Comenius University
Faculty of Mathematics, Physics and Informatics
Department of Astronomy, Physics of the Earth, and Meteorology

(KAFZM FMFI UK)

Peter Moczo

Introduction to Theoretical Seismology

Lecture notes for students of geophysics

http://www.fyzikazeme.sk/mainpage/stud_mat/Introduction_to_Theoretical_Seismology.pdf

Bratislava 2006

Preface

The lecture notes are just transcription of what I originally hand-wrote on transparencies for students of the course *Theory of Seismic Waves* at Universität Wien in 2001. In other words, the material was not and is not intended as a standard introductory text on theoretical seismology.

The material is based on several textbooks, monographs and journal articles. The main sources are: Aki and Richards (1980, 2002) - chapters 1, 2, and 5; Červený and Hron (1980), Červený (1985) - chapter 8, and Novotný (1999) - chapter 6. Though the material is clearly far from well elaborated it can be useful for students who want to learn basics of theory of seismology.

I want to acknowledge help from Peter Pažák as well as technical assistance of Martin Minka and Lenka Molnárová.

Table of Contents

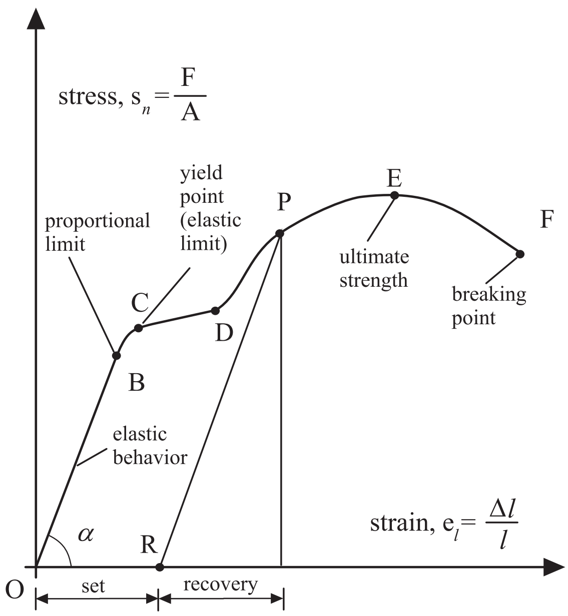
1. BASIC RELATIONS OF CONTINUUM MECHANICS

1.1 Introduction

An application of a force to real object causes some deformation of the object, i.e., change of its shape. If the deformation is negligibly small, we can work with a concept of a rigid body. The rigid body retains a fixed shape under all conditions of applied forces. If the deformations are not negligible, we have to consider the ability of an object to undergo the deformation, i.e., its elasticity, viscosity or plasticity.

Here, we will restrict ourselves to the elastic behavior. For the purpose of the macroscopic description both the rigid and elastic bodies can be defined as a system of material particles (not atoms or molecules!). At the same time we assume a continuous distribution of mass – a continuum. In a continuum we assign values of material parameters to geometric points. Therefore, we can make use of the theory of continuous functions.

A value of a material parameter assigned to a geometric point represents an average value for such a volume of the material in which the real discontinuous (atomic or molecular) structure need not be considered.



In a rigid body, relative coordinates connecting all of the constituent particles remain constant, i.e., the particles do not undergo any relative displacements.

In an elastic body, the particles can undergo relative displacements if forces are applied. Concept of 'continuum' usually is used for description of elastic bodies and fluids. The elastic behavior or objects is a subject of the continuum mechanics.

1.2 Body forces

Non-contact forces proportional to mass contained in a considered volume of a continuum.

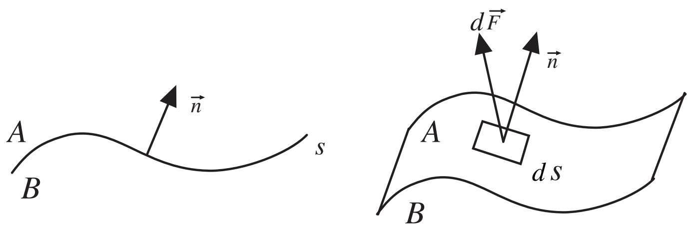
- forces between particles that are not adjacent; e.g., mutual gravitational forces
- forces due to the application of physical processes external to the considered volume; e.g., forces acting on buried particles of iron when a magnet is moving outside the considered volume

Let $\vec{f}(\vec{x},t)$ be a body force acting per unit volume on the particle that was at position $\vec{x}$ at some reference time. An important case of a body force – a force applied impulsively to one particle at $\vec{x}=\vec{\xi}$ and $t=\tau$ in the direction of the x_n-axis

$$\begin{aligned} f_i(\vec{x},t) &= A\delta(\vec{x}-\vec{\xi})\delta(t-\tau)\delta_{in} \\ [f_i]^U &= Nm^{-3}, \quad [\delta(\vec{x}-\vec{\xi})]^U = m^{-3} \\ [A]^U &= Ns, \quad [\delta(t-\tau)]^U = s^{-1} \end{aligned} \tag{1.1}$$

1.3 Stress, traction

If forces are applied at a surface S surrounding some volume of continuum, that volume of continuum is in a condition of stress. This is due to internal contact forces acting mutually between adjacent particles within a continuum. Consider an internal surface S dividing a continuum into part A and part B.



$\vec{n}$ – unit normal vector to S

$\delta\vec{F}$ – an infinitesimal force acting across an infinitesimal area δS
– force due to material A acting upon material B

$$\vec{T}(\vec{n}) = \lim_{\delta S\to 0}\frac{\delta\vec{F}}{\delta S} \tag{1.2}$$

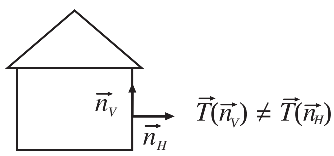
$$[\vec{T}]^U = \mathrm{N\ m}^{-2}$$

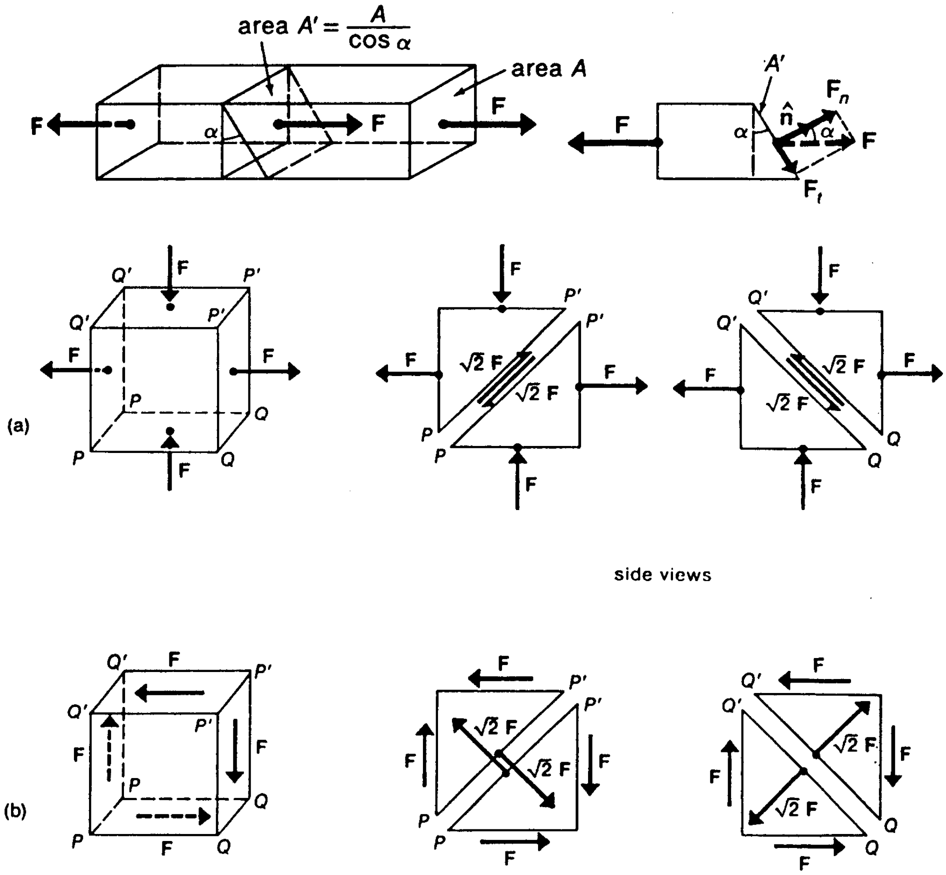
$\vec{T}(\vec{n})$ – traction vector (stress vector)
– force per unit area exerted by the material in the direction of $\vec{n}$ across the surface

The part of $\vec{T}$ – that is normal to the surface – normal stress
– that is parallel to the surface – shear stress

Traction depends on the orientation of the surface element δS across which contract force acts.

Examples:

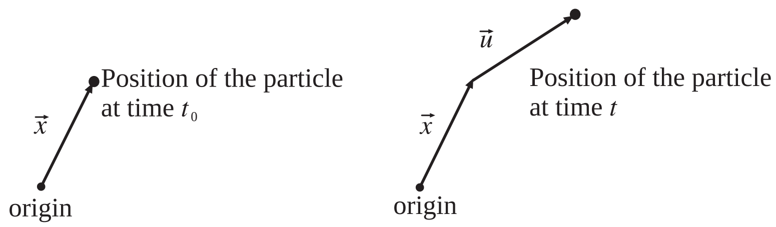




The state of stress at a point has to be described by a tensor.

1.4 Displacement, strain

Lagrangian description follows a particular particle that is specified by its original position at some reference time. Eulerian description follows a particular spatial position and thus whatever particle that happens to occupy that position.
Since a real seismogram is a record of Lagrangian motion, we will use the Lagrangian description.

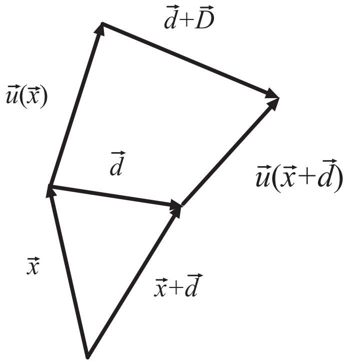


Displacement $\vec{u} = \vec{u}(\vec{x}, t)$ is the vector distance of a particle at time t from the position $\vec{x}$ of the particle at some reference time t_0 . $\vec{X} = \vec{x} + \vec{u}$ is the new position.

$$[\vec{u}]^U = m$$

$$\frac{\partial \vec{u}}{\partial t} \text{ – particle velocity,} \qquad \frac{\partial^2 \vec{u}}{\partial t^2} \text{ – particle acceleration}$$

$\vec{u}$ can generally include both the deformation and rigid–body translation and rotation. To analyze the deformation, we compare displacements of two neighboring particles.



$$\vec{D} = \vec{u}(\vec{x}+\vec{d}) - \vec{u}(\vec{x}) \tag{1.3a}$$
$$D_i = u_i(x_j + d_j) - u_i(x_j) \tag{1.3b}$$

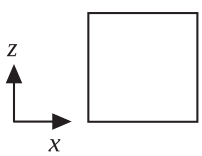
$$u_i(x_j + d_j) \doteq u_i(x_j) + u_{i,j} d_j \tag{1.4a}$$
$$\left(u_{i,j} = \frac{\partial u_i}{\partial x_j} \right)$$
$$\vec{u}(\vec{x}+\vec{d}) \doteq \vec{u}(\vec{x}) + (\vec{d} \cdot \nabla)\vec{u}(\vec{x}) \tag{1.4b}$$
$$\vec{u}(\vec{x}+\vec{d}) = \vec{u}(\vec{x}) + \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \tag{1.4c}$$

$$D_i = u_{i,j} d_j \tag{1.5}$$
$$u_{i,j} = \frac{1}{2}(u_{i,j} + u_{j,i}) + \frac{1}{2}(u_{i,j} - u_{j,i}) \tag{1.6}$$

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad \text{symmetric tensor} \tag{1.7a}$$
$$\Omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) \qquad \text{antisymmetric tensor} \tag{1.8a}$$

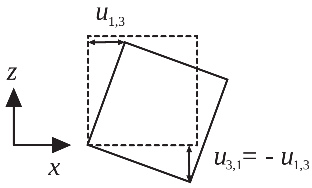
$$e_{ij} = \begin{bmatrix} u_{1,1} & \frac{1}{2}(u_{1,2}+u_{2,1}) & \frac{1}{2}(u_{1,3}+u_{3,1}) \\ \frac{1}{2}(u_{2,1}+u_{1,2}) & u_{2,2} & \frac{1}{2}(u_{2,3}+u_{3,2}) \\ \frac{1}{2}(u_{3,1}+u_{1,3}) & \frac{1}{2}(u_{3,2}+u_{2,3}) & u_{3,3} \end{bmatrix} \tag{1.7b}$$
$$\Omega_{ij} = \begin{bmatrix} 0 & \frac{1}{2}(u_{1,2}-u_{2,1}) & \frac{1}{2}(u_{1,3}-u_{3,1}) \\ \frac{1}{2}(u_{2,1}-u_{1,2}) & 0 & \frac{1}{2}(u_{2,3}-u_{3,2}) \\ \frac{1}{2}(u_{3,1}-u_{1,3}) & \frac{1}{2}(u_{3,2}-u_{2,3}) & 0 \end{bmatrix} \tag{1.8b}$$

$$D_i = e_{ij} d_j + \Omega_{ij} d_j \tag{1.9}$$



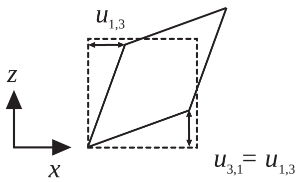
Consider 2D case – a square in the xz-plane.

Let $e_{ij} = 0$. Then $u_{1,3} = -u_{3,1}$ and $\Omega_{ij} = \begin{bmatrix} 0 & u_{1,3} \\ -u_{1,3} & 0 \end{bmatrix}$ $\quad (u_{2,j} = u_{i,2} = 0)$



pure rotation (no deformation)

Let $\Omega_{ij} = 0$ and assume no volume change. Then $u_{1,3} = u_{3,1}$ and $e_{ij} = \begin{bmatrix} 0 & u_{1,3} \\ u_{1,3} & 0 \end{bmatrix}$



shear deformation (no rotation)

Generally,

$$\frac{1}{2}(u_{i,j} - u_{j,i})d_j = \frac{1}{2}\varepsilon_{ijk}\varepsilon_{jlm}u_{m,l}d_k = \frac{1}{2}(\mathrm{rot}\,\vec{u} \times \vec{d})_i \tag{1.10}$$

$\frac{1}{2}$ rot $\vec{u}$ represents a rigid – body rotation if $|u_{i,j}| \ll 1$.
Thus, e_{ij} represents deformation. Therefore, e_{ij} is called the strain tensor. $[e_{ij}]^U = [u_{i,j}]^U = 1$.
This can be also shown by investigating a change of distance between two particles since the change can be only due to deformation

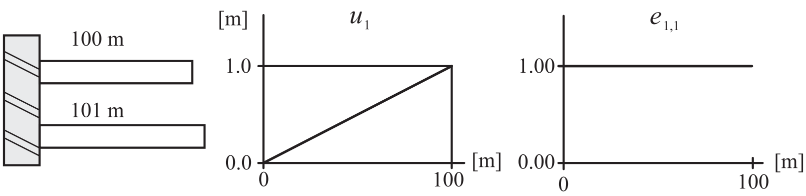
$$|\vec{d}|^2 = d_i d_i \tag{1.11}$$

$$\begin{aligned} |\vec{d} + \vec{D}|^2 &= (d_i + D_i)(d_i + D_i) \\ &= (d_i + u_{i,j}d_j)(d_i + u_{i,j}d_j) \\ &= d_i d_i + 2u_{i,j}d_i d_j + u_{i,j}u_{i,k}d_j d_k \\ &= |\vec{d}|^2 + u_{i,j}d_i d_j + u_{j,i}d_j d_i + u_{k,j}u_{k,i}d_j d_i \\ &= |\vec{d}|^2 + (u_{i,j} + u_{j,i} + \underbrace{u_{k,i}u_{k,j}})d_i d_j \end{aligned} \tag{1.12}$$

we can neglect since $|u_{k,i}| \ll 1$

$$|\vec{d} + \vec{D}|^2 = |\vec{d}|^2 + 2e_{ij}d_i d_j \tag{1.13}$$

Displacement is a local measure of an absolute change in position.
Strain is a local measure of relative change in position and displacement field due to deformation.
Example :



1.5 Stress tensor, equation of motion

Consider a volume V with surface S.
time rate of change of momentum of particles = forces acting on particles

$$\frac{\partial}{\partial t}\iiint\limits_V \rho\frac{\partial\vec{u}}{\partial t}\,dV = \iiint\limits_V \vec{f}\,dV + \iint\limits_S \vec{T}(\vec{n})\,dS \tag{1.14}$$

Since V and S move with the particles (Lagrangian description), ρdV does not change with time and

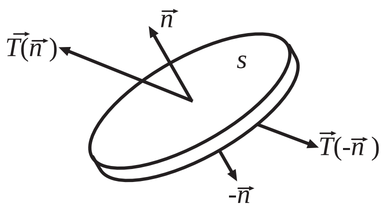
$$\frac{\partial}{\partial t}\iiint\limits_V \rho\frac{\partial\vec{u}}{\partial t}\,dV = \iiint\limits_V \rho\frac{\partial^2\vec{u}}{\partial t^2}\,dV \tag{1.15}$$

Consider a particle P inside the volume V for which none of acceleration, body force and traction have singular value. Shrink V down onto P and compare relative magnitudes of the terms in equation (1.14). Both the volume integrals are of order V while the surface integral is of order $V^{\frac{2}{3}}$. This means that the surface integral approaches zero more slowly than the volume integral does. Then (1.14) $\Big/ \iint\limits_S dS$ leads to

$$\lim_{V\to 0}\frac{|\iint\limits_S \vec{T}\,dS|}{\iint\limits_S dS} = \lim_{V\to 0} O(V^{\frac{1}{3}}) = 0 \tag{1.16}$$

Apply equation (1.16) to two cases.

1st case
Let V be a disc with a negligibly small area of the edge

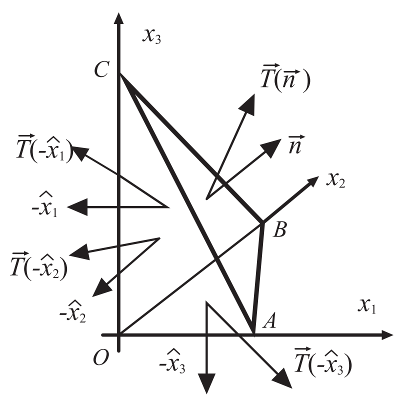


Equation (1.16) $\Rightarrow$

$$\lim_{V\to 0}\frac{[\vec{T}(\vec{n})+\vec{T}(-\vec{n})]S}{2S} = 0$$

$$\Rightarrow \quad \vec{T}(-\vec{n}) = -\vec{T}(\vec{n}) \tag{1.17}$$

2nd case
Let V be a tetrahedron



Equation (1.16) $\Rightarrow$

$$\lim_{V\to 0}\frac{\vec{T}(\vec{n}).ABC+\vec{T}(-\hat{x}_1).OBC+\vec{T}(-\hat{x}_2).OCA+\vec{T}(-\hat{x}_3).OAB}{ABC+OBC+OCA+OAB} = 0 \tag{1.18}$$

$$\begin{aligned} \text{Since} \quad \vec{n} &= (n_1, n_2, n_3); \\ n_1 &= OBC/ABC \;, \quad n_2 = OCA/ABC \;, \quad n_3 = OAB/ABC \end{aligned} \tag{1.19}$$

and

$$\vec{T}(-\hat{x}_i) = -\vec{T}(\hat{x}_i); \quad i = 1, 2, 3$$

we get from (1.18) after dividing it by ABC

$$\lim_{V \to 0} \frac{\vec{T}(\vec{n}) - \vec{T}(\hat{x}_1)n_1 - \vec{T}(\hat{x}_2)n_2 - \vec{T}(\hat{x}_3)n_3}{ABC + OBC + OCA + OAB} = 0$$

and consequently

$$\begin{aligned} \vec{T}(\vec{n}) &= \vec{T}(\hat{x}_j)n_j \\ T_i(\vec{n}) &= T_i(\hat{x}_j)n_j \end{aligned} \tag{1.20}$$

Both properties (1.17) and (1.20) are important since they are valid in a dynamic case. (Their validity in a static case is trivial.)
Equation (1.20) can be written as

$$[T_1(\vec{n}), T_2(\vec{n}), T_3(\vec{n})] = [n_1, n_2, n_3] \begin{bmatrix} T_1(\hat{x}_1) & T_2(\hat{x}_1) & T_3(\hat{x}_1) \\ T_1(\hat{x}_2) & T_2(\hat{x}_2) & T_3(\hat{x}_2) \\ T_1(\hat{x}_3) & T_2(\hat{x}_3) & T_3(\hat{x}_3) \end{bmatrix} \tag{1.21}$$

Define stress tensor τ_{ji}

$$\tau_{ji} = T_i(\hat{x}_j) \tag{1.22}$$

$$[\tau_{ji}]^U = Nm^{-2}$$

Then (1.20) and (1.21) can be rewritten as

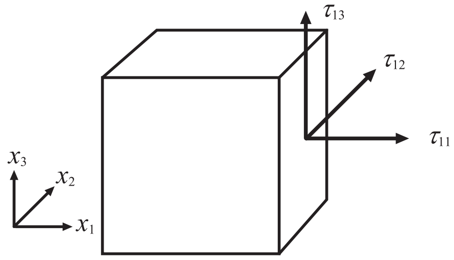
$$T_i(\vec{n}) = \tau_{ji} n_j \qquad \text{– Cauchy's stress formula} \tag{1.23}$$

and

$$[T_1(\vec{n}), T_2(\vec{n}), T_3(\vec{n})] = [n_1, n_2, n_3] \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} \tag{1.24}$$

τ_{ji} is the i-th component of the traction exerted by a material with greater x_j across the plane normal to the j-th axis on material with lesser x_j.

Example:



Stress tensor fully describes a state of stress at a given point.

Now we can apply eq. (1.23) to eq. (1.14). Eq. (1.14) in the index notation is

$$\iiint\limits_V \rho u_{i,tt}\, dV = \iiint\limits_V f_i\, dV + \iint\limits_S T_i(\vec{n})\, dS \tag{1.25}$$

Using eq. (1.23) the surface integral becomes

$$\iint\limits_S \tau_{ji} n_j\, dS = \iint\limits_S \tau_{ji}\, dS_j \tag{1.26}$$

The surface integral can be transformed into a volume integral using Gauss's divergence theorem

$$\iint\limits_S \vec{a}\, d\vec{S} = \iiint\limits_V div\ \vec{a}\, dV$$

$$\iint\limits_S a_j\, dS_j = \iiint\limits_V \frac{\partial a_j}{\partial \xi_j}\, dV(\vec{\xi})$$

In our problem, the particles constituting S have moved from their original positions $\vec{x}$ at the reference time to position $\vec{X} = \vec{x} + \vec{u}$ at time t. Therefore, the spatial differentiation in volume V is $\frac{\partial}{\partial X_j}$. The application of Gauss's theorem thus gives

$$\iint\limits_S \tau_{ji}\, dS_j = \iiint\limits_V \frac{\partial \tau_{ji}}{\partial X_j}\, dV \tag{1.27}$$

Eq. (1.25) can be now written as

$$\iiint\limits_V \left(\rho u_{i,tt} - f_i - \tau_{ji,j}\right)\, dV = 0 \tag{1.28}$$

$$\text{where} \quad \tau_{ji,j} = \partial \tau_{ji} / \partial X_j$$

The integrand in (1.28) must be zero everywhere where it is continuous. Therefore,

$$\rho u_{i,tt} = \tau_{ji,j} + f_i \tag{1.29}$$

This is the equation of motion for the elastic continuum. Look now at the angular momentum of the particles in a volume V.

time rate of change of angular momentum about the origin	=	moment of forces (torque) acting on the particles

$$\frac{\partial}{\partial t}\iiint\limits_V \vec{X}\times\rho\vec{u}_t\,dV = \iiint\limits_V \vec{X}\times\vec{f}\,dV + \iint\limits_S \vec{X}\times\vec{T}\,dS \tag{1.30}$$

$$\tfrac{\partial}{\partial t}(\vec{X}\times\vec{u}_t) = \vec{X}_t\times\vec{u}_t + \vec{X}\times\vec{u}_{tt} = (\underbrace{\vec{x}_t}_{=0} + \underbrace{\vec{u}_t)\times\vec{u}_t}_{=0} + \vec{X}\times\vec{u}_{tt} = \vec{X}\times\vec{u}_{tt} \tag{1.31}$$

Then eq. (1.30) $\Rightarrow$

$$\iiint\limits_V \varepsilon_{ijk}X_j(\rho u_{k,tt} - f_k)dV = \iint\limits_S \varepsilon_{ijk}X_jT_k\,dS \tag{1.32}$$

Eq. (1.29) implies

$$\iiint\limits_V \varepsilon_{ijk}X_j\frac{\partial\tau_{lk}}{\partial X_l}\,dV = \iiint\limits_V \varepsilon_{ijk}X_j(\rho u_{k,tt} - f_k)\,dV \tag{1.33}$$

The right-hand side of eq. (1.33) can be replaced by the right-hand side of eq. (1.32)

$$\begin{aligned}
\iiint\limits_V \varepsilon_{ijk}X_j\frac{\partial\tau_{lk}}{\partial X_l}\,dV &= \iint\limits_S \varepsilon_{ijk}X_jT_k\,dS \\
\text{eq. (1.23)} \Rightarrow\quad &= \iint\limits_S \varepsilon_{ijk}X_j\tau_{lk}n_l\,dS \\
\text{Gauss's theorem} \Rightarrow\quad &= \iiint\limits_V \varepsilon_{ijk}\frac{\partial}{\partial X_l}(X_j\tau_{lk})\,dV \\
\frac{\partial}{\partial X_l}(X_j\tau_{lk}) = \delta_{jl}\tau_{lk} + X_j\frac{\partial\tau_{lk}}{\partial X_l} &= \tau_{jk} + X_j\frac{\partial\tau_{lk}}{\partial X_l}
\end{aligned} \tag{1.34}$$

Then eq. (1.34) becomes

$$\iiint\limits_V \varepsilon_{ijk}X_j\frac{\partial\tau_{lk}}{\partial X_l}\,dV = \iiint\limits_V \left(\varepsilon_{ijk}\tau_{jk} + \varepsilon_{ijk}X_j\frac{\partial\tau_{lk}}{\partial X_l}\right)\,dV$$

This gives

$$\iiint\limits_V \varepsilon_{ijk}\;\tau_{jk}\;dV = 0 \tag{1.35}$$

Since eq. (1.35) applies to any volume

$$\varepsilon_{ijk}\;\tau_{jk} = 0 \tag{1.36}$$

and consequently

$$\tau_{jk} = \tau_{kj} \tag{1.37}$$

which means that the stress tensor is symmetric. This is a very important property meaning that the stress tensor has only 6 independent components. The state of stress at a given point is thus fully described by 6 independent components of the stress tensor.

We can now rewrite relation for traction (1.23) and equation of motion (1.29) as

$$T_i = \tau_{ij} n_j \tag{1.38}$$

$$\rho u_{i,tt} = \tau_{ij,j} + f_i \tag{1.39}$$

Strictly, $\tau_{ij,j} = \frac{\partial \tau_{ij}}{\partial X_j}$. In the case of seismic wave propagation, displacement, strain, acceleration and stress vary over distances much larger than the amplitude of particle displacement and the other quantities. Therefore, differentiation with respect to x_j gives a very good approximation of differentiation with respect to X_j. In other words, the difference between derivative evaluated for a particular particle ($\sim$ Lagrangian description) and derivative evaluated at a fixed position ($\sim$ Eulerian description) is negligible.

1.6 Stress - strain relation. Strain - energy function.

The mechanical behavior of a continuum is defined by the relation between the stress and strain. If forces are applied to the continuum, the stress and strain change together according to the stress–strain relation. Such the relation is called the constitutive relation.
A linear elastic continuum is described by Hooke's law which in Cauchy's generalized formulation reads

$$\tau_{ij} = c_{ijkl} e_{kl} \tag{1.40}$$

Each component of the stress tensor is a linear combination of all components of the strain tensor. c_{ijkl} is the 4th – order tensor of elastic coefficients and has $3^4 = 81$ components.

$$\tau_{ij} = \tau_{ji} \quad \Rightarrow \quad c_{ijkl} = c_{jikl} \tag{1.41}$$

$$e_{kl} = e_{lk} \quad \Rightarrow \quad c_{ijkl} = c_{ijlk} \tag{1.42}$$

The symmetry of the stress and strain tensors reduces the number of different coefficients to $6 \times 6 = 36$. A further reduction of the number of coefficients follows from the first law of thermodynamics which will also give a formula for the strain-energy function.

$$\begin{aligned} &\text{Rate of mechanical work} + \text{Rate of heating} \\ &= \text{Rate of increase of kinetic and internal energies} \end{aligned} \tag{1.43}$$

Rate of mechanical work

$$\begin{aligned}
\iiint\limits_V \vec{f}\cdot\dot{\vec{u}}dV + \iint\limits_S \vec{T}\cdot\dot{\vec{u}}dS &= \iiint\limits_V f_i\dot{u}_i dV + \iint\limits_S \tau_{ij}\dot{u}_i n_j dS \\
\text{Gauss's theorem} \Rightarrow \quad & \iiint\limits_V f_i\dot{u}_i dV + \iiint\limits_V (\tau_{ij}\dot{u}_i)_{,j} dV \\
\text{equation of motion(1.39)} \Rightarrow &= \iiint\limits_V (\underbrace{f_i\dot{u}_i + \tau_{ij,j}\dot{u}_i}_{\dot{u}_i\rho\ddot{u}_i} + \tau_{ij}\dot{u}_{i,j})dV \\
&= \iiint\limits_V (\rho\dot{u}_i\ddot{u}_i + \tau_{ij}\dot{u}_{i,j})dV \\
&= \frac{\partial}{\partial t}\iiint\limits_V \frac{1}{2}\rho\dot{u}_i\dot{u}_i dV + \iiint\limits_V \tau_{ij}\dot{e}_{ij}dV \qquad (1.44)
\end{aligned}$$

since $\tau_{ij}\dot{u}_{i,j} = \tau_{ij}\dot{e}_{ij}$ (antisymmetric part of $\dot{u}_{i,j}$ does not contribute)

Rate of heating

Let $\vec{h}(\vec{x},t)$ be the heat flux per unit area and $Q(\vec{x},t)$ the heat input per unit volume. Then

$$\begin{aligned}
-\iint\limits_S \vec{h}\cdot\vec{n}dS &= \frac{\partial}{\partial t}\iiint\limits_V QdV \\
-\iint\limits_S h_i n_i dS & \\
-\iiint\limits_V h_{i,i}dV &= \iiint\limits_V \dot{Q}dV \qquad (1.45)
\end{aligned}$$

$$-h_{i,i} = \dot{Q} \qquad \text{or} \qquad -\nabla\cdot\vec{h} = \dot{Q} \qquad (1.46)$$

Rate of increase of kinetic energy

$$\frac{\partial}{\partial t}\iiint\limits_V \frac{1}{2}\rho\dot{u}_i\dot{u}_i dV \qquad (1.47)$$

Rate of increase of internal energy

Let $\mathcal{U}$ be the internal energy per unit volume. Then the rate is

$$\frac{\partial}{\partial t}\iiint\limits_V \mathcal{U}dV \qquad (1.48)$$

Inserting (1.44), (1.45), (1.47) and (1.48) into (1.43) we get

$$\begin{aligned}
&\frac{\partial}{\partial t}\iiint\limits_V \frac{1}{2}\rho\dot{u}_i\dot{u}_i dV + \iiint\limits_V \tau_{ij}\dot{e}_{ij}dV - \iiint\limits_V h_{i,i}dV \\
&\qquad = \frac{\partial}{\partial t}\iiint\limits_V \frac{1}{2}\rho\dot{u}_i\dot{u}_i dV + \frac{\partial}{\partial t}\iiint\limits_V \mathcal{U}dV
\end{aligned}$$

This gives

$$\dot{\mathcal{U}} = -h_{i,i} + \tau_{ij}\dot{e}_{ij} \tag{1.49a}$$

or (due to (1.46))

$$\dot{\mathcal{U}} = \dot{Q} + \tau_{ij}\dot{e}_{ij} \tag{1.49b}$$

For small perturbations of the thermodynamic equilibrium (1.49b) gives

$$d\mathcal{U} = dQ + \tau_{ij} de_{ij} \tag{1.50}$$

For reversible processes (1.50) implies

$$d\mathcal{U} = T d\mathcal{S} + \tau_{ij} de_{ij} \tag{1.51}$$

where T is the absolute temperature and $\mathcal{S}$ entropy per unit volume.
It follows from (1.51) that the entropy and strain–tensor components completely and uniquely determine the internal energy, i.e., they are the state variables.
Define the strain-energy function W such that

$$\tau_{ij} = \frac{\partial W}{\partial e_{ij}} \tag{1.52}$$

Then (1.51) implies

$$\tau_{ij} = \left(\frac{\partial \mathcal{U}}{\partial e_{ij}}\right)_{\mathcal{S}} \tag{1.53}$$

If the process of deformation is adiabatic, i.e., if $\vec{h} = 0$ and $\dot{Q} = 0$, the entropy $\mathcal{S}$ is constant and the internal energy $\mathcal{U}$ can be taken as the strain energy function : $W = \mathcal{U}$.

