# Introduction to the Theory of Plates

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## Stretching and Bending of Plates - Fundamentals

### Introduction

A plate is a structural element which is thin and flat. By "thin," it is meant that the plate's transverse dimension, or thickness, is small compared to the length and width dimensions. A mathematical expression of this idea is:

$$t/L \ll 1$$

where t represents the plate's thickness, and L represents a representative length or width dimension. (See Fig. [See Plate and associated (x, y, z) coordinate system...](#)) More exactly, L represents the minimum wave length of deformation, which can be much smaller than the plate minimum lateral dimension for problems of localized loading, dynamics and stability. Plates might be classified as very thin if $Łt > 100$, moderately thin if $20 < Łt < 100$, thick if $3 < Łt < 20$, and very thick if $Łt < 3$. The "classical" theory of plates is applicable to very thin and moderately thin plates, while "higher order theories" for thick plates are useful. For the very thick plates, however, it becomes more difficult and less useful to view the structural element as a plate - a description based on the three-dimensional theory of elasticity is required.
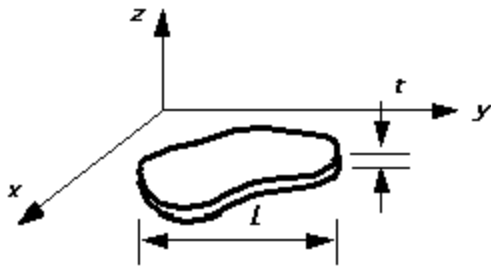
**FIGURE 1.1. Plate and associated (x, y, z) coordinate system.**

In this chapter, we derive the basic equations which describe the behavior of plates taking advantage of the plate's thin, planar character. The approach is a generalization of the one-dimensional Euler-Bernoulli beam theory, which exploits the slender shape of a beam. We will develop a two-dimensional plate theory which employs the in-plane coordinates x and y in See Plate and associated (x, y, z) coordinate system.. as independent variables. Of particular interest are the plate's "stretching" behavior - associated with in-plane loads in the x- and y-directions, and the plate's bending behavior - associated with moment and shear loads applied to the plate's edges, and transverse loads in the z- direction.

Figure See Plate and associated (x, y, z) coordinate system.. depicts a plate and an associated (x, y, z) coordinate system. The top and bottom surfaces lie at z = ±t/2. The flat surface z = 0 is the plate midsurface, which provides a convenient reference plane for the derivation of the governing equations for the plate.

## Three-Dimensional Considerations

Many things are easier if we begin with the three-dimensional equations of linear elasticity. The solutions for three-dimensional problems is generally difficult and time-consuming for a computation. In many respects, however, the derivation of the three-dimensional theory is more straight-forward than the reduced approximate beam, plate and shell theories. Indeed, the three-dimensional theory is the basis for all approximate theories. The equations can be found in many texts, including Timoshenko and Goodier, 1970See Timoshenko SP and Goodier N (1970). Theory of Elasticity, Third Edition, (McGraw-Hill, New York)...

**Stress components**

The application of external forces to a body produces an internal state of stress. Stress is measured in units of force per unit area, and can be thought of as the intensity of the internal forces acting at a particular point in the body. Figure See Stress components acting on the surfaces of a three-dimensional element in cartesian coordinates.. depicts the stresses which act on the surfaces of a three-dimensional element of a solid body.
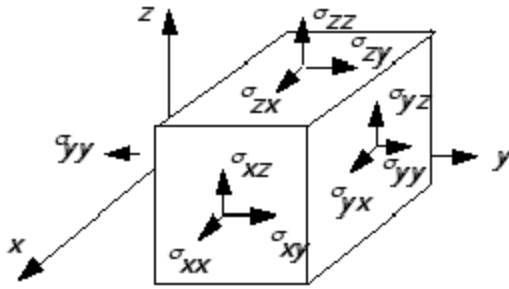
**FIGURE 1.2. Stress components acting on the surfaces of a three-dimensional element in cartesian coordinates.**

The components in Fig. [See Stress components acting on the surfaces of a three-dimensional element in cartesian coordinates..](#) are identified by two indices. The first index denotes the direction of the outward normal to the surface being acted upon, and the second index denotes the direction in which the force acts. For example, $\sigma zx$ is the force per unit area of the surface with outward normal in the positive z-direction and acting in the positive x-direction. The stress components $\sigma xx$, $\sigma yy$, and $\sigma zz$ act normally to the element faces, while the remaining six components $\sigma xy$, $\sigma xz$, $\sigma yx$, $\sigma yz$, $\sigma zx$, and $\sigma zy$ are the components of shear stresses, which act tangentially to the element faces. On the "back" faces not shown in Fig. [See Stress components acting on the surfaces of a three-dimensional element in cartesian coordinates..](#), i.e., the faces with outward normals in the negative directions of the coordinate axes, the directions of all the components are reversed, as indicated by the component $\sigma yy$. With this sign convention, tensile forces have positive values of stress and compressive forces negative values.

Some authors choose the opposite order for the indices, with the first index giving the direction of force and the second the face. However, the majority of publications on plates and shells use the notation in Fig. [See Stress components acting on the surfaces of a three-dimensional element in cartesian coordinates...](#).

**Equilibrium**

**Divergence theorem**

Consider the cube in Fig. [See Stress components acting on the surfaces of a three-dimensional element in cartesian coordinates..](#) to be located in the region of a body with the coordinates:

$$x_1 \leq x \leq x_2 \; ; \; y_1 \leq y \leq y_2 \; ; \; z_1 \leq z \leq z_2$$

The total force in the x-direction can be obtained by integrating the components in the x-direction on the six faces, with the result:

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}[\sigma_{xx}(x_2,y,z)-\sigma_{xx}(x_1,y,z)]dy\,dz + \int_{x_1}^{x_2}\int_{z_1}^{z_2}[\sigma_{yx}(x,y_2,z)-\sigma_{yx}(x,y_1,z)]dz\,dx$$

$$\int_{y_o}^{y_n}\int_{x_o}^{x_n}[\sigma_{zx}(x,y,z_2)-\sigma_{zx}(x,y,z_1)]dxdy + \int_{x_o}^{x_n}\int_{y_o}^{y_n}\int_{z_o}^{z_n}b_x(x,y,z)dzdy\,dx = 0$$

in which bx is the component in the x-direction of the body force per unit of volume. However, the first surface integrals can be rewritten as volume integrals. (This is the divergence theorem for rectangular coordinates.) Thus the equation becomes:

$$\int_{x_o}^{x_n}\int_{y_o}^{y_n}\int_{z_o}^{z_n}\left[\frac{\partial}{\partial x}\sigma_{xx}(x,y,z)+\frac{\partial}{\partial y}\sigma_{yx}(x,y,z)+\frac{\partial}{\partial z}\sigma_{zx}(x,y,z)+b_x(x,y,z)\right]dxdydz = 0$$

Within the body, the subregion can be chosen arbitrarily. For the result of the integration to be zero for any arbitrary subregion, no matter how small or large, the integrand must be identically zero. Thus the partial differential equations for equilibrium are obtained:

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}+b_x = 0$$

$$\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{zy}}{\partial z}+\frac{\partial\sigma_{xy}}{\partial x}+b_y = 0$$

$$\frac{\partial\sigma_{zz}}{\partial z}+\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}+b_z = 0$$

in which bx, by, and bz are the components of the body force per unit of volume.

Moment equilibrium of the three-dimensional element in Fig. [See Stress components acting on the surfaces of a three-dimensional element in cartesian coordinates..](#) around each of the coordinate axes leads to the condition of symmetry of the shear stress components for the classical theory of elasticity:

$$\sigma_{xy} = \sigma_{yx} \; ; \; \sigma_{yz} = \sigma_{zy} \; ; \; \sigma_{zx} = \sigma_{xz}$$

We mention that for "nonclassical" elasticity, in addition to a force per unit area, the possibility of a torque per unit area acting on the body is included. For such theories, Eq. [See ; :.](#) does not hold, and the stress components are not symmetric.

**Expansion method**

The above derivation is exact. However, a direct method which is often used is to consider the element in [See Increments of the components of stress in the x-direction...](#) The summation of the forces in the x-direction gives:

$$[\sigma_{xx}(x+\Delta x,y,z)-\sigma_{xx}(x,y,z)]\Delta y\,\Delta z + [\sigma_{yx}(x,y+\Delta y,z)-\sigma_{yx}(x,y,z)]\Delta z\Delta x$$

$$[\sigma_{zx}(x,y,z+\Delta z)-\sigma_{zx}(x,y,z)]\Delta x\Delta y+b_x\Delta x\Delta y\Delta z = 0$$

Dividing by $\Delta$ x $\Delta$ y $\Delta$ z and taking the limit yields Eq. [See .](#). The same procedure yields the other two equations of equilibrium.
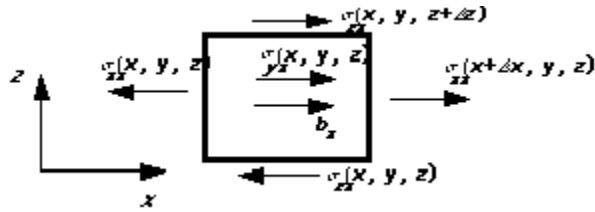
**FIGURE 1.3. Increments of the components of stress in the x-direction.**

### Strain and Displacement

The components of displacement in the x-, y-, and z- directions is often denoted by u, v, and w. The strain (engineering) is the change in length divided by the original length, which gives the strain components in the three directions:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \; ; \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} \; ; \quad \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

and the components of shear strain:

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \; ; \quad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \; ; \quad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

If two lines that intersect at 90° are scribed on the body, then the change in the angle of intersection is the shear strain.

### Constitutive Relations

The stress components are related to the strain components by the generalized Hooke's law for an isotropic material:

$$\varepsilon_{xx} = \frac{1}{E}[\sigma_{xx} - v(\sigma_{yy} + \sigma_{zz})]$$

$$\varepsilon_{yy} = \frac{1}{E}[\sigma_{yy} - v(\sigma_{zz} + \sigma_{xx})]$$

$$\varepsilon_{zz} = \frac{1}{E}[\sigma_{zz} - v(\sigma_{xx} + \sigma_{yy})]$$

where E is the Young's modulus and $v$ is the Poisson's ratio. The shearing strain components are:

$$\gamma_{xy} = \frac{\sigma_{xy}}{G}$$

$$\gamma_{yz} = \frac{\sigma_{yz}}{G}$$

$$\gamma_{zx} = \frac{\sigma_{zx}}{G}$$

where G is the shear modulus. For the isotropic material, the shear modulus is determined by the Young's modulus and Poisson's ratio:

$$G = \frac{E}{2(1+v)}$$

So both the shear stress and shear strain components are symmetric in the two indices. For a general anisotropic material, all the stress and strain components are related.

