), we may break up the solution y(x) into the two sections as follows:

$$y(x) = \frac{c_1 x^2}{2} + \frac{c_2 x^3}{6} \quad for\ 0 \le x < L \qquad \text{(f1)}$$

$$y(x) = \frac{c_1 x^2}{2} + \frac{c_2 x^3}{6} - \frac{P}{6EI}(x-L)^3 \quad for\ \ x = L \qquad \text{(f2)}$$

By applying the two unused end conditions y"(L) = 0 and y"'(L) =0 in Equation (f2), we may determine both constants c_1 and c_2 to be:

$$c_1 = -\frac{PL}{EI} \quad and \quad c_2 = \frac{P}{EI}$$

which leads to the solution of the deflection of the beam y(x) in the following form:

$$y(x) = -\frac{PL}{2EI}x^2 + \frac{P}{6EI}x^3 = -\frac{P}{6EI}\left(-x^3 + 3Lx^2\right)$$

6.7 Solution of Partial Differential Equations Using Laplace Transforms

We have learned to transform partial derivatives in Section 6.5.2 (p.181) and derived the expressions in terms of the Laplace transform parameter, i.e., F*(x,s) for the function f(x,t) with variable t being transformed to the s- parametric domain such that:

$$L_t\left[\frac{\partial f(x,t)}{\partial x}\right]=\frac{\partial F^*(x,s)}{\partial x}=\frac{dF^*(x,s)}{dx} \qquad (6.17)$$

and

$$L_t\left[\frac{\partial f(x,t)}{\partial t}\right]=sF^*(x,s)-f(x,0) \qquad (6.20)$$

We may use these formulations to solve partial differential equations, as will be demonstrated in the following example.

Example 6.18 (p.193)

Solve the following partial differential equation using Laplace transform method.

$$\frac{\partial U(x,t)}{\partial x}=2\frac{\partial U(x,t)}{\partial t}+U(x,t) \quad with \quad 0\le t<\infty \qquad \text{(a)}$$

with the initial and boundary conditions:

$$U(x,t)\big|_{t=0}=U(x,0)=6e^{-3x} \qquad \text{(b)}$$

and the function U(x,t) is bonded for x > 0 and t > 0

Example 6.18 – Cont'd

Solution:

We will apply the Laplace transform of function U(x,t) defined as: $L_t[U(x,t)] = \int_0^\infty e^{-st}U(x,t)dt = U^*(x,s)$

to the partial differential equation in (a) as:

$$L_t\left[\frac{\partial U(x,t)}{\partial x}\right] = 2L_t\left[\frac{\partial U(x,t)}{\partial t}\right] = L_t[U(x,t)]$$

From which and those defined in Equations (6.17) and (6.20), we will have:

$$L_t\left[\frac{\partial U(x,t)}{\partial x}\right] = \frac{dU^*(x,s)}{dx} \quad \text{and} \quad L_t\left[\frac{\partial U(x,t)}{\partial t}\right] = -U(x,0) + sU^*(x,s)$$

Applying the Laplace transform to all terms in Equation (a) will result in the following first order differential equation for U*(x,s):

$$\frac{dU^*(x,s)}{dx} = 2\left[-U(x,0) + sU^*(x,s)\right] + U^*(x,s)$$

Substituting the initial condition U(x,0) = 6e^{-3x} in Equation (b) in the above expression, we will establish the differential equation of U*(x,s) – now a first order ordinary DE to be:

$$\frac{dU^*(x,s)}{dx} = 2\left[-6e^{-3x} + sU^*(x,s)\right] + U^*(x,s) \tag{d}$$

The solution of the above first order ordinary DE is: $U^*(x,s) = \frac{6}{s+2}e^{-3x}$ (e)

The solution U(x,t) in Equation (a) is obtained by inverting the above Laplace transformexpression U*(x,s) in Equation (e), resulting in:

$$U(x,t) = L_t^{-1}\left[U^*(x,s)\right] = L_t^{-1}\left[\frac{6}{s+2}e^{-3x}\right] = 6e^{-2t-3x}$$

6.7 **Solution of Partial Differential Equations Using Laplace Transforms –** cont'd

Example 6.19 (p.194)

Solve the following partial differential equation using the Laplace transform method.

$$\frac{\partial^2 U(x,t)}{\partial x^2} = \frac{\partial U(x,t)}{\partial t} \qquad \text{with } 0 < x < \infty \text{ and } 0 < t < \infty \qquad \text{(a)}$$

with the initial condition (IC): U(x,0) = 0 (b1)

and the boundary condition (BC): U(0,t) = 1 (b2)

Solution

Because both the variables x and t in function U(x,t) are not bounded in both space and time as shown in Equation (a) with $0 < x < \infty$ and $0 < t < \infty$, we may perform Laplace transform of this function U(x,t) with either of these variables. However, we will perform the following computations based on Laplace transform on the variable t. Consequently the definition of the Laplace transform in Equation (c) in Example 6.18 is adopted in this example. Thus, by apply the Laplace transform on both sides of Equation (a) resulting in the following expression:

$$\frac{\partial^2 U^*(x,s)}{\partial x} = sU^*(x,s) - U(x,0) \qquad \text{(c)}$$

We realize that Equation (c) is a second order ordinary differential equation with the remaining variable x, in the form:.

$$\frac{d^2 U^*(x,s)}{dx^2} - sU^*(x,s) = 0 \qquad \text{(d)}$$

Example 6.19 (p.194)-Cont'd

The complete solution of the 2nd order ODE in Equation (d) requires the form of the Laplace transform available in Section form of the specified BC of Equation (a) to be: U(0,t) = 1 in Equation (b2), as follows:

$$L_t[U(0,t)] = L_t(1) = \int_0^\infty e^{-st}(1)dt = \frac{1}{s} \quad at\ x = 0$$

from which we will have the boundary condition for the DE in Equation (a) to be: U*(0,s) = 1/s

The general solution of the second order differential equation in Equation (d) may be obtained by methods such as presented in Section 8.2 (p.243) in the following form:

$$U^*(x,s) = c_1 e^{\sqrt{s}\,x} + c_2 e^{-\sqrt{s}\,x} \qquad \text{(f)}$$

where c_1 and c_2 in Equation (f) are arbitrary constants to be determined by explicit condition in Equation (e) and another implicit condition as will be described as follows:

We realize from Equation (f) that x→∞ will lead to U*(x,s)→∞, and hence $U(x,t)\big|_{x\to\infty} \to \infty$, which is not a realistic solution for the problems if this nature in engineering analysis. The only way that this situation may be avoided is to set the constant $c_1 = 0$ in Equation (f). Consequently, the solution of U*(x,s) in Equation (f) has been reduced to the form:

$$U^*(x,s) = c_2 e^{-\sqrt{s}\,x} \qquad \text{(g)}$$

The remaining constant c_2 may be determined by the condition expressed in Equation (e) with c_2 = 1/s. We thus have the solution U*(x,s) to be: $U^*(x,s) = \frac{1}{s}e^{-\sqrt{s}\,x}$ (h)

The solution of Equation (a) is obtained by the inverse of the Laplace transform of U*(x,s) in Equation (h), resulting in:

$$U(x,t) = L_t^{-1}[U^*(x,s)] = L_t^{-1}\left(\frac{e^{-\sqrt{s}\,x}}{s}\right) = erfc\left(\frac{x}{2\sqrt{t}}\right) \qquad \text{(j)}$$

where erfc(X) is the complimentary error function as defined in Section 2.4.1.1 (p.55)