Since the time constant of thermal diffusion in the Earth is very much longer than the period of seismic waves, the process of deformation due to passage of seismic waves can be considered adiabatic.

Note that the free energy $\mathcal{F} = \mathcal{U} - \mathcal{T}\mathcal{S}$ would be a proper choice for W in the case of an isothermal process – such as a tectonic process in which the deformation is very slow.

(1.40) and (1.52) :

$$\frac{\partial W}{\partial e_{ij}} = \tau_{ij} = c_{ijkl} e_{kl} \tag{1.54}$$

$$\frac{\partial^2 W}{\partial e_{kl} \partial e_{ij}} = c_{ijkl}$$

$$\frac{\partial W}{\partial e_{kl}} = \tau_{kl} = c_{klij} e_{ij}$$

$$\frac{\partial^2 W}{\partial e_{ij} \partial e_{kl}} = c_{klij}$$

$$\Rightarrow \qquad c_{ijkl} = c_{klij} \tag{1.55}$$

Thus, the first law of thermodynamics implies further reduction of the number of independent coefficients – to 21. This is because :
In 6 cases there is $ij = kl$ and relation (1.55) is identically satisfied.

The remaining 30 $(= 36 - 6)$ coefficients satisfy 15 relations (1.55) which means that only 15 are independent. Thus, $6 + 15 = 21$.
21 independent elastic coefficients describe an anisotropic continuum in which material parameters at a point depend on direction.
In the simplest, isotropic, continuum, material parameters depend only on position – they are the same in all directions at a given point.

In the isotropic continuum c_{ijkl} must be isotropic[1] :

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{1.56}$$

and λ and μ are the only two independent elastic coefficients.
Since c_{ijkl} do not depend on strain, they are also called elastic constants.
λ and μ are Lamè constants.
Inserting (1.56) into (1.40) gives

$$\tau_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij} \tag{1.57}$$

which is the Hooke's law for the isotropic continuum.

Now return to the strain–energy function.
Since all the first derivatives of W are homogeneous functions (of order one) in the strain–tensor components, and W can be taken as zero in the natural state[2] , W itself has to be homogeneous (of order two)

$$W = d_{ijkl}e_{ij}e_{kl} \tag{1.58}$$

(1.58) $\Rightarrow$

$$\begin{aligned}
\tau_{ij} = \frac{\partial W}{\partial e_{ij}} &= d_{ijkl}\left(e_{kl} + e_{ij}\frac{\partial e_{kl}}{\partial e_{ij}}\right) \\
&= d_{ijkl}(e_{kl} + e_{ij}\delta_{ik}\delta_{jl}) \\
&= d_{ijkl}(e_{kl} + e_{kl}) \\
&= 2d_{ijkl}e_{kl}
\end{aligned}$$

(1.40) $\Rightarrow$ $d_{ijkl} = \frac{1}{2}c_{ijkl}$

$$\begin{aligned}
W &= \frac{1}{2}c_{ijkl}e_{ij}e_{kl} \\
W &= \frac{1}{2}\tau_{ij}e_{ij}
\end{aligned} \tag{1.59}$$

[1] general isotropic tensor : $c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}$; due to symmetry of c_{ijkl} $\mu = \nu$
[2] natural state: $e_{ij} = 0$ and $W = 0$ $\Rightarrow$ $W = d_{ijkl}e_{ij}e_{kl} + \underbrace{\text{constant}}_{=0}$

Self-gravitation in the Earth causes pressures of up to $\approx 10^{11}Pa = 10^{11}Nm^{-2}$ (the pressure in the center of the Earth reaches $4 \cdot 10^{11}Pa$) and large strains. A finite strain and nonlinear stress–strain relation would be appropriate.

If we want to use theory based on the assumption of small perturbations of a reference state with zero stress and strain (i.e., the assumption used above), we can consider the static equilibrium prior to an earthquake as a reference state. Then we assume zero strain with nonzero initial stress σ_{ij}^0. Nonzero strain e_{kl} is then due to incremental stress τ_{ij} (the total stress being $\sigma_{ij}^0 + \tau_{ij}$) and $\tau_{ij} = c_{ijkl}e_{kl}$.
Unless said otherwise, we will neglect the initial stress σ_{ij}^0.

1.7 Uniqueness theorem

The displacement $\vec{u} = \vec{u}(\vec{x}, t)$ in the volume V with surface S at any time $t > t_0$ is uniquely determined by

I. $\vec{u}(\vec{x}, t_0)$ and $\dot{\vec{u}}(\vec{x}, t_0)$ – initial displacement and particle velocity
II. $\vec{f}(\vec{x}, t)$ and $Q(\vec{x}, t)$ – body forces and supplied heat
III. $\vec{T}(\vec{x}, t)$ over any part S_1 of S – traction
IV. $\vec{u}(\vec{x}, t)$ over S_2 where $S_1 + S_2 = S$ – displacement

Proof
Let $\vec{u}_1(\vec{x}, t)$ and $\vec{u}_2(\vec{x}, t)$ be any solutions satisfying the same conditions I.–IV. Then, obviously, displacement $\vec{U}(\vec{x}, t) \equiv \vec{u}_1(\vec{x}, t) - \vec{u}_2(\vec{x}, t)$ has zero initial conditions and is set up by zero body forces, zero heating, zero traction over S_1 and zero displacement over S_2.
We have to show that $\vec{U}(\vec{x}, t) = 0$ in the volume V for times $t > t_0$.
The rate of mechanical work is obviously zero in V and on S for $t > t_0$ ($\vec{f} \equiv 0$, $\vec{T} \equiv 0$ on S_1 and $\dot{\vec{u}} \equiv 0$ on S_2 for $\vec{U}$), i.e., according to eq. (1.44)

$$\frac{\partial}{\partial t} \iiint\limits_V \frac{1}{2}\rho \dot{U}_i \dot{U}_i dV + \iiint\limits_V \tau_{ij}\dot{e}_{ij} dV = 0$$

Integrate the equation from t_0 to t:

$$\int\limits_{t_0}^{t} \left[\frac{\partial}{\partial t} \iiint\limits_V \frac{1}{2}\rho \dot{U}_i \dot{U}_i dV \right] dt = \left[\iiint\limits_V \frac{1}{2}\rho \dot{U}_i \dot{U}_i dV \right]_{t_0}^{t} = \iiint\limits_V \frac{1}{2}\dot{U}_i(t)\rho \dot{U}_i(t) dV$$

$$\int\limits_{t_0}^{t} \left[\iiint\limits_V \tau_{ij}\dot{e}_{ij} dV \right] dt = \iiint\limits_V \left[\int\limits_{t_0}^{t} \tau_{ij}\dot{e}_{ij} dt \right] dV$$

$$\begin{aligned}
\int_{t_0}^{t} \tau_{ij}\dot{e}_{ij}dt &= [\tau_{ij}e_{ij}]_{t_0}^{t} && - \int_{t_0}^{t} \dot{\tau}_{ij}e_{ij}dt \\
&= [c_{ijkl}e_{kl}e_{ij}]_{t_0}^{t} && - \int_{t_0}^{t} c_{ijkl}\dot{e}_{kl}e_{ij}dt \quad /c_{ijkl} = c_{klij} \\
& && - \int_{t_0}^{t} c_{klij}\dot{e}_{kl}e_{ij}dt \quad /i \leftrightarrow k, j \leftrightarrow l \\
& && - \int_{t_0}^{t} c_{ijkl}\dot{e}_{ij}e_{kl}dt \\
& && - \int_{t_0}^{t} c_{ijkl}e_{kl}\dot{e}_{ij}dt \\
& && - \int_{t_0}^{t} \tau_{ij}\dot{e}_{ij}dt \\
\int_{t_0}^{t} \tau_{ij}\dot{e}_{ij}dt &= [c_{ijkl}e_{kl}e_{ij}]_{t_0}^{t} && - \int_{t_0}^{t} \tau_{ij}\dot{e}_{ij}dt
\end{aligned}$$

$$\begin{aligned}
\int_{t_0}^{t} \tau_{ij}\dot{e}_{ij}dt &= \frac{1}{2}\left[c_{ijkl}e_{kl}e_{ij}\right]_{t_0}^{t} \\
&= \frac{1}{2}\left[c_{ijkl}U_{k,l}U_{i,j}\right]_{t_0}^{t} \\
&= \frac{1}{2}c_{ijkl}U_{k,l}(\vec{x},t)U_{i,j}(\vec{x},t)
\end{aligned}$$

The integrated equation gives

$$\iiint_V \frac{1}{2}\rho\dot{U}_i\dot{U}_i dV + \iiint_V \frac{1}{2}c_{ijkl}U_{k,l}U_{i,j}dV = 0$$

Since both the kinetic and strain energies are positive, $\dot{U}_i(\vec{x},t) = 0$ for $t \geq t_0$. Since $U_i(\vec{x},t_0) = 0$, $\vec{U}(\vec{x},t) = 0$ in V for $t > t_0$.

1.8 Reciprocity theorem

Consider volume V with surface S.
Let $\vec{u} = \vec{u}(\vec{x}, t)$ be displacement due to body force $\vec{f}$, boundary conditions on S, and initial conditions at $t = 0$.
Let $\vec{v} = \vec{v}(\vec{x}, t)$ be displacement due to body force $\vec{g}$, boundary conditions on S, and initial conditions at $t = 0$. Both the boundary and initial conditions are in general different from those for $\vec{u}$.
Let $\vec{T}(\vec{u}, \vec{n})$ and $\vec{T}(\vec{v}, \vec{n})$ be tractions due to $\vec{u}$ and $\vec{v}$, respectively, acting across surface with the normal $\vec{n}$.
Then (Betti's theorem)

$$\begin{aligned} & \iiint\limits_V \left(\vec{f} - \rho\ddot{\vec{u}}\right) \cdot \vec{v} dV \quad + \quad \iint\limits_S \vec{T}(\vec{u}, \vec{n}) \cdot \vec{v} dS \\ = & \iiint\limits_V \left(\vec{g} - \rho\ddot{\vec{v}}\right) \cdot \vec{u} dV \quad + \quad \iint\limits_S \vec{T}(\vec{v}, \vec{n}) \cdot \vec{u} dS \end{aligned} \tag{1.60}$$

Proof

$$\underbrace{\iiint\limits_V (f_i - \rho\ddot{u}_i)\, v_i dV}_{-\iiint\limits_V \tau_{ij,j} v_i dV \quad \text{(Eq. of motion (1.39))}} + \iint\limits_S T_i(\vec{u}, \vec{n}) v_i dS \quad \boxed{=}$$

Eq. (1.38) $\Rightarrow$

$$\begin{aligned} \iint\limits_S T_i(\vec{u}, \vec{n}) v_i dS &= \iint\limits_S \tau_{ij} n_j v_i dS \\ &= \iiint\limits_V (\tau_{ij} v_i)_{,j} dV \\ &= \iiint\limits_V \tau_{ij,j} v_i dV + \iiint\limits_V \tau_{ij} v_{i,j} dV \end{aligned}$$

$$\begin{aligned} \boxed{=} \quad & \iiint\limits_V \tau_{ij} v_{i,j} dV \\ = \quad & \iiint\limits_V c_{ijkl} e_{kl} v_{i,j} dV \\ = \quad & \iiint\limits_V c_{ijkl} u_{k,l} v_{i,j} dV \end{aligned}$$

Analogously, it can be shown that the right–hand side of eq. (1.60) is equal to

$$\begin{aligned}
& \iiint_V c_{ijkl} u_{i,j} v_{k,l} dV \\
= & \iiint_V c_{klij} u_{i,j} v_{k,l} dV \qquad \leftarrow c_{ijkl} = c_{klij} \qquad \text{interchanging indices} \\
= & \iiint_V c_{ijkl} u_{k,l} v_{i,j} dV
\end{aligned}$$

which is the same as the left–hand side of eq. (1.60)
It is important that
– the theorem does not involve the initial conditions,
– $\vec{u}$, $\ddot{\vec{u}}$, $\vec{T}(\vec{u}, \vec{n})$ and $\vec{f}$ may relate to time t_1, while $\vec{v}$, $\ddot{\vec{v}}$, $\vec{T}(\vec{v}, \vec{n})$ and $\vec{g}$ may relate to time $t_2 \neq t_1$

Let $t_1 = t$ and $t_2 = \tau - t$.
Integrate Betti's theorem (1.60) from 0 to τ, integrate first the acceleration terms:

$$\begin{aligned}
& \int_0^\tau \rho \left[\ddot{\vec{u}}(t) \cdot \vec{v}(\tau - t) - \vec{u}(t) \cdot \ddot{\vec{v}}(\tau - t) \right] dt \\
= \; & \rho \int_0^\tau \frac{\partial}{\partial t} \left[\dot{\vec{u}}(t) \cdot \vec{v}(\tau - t) - \vec{u}(t) \cdot \dot{\vec{v}}(\tau - t) \right] dt \\
= \; & \rho \left[\dot{\vec{u}}(\tau) \cdot \vec{v}(0) - \dot{\vec{u}}(0) \cdot \vec{v}(\tau) + \vec{u}(\tau) \cdot \dot{\vec{v}}(0) + \vec{u}(0) \cdot \dot{\vec{v}}(\tau) \right] \qquad (1.61)
\end{aligned}$$

After the integration, the acceleration terms depend only on the initial ($t = 0$) and final ($t = \tau$) values.
Let $\vec{u} = 0$ and $\vec{v} = 0$ for $\tau \leq \tau_0$. Consequently, also $\dot{\vec{u}} = 0$ and $\dot{\vec{v}} = 0$ for $\tau \leq \tau_0$.
Then it follows from eq. (1.61) that

$$\int_{-\infty}^{\infty} \rho \left[\ddot{\vec{u}}(t) \cdot \vec{v}(\tau - t) - \vec{u}(t) \cdot \ddot{\vec{v}}(\tau - t) \right] dt = 0 \qquad (1.62)$$

Integrating Betti's theorem (1.60) from $-\infty$ to ∞ and applying eq. (1.62) we obtain

$$\begin{aligned}
& \int_{-\infty}^{\infty} dt \iiint_V \left[\vec{u}(\vec{x}, t) \cdot \vec{g}(\vec{x}, \tau - t) - \vec{v}(\vec{x}, \tau - t) \cdot \vec{f}(\vec{x}, t) \right] dV \\
= & \int_{-\infty}^{\infty} dt \iint_S \left[\vec{v}(\vec{x}, \tau - t) \cdot \vec{T}(\vec{u}(\vec{x}, t), \vec{n}) - \vec{u}(\vec{x}, t) \cdot \vec{T}(\vec{v}(\vec{x}, \tau - t), \vec{n}) \right] dS \qquad (1.63)
\end{aligned}$$

This is the important reciprocity theorem for displacements $\vec{u}$ and $\vec{v}$ with a quiescent past.

1.9 Green's function

Let the unit impulse force in the direction of the x_n – axis be applied at point $\vec{\xi}$ and time τ (see definition (1.1)):

$$f_i(\vec{x},t) = A\delta(\vec{x}-\vec{\xi})\delta(t-\tau)\delta_{in}$$

Then the equation of motion is

$$\begin{aligned} \rho\ddot{u}_i &= (c_{ijkl}u_{k,l})_{,j} + A\delta(\vec{x}-\vec{\xi})\delta(t-\tau)\delta_{in} \\ \rho\frac{\ddot{u}_i}{A} &= \left(c_{ijkl}\left(\tfrac{u_k}{A}\right)_{,l}\right)_{,j} + \delta(\vec{x}-\vec{\xi})\delta(t-\tau)\delta_{in} \end{aligned}$$

Define Green's function $G_{in}(\vec{x},t;\vec{\xi},\tau)$:

$$G_{in}(\vec{x},t;\vec{\xi},\tau) = \frac{u_i}{A} \qquad ; \qquad [G_{in}]^U = \frac{m}{Ns} = \frac{s}{kg}$$

Green's function satisfies equation

$$\rho\ddot{G}_{in} = (c_{ijkl}G_{kn,l})_{,j} + \delta(\vec{x}-\vec{\xi})\delta(t-\tau)\delta_{in} \tag{1.64}$$

Let $A = 1Ns$. Then the value of $G_{in}(\vec{x},t;\vec{\xi},\tau)$ is equal to the value of the i–th component of the displacement at $(\vec{x},t)$ due to the unit impulse force applied at $(\vec{\xi},\tau)$ in the direction of axis x_n. To specify G_{in} uniquely, we have to specify initial conditions and boundary conditions on S.

Initial conditions:

$$G_{in}(\vec{x},t,\vec{\xi},\tau) = 0 \text{ and } \dot{G}_{in}(\vec{x},t,\vec{\xi},\tau) = 0 \quad \text{for} \quad t \le \tau, \vec{x} \ne \vec{\xi}$$

Boundary conditions on S:

Time independent b.c.
$\Rightarrow$ The time origin can obviously be arbitrarily shifted. Then eq. (1.64) implies

$$G_{in}(\vec{x},t;\vec{\xi},\tau) = G_{in}(\vec{x},t-\tau;\vec{\xi},0) = G_{in}(\vec{x},-\tau;\vec{\xi},-t) \tag{1.65}$$

Homogeneous boundary conditions (either the displacement or the traction is zero at every point of the surface)
Recall the reciprocity theorem (1.63):

$$\begin{aligned} &\int_{-\infty}^{\infty} dt \iiint_V \left[\vec{u}(\vec{x},t)\cdot\vec{g}(\vec{x},\tau-t) - \vec{v}(\vec{x},\tau-t)\cdot\vec{f}(\vec{x},t)\right] dV \\ = &\int_{-\infty}^{\infty} dt \iint_S \left[\vec{v}(\vec{x},\tau-t)\cdot\vec{T}(\vec{u}(\vec{x},t),\vec{n}) - \vec{u}(\vec{x},t)\cdot\vec{T}(\vec{v}(\vec{x},\tau-t),\vec{n})\right] dS \end{aligned}$$

Let $\vec{f}$ and $\vec{g}$ be unit impulse forces

$$
\begin{aligned}
f_i(\vec{x},t) &= A\delta(\vec{x}-\vec{\xi}_1)\delta(t-\tau_1)\delta_{im} && (1.66a)\\
g_i(\vec{x},t) &= A\delta(\vec{x}-\vec{\xi}_2)\delta(t+\tau_2)\delta_{in} \qquad \mathrm{A} = 1\ \mathrm{Ns} && (1.66b)
\end{aligned}
$$

Then the displacements $\vec{u}$ due to $\vec{f}$ and $\vec{v}$ due to $\vec{g}$ are

$$
\begin{aligned}
u_i(\vec{x},t) &= AG_{im}(\vec{x},t;\vec{\xi}_1,\tau_1) && (1.67a)\\
v_i(\vec{x},t) &= AG_{in}(\vec{x},t;\vec{\xi}_2,-\tau_2) && (1.67b)
\end{aligned}
$$

$$
\begin{aligned}
\text{Eq.(1.66b)} \quad &\Rightarrow \quad g_i(\vec{x},\tau-t) = A\delta(\vec{x}-\vec{\xi}_2)\delta(\tau-t+\tau_2)\delta_{in} && (1.68)\\
\text{Eq. (1.67b)} \quad &\Rightarrow \quad v_i(\vec{x},\tau-t) = AG_{in}(\vec{x},\tau-t;\vec{\xi}_2,-\tau_2) && (1.69)
\end{aligned}
$$

Insert (1.66a), (1.67a), (1.68) and (1.69) into (1.63)

$$
\int_{-\infty}^{\infty} dt \iiint_V \Big[G_{im}(\vec{x},t;\vec{\xi}_1,\tau_1)\delta(\vec{x}-\vec{\xi}_2)\delta(\tau-t+\tau_2)\delta_{in}
$$
$$
-G_{in}(\vec{x},\tau-t;\vec{\xi}_2,-\tau_2)\delta(\vec{x}-\vec{\xi}_1)\delta(t-\tau_1)\delta_{im}\Big]\, dV = 0
$$

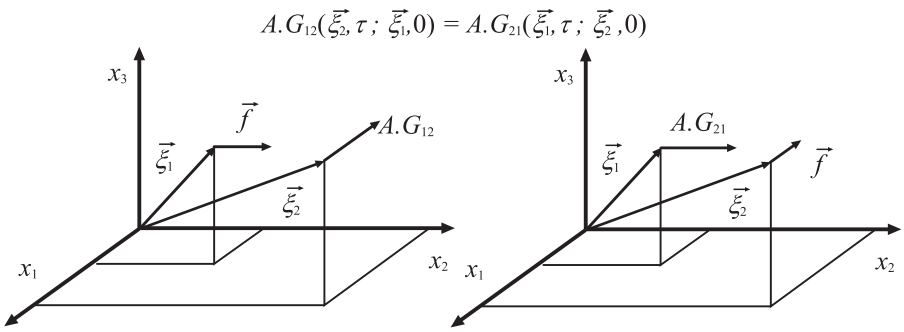
(The integral over S in (1.63) is zero due to homogeneous boundary conditions.)

$$
\begin{aligned}
G_{nm}(\vec{\xi}_2,\tau+\tau_2;\vec{\xi}_1,\tau_1) &- G_{mn}(\vec{\xi}_1,\tau-\tau_1;\vec{\xi}_2,-\tau_2) = 0\\
G_{nm}(\vec{\xi}_2,\tau+\tau_2;\vec{\xi}_1,\tau_1) &= G_{mn}(\vec{\xi}_1,\tau-\tau_1;\vec{\xi}_2,-\tau_2) && (1.70)
\end{aligned}
$$

Let $\tau_1 = \tau_2 = 0$. Then eq. (1.70) implies

$$
G_{nm}(\vec{\xi}_2,\tau;\vec{\xi}_1,0) = G_{mn}(\vec{\xi}_1,\tau;\vec{\xi}_2,0) \tag{1.71}
$$

Relation (1.71) gives a purely spatial reciprocity of Green's function.
Example:



Let $\tau = 0$. Then eq. (1.70) implies

$$
G_{nm}(\vec{\xi}_2,\tau_2;\vec{\xi}_1,\tau_1) = G_{mn}(\vec{\xi}_1,-\tau_1;\vec{\xi}_2,-\tau_2) \tag{1.72}
$$

Relation (1.72) gives a space–time reciprocity of Green's function.

1.10 Representation theorem

Find displacement $\vec{u}$ due to body forces $\vec{f}$ in volume V and to boundary conditions on surface S assuming

$$g_i(\vec{x},t) = A\delta(\vec{x}-\vec{\xi})\delta(t)\delta_{in} \tag{1.73}$$

and corresponding displacement

$$v_i(\vec{x},t) = AG_{in}(\vec{x},t;\vec{\xi},0) \tag{1.74}$$

Insert (1.73) and (1.74) into the reciprocity theorem (1.63)

$$\int_{-\infty}^{\infty} dt \iiint_V \left[u_i(\vec{x},t)A\delta(\vec{x}-\vec{\xi})\delta(\tau-t)\delta_{in} \quad - \quad AG_{in}(\vec{x},\tau-t;\vec{\xi},0)f_i(\vec{x},t)\right] dV =$$

$$\int_{-\infty}^{\infty} dt \iint_S \left[AG_{in}(\vec{x},\tau-t;\vec{\xi},0)T_i\left(\vec{u}(\vec{x},t),\vec{n}\right) \quad - \quad u_i(\vec{x},t)T_i\left(AG_{kn}(\vec{x},\tau-t;\vec{\xi},0),\vec{n}\right)\right] dS \tag{1.75}$$

$$T_i\left(AG_{kn}(\vec{x},\tau-t;\vec{\xi},0),\vec{n}\right) = \tau_{ij}n_j = c_{ijkl}AG_{kn,l}(\vec{x},\tau-t;\vec{\xi},0)n_j \tag{1.76}$$

Inserting (1.76) into (1.75) we obtain

$$\begin{aligned} u_n(\vec{\xi},\tau) \quad = \quad & \int_{-\infty}^{\infty} dt \iiint_V f_i(\vec{x},t)G_{in}(\vec{x},\tau-t;\vec{\xi},0)dV \\ + \quad & \int_{-\infty}^{\infty} dt \iint_S \left[G_{in}(\vec{x},\tau-t;\vec{\xi},0)\ T_i\left(\vec{u}(\vec{x},t),\vec{n}\right)\right. \\ & \left. - \ u_i(\vec{x},t)c_{ijkl}n_j\frac{\partial G_{kn}(\vec{x},\tau-t;\vec{\xi},0)}{\partial \xi_l}\right] dS \end{aligned}$$

Interchanging formally $\vec{x}$ and $\vec{\xi}$ as well as t and τ we have

$$\begin{aligned} \vec{u}_n(\vec{x},t) \quad = \quad & \int_{-\infty}^{\infty} d\tau \iiint_V f_i(\vec{\xi},\tau)G_{in}(\vec{\xi},t-\tau;\vec{x},0)dV(\vec{\xi}) \\ + \quad & \int_{-\infty}^{\infty} d\tau \iint_S \left[G_{in}(\vec{\xi},t-\tau;\vec{x},0)T_i\left(\vec{u}(\vec{\xi},\tau),\vec{n}\right)\right. \\ & \left. - \ u_i(\vec{\xi},\tau)c_{ijkl}(\vec{\xi})n_jG_{kn,l}(\vec{\xi},t-\tau;\vec{x},0)\right] dS(\vec{\xi}) \end{aligned} \tag{1.77}$$

Relation (1.77) gives displacement $\vec{u}$ at a point $\vec{x}$ and time t in terms of contributions due to body force $\vec{f}$ in V, to traction $\vec{T}$ on S and to the displacement $\vec{u}$ itself on S. A disadvantage of

the representation (1.77) is that the involved Green's function corresponds to the impulse source at $\vec{x}$ and observation point at $\vec{\xi}$.
The reciprocity relations for Green's function can be used to replace Green's function in (1.77) by that corresponding to a source at $\vec{\xi}$ and observation point at $\vec{x}$.
Let S be a rigid boundary, i.e., a boundary with zero displacement:

$$G_{in}^{\text{rigid}}(\vec{\xi}, t-\tau; \vec{x}, 0) = 0 \quad \text{for } \vec{\xi} \text{ in } S.$$

The above condition is a homogeneous condition. Therefore, the spatial reciprocity relation (1.71) can be applied:

$$G_{in}^{\text{rigid}}(\vec{\xi}, t-\tau; \vec{x}, 0) = G_{ni}^{\text{rigid}}(\vec{x}, t-\tau; \vec{\xi}, 0)$$

Inserting this into representation relation (1.77) we get

$$\begin{aligned} u_n(\vec{x}, t) &= \int\limits_{-\infty}^{\infty} d\tau \iiint\limits_V f_i(\vec{\xi}, \tau) G_{ni}^{\text{rigid}}(\vec{x}, t-\tau; \vec{\xi}, 0) dV(\vec{\xi}) \\ &- \int\limits_{-\infty}^{\infty} d\tau \iint\limits_S u_i(\vec{\xi}, \tau) c_{ijkl}(\vec{\xi}) n_j \frac{\partial G_{nk}^{\text{rigid}}}{\partial \xi_l}(\vec{x}, t-\tau; \vec{\xi}, 0) dS(\vec{\xi}) \end{aligned} \tag{1.78}$$

Let S be a free surface, i.e., a surface with zero traction:

$$c_{ijkl} n_j \frac{\partial G_{kn}^{\text{free}}(\vec{\xi}, t-\tau; \vec{x}, 0)}{\partial \xi_l} = 0 \quad \text{for } \vec{\xi} \text{ in } S.$$

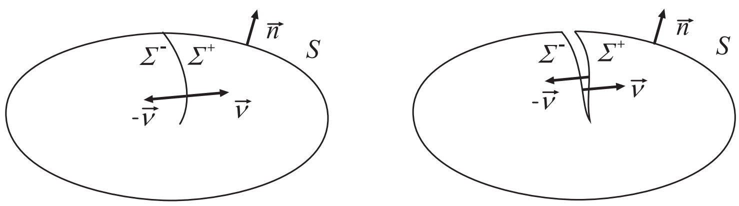
This is again a homogeneous condition and relation (1.71) can be applied. It follows from (1.77) that

$$\begin{aligned} u_n(\vec{x}, t) &= \int\limits_{-\infty}^{\infty} d\tau \iiint\limits_V f_i(\vec{\xi}, \tau) G_{ni}^{\text{free}}(\vec{x}, t-\tau; \vec{\xi}, 0) dV(\vec{\xi}) \\ &+ \int\limits_{-\infty}^{\infty} d\tau \iint\limits_S G_{ni}^{\text{free}}(\vec{x}, t-\tau; \vec{\xi}, 0)\ T_i\left(\vec{u}(\vec{\xi}, \tau), \vec{n}\right) dS(\vec{\xi}) \end{aligned} \tag{1.79}$$

2. SEISMIC SOURCE

2.1 Representation theorems for an internal surface

Consider volume V with an external surface S and two adjacent internal surfaces Σ^+ and Σ^-. In order to understand what are unit normals to surfaces Σ^+ and Σ^-, unfold imaginary the surface $S + \Sigma^+ + \Sigma^-$:



It is clear from the figure that the surfaces and their normal vectors are

$$S : \vec{n} \qquad \Sigma^+ : -\vec{\nu} \qquad \Sigma^- : \vec{\nu}$$

Assume now that Σ^+ and Σ^- are opposite faces of an earthquake fault and a slip on the fault can occur leading to a discontinuity in the displacement (displacements on the Σ^+ side of the internal surface Σ may be different from displacements on the Σ^- side of the surface). Denote the displacement discontinuity as $\left[\vec{u}(\vec{\xi},\tau)\right]$ for $\vec{\xi}$ on Σ and define

$$\left[\vec{u}(\vec{\xi},\tau)\right] = \vec{u}(\vec{\xi},\tau)|_{\Sigma^+} - \vec{u}(\vec{\xi},\tau)|_{\Sigma^-} \tag{2.1}$$

Since the displacement inside V is discontinuous, the equation of motion is not satisfied inside V, i.e., in the interior of surface S. It is, however, satisfied in the interior of the surface $S+\Sigma^+ +\Sigma^-$. Then the above representation relations (1.77) – (1.79) can be applied to this interior.
The surface S may represent the Earth's free surface. We will assume that both $\vec{u}$ and G_{in} satisfy homogeneous boundary conditions on S. Therefore, the surface integral over S in relation (1.77) will be zero. Then

$$\begin{aligned} u_n(\vec{x},t) &= \int\limits_{-\infty}^{\infty} d\tau \iiint\limits_V f_i(\vec{\eta},\tau) G_{ni}(\vec{x},t-\tau;\vec{\eta},0) dV(\vec{\eta}) \\ &+ \int\limits_{-\infty}^{\infty} d\tau \left\{ \iint\limits_{\Sigma^+} G_{ni}(\vec{x},t-\tau;\vec{\xi},0)\; T_i\left(\vec{u}(\vec{\xi},\tau),-\vec{\nu}\right) d\Sigma^+ \right. \\ &\qquad + \iint\limits_{\Sigma^-} G_{ni}(\vec{x},t-\tau;\vec{\xi},0)\; T_i\left(\vec{u}(\vec{\xi},\tau),\vec{\nu}\right) d\Sigma^- \end{aligned}$$

$$- \iint_{\Sigma^+} u_i(\vec{\xi},\tau) c_{ijkl} (-\nu_j) \frac{\partial G_{nk}(\vec{x}, t-\tau; \vec{\xi}, 0)}{\partial \xi_l} d\Sigma^+$$
$$\left. - \iint_{\Sigma^-} u_i(\vec{\xi},\tau) c_{ijkl}\ \nu_j \frac{\partial G_{nk}(\vec{x}, t-\tau; \vec{\xi}, 0)}{\partial \xi_l} d\Sigma^- \right\}$$

$$\begin{aligned} u_n(\vec{x},t) &= \int_{-\infty}^{\infty} d\tau \iiint_V f_i(\vec{\eta},\tau) G_{ni}(\vec{x}, t-\tau; \vec{\eta}, 0) dV(\vec{\eta}) \\ &+ \int_{-\infty}^{\infty} d\tau \iint_{\Sigma} \left\{ \left[u_i(\vec{\xi},\tau) c_{ijkl} \nu_j \frac{\partial G_{nk}(\vec{x}, t-\tau; \vec{\xi}, 0)}{\partial \xi_l} \right] \right. \\ &- \left. \left[G_{ni}(\vec{x}, t-\tau; \vec{\xi}, 0)\ T_i\left(\vec{u}(\vec{\xi},\tau), \vec{\nu} \right) \right] \right\} d\Sigma \end{aligned} \tag{2.2}$$

Brackets [ ] are used for the difference between values on Σ^+ and Σ^-.
Consider now the boundary conditions on Σ. We know (see the reciprocity theorem 1.60) that $\vec{u}$ and G_{ni} may be due to different initial and boundary conditions. While conditions for $\vec{u}$ must be appropriate for a fault, conditions for G_{ni} can be chosen arbitrarily. Obviously, we choose them so that they can be useful for representation of $\vec{u}$.
Conditions for $\vec{u}$ and $\vec{T}(\vec{u},\vec{\nu})$ on Σ:
slip on the fault $\quad\Rightarrow\quad [\vec{u}] \neq 0$
spontaneous rupture on the fault $\quad\Rightarrow\quad \left[\vec{T}(\vec{u},\vec{\nu})\right] = 0$
Conditions for G_{ni} on Σ:
Let Σ be transparent for G_{ni}, i.e., let G_{ni} satisfy the equation of motion (1.64) even on Σ. Then

$$\left[G_{ni}(\vec{x}, t-\tau; \vec{\xi}, 0) \right] = 0$$

and

$$\left[\frac{\partial G_{nk}(\vec{x}, t-\tau; \vec{\xi}, 0)}{\partial \xi_l} \right] = 0$$

i.e., G_{ni} and its derivatives are continuous on Σ.
In addition to the above boundary conditions let us assume zero body forces for $\vec{u}$:

$$\vec{f}(\vec{\eta},\tau) = 0 \quad \text{in} \quad V.$$

Then we get from (2.2)

$$u_n(\vec{x},t) = \int_{-\infty}^{\infty} d\tau \iint_{\Sigma} \left[u_i(\vec{\xi},\tau) \right] c_{ijkl} \nu_j \frac{\partial G_{nk}(\vec{x}, t-\tau; \vec{\xi}, 0)}{\partial \xi_l} d\Sigma \tag{2.3}$$

Relations (2.2) and (2.3) are important representation relations expressing displacement $\vec{u}$ inside the volume V. The first one, (2.2) is general and allows to consider discontinuous $\vec{u}$ and $\vec{T}(\vec{u},\vec{\nu})$

on Σ as well as G_{ni} and its derivative, and nonzero body forces in V. The second one assumes transparency of Σ for G_{ni} and no body forces for $\vec{u}$. It expresses displacement $\vec{u}$ at some point $\vec{x}$ and time t as an integral superposition of spatial derivatives of the Green's function weighted by displacement discontinuity over the fault surface Σ.