## Force and Moment Resultants

The nine stress components shown in Fig. See Stress components acting on the surfaces of a three-dimensional element in cartesian coordinates.. uniquely define the state of stress at a point in a three-dimensional body, and are in general functions of x, y, and z. In the theory for a thin plate, however, we would like to work with force quantities which depend on x and y alone. This can be achieved by integrating the stresses in the z-direction, through the plate's thickness, in order to obtain the following stress resultant quantities:

$$N_x = \int_{-t/2}^{t/2} \sigma_{xx}\,dz \quad N_{xy} = \int_{-t/2}^{t/2} \sigma_{xy}\,dz \quad Q_x = \int_{-t/2}^{t/2} \sigma_{xz}\,dz$$
$$;$$

$$N_y = \int_{-t/2}^{t/2} \sigma_{yy}\,dz \quad N_{yx} = \int_{-t/2}^{t/2} \sigma_{yx}\,dz \quad Q_y = \int_{-t/2}^{t/2} \sigma_{yz}\,dz$$
$$;$$

and the moment resultant quantities:

$$M_x = \int_{-t/2}^{t/2} \sigma_{xx}z\,dz \quad M_{xy} = \int_{-t/2}^{t/2} \sigma_{xy}z\,dz$$
$$;$$

$$M_y = \int_{-t/2}^{t/2} \sigma_{yy}z\,dz \quad M_{yx} = \int_{-t/2}^{t/2} \sigma_{yx}z\,dz$$
$$;$$

The stress resultants given by Eq. See ; ;. have the units of force per unit length, and act in the directions shown in Fig. See Force resultants acting on a two-dimensional plate element... Nx and Ny are in-plane tensile (when positive) stress resultants in the x- and y- directions; Nxy and Nyx are in-plane shear stress resultants; and Qx and Qy are transverse shear stress resultants.

Loads on the plate in the x-, y-, and z -directions are denoted by px, py, and pz in Fig. See Force resultants acting on a two-dimensional plate element... These are the external forces acting on the plate at a given point. External loads can be applied as body forces, such as gravity, and they can be applied as surface tractions, in which case they correspond to the stresses $\sigma$zx, $\sigma$zy, and $\sigma$zz, evaluated at z = ± t/2, as shown in the next section. The loads have the units of force per unit area and
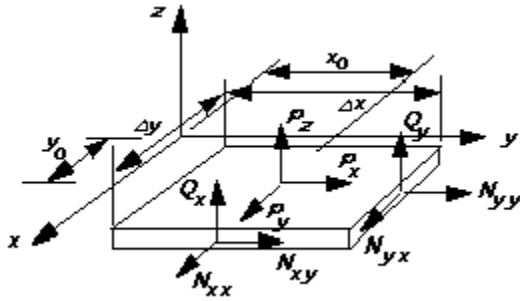
are functions of x and y.

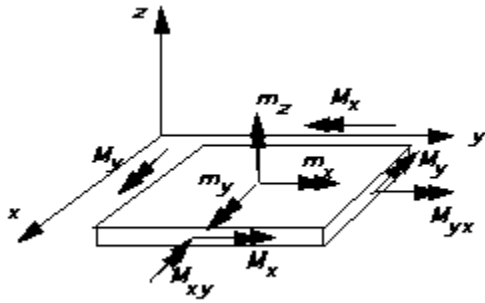**FIGURE 1.4. Force resultants acting on a two-dimensional plate element.**

**FIGURE 1.5. Moment resultants acting on the two-dimensional plate element.**

The moment resultants given by Eq. See :. have the units of moment per unit length, and act in the directions shown in Fig. See Moment resultants acting on the two-dimensional plate element... Mx and My are bending moment resultants. The directions of these moments are determined from Eq. See :., so the right-hand vector representation is as seen in Fig. See Moment resultants acting on the two-dimensional plate element... Thus the vector Mx is in the y-direction, and My is in the negative x-direction. If one remembers the basic definitions Eqs. See : :. and See :., then the directions of all the resultants can be easily obtained. A positive value of the stress component $\sigma$xx at the upper plate surface (z = + t/2) causes a positive value of Mx. The vectors for Mx and My for the back faces are shown in Fig. See Moment resultants acting on the two-dimensional plate element.., to emphasize that, unlike the shear stress resultants, the moment resultants `chase each other around the plate' (W. Flügge). Earlier authors used various definitions for the moment resultants, but Eq. See :. seems to be standard today. The quantities Mxy and Myx are the twisting moment resultants, which tend to twist the edge of the plate around the x- or y-axis. The vector directions of the twisting moments can be obtained by considering a positive value of $\sigma$xy and $\sigma$yx at the upper surface. For completeness, a prescribed moment per unit area acting on the surface is shown in Fig. See Moment resultants acting on the two-dimensional plate element.., with the components mx, my, and mz.

The stress and moment resultants provide convenient force quantities for the analysis of plates, just as moment, shear, and net tensile force are convenient in the analysis of beams. Note that the edge of the plate has a force resultant with components in all three directions (x, y, z) but the moment resultant has components only in the in-plane directions (x, y). This is even with the moment component mz applied to the plate surface.

## Equilibrium

When a plate carries a static load, the plate must be in equilibrium, which means that the forces and moments acting on any arbitrary element of the plate must sum to zero. In general, there will be six equilibrium conditions: one force balance equation and one moment balance equation for each of the three coordinate directions. We consider two approaches to the derivation of the equations of equilibrium.

### Direct from Resultants

The equations of equilibrium can be obtained from the Taylor series expansion of the resultants for the rectangular sub-region See Force resultants acting on a two-dimensional plate element.., as follows. The forces in the x-direction are:

$$\left(N_x + \frac{\partial N_x}{\partial x}\Delta x\right)\Delta y - N_x \Delta y$$

$$+ \left(N_{yx} + \frac{\partial N_{yx}}{\partial y}\Delta y\right)\Delta x - N_{yx}\Delta x + p_x \Delta x \Delta y = 0$$

Dividing by $\Delta x \Delta y$ and taking the limit yields the equation for force equilibrium in the x-direction:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} + p_x = 0$$

Similarly, we obtain the equation for force equilibrium in the y-direction:

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} + p_y = 0$$

the force equilibrium in the z-direction:

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p_z = 0$$

the moment equilibrium in the y-direction:

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} - Q_x + m_y = 0$$

the moment equilibrium in the x-direction:

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y - m_x = 0$$

and the moment equilibrium in the z-direction:

$$N_{xy} - N_{yx} + m_z = 0$$

However, for classical elasticity, the three-dimensional stresses are symmetric Eq. See ; ;., so that the definition of the resultants Eq. See ; ;. gives Nxy = Nyx, Consequently, Eq. See . shows that mz = 0 for a plate theory consistent with classical elasticity theory. Because of Eq. See ; ;., it follows from Eq. See ;. that the twisting moment is also symmetric:

$$M_{xy} = M_{yx}$$

In order to simplify the derivation of Eqs. See .-See ., it has been assumed that the forces can be treated as acting on the undeformed plate element. The equations therefore apply only when the deformation of the plate is small, i.e., for strains small relative to unity and normal displacement small relative to the thickness t. The analysis for larger deformation is considered later.

**Integrate 3-Dimensional Equations**

An alternate method to obtain the equations of equilibrium is to integrate the three-dimensional equations directly. This works particularly well for the flat plate under present consideration. Integrating Eq. See . yields Eq. See ., in which the effective pressure is precisely related to the 3-dimensional stress on the surface and the body force:

$$p_x(x,y) = \sigma_{zx}(x,y,t/2) - \sigma_{zx}(x,y,-t/2) + \int_{-t/2}^{t/2} b_x dz$$

Similarly, integrating Eq. See . yields Eq. See . in which:

$$p_y(x,y) = \sigma_{zy}(x,y,t/2) - \sigma_{zy}(x,y,-t/2) + \int_{-t/2}^{t/2} b_y dz$$

Integrating Eq. See . yields Eq. See ., in which:

$$p_z(x,y) = \sigma_{zz}(x,y,t/2) - \sigma_{zz}(x,y,-t/2) + \int_{-t/2}^{t/2} b_z dz$$

Multiplying Eq. See . and Eq. See . by z and integrating yields Eq. See . and Eq. See . in which:

$$m_y(x,y) = \frac{t}{2}[\sigma_{zx}(x,y,t/2) + \sigma_{zx}(x,y,-t/2)] + \int_{-t/2}^{t/2} b_x z dz$$

$$m_x(x,y) = -\frac{t}{2}[\sigma_{zy}(x,y,t/2) + \sigma_{zy}(x,y,-t/2)] - \int_{-t/2}^{t/2} b_y z dz$$

Something new is the result of multiplying Eq. See . by z and integrating. The result is:

$$\frac{\partial M_{xz}}{\partial x} + \frac{\partial M_{yz}}{\partial y} - N_z + m_{z2} = 0$$

where the new "resultants" are:

$$M_{xz} = \int_{-t/2}^{t/2} \sigma_{xz} z \, dz \quad ; \quad M_{yz} = \int_{-t/2}^{t/2} \sigma_{yz} z \, dz \quad ; \quad N_z = \int_{-t/2}^{t/2} \sigma_{zz} dz$$

which actually correspond to a self-equilibrating distribution of the 3-dimensional stress in the plate. Thus these are not actually resultants. The other new term in Eq. See . is:

$$m_{z2}(x,y) = \frac{t}{2}[\sigma_{zz}(x,y,t/2) + \sigma_{zz}(x,y,-t/2)] + \int_{-t/2}^{t/2} b_z z \, dz$$

which is also a self-equilibrating quantity and not actually a resultant moment. So integrating the 3-dimensional equations directly yields only five equations Eqs. See .-See . with actual resultants and a supplementary condition Eq. See ., that is useful in understanding effects of higher order. In particular, from Eq. See . with the normal stress from the two surfaces added, it is clear that Eq. See . is a statement regarding the stretching of the plate in the z-direction.

## Plate Kinematics and Constitutive Relations

The equilibrium equations (See .-See .) alone are insufficient to determine the response of a plate to a particular load, because they contain too many unknown resultant components. Eqs. See .,See . are two equations in three unknown in-plane stress resultants; and Eqs. See .-See . are three equations in three unknown moment resultants and two unknown transverse shear resultants. The plate is therefore statically indeterminate. In the analysis of beams and beam structures, there are many significant problems that are directly statically determinant, so that the state of stress can be obtained only from the equations of equilibrium. In contrast, there are few plate problems for which a useful solution can be obtained directly from the equilibrium equations. Hence generally, it is necessary to consider the displacements for the plate.

The kinematic relations Eqs. See ; ;. and See ; ;. and constitutive relations Eq. See .-See . are expressed in terms of the three-dimensional strains and stresses, but the equilibrium equations are expressed in terms of stress and moment resultants that are functions of just x and y. Equations See ; ;., See ; ;. and See .-See . must therefore be simplified to fit with our formulation in terms of the midsurface coordinates (x, y). This requires the introduction of some approximations.