2.2 Body-force equivalents

The representation (1.82) does not directly involve any body forces. However it gives displacement at $(\vec{x}, t)$ as an integral over contributing Green's functions and each of the Green's functions is set up by a body force. Therefore, there must by some sense in which an active fault surface can be represented as a surface distribution of body forces.

Making no assumptions on $[\vec{u}]$ and $[\vec{T}(\vec{u},\ \vec{n})]$ across Σ and assuming Σ transparent to G_{ni} we have from (1.81)

$$\begin{aligned} \vec{u}_n(\vec{x},\ t) &= \int_{-\infty}^{\infty} \mathrm{d}\tau \iiint\limits_V f_p(\vec{\eta},\ \tau) G_{np}(\vec{x}, t-\tau, \vec{\eta}, 0) \mathrm{d}^3\vec{\xi} \\ &+ \int_{-\infty}^{\infty} \mathrm{d}\tau \iint\limits_\Sigma [u_i(\vec{\xi},\ \tau)] c_{ijpq} \nu_j \frac{\partial G_{np}(\vec{x}, t-\tau, \vec{\xi}, 0)}{\partial \xi_q} \mathrm{d}^2\vec{\xi} \\ &- \int_{-\infty}^{\infty} \mathrm{d}\tau \iint\limits_\Sigma [T_p(\vec{u}(\vec{\xi}, \tau), \vec{\nu})] G_{np}(\vec{x}, t-\tau, \vec{\xi}, 0) \mathrm{d}^2\vec{\xi} \end{aligned} \tag{2.4}$$

The discontinuities on Σ can be localized within V by using the delta function $\delta(\vec{\eta} - \vec{\xi})$. For example, $[\vec{T}]\mathrm{d}^2\vec{\xi}$ has the dimension of force, and its body-force distribution (i.e., force / unit-volume) is $[\vec{T}]\delta(\vec{\eta} - \vec{\xi})\mathrm{d}^2\vec{\xi}$ as $\vec{\eta}$ varies throughout V.

$$\begin{aligned} &- \iint\limits_\Sigma [T_p(\vec{u}(\vec{\xi}, \tau), \vec{\nu})] G_{np}(\vec{x}, t-\tau, \vec{\xi}, 0) \mathrm{d}^2\vec{\xi} \\ = &- \iint\limits_\Sigma \left\{ \iiint\limits_V [T_p(\vec{u}(\vec{\xi}, \tau), \vec{\nu})] \delta(\vec{\eta} - \vec{\xi}) G_{np}(\vec{x}, t-\tau, \vec{\eta}, 0) \mathrm{d}^3\vec{\eta} \right\} \mathrm{d}^2\vec{\xi} \\ = &\iiint\limits_V \left\{ - \iint\limits_\Sigma [T_p(\vec{u}(\vec{\xi}, \tau), \vec{\nu})] \delta(\vec{\eta} - \vec{\xi}) \mathrm{d}^2\vec{\xi} \right\} G_{np}(\vec{x}, t-\tau, \vec{\eta}, 0) \mathrm{d}^3\vec{\eta} \end{aligned}$$

Thus the traction discontinuity contributes the displacement

$$\begin{aligned} \vec{u}_n^{[\vec{T}]}(\vec{x},\ t) &= \int_{-\infty}^{\infty} \mathrm{d}\tau \iiint\limits_V \left\{ - \iint\limits_\Sigma [T_p(\vec{u}(\vec{\xi}, \tau), \vec{\nu})] \delta(\vec{\eta} - \vec{\xi}) \mathrm{d}^2\vec{\xi} \right\} G_{np}(\vec{x}, t-\tau, \vec{\eta}, 0) \mathrm{d}^3\vec{\eta} \\ &= \int_{-\infty}^{\infty} \mathrm{d}\tau \iiint\limits_V f_p^{[\vec{T}]}(\vec{\eta}, \tau) G_{np}(\vec{x}, t-\tau, \vec{\eta}, 0) \mathrm{d}^3\vec{\eta} \end{aligned}$$

where

$$f_p^{[\vec{T}]}(\vec{\eta},\tau) = -\iint\limits_{\Sigma} [T_p(\vec{u}(\vec{\xi},\tau),\vec{\nu})]\delta(\vec{\eta}-\vec{\xi})\mathrm{d}^2\vec{\xi} \tag{2.5}$$

(2.5) has exactly the same form as the first term in the right-hand side od eq. (2.4). Therefore, $\vec{f}^{[\vec{T}]}(\vec{\eta},\tau)$ is the body-force equivalent of a traction discontinuity on Σ.

Because the term with the displacement discontinuity in eq. (2.4) contains spatial derivative of G_{np}, we have to use the derivative of a delta function, $\partial\delta(\vec{\eta}-\vec{\xi})/\partial\eta_q$, to localize Σ within V. The derivative has the property

$$\frac{\partial G_{np}(\vec{x},t-\tau,\vec{\xi},0)}{\partial\xi_q} = -\iiint\limits_{V} \frac{\partial\delta(\vec{\eta}-\vec{\xi})}{\partial\eta_q} G_{np}(\vec{x},t-\tau,\vec{\eta},0)\mathrm{d}^3\vec{\eta} \tag{2.6a}$$

Then

$$\begin{aligned}
&\iint\limits_{\Sigma} [u_i(\vec{\xi},\tau)]c_{ijpq}\nu_j \frac{\partial G_{np}(\vec{x},t-\tau,\vec{\xi},0)}{\partial\xi_q}\mathrm{d}^2\vec{\xi} \\
=\;& \iint\limits_{\Sigma}\left\{-\iiint\limits_{V}[u_i(\vec{\xi},\tau)]c_{ijpq}\nu_j\frac{\partial\delta(\vec{\eta}-\vec{\xi})}{\partial\eta_q}G_{np}(\vec{x},t-\tau,\vec{\eta},0)\mathrm{d}^3\vec{\eta}\right\}\mathrm{d}^2\vec{\xi} \\
=\;& \iiint\limits_{V}\left\{-\iint\limits_{\Sigma}[u_i(\vec{\xi},\tau)]c_{ijpq}\nu_j\frac{\partial\delta(\vec{\eta}-\vec{\xi})}{\partial\eta_q}\mathrm{d}^2\vec{\xi}\right\}G_{np}(\vec{x},t-\tau,\vec{\eta},0)\mathrm{d}^3\vec{\eta}
\end{aligned}$$

The displacement discontinuity contributes the displacement

$$u_n^{[\vec{u}]}(\vec{x},\,t) = \int_{-\infty}^{\infty}\mathrm{d}\tau\iiint\limits_{V}\left\{-\iint\limits_{\Sigma}[u_i(\vec{\xi},\tau)]c_{ijpq}\nu_j\frac{\partial\delta(\vec{\eta}-\vec{\xi})}{\partial\eta_q}\mathrm{d}^2\vec{\xi}\right\}G_{np}(\vec{x},t-\tau,\vec{\eta},0)\mathrm{d}^3\vec{\eta} \tag{2.6b}$$

Denote

$$f_p^{[\vec{u}]}(\vec{\eta},\tau) = -\iint\limits_{\Sigma}[u_i(\vec{\xi},\tau)]c_{ijpq}\nu_j\frac{\partial\delta(\vec{\eta}-\vec{\xi})}{\partial\eta_q}\mathrm{d}^2\vec{\xi} \tag{2.6c}$$

Then

$$u_n^{[\vec{u}]}(\vec{x},\,t) = \int_{-\infty}^{\infty}\mathrm{d}\tau\iiint\limits_{V} f_p^{[\vec{u}]}(\vec{\eta},\tau)G_{np}(\vec{x},t-\tau,\vec{\eta},0)\mathrm{d}^3\vec{\eta}$$

has exactly the same form as the first term in the right-hand side of eq. (2.4). Therefore $\vec{f}^{[\vec{u}]}(\vec{\eta},\tau)$ is the body-force equivalent of a displacement discontinuity on Σ.

The body-force equivalent of the discontinuity can be expressed in an alternative way.

$$\frac{\partial\delta(\vec{\eta}-\vec{\xi})}{\partial\eta_q} = -\frac{\partial\delta(\vec{\eta}-\vec{\xi})}{\partial\xi_q}$$

Define symbolic surface δ-function $\delta(\Sigma)$ by

$$\iint\limits_{\Sigma}\ldots\mathrm{d}^2\vec{\xi} = \iiint\limits_{V}\ldots\delta(\Sigma)\mathrm{d}^3\vec{\xi}\quad([\delta(\Sigma)]^U = \mathrm{m}^{-1})$$

Then the surface integral in (2.6b) can be rewritten as

$$\iiint_V [u_i(\vec{\xi},\tau)]c_{ijpq}\nu_j \frac{\partial\delta(\vec{\eta}-\vec{\xi})}{\partial\xi_q}\delta(\Sigma)\mathrm{d}^3\vec{\xi}$$

Now using property (2.6a) with interchanged $\vec{\eta}$ and $\vec{\xi}$

$$\frac{\partial F(\vec{\eta})}{\partial\eta_q} = -\iiint_V \frac{\partial\delta(\vec{\xi}-\vec{\eta})}{\partial\xi_q}F(\vec{\xi})\mathrm{d}^3\vec{\xi}$$

we rewrite the surface integral and rewrite the expression for the contribution to the displacement

$$u_n^{[\vec{u}]}(\vec{x},\,t) = \int_{-\infty}^{\infty}\mathrm{d}\tau\iiint_V -\frac{\partial}{\partial\eta_q}\Big([u_i(\vec{\eta},\tau)]c_{ijpq}\nu_j\delta(\Sigma)\Big)G_{np}(\vec{x},t-\tau,\vec{\eta},0)\mathrm{d}^3\vec{\eta}$$

We see now that

$$f_p^{[\vec{u}]}(\vec{\eta},\,\tau) = -\frac{\partial}{\partial\eta_q}\Big([u_i(\vec{\eta},\tau)]c_{ijpq}\nu_j\delta(\Sigma)\Big) \tag{2.6d}$$

The body-force equivalents (2.5) and (1.85) are valid for a general heterogeneous anisotropic medium. They depend on properties of the medium only at the fault surface itself. Since faulting within the volume V is an internal process, the total momentum and total angular momentum must be conserved. We can verify that the total force and total moment of the forces are equal to zero :

$$\iiint_V \vec{f}^{[\vec{u}]}(\vec{\eta},\,\tau)\mathrm{d}^3\vec{\xi} = \vec{0} \qquad \text{for all } \tau \tag{2.7}$$

$$\iiint_V (\vec{\eta}-\vec{\eta}^*)\times\vec{f}^{[\vec{u}]}(\vec{\eta},\,\tau)\mathrm{d}^3\vec{\eta} = \vec{0} \qquad \text{for all } \tau \text{ and fixed } \vec{\eta}^* \tag{2.8}$$

Verify (2.7). Inserting (2.6c) we get

$$\begin{aligned}
&\iiint_V\Big\{-\iint_\Sigma [u_i(\vec{\xi},\tau)]c_{ijpq}\nu_j\frac{\partial\delta(\vec{\eta}-\vec{\xi})}{\partial\eta_q}\mathrm{d}^2\vec{\xi}\Big\}\mathrm{d}^3\vec{\eta}\\
= \quad & -\iint_\Sigma [u_i(\vec{\xi},\tau)]c_{ijpq}\nu_j\underbrace{\Big\{\iiint_V\frac{\partial\delta(\vec{\eta}-\vec{\xi})}{\partial\eta_q}\mathrm{d}^3\vec{\eta}\Big\}}_{=\iint_S\delta(\vec{\eta}-\vec{\xi})\eta_q\mathrm{d}^2\vec{\eta}}\mathrm{d}^2\vec{\xi}
\end{aligned}$$

this integral vanishes because S and Σ have at most a common curve, not surface

Verify (2.8). In the index notation (2.8) reads

$$\iiint_V \varepsilon_{mnp}(\eta_n-\eta_n^*)f_p^{[\vec{u}]}(\vec{\eta},\,\tau)\mathrm{d}^3\vec{\eta} = \vec{0} \tag{2.8a}$$

Inserting (2.6c) into (2.8a) we get

$$\iiint\limits_V \varepsilon_{mnp}(\eta_n - \eta_n^*)\Big\{-\iint\limits_\Sigma [u_i(\vec{\xi},\tau)]c_{ijpq}\nu_j \frac{\partial\delta(\vec{\eta}-\vec{\xi})}{\partial\eta_q}\mathrm{d}^2\vec{\xi}\Big\}\mathrm{d}^3\vec{\eta}$$
$$= -\iint\limits_\Sigma [u_i(\vec{\xi},\tau)]c_{ijpq}\nu_j\Big\{\iiint\limits_V \varepsilon_{mnp}(\eta_n - \eta_n^*)\frac{\partial\delta(\vec{\eta}-\vec{\xi})}{\partial\eta_q}\mathrm{d}^3\vec{\eta}\Big\}\mathrm{d}^2\vec{\xi}$$

Due to the property of the delta function derivative

$$\iiint\limits_V \varepsilon_{mnp}(\eta_n - \eta_n^*)\frac{\partial\delta(\vec{\eta}-\vec{\xi})}{\partial\eta_q}\mathrm{d}^3\vec{\eta} = \frac{\partial}{\partial\xi_q}\Big(\varepsilon_{mnp}(\xi_n - \xi_n^*)\Big) = \varepsilon_{mnp}\delta_{nq} = \varepsilon_{mqp}$$

Then

$$-\iint\limits_\Sigma [u_i(\vec{\xi},\tau)]c_{ijpq}\nu_j\Big\{\iiint\limits_V \varepsilon_{mnp}(\eta_n - \eta_n^*)\frac{\partial\delta(\vec{\eta}-\vec{\xi})}{\partial\eta_q}\mathrm{d}^3\vec{\eta}\Big\}\mathrm{d}^2\vec{\xi}$$
$$= -\iint\limits_\Sigma [u_i(\vec{\xi},\tau)]c_{ijpq}\nu_j\varepsilon_{mqp}\mathrm{d}^2\vec{\xi}$$

and

$$\begin{aligned}\varepsilon_{mpq}c_{ijpq} &= \frac{1}{2}(\varepsilon_{mpq}c_{ijpq} + \varepsilon_{mpq}c_{ijpq}) = \frac{1}{2}(-\varepsilon_{mqp}c_{ijpq} + \varepsilon_{mpq}c_{ijpq}) = \\ &\frac{1}{2}(-\varepsilon_{mqp}c_{ijqp} + \varepsilon_{mpq}c_{ijpq}) = \frac{1}{2}(-\varepsilon_{mpq}c_{ijpq} + \varepsilon_{mpq}c_{ijpq}) = 0\end{aligned}$$

and the expression vanishes.

An example of a body force that is equivalent to the discontinuity :
Consider a point force with magnitude F applied at $(0, 0, h)$ at time $\tau = 0$ in a vertical direction and held steady :

$$\vec{f}(\vec{\eta},\tau) = (0,\ 0,\ F)\delta(\eta_1)\delta(\eta_2)\delta(\eta_3 - h)H(\tau) \tag{2.9}$$

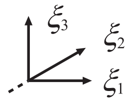
(compare with eq. 1.1)
Force (2.9) can be interpreted as a discontinuity in traction across one point of the plane $\xi_3 = h$ with

$$[\vec{T}(\vec{\xi},\tau)]^{\vec{\xi}=(\xi_1,\xi_2,h^+)}_{\vec{\xi}=(\xi_1,\xi_2,h^-)} = -(0,\ 0,\ F)\delta(\xi_1)\delta(\xi_2)H(\tau) \tag{2.10}$$

i.e., τ_{13} and τ_{23} are continuous, and the jump is in τ_{33}. The equivalence of (2.10) and (2.9) can be shown by inserting (2.10) into (2.5) :

$$\begin{aligned}f_p^{[\vec{T}]}(\vec{\eta},\tau) &= -\iint\limits_\Sigma [T_p(\vec{u}(\vec{\xi},\tau),\vec{\nu})]\delta(\vec{\eta}-\vec{\xi})\mathrm{d}^2\vec{\xi} \\ &= -\iint\limits_\Sigma \Big\{-(0,\ 0,\ F)\delta(\xi_1)\delta(\xi_2)H(\tau)\Big\}\delta(\vec{\eta}-\vec{\xi})\mathrm{d}^2\vec{\xi} \\ &= -\iint\limits_\Sigma \Big\{-(0,\ 0,\ F)\delta(\xi_1)\delta(\xi_2)H(\tau)\Big\}\delta(\eta_1-\xi_1)\delta(\eta_2-\xi_2)\delta(\eta_3-\xi_3)\mathrm{d}^2\vec{\xi} \\ &= (0,\ 0,\ F)\delta(\eta_1)\delta(\eta_2)\delta(\eta_3 - h)H(\tau)\end{aligned}$$

A simple example of slip on a buried fault



fault surface Σ lies in the plane $\xi_3 = 0$
$\vec{\nu} = (0, 0, 1)$, $\Sigma^+ = \xi_3 = 0^+$,$\Sigma^- = \xi_3 = 0^-$
assume a "fault slip" i.e., $[\vec{u}]$ parallel to Σ : $[\vec{u}] = ([u_1], 0, 0)$
Specify representation (1.82) for our fault slip :

$$u_n(\vec{x}, t) = \int\limits_{-\infty}^{\infty} \mathrm{d}\tau \iint\limits_{\Sigma} [u_1] c_{13kl} \frac{\partial G_{nk}(\vec{x}, t-\tau, \vec{\xi}, 0)}{\partial \xi_l} \mathrm{d}\Sigma$$

Recall Hooke's law (e.q. (1.56)) for the isotropic medium :

$$\begin{aligned} c_{ijkl} &= \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ c_{13kl} &= \mu(\delta_{1k}\delta_{3l} + \delta_{1l}\delta_{3k}) \\ c_{1313} &= \mu \\ c_{1331} &= \mu \end{aligned}$$

Then

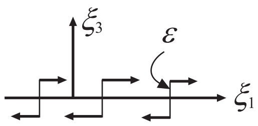
$$u_n(\vec{x}, t) = \int\limits_{-\infty}^{\infty} \mathrm{d}\tau \iint\limits_{\Sigma} \mu[u_1] \left(\frac{\partial G_{n1}}{\partial \xi_3} + \frac{\partial G_{n3}}{\partial \xi_1} \right) \mathrm{d}\Sigma \tag{2.11}$$

By definition

$$\frac{\partial G_{n1}(\vec{x}, t-\tau, \vec{\xi}, 0)}{\partial \xi_3} = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \{ G_{n1}(\vec{x}, t-\tau, \vec{\xi} + \varepsilon\hat{\xi}_3, 0) - G_{n1}(\vec{x}, t-\tau, \vec{\xi} - \varepsilon\hat{\xi}_3, 0) \} \tag{2.12}$$

where $\hat{\xi}_3$ is a unit vector in the ξ_3-direction.

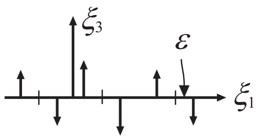
The expression in (2.12) represents the single-couple distribution



Similarly

$$\frac{\partial G_{n3}(\vec{x}, t-\tau, \vec{\xi}, 0)}{\partial \xi_1} = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \{ G_{n3}(\vec{x}, t-\tau, \vec{\xi} + \varepsilon\hat{\xi}_1, 0) - G_{n3}(\vec{x}, t-\tau, \vec{\xi} - \varepsilon\hat{\xi}_1, 0) \} \tag{2.13}$$

The expression in (2.13) represents the single-couple distribution

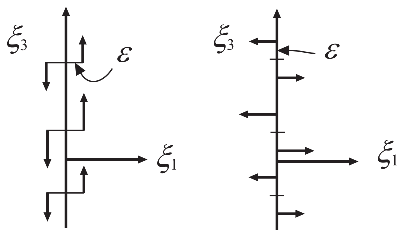


The two single-couple distributions taken together are equivalent (in the sense of radiating the same waves) to the fault slip. Note that there is no net couple and no net force acting on any element of area in the fault plane $\xi_3 = 0$.

Now consider that the fault surface Σ lies in the plane $\xi_1 = 0$
($\vec{\nu} = (1, 0, 0)$ and $[\vec{u}] = (0, 0, [u_3])$)
Then from (1.82) we get

$$u_n(\vec{x}, t) = \int_{-\infty}^{\infty} \mathrm{d}\tau \iint_{\Sigma} \mu[u_3]\Big(\frac{\partial G_{n3}}{\partial \xi_1} + \frac{\partial G_{n1}}{\partial \xi_3}\Big)\mathrm{d}\Sigma \tag{2.14}$$

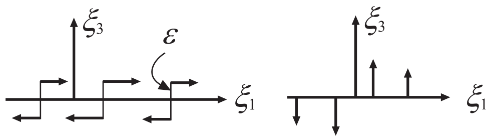
Compare (2.14) with (2.11) and the two single-couple distributions.



The body-force representation (2.11) and (2.13) is not unique. This can be illustrated if (e.g.) the second term in eq. (2.11) is integrated by parts :

$$u_n(\vec{x}, t) = \int_{-\infty}^{\infty} \mathrm{d}\tau \Big\{ \iint_{\Sigma} \mu[u_1]\frac{\partial G_{n1}}{\partial \xi_3}\mathrm{d}\Sigma + \int_{\xi_2} \underbrace{(\mu[u_1]G_{n3})|_{\xi_1'}^{\xi_1''}}_{=0} \mathrm{d}\xi_2 - \iint_{\Sigma} \frac{\partial \mu[u_1]}{\partial \xi_1} G_{n3}\mathrm{d}\Sigma \Big\} \tag{2.15}$$

This force system is illustrated in figure



In general (in our example), there is always a single couple made up of in the same direction fault-surface displacement. However, a complete equivalent to fault slip as another part, which may be a single force, a single couple, or an appropriate linear combination of these alternatives.

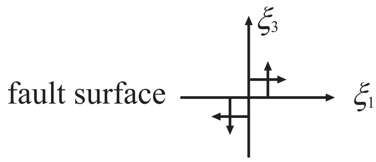
The above findings illustrate the limited utility of force equivalents for studying the dynamics of fault slip. It is the whole fault surface that radiates seismic waves and we cannot assess from (2.11) or (2.15) the actual contribution made to the radiation by individual elements of fault area. This makes sense in physical terms, because individual elements of fault area do not move dynamically in isolation from other parts of the fault.

Force equivalents (usually chosen as the double-couple distribution) find their main use only when the slip function $\vec{u}[\vec{\xi}, \tau]$ is determined (or guessed), and then they are important, because they enable one to compute the radiation by weighting Green's functions.

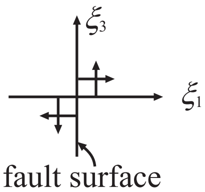
2.3 Effective point source

Only waves with wavelengths much larger than linear dimensions of Σ are observed at large distances. Higher-frequency motion is relatively weak even close to the fault and is more attenuated during propagation. Therefore, at large distances, fault surface Σ acts like a point source.

Now return to our example of the fault slip on the plane perpendicular to ξ_3 - see eq. (2.11). In the case of the point-source approximation, the two single-couple distributions become two single couples, i.e., one double couple, as it is illustrated in the figure.



Consider now the fault slip represented by eq. (2.14). In the case of the point-source approximation, the two single couple distributions become exactly the same double couple as in the above example. The difference is in the orientation of the fault surface.



We see that, in principle, it is impossible to identify which is the fault plane and which its auxiliary plane perpendicular to the fault plane (and slip).

2.4 Moment density tensor

Recall the displacement due to a displacement discontinuity given by eq. (1.82) and use symbol $*$ for a time convolution :

$$\vec{u}_n(\vec{x}, t) = \iint_{\Sigma} c_{ijpq}\nu_j[u_i(\vec{\xi}, t)] * \frac{\partial G_{np}(\vec{x}, t, \vec{\xi}, 0)}{\partial \xi_q} \mathrm{d}^2\vec{\xi} \tag{2.16}$$

From the previous analysis we know, that an expression in the form $S_{pq} * \partial G_{np}/\partial \xi_q \mathrm{d}^2\vec{\xi}$ is a contribution to the n-th component of displacement at $\vec{x}$ and can be represented as a single couple at $\vec{\xi}$ with an arm in the direction of the derivative - q and forces in the p direction, in general there are 9 such couples. Each such expression in (2.16) is weighted by a coefficient : $S_{pq} = [u_i]\nu_j c_{ijpq}$ which is the strength of the couple.

$$\Big[[u_i]\nu_j c_{ijpq}\Big]^U = \mathrm{Nm.m}^{-2} \qquad \text{i.e., moment per unit area}$$

This is understandable, because the contribution from $\vec{\xi}$ has to be a surface density weighted by the infinitezimal area element $\mathrm{d}\Sigma$ to give a moment contribution.

Therefore, define the moment density tensor

$$m_{pq} \equiv [u_i]\nu_j c_{ijpq} \tag{2.17}$$

Tensor m_{pq} is symmetric due to symmetry $c_{ijpq} = c_{ijqp}$ (see eq. (1.42)).
Equation (2.16) can be written as

$$\vec{u}_n(\vec{x}, t) = \iint_{\Sigma} m_{pq} * G_{np,q}\, \mathrm{d}\Sigma \tag{2.18}$$

Consider now an isotropic medium. It follows from (1.56) and (2.17) that

$$m_{pq} = [u_k]\nu_k\lambda\delta_{pq} + \mu([u_q]\nu_p + [u_p]\nu_q) \tag{2.19}$$

Assuming a slip parallel to Σ, i.e., $[\vec{u}].\vec{\nu} = [u_k]\nu_k = 0$,

$$m_{pq} = \mu([u_q]\nu_p + [u_p]\nu_q) \tag{2.20}$$

Examples :
Let Σ lie in the plane $\xi_3 = 0$, i.e., $\vec{\nu} = (0, 0, 1)$, and fault slip only in the ξ_1 direction, i.e., $[\vec{u}] = ([u_1], 0, 0)$. As we saw before, the (1,3)_(3,1) double couple is equivalent to this discontinuity in displacement. Eq. (2.20) gives

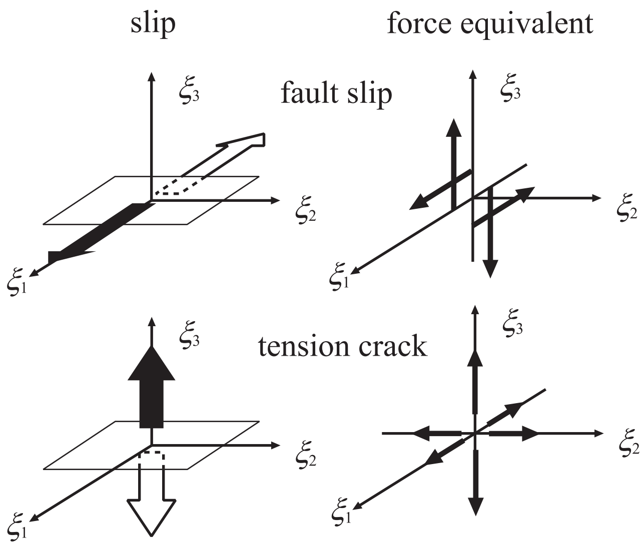
$$\bar{\bar{m}} = \begin{pmatrix} 0 & 0 & \mu[u_1] \\ 0 & 0 & 0 \\ \mu[u_1] & 0 & 0 \end{pmatrix} \tag{2.21}$$

Let Σ lie in the plane $\xi_3 = 0$, $\vec{\nu} = (0, 0, 1)$, and only the $[u_3]$ component of slip $[\vec{u}] = (0, 0, [u_3])$ is nonzero. This is the case of a tension crack for which eq. (2.20) gives

$$\bar{\bar{m}} = \begin{pmatrix} \lambda[u_3] & 0 & 0 \\ 0 & \lambda[u_3] & 0 \\ 0 & 0 & (\lambda + 2\mu)[u_3] \end{pmatrix} \tag{2.22}$$

We can see that the tension crack is equivalent to a superposition of three vector dipoles, (1,1)_(2,2)_(3,3), with magnitude in the ratio $1 : 1 : 1 + 2\mu/\lambda$.

Illustration to the two examples :



2.5 Effective point source and scalar seismic moment

Recall representation (2.18)

$$u_n(\vec{x},t) = \iint\limits_{\Sigma} m_{pq} * G_{np,q} \mathrm{d}\Sigma$$

In the point-source approximation

$$u_n(\vec{x},t) \doteq G_{np,q} * \iint\limits_{\Sigma} m_{pq} \mathrm{d}\Sigma$$

Define the moment tensor M_{pq} :

$$M_{pq} = \iint\limits_{\Sigma} m_{pq} \mathrm{d}\Sigma \tag{2.23a}$$

$$\text{i.e.,} \quad m_{pq} = \frac{\mathrm{d}M_{pq}}{\mathrm{d}\Sigma} \tag{2.23b}$$

Then,

$$u_n(\vec{x},t) \doteq M_{pq} * G_{np,q} \tag{2.24}$$

Consider a fault slip, eq. (2.21), and assume an average shear modulus $\bar{\mu}$ on the fault plane and average slip $\overline{[u_1]}$. Then

$$M_{31} = M_{13} = \iint\limits_{\Sigma} \mu[u_1] \mathrm{d}\Sigma \doteq \bar{\mu}\overline{[u_1]}A$$

where $A = \iint\limits_{\Sigma} \mathrm{d}\Sigma$.

Define a scalar seismic moment M_0 :

$$M_0 = \bar{\mu}\overline{[u_1]}A \tag{2.25}$$

$$[M_0]^U = \frac{\mathrm{N}}{\mathrm{m}^2}\mathrm{m.m}^2 = \mathrm{N.m}$$

Then the moment tensor for an effective point source is

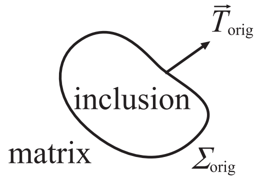
$$\bar{\bar{M}} = \begin{pmatrix} 0 & 0 & M_0 \\ 0 & 0 & 0 \\ M_0 & 0 & 0 \end{pmatrix} \tag{2.26}$$

The representation (2.18) can be interpreted as an areal distribution of point sources, each having the moment tensor $\bar{\bar{m}}\mathrm{d}\Sigma$

2.6 Volume Source

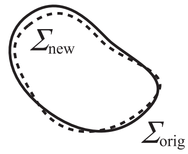
Procedure of imaginary cutting, straining and welding :

1. Separate the source
   - ∘ cut along Σ enclosing the source
   - ∘ remove the source volume

   The removed material is held in its original shape by tractions having the same value over Σ as the tractions imposed across Σ by the matrix before the cutting operation.

2. Let the source undergo transformational strain Δe_{rs}
   - without changing the stress within the inclusion.

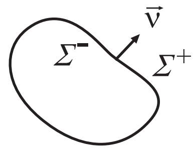
   It is this stress-free strain that characterizes the source. (Examples of stress-free strain processes : phase transition, thermal expansion, some plastic deformations.) The stress-free strain is a static concept.

3. Apply additional surface tractions that restore the inclusion to its original shape. This results in an additional stress field $-\Delta\tau_{pq} = -c_{pqrs}\Delta e_{rs}$ throughout the inclusion. The additional tractions applied on Σ are $-c_{pqrs}\Delta e_{rs}\nu_q$. Since $\Delta\tau_{pq}$ is a static field,

$$\rho \underbrace{\Delta u_{p,tt}}_{=0} = \Delta\tau_{pq,q} + \underbrace{f_p}_{=0} \qquad \Rightarrow \qquad \Delta\tau_{pq,q} = 0$$

   The stress in the matrix is still unchanged, being held by tractions imposed across the internal surface Σ (having the same value as tractions imposed on the matrix by the inclusion before it was cut out).

4. Put the inclusion back in its hole (which has exactly the correct shape) and weld the material across the cut.
   Due to the additional traction on Σ (the surface of the inclusion; step 3), there is a traction discontinuity across Σ :

$$T_p|_{\Sigma^+} - T_p|_{\Sigma^-} = T_p|_{\Sigma} - (T_p|_{\Sigma} - c_{pqrs}\Delta e_{rs}\nu_q) = +c_{pqrs}\Delta e_{rs}\nu_q$$

   (The traction on Σ^- is due to the applied (in step 3) surface forces, that are external to the source and which act on the inclusion to maintain its correct shape.)

5. Release the applied surface forces over Σ^-. Since traction is actually continuous across Σ, this amounts to imposing an apparent traction discontinuity of $-(c_{pqrs}\Delta e_{rs})\nu_q$. The elastic field produced in the matrix by the whole process is that due to the apparent traction discontinuity across Σ.

The above procedure can be extended to a dynamic case of seismic wave generation, since at a given time, a transformational strain can be defined for the unrestrained material. For each instant it is still true, that $\Delta\tau_{pq,q} = 0$. The displacement generated by the traction discontinuity is given by the last term in eq. (2.4). Putting

$$[T_p] = -(c_{pqrs}\Delta e_{rs})\nu_q$$

in (2.4) we get

$$u_n(\vec{x},t) = \int_{-\infty}^{\infty} \mathrm{d}\tau \iint_{\Sigma} c_{pqrs}\Delta e_{rs}\nu_q G_{np}(\vec{x},t-\tau,\vec{\xi},0)\mathrm{d}^2\vec{\xi} \tag{2.27}$$

If the integrand and its derivatives with respect to $\vec{\xi}$ are continuous, we can apply the Gauss theorem to obtain

$$u_n(\vec{x},t) = \int_{-\infty}^{\infty} \mathrm{d}\tau \iiint_{V} \frac{\partial}{\partial \xi_q}\Big\{c_{pqrs}\Delta e_{rs} G_{np}(\vec{x},t-\tau,\vec{\xi},0)\Big\}\mathrm{d}^3\vec{\xi} \tag{2.28}$$

Here, V refers only to the volume of the inclusion.
Using

$$\frac{\partial(c_{pqrs}\Delta e_{rs})}{\partial \xi_q} = \Delta\tau_{pq,q} = 0$$

we can rewrite (2.28) as

$$u_n(\vec{x},t) = \iiint_{V} c_{pqrs}\Delta e_{rs} * G_{np,q}\mathrm{d}^3\vec{\xi} \tag{2.29}$$

Comparing this volume integral with the surface integral in (2.16), i.e.,

$$u_n(\vec{x},t) = \iint_{\Sigma} [u_i]\nu_j c_{ijpq} * G_{np,q}\mathrm{d}\Sigma = \iint_{\Sigma} m_{pq} * G_{np,q}\mathrm{d}\Sigma$$

we see that it is natural to introduce a moment density tensor

$$\frac{\mathrm{d}M_{pq}}{\mathrm{d}V} = c_{pqrs}\Delta e_{rs} \tag{2.30}$$

$$\left[\frac{\mathrm{d}M_{pq}}{\mathrm{d}V}\right] = \frac{\mathrm{Nm}}{\mathrm{m}^3} = \frac{\mathrm{N}}{\mathrm{m}^2}$$

Eq. (2.29) is then

$$u_n(\vec{x},t) = \iiint_{V} \frac{\mathrm{d}M_{pq}}{\mathrm{d}V} * G_{np,q}\mathrm{d}V \tag{2.31}$$

Note that $\Delta\tau_{pq} = \frac{\mathrm{d}M_{pq}}{\mathrm{d}V}$ is not a stress drop (the difference between the initial equilibrium stress and the final equilibrium stress in the source region). The stress drop is not limited to the source volume, but $\Delta\tau_{pq}$ vanishes outside the source volume. $\Delta\tau_{pq}$ is called the "stress glut" by Backus & Mulcahy (1976).

3. METHODS OF SOLUTION OF THE EQUATION OF MOTION

3.1 Equations of motion – 3D problem

Consider a perfectly elastic unbounded heterogeneous continuum. The equation of motion (or elastodynamic equation) for such a medium is eq. (1.39)

$$\rho\, \ddot{u}_i = \tau_{ij,j} + f_i \tag{3.1}$$

Let us restrict ourselves to an isotropic medium. Then the stress tensor is given by Hooke's law for the isotropic continuum eq. (1.57)

$$\tau_{ij} = \lambda\, \delta_{ij} e_{kk} + 2\mu\, e_{ij} \tag{3.2}$$

where λ and μ are Lamè elastic coefficients and e_{ij} is the strain tensor eq. (1.7a)

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{3.3}$$

Inserting eq. (3.3) into (3.2) we have

$$\tau_{ij} = \lambda\, \delta_{ij} e_{kk} + \mu(u_{i,j} + u_{j,i}) \tag{3.4}$$

Equations (3.1) and (3.4) can be called together the displacement–stress formulation of the equation of motion for the perfectly elastic, unbounded isotropic heterogeneous continuum.
Inserting (3.4) into (3.1) and using $e_{kk} = u_{k,k}$ and $(\lambda\, \delta_{ij}\, u_{k,k})_{,j} = (\lambda\, u_{k,k})_{,i}$ leads to

$$\rho\, \ddot{u}_i = (\lambda\, u_{k,k})_{,i} + (\mu\, u_{i,j})_{,j} + (\mu\, u_{j,i})_{,j} + f_i \tag{3.5}$$

which can be called the displacement formulation of the equation of motion.
The r.h.s. of eq. (3.5) can be rewritten

$$\rho\, \ddot{u}_i = \lambda_{,i}\, u_{k,k} + \lambda\, u_{k,ki} + \mu_{,j}\, u_{i,j} + \mu\, u_{i,jj} + \mu_{,j}\, u_{j,i} + \mu\, u_{j,ij} + f_i \tag{3.6}$$

Using the nabla operator ∇ in eq. (3.6) we have

$$\rho\, \ddot{\vec{u}} = \nabla\lambda(\nabla\cdot\vec{u}) + \lambda\nabla\nabla\cdot\vec{u} + \nabla\mu\cdot\nabla\vec{u} + \mu\nabla^2\vec{u} + \nabla\mu\cdot(\nabla\vec{u})^T + \mu\nabla\nabla\cdot\vec{u} + \vec{f} \tag{3.7}$$

$$\rho\, \ddot{\vec{u}} = \nabla\lambda(\nabla\cdot\vec{u}) + (\lambda+\mu)\nabla\nabla\cdot\vec{u} + \nabla\mu\cdot\left[\nabla\vec{u} + (\nabla\vec{u})^T\right] + \mu\nabla^2\vec{u} + \vec{f} \tag{3.8}$$

Applying the vector identity

$$\nabla\times\nabla\times\vec{u} = \nabla\nabla\cdot\vec{u} - \nabla^2\vec{u}$$

to eq. (3.8) we obtain

$$\rho\,\ddot{\vec{u}} = \nabla\lambda(\nabla\cdot\vec{u}) + \nabla\mu\cdot\left[\nabla\vec{u} + (\nabla\vec{u})^T\right] + (\lambda+2\mu)\,\nabla\nabla\cdot\vec{u} - \mu\nabla\times\nabla\times\vec{u} + \vec{f} \tag{3.9}$$

If the medium is homogeneous, i.e., if Lamè coefficients λ and μ as well as density ρ are spatial constants, eq. (3.9) reduces to the form

$$\rho\,\ddot{\vec{u}} = (\lambda+2\mu)\,\nabla\nabla\cdot\vec{u} - \mu\nabla\times\nabla\times\vec{u} + \vec{f} \tag{3.10}$$

Similarly, eqs. (3.5) and (3.6) reduce to the form

$$\rho\,\ddot{u}_i = (\lambda+\mu)\,u_{k,ki} + \mu\,u_{i,jj} + f_i \tag{3.11}$$

Alternatively, we have eq. (3.10) in the index notation

$$\rho\,\ddot{u}_i = (\lambda+2\mu)u_{k,ki} - \mu\varepsilon_{ijk}\,\varepsilon_{klm}\,u_{m,lj} + f_i \tag{3.12}$$

Instead of a concise index notation it is sometimes useful to use an alternative notation:

$$\begin{array}{lllllll}
x = x_1 & , & y = x_2 & , & z = x_3 & & \\
u = u_1 & , & v = u_2 & , & w = u_3 & & \\
\tau_{xx} = \tau_{11} & , & \tau_{xy} = \tau_{12} & , & \dots & & \\
f_x = f_1 & , & f_y = f_2 & , & f_z = f_3 & & \\
u_x = u_{1,1} & , & u_y = u_{1,2} & , & \dots & ; & u_{xx} = u_{1,11} \quad , \quad \dots \\
\tau_{xx,x} = \tau_{11,1} & , & \tau_{xy,x} = \tau_{12,1} & , & \dots & &
\end{array}$$

Then the equation of motion in the displacement–stress formulation is

$$\begin{aligned}
\rho\,\ddot{u} &= \tau_{xx,x} + \tau_{xy,y} + \tau_{xz,z} + f_x \\
\rho\,\ddot{v} &= \tau_{xy,x} + \tau_{yy,y} + \tau_{yz,z} + f_y \\
\rho\,\ddot{w} &= \tau_{xz,x} + \tau_{yz,y} + \tau_{zz,z} + f_z
\end{aligned} \tag{3.13a}$$

$$\begin{aligned}
\tau_{xx} &= (\lambda+2\mu)\,u_x + \lambda\,v_y + \lambda\,w_z \\
\tau_{yy} &= \lambda\,u_x + (\lambda+2\mu)\,v_y + \lambda\,w_z \\
\tau_{zz} &= \lambda\,u_x + \lambda\,v_y + (\lambda+2\mu)\,w_z \\
\tau_{xy} &= \mu(u_y + v_x) \\
\tau_{yz} &= \mu(v_z + w_y) \\
\tau_{xz} &= \mu(u_z + w_x)
\end{aligned} \tag{3.13b}$$

The displacement formulation is

$$\begin{aligned}
\rho\,\ddot{u} &= ([\lambda+2\mu]u_x)_x + (\mu\,u_y)_y + (\mu\,u_z)_z + (\lambda\,v_y)_x \\
&+ (\lambda\,w_z)_x + (\mu\,v_x)_y + (\mu\,w_x)_z + f_x \\
\rho\,\ddot{v} &= (\mu\,v_x)_x + ([\lambda+2\mu]v_y)_y + (\mu\,v_z)_z + (\mu\,u_y)_x \\
&+ (\lambda\,u_x)_y + (\lambda\,w_z)_y + (\mu\,w_y)_z + f_y \\
\rho\,\ddot{w} &= (\mu\,w_x)_x + (\mu\,w_y)_y + ([\lambda+2\mu]w_z)_z + (\mu\,u_z)_x \\
&+ (\mu\,v_z)_y + (\lambda\,u_x)_z + (\lambda\,v_y)_z + f_z
\end{aligned} \tag{3.14}$$

In the case of a homogeneous medium, the displacement–stress formulation does not simplify in the form. The displacement formulation takes a simpler form:

$$\begin{aligned}
\rho\,\ddot{u} &= (\lambda+2\mu)u_{xx} + \mu\,u_{yy} + \mu\,u_{zz} + \lambda\,v_{yx} + \lambda\,w_{zx} + \mu\,v_{xy} + \mu\,w_{xz} + f_x \\
\rho\,\ddot{v} &= \mu\,v_{xx} + (\lambda+2\mu)v_{yy} + \mu\,v_{zz} + \mu\,u_{yx} + \lambda\,u_{xy} + \lambda\,w_{zy} + \mu\,w_{yz} + f_y \\
\rho\,\ddot{w} &= \mu\,w_{xx} + \mu\,w_{yy} + (\lambda+2\mu)w_{zz} + \mu\,u_{zx} + \mu\,v_{zy} + \lambda\,u_{xz} + \lambda\,v_{yz} + f_z
\end{aligned} \tag{3.15}$$

3.2 1D Problems

Coordinate system can always be rotated to get one of the following two cases.
Material parameters are functions of one coordinate: $\rho = \rho(x), \lambda = \lambda(x), \mu = \mu(x)$

P waves

$$\begin{aligned}
\vec{u} &= (\ u(x,t),\ 0,\ 0\ ) \\
\tau_{xx} &= \tau(x,t) \\
\tau_{ij} &= 0 \qquad (i,j) \neq (1,1) \\
\vec{f} &= (\ f(x,t),\ 0,\ 0\ )
\end{aligned}$$

S waves

$$\begin{aligned}
\vec{u} &= (\ 0,\ v(x,t),\ 0\ ) \\
\tau_{xy} &= \tau(x,t) \\
\tau_{ij} &= 0 \qquad (i,j) \neq (1,2) \\
\vec{f} &= (\ 0,\ f(x,t),\ 0\ )
\end{aligned}$$

Inserting these assumptions into equations (3.13a) and (3.13b) we get:
Displacement–stress formulation

$$\rho\ddot{u} = \tau_{,x} + f \qquad \tau = (\lambda+2\mu)\ u_x \qquad\qquad \rho\ddot{v} = \tau_{,x} + f \qquad \tau = \mu\ v_x$$

Displacement formulation

$$\rho\ddot{u} = ((\lambda+2\mu)\ u_x)_{,x} + f \qquad\qquad \rho\ddot{v} = (\mu\ v_x)_{,x} + f$$

3.3 2D Problems

P–SV Problem
Consider the following problem:

$$\begin{aligned}
\vec{u} &= (\ u(x,z,t),\ 0,\ w(x,z,t)\ ) \\
\tau_{\xi\eta} &= \tau_{\xi\eta}(x,z,t) \quad ; \quad \xi,\eta \in \{x,z\} \\
\tau_{xy} &= \tau_{yz} = 0 \qquad (\tau_{yy} = \tau_{yy}(x,z,t), \quad \text{eq. satisfied identically}) \\
\vec{f} &= (\ f_x(x,z,t),\ 0,\ f_z(x,z,t)\ ) \\
\rho &= \rho(x,z), \qquad \lambda = \lambda(x,z), \qquad \mu = \mu(x,z)
\end{aligned}$$

Then the equations of motion are:

Displacement–stress formulation

$$\begin{aligned}\rho\,\ddot{u} &= \tau_{xx,x} + \tau_{xz,z} + f_x \\ \rho\,\ddot{w} &= \tau_{xz,x} + \tau_{zz,z} + f_z\end{aligned} \tag{3.16a}$$

$$\begin{aligned}\tau_{xx} &= (\lambda + 2\mu)u_x + \lambda\, w_z \\ \tau_{zz} &= \lambda\, u_x + (\lambda + 2\mu)w_z \\ \tau_{xz} &= \mu(u_z + w_x)\end{aligned} \tag{3.16b}$$

Displacement formulation

$$\begin{aligned}\rho\,\ddot{u} &= ([\lambda + 2\mu]u_x)_x + (\mu\, u_z)_z + (\lambda\, w_z)_x + (\mu\, w_x)_z + f_x \\ \rho\,\ddot{w} &= (\mu w_x)_x + ([\lambda + 2\mu]w_z)_z + (\mu\, u_z)_x + (\lambda\, u_x)_z + f_z\end{aligned} \tag{3.17}$$

In the case of a homogeneous medium eqs. (3.17) become

$$\begin{aligned}\rho\,\ddot{u} = (\lambda + 2\mu)u_{xx} + \mu\, u_{zz} + \lambda\, w_{zx} + \mu\, w_{xz} + f_x \\ \rho\,\ddot{w} = \mu\, w_{xx} + (\lambda + 2\mu)w_{zz} + \mu\, u_{zx} + \lambda\, u_{xz} + f_z\end{aligned} \tag{3.18}$$

SH Problem

Consider the following problem:

$$\begin{aligned}\vec{u} &= (0, v(x,z,t), 0) \\ \tau_{xy} &= \tau_{xy}(x,z,t) \quad , \quad \tau_{yz} = \tau_{yz}(x,z,t) \\ \tau_{xx} &= \tau_{yy} = \tau_{zz} = \tau_{xz} = 0 \\ \vec{f} &= (0, f_y(x,z,t), 0) \\ \rho &= \rho(x,z), \qquad \lambda = \lambda(x,z), \qquad \mu = \mu(x,z)\end{aligned}$$

Then the equations of motion are:
Displacement–stress formulation

$$\rho\,\ddot{v} = \tau_{xy,x} + \tau_{yz,z} + f_y \tag{3.19a}$$

$$\begin{aligned}\tau_{xy} &= \mu\, v_x \\ \tau_{yz} &= \mu\, v_z\end{aligned} \tag{3.19b}$$

Displacement formulation

$$\rho\,\ddot{v} = (\mu\, v_x)_x + (\mu\, v_z)_z + f_y \tag{3.20}$$

In the case of a homogeneous medium eq. (3.20) becomes

$$\rho\,\ddot{v} = \mu\, v_{xx} + \mu\, v_{zz} + f_y \tag{3.21}$$

3.4 Solving equations of motion in the time and frequency domains

Depending on a problem one can choose to solve the equation of motion in the time or frequency domain since one of the two approaches can be easier or more advantageous.

Denote the r.h.s. (except the body force) of any of the above equations of motion symbolically as $\vec{L}(\vec{u})$. $\vec{L}$ denotes a vector linear differential operator.
Then the equation of motion can be written as

$$\rho\,\ddot{\vec{u}} = \vec{L}(\vec{u}) + \vec{f} \tag{3.22}$$

where $\vec{u} = \vec{u}(x, y, z, t)$ and $\vec{f} = \vec{f}(x, y, z, t)$.
Apply the Fourier transform to eq. (3.22). We obtain

$$-\rho(2\pi f)^2\vec{U} = \vec{L}(\vec{U}) + \vec{F} \tag{3.23}$$

where

$$\vec{U}(x, y, z; f) = \int_{-\infty}^{\infty} \vec{u}(x, y, z, \tau)\, \exp(\mathrm{i}2\pi f\tau)d\tau \tag{3.24}$$

$$\vec{F}(x, y, z; f) = \int_{-\infty}^{\infty} \vec{f}(x, y, z, \tau)\, \exp(\mathrm{i}2\pi f\tau)d\tau \tag{3.25}$$

Transformation of the original equation of motion (3.22) in the time domain into the frequency domain reduces the number of independent variables – the dependence of the solution on time was removed. The new unknown $\vec{U}$ depends on frequency f but f is only a parameter since eq. (3.23) does not contain any derivative with respect to f. The reduction of the number of independent variables usually significantly simplifies the problem of finding a solution.

After we find the solution of eq. (3.23) for all (or, sufficient number of) frequencies f we can apply the inverse Fourier transform (or, in practice, the discrete inverse Fourier tr.)

$$\vec{u}(x, y, z, t) = \int_{-\infty}^{\infty} \vec{U}(x, y, z; f)\, \exp(-\mathrm{i}2\pi f t)df \tag{3.26}$$

to obtain the solution in the time domain, i.e., the solution of the original eq. (3.22).

In many cases we simply have a reason to find a harmonic solution of eq.(3.22). Then we assume that

$$\vec{u}_H(x, y, z, t; f) = \vec{U}(x, y, z; f)\exp(-\mathrm{i}2\pi f t)$$

solves eq. (3.22). Inserting this into eq. (3.22) we obtain

$$-\rho(2\pi f)^2\vec{U} = \vec{L}(\vec{U}) + \vec{F}$$

which is the equation of the same form as eq. (3.23)

3.5 Methods of solving the equation of motion

There are tens of methods developed for solving the equation of motion. In principle, all methods can be divided into two groups:

– exact
– approximate

The exact (also wave, analytic) methods are applicable in the case of a homogeneous medium or simple heterogeneous models – e.g., 1D vertically heterogeneous models or radially (spherically) symmetrical models. The separation of variables or matrix methods can typically be applied.

The approximate methods can be roughly divided into the high–frequency methods and low–frequency methods. The most important high–frequency method is the ray method (or, the asymptotic ray theory – ART). The h.–f. methods are crucially important in the structural seismology and in the seismic oil exploration. The low–frequency methods are important mainly for simulating earthquake ground motion (also seismic ground motion). They can be divided into three groups –

– domain methods (e.g., FDM, FEM, SPEM, ADER-DGM, ...)
– boundary methods (e.g., BIEM, BEM, DWNM, ...)
– hybrid methods (e.g., FD–FEM, DWN–FDM, A–MM, ...)

The boundary methods are generally more accurate than the domain methods. However, they are practically applicable to models with two or three homogeneous layers / blocks since computer memory and time requirements in the case of more complex models are too large. The domain methods are generally less accurate than the boundary methods but allow to compute seismic motion in relatively complex models. This is confirmed by a dominant role of the FDM in the recent modeling of earthquake ground motion in large sedimentary basins (as, e.g., the LA basin and Osaka basin). The hybrid methods combine two or three methods in order to eliminate drawbacks of individual methods. The hybrid methods use one particular method to solve dependence on some of the independent variables and other method to solve dependence on the remaining independent variables or they use one method for one part of a computational region and other method for the remaining part of the computational region. Therefore, they usually are more computationally efficient but imply more difficult computational algorithm.

FDM	finite-difference method
FEM	finite element method
SPEM	spectral element method
ADER-DGM	arbitrary high-order derivative discontinuous Galerkin method
BIEM	boundary integral equation method
BEM	boundary element method
DWNM	discrete-wavenumber method
A-MM	Alekseev - Mikhailenko method

4. ELASTIC WAVES IN UNBOUNDED HOMOGENEOUS ISOTROPIC MEDIUM

4.1 Wave potentials and separation of the equation of motion. Wave equations for P and S waves

Consider an unbounded homogeneous isotropic medium. The equation of motion is (see eq. 3.11)

$$\rho \ddot{u}_i = (\lambda + \mu) u_{j,ji} + \mu u_{i,jj} \tag{4.1}$$

where we omitted the body-force term, and ρ , λ and μ are spatial constants.

Apply now the Helmholtz decomposition to the displacement vector $\vec{u}$:

$$u_i = \mathit{\Phi}_{,i} + \varepsilon_{ilk} \mathit{\Psi}_{k,l} \tag{4.2a}$$

$$\text{i.e.,} \quad \vec{u} = \nabla \mathit{\Phi} + \nabla \times \vec{\mathit{\Psi}} \tag{4.2b}$$

$\mathit{\Phi}$ and $\vec{\mathit{\Psi}}$ are scalar and vector Helmholtz potentials. Find the divergence of $\vec{u}$. Apply the divergence to eq. (4.2a):

$$u_{i,i} = \mathit{\Phi}_{,ii} + \varepsilon_{ilk} \mathit{\Psi}_{k,li}$$

$$\text{Since} \quad \varepsilon_{ilk}\, \mathit{\Psi}_{k,li} = 0 \quad (\nabla . \nabla \times \vec{\mathit{\Psi}} = 0) \tag{4.3}$$

$$u_{i,i} = \mathit{\Phi}_{,ii} \quad (\nabla . \vec{u} = \nabla . \nabla \mathit{\Phi}) \tag{4.4}$$

Find now the rotation of $\vec{u}$. Apply the rotation to eq. (4.2a):

$$\varepsilon_{mni}\, u_{i,n} = \varepsilon_{mni}\, \mathit{\Phi}_{,in} + \varepsilon_{mni}\, \varepsilon_{ilk}\, \mathit{\Psi}_{k,ln}$$

$$\varepsilon_{mni}\, \mathit{\Phi}_{,in} = \frac{1}{2}(\varepsilon_{mni}\, \mathit{\Phi}_{,in} + \varepsilon_{min}\, \mathit{\Phi}_{,ni}) = \frac{1}{2}(\varepsilon_{mni}\, \mathit{\Phi}_{,in} - \varepsilon_{mni}\, \mathit{\Phi}_{,in}) = 0 \tag{4.5a}$$

$$(\nabla \times \nabla\, \mathit{\Phi} = 0) \tag{4.5b}$$

$$\varepsilon_{mni}\, u_{i,n} = \varepsilon_{mni} \varepsilon_{ilk}\, \mathit{\Psi}_{k,ln} \quad (\nabla \times \vec{u} = \nabla \times \nabla \times \vec{\mathit{\Psi}}) \tag{4.6}$$

Since the divergence of $\vec{u}$ equals $\nabla . \nabla \mathit{\Phi} \neq 0$ and the rotation of $\vec{u}$ equals $\nabla \times \nabla \times \vec{\mathit{\Psi}}$, the Helmholtz decomposition of $\vec{u}$ means the decomposition into the part $(\nabla \mathit{\Phi})$ which causes only volume changes and the part $(\nabla \times \vec{\mathit{\Psi}})$ which causes only shear changes without any volume change (i.e., changes in form).

Since eq. (4.2a assigns 4 functions $\mathit{\Phi}$, $\mathit{\Psi}_1$, $\mathit{\Psi}_2$ and $\mathit{\Psi}_3$ to 3 components of the displacement vector, one additional condition for the four functions is necessary. We can use

$$\mathit{\Psi}_{i,i} = 0 \quad (\nabla . \vec{\mathit{\Psi}} = 0) \tag{4.7}$$

since the use of the rotation in eq. (4.2a) means, in fact, that we give up any part of $\vec{\Psi}$ which would have a nonzero divergence.
Insert now decomposition (4.2a) into eq. (4.1):

$$\begin{aligned}
\rho\,\ddot{\Phi}_{,i} + \rho\,\varepsilon_{ilk}\,\ddot{\Psi}_{k,l} &= (\lambda+\mu)(\Phi_{,jji} + \varepsilon_{jlk}\,\Psi_{k,lji}) + \mu(\Phi_{,ijj} + \varepsilon_{ilk}\,\Psi_{k,ljj}) \\
\varepsilon_{jlk}\,\Psi_{k,lji} &= (\varepsilon_{jlk}\,\Psi_{k,lj})_{,i} = (0)_{,i} = 0 \\
\varepsilon_{ilk}\,\Psi_{k,ljj} &= \varepsilon_{ilk}(\Psi_{k,jj})_{,l}
\end{aligned}$$

$$\begin{aligned}
\rho\,\ddot{\Phi}_{,i} + \rho\,\varepsilon_{ilk}\,\ddot{\Psi}_{k,l} &= (\lambda+2\mu)\Phi_{,jji} + \mu\,\varepsilon_{ilk}(\Psi_{k,jj})_{,l} \\
(\rho\,\ddot{\Phi} - (\lambda+2\mu)\Phi_{,jj})_{,i} &+ \varepsilon_{ilk}(\rho\,\ddot{\Psi}_k - \mu\,\Psi_{k,jj})_{,l} = 0
\end{aligned}$$

This equation (together with appropriate boundary conditions) implies that

$$\rho\,\ddot{\Phi} - (\lambda+2\mu)\Phi_{,jj} = 0 \tag{4.8a}$$

and

$$\rho\,\ddot{\Psi}_k - \mu\,\Psi_{k,jj} = 0 \tag{4.9a}$$

i.e.,

$$\rho\,\ddot{\Phi} - (\lambda+2\mu)\nabla^2\Phi = 0 \tag{4.8b}$$

$$\rho\,\ddot{\vec{\Psi}} - \mu\nabla^2\vec{\Psi} = 0 \tag{4.9b}$$

Define

$$\alpha = \sqrt{\frac{\lambda+2\mu}{\rho}} \tag{4.10}$$

and

$$\beta = \sqrt{\frac{\mu}{\rho}} \tag{4.11}$$

Then eq. (4.8a, 4.9a) and (4.8b, 4.9b) become

$$\ddot{\Phi} = \alpha^2\,\Phi_{,jj} \tag{4.12a}$$

$$\ddot{\Psi}_k = \beta^2\,\Psi_{k,jj} \tag{4.13a}$$

and

$$\ddot{\Phi} = \alpha^2\nabla^2\Phi \tag{4.12b}$$

$$\ddot{\vec{\Psi}} = \beta^2\nabla^2\vec{\Psi} \tag{4.13b}$$

Equations (4.12a) and (4.13a) are the wave equations. Eq. (4.12a, 4.12b) describes propagation of a wave with speed α whereas eq. (4.13a, 4.13b) describes propagation of a wave with speed β. Since $\alpha > \beta$, the wave propagation with speed α is faster and arrives in a given place as the first of the two waves. Therefore, it is called the P wave according to the Latin word primae. The other type of wave is called the S wave according to the Latin word secundae.

We have found an interesting and important result:

- The equation of motion for an unbounded homogeneous isotropic medium can be separated into two wave equations.
- Two independent types of waves can propagate in the unbounded homogeneous isotropic medium - one with speed α - the P wave, and the other with speed β - the S wave. Propagation of the P wave is accompanied with changes in volume, propagation of the S wave accompanied with changes in form.