**Basics**

**Plane Stress**

From our experience with beams, we know that a beam carries loads primarily through axial and bending stress components in the direction parallel to the beam's axis. Transverse shear stresses and stresses acting normal to the beam's axis are relatively small. The generalization of this behavior to the case of a thin plate suggests the following approximation:

$$|\sigma_{zz}|, |\sigma_{xz}|, |\sigma_{yz}| \ll |\sigma_{xx}|, |\sigma_{yy}|, |\sigma_{xy}|$$

Thus the constitutive relations Eqs. See ., See ., and See . reduce to the plane stress approximation:

$$
\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -v & 0 \\ -v & 1 & 0 \\ 0 & 0 & 2(1+v) \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}
$$

which has the inverse:

$$
\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-v^2} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{(1-v)}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}
$$

and the additional conditions:

$$\varepsilon_{zz} \equiv -\frac{v}{E}(\sigma_{xx}+\sigma_{yy}) \; ; \; \gamma_{yz} \equiv 0 \; ; \; \gamma_{xz} \equiv 0$$

**Plane Strain**

The approximation Eq. See : :. is widely used for thin beam, plate and shell theories. However, in certain situations, the plate or shell wall consists of stiff reinforcement in the transverse direction. An example is the biological membrane. The appropriate approximation is then that the strain in the radial direction is small:

$$|\varepsilon_{zz}| \ll |\varepsilon_{xx}|, |\varepsilon_{yy}|$$

The three-dimensional equations Eqs. See .-See . then reduce to the plane strain relations:

$$
\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \frac{1+v}{E} \begin{bmatrix} 1-v & -v & 0 \\ -v & 1-v & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}
$$

which has the inverse:

$$
\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1-2v)(1+v)} \begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{(1-2v)}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}
$$

**Expansion for displacements**

The displacements may be expanded in a Taylor series in the z-direction:

$$u(x,y,z) = u(x,y,0) + z\frac{\partial}{\partial z}u(x,y,0) + \frac{z^2}{2}\frac{\partial^2}{\partial x^2}u(x,y,0) + \dots$$

$$v(x,y,z) = v(x,y,0) + z\frac{\partial}{\partial z}v(x,y,0) + \frac{z^2}{2}\frac{\partial^2}{\partial x^2}v(x,y,0) + \dots$$

$$w(x,y,z) = w(x,y,0) + z\frac{\partial}{\partial z}w(x,y,0) + \frac{z^2}{2}\frac{\partial^2}{\partial x^2}w(x,y,0) + \dots$$

So different levels of approximation are obtained by retaining a different number of terms in the expansion in the z-direction.

## Classical Theory (Kirchhoff)

The `classical' theory is obtained by retaining the linear terms in the in-plane displacement components and only the constant in the normal displacement:

$$u(x,y,z) \equiv u^0(x,y) + z\beta_x(x,y)$$

$$v(x,y,z) \equiv v^0(x,y) + z\beta_y(x,y)$$

$$w(x,y,z) \equiv w^0(x,y)$$

in which the superscript 0 denotes the displacement of the midsurface ($z = 0$), and the coefficients of the linear terms $\beta$x and $\beta$y can be interpreted as the rotations of the midsurface in the x- and y-directions. From the condition Eq. <u>See ; ;.</u> that the transverse shear strain components be small, the rotations of the midsurface are related to the rotations of the unit normal to the surface:

$$\frac{\partial w^0}{\partial x} = -\beta_x \; ; \; \frac{\partial w^0}{\partial y} = -\beta_y$$

Substituting Eqs. <u>See .</u>-<u>See ;.</u> into Eqs. <u>See ; ;.</u>,<u>See ; ;.</u> yields the in-plane strain components:

$$\varepsilon_{xx} = \frac{\partial u^0}{\partial x} + z\frac{\partial \beta_x}{\partial x} = \varepsilon^0_{xx} + z\kappa_x$$

$$\varepsilon_{yy} = \frac{\partial v^0}{\partial y} + z\frac{\partial \beta_y}{\partial y} = \varepsilon^0_{yy} + z\kappa_y$$

$$\gamma_{xy} = \frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x} + z\left(\frac{\partial \beta_y}{\partial x} + \frac{\partial \beta_x}{\partial y}\right) = \gamma^0_{xy} + 2z\kappa_{xy}$$

So each strain component has a constant term, corresponding to the midsurface strain, and a term varying with z, corresponding to the change in curvature of the midsurface. Because of Eq. <u>See ;.</u>, the

changes in curvature can be written in terms of the derivatives of the normal displacement:

$$\kappa_x = \frac{\partial^2 w^0}{\partial x^2} \; ; \quad \kappa_y = \frac{\partial^2 w^0}{\partial y^2} \; ; \quad \kappa_{xy} = -\frac{\partial}{\partial y}\frac{\partial}{\partial x} w^0$$

With Eqs. [See .](#)-[See .](#), the three-dimensional strain is approximated by quantities dependent only on the displacement of the midsurface, referred to as strain measures. Therefore, with the constitutive relation Eq. [See .](#) and the definitions Eqs. [See ; :.](#), [See :.](#), the stress resultants can be obtained in terms of the strain measures. For example, the x-components of force resultant is:

$$N_x = \int_{-t/2}^{t/2} \sigma_{xx}\,dz = \int_{-t/2}^{t/2} \frac{E}{1-v^2}[\varepsilon^0{}_{xx} + v\varepsilon^0{}_{yy} + z(\kappa_x + v\kappa_y)]\,dz$$

and the moment is:

$$M_x = \int_{-t/2}^{t/2} \sigma_{xx}z\,dz = \int_{-t/2}^{t/2} \frac{E}{1-v^2}[\varepsilon^0{}_{xx} + v\varepsilon^0{}_{yy} + z(\kappa_x + v\kappa_y)]z\,dz$$

So for the elastic properties E and $v$ independent of the thickness coordinate z, the integrations reduce substantially because the curvature terms drop out of the expression for the in-plane force resultant Eq. [See .](#) and the in-plane strains drop out of the expression for the bending moment resultant Eq. [See .](#):

$$N_x = \frac{Et}{1-v^2}[\varepsilon^0{}_{xx} + v\varepsilon^0{}_{yy}]$$

$$M_x = \frac{Et^3}{12(1-v^2)}[\kappa_x + v\kappa_y]$$

The complete system for the in-plane force resultants is:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \frac{Et}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{(1-v)}{2} \end{bmatrix}\begin{bmatrix} \varepsilon^0{}_{xx} \\ \varepsilon^0{}_{yy} \\ \gamma^0{}_{xy} \end{bmatrix}$$

with the inverse:

$$\begin{bmatrix} \varepsilon^0{}_{xx} \\ \varepsilon^0{}_{yy} \\ \gamma^0{}_{xy} \end{bmatrix} = \frac{1}{Et}\begin{bmatrix} 1 & -v & 0 \\ -v & 1 & 0 \\ 0 & 0 & 2(1+v) \end{bmatrix}\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix}$$

and for the bending resultants:

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \frac{Et^3}{12(1-v^2)} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & (1-v) \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$$

with the inverse:

$$\begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \frac{12}{Et^3} \begin{bmatrix} 1 & -v & 0 \\ -v & 1 & 0 \\ 0 & 0 & (1+v) \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix}$$

The bending stiffness coefficient is often denoted by the symbol D:

$$D = \frac{Et^3}{12(1-v^2)}$$

or alternatively as:

$$D = Etc^2$$

where c is a reduced thickness:

$$c = \frac{t}{\sqrt{12(1-v^2)}}$$

Some notes are the following:

(i) The shear strain $\gamma^0$ xy in Eq. <u>See</u> . is the total while the twist $\kappa$ xy is half. This is convenient since the twist is just the mixed second derivative of the displacement Eq. <u>See ; :</u>..

(ii) The transverse shear strains and stresses are assumed to be small (Eq. <u>See ; :</u>.); however, this does not mean that the transverse shear force resultants Qx and Qy should be set to zero. These are very significant in the equations of equilibrium (Eqs. <u>See</u> .-<u>See</u> .).

(iii) The reduction is shown for constant E and $v$ . However, the key is the explicit form for the dependence on z of the strains (Eqs. <u>See</u> .-<u>See</u> .). Thus the same procedure holds for general layered anisotropic plates, yielding the resultants in terms of the strain measures defined on the midsurface. Generally, however, the in-plane quantities Eq. <u>See</u> . and the bending quantities Eq. <u>See</u> . will be coupled. In the design of layered composite plates, a "balanced" design is often used, for which the bending and stretching are uncoupled, and furthermore, a "quasi-isotropic" layup of the fibers is used, for which Eqs. <u>See</u> . and <u>See</u> . are valid with effective values of E and $v$ .

The assumptions made are summarized as:

(i) The displacements of the plate are small in comparison to the plate's thickness, and the strains and midsurface slopes are much less than unity.

(ii) The displacements u and v vary linearly through the thickness of the cross section Eq. See . and Eq. See ., or in other words, "normals remain straight." This is analogous to the assumption that "plane sections remain plane" in elementary beam theory (Euler-Bernoulli beam theory). However, the plate kinematic model permits warping of an initially plane section, so that plane sections do not necessarily remain plane.

(iii) The normal stress $\sigma_{zz}$ is small in comparison with the in-plane components $\sigma_{xx}$, $\sigma_{yy}$ and $\sigma_{xy}$ . This yields the plane stress approximation Eq. See ..

(iv) The normal displacement w is essentially constant through the thickness Eq. See ..

(v) The transverse shear stresses $\sigma_{xz}$ and $\sigma_{yz}$ are small in comparison with the in-plane components $\sigma_{xx}$, $\sigma_{yy}$ and $\sigma_{xy}$ . Hence the transverse shear strains $\gamma_{xz}$ and $\gamma_{yz}$ are small and the slope of the mid-surface is the same as the rotation of the normals Eq. See ;.. In other words, "normals remain normal."