The separation of the equation of motion into two wave equations is possible also in the case of presence of the body-force term in the equation,

$$\rho\,\ddot{u}_i = (\lambda+\mu)u_{j,ji} + \mu\,u_{i,jj} + f_i \tag{4.14}$$

The Helmholtz decomposition can be applied also to $\vec{f}$:

$$f_i = g_{,i} + \varepsilon_{ilk}\,q_{k,l} \tag{4.15a}$$

$$\text{i.e.,}\qquad \vec{f} = \nabla g + \nabla\times\vec{q} \tag{4.15b}$$

Then we can get (analogously with the previous case)

$$\ddot{\Phi} = \alpha^2\,\Phi_{,jj} + \frac{1}{\rho}\,g \tag{4.16a}$$

$$\ddot{\Psi}_k = \beta^2\,\Psi_{k,jj} + \frac{1}{\rho}\,q_k \tag{4.17a}$$

and

$$\ddot{\Phi} = \alpha^2\nabla^2\Phi + \frac{1}{\rho}\,g \tag{4.16b}$$

$$\ddot{\vec{\Psi}} = \beta^2\nabla^2\vec{\Psi} + \frac{1}{\rho}\,\vec{q} \tag{4.17b}$$

Recalling decomposition (4.2a),

$$u_i = \Phi_{,i} + \varepsilon_{ilk}\,\Psi_{k,l}$$

define $\vec{u}^P$ and $\vec{u}^S$:

$$u_i^P = \Phi_{,i}\ ,\quad u_i^S = \varepsilon_{ilk}\,\Psi_{k,l} \tag{4.18}$$

$$u_i = u_i^P + u_i^S \tag{4.19}$$

Then it follows from the wave equations (4.12a) and (4.13a) that $\vec{u}^P$ and $\vec{u}^S$ also satisfy wave equations

$$\ddot{u}_i^P = \alpha^2\,u_{i,jj}^P \tag{4.20}$$

$$\ddot{u}_i^S = \beta^2\,u_{i,jj}^S \tag{4.21}$$

Relations (4.4) and (4.6) mean that

$$u^S_{i,i} = 0 \qquad (\nabla.\vec{u}^S = 0) \tag{4.22}$$

and

$$\varepsilon_{mni}\, u^P_{i,n} = 0 \qquad (\nabla \times \vec{u}^P = 0) \tag{4.23}$$

Recall now equation of motion in the form (3.12) with $\vec{f} = 0$

$$\rho\, \ddot{u}_i = (\lambda + 2\mu) u_{k,ki} - \mu \varepsilon_{ijk}\, r_{k,j} \tag{4.24}$$

where $r_k = \varepsilon_{klm}\, u_{m,l}$. Apply the divergence to the above equation (4.24). We get

$$\rho\, \ddot{u}_{i,i} = (\lambda + 2\mu) u_{k,kii}$$

since $\varepsilon_{ijk}\, r_{k,ji} = 0$.

$$\rho (u_{i,i})_{,tt} = (\lambda + 2\mu)(u_{i,i})_{,kk}$$

or

$$(u_{i,i})_{,tt} = \alpha^2 (u_{i,i})_{,kk} \tag{4.25a}$$

i.e.,

$$(\nabla.\vec{u})_{,tt} = \alpha^2 \nabla^2 (\nabla.\vec{u}) \tag{4.25b}$$

Recall now equation of motion in the form (4.1 or 3.11 with $\vec{f} \equiv 0$).

$$\rho\, \ddot{u}_i = (\lambda + \mu) u_{j,ji} + \mu\, u_{i,jj}$$

Apply the rotation to the equation:

$$\rho\, \varepsilon_{mni}\, \ddot{u}_{i,n} = (\lambda + \mu) \varepsilon_{mni}\, u_{j,jin} + \mu\, \varepsilon_{mni}\, u_{i,jjn}$$

since $\varepsilon_{mni}\, u_{j,jin} = 0$

$$\rho (\varepsilon_{mni}\, u_{i,n})_{,tt} = \mu (\varepsilon_{mni}\, u_{i,n})_{,jj}$$

or

$$(\varepsilon_{mni}\, u_{i,n})_{,tt} = \beta^2 (\varepsilon_{mni}\, u_{i,n})_{,jj} \tag{4.26a}$$

i.e.,

$$(\nabla \times \vec{u})_{,tt} = \beta^2 \nabla^2 (\nabla \times \vec{u}) \tag{4.26b}$$

We see that eqs. (4.25a, 4.25b) and (4.26a, 4.26b) are the wave equations for $\nabla.\vec{u}$ and $\nabla \times \vec{u}$, respectively.
Using definitions (4.10) and (4.11) we can rewrite e.g. eq. (3.12) in the form

$$\ddot{u}_i = \alpha^2\, u_{k,ki} - \beta^2\, \varepsilon_{ijk} \varepsilon_{klm}\, u_{m,lj} + \frac{f_i}{\rho} \tag{4.27}$$

The ratio between the P-wave and S-wave speeds is

$$\frac{\alpha}{\beta} = \sqrt{\frac{\lambda + 2\mu}{\mu}} \tag{4.28}$$

More important than relation (4.28) is the relation between α/β and Poisson's ratio σ . Poisson's ratio is the ratio of the radial to axial strain when an uniaxial stress is applied. For example

$$\tau_{11} \neq 0\,, \quad \tau_{22} = \tau_{33} = 0$$
$$e_{22} = e_{33}$$
$$\sigma = \tfrac{-e_{22}}{e_{11}} = \tfrac{\lambda}{2(\lambda+\mu)} \tag{4.29}$$

It follows from (4.28) and (4.29) that

$$\frac{\alpha}{\beta} = \sqrt{\frac{2(1-\sigma)}{1-2\sigma}} \tag{4.30}$$

If $\mu = 0$ (fluid, no shear resistance), $\sigma = 0.5$.
If the solid has an infinite shear resistance, $\sigma = 0$.
Thus

$$0 < \sigma < 0.5 \tag{4.31}$$

Relations (4.30) and (4.31) imply a very important relation

$$\frac{\alpha}{\beta} > \sqrt{2} \tag{4.32}$$

The case of $\mu = \lambda$ defines Poisson's body. Then

$$\frac{\alpha}{\beta} = \sqrt{3} \quad \text{and} \quad \sigma = 0.25 \tag{4.33}$$

4.2 Plane waves

Consider the wave equation for the P wave

$$\ddot{\Phi} = \alpha^2\, \Phi_{,jj} \tag{4.34}$$
$$\text{and} \qquad u_i^P = \Phi_{,i} \tag{4.35}$$

Assume a solution in a form

$$\Phi(x_i, t) = A\, f(\vartheta) \tag{4.36a}$$
$$\text{where} \qquad \vartheta(x_i, t) = t - \tau(x_i) \tag{4.36b}$$
$$\text{and} \qquad \tau(x_i) = p_l x_l \tag{4.36c}$$

and A and p_l $(l = 1, 2, 3)$ are real constants. ϑ is the phase function or phase. Let $\vartheta = \vartheta_0$ for time $t = t_0$. Then

$$\vartheta_0 = t_0 - p_l x_l \tag{4.37}$$

or

$$p_l x_l + \vartheta_0 - t_0 = 0 \tag{4.38}$$

Since eq. (4.38) is the equation of a plane, the surface of constant phase ϑ_0 at time t_0 is a plane. Therefore, solution (4.36a) of the equation (4.34) is called the plane P wave. Let $f(\vartheta) \neq 0$ for

$\vartheta \in< \vartheta_1, \vartheta_2 >$ and $f(\vartheta) = 0$ outside the interval. Then the plane $\vartheta(x_i, t) = \vartheta_1$ is called the wavefront since it separates the region in motion from the region which is in rest.
Insert (4.36a) into eq. (4.34):

$$A\ f'' = \alpha^2 A\ f''(p_1^2 + p_2^2 + p_3^2)$$

where

$$\begin{aligned} f'' &= \frac{d^2 f}{d\vartheta^2} \\ p_1^2 + p_2^2 + p_3^2 &= \frac{1}{\alpha^2} \end{aligned} \tag{4.39}$$

Define

$$\nu_i = \alpha p_i \tag{4.40}$$

Then

$$\nu_1^2 + \nu_2^2 + \nu_3^2 = 1 \tag{4.41}$$

It follows from (4.41) that ν_i is a directional cosine of some vector $\vec{N}$.
Relation (4.36c) implies

$$\tau(x_i) = \frac{1}{\alpha}\ \nu_j\ x_j \tag{4.42}$$

Consider now a scalar product of $\nabla\tau$ and $\vec{N}$:

$$\begin{aligned} \tau_{,i}\ \nu_i &= \frac{1}{\alpha}\ \nu_j\ x_{j,i}\ \nu_i = \frac{1}{\alpha}\ \nu_j\ \nu_i\ \delta_{ij} \\ &= \frac{1}{\alpha}\nu_i\ \nu_i = \frac{1}{\alpha} \end{aligned}$$

Consider now a vector product of $\nabla\tau$ and $\vec{N}$:

$$\begin{aligned} \varepsilon_{ikl}\ \tau_{,k}\ \nu_l &= \varepsilon_{ikl}\ \frac{1}{\alpha}\ \nu_j\ x_{j,k}\ \nu_l \\ &= \frac{1}{\alpha}\ \varepsilon_{ikl}\ \delta_{jk}\ \nu_j\ \nu_l \\ &= \frac{1}{\alpha}\ \varepsilon_{ijl}\ \nu_j\ \nu_l = 0 \end{aligned}$$

We have found that vector $\vec{N}$ is parallel to $\nabla\tau$, i.e., it is perpendicular to the surface $\tau(x_i) =$ const. Then it follows from (4.36b) that $\vec{N}$ is perpendicular to the surface (plane) of the constant phase.
Vector $\vec{p} = (p_1, p_2, p_3)$; $|\vec{p}| = \frac{1}{\alpha}$ is frequently used in the theory of elastic waves and is called the slowness vector.
Relations (4.35) and (4.36a) imply

$$\begin{aligned} u_i^P &= (A\ f(t - \frac{1}{\alpha}\ \nu_j\ x_j))_{,i} \\ u_i^P &= -A\ \frac{1}{\alpha} f'(t - \frac{1}{\alpha}\ \nu_j\ x_j)\nu_i \end{aligned} \tag{4.43}$$

Relation (4.43) means that the displacement vector of the plane P wave has the direction of vector $\vec{N}$ and is perpendicular to the plane of a constant phase (i.e., a wavefront).

Consider now a special case:

$$\vec{N} = \vec{N}(1,0,0)$$

$$\text{Then} \quad \Phi = A\ f\,(t - \tfrac{x_1}{\alpha})$$

$$\text{and} \quad \vartheta = t - \tfrac{x_1}{\alpha}$$

The plane of the constant phase $\vartheta_0 = t - \frac{x_1}{\alpha}$ propagates in the direction x_1 with speed α since

$$\frac{dx_1}{dt} = \alpha$$

It is easy to verify that

$$\Phi(x_i,t) = A\ f(t - \frac{1}{\alpha}\ \nu_i\ x_i) + B\ g(t + \frac{1}{\alpha}\ \nu_i\ x_i) \tag{4.44}$$

also satisfies eq. (4.34). $\Phi(x_i,t)$ represents two plane waves. The first one, described by

$$A\ f(t - \frac{1}{\alpha}\ \nu_i\ x_i)$$

propagates in the direction of $\vec{N}$ with speed α, the second one, described by

$$B\ g(t + \frac{1}{\alpha}\ \nu_i\ x_i)$$

propagates in the direction of $-\vec{N}$ with speed α.
Consider now the wave equation for the S wave

$$\ddot{\Psi}_i = \beta^2\ \Psi_{i,jj} \tag{4.45}$$

and

$$u_i^S = \varepsilon_{ilk}\ \Psi_{k,l} \tag{4.46}$$

Assume solution in the form of the plane wave

$$\Psi_i(x_j,t) \quad = \quad C_i\ f(\vartheta) \tag{4.47a}$$

$$\vartheta(x_j,t) \quad = \quad t - \tau(x_j) \tag{4.47b}$$

$$\tau(x_j) \quad = \quad p_l x_l \tag{4.47c}$$

Insert (4.47a) into eq. (4.45):

$$C_i\ f'' = \beta^2\ C_i\ f''(p_1^2 + p_2^2 + p_3^2)$$

$$p_1^2 + p_2^2 + p_3^2 = \tfrac{1}{\beta^2} \tag{4.48}$$

Define

$$\nu_i = \beta p_i \tag{4.49}$$

Then

$$\nu_1^2 + \nu_2^2 + \nu_3^2 = 1 \tag{4.50}$$

It follows from eq. (4.46) that

$$u_i^S \quad = \quad \varepsilon_{ilk}(C_k\ f(t - \frac{1}{\beta}\ \nu_j\ x_j))_{,l}$$

$$u_i^S \quad = \quad -\frac{1}{\beta}\ \varepsilon_{ilk}\ C_k\ f'(t - \frac{1}{\beta}\ \nu_j\ x_j)\nu_l \tag{4.51}$$

Relation (4.51) means that the displacement of the plane S wave is perpendicular to the direction of propagation (i.e., particles oscillate perpendicularly to the direction of propagation).

4.3 Harmonic plane wave

The harmonic solution of the wave equation for the P wave is

$$\Phi(x_i, t, \omega) = \overline{\Phi}(x_i, \omega) \exp[-\mathrm{i}\omega t] \tag{4.52}$$

where $\overline{\Phi}$ is the harmonic potential. (To distinguish between the spatial index i and imaginary unit, we will denote the latter by i.)
The harmonic plane P wave is

$$\Phi(x_i, t, \omega) = A \exp[-\mathrm{i}\omega(t - \tau(x_i))] \tag{4.53}$$

where

$$\tau(x_i) = \frac{1}{\alpha}\nu_j x_j$$

and A may depend on frequency.
The harmonic wave is periodic both in time and space. Therefore it is reasonable to define the wavenumber vector $\vec{k}$.
Equation (4.53) can be rewritten as

$$\Phi(x_i, t, \omega) = A \exp[-\mathrm{i}(\omega t - \frac{\omega}{\alpha}\nu_j x_j)] \tag{4.54}$$

Define the wavenumber vector $\vec{k}$ as

$$k_i = \frac{\omega}{\alpha}\nu_i \tag{4.55}$$

The wavenumber vector $\vec{k}$ has the direction of vector $\vec{N}$, i.e., the direction of propagation of the wave. The absolute value of $\vec{k}$ is

$$|\vec{k}| = k = \frac{\omega}{\alpha} \tag{4.56}$$

The absolute value k is called the wavenumber. Using the wavenumber vector, the harmonic plane wave is

$$\Phi(x_i, t, \omega) = A \exp[-\mathrm{i}(\omega t - k_j x_j)] \tag{4.57}$$

It follows from eq. (4.57) that k has the meaning of the spatial frequency. The harmonic wave is periodic in space with period λ which is the wavelength. Since

$$\begin{aligned} \lambda &= \frac{\alpha}{f} = \frac{2\pi\alpha}{\omega} = \frac{2\pi}{k} \\ k &= \frac{2\pi}{\lambda} \end{aligned} \tag{4.58}$$

which means that the wavenumber is the spatial frequency.

The concept of plane wave has a great importance in the theory of elastic and seismic waves despite obvious fact that plane waves do not exist in reality. Their importance is mainly due to two properties. A nonplanar wave-front can be locally sufficiently well approximated by a planar wavefront if the surface of the wave is sufficiently distant and the medium is weakly (smoothly) heterogeneous. Even more important property of the plane waves is that an arbitrary wave can be correctly expressed as an integral superposition of infinite number of plane waves. Each plane wave represents propagation in one direction.

Let us note, however, that the decomposition of an arbitrary wave into plane waves also includes complex inhomogeneous waves.
Assume now that A and p_k in relations (4.36a,4.36c) are complex and relation (4.39), i.e.,

$$p_k\, p_k = \frac{1}{\alpha^2} \tag{4.59}$$

is valid. Denote real and imaginary parts of τ and p_k as τ^R, τ^I, p_k^R and p_k^I,

$$\tau = \tau^R + \mathrm{i}\tau^I \ , \quad p_k = p_k^R + \mathrm{i}p_k^I$$

Since (eq. 4.36c) $\tau = p_k x_k$

$$\tau^R = p_k^R x_k \quad , \quad \tau^I = p_k^I x_k \tag{4.60}$$

Equation (4.59) implies

$$p_k^R p_k^R - p_k^I p_k^I = \frac{1}{\alpha^2} \tag{4.61}$$

and

$$p_k^R p_k^I = 0 \tag{4.62}$$

Equation (4.62) means that $\vec{p}^R$ is orthogonal to $\vec{p}^I$.
The harmonic plane wave can be written in the form

$$\Phi(x_i, t, \omega) = A \exp[-\omega\tau^I] \exp[-\mathrm{i}\omega(t - \tau^R)] \tag{4.63}$$

It is clean from eq. (4.63) that the amplitude of the wave decreases exponentially with increasing ω.
Since $\vec{p}^R$ and $\vec{p}^I$ are orthogonal, relations (4.60) imply that a surface of a constant phase $t - \tau^R(x_k) = C_1$ is perpendicular to a surface of a constant amplitude $\tau^I(x_k) = C_2$. This means that the amplitude does not change in the direction of propagation and changes fastest along the surface of the constant phase.
Also note that the surface of the constant phase propagates with speed $\sqrt{\frac{1}{p_k^R p_k^R}} < \alpha$.

4.4 Spherical waves

Assume now that the solution of the wave equation (4.34) only depends on a distance r from the origin, $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$.
We can rewrite the wave equation using r.

$$\Phi_{,i} = \Phi_{,r}\, r_{,i} = \tfrac{x_i}{r}\Phi_{,r}$$
$$\Phi_{,11} = \tfrac{x_1^2}{r^2}\Phi_{,rr} + \tfrac{1}{r}\Phi_{,r} - \tfrac{x_1^2}{r^3}\Phi_{,r}$$

and analogous formulas hold for $\Phi_{,22}$ and $\Phi_{,33}$.
For $\nabla^2\Phi = \Phi_{,jj}$ we get

$$\nabla^2\Phi = \Phi_{,rr} + \frac{2}{r}\Phi_{,r} = \frac{1}{r}(r\Phi)_{,rr}$$

Then the wave equation can be written in the form

$$\ddot{\Phi} = \frac{\alpha^2}{r}(r\Phi)_{,rr}$$

and, consequently,

$$(r\Phi)_{,tt} = \alpha^2 (r\Phi)_{,rr} \tag{4.64}$$

Equation (4.64) has the form of wave equation (4.34) with $r\Phi$ instead of Φ. We know that, e.g.,

$$\Phi = Af\left(t - \frac{x_1}{\alpha}\right)$$

solves wave equation (4.34). Therefore,

$$r\Phi = Af\left(t - \frac{r}{\alpha}\right)$$

solves eq. (4.64). Consequently,

$$\Phi = \frac{A}{r} f\left(t - \frac{r}{\alpha}\right) \tag{4.65}$$

A surface of a constant phase, $\vartheta_0 = t - \frac{r}{\alpha_0}$ is a sphere with radius r. Therefore, solution (4.65) is called the spherical wave. It is important that the amplitude of the spherical wave, $\frac{A}{r}$, decreases with an increasing distance from the origin.

The importance of the spherical wave is obvious. The spherical wave is radiated by a point source in a homogeneous medium. The point sourceis a sufficient approximation to real sources (earthquakes, explosions) at sufficient distances.

5. REFLECTION AND TRANSMISSION OF PLANE WAVES AT A PLANE INTERFACE

5.1 Conditions at interface

Let S, F and V denote a solid, fluid and vacuum. Then there are five nontrivial types of interface:

$$\frac{S}{S} \qquad \frac{F}{S} \qquad \frac{V}{S} \qquad \frac{F}{F} \qquad \frac{V}{F}$$

We will consider the simplest interface - a plane interface between two homogeneous halfspaces.

A solution of any problem has to obey the equation of motion in the solid and fluid halfspaces and, at the same time, satisfy conditions at interface - the boundary conditions.

Conditions for displacement

$\frac{S}{S}$,	$\frac{F_{\mathrm{viscous}}}{S}$	continuity of displacement
	$\frac{F_{nonviscous}}{S}$	continuity of normal component of displacement

Note that in the Earth there are two fluids to be considered - the oceanic water and the outer core. Both fluids behave as nonviscous fluids for typical wavelengths (kilometers) and periods (seconds) of seismic waves. This means, in fact, that their viscosity is so low that the thickness of the viscously dragged layer is only a negligible fraction of a wavelength. Therefore, the tangential component of displacement may be discontinuous at the F/S interface. The condition of continuity of the normal component of displacement is due to strong compressive stresses. There is no need to consider diffusion of fluid into solid since it would require time much larger than the period of seismic waves.

Conditions for traction

$\frac{S}{S}, \frac{F}{S}, \frac{F}{F}$	continuity of traction
$\frac{V}{S}, \frac{V}{F}$	zero traction

Note that the zero traction at the V/F and V/S interfaces are spatial cases of the continuity of traction. Vacuum can be a good approximation to the real atmosphere because the elastic constants of the atmosphere are several orders of magnitude less than the elastic constants of rock or the bulk modulus of the oceanic water. The condition of zero traction is often called the free-surface condition.

5.2 Reflection of the plane P and S waves at a free surface

Consider a homogeneous elastic halfspace with the free surface at $z = 0$. Then the zero traction at the free surface implies

$$\tau_{zx} = \tau_{zy} = \tau_{zz} = 0$$

The case of an incident P wave

Assume a plane P wave propagation with horizontal slowness in the direction of increasing x. Then the displacement of the P waves is

$$\vec{u} = (\ \Phi_{,x},\ 0,\ \Phi_{,z}\ ) \tag{5.1}$$

The associated $\vec{T}(\vec{u},\vec{n})$ where $\vec{n} = (0,0,1)$ is

$$\begin{aligned} \vec{T} &= (\ \tau_{zx},\ \tau_{zy},\ \tau_{zz}\ ) \\ &= (\ 2\mu\Phi_{,zx},\ 0,\ \lambda\nabla^2\Phi + 2\mu\Phi_{,zz}\ ) \end{aligned} \tag{5.2}$$

Since $\tau_{zy} = 0$ no horizontally polarized S wave (i.e., the SH wave) can propagate. Therefore, only reflected P and vertically polarized S wave (i.e., the SV wave) can be assumed from the incidence of the plane P wave on the free surface as it is shown in figure 4.1.

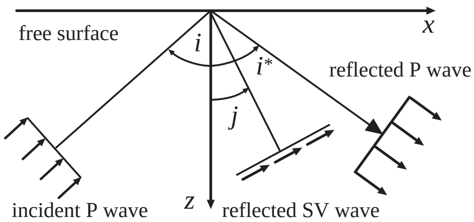


Fig. 4.1

The displacement of the SV wave is

$$\vec{u} = (-\Psi_{,z}, 0, \Psi_{,x}) \tag{5.3}$$

and the associated traction is

$$\begin{aligned} \vec{T} &= (\ \tau_{zx},\ \tau_{zy},\ \tau_{zz}\ ) \\ \vec{T} &= (\ \mu(\Psi_{,xx} - \Psi_{,zz}),\ 0,\ 2\mu\Psi_{,zx}\ ) \end{aligned} \tag{5.4}$$

Note that only scalar potential Ψ is needed here for the SV wave.

Both Φ and Ψ have the form

$$A \exp[-\mathrm{i}\omega(t - p_i x_i)]$$

The slowness vector $\vec{p}$ is

$$\left(\frac{\sin i}{\alpha}, 0, \frac{-\cos i}{\alpha}\right) \quad \text{for the incident P wave}$$
$$\left(\frac{\sin i^*}{\alpha}, 0, \frac{\cos i^*}{\alpha}\right) \quad \text{for the reflected P wave}$$
$$\left(\frac{\sin j}{\beta}, 0, \frac{\cos j}{\beta}\right) \quad \text{for the reflected S wave}$$

The total potential Φ is
$$\Phi = \Phi^I + \Phi^R \tag{5.5}$$
where Φ^I and Φ^R denote the incident and reflected components, respectively.

$$\Phi^I = A\exp[-\mathrm{i}\omega(t - \frac{\sin i}{\alpha}x + \frac{\cos i}{\alpha}z)] \tag{5.6}$$
$$\Phi^R = B\exp[-\mathrm{i}\omega(t - \frac{\sin i^*}{\alpha}x + \frac{\cos i^*}{\alpha}z)] \tag{5.7}$$

where A and B are constants.

The total potential Ψ is only made up from the reflected component

$$\Psi = \Psi^R \tag{5.8}$$
$$\Psi^R = C\exp[-\mathrm{i}\omega(t - \frac{\sin j}{\beta}x - \frac{\cos j}{\beta}z)] \tag{5.9}$$

It follows from eqs. (5.2), (5.4) and (5.5) - (5.9) that τ_{zx} and τ_{zz} are sums of three contributions involving factors of the type

$$\exp[-\mathrm{i}\omega(t - \frac{\sin i}{\alpha}x)] \quad \text{or} \quad \exp[-\mathrm{i}\omega(t - \frac{\sin i^*}{\alpha}x)] \quad \text{or} \quad \exp[-\mathrm{i}\omega(t - \frac{\sin j}{\beta}x)]$$

The boundary conditions on the free surface hold for all values of x and t. Therefore, the above exponential factors, which control the propagation in the horizontal direction, must be all the same. Consequently,

$i = i^*$ — i.e., the angles of incidence and reflection of the P wave are equal and

$\frac{\sin i}{\alpha} = \frac{\sin j}{\beta}$ — i.e., the horizontal component of the slowness of the incident wave is preserved on both reflection to the P wave and conversion to the SV wave.

The ratio $\frac{\sin i}{\alpha} = \frac{\sin j}{\beta}$ is often denoted as p and called the ray parameter. The ray parameter is very important because it is the same for the whole system of waves setup by reflection and transmission of plane waves in plane-layered media.

Having
$$p = \frac{\sin i}{\alpha} = \frac{\sin i^*}{\alpha} = \frac{\sin j}{\beta} \tag{5.10}$$
we can perform the x-derivatives in relations (5.1) and (5.2) using eqs. (5.5) - (5.9):

$$\begin{aligned} \text{P:} \quad \vec{u} &= (\mathrm{i}\omega p\Phi,\ 0,\ \Phi_{,z}) \\ \vec{T} &= (2\rho\beta^2\mathrm{i}\omega p\Phi_{,z},\ 0,\ -\rho(1-2\beta^2p^2)\omega^2\Phi) \end{aligned} \tag{5.11}$$

$$\begin{aligned} \text{SV:} \quad \vec{u} &= (-\Psi_{,z},\ 0,\ \mathrm{i}\omega p\Psi) \\ \vec{T} &= (\rho(1-2\beta^2p^2)\omega^2\Psi,\ 0,\ 2\rho\beta^2\mathrm{i}\omega p\Psi_{,z}) \end{aligned} \tag{5.12}$$

We want to find expression for the amplitude ratios B/A and C/A. Let us use the traction free condition

$$\tau_{zx} = 0 \quad \text{and} \quad \tau_{zz} = 0 \quad \text{at} \quad z = 0$$

From eqs. (5.11), (5.12), (5.5) and (5.8) we obtain

$$
\begin{aligned}
\tau_{zx} &= 2\rho\beta^2 \mathrm{i}\omega p(\Phi^I_{,z} + \Phi^R_{,z}) + \rho(1 - 2\beta^2 p^2)\omega^2 \Psi^R = 0 \\
\tau_{zz} &= -\rho(1 - 2\beta^2 p^2)\omega^2(\Phi^I + \Phi^R) + 2\rho\beta^2 \mathrm{i}\omega p \Psi^R_{,z} = 0
\end{aligned}
\tag{5.13}
$$

Inserting eqs. (5.6), (5.7) and (5.9) into eqs. (5.13) leads to

$$
\begin{aligned}
2\rho\beta^2 p\frac{\cos i}{\alpha}(A - B) + \rho(1 - 2\beta^2 p^2)C = 0 \\
\rho(1 - 2\beta^2 p^2)(A + B) + 2\rho\beta^2 p\frac{\cos j}{\beta}C = 0
\end{aligned}
$$

After some algebra we can obtain

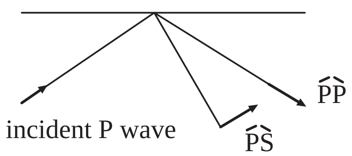
$$\frac{B}{A} = \frac{4\beta^4 p^2 \frac{\cos i}{\alpha}\frac{\cos j}{\beta} - (1 - 2\beta^2 p^2)^2}{4\beta^4 p^2 \frac{\cos i}{\alpha}\frac{\cos j}{\beta} + (1 - 2\beta^2 p^2)^2} \tag{5.14}$$

$$\frac{C}{A} = \frac{-4\beta^2 p\frac{\cos i}{\alpha}(1 - 2\beta^2 p^2)}{4\beta^4 p^2 \frac{\cos i}{\alpha}\frac{\cos j}{\beta} + (1 - 2\beta^2 p^2)^2} \tag{5.15}$$

Note that we could use, e.g., $\cos 2j = 1 - 2\beta^2 p^2$. We used p, $\frac{\cos i}{\alpha}$ and $\frac{\cos j}{\beta}$ because the above formulas can be easily generalized for the case of a vertically heterogeneous medium.

Ratios B/A and C/A are reflection coefficients for potentials. We are more interested in coefficients for displacements. Since the amplitude of displacement of the P wave is equal to $\frac{\omega}{\alpha}$ times the amplitude of Φ potential, B/A is also equal to the reflection coefficient PP for the displacement. In the case of the reflected S wave the amplitude of displacement is equal to $\frac{\omega}{\beta}$ times the amplitude of the Ψ potential. Therefore, $\frac{\alpha C}{\beta A}$ is equal to the reflection coefficient PS for the displacement.

Before we summarize the case of the incident P wave, define the notation and sign convention for the reflection coefficients.



A motion is positive if its component in the horizontal direction of propagation has the same phase as the propagation factor $\exp[-\mathrm{i}\omega(t - px)]$.
Let S be the amplitude of the incident wave. Then the displacements of the incident P wave, reflected P wave and reflected SV waves are

$$\text{inc. P:} \quad \vec{u} = S(\sin i, 0, -\cos i)\exp[-\mathrm{i}\omega(t - \frac{\sin i}{\alpha}x + \frac{\cos i}{\alpha}z)] \tag{5.16}$$

$$\text{refl. P:} \quad \vec{u} = S(\sin i, 0, \cos i)\ \acute{P}\grave{P}\ \exp[-\mathrm{i}\omega(t - \frac{\sin i}{\alpha}x - \frac{\cos i}{\alpha}z)] \tag{5.17}$$

$$\text{refl. SV:} \quad \vec{u} = S(\cos j, 0, -\sin j)\ \acute{P}\grave{S}\ \exp[-\mathrm{i}\omega(t - \frac{\sin j}{\beta}x - \frac{\cos j}{\beta}z)] \tag{5.18}$$

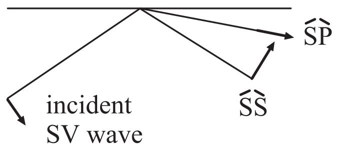
with the reflection coefficients

$$\acute{P}\grave{P} = \frac{-\left(\frac{1}{\beta^2}-2p^2\right)^2+4p^2\ \frac{\cos i}{\alpha}\frac{\cos j}{\beta}}{\left(\frac{1}{\beta^2}-2p^2\right)^2+4p^2\ \frac{\cos i}{\alpha}\frac{\cos j}{\beta}} \tag{5.19}$$

$$\acute{P}\grave{S} = \frac{4\frac{\alpha}{\beta}p\frac{\cos i}{\alpha}\left(\frac{1}{\beta^2}-2p^2\right)}{\left(\frac{1}{\beta^2}-2p^2\right)^2+4p^2\ \frac{\cos i}{\alpha}\frac{\cos j}{\beta}} \tag{5.20}$$

The case of an incident SV wave

Consistently with the case of the incident P wave define the rotation and sign convention for the reflection coefficients:



Let S be the amplitude of the incident SV wave. Then the displacements of the incident SV wave, reflected P wave and reflected SV wave are

$$\text{inc. SV:}\quad \vec{u} = S(\cos j, 0, \sin j)\exp[-\mathrm{i}\omega(t-\frac{\sin j}{\beta}x+\frac{\cos j}{\beta}z)] \tag{5.21}$$

$$\text{refl. P:}\quad \vec{u} = S(\sin i, 0, \cos i)\ \acute{S}\grave{P}\exp[-\mathrm{i}\omega(t-\frac{\sin i}{\alpha}x-\frac{\cos i}{\alpha}z)] \tag{5.22}$$

$$\text{refl. SV:}\quad \vec{u} = S(\cos j, 0, -\sin j)\ \acute{S}\grave{S}\exp[-\mathrm{i}\omega(t-\frac{\sin j}{\beta}x-\frac{\cos j}{\beta}z)] \tag{5.23}$$

Conditions $\tau_{zx}=0$ and $\tau_{zz}=0$ at $z=0$ lead to

$$2p\alpha\beta\frac{\cos i}{\alpha}\ \acute{S}\grave{P} + (1-2\beta^2p^2)(1-\acute{S}\grave{S}) = 0$$

and

$$-(1-2\beta^2p^2)\ \acute{S}\grave{P} + \frac{2\beta^3p}{\alpha}\frac{\cos j}{\beta}(1+\acute{S}\grave{S}) = 0$$

from which we can obtain

$$\acute{S}\grave{P} = \frac{4\frac{\beta}{\alpha}p\frac{\cos j}{\beta}\left(\frac{1}{\beta^2}-2p^2\right)}{\left(\frac{1}{\beta^2}-2p^2\right)^2+4p^2\ \frac{\cos i}{\alpha}\frac{\cos j}{\beta}} \tag{5.24}$$

$$\acute{S}\grave{S} = \frac{\left(\frac{1}{\beta^2}-2p^2\right)^2-4p^2\ \frac{\cos i}{\alpha}\frac{\cos j}{\beta}}{\left(\frac{1}{\beta^2}-2p^2\right)^2+4p^2\ \frac{\cos i}{\alpha}\frac{\cos j}{\beta}} \tag{5.25}$$

A matrix of all four possible reflection coefficients for the P-SV problem on the free surface,

$$\begin{pmatrix} \acute{P}\grave{P} & \acute{S}\grave{P} \\ \acute{P}\grave{S} & \acute{S}\grave{S} \end{pmatrix}$$

is called a scattering matrix.

5.3 Reflection and transmission of the plane SH waves at a solid-solid interface

The SH case is defined by

$$\begin{aligned} \vec{u} &= (0,\ v(x,z)\ ,0) \\ \vec{T}(\vec{u},\vec{n}) &= (0\ ,\ \mu v_{,z}\ ,\ 0) \end{aligned}$$

incident SH wave

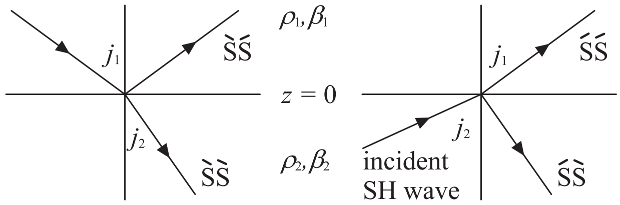


Fig. 4.4

Boundary conditions on $z = 0$: continuity of displacement
continuity of traction

Scattering matrix
$$\begin{pmatrix} \overset{\backslash}{S}\overset{/}{S} & \overset{/}{S}\overset{/}{S} \\ \overset{\backslash}{S}\overset{\backslash}{S} & \overset{/}{S}\overset{\backslash}{S} \end{pmatrix}$$

$$\text{inc. } \overset{\backslash}{S}: \quad \vec{u} = (0,S,0)\exp[-\mathrm{i}\omega(t - px - \frac{\cos j_1}{\beta_1}z)] \tag{5.26}$$

$$\text{refl. } \overset{/}{S}: \quad \vec{u} = (0,S,0)\ \overset{\backslash}{S}\overset{/}{S}\exp[-\mathrm{i}\omega(t - px + \frac{\cos j_1}{\beta_1}z)] \tag{5.27}$$

$$\text{transm. } \overset{\backslash}{S}: \quad \vec{u} = (0,S,0)\ \overset{\backslash}{S}\overset{\backslash}{S}\exp[-\mathrm{i}\omega(t - px - \frac{\cos j_2}{\beta_2}z)] \tag{5.28}$$

$$\text{inc. } \overset{/}{S}: \quad \vec{u} = (0,S,0)\exp[-\mathrm{i}\omega(t - px + \frac{\cos j_2}{\beta_2}z)] \tag{5.29}$$

$$\text{refl. } \overset{\backslash}{S}: \quad \vec{u} = (0,S,0)\ \overset{/}{S}\overset{\backslash}{S}\exp[-\mathrm{i}\omega(t - px - \frac{\cos j_2}{\beta_2}z)] \tag{5.30}$$

$$\text{transm. } \overset{/}{S}: \quad \vec{u} = (0,S,0)\ \overset{/}{S}\overset{/}{S}\exp[-\mathrm{i}\omega(t - px + \frac{\cos j_1}{\beta_1}z)] \tag{5.31}$$

$$\begin{aligned} \overset{\backslash}{S}\overset{/}{S} &= \frac{\rho_1\beta_1\cos j_1 - \rho_2\beta_2\cos j_2}{\Delta} \\ \overset{/}{S}\overset{\backslash}{S} &= -\ \overset{\backslash}{S}\overset{/}{S} \\ \overset{/}{S}\overset{/}{S} &= \frac{2\rho_2\beta_2\cos j_2}{\Delta} \quad , \quad \overset{\backslash}{S}\overset{\backslash}{S} = \frac{2\rho_1\beta_1\cos j_1}{\Delta} \end{aligned} \tag{5.32}$$

$$\Delta = \rho_1\beta_1\cos j_1 + \rho_2\beta_2\cos j_2$$

The case of a vertical incidence : $j_1 = j_2 = 0$

$$\overset{\backslash}{S}\overset{/}{S} = \frac{\rho_1\beta_1 - \rho_2\beta_2}{\rho_1\beta_1 + \rho_2\beta_2} \quad , \quad \overset{\backslash}{S}\overset{\backslash}{S} = \frac{2\rho_1\beta_1}{\rho_1\beta_1 + \rho_2\beta_2} \tag{5.33}$$

$\rho\beta$ is called the wave impedance.

5.4 The case of the critical incidence

Let $\beta_1 < \beta_2$. Consider a downgoing incident SH wave (see Fig. 4.4). The angle of incidence j_1, and angle of transmission j_2 are related through Snell's law

$$\frac{\sin j_1}{\sin j_2} = \frac{\beta_1}{\beta_2} \tag{5.34}$$

If $j_1 = j_c$ such that

$$\sin j_c = \frac{\beta_1}{\beta_2} \tag{5.35}$$

the angle of transmission j_2 is $j_2 = 90^{\text{o}}$
Angle j_c is called the critical angle. It follows from (5.32) that

$$\overset{\backslash}{S}\overset{/}{S}\,(j_1 = j_c) = 1 \tag{5.36}$$

$$\text{and} \qquad \overset{\backslash}{S}\overset{\backslash}{S}\,(j_1 = j_c) = 2$$

Consider the transmitted wave (eq. 5.28)

$$\vec{u} = (0, S, 0)\ \overset{\backslash}{S}\overset{\backslash}{S}\ \exp[-\mathrm{i}\omega(t - \frac{\sin j_2}{\beta_2}x - \frac{\cos j_2}{\beta_2}z)] \tag{5.37}$$

$$\begin{aligned} \cos j_2 &= \sqrt{1 - \sin^2 j_2} = \sin j_2 \sqrt{\frac{1}{\sin^2 j_2} - 1} = \\ &= \sin j_2 \sqrt{\frac{\beta_2^2}{\beta_2^2 \sin^2 j_2} - 1} \end{aligned}$$

$\frac{\beta_1}{\sin j_1} = \frac{\beta_2}{\sin j_2} = c_x$ – apparent velocity in the x-direction

$$\cos j_2 = \sin j_2 \sqrt{\frac{c_x^2}{\beta_2^2} - 1} \tag{5.38}$$

Substituting $\cos j_2$ from eq.(5.38) into relation (5.37) we have

$$\vec{u} = (0, S, 0)\ \overset{\backslash}{S}\overset{\backslash}{S}\ \exp\left[-\mathrm{i}\omega\left(t - \frac{\sin j_2}{\beta_2}x - \frac{\sin j_2}{\beta_2}\sqrt{\frac{c_x^2}{\beta_2^2} - 1}\, z\right)\right]$$

Compare now c_x with β_2 for three different angles of incidence:

$$\begin{aligned} j_1 < j_c \quad &\Rightarrow \quad c_x > \beta_2 \\ j_1 = j_c \quad &\Rightarrow \quad c_x = \beta_2 \qquad \text{since } j_2 = 90^{\text{O}} \\ j_1 > j_c \quad &\Rightarrow \quad \sin j_1 > \sin j_c \end{aligned}$$

$$\frac{\beta_1}{\sin j_1} \quad < \quad \frac{\beta_1}{\sin j_c}$$

$$\text{Since} \quad \frac{\beta_1}{\sin j_1} = c_x \quad \text{and} \quad \frac{\beta_1}{\sin j_c} = \beta_2$$

$$c_x < \beta_2$$

$$\sqrt{\frac{c_x^2}{\beta_2^2} - 1} \quad \text{is imaginary}$$

$$\sqrt{\frac{c_x^2}{\beta_2^2} - 1} = \mathrm{i}\sqrt{1 - \frac{c_x^2}{\beta_2^2}}$$

Then

$$\vec{u} = (0, S, 0)\ \grave{S}\grave{S}\ \exp\left[-\omega\frac{\sin j_2}{\beta_2}\sqrt{1 - \frac{c_x^2}{\beta_2^2}}\, z\right] \exp\left[-\mathrm{i}\omega(t - \frac{\sin j_2}{\beta_2}x)\right]$$

We see that in the case of an overcritical incidence, $j_1 > j_c$, we obtain an inhomogeneous wave propagating in the x-direction and attenuating exponentially away from the interface (in the z-direction).

Note on the P-SV transmission across the solid-solid interface.

Consider a configuration in figure 4.5

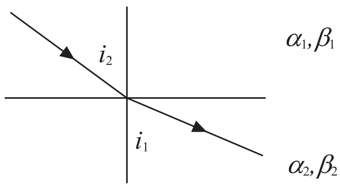


Fig. 4.5

Snell's law:
$$\frac{\sin i_1}{\alpha_1} = \frac{\sin i_2}{\alpha_2}$$

Let $\alpha_2 > \alpha_1$. Then the critical angle i_c is given by

$$\sin i_c = \frac{\alpha_1}{\alpha_2} \qquad \text{and} \qquad i_2 = 90^{\circ}$$

Since

$$\frac{\sin i_1}{\alpha_1} = \frac{\sin j_2}{\beta_2}$$

a transmitted SV wave is generated even for $i_1 > i_c$.

Let $\beta_2 > \alpha_1$. Then there is the second critical angle i_{c2} given by

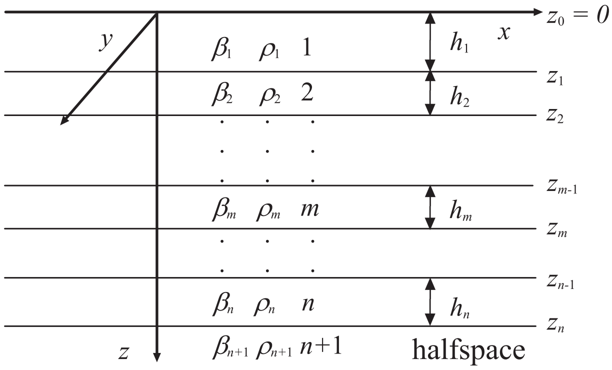
$$\sin i_{c2} = \frac{\alpha_1}{\beta_2}$$

For angles $i_1 > i_{c2}$ there is no transmitted wave.

6. SURFACE WAVES

6.1 Love waves in a layered halfspace

Consider a system of homogeneous, isotropic, perfectly elastic, horizontal planparallel layers over homogeneous, isotropic, perfectly elastic halfspace.



We will investigate propagation of a plane harmonic wave with speed c in the direction of axis x assuming that the wave is polarized in the direction of axis y and the amplitude in layer m is a function of coordinate z. Thus, displacement in layer m is $\vec{u} = (0, v_m, 0)$ and

$$v_m = A_m(z) \exp\left[-\mathrm{i}\omega\left(t - \frac{x}{c}\right)\right] \tag{6.1}$$

Inside layer m, v_m has to satisfy the equation of motion

$$v_{m,xx} + v_{m,zz} = \frac{1}{\beta_m^2}\ddot{v}_m \tag{6.2}$$

Inserting (6.1) into eq. (6.2) we obtain

$$A_{m,zz} + \frac{\omega^2}{c^2}\left(\frac{c^2}{\beta_m^2} - 1\right) A_m = 0 \tag{6.3}$$

Define

$$\begin{aligned} P_m &= \sqrt{\frac{c^2}{\beta_m^2} - 1}\,; \quad c > \beta_m \\ &= \mathrm{i}\sqrt{1 - \frac{c^2}{\beta_m^2}}\,; \quad c \le \beta_m \end{aligned} \tag{6.4}$$

Denoting wavenumber by $k = \frac{\omega}{c}$ and using (6.4) in eq. (6.3) we get

$$A_{m,zz} + k^2 P_m^2 A_m = 0 \tag{6.5}$$

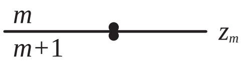
The solution of eq. (6.5) can be found in a form

$$A_m = \tilde{v}'_m \exp[\mathrm{i}kP_m z] + \tilde{v}''_m \exp[-\mathrm{i}kP_m z]$$

As we will see later it is advantageous to rewrite the above solution in a form

$$A_m = v'_m \exp[\mathrm{i}kP_m(z - z_{m-1})] + v''_m \exp[-\mathrm{i}kP_m(z - z_{m-1})] \tag{6.6}$$

In order to determine $2n + 2$ unknown constants v'_m and v''_m ; $m = 1, 2, \ldots, n + 1$, we need $2n + 2$ boundary conditions. Continuity of displacement and traction at internal interfaces give $2n$ conditions:



$$v_{m+1}(z_m) = v_m(z_m) \; ; m = 1, 2, \ldots, n \tag{6.7}$$

$$\vec{T}_{m+1}(z_m) = \vec{T}_m(z_m) \; ; m = 1, 2, \ldots, n \tag{6.8}$$

In the problem under consideration, continuity of traction implies continuity of the τ_{zy} stress-tensor component:

$$\tau_{zy}^{m+1}(z_m) = \tau_{zy}^{m}(z_m) \; ; m = 1, 2, \ldots, n \tag{6.9}$$

Since $\tau_{zy} = \mu v_{,z}$

$$\tau_{zy}^m = ik\mu_m P_m \{v'_m \exp[\mathrm{i}kp_m(z - z_{m-1})] - v''_m \exp[-\mathrm{i}kP_m(z - z_{m-1})]\} \exp\left[-\mathrm{i}\omega\left(t - \frac{x}{c}\right)\right] \tag{6.10}$$

Zero traction on the free surface ($z_0 = 0$) gives the $(2n + 1)$-th condition

$$\vec{T}_1(z_0) = 0 \tag{6.11}$$

$$\text{i.e.,} \qquad \tau_{zy}^1(z_0) = 0 \tag{6.12}$$

It follows from eqs. (6.10) and (6.12) that

$$v'_1 = v''_1 \tag{6.13}$$

The remaining $(2n+2)$-nd condition can be found by investigating displacement in the halfspace $(m = n + 1)$.

Assume that $\qquad c \leq \beta_{n+1}$

Then

$$P_{n+1} = \mathrm{i}\sqrt{1 - \frac{c^2}{\beta_{n+1}^2}}$$

and

$$\begin{aligned} v_{n+1} \;=\; & \left\{ v'_{n+1} \exp\left[-k\sqrt{1 - \frac{c^2}{\beta_{n+1}^2}}(z - z_n)\right] + \right. \\ & \left. v''_{n+1} \exp\left[k\sqrt{1 - \frac{c^2}{\beta_{n+1}^2}}(z - z_n)\right]\right\} \exp\left[-\mathrm{i}\omega\left(t - \frac{x}{c}\right)\right] \end{aligned}$$

Obviously, the amplitude would grow exponentialy with increasing z. Therefore,

$$v''_{n+1} = 0$$

Since the assumption of $c > \beta_{n+1}$ would lead to a wave not propagating in the x-direction, we conclude that

$$c \leq \beta_{n+1} \quad \text{and} \quad v''_{n+1} = 0 \tag{6.14}$$

Further we will omit factor $\exp[-\mathrm{i}\omega\,(t - \frac{x}{c})]$. Denote for brevity

$$\tau_m \equiv \tau^m_{zy} \tag{6.15}$$

$$q_m \equiv kP_m h_m \tag{6.16}$$

Compare displacements and stress at the top and bottom of a layer m:

$$\begin{aligned} v_m(z_{m-1}) &= v'_m + v''_m \\ \tau_m(z_{m-1}) &= \mathrm{i}k\mu_m P_m(v'_m - v''_m) \end{aligned} \tag{6.17}$$

$$\begin{aligned} v_m(z_m) &= v'_m \exp(\mathrm{i}q_m) + v''_m \exp(-\mathrm{i}q_m) \\ \tau_m(z_m) &= \mathrm{i}k\mu_m P_m[v'_m \exp(\mathrm{i}q_m) - v''_m \exp(-\mathrm{i}q_m)] \\ v_m(z_m) &= (v'_m + v''_m)\cos q_m + \mathrm{i}(v'_m - v''_m)\sin q_m \\ \tau_m(z_m) &= \mathrm{i}k\mu_m P_m[\mathrm{i}(v'_m + v''_m)\sin q_m + (v'_m - v''_m)\cos q_m] \end{aligned} \tag{6.18}$$

Inserting eqs. (6.17) into eqs. (6.18) gives

$$\begin{aligned} v_m(z_m) &= v_m(z_{m-1})\cos q_m &+& \ \tau_m(z_{m-1})\frac{\sin q_m}{k\mu_m P_m} \\ \tau_m(z_m) &= v_m(z_{m-1})(-k\mu_m P_m \sin q_m) &+& \ \tau_m(z_{m-1})\cos q_m \end{aligned}$$

It is obvious that the above equations may be written in a matrix form.

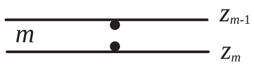
Define vector $S_m(z_m)$

$$S_m(z_m) = \begin{bmatrix} v_m(z_m) \\ \tau_m(z_m) \end{bmatrix} \tag{6.19}$$

and layer matrix C_m

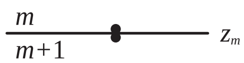
$$C_m = \begin{bmatrix} \cos q_m & \frac{\sin q_m}{k\mu_m P_m} \\ -k\mu_m P_m \sin q_m & \cos q_m \end{bmatrix} \tag{6.20}$$

Then we have



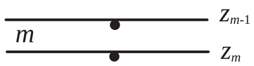
$$S_m(z_m) = C_m S_m(z_{m-1}) \tag{6.21}$$

Continuity of displacement and traction at interface z_m imply



$$S_{m+1}(z_m) = S_m(z_m) \tag{6.22}$$

Combining eqs. (6.21) and (6.22) we obtain



$$S_{m+1}(z_m) = C_m S_m(z_{m-1}) \tag{6.23}$$

Applying relation (6.23) recurrently to all layers we get

$$S_{n+1}(z_n) = CS_1(z_0) \tag{6.24}$$

where

$$C = C_n C_{n-1} \dots C_1 \tag{6.25}$$

From eqs. (6.17), (6.14) and (6.19) we have

$$S_{n+1}(z_n) = \begin{bmatrix} v'_{n+1} \\ \mathrm{i}k\mu_{n+1}P_{n+1}v'_{n+1} \end{bmatrix} \tag{6.26}$$

From eqs. (6.17), (6.13) and (6.19) we have

$$S_1(z_0) = \begin{bmatrix} 2v'_1 \\ 0 \end{bmatrix} \tag{6.27}$$

Inserting eqs. (6.26) and (6.27) into (6.24) we obtain

$$\begin{aligned} v'_{n+1} &= 2C_{11}v'_1 \\ \mathrm{i}k\mu_{n+1}P_{n+1}v'_{n+1} &= 2C_{21}v'_1 \end{aligned} \tag{6.28}$$

A nontrivial solution of the system of equations exists if determinant of the system is equal to zero:

$$\begin{vmatrix} 1 & -2C_{11} \\ ik\mu_{n+1}P_{n+1} & -2C_{21} \end{vmatrix} = 0$$

This gives

$$C_{21} - \mathrm{i}k\mu_{n+1}P_{n+1}C_{11} = 0$$

It follows from eqs. (6.4) and (6.14) that

$$P_{n+1} = \mathrm{i}\sqrt{1 - \frac{c^2}{\beta_{n+1}^2}} \tag{6.29}$$

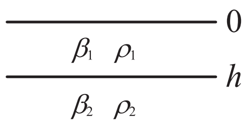
Then the above equation becomes

$$C_{21} + k\mu_{n+1}\sqrt{1 - \frac{c^2}{\beta_{n+1}^2}}\,C_{11} = 0 \tag{6.30}$$

It follows from the equation that velocity c is a function of frequency, material parameters of the medium and thicknesses of the layers. Due to dependence of c on frequency, the equation is called the dispersion equation. We can conclude that the plane, horizontally polarized wave may propagate in the considered layered halfspace in the x-direction if its speed c satisfies the dispersion equation (6.30). Such the wave is called Love wave.

6.2 Love waves in a single layer over halfspace

Consider the case of one layer over halfspace:



Matrix C is equal to the layer matrix C_1. Its C_{11} and C_{21} components are (see 6.20)

$$C_{11} = \cos q_1 \quad \text{and} \quad C_{21} = -k\mu_1 P_1 \sin q_1$$

where $q_1 = khP_1$
$c \le \beta_2$ (due to 6.14). Assume first that $c \le \beta_1$. Then

$$P_1 = \mathrm{i}\sqrt{1 - \frac{c^2}{\beta_1^2}} \equiv \mathrm{i}P_1^+$$

Dispersion equation (6.30) then becomes

$$-k\mu_1 \mathrm{i}P_1^+ \sin(kh\mathrm{i}P_1^+) + k\mu_2\sqrt{1 - \frac{c^2}{\beta_2^2}}\cos(kh\mathrm{i}P_1^+) = 0$$

$$\begin{aligned}
\frac{\sin(\mathrm{i}khP_1^+)}{\cos(\mathrm{i}khP_1^+)} &= \frac{\mu_2}{\mu_1}\frac{\sqrt{1-\frac{c^2}{\beta_2^2}}}{\mathrm{i}P_1^+} \\
\sin(\mathrm{i}khP_1^+) &= \mathrm{i}\sinh(khP_1^+) \\
\cos(\mathrm{i}khP_1^+) &= \cosh(khP_1^+) \\
\frac{\mathrm{i}\sinh(khP_1^+)}{\cosh(khP_1^+)} &= \mathrm{i}\tanh(khP_1^+) \\
-\tanh(khP_1^+) &= \frac{\mu_2}{\mu_1}\frac{\sqrt{1-\frac{c^2}{\beta_2^2}}}{P_1^+}
\end{aligned}$$

Since $\tanh(khP_1^+) > 0$ and also the right-hand side is positive, the equation is not possible. Therefore, we conclude that $c > \beta_1$, thus we have

$$\beta_1 < c \le \beta_2 \tag{6.31}$$

Dispersion equation is, taking $P_1 = \sqrt{\frac{c^2}{\beta_1^2} - 1}$,

$$-k\mu_1\sqrt{\frac{c^2}{\beta_1^2} - 1}\,\sin\left[kh\sqrt{\frac{c^2}{\beta_1^2} - 1}\right] + k\mu_2\sqrt{1 - \frac{c^2}{\beta_2^2}}\,\cos\left[kh\sqrt{\frac{c^2}{\beta_1^2} - 1}\right] = 0$$

$$\mathrm{tg}\left[kh\sqrt{\frac{c^2}{\beta_1^2} - 1}\right] = \frac{\mu_2}{\mu_1}\frac{\sqrt{1 - \frac{c^2}{\beta_2^2}}}{\sqrt{\frac{c^2}{\beta_1^2} - 1}} \tag{6.32}$$

Eq. (6.32) is the dispersion equation of Love waves in a layer over halfspace. There is no analytical solution to the dispersion equation. Values of $c(f)$ have to be found numerically.
It follows from the equation that

$$f = \frac{c}{2\pi h\sqrt{\frac{c^2}{\beta_1^2}-1}}\left\{\mathrm{arctg}\left[\frac{\mu_2}{\mu_1}\frac{\sqrt{1-\frac{c^2}{\beta_2^2}}}{\sqrt{\frac{c^2}{\beta_1^2}-1}}\right]+l\pi\right\} \tag{6.33}$$

It is obvious from eq. (6.33) that there is an infinite number of solution to the dispersion relations. Each of them corresponds to a mode of propagation. $l = 0$ corresponds to the fundamental mode, $l = 1$ corresponds to the 1-st higher mode, and so on.
Since, relation (6.31), $\beta_1 < c \leq \beta_2$, find frequencies fc_l such that $c(fc_l) = \beta_2$. From eq. (6.33) we get

$$f_{cl} = \frac{l\beta_2}{2h\sqrt{\frac{\beta_2^2}{\beta_1^2}-1}} \tag{6.34}$$

Frequencies f_{c_l} are called cut-off frequencies since the l-th mode does not exist for frequencies lower than f_{c_l}. The only mode which exist for all frequencies larger than 0 is the fundamental mode since $f_{c_0} = 0$. Dependence of the phase velocity c on frequency implies the existence and frequency dependence of a group velocity

$$v_g = \frac{c}{1-\frac{f}{c}\frac{dc}{df}} \tag{6.35}$$

It follows from eq. (6.35) that $v_g \leq c$.

Graphically displayed dependencies $c(f)$ and $v_g(f)$ [or $c(T)$ and $v_g(T)$] are dispersion curves of the phase and group velocities. Dispersion curves are important tools for studying the layered structure of the Earth's interior. Dispersion curves obtained from recorded seismograms are compared with those calculated for an assumed model. A curve-fitting procedure then leads to a possible layered model of the investigated region.

It can be shown that Love waves are formed by the constructive interference of the SH waves with supercritical incidence in the layer and inhomogeneous waves in the halfspace. Since the dispersion is due to interference nature of Love waves, the dispersion is called geometrical dispersion.
It follows from eq. (6.1), (6.6) and (6.13) that

$$\begin{aligned} v_1(z) &= v_1'[\exp(\mathrm{i}kP_1z)+\exp(-\mathrm{i}kP_1z)]\exp\left[-\mathrm{i}\omega\left(t-\frac{x}{c}\right)\right] \\ v_1(z) &= 2v_1'\cos(k\sqrt{\frac{c^2}{\beta_1^2}-1}\;z)\exp\left[-\mathrm{i}\omega\left(t-\frac{x}{c}\right)\right] \end{aligned} \tag{6.36}$$

Eq. (6.36) means that the amplitude of a mode of Love waves inside a layer oscillates with depth.

Eqs. (6.1), (6.4), (6.6) and (6.14) imply

$$v_2(z) = v_2'\exp[-k\sqrt{1-\frac{c^2}{\beta_2^2}}(z-h)]\exp\left[-\mathrm{i}\omega\left(t-\frac{x}{c}\right)\right] \tag{6.37}$$

The amplitude in the halfspace exponentially decreases with depth.

7. SEISMIC RESPONSE OF A SYSTEM OF HORIZONTAL LAYERS OVER A HALFSPACE TO A VERTICALLY INCIDENT PLANE SH WAVE

7.1 The case of n layers over halfspace

Consider the same model of the medium as in the case of Love waves. However, assume a harmonic plane SH wave propagating in the vertical direction.
Wavefield $\vec{u}_m = (0, v_m, 0)$ in layer m is a superposition of waves propagating in the upward direction and waves propagating in the downward direction:

$$v_m = \tilde{v}'_m \exp\left[-\mathrm{i}\omega\left(t - \frac{z}{\beta_m}\right)\right] + \tilde{v}''_m \exp\left[-\mathrm{i}\omega\left(t + \frac{z}{\beta_m}\right)\right]$$

where $\tilde{v}'_m$ and $\tilde{v}''_m$ are unknown coefficients.
Modify the above expression similarly as in the case of Love waves:

$$v_m = \{v'_m \exp[\mathrm{i}\omega\beta_m^{-1}(z - z_{m-1})] + v''_m \exp[-\mathrm{i}\omega\beta_m^{-1}(z - z_{m-1})]\} \exp(-\mathrm{i}\omega t) \qquad (7.1)$$

In order to determine $2n+2$ undetermined coefficients v'_m and v''_m ; $m = 1, 2, \ldots, n+1$, we need $2n+2$ boundary conditions. Except for the condition in the halfspace, they are the same as in the case of Love waves. Here we assume a plane wave inciding from the halfspace, i.e., we know v''_{n+1}.

The stress-tensor component we need is $\tau_{zy} = \mu v_{,z}$. τ_{zy} in layer m is

$$\tau^m_{zy} = \mathrm{i}\omega q_m \{v'_m \exp[\mathrm{i}\omega\beta_m^{-1}(z - z_{m-1})] - v''_m \exp[-\mathrm{i}\omega\beta_m^{-1}(z - z_{m-1})]\} \exp(-\mathrm{i}\omega t) \qquad (7.2)$$

Further we will use τ_m instead of τ^m_{zy} and omit factor $\exp(-\mathrm{i}\omega t)$. In eq. (7.2) we use q_m, the wave impedance defined as

$$q_m = \mu_m \beta_m^{-1} = \rho_m \beta_m \qquad (7.3)$$

Denote also

$$b_m \equiv \omega \beta_m^{-1} h_m \qquad (7.4)$$

Compare displacement and stress at the top and bottom of layer m:

$$\begin{aligned} v_m(z_{m-1}) &= v'_m + v''_m \\ \tau_m(z_{m-1}) &= \mathrm{i}\omega q_m (v'_m - v''_m) \end{aligned} \qquad (7.5)$$

$$\begin{aligned} v_m(z_m) &= v'_m \exp(\mathrm{i}b_m) + v''_m \exp(-\mathrm{i}b_m) \\ \tau_m(z_m) &= \mathrm{i}\omega q_m [v'_m \exp(\mathrm{i}b_m) - v''_m \exp(-\mathrm{i}b_m)] \end{aligned}$$

$$\begin{aligned} v_m(z_m) &= (v'_m + v''_m)\cos b_m + \mathrm{i}(v'_m - v''_m)\sin b_m \\ \tau_m(z_m) &= \mathrm{i}\omega q_m [\mathrm{i}(v'_m + v''_m)\sin b_m + (v'_m - v''_m)\cos b_m] \end{aligned} \qquad (7.6)$$

Inserting eqs. (7.5) into eqs. (7.6) gives

$$\begin{aligned} v_m(z_m) &= v_m(z_{m-1})\cos b_m &+& \tau_m(z_{m-1})\tfrac{\sin b_m}{\omega q_m} \\ \tau_m(z_m) &= v_m(z_{m-1})(-\omega q_m \sin b_m) &+& \tau_m(z_{m-1})\cos b_m \end{aligned} \tag{7.7}$$

The above equations may be written in a matrix form. Define vector $S_m(z)$

$$S_m(z) = \begin{bmatrix} v_m(z) \\ \tau_m(z) \end{bmatrix} \quad ; \qquad z_{m-1} \leq z \leq z_m \tag{7.8}$$

and layer matrix B_m

$$B_m = \begin{bmatrix} \cos b_m & \frac{\sin b_m}{\omega q_m} \\ -\omega q_m \sin b_m & \cos b_m \end{bmatrix} \tag{7.9}$$

Then eqs. (7.7) give

$$S_m(z_m) = B_m S_m(z_{m-1}) \tag{7.10}$$

Applying continuity of displacement and traction at interface z_m to eq. (7.10) gives

$$S_{m+1}(z_m) = B_m S_m(z_{m-1}) \tag{7.11}$$

Applying relation (7.11) recurrently to all layers we get

$$S_{n+1}(z_n) = BS_1(z_0) \tag{7.12}$$

where

$$B = B_n.B_{n-1}\ldots B_1 \tag{7.13}$$

Since $\tau_1(z_0) = 0$,

$$S_1(z_0) = \begin{bmatrix} 2v_1' \\ 0 \end{bmatrix} = \begin{bmatrix} v_1(z_0) \\ 0 \end{bmatrix}$$

and eq. (7.12) gives

$$\begin{aligned} (v_{n+1}' + v_{n+1}'')\exp(-\mathrm{i}\omega t) &= B_{11}v_1 \\ \mathrm{i}\omega q_{n+1}(v_{n+1}' - v_{n+1}'')\exp(-\mathrm{i}\omega t) &= B_{21}v_1 \end{aligned}$$

$$2v_{n+1}''\exp(-\mathrm{i}\omega t) = \left(B_{11} - \frac{B_{21}}{\mathrm{i}\omega q_{n+1}}\right)v_1$$

$$v_1(z_0) = \frac{2}{B_{11} + \mathrm{i}\frac{B_{21}}{\omega q_{n+1}}}v_{n+1}''\exp(-\mathrm{i}\omega t) \tag{7.14}$$

Assume $v_{n+1}'' = 1$ ($\sim$ unit amplitude of the wave inciding from the halfspace) and define

$$H(\omega) = \frac{2}{B_{11} + \mathrm{i}\frac{B_{21}}{\omega q_{n+1}}} \tag{7.15}$$

Then eq. (7.14) becomes

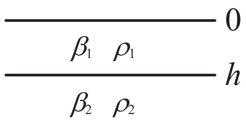
$$v_1(z_0) = H(\omega)\exp(-\mathrm{i}\omega t) \tag{7.16}$$

Formula (7.16) gives displacement of a harmonic plane SH wave at the free surface of a layered halfspace assuming that a plane harmonic SH wave with unit amplitude is inciding from the halfspace. $H(\omega)$ or, alternatively, $H(f)$, gives the amplitude of the harmonic wave at the free surface. $H(f)$ is a function of material parameters and layer thickness for a given frequency f. It fully characterizes transfer properties of a medium at frequency f. If we calculate $H(f)$ for all frequencies we get the transfer function (spectral characteristics) of the considered medium. This means that the displacement $d(z_0)$ at the free surface due to vertically incident plane SH wave with an arbitrary time function $s(t)$ can be computed as

$$d(z_0) = \mathcal{F}^{-1}\{S(f)H(f)\} \tag{7.17}$$

where $S(f) = \mathcal{F}\{s(t)\}$.