**Shear Deformation Theory (Mindlin-Reissner)**

Substantially better results for the thicker plates can be obtained by including the effect of transverse shear deformation. Consistent treatments were introduced by Reissner (1945)See Reissner E (1945). The effect of transverse shear deformation on the bending of elastic plates, J. Appl. Mech. 12 (2): 69-77.. and Mindlin (1951)See Mindlin RD (1951). Influence of rotatory inertia and shear on flexural motions of isotropic, elastic plates, J. Appl. Mech. 18 (1):31-38... Eqs. See .-See . remain valid. However, assumption (v) above, giving Eq. See ;., is not made. Instead, the transverse shearing strains are computed from Eqs. See ; ;.:

$$\gamma_{xz} = \beta_x + \frac{\partial w^0}{\partial x} \quad ; \quad \gamma_{yz} = \beta_y + \frac{\partial w^0}{\partial y}$$

and the transverse shearing force resultants from Eq. See ; ;. are:

$$Q_y = \int_{-t/2}^{t/2} \sigma_{yz} ds = \frac{1}{\mu} \int_{-t/2}^{t/2} G \gamma_{yz} ds = \frac{Gt}{\mu}\left(\beta_y + \frac{\partial w^0}{\partial y}\right) \quad ; \quad Q_x = \frac{Gt}{\mu}\left(\beta_x + \frac{\partial w^0}{\partial x}\right)$$

A more careful consideration of the three-dimensional shear stress distribution, yields the correction factor $\mu$ =6/5 (Reissner, 1945See Reissner E (1945). The effect of transverse shear deformation on the bending of elastic plates, J. Appl. Mech. 12 (2): 69-77..) to Eq. See ;.. The other relations Eqs. See ., See . remain the same as in the Kirchhoff theory. For the Kirchhoff theory, the unknown quantities are the displacement components u0, v0, and w0. For the shear deformation theory, the rotation components $\beta_x$ and $\beta_y$ are also unknown. For a fiber reinforced, composite plate it is usually the situation that the matrix is soft in comparison with the fibers. In this case the transverse shear stiffness is determined mainly by the matrix modulus, while the effective E and $\nu$ are determined by the fibers. If we keep the relation Eq. See ., then the reduced matrix modulus can be taken into account by an increased value of $\mu$ . Typically for composites, $5 < \mu < 50$.

**Higher Order Theories**

There are a variety of techniques for including additional terms in the expansion for the displacements, such as discussed by Lo, Christensen, and Wu (1977)See Lo KH, Christensen RM, and Wu EM (1977). A high-order theory of plate deformation, J. Appl. Mech, 44: 663-676.., and in countless subsequent publications. For the layered plate, however, the best approach is to use a "zig-zag" displacement field, in which the preceding shear deformation theory is used for each laminate, with the continuity of displacement enforced at each interface. When the elastic properties of the laminates are substantially different, this approach is necessary. For all such theories, the degree of complexity is substantially increased. For problems involving constraint conditions at the plate surface, the normal stress $\sigma_{zz}$ can be significant, and a more refined representation is required for the normal displacement w. (See, e.g., Essenburg, 1975See Essenburg, F (1975). On the significance of the inclusion of the effect of transverse normal strain in problems involving beams with surface constraints. J. App. Mech., 42: 127-132...)

## Stresses in Plates

After the solution for the resultants is obtained, it is necessary to determine the three-dimensional stresses. The relation is obtained by substituting the strain distributions Eqs. See .-See . into the stress-strain relations Eq. See ., and make use of the resultants Eqs. See ., See ., with the result:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{1}{t}\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} + \frac{12z}{t^3}\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix}$$

Within the framework of the present assumptions, two contributions to the in-plane stresses are identified:

(i) Direct stresses, which are proportional to the in-plane stress resultants $N_x$, $N_y$, and $N_{xy}$. and are uniformly distributed through the thickness.

(ii) Bending stresses, which are proportional to the moment resultants $M_x$, $M_y$, and $M_{xy}$, and vary linearly through the thickness.

Equation See . is the two-dimensional analog of the formula for the stress in a beam subjected to combined tension and bending. For extremely thin plates, or membranes, loads are carried predominantly by means of the direct stresses. Thus the direct stresses are also referred to as membrane stresses.

In addition to the three stress components given by Eq. See ., it may be required to know the values of the transverse shear stresses $\sigma_{xz}$ and $\sigma_{yz}$, as well as the normal stress $\sigma_{zz}$. Recall that these stresses are generally much smaller than $\sigma_{xx}$, $\sigma_{yy}$, and $\sigma_{xy}$. It is interesting that the smaller quantities require more effort to estimate. After substitution for $\sigma_{xx}$, $\sigma_{yy}$, and $\sigma_{xy}$ from Eq. See ., the first two equations Eqs See . and See . may be integrated with respect to z to obtain:

$$\sigma_{xz} = \sigma^0_{xz} - \frac{z}{t}\left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y}\right) - \frac{6z^2}{t^3}\left(\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y}\right)$$

$$\sigma_{yz} = \sigma^0_{yz} - \frac{z}{t}\left(\frac{\partial N_y}{\partial y} + \frac{\partial N_{yx}}{\partial x}\right) - \frac{6z^2}{t^3}\left(\frac{\partial M_y}{\partial y} + \frac{\partial M_{yx}}{\partial x}\right)$$

where $\sigma 0xz$ and $\sigma 0yz$ are integration constants. The equilibrium equations (Eqs. See ., See ., See ., and See .) can be used to rewrite Eqs. See ., See . as:

$$\sigma_{xz}(x,y,z) = \frac{3Q_x}{2t}\left[1 - \left(2\frac{z}{t}\right)^2\right] + R_x$$

$$\sigma_{yz}(x,y,z) = \frac{3Q_y}{2t}\left[1 - \left(2\frac{z}{t}\right)^2\right] + R_y$$

where the terms with zero average value are collected into Rx and Ry:

$$R_x = \sigma_{xz}\left(x,y,-\frac{t}{2}\right) - \frac{3m_y}{2t}\left[1 - \left(2\frac{z}{t}\right)^2\right] + \frac{1}{2}\left(1 + \frac{2z}{t}\right)p_x - \int_{-t/2}^{z} b_x dz$$

$$R_y = \sigma_{yz}\left(x,y,-\frac{t}{2}\right) + \frac{3m_x}{2t}\left[1 - \left(2\frac{z}{t}\right)^2\right] + \frac{1}{2}\left(1 + \frac{2z}{t}\right)p_y - \int_{-t/2}^{z} b_y dz$$

where the constants of integration have been chosen to satisfy the surface conditions at z = ±t/2. The significant term is that due to the transverse shear resultant, which causes a transverse shear stress that varies parabolically through the thickness and has a maximum at the midsurface z = 0.

The normal stress is calculated from the third equation of equilibrium Eq. See . in a similar fashion. The shear resultants Qx and Qy can be eliminated with Eq. See .. The result is completely independent of the resultants! So we have a "statically determinate" result of a stress component determined directly from the surface loading. The distribution that satisfies the conditions on the faces z = ±t/2 is:

$$\sigma_{zz}(x,y,z) = \sigma_{zz}\left(x,y,-\frac{t}{2}\right) + \frac{3}{4}\left[\frac{2}{3} + \frac{2z}{t} - \frac{1}{3}\left(\frac{2z}{t}\right)^3\right]p_z - \int_{-t/2}^{z} b_z dz - \int_{-t/2}^{z}\left(\frac{\partial R_x}{\partial x} + \frac{\partial R_y}{\partial y}\right)dz$$

The significant effect is the loading in the z-direction, which according to Eq. See . causes a normal stress $\sigma zz$ that varies as a cubic polynomial in z. Because the total integrals of Rx and Ry are zero, the last term in Eq. See . is zero at both faces.

The approximate kinematic conditions and the constitutive relations are based on the assumption that the stresses Eqs. See ., See ., and See . are small in comparison with Eq. See .. Therefore these can be calculated for a verification.

## Plate Differential Equations

The preceding equations can be reduced to partial differential equations of a standard form. Until this point the partial derivatives have been written fully; subsequently the following shorter notation will be used, in which the comma before the index indicates the partial derivative:

$$\frac{\partial F}{\partial x} = F_{,x}$$

The particular advantage of the linear solution for the homogeneous, isotropic plate, or the balanced composite plate, is that the in-plane and the out-of-plane bending problems are uncoupled.

**In-Plane Equation**

A convenient approach to the in-plane problem is to introduce the Airy stress function $\varphi$. In terms of the scalar function $\varphi = \varphi(x, y)$ the in-plane resultants are:

$$N_x = \varphi_{,yy} - \int_a^x p_x dx \ ; \ N_y = \varphi_{,xx} - \int_b^y p_y dy \ ; \ N_{xy} = -\varphi_{,xy}$$

It is easily verified that the relations Eq. See ; ;. satisfy identically the equations for in-plane equilibrium Eqs. See . and See .. the strains are then easily computed from Eq. See .:

$$\varepsilon_{xx}^0 = \frac{1}{Et}\left(\varphi_{,yy} - v\varphi_{,xx} - \int_a^x p_x dx + v\int_b^y p_y dy\right)$$

$$\varepsilon_{yy}^0 = \frac{1}{Et}\left(\varphi_{,xx} - v\varphi_{,yy} - \int_b^y p_y dy + v\int_a^x p_x dx\right)$$

$$\gamma_{xy}^0 = \frac{2(1+v)}{Et}\varphi_{,xy}$$

The function $\varphi$ is however, restricted since the strains must yield a consistent displacement distribution. The method for this is to consider the strain-displacement relations from Eqs. See .-See .:

$$\varepsilon_{xx}^0 = u_{,x}^0 \ ; \ \varepsilon_{yy}^0 = v_{,y}^0 \ ; \ \gamma_{xy}^0 = u_{,y}^0 + v_{,x}^0$$

For the strains prescribed, Eq. See ; ;. gives three equations for the two displacement components u and v. Consequently, there must be a condition on the strain components to permit a solution for the displacements. The condition, referred to as compatibility of strain, is:

$$\varepsilon_{xx,yy}^0 + \varepsilon_{yy,xx}^0 - \gamma_{xy,xy}^0 = 0$$

It is easily seen that substitution of the strains in terms of displacements Eq. See ; ;. satisfies Eq. See .. Now, substituting the strains in terms of the stress function Eqs. See .-See . into Eq. See . yields the equation for $\varphi$:

$$\Delta\Delta\varphi = \int_a^x (p_{x,yy} - vp_{x,xx})dx + \int_b^y (p_{y,xx} - vp_{y,yy})dy$$

in which $\Delta$ is the Laplacian, or harmonic operator:

$$\Delta\varphi = \varphi_{,xx} + \varphi_{,yy}$$

The differential operator $\Delta\Delta$ is often referred to as the biharmonic operator. Since the Laplacian can be interpreted as the divergence $\nabla$ of the gradient $\nabla$, the operators are also written as:

$$\Delta = \nabla \cdot \nabla = \nabla^2$$

$$\Delta^2 = \nabla^2 \nabla^2 = \nabla^4$$

**Bending Equation (Kirchhoff)**

The basic approximation that normals remain normal Eq. See :. gives the rotations in terms of the displacement of the midsurface. Equations See .-See . give the curvature change measures in terms of the rotations, and Eqs. See . give the moments in terms of the curvature changes. Therefore, the transverse shear Eq. See . can be written in terms of the displacement:

$$Q_x = m_y + M_{x,x} + M_{yx,y}$$

$$= m_y + D\left[(\beta_{x,x} + \nu\beta_{y,y})_{,x} + \frac{(1-\nu)}{2}(\beta_{x,y} + \beta_{x,y})_{,y}\right]$$

$$= m_y - D[(w_{,xx} + \nu w_{,yy})_{,x} + (1-\nu)w_{,xy,y}]$$

$$= m_y - D(w_{,xx} + w_{,yy})_{,x}$$

$$= m_y - D\Delta w_{,x}$$

For this reduction, the plate bending stiffness D is constant. Substituting Eq. See . and the similar equation for the y-component into the equation of equilibrium Eq. See . yields the final equation:

$$(m_y - D\Delta w_{,x})_{,x} + (-m_x - D\Delta w_{,y})_{,y} + p_z = 0$$

which can be rewritten as:

$$D\Delta\Delta w = p_z + m_{y,x} - m_{x,y}$$

Thus both the in-plane and the out-of-plane problems reduce to the biharmonic operator.

**Plate Bending with Transverse Shear Deformation (Mindlin-Reissner)**

For the consideration of the transverse shear deformation, the components of the rotation of the normal $\beta$ x and $\beta$ y can be rewritten in terms of two functions $\Phi$ and $\Psi$:

$$\beta_x = -\Phi_{,x} + \Psi_{,y}$$

$$\beta_y = -\Phi_{,y} - \Psi_{,x}$$

In terms of vector analysis, we are using the standard procedure of writing the rotation vector as the sum of the gradient of the scalar $\Phi$ and the curl of a vector with the z-component $\Psi$. Since we are dealing with a linear system of equations, the equations for each component can be handled separately. It is also necessary to split the load terms into two parts:

$$m_y = m_{G,x} + m_{C,y}$$

$$m_x = -m_{G,y} + m_{C,x}$$

where mG and mC are two scalar functions. Note that the curl part mC drops out of the equation See .:

**Gradient part**

The substitution of the $\Phi$ part of Eqs. See , and See , into Eq. See , yields instead of Eq. See . the results:

$$M_x = -D(\Phi_{,xx} + v\Phi_{,yy})$$

$$M_y = -D(\Phi_{,yy} + v\Phi_{,xx})$$

$$M_{xy} = -D(1 - v)\Phi_{,xy}$$

$$Q_x = m_{G,x} - D\Delta\Phi_{,x}$$

$$Q_y = m_{G,y} - D\Delta\Phi_{,y}$$

So, instead of Eq. See . the result is exactly the same equation for $\Phi$:

$$D\Delta\Delta\Phi = p_z + m_{y,x} - m_{x,y} = p_z + \Delta m_G$$

The equation for the normal displacement w comes from Eq. See ;.:

$$w_{,x} = \Phi_{,x} + \frac{\mu Q_x}{Gt}$$

so from Eq. See .:

$$w_{,x} = \Phi_{,x} + \frac{\mu}{Gt}(m_{G,x} - D\Delta\Phi_{,x})$$

$$w_{,y} = \Phi_{,y} + \frac{\mu}{Gt}(m_{G,y} - D\Delta\Phi_{,y})$$

so the integral of each is the same:

$$w = \Phi + \frac{\mu}{Gt}(m_G - D\Delta\Phi)$$

By Eq. <u>See</u>. the effect of the transverse shear deformation can be estimated. The Laplacian term has the coefficient $\delta$G2, where:

$$\delta_G = \sqrt{\frac{\mu}{Gt}D} = t\sqrt{\frac{\mu}{6(1-v^2)}}$$

If the solution $\Phi$ of Eq. <u>See</u>. varies significantly over the distance L, then each derivative has the magnitude:

$$|\Phi_{,x}|, |\Phi_{,x}| < \frac{|\Phi|}{L}$$

The Laplacian term in Eq. <u>See</u>. is small for $L \gg \delta$G, e.g., for a deformation with the variation distance L that is large in comparison with the plate thickness. A similar argument can be made for the mG term. When both terms are neglected in Eq. <u>See</u>., we have the Kirchhoff result Eq. <u>See</u>. with $\Phi =$ w.

**Curl Part**

the component $\Psi$ in Eqs. <u>See</u>. and <u>See</u>. leads to a rather different type of equation. The bending moment and transverse shears components are:

$$M_x = (1-v)D\Psi_{,xy}$$

$$M_y = -(1-v)D\Psi_{,xy}$$

$$M_{xy} = \frac{1}{2}D(1-v)(\Psi_{,yy} - \Psi_{,xx})$$

$$Q_x = \left[D\frac{(1-v)}{2}\Delta\Psi + m_C\right]_{,y}$$

$$Q_y = -\left[D\frac{(1-v)}{2}\Delta\Psi + m_C\right]_{,x}$$

With these shear components, the equation of transverse force equilibrium Eq. <u>See</u>. is identically satisfied, and the derivatives of the normal displacement are:

$$w_{,x} = \left[-\Psi + \frac{\mu}{Gt}\left(D\frac{(1-v)}{2}\Delta\Psi + m_C\right)\right]_{,y}$$

$$w_{,y} = -\left[-\Psi + \frac{\mu}{Gt}\left(D\frac{(1-v)}{2}\Delta\Psi + m_C\right)\right]_{,x}$$

So, the only solution for these two equations is for w to be identically zero, and for $\Psi$ to satisfy the

partial differential equation:

$$-\frac{\mu}{Gt}D\frac{(1-v)}{2}\Delta\Psi+\Psi=\frac{\mu}{Gt}m_C$$

The solution of this equation consists of a particular solution, due to the nonhomogeneous load term on the right-hand side, and a complementary solution of the homogeneous equation, (with zero right-hand side). The coefficient of the derivative terms is $\delta C2$, where:

$$\delta_C = \sqrt{\frac{\mu}{Gt}D\frac{(1-v)}{2}} = t\sqrt{\frac{\mu}{12(1-v^2)}}$$

So $\delta C$ is equal to a length less than the thickness. For a plate with lateral dimensions large in comparison with the thickness, this is called a singular perturbation problem. The solution of the homogeneous equation consists of terms which decrease exponentially within the distance $\delta C$ from the boundary of the plate, and so are referred to as boundary layer solutions. Therefore, the effect of this extra work in including the transverse shear deformation consists of a little correction to the displacement w in the potential solution Eq. See . and a boundary layer correction within a thickness of the boundary from the curl part Eq. See .. Consequently, an excellent understanding of the general behavior of the plate can be obtained by considering only the reduced Kirchhoff theory.

### Stretching of Normal

Equation See . provides the solution for the distribution of the stress in the normal direction in terms of the surface loading. However, insight into the effects of higher order can be obtained by considering the stretching of the normal. Instead of Eq. See .-See ., we add a term to the normal displacement:

$$u(x,y,z) \equiv u^0(x,y)+z\beta_x(x,y)$$

$$v(x,y,z) \equiv v^0(x,y)+z\beta_y(x,y)$$

$$w(x,y,z) \equiv w^0(x,y)+z\beta_z(x,y)$$

The strains from Eqs. See ; ;. and See ; ;. are:

$$\varepsilon_{zz} = w_{,z} = \beta_z$$

$$\gamma_{xz} = w_{,x}+u_{,z} = z\beta_{z,x}+\beta_x$$

$$\gamma_{yz} = w_{,y}+v_{,z} = z\beta_{z,y}+\beta_y$$

which give the resultants from Eqs. See ; ;.:

$$M_{xz} = \int_{-t/2}^{t/2} \sigma_{xz} z \, dz = \frac{Gt^3}{12}\beta_{z,x}$$

$$M_{yz} = \int_{-t/2}^{t/2} \sigma_{yz} z \, dz = \frac{Gt^3}{12}\beta_{z,y}$$

$$N_z = \int_{-t/2}^{t/2} \sigma_{zz} \, dz = Et\beta_z + \nu(N_x + N_y)$$

Thus Eq. <u>See</u> . yields the equation for $\beta$ z:

$$-\frac{Gt^3}{12Et}\Delta\beta_z + \beta_z = \frac{1}{Et}[m_{z2} - \nu(N_x + N_y)]$$

which is a second order equation for the average normal strain $\beta$ z. The coefficient of the Laplacian operator is $\delta$ z2, where:

$$\delta_z = \frac{t}{\sqrt{12}}$$

Once again, for a solution with the variation on the distance L $\gg$ $\delta$ z, the Laplacian term is small and the average strain is:

$$\beta_z \cong \frac{1}{Et}[m_{z2} - \nu(N_x + N_y)]$$

which is the condition that the normal stress be small. The solution of Eq. <u>See</u> . with zero right-hand side, however, permits the prescription of the normal strain on the boundary. This is again a boundary layer effect, with the decay distance $\delta$ z. This decay distance is about the same as for the transverse shear layer Eq. <u>See</u> . Therefore, in general, the solution for the plate separates into global effects described by the classical Kirchhoff theory and three dimensional effects that are restricted to the region of about one thickness near the boundaries.

## Boundary Conditions

The boundary conditions are perhaps most easily understood for the rectangular, three-dimensional body in Fig. <u>See Stress components acting on the surfaces of a three-dimensional element in cartesian coordinates..</u>. For the face with outward normal in the positive x-direction, the tractions, or components of stress, that can be prescribed are $\sigma$ xx, $\sigma$ xy, and $\sigma$ xz. The displacements in the directions of the stress components are u, v, and w. So the differential work that the boundary tractions perform on the body is the product of the stress components and the corresponding differential displacements. Therefore, the rate of work on the face is:

$$\dot{W} = \int_{y_m}^{y_M}\int_{-t/2}^{t/2} [\sigma_{xx}\dot{u} + \sigma_{xy}\dot{v} + \sigma_{xz}\dot{w}]\,dz\,dy$$

where the dots denote the time derivative. (Alternately, the dots could denote the virtual displacements.) With the approximation for the displacements Eq. See .-See ., the expression for the work becomes:

$$\dot{W} = \int_{y_w}^{y_w} \int_{-t/2}^{t/2} [\sigma_{xx}(\dot{u}^0 + z\dot{\beta}_x) + \sigma_{xy}(\dot{v}^0 + z\dot{\beta}_y) + \sigma_{xz}(\dot{w}^0 + z\dot{\beta}_z)] dz\, dy$$

which from the definitions of the resultants Eqs. See ; ;. and See ;. reduces to the plate quantities:

$$\dot{W} = \int_{y_w}^{y_w} [N_x \dot{u}^0 + M_x \dot{\beta}_x + N_{xy} \dot{v}^0 + M_{xy} \dot{\beta}_y + Q_x \dot{w}^0 + N_z \dot{\beta}_z] dy$$

Hence, the edge tractions on the face of the three-dimensional body are reduced to the plate resultants which can be prescribed on the edge of the plate:

Nx, Nxy, Mz, Mx, Mxy, Qx

and the corresponding displacement and rotation quantities:

u0, v0, $\beta$z, $\beta$x, $\beta$y, w0

For boundary conditions, either the force quantities Eq. See Nx, Nxy, Mz, Mx, Mxy, Qx., or the displacement quantities Eq. See u0, v0, βz, βx, βy, w0., may be prescribed on the edge. Another possibility is to prescribe a linear combination of the force and displacement quantities which includes a mix of some displacement and some force quantities. For the isotropic plate, the in-plane and bending problems are uncoupled.