7.2 The case of a single layer over halfspace



$$H(f) = \frac{2}{\cos b_1 - \mathrm{i}\frac{q_1}{q_2}\sin b_1} \tag{7.18}$$

$$|H(f)| = \frac{2}{\left[\cos^2(\frac{2\pi h_m}{\beta_m}f) + \frac{q_1^2}{q_2^2}\sin^2(\frac{2\pi h_m}{\beta_m}f)\right]^{1/2}} \tag{7.19}$$

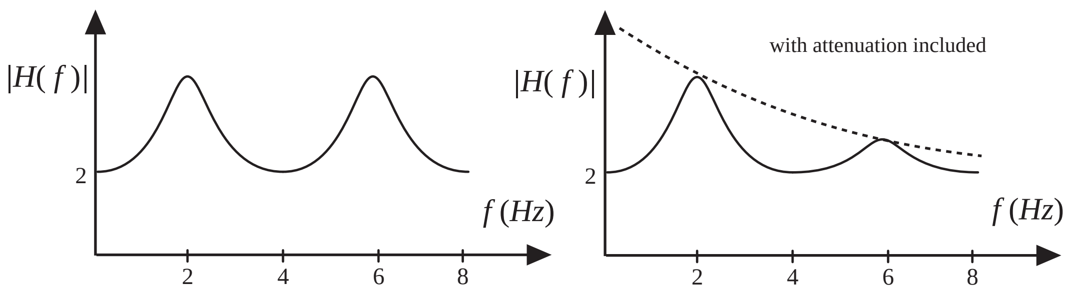
$$\frac{\partial|H(f)|}{\partial f} = 0 \quad \Rightarrow \quad \cos(\tfrac{2\pi h_m}{\beta_m}f)\sin(\tfrac{2\pi h_m}{\beta_m}f)\left(1 - \frac{q_1^2}{q_2^2}\right) = 0$$

The latter equation is satisfied if either $\cos(\frac{2\pi h_m}{\beta_m}f) = 0$ or $\sin(\frac{2\pi h_m}{\beta_m}f) = 0$. The second option is wrong because the value of local maximum would not depend on the ratio of wave impedances. The first option leads to

$$f_n = \frac{2n-1}{4}\frac{\beta_1}{h} \; ; n = 1, 2, \ldots \tag{7.20}$$

f_n is a resonant frequency of the n-th mode of the so-called 1D vertical resonance in a soft surface layer. Formula (7.20) has great importance in earthquake engineering since it gives an estimate of frequency at which earthquake ground motion can be amplified if the local surface geological structure can be approximately described as a soft surface layer over a harder soil or rock.

Example: $\beta_1 = 400\mathrm{m/s}$, $h = 50\mathrm{m}$



8. THE RAY METHOD

8.1 The ray series in the frequency domain

Consider a perfectly elastic, isotropic, smoothly heterogeneous medium (λ, μ and ρ are continuous functions of spatial coordinates together with their first derivatives, let second derivatives be piece–wise continuous).
Consider a harmonic wave

$$\vec{u}_H(x_i, t, \omega) = \vec{U}_H(x_i, \omega) \exp(-\mathrm{i}\omega t) \tag{8.1}$$

The complex vector $\vec{U}_H$ varies rapidly with spatial coordinates, especially if ω is large. We know that in a homogeneous medium the changes are caused by a factor $\exp[\mathrm{i}\omega\tau(x_i)]$.[1] An example is a plane wave propagating in the x–direction: $\tau(x_i) = \frac{x}{v}$, v being a speed. Thus, it is useful to rewrite $\vec{U}_H$ in a form

$$\vec{U}_H(x_i, \omega) = \vec{A}(x_i, \omega) \exp\left[\mathrm{i}\omega\tau(x_i)\right]$$

Here, $\vec{A}$ still depends on frequency ω but with an increasing frequency, factor $\exp[\mathrm{i}\omega\tau(x_i)]$ is more and more responsible for the changes. Therefore, for a sufficiently large ω we can approximate $\vec{A}(x_i, \omega)$ by $\vec{U}^{(0)}(x_i)$.
Then eq. (8.1) becomes

$$\vec{u}_H(x_i, t, \omega) = \vec{U}^{(0)}(x_i) \exp[-\mathrm{i}\omega(t - \tau(x_i))] \tag{8.2}$$

Eq. (8.2) gives the form of the high-frequency (HF) solution to the equation of motion. The above derivation is, of course, only intuitive – it itself does not prove anything. It was suggested for electromagnetic waves by Sommerfeld and Runge, and was only a posteriori justified by the working theory built on the assumption of (8.2).
The accuracy of the approximation can be increased by the asymptotic expansion in powers of ω^{-1}:

$$\vec{A}(x_i, \omega) \sim \vec{U}^{(0)}(x_i) + \frac{\vec{U}^{(1)}(x_i)}{(-i\omega)} + \frac{\vec{U}^{(2)}(x_i)}{(-i\omega)^2} + \cdots \tag{8.3}$$

Then the solution (8.2) becomes

$$\vec{u}_H(x_i, t, \omega) = \exp[-\mathrm{i}\omega(t - \tau(x_i))] \sum_{k=0}^{\infty} (-\mathrm{i}\omega)^{-k} \vec{U}^{(k)}(x_i) \tag{8.4}$$

For sufficiently large ω it is enough to consider only the first term in the series : $k = 0$. The approximation is then given by eq. (8.2), it is called the zero-order approximation or the ray approximation.

[1] impulse response in homogeneous medium for point source : $\vec{u}(\vec{x}, t) = \frac{\vec{A}}{r}\delta(t - \tau(\vec{x})) = \frac{\vec{A}}{r}\mathcal{F}\{e^{i\omega\tau(\vec{x})}\}$

Notes:

1. We could consider factor $\exp(\mathrm{i}\omega t)$ instead of $\exp(-\mathrm{i}\omega t)$. The difference between the two sign conventions are only in the complex quantities. The complex quantities in one convention are replaced by the complex-conjugate quantities in the other convention.
2. The assumption of large ω, the high-frequency assumption, is symbolically expressed as $\omega \to \infty$ or $\omega \gg 1$. Since $\lambda = 2\pi v/\omega$, high frequency means a short wave length – $\lambda \ll 1$. Both the high frequency and short wavelength are relative concepts. Frequency and wavelength have to be compared to quantities that measure heterogeneity of a medium. For example, λ is to be compared with a radius of curvature R of an interface or $\frac{v}{|\nabla v|}$. Frequency can be compared with $|\nabla v|$. A high-frequency or short-wave length assumption then means $\lambda \ll R$, $\lambda \ll \frac{v}{|\nabla v|}$, $\omega \gg |\nabla v|$.

8.2 The ray series in the time domain

The application of the Fourier transform to the ray series (8.4)[1] gives the ray series in the time domain. Before transforming the series (8.4), we can multiply it by some high-frequency function $S(\omega)$ which represents a source-time function $s(t) : S(\omega) = \mathcal{F}[s(t)]$.[2] The ray series in the time domain has then the form

$$u_i(x_j, t) = \mathrm{Re} \sum_{k=0}^{\infty} U_i^{(k)}(x_j) F_k(t - \tau(x_j)) \tag{8.5}$$

where

$$\begin{aligned} F_k(t - \tau(x_j)) &= \frac{1}{\pi} \int_{\omega_0}^{\infty} (-\mathrm{i}\omega)^{-k}\, S(\omega)\, \exp[-\mathrm{i}\omega(t - \tau(x_j))] d\omega \\ k &= 0, 1, 2, \ldots \end{aligned} \tag{8.6}$$

The integration is performed from ω_0 because we assume that $S(\omega) = 0$ for $\omega < \omega_0$. We use (8.6) to define also F_{-2} and F_{-1}.

Properties of the function F_k:

1. F_k is a high–frequency function
2.

$$\begin{aligned} F_k'(\xi) = \frac{dF_k(\xi)}{d\xi} &= \frac{1}{\pi} \int_{\omega_0}^{\infty} (-\mathrm{i}\omega)^{-k+1} S(\omega) \exp(-\mathrm{i}\omega\xi) d\omega \\ F_k'(\xi) &= F_{k-1}(\xi) \end{aligned} \tag{8.7}$$

3. Under certain conditions

$$\int_{-\infty}^{\xi} F_k(\eta) d\eta = \frac{1}{\pi} \int_{\omega_0}^{\infty} \frac{(-\mathrm{i}\omega)^{-k}}{(-\mathrm{i}\omega)} S(\omega) \exp(-\mathrm{i}\omega\xi) d\omega$$

[1] precisely we create an integral composition $\vec{u}(\vec{x}, t) = \int_{-\infty}^{\infty} ... d\omega$

[2] if multiplied by $S(\omega) = 1 = \mathcal{F}[\delta(t)]$ we get an impule response

$$\begin{aligned}
\frac{1}{\omega_0}\left|\frac{1}{\pi}\int\limits_{\omega_0}^{\infty}(-\mathrm{i}\omega)^{-k}S(\omega)\exp(-\mathrm{i}\omega\xi)d\omega\right| &\geq \left|\frac{1}{\pi}\int\limits_{\omega_0}^{\infty}\frac{(-\mathrm{i}\omega)^{-k}}{(-\mathrm{i}\omega)}S(\omega)\exp(-\mathrm{i}\omega\xi)d\omega\right| \\
\frac{1}{\omega_0}\,|F_k(\xi)| &\geq \left|\int\limits_{-\infty}^{\xi}F_k(\eta)d\eta\right|
\end{aligned} \tag{8.8}$$

i.e., the integral of F_k can be neglected compared to F_k itself. Due to relation (8.7) this means that F_k can be neglected compared to F_{k-1}.

4. $F_k(\xi)$ is an analytical signal – its real and imaginary parts form a Hilbert pair.

Function $F_k(\xi)$ is called the HF signal. $\vec{U}^{(k)}$ is the complex vectorial amplitude. The scalar real-valued τ is called the eikonal or the phase function. The surface of constant $\tau : \tau(x_i) = t_0$, represents the wavefront for a specified time t_0.

8.3 The basic system of equations of the ray method

The equation of motion is

$$(\lambda+\mu)u_{j,ij} + \mu\, u_{i,jj} + \lambda_{,j}\, u_{j,j} + \mu_{,j}\,(u_{i,j}+u_{j,i}) = \rho\, u_{i,tt} \tag{8.9}$$

The ray series in the time domain is

$$u_i(x_j,t) = \sum_{k=0}^{\infty} U_i^{(k)}(x_j)\, F_k(t-\tau(x_j)) \tag{8.10}$$

We insert solution (8.10) into the equation (8.9).
Differentiating u_i includes the following terms:

$$\begin{aligned}
u_{i,tt} &= \sum_{k=0}^{\infty} U_i^{(k)}\, F_k'' \\
u_{i,j} &= \sum_{k=0}^{\infty}\left(U_{i,j}^{(k)}\, F_k - U_i^{(k)}\, F_k'\tau_{,j}\right) \\
u_{i,jm} &= \sum_{k=0}^{\infty}\left(U_{i,jm}^{(k)}\, F_k - U_{i,j}^{(k)}\, F_k'\,\tau_{,m} - U_{i,m}^{(k)}\, F_k'\,\tau_{,j} + U_i^{(k)}\, F_k''\,\tau_{,j}\tau_{,m} - U_i^{(k)}\, F_k'\,\tau_{,jm}\right)
\end{aligned}$$

Define the vector operators N_i, M_i and L_i:

$$N_i\left[\vec{U}^{(k)}\right] = -\rho\, U_i^{(k)} + (\lambda+\mu)\, U_j^{(k)}\,\tau_{,i}\,\tau_{,j} + \mu\, U_i^{(k)}\,\tau_{,j}\,\tau_{,j} \tag{8.11}$$

$$\begin{aligned}
M_i\left[\vec{U}^{(k)}\right] &= (\lambda+\mu)\left(U_{j,i}^{(k)}\,\tau_{,j} + U_{j,j}^{(k)}\,\tau_{,i} + U_j^{(k)}\,\tau_{,ij}\right) + \mu\left(2U_{i,j}^{(k)}\,\tau_{,j} + U_i^{(k)}\,\tau_{,jj}\right) \\
&+ \lambda_{,i}\, U_j^{(k)}\,\tau_{,j} + \mu_{,j}\, U_i^{(k)}\,\tau_{,j} + \mu_{,j}\, U_j^{(k)}\,\tau_{,i}
\end{aligned} \tag{8.12}$$

$$L_i\left[\vec{U}^{(k)}\right] = (\lambda+\mu)\, U_{j,ij}^{(k)} + \mu\, U_{i,jj}^{(k)} + \lambda_{,i}\, U_{j,j}^{(k)} + \mu_{,j}\, U_{i,j}^{(k)} + \mu_{,j}\, U_{j,i}^{(k)} \tag{8.13}$$

The equation of motion may be then written in a form

$$\sum_{k=0}^{\infty} \left\{ F_k'' N_i \left[\vec{U}^{(k)}\right] - F_k' M_i \left[\vec{U}^{(k)}\right] + F_k L_i \left[\vec{U}^{(k)}\right] \right\} = 0$$

Since $F_k'' = F_{k-2}$ and $F_k' = F_{k-1}$, the above equation gives

$$\sum_{k=0}^{\infty} \left\{ F_{k-2} N_i \left[\vec{U}^{(k)}\right] - F_{k-1} M_i \left[\vec{U}^{(k)}\right] + F_k L_i \left[\vec{U}^{(k)}\right] \right\} = 0$$

Formally we can consider $\vec{U}^{(-1)}$ and $\vec{U}^{(-2)}$ both equal to 0. Then nothing is changed if we add

$$\begin{aligned} -F_{-2} M_i \left[\vec{U}^{(-1)}\right] &+ F_{-1} L_i \left[\vec{U}^{(-1)}\right] \\ &+ F_{-2} L_i \left[\vec{U}^{(-2)}\right] \end{aligned}$$

to the left–hand side of the above equation. Then, however, the equation may be rearranged as

$$\sum_{k=0}^{\infty} F_{k-2} \left\{ N_i \left[\vec{U}^{(k)}\right] - M_i \left[\vec{U}^{(k-1)}\right] + L_i \left[\vec{U}^{(k-2)}\right] \right\} = 0$$

The equation will be identically satisfied if

$$\begin{gathered} N_i \left[\vec{U}^{(k)}\right] - M_i \left[\vec{U}^{(k-1)}\right] + L_i \left[\vec{U}^{(k-2)}\right] = 0 \\ k = 0, 1, 2, \dots \text{ and } \vec{U}^{(-2)} = \vec{U}^{(-1)} = 0 \end{gathered} \tag{8.14}$$

System of equations (8.14) is the basic system of equations in the ray method. It is a recurrent system and can used to determine the amplitude coefficients $\vec{U}^{(k)}$ and phase function τ.

8.4 The first equations in the basic system

The first vectorial equation in the system (8.14) is

$$N_i \left[\vec{U}^{(0)}\right] = 0 \tag{8.15}$$

According to definition (8.11) the equation may be written as

$$-\rho U_i^{(0)} + (\lambda + \mu) U_j^{(0)} \tau_{,i} \tau_{,j} + \mu U_i^{(0)} \tau_{,j} \tau_{,j} = 0 \qquad i = 1, 2, 3 \tag{8.16}$$

System (8.16) is the system of three linear equations for three unknown components $U_1^{(0)}, U_2^{(0)}$ and $U_3^{(0)}$.
A nontrivial solution of the system exists if the determinant of the system is equal to zero.

$$\begin{vmatrix} -\rho + (\lambda+\mu)\tau_{,1}^2 + \mu\tau_{,j}\tau_{,j} & (\lambda+\mu)\tau_{,1}\tau_{,2} & (\lambda+\mu)\tau_{,1}\tau_{,3} \\ (\lambda+\mu)\tau_{,1}\tau_{,2} & -\rho + (\lambda+\mu)\tau_{,2}^2 + \mu\tau_{,j}\tau_{,j} & (\lambda+\mu)\tau_{,2}\tau_{,3} \\ (\lambda+\mu)\tau_{,1}\tau_{,3} & (\lambda+\mu)\tau_{,2}\tau_{,3} & -\rho + (\lambda+\mu)\tau_{,3}^2 + \mu\tau_{,j}\tau_{,j} \end{vmatrix} = 0$$

The above equation may be rewritten as

$$\begin{aligned} -\rho^3 + (\tau_{,i}\tau_{,i})\rho^2[(\lambda+2\mu)+2\mu] &- (\tau_{,i}\tau_{,i})^2 \rho\mu[2(\lambda+2\mu)+\mu] \\ &+ (\tau_{,i}\tau_{,i})^3 \mu^2 (\lambda+2\mu) = 0 \end{aligned} \tag{8.17}$$

Define

$$\alpha(x_i) = \sqrt{\frac{\lambda(x_i) + 2\mu(x_i)}{\rho(x_i)}} \tag{8.18}$$

and

$$\beta(x_i) = \sqrt{\frac{\mu(x_i)}{\rho(x_i)}} \tag{8.19}$$

Then eq. (8.17) can be simplified to

$$\left(\tau_{,i}\tau_{,i} - \frac{1}{\alpha^2}\right)\left(\tau_{,i}\tau_{,i} - \frac{1}{\beta^2}\right)^2 = 0 \tag{8.20}$$

A nontrivial solution exists if and only if one of the following conditions is satisfied

$$\tau_{,i}\tau_{,i} = \frac{1}{\alpha^2} \tag{8.21}$$

$$\tau_{,i}\tau_{,i} = \frac{1}{\beta^2} \tag{8.22}$$

Equations (8.21) and (8.22) are the eikonal equations. The first one relates to the P wave propagating with speed α, the second one relates to the S wave propagating with speed β. Note, however, that both α and β are not constant – they change with spatial coordinates. The eikonal equation is, in fact, a mathematical formulation of Huyghens' principle. This means that it describes propagation of the wavefront (therefore, it is sometimes called the equation of wavefronts).

The eikonal equations represent a very important result. We have found that in the HF approximation two independent waves, P and S, can propagate in a heterogeneous isotropic medium. In other words, under the HF assumption we approximately separated the equation of motion into two equations describing propagation of HF P and S waves. The higher frequency ($\sim \omega_0$), the better separation of the two waves. Generally, the wave process in the heterogeneous medium is complicated and includes properties of both the P and S waves.

Note the important difference between the homogeneous and heterogeneous medium. The equation of motion for the homogeneous medium can be mathematically strictly separated into two wave equations – one describing the P wave, the other describing the S wave. The P wave is strictly independent of the S wave and vice versa.

We can also use eqs. (8.16) to determine polarization of vector $\vec{U}^{(0)}$. First, multiply (8.16) by $\nabla\tau$ in a scalar product. We get

$$\begin{aligned} -\rho U_i^{(0)}\tau_{,i} + (\lambda+\mu)U_j^{(0)}\tau_{,i}\tau_{,j}\tau_{,i} + \mu U_i^{(0)}\tau_{,j}\tau_{,j}\tau_{,i} &= 0 \\ \left(U_i^{(0)}\tau_{,i}\right)\left(-\rho + (\lambda+\mu)\tau_{,j}\tau_{,j} + \mu\tau_{,j}\tau_{,j}\right) &= 0 \\ \left(\vec{U}^{(0)}\cdot\nabla\tau\right)\left(\tau_{,j}\tau_{,j} - \frac{1}{\alpha^2}\right) &= 0 \qquad (8.23) \end{aligned}$$

Now we apply the vector product to eqs. (8.16). It is convenient to use the ∇ symbolics. Eq. (8.16) is

$$
\begin{aligned}
-\rho\vec{U}^{(0)} + (\lambda+\mu)\left(\vec{U}^{(0)}\cdot\nabla\tau\right)\nabla\tau + \mu\vec{U}^{(0)}\left(\nabla\tau\cdot\nabla\tau\right) &= 0 \\
-\rho\vec{U}^{(0)}\times\nabla\tau + (\lambda+\mu)\left(\vec{U}^{(0)}\cdot\nabla\tau\right)\nabla\tau\times\nabla\tau + \mu\vec{U}^{(0)}\times\nabla\tau(\nabla\tau\cdot\nabla\tau) &= 0 \\
\left(\vec{U}^{(0)}\times\nabla\,\tau\right)\left(\tau_{,j}\tau_{,j} - \frac{1}{\beta^2}\right) &= 0
\end{aligned} \tag{8.24}
$$

Equations (8.23) and (8.24) apply to both waves. Consider first the P wave. Since $\tau_{,j}\tau_{,j} = \frac{1}{\alpha^2}$, we have

$$\vec{U}^{(0)}\times\nabla\,\tau = 0$$

i.e., the displacement vector of the P wave in the zero–order approximation is perpendicular to the wavefront.
Consider now the S wave. Since $\tau_{,j}\tau_{,j} = \frac{1}{\beta^2}$, we have

$$\vec{U}^{(0)}\cdot\nabla\,\tau = 0$$

i.e., the displacement vector of the S wave in the zero–order approximation is parallel to the wavefront.

8.5 Rays and ray fields

The eikonal equations are nonlinear partial differential equations. They can be solved by the method of the characteristic curves that enables to find solution by means of a system of ordinary differential equations. The characteristic curves of the eikonal equation are rays and the corresponding set of 6 ordinary differential equations, so-called ray tracing system, is

$$\frac{dx_i}{d\tau} = v^2 p_i \quad , \quad \frac{dp_i}{d\tau} = -\frac{1}{v}\frac{\partial v}{\partial x_i} \tag{8.25}$$

Here, τ means the travel time along the ray measured from same reference point, v is either α or β, and p_i are the Cartesian components of the slowness vector $\vec{p}$, which is perpendicular to the wavefront and

$$\vec{p} = \frac{1}{v}\vec{t} = \nabla\tau \tag{8.26}$$

where $\vec{t}$ is the unit vector perpendicular to the wavefront. Eq. (8.26) means that

$$p_i = \tau_{,i} \tag{8.27}$$

It follows from the first equation of (8.25) that $\vec{t}$ and $\vec{p}$ are tangent to the ray.
If we consider the arc length s along the ray instead of the travel time τ, $ds = v d\tau$, we rewrite the system (8.25) as

$$\frac{d\vec{x}}{ds} = \vec{t} \quad , \quad \frac{d\vec{p}}{ds} = \nabla\left(\frac{1}{v}\right) \tag{8.28}$$

Solving the ray tracing system (8.25 or 8.28) we get $x_i = x_i(\tau)$ or $x_i = x_i(s)$, i.e., the ray, and $p_i = p_i(\tau)$ or $p_i = p_i(s)$, i.e., the slowness vector at any point of the ray.

The ray tracing system can be solved analytically only for relatively simple models of medium. Numerical methods are necessary to solve the system in more realistic models.
Note: Rays as extremals of Fermat's functional.
Rays can be introduced using Fermat's principle. The Fermat's principle is applied to the P and S waves as to two independent waves. Obviously, this itself is only an intuitive a priori assumption. This is the weakness of this way to introduce rays.
Assume that $v(x_i)$ and its first derivatives are continuous functions of coordinates. The Fermat's functional is

$$\tau = \int_{M_0}^{M} \frac{ds}{v} \tag{8.29}$$

Fermat's principle: A signal propagates from point M_0 to point M along a curve which renders the integral (8.29) stationary. The curve is called the extremal curve of the integral.
The integral can be written as

$$\tau = \int_{M_0}^{M} \frac{1}{v}\sqrt{(x_1')^2 + (x_2')^2 + (x_3')^2}\,ds \tag{8.30}$$

where $x_i' = \frac{dx_i}{ds}$ are directional cosines. Euler's equations for the extremal of integral (8.30) are

$$\frac{d}{ds}\left(\frac{1}{v}\frac{dx_i}{ds}\right) + \frac{1}{v^2}\frac{\partial v}{\partial x_i} = 0 \tag{8.31}$$

If we use

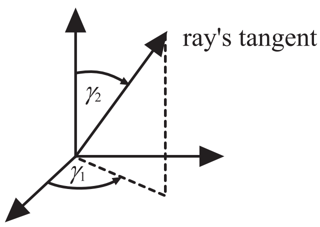
$$p_i = \frac{1}{v}\frac{dx_i}{ds}$$

and substitute it into eqs. (8.31) we get the ray tracing system (8.28).

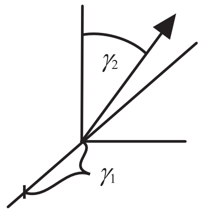
8.6 Ray parameters

Consider a two-parameter system of rays. Each ray is determined by parameters γ_1 and γ_2, the so-called ray parameters. Examples:

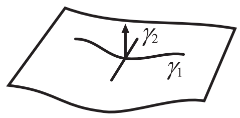
1. Point source



2. Line source



3. Wavefront at time t_0: $\tau(x_i) = t_0$. γ_1 and γ_2 are curvilinear coordinates on the wavefront

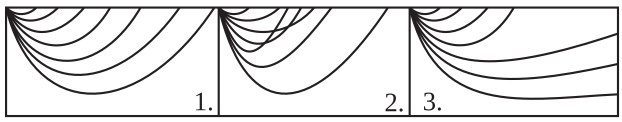


8.7 Ray coordinates

(s, γ_1, γ_2) or $(\tau, \gamma_1, \gamma_2)$ are the ray coordinates. A parametric equation $\vec{x} = \vec{x}(\tau, \gamma_1, \gamma_2)$ is the equation of a ray.

8.8 Function *J*

Consider e.q. a 2D heterogeneous medium and three examples of a ray diagram



The 3 pictures show 3 different situations in the ray field. In the first one each point is intersected by a single ray. In the second one neighboring rays are intersecting each other. A point of intersection is thus described by two different sets of ray coordinates and therefore the ray coordinates are not single-valued at it. In the third figure, there is a shadow region to which no rays penetrate. All the three situations, generally in 3D can be classified in terms of the Jacobian of the transformation from the ray to Cartesian coordinates:

$$J = \frac{D(x, y, z)}{D(s, \gamma_1, \gamma_2)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial \gamma_1} & \frac{\partial y}{\partial \gamma_1} & \frac{\partial z}{\partial \gamma_1} \\ \frac{\partial x}{\partial \gamma_2} & \frac{\partial y}{\partial \gamma_2} & \frac{\partial z}{\partial \gamma_2} \end{vmatrix} \tag{8.32}$$

Since in the isotropic medium $ds = v d\tau$, v being the speed,

$$J^{\tau} = \frac{D(x, y, z)}{D(\tau, \gamma_1, \gamma_2)} = vJ \tag{8.33}$$

The ray field is regular, if J exists and $J \neq 0$. An example is in the first picture. The ray field is singular at a point M, if $J(M) = 0$ (the rays intersect each other at point M; the second picture) or if J is not defined at point M (the shadow region; the third picture).

8.9 The ray tube

In order to characterize the density of the rays, we can introduce the ray tube and the cross-sectional area of the ray tube. The narrower is the ray tube, i.e., the denser is the ray field, the higher are the ray amplitudes, and vice versa. The ray tube is defined as a system of rays with the ray parameters in the interval

$$(\gamma_1, \gamma_1 + d\gamma_1) \times (\gamma_2, \gamma_2 + d\gamma_2)$$

The cross-sectional area is the part of the surface s = const (or τ = const) cut out by the ray tube.
Consider the parametric equation

$$\vec{x} = \vec{x}\,(\tau, \gamma_1, \gamma_2)$$

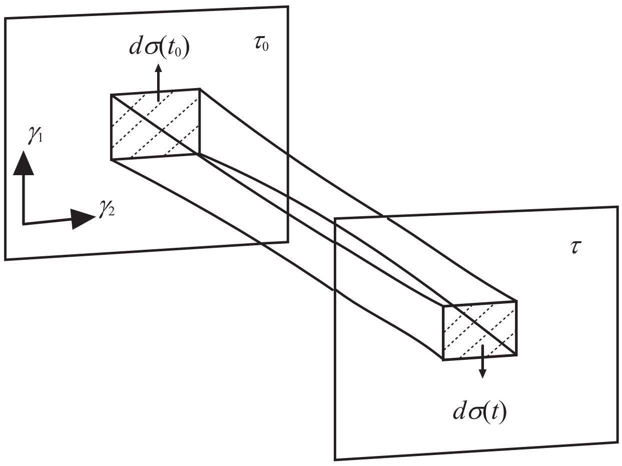
It is for a fixed τ the equation of wavefront. For fixed γ_2, $\vec{x} = \vec{x}(\gamma_1)$ is the equation of a curve on the wavefront. Vector $\frac{\partial \vec{x}}{\partial \gamma_1}$ is tangent to the curve $\vec{x} = \vec{x}(\gamma_1)$ and thus to the wavefront. Similarly, vector $\frac{\partial \vec{x}}{\partial \gamma_1}$ for fixed γ_1 is tangent to the wavefront. Therefore, vector

$$d\vec{\sigma} = \left(\frac{\partial \vec{x}}{\partial \gamma_1} \times \frac{\partial \vec{x}}{\partial \gamma_2} \right) d\gamma_1 d\gamma_2$$

is perpendicular to the wavefront. If $\vec{t} = \frac{d\vec{x}}{ds}$ is a tangent to the ray,

$$d\sigma = d\vec{\sigma} \cdot \vec{t} = \left(\frac{\partial \vec{x}}{\partial\, \gamma_1} \times \frac{\partial \vec{x}}{\partial\, \gamma_2} \right) \cdot \vec{t}\, d\gamma_1\, d\gamma_2 \tag{8.34}$$

is the cross–sectional area of the ray tube.



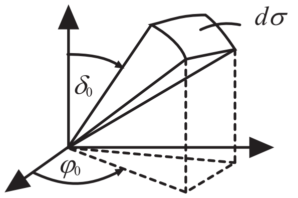
If vector $\vec{t}$ has the same direction as vector $\frac{\partial \vec{x}}{\partial \gamma_1} \times \frac{\partial \vec{x}}{\partial \gamma_2}$, $d\sigma > 0$. In the case of the opposite directions, $d\sigma < 0$. $d\sigma = 0$ at a point where the rays intersect each other. Such points are called the caustic points.