## Isotropic Plate In-Plane

Equation See . governs the in-plane behavior of the plate, also referred to as the plane stress problem. This is a fourth order partial differential equation for the stress function $\varphi$ . In the classification of partial differential equations, this is referred to as elliptic. Consequently, exactly two scalar boundary conditions must be prescribed at each point of the boundary. The two force quantities are obviously:

Nx, Nxy,

and the two corresponding displacement quantities are:

u0, v0

From the relation of the stress resultants and the stress function Eq. See ; ;., prescribing the resultants Eq. See Nx, Nxy,. is equivalent to prescribing the value of $\varphi$ and its normal derivative on each point of the boundary. This convenient treatment for prescribed forces on the boundary is an advantage of the stress function approach. Displacements require more effort, since the strain displacement relations Eq. See ; ;. must be integrated, and no simple relation between the stress function and the displacement components can be found. However, a combination of displacements can be considered, namely the strain along the boundary and the change in curvature of the boundary. These are quantities that are independent of rigid body translation and rotation. For the positive x- edge, the strain along the edge is:

$$\overset{0}{\varepsilon}_{yy} = v_{,y} = \frac{1}{Et}\left(\varphi_{,xx} - v\varphi_{,yy} - \int_b^y p_y \, dy + v\int_a^x p_x \, dx\right)$$

and the curvature change of the edge is:

$$\kappa_g = -\overset{0}{u}_{,yy} = -(\overset{0}{u}_{,y} + \overset{0}{v}_{,x})_{,y} + (\overset{0}{v}_{,y})_{,x} = -\gamma_{xy,y} + \overset{0}{\varepsilon}_{yy,x}$$

Writing the strain quantities in terms of the stress function yields:

$$\kappa_g = \frac{1}{Et}\left[\Delta\varphi_{,x} + (1+v)\varphi_{,xyy} - \int_b^y p_{y,x} \, dy + vp_x\right]$$

Thus the derivatives of the displacements along the boundary yield results in terms of the derivatives of the stress function, which is convenient for many problems.

The effect of stretching of the normal can be added. Equation <u>See .</u> is a second order equation for $\beta z$, once the resultants Nx and Ny are known. Thus one scalar condition is required on each point of the boundary, either the average strain $\beta z$ or the "resultant" Mxz in terms of the derivative of $\beta z$ given by Eq. <u>See .</u>.

**Isotropic Plate Bending with Mindlin-Reissner Theory**

After deleting the in-plane quantities, three force and three displacement quantities remain to be satisfied Eqs. <u>See Nx, Nxy, Mz, Mx, Mxy, Qx.</u> and <u>See u0, v0, βz, βx, βy, w0.</u>. The gradient part of plate bending Eq. <u>See .</u> is a fourth order equation and the curl part Eq. <u>See .</u> is a second order equation. So together, three conditions on each point of the boundary can be prescribed. The in-plane resultants and the stretching, discussed in the previous section, can be handled separately. It is unfortunate that the gradient and curl parts of the bending problem are mixed together. The displacement quantities that can be prescribed on the positive x-edge are repeated from Eqs. <u>See .</u>, <u>See .</u>, and <u>See .</u>:

$$\beta_x = -\Phi_{,x} + \Psi_{,y}$$

$$\beta_y = -\Phi_{,y} - \Psi_{,x}$$

$$w = \Phi + \frac{\mu}{Gt}(m_G - D\Delta\Phi)$$

and the corresponding resultants are from Eqs. <u>See .</u>, <u>See .</u>, <u>See .</u>, <u>See .</u>, <u>See .</u>, and <u>See .</u>:

$$M_x = -D[\Phi_{,xx} + v\Phi_{,yy} - (1-v)\Psi_{,xy}]$$

$$M_{xy} = -D(1-v)\left[\Phi_{,xy} - \frac{1}{2}(\Psi_{,yy} - \Psi_{,xx})\right]$$

$$Q_x = m_{G,x} + m_{C,y} - D\Delta\left[\Phi_{,x} - \frac{(1-v)}{2}\Psi_{,y}\right]$$

Apparently no exact separation of the $\Phi$ and $\Psi$ contributions can be made. However, because of the boundary layer nature of the $\Psi$ contribution, an approximate uncoupling is possible. For a computational finite element approach, a separation of the problem into the $\Phi$ and $\Psi$ contributions is not made, and the formulation in terms of the basic quantities Eq. See . is straightforward.

**Isotropic Plate Bending for Kirchhoff Theory**

The preceding shear deformation theory is satisfying from several aspects. However, it is also more elaborate than is necessary for the general problem of plate bending. The simpler classical theory of Kirchhoff (1850)See Kirchhoff G (1850). Über das Gleichgewicht und die Bewegung einer elastischen Scheibe, Journal für reine und angewandte Mathematik, 40: 51-88.. is widely used. The biharmonic Eq. See . with the displacement w governs the problem. This is a fourth order equation, so that two independent quantities must be prescribed at each point of the boundary. A problem emerges, however, since the three displacement quantities (the displacement w, and the two rotation components $\beta$x and $\beta$y) or the three force quantities (Qx, Mx, and Mxy) can be prescribed in the preceding shear deformation theory. the proper reduction to only two boundary conditions can be obtained from the work on the edge Eq. See . For the Kirchhoff theory the rotation components are related directly to the derivatives of the normal displacement Eq. See ;.. Thus the edge work rate for the bending problem becomes:

$$\dot{W} = \int_{y_{\infty}}^{y_0} [M_x \dot{\beta}_x + M_{xy} \dot{\beta}_y + Q_x \dot{w}] dy$$

$$= \int_{y_{\infty}}^{y_0} [-M_x \dot{w}_{,x} - M_{xy} \dot{w}_{,y} + Q_x \dot{w}] dy$$

However, the second term may be integrated by parts, giving:

$$= (-M_{xy} \dot{w})\Big|_{y_{\infty}}^{y_0} + \int_{y_{\infty}}^{y_0} [-M_x \dot{w}_{,x} + (M_{xy,y} + Q_x) \dot{w}] dy$$

Thus the work done by the normal displacement w depends on the magnitude of the combined term Vx = Qx + Mxy,y. This represents an effective transverse shear force on the edge. Thus the Kirchhoff theory uses the force boundary quantities of only:

Mx and Vx = Qx + Mxy,y

and the corresponding displacement and rotation quantities:

$\beta$x = -w,x and w

Writing the boundary quantities See Mx and Vx = Qx + Mxy,y. in terms of the displacement gives:

$$M_x = -D(w_{,xx} + \nu w_{,yy})$$

$$V_x = m_{G,x} + m_{C,y} - D(\Delta w_{,x} + (1 - \nu) w_{,xyy})$$

So it is interesting that without surface loads, the in-plane stress problem, with the boundary conditions on the strain See . and curvature change See ., is exactly analogous to the Kirchoff plate bending problem, with the boundary conditions on moment Mx See . and effective transverse shear Vx. See ., except for a change in the sign of Poisson's ratio. Similarly, the in-plane problem, with boundary conditions on both stress components, is analogous to the bending problem of prescribed displacement and rotation.

The extra term in the work Eq. See . from the integration is of considerable importance. This shows that the twisting moments Mxy at the corners y = y1 and y = y2 has the effect of a concentrated force. This is called the corner force. For the rectangular plate, There are also contribution from the edges x = x1 and x2. So the total corner force is equal to 2Mxy and, for positive Mxy, acts in the negative z-direction on the corners (x1, y1) and (x2, y2). On the corners (x1, y2) and (x2, y1) the force 2Mxy is in the positive z-direction, as shown in See Effective resultants from Kirchhoff theory acting on the rectangular plate...
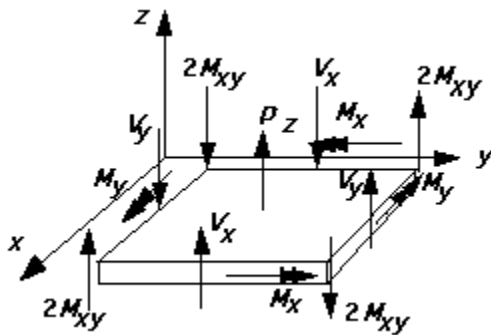


**FIGURE 1.6. Effective resultants from Kirchhoff theory acting on the rectangular plate.**

## Cylindrical Bending of Plates

The plate equations are two-dimensional, in the x-y plane. However, it is of value to consider the situation for which the solution depends on only one variable, that we take to be the x-coordinate. The displacement in the y-direction is taken as identically zero. Thus the exact solution for the three-dimensional body reduces to the plane strain problem in the x-z plane. The plate equation reduces to a one-dimensional equation in the x-coordinate, exactly that of a straight beam, but with the plate stiffness term which has dependence on Poisson's ratio. The following problems are of interest because an exact solution of the plane strain equation can be compared with the approximate plate solution. For convenience, the plate equations necessary for the cylindrical bending are summarized here. The equilibrium equations See . and See . reduce to:

$$Q_{x,x} = -p_z$$

$$M_{x,x} = Q_x - m_y$$

and the moment-curvature relations See . give:

$$M_x = D\kappa_x$$

$$M_y = \nu D\kappa_x$$

The curvature-rotation relation See . is:

$$\kappa_x = \beta_{x,x}$$

and the rotation- displacement relation from the Kirchhoff theory See ;. is:

$$w^0_{,x} = -\beta_x$$

while that from the Mindlin-Reissner theory See ;. is:

$$w^0_{,x} = -\beta_x + \frac{\mu Q_x}{Gt}$$

For statically determinate problems, Qx and Mx are known at some point. In this case, the shear diagram is the integral of See . and the moment diagram is the integral of See .. The rotation is obtained from the integral of See . and the displacement from the integral of See ., or See . for more accuracy. For statically indeterminate problems, the boundary conditions may be in terms of the displacement and rotation. The elimination of the stress resultants in the system of equations leads to the equations for Mindlin-Reissner theory:

$$(D\beta_{x,x})_{,xx} = -p_z - m_{y,x}$$

$$w_{,x} = -\beta_x + \frac{\mu}{Gt}[(D\beta_{x,x})_{,x} + m_y]$$

while for the Kirchhoff theory the final equation reduces to:

$$(Dw_{,xx})_{,xx} = p_z + m_{y,x}$$

For constant bending stiffness D, this is the same as See . with the dependence on the y-coordinate removed.

## Pure Bending

This is the most simple problem in beam and plate bending.

In See Pure bending of plate.. is a sketch of a plate clamped to a wall at x = - L and loaded at the end x = 0 by a moment ML. The z-coordinate is chosen in the downward direction, so that the positive moment from See ;. is in the direction corresponding to tension in the x-direction at positive values of z.



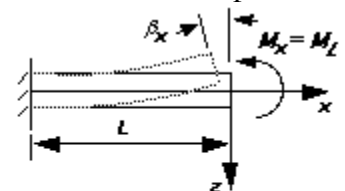**Plate Solution**

For zero transverse loading pz, zero distributed moment my, and zero end shear Qx, the solution of <u>See</u> . is Qx identically zero, and from <u>See</u> . that Mx is constant. Thus:

Figure 2.1 Pure bending of plate.

$$M_x \equiv M_{\bar{z}} = D\kappa_{\bar{x}} = D\beta_{x,x}$$

In addition, <u>See</u> . shows that the moment in the y-direction is nonzero:

$$M_y = \nu D\kappa_{\bar{x}}$$

Therefore, the stress components from <u>See</u> ., <u>See</u> ., and <u>See</u> . are:

$$\sigma_{xx} = \frac{12M_{\bar{z}}}{t^2} z$$

$$\sigma_{xz} = \sigma_{zz} = 0$$

Since the transverse shear Qx is zero, the rotation is related to the derivative of the displacement in both the Kirchhoff theory <u>See</u> . and in the shear deformation theory <u>See</u> .. Thus the integral of <u>See</u> . gives the rotation:

$$\beta_x = \frac{M_{\bar{z}}}{D}(x+L) = -w^0{}_{,x}$$

with the constant of integration chosen so that the rotation is zero at the clamped end. So <u>See</u> . gives the tangential displacement:

$$u = \frac{M_{\bar{z}}}{D}(x+L)z$$

and the integration of <u>See</u> . gives the transverse displacement:

$$w^0 = \frac{M_{\bar{z}}}{2D}(x+L)^2$$

**Exact Solution**

The body in <u>See Pure bending of plate.</u>. can be considered as a plane strain in the x-z plane. So the <u>See</u> . is valid with the y-coordinate replaced with z. The exact solution has the stress function:

$$\varphi = \frac{2M_{\bar{z}}}{t^3} z^3$$

which yields exactly the stress components <u>See</u> . and <u>See</u> .. The displacements are:

$$u = \frac{M_{\bar{z}} L}{D}\left(1 + \frac{x}{L}\right)z$$

$$w = -\frac{M_L L}{2D}\left[\left(1+\frac{x}{L}\right)^2 + \frac{t^2}{L^2}\frac{v}{4(1+v)}\left(2\frac{z}{t}\right)^2\right]$$

Therefore in the case of pure bending, the plate solution gives the exact stress distribution and the exact tangential displacement u. The exact transverse displacement w See . agrees with the plate solution on the midsurface z = 0 See ., but changes with the distance from the midsurface. Note also that the exact condition of zero w at every point of the clamped end is not satisfied by this exact solution. (In fact, to satisfy this condition a singular stress state is required at the clamped corners.) The maximum error at the end x = 0 is however of the magnitude:

$$\text{Error} = \frac{v}{4(1+v)}\left(\frac{t}{L}\right)^2$$

So for a beam longer than the thickness, the additional displacement in See . is rather small.

## Pure Shear Loading

Now we consider the next most basic problem in beam and plate bending. In See Pure shear loading of plate.. is a sketch of a plate clamped to a wall at x = - L and loaded at the end x = 0 by a shear force QL. The solution is that the shear is constant but that the moment varies along the beam.
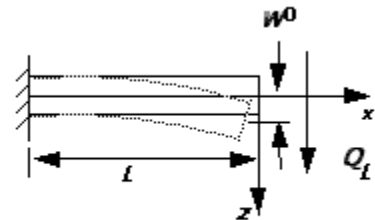


Figure 2.2 Pure shear loading of plate.

### Plate Solution

For zero pz and my, and nonzero shear at z = 0, See . and See . are satisfied by a constant transverse shear and a linearly varying bending moment:

$$Q_x \equiv Q_L$$

$$M_x = D\kappa_x = D\beta_{x,x} = Q_L x$$

The constant of integration is chosen to give the moment Mx that is zero at the end x = 0. As before, See . shows that the moment in the y-direction is nonzero:

$$M_y = vD\kappa_x$$

Therefore, the tangential stress component is from See .:

$$\sigma_{xx} = \frac{12Q_L}{t^3}zx$$

while the transverse shear stress is computed from See .:

$$\sigma_{xz} = \frac{Q_z}{t} \frac{3}{2} \left[ 1 - \left( \frac{2z}{t} \right)^2 \right]$$

and the normal stress is zero from See .:

$$\sigma_{zz} = 0$$

The rotation is the integral of See ., with the constant of integration chosen to give zero rotation at x = - L:

$$\beta_x = \frac{Q_z L^2}{2D} \left( 1 - \frac{x^2}{L^2} \right)$$

so See . gives the tangential displacement:

$$u = \frac{Q_z L^2}{2D} \left( 1 - \frac{x^2}{L^2} \right) z$$

Another integration gives the transverse displacement. For the Kirchhoff theory See :. gives:

$$w^0 = \frac{Q_z L^3}{3D} \left( \frac{x}{L} + 1 \right) \left( 1 + \frac{x}{2L} - \frac{x^2}{2L^2} \right)$$

while the Mindlin-Reissner theory See :. gives an additional displacement contribution:

$$w^0 = \frac{Q_z L^3}{3D} \left( \frac{x}{L} + 1 \right) \left( 1 + \frac{x}{2L} - \frac{x^2}{2L^2} + \frac{3\mu D}{G t L^2} \right)$$

Writing out the additional contribution gives:

$$\frac{3\mu D}{G t L^2} = \frac{\mu t^2}{2(1 - \nu) L^2}$$

So for the isotropic plate, with $\mu = 6/5$, the shear deformation correction is small when the length is greater than the thickness.

**Exact Solution**

The body in See Pure shear loading of plate.. can be considered as a plane strain in the x-z plane. So See . is valid with the y-coordinate replaced with z. The exact solution has the stress function:

$$\varphi = \frac{12 Q_z}{t^2} \left[ z^2 - 3 \left( \frac{t}{2} \right)^2 \right] xz$$

which yields exactly the stress components See .-See .. The displacement in the tangential direction is:

$$u = \frac{Q_z L^2}{2D}\left[\frac{x^2}{L^2} - 1 + \frac{(2-v)t^2}{4(1-v)L^2}\left[\frac{1}{5} - \frac{1}{3}\left(\frac{2z}{t}\right)^2\right]\right]z$$

A two-dimensional effect appears, in the additional z-dependence in the exact solution compared with the plate solution See .. Since the behavior is cubic in the z-coordinate, this is referred to as warping of the normal. The effect is, however, also small when the length L is greater than the thickness t. However, using See . to compute an averaged tangential displacement u,defined by:

$$\bar{u}(x) = \frac{1}{M_x}\int_{-\frac{t}{2}}^{\frac{t}{2}} u(x,z)\sigma_{xx}(x,z)dz$$

yields exactly the same as the plate solution See .. The exact transverse displacement is:

$$w = -\frac{Q_z L^3}{3D}\left[\left(\frac{x}{L}+1\right)\left(1+\frac{x}{2L}-\frac{x^2}{2L^2}\right) - \frac{t^2 x}{L^3}\left[\left(\frac{2z}{t}\right)^2 - \frac{1}{5}\right]\frac{3v}{8(1-v)}\right]$$

However, the averaged displacement w, defined by:

$$\bar{w}(x) = \frac{1}{Q_x}\int_{-\frac{t}{2}}^{\frac{t}{2}} w(x,z)\sigma_{xz}(x,z)dz$$

yields:

$$\bar{w} = \frac{Q_z L^3}{3D}\left(\frac{x}{L}+1\right)\left(1+\frac{x}{2L}-\frac{x^2}{2L^2}+\frac{t^2}{L^2}\frac{3}{5(1-v)}\right)$$

which is exactly the plate solution See . for $\mu = 6/5$, which is a justification for the use of this value of $\mu$. So for the pure shear problem, as well as the pure bending problem, the difference between the Kirchhoff theory, the Mindlin-Reissner theory and the exact solutions is in terms that have the size of t2/L2. Also to note is the ratio of the maximum of the transverse shear stress See . to the maximum of the tangential stress See .:

$$\frac{|\sigma_{xy}|_{max}}{|\sigma_{xx}|_{max}} = \frac{t}{4L}$$

So for the thin plate, the transverse stress is small in comparison with the tangential. This is the justification for the Kirchhoff approximation See ; :. that the transverse shear strain in small in comparison with the tangential.

## Transverse Loading

Now we consider a problem with a uniform transverse loading of the plate with the nonzero magnitude pz. In See Uniform loading of plate.. is the sketch of the plate clamped to a wall at x = - L and free
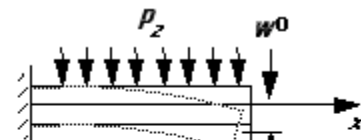
at the end $x = 0$. The solution has a resultant shear that varies linearly and a resultant moment that varies quadratically along the beam. The exact solution has a nonzero transverse normal stress $\sigma zz$ that is of interest to compare with the plate approximation. To obtain a simple exact solution, however, a special distribution of body force is used:
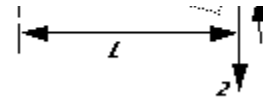
**Figure 2.3 Uniform loading of plate.**

$$b_z = \frac{p_z}{t^3}\frac{12(1-\nu)}{\nu}z^2$$

with the surface tractions:

$$\sigma_{zz}\left(x,\frac{t}{2}\right) = -p_z\left(\frac{1}{2\nu}-1\right)$$

$$\sigma_{zz}\left(x,-\frac{t}{2}\right) = p_z\left(\frac{1}{2\nu}-1\right)$$

It can be verified that substituting See .-See . into See . yields pz, the resultant force per unit length of the midsurface. The dependence of the body force and tractions on Poisson's ratio is strange, but it works.

**Plate Solution**

The equilibrium equations See . and See . are satisfied by the distribution of transverse shear and bending moment:

$$Q_x = -p_z x$$

$$M_x = D\kappa_x = D\beta_{x,x} = \frac{p_z x^2}{2}$$

The constants of integration are chosen so that both the moment Mx and the transverse shear are zero at the end $x = 0$. As before, See . shows that the moment in the y-direction is nonzero:

$$M_y = \nu D\kappa_x$$

Therefore, the tangential stress component is from See .:

$$\sigma_{xx} = -\frac{6p_z}{t^3}zx^2$$

while the transverse shear stress is computed from See .:

$$\sigma_{xz} = -\frac{3p_z}{2t}\left[1-\left(\frac{2z}{t}\right)^2\right]x$$

For this case, the normal stress from See . is not equal to zero:

$$\sigma_{zz} = -\frac{p_z}{4}\left(\frac{2z}{t}\right)\left[\left(\frac{2}{v}-1\right)\left(\frac{2z}{t}\right)^2 - 3\right]$$

The rotation is the integral of See ., with the constant of integration chosen to give zero rotation at x = - L:

$$\beta_x = -\frac{p_z L^3}{6D}\left(1 + \frac{x^3}{L^3}\right)$$

so See . gives the tangential displacement:

$$u = -\frac{p_z L^3}{6D}\left(1 + \frac{x^3}{L^3}\right)z$$

Another integration gives the transverse displacement. For the Kirchhoff theory See :. gives:

$$w^0 = \frac{p_z L^4}{8D}\left(\frac{x}{L}+1\right)\left[1 + \frac{x}{3L} - \frac{x^2}{3L^2} + \frac{x^3}{3L^3}\right]$$

while the Mindlin-Reissner theory See :. gives an additional displacement contribution, again of the relative size of t2/L2:

$$w^0 = \frac{p_z L^4}{8D}\left(\frac{x}{L}+1\right)\left[1 + \frac{x}{3L} - \frac{x^2}{3L^2} + \frac{x^3}{3L^3} + \frac{t^2}{L^2}\frac{2\mu}{3(1-v)}\left(1-\frac{x}{L}\right)\right]$$

## Exact Solution

The body in See Uniform loading of plate.. with the body force See . and surface tractions See . and See . has the plane strain solution given by the stress function:

$$\varphi = -\frac{p_z}{t^3}\left[z^2 - 3\left(\frac{t}{2}\right)^2\right]x^2 z$$

which yields exactly the stress components See .-See .. The averaged tangential displacement u, defined by See . yields the result:

$$\bar{u} = -\frac{p_z L^3}{6D}\left[1 + \frac{x^3}{L^3} - \frac{3(1-3v)t^2}{10(1-v)L^2}\left(1+\frac{x}{L}\right)\right]z$$

Thus for this problem, the averaged tangential displacement differs from the plate result See . by terms that are the size of t2/L2. Note that the difference is zero for $v = 1/3$, a typical value for many materials. The total displacement in the tangential direction is:

$$u = \bar{u} + \frac{p_z L^3}{6D}\left[\frac{t^2}{L^2}\frac{(2-v)x}{24(1-v)L}\left[\frac{3}{5} - \left(\frac{2z}{t}\right)^2\right]\right]z$$

which has the cubic variation in the z-direction. The averaged displacement in the z-direction w, defined by See . yields:

$$\bar{w} = \frac{p_z L^4}{8D}\left(\frac{x}{L}+1\right)\left(1 + \frac{x}{3L} - \frac{x^2}{3L^2} + \frac{x^3}{3L^3} + \frac{t^2}{L^2}\frac{1}{5(1-v)}\left[(5-3v)\left(1-\frac{x}{L}\right) - 2(1-3v)\right]\right)$$

Which is exactly the plate solution See . for $\mu = 6/5$, when $v = 1/3$. The exact transverse displacement has additional terms:

$$w = \bar{w} + \frac{p_z L^4}{8D}\left\{\frac{t^2}{L^2}\frac{v}{2(1-v)}\frac{x^2}{L^2}\left[\left(\frac{2z}{t}\right)^2 - \frac{1}{5}\right]\right\}$$

$$+ \frac{p_z L^4}{8D}\left\{\frac{t^4}{16L^4}\left[\frac{1}{3}\left(\frac{2}{v}-1\right)\left\{\left(\frac{2z}{t}\right)^4 - \frac{3}{35}\right\} - 2\left\{\left(\frac{2z}{t}\right)^2 - \frac{1}{5}\right\}\right]\right\}$$

So for this particular transverse load problem, as well as the pure bending and pure shear problems, the difference between the Kirchhoff theory, the Mindlin-Reissner theory and the exact solution is in terms that have the size of t2/L2.
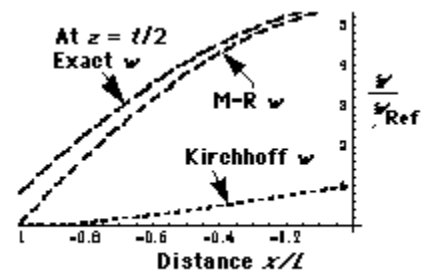


In See Transverse load, Łt = 0.5.. is shown a comparison of the Kirchhoff solution See ., the Mindlin-Reissner solution See ., and the exact solution See . at the surface z = t/2, for a very short beam, Łt = 0.5. Even for this short beam, the Mindlin-Reissner correction gives a reasonable approximation for the normal displacement. In See Transverse load, Łt = 2.. are the results for Łt = 2, for which the Mindlin-Reissner approximation is fairly good. For Łt = 5 in See Transverse load, Łt = 5.., the Kirchhoff solution is not too far in error. For higher values of Łt the three are indistinguishable.



Transverse load, Łt = 2.
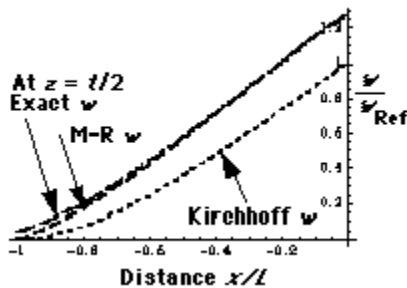
**Transverse load, Łt = 5.**
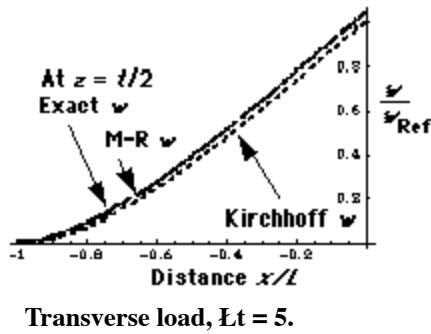
Also note that the ratio of maximum shear stress to maximum tangential stress is:

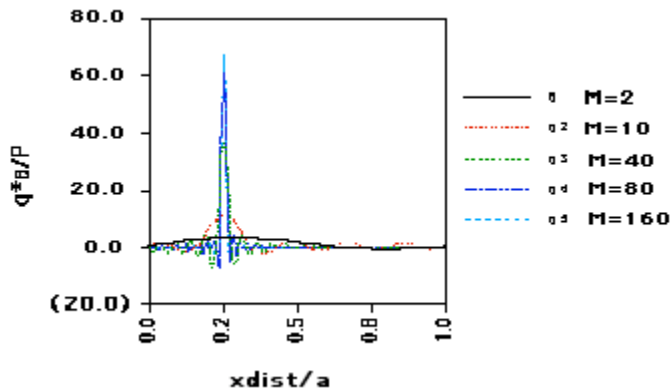$$\frac{|\sigma_{xz}|_{max}}{|\sigma_{xx}|_{max}} = \frac{t}{2L}$$

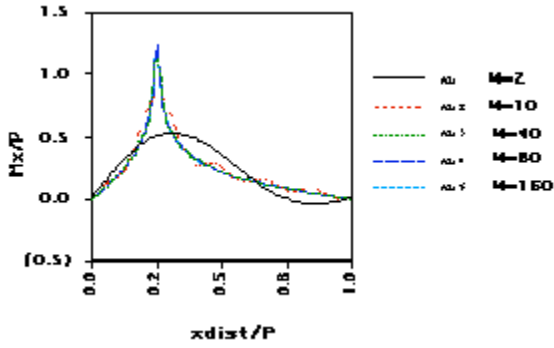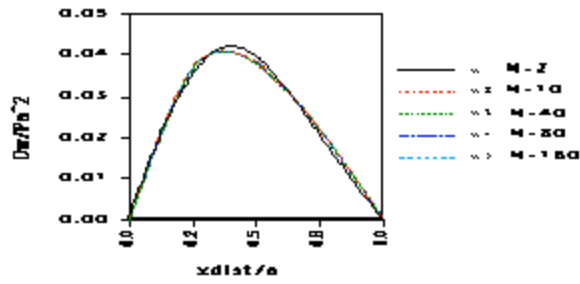and the ratio of maximum transverse stress to maximum tangential stress, for $v = 1/3$, is:

$$\frac{|\sigma_{zz}|_{max}}{|\sigma_{xx}|_{max}} = \frac{t^2}{6L^2}$$

These results of relatively small transverse shear and normal stress are consistent with the kinematic approximation of the Kirchhoff theory.
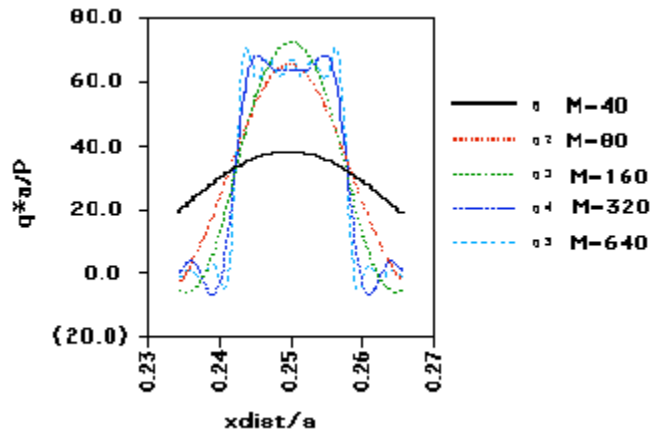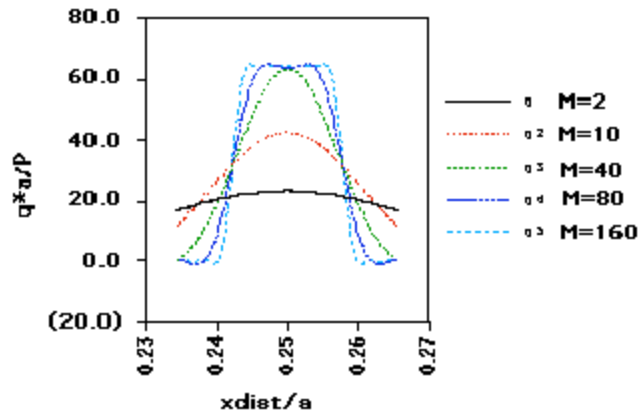
## Rectangular Plates

Results for plate strip with shear load on end: loadLoc/a = 1/4, loadWidth/a = 1/64

Local distribution for q, for loadLoc/a = 1/4, loadWidth/a = 1/64

## Circular Plates

## Moderate Rotation Theory

## Fourier Series Method for Polygonal Plates

## Bending of Circular Plates with Circular Holes

## Fracture Mechanics with Fourier Series

## Example Problems

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