Since

$$\left(\frac{\partial\vec{x}}{\partial\gamma_1}\times\frac{\partial\vec{x}}{\partial\gamma_2}\right)\cdot\vec{t} \;=\; \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial\gamma_1} & \frac{\partial y}{\partial\gamma_1} & \frac{\partial z}{\partial\gamma_1} \\ \frac{\partial x}{\partial\gamma_2} & \frac{\partial y}{\partial\gamma_2} & \frac{\partial z}{\partial\gamma_2} \end{vmatrix} = J \tag{8.35}$$

$$d\sigma \;=\; J d\gamma_1 d\gamma_2 \tag{8.36}$$

i.e., function J measures the cross–sectional area of the ray tube – its expansion and contraction. Obviously, $J(s) = 0$ at the caustic points.
Example of function J: a point source in a homogeneous medium



$$\begin{aligned} d\sigma &= R^2 \sin\delta_0\, d\varphi_0\, d\delta_0 \\ J &= R^2 \sin\delta_0 \end{aligned} \tag{8.37}$$

8.10 Relation between J and $\Delta\tau$

Consider an element of the ray tube between τ and $\tau + d\tau$.
The element has the volume

$$\Omega = J d\gamma_1\, d\gamma_2\, ds$$

and surface

$$S = J\, d\gamma_1\, d\gamma_2$$

with an outer normal $\vec{n}$. Since

$$\Delta\tau = \nabla^2\tau = \operatorname{div}\operatorname{grad}\tau$$

consider the integral definition of div $\vec{a}$

$$\operatorname{div}\vec{a} = \lim_{\Omega\to 0}\frac{1}{\Omega}\int\!\!\int_S \vec{a}\cdot d\vec{S} \qquad ; \qquad d\vec{S} = \vec{n}dS$$

Then

$$\Delta\tau = \lim_{\Omega\to 0}\frac{1}{\Omega}\int\!\!\int_S \frac{\partial\tau}{\partial n}dS$$

Obviously, $\frac{\partial\tau}{\partial n} \neq 0$ only on the element's cross sections and

$$\left.\frac{\partial\tau}{\partial n}\right|_{\tau} = -\frac{1}{v} \quad , \quad \left.\frac{\partial\tau}{\partial n}\right|_{\tau+d\tau} = \frac{1}{v}$$

Then

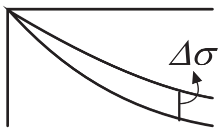
$$\Delta\tau = \lim_{\Omega\to 0}\frac{1}{J\,d\gamma_1\,d\gamma_2\,ds}\left\{\left.\frac{J\,d\gamma_1\,d\gamma_2}{v}\right|_{\tau+d\tau} - \left.\frac{J\,d\gamma_1\,d\gamma_2}{v}\right|_{\tau}\right\} = \lim_{\Omega\to 0}\frac{1}{Jvd\tau}\left\{\left.\frac{1}{v}\right|_{\tau+d\tau} - \left.\frac{J}{v}\right|_{\tau}\right\}$$

and

$$\Delta\tau = \frac{1}{Jv}\frac{d}{d\tau}\left(\frac{J}{v}\right) \tag{8.38}$$

8.11 Determination of function J

Function J can be evaluated along a ray in several ways. Formula (8.38) is, in principle, one of them. The simplest but not the best is to measure the cross–sectional area of the ray tube and use relation (8.36). In the 2D case we have approximately



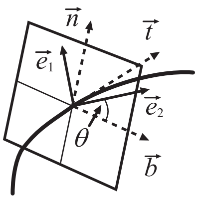
$$J = \frac{\Delta\sigma}{\Delta\gamma}$$

where γ is an angle between two rays.
The most accurate way to determine function J is to solve the dynamic ray tracing system of equations. The system consists of 12 linear ordinary differential equations and allows to evaluate function J along one ray.

8.12 The ray-centered coordinate system

In smoothly heterogeneous medium, ray is a 3D curve, for which in its every point tangential, normal and binormal vector can be determined. However, coordinate system with these vectors as a basis leads to a coplicated form of the transport equations (the next section). It is useful to create another basis - so called ray-centered coordinate system.

The vector basis of the ray-centered coordinate system connected with a ray is formed at an arbitrary point of the ray by a right-handed system of three unit vectors $\vec{e}_1(s)$, $\vec{e}_2(s)$ and $\vec{t}(s)$ where $\vec{t}(s)$ is the unit tangent to the ray and



$$\begin{aligned}\vec{e}_1 &= \vec{n}\cos\theta - \vec{b}\sin\theta\\ \vec{e}_2 &= \vec{n}\sin\theta + \vec{b}\cos\theta\end{aligned} \tag{8.39}$$

where $\vec{n}$ and $\vec{b}$ are the unit normal and binormal to the ray, respectively, and

$$\theta(s) = \theta(s_0) + \int_{s_0}^{s} T(s)ds \tag{8.40}$$

where T is the torsion of the ray[1]. The integral is taken along the ray. For fixed value of s, vectors

[1] $\vec{e}_1, \vec{e}_2$ are created from $\vec{n}, \vec{b}$ to eliminate rotations around $\vec{t}$

$\vec{e}_1$ and $\vec{e}_2$ determine a plane perpendicular to the ray. For an arbitrary point in the vicinity of the ray we define the ray-centered coordinates s, q_1, q_2 such that its position vector is

$$\vec{r}(s, q_1, q_2) = \vec{r}(s, 0, 0) + q_1\vec{e}_1(s) + q_2\vec{e}_2(s) \tag{8.41}$$

The ray-centered coordinate system is orthogonal.

8.13 Transport equations

Since we assume the high-frequency propagation, the P and S waves propagate approximately independently. Therefore we will separately treat the vectorial amplitudes for the P and S waves. Thus, let $\vec{U}^{(k)}$ is the amplitude of either P and S wave. Then

$$\vec{U}^{(k)} = \underbrace{U_p^{(k)}\vec{t}}_{\vec{U}_{\parallel}^{(k)}} + \underbrace{U_{s_1}^{(k)}\vec{e}_1 + U_{s_2}^{(k)}\vec{e}_2}_{\vec{U}_{\perp}^{(k)}} \tag{8.42}$$

We will call $\vec{U}_{\parallel}^{(k)}$ and $\vec{U}_{\perp}^{(k)}$ the principal and additional components of the amplitude coefficient $\vec{U}^{(k)}$, respectively, if the wave is the P wave. In the case of the S wave, $\vec{U}_{\perp}^{(k)}$ will be the principal and $\vec{U}_{\parallel}^{(k)}$ the additional component.

	Principal comp.	Additional comp.
P wave	$\vec{U}_{\parallel}^{(k)}$; $U_p^{(k)}$	$\vec{U}_{\perp}^{(k)}$; $U_{s_1}^{(k)}, U_{s_2}^{(k)}$
S wave	$\vec{U}_{\perp}^{(k)}$; $U_{s_1}^{(k)}, U_{s_2}^{(k)}$	$\vec{U}_{\parallel}^{(k)}$; $U_p^{(k)}$

Assuming decomposition (8.42), the basic system of equations (8.14) can be solved for the principal and additional components of the P and S waves.

Principal components

Solving the basic system for the principal components we get the transport equations

$$\frac{dU^{(k)}}{d\tau} + \frac{1}{2}U^{(k)}\left(v^2\Delta\tau + \frac{d\ln(\rho v^2)}{d\tau}\right) = g^{(k)}(\tau) \tag{8.43}$$

where

	$U^{(k)}$	v	$g^{(k)}$
P wave	$U_p^{(k)}$	α	$g_0^{(k)}$
S wave	$U_{s_1}^{(k)}$	β	$g_1^{(k)}$
	$U_{s_2}^{(k)}$	β	$g_2^{(k)}$

and

$$\begin{aligned}
g_0^{(k)}(\tau) &= \frac{\alpha}{2\rho}\left[L_i\left(\vec{U}^{(k-1)}\right) - M_i\left(\vec{U}_{\perp}^{(k)}\right)\right]\tau_{,i} \\
g_1^{(k)}(\tau) &= \frac{1}{2\rho}\left[L_i\left(\vec{U}^{(k-1)}\right) - M_i\left(\vec{U}_{\parallel}^{(k)}\right)\right]e_{1i} \\
g_2^{(k)}(\tau) &= \frac{1}{2\rho}\left[L_i\left(\vec{U}^{(k-1)}\right) - M_i\left(\vec{U}_{\parallel}^{(k)}\right)\right]e_{2i}
\end{aligned} \tag{8.44}$$

Additional components

P wave

Multiplication of eq. (8.11) by $\vec{e}_1$ and $\vec{e}_2$ gives

$$N_i\left(\vec{U}^{(k)}\right)e_{1i} = (-\rho + \mu\,\alpha^{-2})U_{s_1}^{(k)}$$
$$N_i\left(\vec{U}^{(k)}\right)e_{2i} = (-\rho + \mu\,\alpha^{-2})U_{s_2}^{(k)}$$

Then the scalar products of eq. (8.14) with $\vec{e}_1$, and $\vec{e}_2$ lead to the formulas

$$\begin{aligned} U_{s_1}^{(k)} &= \left(-\rho + \mu\,\alpha^{-2}\right)^{-1}\left[M_i\left(\vec{U}^{(k-1)}\right) - L_i\left(\vec{U}^{(k-2)}\right)\right]e_{1i} \\ U_{s_2}^{(k)} &= \left(-\rho + \mu\,\alpha^{-2}\right)^{-1}\left[M_i\left(\vec{U}^{(k-1)}\right) - L_i\left(\vec{U}^{(k-2)}\right)\right]e_{2i} \end{aligned} \tag{8.45}$$

Eqs. (8.45) mean that the additional components for the P wave are obtained as a result of differentiation carried on the amplitude coefficients of lower order – $\vec{U}^{(k-1)}$ and $\vec{U}^{(k-2)}$.
It follows from eq. (8.45) that

$$U_{s_1}^{(0)} = U_{s_2}^{(0)} = 0 \qquad \text{for the P wave}$$

This is, however, already known result – see eqs.(8.23) and (8.24) and the polarization of $\vec{U}^{(0)}$.

S wave

Multiply eq. (8.14) by $\vec{t}$. We get

$$N_i\left(\vec{U}^{(k)}\right)t_i = \left[M_i\left(\vec{U}^{(k-1)}\right) - L_i\left(\vec{U}^{(k-2)}\right)\right]t_i$$

Since

$$\begin{aligned} U_i^{(k)}t_i &= U_P^{(k)} \\ N_i\left(\vec{U}^{(k)}\right)t_i &= (\lambda + \mu)\,\beta^{-2}U_P^{(k)} \end{aligned}$$

Then

$$U_P^{(k)} = \frac{\beta^2}{\lambda + \mu}\left[M_i\left(\vec{U}^{(k-1)}\right) - L_i\left(\vec{U}^{(k-2)}\right)\right]t_i \tag{8.46}$$

Formula (8.46) means that also the additional component of the S wave is obtained as a result of differentiation of the amplitude coefficients of lower order.

8.14 Solution of transport equations

The transport equation can be easily solved if we substitute $\Delta\tau$ from eq. (8.38). We will restrict ourselves to the zero–order term $\vec{U}^{(0)}$. As we found earlier,

$$\vec{U}^{(0)} = U_p^{(0)}\vec{t} \qquad \text{for the P wave}$$

and

$$\vec{U}^{(0)} = U_{s_1}^{(0)}\vec{e}_1 + U_{s_2}^{(0)}\vec{e}_2 \qquad \text{for the S wave}$$

Further we will denote $U_p^{(0)}$ by U_p and $U_{s_1}^{(0)}$ and $U_{s_2}^{(0)}$ by U_{s_1} and U_{s_2}, respectively.
Let $U = U_p$ if $v = \alpha$
and $U = U_{s_1}$ or $U = U_{s_2}$ if $v = \beta$
Eq. (8.38) can be written as

$$\Delta\tau = \frac{1}{J}\frac{d}{ds}\left(\frac{J}{v}\right)$$

Then the transport equation (8.43) becomes

$$\frac{dU}{ds} + \frac{1}{2}U\left(\frac{v}{J}\frac{d}{ds}\left(\frac{J}{v}\right) + \frac{d}{ds}\ln(\rho v^2)\right) = 0$$

The above equation leads to

$$\frac{d}{ds}\left(U\sqrt{J\rho v}\right) = 0 \tag{8.47}$$

The solution of this equation is

$$U(s) = \frac{\psi(\gamma_1,\gamma_2)}{\sqrt{J(s)\,\rho(s)\,v(s)}} \tag{8.48}$$

where ψ is constant for a given ray and different for $U_p, U_{s_1}$and U_{s_2}.
Instead of solution in form given by eq. (8.48) we can write a solution of eq. (8.47) as

$$U(s) = U(s_0)\sqrt{\frac{J(s_0)\,\rho(s_0)\,v(s_0)}{J(s)\,\rho(s)\,v(s)}} \tag{8.49}$$

where s_0 refers to some reference point on the ray.

Example: a point source
The point source radiates, in a homogeneous medium, a spherical wave, e.g.,

$$u(r,t) = \frac{g\left(\delta_0,\varphi_0\right)}{r}f\left(t - \frac{r}{\alpha}\right) \tag{8.50}$$

where $g\left(\delta_0,\varphi_0\right)$ is the radiation pattern, and $\gamma_1 = \delta_0$, $\gamma_2 = \varphi_0$. The spherical wave described by eq. (8.50) has the source at $r = 0$. Assume now that a certain vicinity of the source is homogeneous, i.e., $\rho = \rho_0$ and $\alpha = \alpha_0$ both at $r = 0$ and $r = r_0 \neq 0$. Then we can express the amplitude at distance r_0 either by formula (8.48) or (8.50) and

$$\frac{\psi\left(\delta_0,\varphi_0\right)}{\sqrt{J_0\,\rho_0\,\alpha_0}} = \frac{g\left(\delta_0,\varphi_0\right)}{r_0}$$

Since

$$J_0 = r_0^2 \sin\delta_0 \qquad \text{(see eq. 8.37)},$$
$$\psi(\delta_0, \varphi_0) = g(\delta_0, \varphi_0)\sqrt{\rho_0\,\alpha_0\,\sin\delta_0}$$

Then eq. (8.48) gives

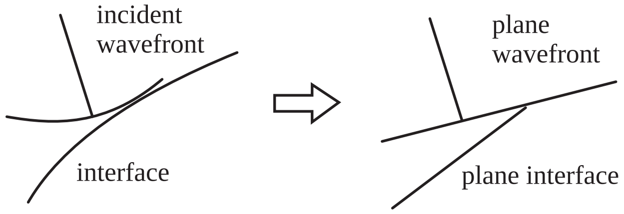
$$U(S) = \frac{g(\delta_0, \varphi_0)}{L}\sqrt{\frac{\rho_0\alpha_0}{\rho(s)\,\alpha(s)}} \qquad ; \qquad L = \sqrt{\frac{J(s)}{\sin\delta_0}} \tag{8.51}$$

Factor L is called the geometrical spreading.
Example: a line source

$$\psi(\delta_0, L_0) = \bar{g}(\delta_0, L_0)\sqrt{\rho_0 v_0} \tag{8.52}$$

8.15 Medium with interfaces

Consider now medium with interfaces, i.e., material discontinuities. We say that an interface is of the $(n+1)$-st order if the n-th derivative of elastic modulus or density is discontinuous across the interface. Discontinuity of the elastic modulus or density themselves means an interface of the 1st order. The zero-order approximation of the ray method is discontinuous at the interface of the 1st order.
In principle, we can choose one of two approaches to apply the ray method to medium with interfaces. In the first one we solve the equation of motion with boundary conditions at the interfaces. Obviously, this approach is generally very complicated. In the other approach we apply the local principle:
In the zero-order approximation, reflection and transmission at the interface is determined only by a small vicinity of the reflection/transmission point.
Note that the zero-order approximation is sufficiently accurate, if the principal radii of curvature of the interface are substantially larger than the dominant wavelength.
The local principle implies that the reflection/transmission of waves with curved wavefronts at the curved interface can be well approximated by the reflection/transmission of a plane wave at a plane interface if the plane wavefront is tangential to the curved wavefront at the point of the reflection/transmission and the plane interface is tangential to the curved interface at the point of reflection/transmission. At the same time we assume that the plane interface separates two homogeneous halfspaces with elastic moduli and densities equal to the elastic moduli and density at the point of incidence, and the elastic moduli and density at the point of reflection/transmission, respectively.



8.16 Ray tracing across an interface

If a ray intersects an interface, the slowness vector $\vec{p}$ changes discontinuously. Therefore it is necessary to find new initial conditions for the ray tracing of the reflected/transmitted wave.

Let 0 be a point of incidence and $\vec{n}$ normal to the interface at 0. Let the plane determined by vectors $\vec{n}(O)$ and $\vec{p}^I(O)$,be the plane of incidence ($\vec{p}^I(O)$ being the slowness vector of the incident wave). Then the initial conditions for the reflected/transmitted wave are determined by two conditions:

a) the slowness vector of the reflected/transmitted wave $\vec{p}^R(O)$ lies in the plane of incidence
b) the angle of incidence ϑ^I (the angle between $\vec{n}(O)$ and $\vec{p}^I(O)$) and the angle of reflection/transmission ϑ^R (the angle between $\vec{n}(O)$ and $\vec{p}^R(O)$) satisfy Snell's law

$$\frac{\sin(\vartheta^I)}{v^I(O)} = \frac{\sin(\vartheta^R)}{v^R(O)} \tag{8.53}$$

where $v^I(O)$ and $v^R(O)$ denote speeds of the incident and reflected/transmitted waves.

Note: In fact, either positive or negative direction of the slowness vector has to be considered in order to have $\vartheta^I < \frac{\pi}{2}$, $\quad \vartheta^R \leq \frac{\pi}{2}$.

8.17 Amplitudes in a medium with interfaces

Displacement vector of the P wave has the direction of the tangent to the ray, i.e., it lies in the plane of incidence. Displacement vector of the S wave can be decomposed into a component in the plane of incidence and component perpendicular to the plane of incidence, say $U_{s\|}$ and $U_{s\perp}$, respectively. From the theory of reflection/transmission of plane waves at a plane interface we know that the incidence of the P or $S_{\|}$ wave can generate reflected P and $S_{\|}$ waves and transmitted P and $S_{\|}$ waves. The incidence of the $S_\perp$ can generate reflected $S_\perp$ and transmitted $S_\perp$ waves. For all cases we know coefficients of reflection/transmission. (Note that $S_{\|}$ = SV and $S_\perp$ = SH if $\vec{n}$ has the vertical direction.)

Let us recall that along a ray we know $U_p(\sim \vec{t})$, $U_{s_1}(\sim \vec{e}_1)$ and $U_{s_2}(\sim \vec{e}_2)$ since we decompose the displacement vector (or vector amplitude) in the local ray-centered system.

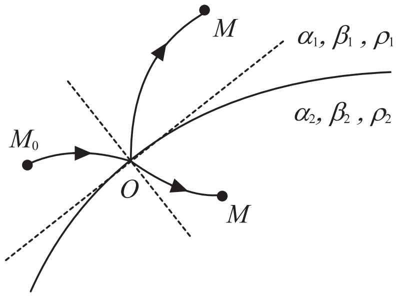
In the case of an incident P wave, the reflection/transmission coefficient R is directly applicable to U_p :

$$U_p^R = RU_p^I$$

where U_p^R and U_p^I denote amplitudes of the reflected/transmitted wave and incident wave, respectively.

In the case of an incident S wave, the reflection/transmission coefficients are not directly applicable to the U_{s_1} and U_{s_2} components since, in general, $U_{s_1} \neq U_{s\|}$ or $U_{s\perp}$ and $U_{s_2} \neq U_{s\perp}$ or $U_{s\|}$. U_{s_1} and U_{s_2} have to be transformed into $U_{s\|}$ and $U_{s\perp}$ components. If, however, the ray is a plane curve, $\vec{e}_1 = \vec{n}$, $\vec{e}_2 = \vec{b}_1$ and $U_{s_1} = U_{s\|}$ and $U_{s_2} = U_{s\perp}$.

Consider now the case shown in the figure:



Let U be an appropriate component of the vector amplitude.
Then

$$U^I(O) = U^I(M_0)\sqrt{\frac{J(M_0)\rho(M_0)v(M_0)}{J(O)\rho(O)v(O)}} \tag{8.54}$$

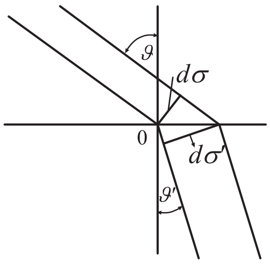
$$U^R(O) = RU^I(O) \tag{8.55}$$

$$U^R(M) = U^R(O)\sqrt{\frac{J'(O)\rho'(O)v'(O)}{J(M)\rho(M)v(M)}} \tag{8.56}$$

The apostrophe denotes quantities at the side of the reflected/transmitted wave. It follows from eqs. (8.54 - 8.56) that

$$U^R(M) = RU^I(M_0)\sqrt{\frac{J(M_0)\rho(M_0)v(M_0)}{J(O)\rho(O)v(O)}\frac{J'(O)\rho'(O)v'(O)}{J(M)\rho(M)v(M)}}$$

The ratio $J'(O)/J(O)$ can be simplified. Consider the ray tube at the interface (a 2D case in the figure).



$$\begin{aligned}\cos\vartheta &= \frac{d\sigma}{dx}\\ \cos\vartheta' &= \frac{d\sigma'}{dx}\end{aligned}$$

$$\begin{aligned}d\sigma &= Jd\gamma_1 d\gamma_2\\ d\sigma' &= J'd\gamma_1 d\gamma_2\end{aligned}$$

$$\frac{J'(O)}{J(O)} = \frac{\cos\vartheta'(O)}{\cos\vartheta(O)} \tag{8.57}$$

Then we get

$$U^R(M) = U^I(M_0)\sqrt{\frac{J(M_0)\rho(M_0)v(M_0)}{J(M)\rho(M)v(M)}}\, R\sqrt{\frac{\rho'(O)v'(O)\cos\vartheta'(O)}{\rho(O)v(O)\cos\vartheta(O)}} \tag{8.58}$$

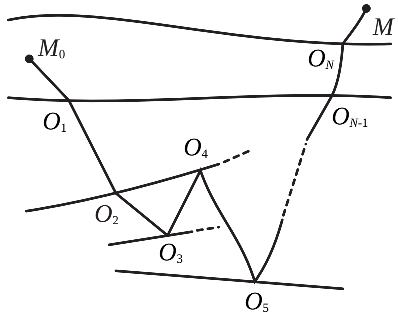
Example: a point source at M_0

$$U^R(M) = \frac{g(\delta_0,\varphi_0)}{L}\sqrt{\frac{\rho(M_0)v(M_0)}{\rho(M)v(M)}}\,R\sqrt{\frac{\rho'(O)v'(O)}{\rho(O)v(O)}} \tag{8.59}$$

where L, the geometrical spreading, is

$$L = \sqrt{\frac{J(M)}{\sin\delta_0}\frac{\cos\vartheta(O)}{\cos\vartheta'(O)}} \tag{8.60}$$

Consider now N reflections/transmissions. An example is given in the figure:



For the amplitude component $U(M)$ we get easily

$$U(M) = U(M_0)\sqrt{\frac{\rho(M_0)v(M_0)J(M_0)}{\rho(M)v(M)J(M)}}\prod_{j=1}^{N} R_j\sqrt{\frac{\rho'(O_j)\alpha'(O_j)\cos\vartheta'(O_j)}{\rho(O_j)\alpha(O_j)\cos\vartheta'(O_j)}} \tag{8.61}$$

8.18 Elementary seismogram

Considering the zero-order approximation, the displacement is given by

$$\vec{u}(x_j,t) = \mathrm{Re}\{\vec{U}^{(0)}(x_j)F_0(t-\tau(x_j))\} \tag{8.62}$$

In practice we are usually interested in the Cartesian components of the displacement. Let $c \in \{x,y,z\}$, $U \in \{U_p^{(0)}, U_{s1}^{(0)}, U_{s2}^{(0)}\}$ and q_c be the c-component of the corresponding unit vector (e.g., the c-component of $\vec{t}$ in the case of the P wave). Then

$$u_c = \mathrm{Re}\{UF_0(t-\tau)q_c\} \tag{8.63}$$

(If u_c should be a displacement component at the free surface, q_c has to be a coefficient of conversion.)

Since Uq_c is generally complex

$$Uq_c = A_c\exp{(\mathrm{i}\psi)} \tag{8.64}$$

Since F_0 is an analytical signal,

$$F_0(t-\tau) = f_0(t-\tau) + ih_0(t-\tau) \tag{8.65}$$

where $h_0 = \mathcal{H}(f_0)$, we get from eqs. (8.63 - 8.65)

$$\begin{aligned} u_c &= \mathrm{Re}\{A_c(\cos\psi + \mathrm{i}\sin\psi)(f_0 + \mathrm{i}h_0)\} \\ u_c &= A_c[f_0(t-\tau)\cos\psi - h_0(t-\tau)\sin\psi] \end{aligned} \tag{8.66}$$

If we know the source-time function $f_0(t)$, the elementary seismogram is determined by the propagation time τ, ray amplitude A_c and phase shift ψ. The elementary seismogram represents contribution (to the displacement at a receiver) due to wave propagation along one ray.

8.19 Ray synthetic seismogram

The total displacement at a receiver is given by a superposition of contributions along all rays connecting the source and receiver. In terms of types of waves, the total displacement is given by a superposition of all (computed) types of waves arriving in the receiver - e.g., direct P and S waves, reflected/transmitted waves, multiply reflected/transmitted waves, converted waves, etc. Let us stress that not all types of waves can be computed since not all types of waves are the HF waves. Some types of waves can be well approximated by the zero-order approximation, some require the higher-order approximation.

One type of wave may be represented by contributions a long several rays.

A synthetic seismogram is given by

$$u_c^s(x_j,t) = \sum_i u_c^{(i)}(x_j,t) \tag{8.67}$$

where $u_c^{(i)}$ denotes the i-th elementary seismogram and summation includes all rays connecting the source and receiver.

Procedure to construct the synthetic seismogram in time window $< t_B, t_E >$:

1. Generate numerical code of a type of wave (elementary wave).
   - No code - go to 8.
   - Code available - continue.
2. Find a ray and the corresponding propagation time $\tau^{(j)}$
   - ray not found - go to 1.
   - ray found - continue
3. Compare $\tau^{(j)}$ with $< t_B, t_E >$
   - ray not included - go to 2.
   - ray included - continue
4. Calculate $A^{(j)}$, $\psi^{(j)}$
5. Calculate elementary seismogram $u^{(j)}$
6. Add $u^{(j)}$ to u_c - synthetic seismogram
7. Go to 2.
8. End

8.20 Elementary and synthetic seismograms - computation in the frequency domain

Displacement of a harmonic wave in the zero-order approximation is

$$\vec{u}_H(x_j,t,\omega) = \vec{U}^{(0)}(x_j)\exp[-\mathrm{i}\omega(t-\tau(x_j))]$$

Using the same rotation as in the time domain we have for the Cartesian component of displacement

$$u_{H,c} = U\exp(\mathrm{i}\omega\tau)q_c\exp(-\mathrm{i}\omega t)$$

The harmonic amplitude is

$$U_H = U\exp(i\omega\tau)q_c = A_c\exp[i(\psi+\omega\tau)]$$

$U_H(\omega)$ represents a transfer function. Let $S(\omega) = \mathcal{F}[f_0(t)]$ be the spectrum of a source-time function $f_0(t)$. Then

$$u_c(x_j, t) = \mathcal{F}^{-1}[U_H(\omega)S(\omega)] \tag{8.68}$$

A synthetic seismogram is given by

$$u_c^s(x_j, t) = \mathcal{F}^{-1}[U_H^s(\omega)S(\omega)] \tag{8.69}$$

where

$$U_H^s(\omega) = \sum_i U_H^{(i)}(\omega) \tag{8.70}$$

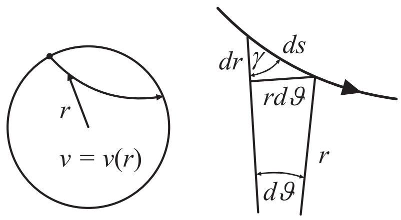
and

$$U_H^{(i)}(\omega) = A_c^{(i)} \exp[\mathrm{i}(\psi^{(i)} + \omega\tau^{(i)})] \tag{8.71}$$

with summation over all (computed) rays connecting the source and receiver.

8.21 Rays in a radially symmetric medium

Let speed v of a wave (the P wave or S wave) be a function of the distance from the center - as in the figure



A propagation time between two points, r_0 and r_1, is given by $\tau = \int\limits_{r_0}^{r_1} \frac{ds}{v(r)}$

The element of the arc length can be expressed as

$$\begin{aligned} ds &= \sqrt{[(dr)^2 + (r\,d\vartheta)^2]} \\ ds &= \sqrt{r'^2 + r^2}\,d\vartheta \\ \text{where} \quad r' &= \frac{dr}{d\vartheta} \end{aligned}$$

Then

$$\tau = \int\limits_{\vartheta_0}^{\vartheta_1} \frac{\sqrt{r'^2 + r^2}}{v(r)} d\vartheta \tag{8.72}$$

or

$$\tau = \int\limits_{\vartheta_0}^{\vartheta 1} F(r, r') d\vartheta \tag{8.73}$$

with

$$F(r, r') = \frac{\sqrt{r'^2 + r^2}}{v(r)} \tag{8.74}$$

Euler's equation for functional F is $\frac{d}{d\vartheta}\left(\frac{\partial F}{\partial r'}\right) - \frac{\partial F}{\partial r} = 0$
It is easy to prove that[1]

$$\frac{d}{d\vartheta}\left(F - r'\frac{\partial F}{\partial r'}\right) = 0$$

$$\begin{aligned}
\frac{d}{d\vartheta}\left(F - r'\frac{\partial F}{\partial r'}\right) &= \frac{\partial F}{\partial \vartheta} + \frac{\partial F}{\partial r}r' + \frac{\partial F}{\partial r'}r'' - r''\frac{\partial F}{\partial r'} - r'\frac{d}{d\vartheta}\frac{\partial F}{\partial r'} \\
&= r'\frac{\partial F}{\partial r} - r'\frac{d}{d\vartheta}\frac{\partial F}{\partial r'} = r'\left(\frac{\partial F}{\partial r} - \frac{d}{d\vartheta}\frac{\partial F}{\partial r'}\right) = 0
\end{aligned}$$

Then we can define a constant (for a given ray) p :

$$p = F - r'\frac{\partial F}{\partial r'} \tag{8.75}$$

Inserting from eq. (8.74)

$$\begin{aligned}
p &= \frac{\sqrt{r'^2 + r^2}}{v} - r'\frac{r'}{v\sqrt{r'^2 + r^2}} \\
p &= \frac{r^2}{v\sqrt{r'^2 + r^2}} \qquad (8.76)
\end{aligned}$$

Since

$$\begin{aligned}
\sin\gamma &= \frac{r d\vartheta}{ds} = \frac{r d\vartheta}{\sqrt{r'^2 + r^2}d\vartheta} \\
\sin\gamma &= \frac{r}{\sqrt{r'^2 + r^2}} \qquad (8.77)
\end{aligned}$$

we can rewrite eq. (8.76) as

$$p = \frac{r\sin\gamma(r)}{v(r)} \tag{8.78}$$

Equation (8.78) is the equation of a ray in a radially symmetric medium or Snell's law for the ray in the radially symmetric medium. Constant p is the parameter of the ray.
Let us find $\vartheta = \vartheta(r)$. A square of eq. (8.76) gives

$$\begin{aligned}
p^2 &= \frac{r^4}{v^2 r'^2 + v^2 r^2} \\
r'^2 &= \frac{r^2(r^2 - v^2p^2)}{v^2p^2} \\
\frac{1}{r'} &= \pm\frac{vp}{r\sqrt{r^2 - v^2p^2}} \quad ; \qquad r' = \frac{dr}{d\vartheta} \qquad (8.79) \\
\vartheta(r_1) &= \vartheta(r_0) \pm \int\limits_{r_0}^{r_1} \frac{vp\, dr}{r\sqrt{r^2 - v^2p^2}} \qquad (8.80)
\end{aligned}$$

[1] conservation law for the cyclic coordinate ϑ

Let us find $\tau = \tau(r)$. Eq. (8.72) implies

$$\frac{d\tau}{d\vartheta} = \frac{\sqrt{r'^2 + r^2}}{v}$$

Substituting from eq. (8.76) we have

$$\frac{d\tau}{d\vartheta} = \frac{r^2}{v^2 p}$$

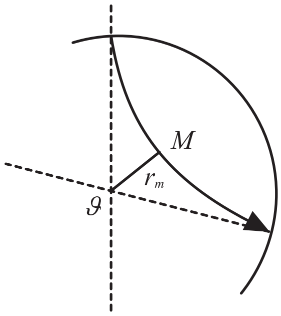
Substituting derivative $d\vartheta/dr$ from eq. (8.79) we get

$$\frac{d\tau}{dr} = \pm \frac{r}{v\sqrt{r^2 - v^2p^2}}$$

and

$$\tau(r_1) = \tau(r_0) \pm \int_{r_0}^{r_1} \frac{r dr}{v\sqrt{r^2 - v^2p^2}} \tag{8.81}$$

Find a condition for the minimum of the ray.



$$dr < 0 \qquad \rightarrow d\gamma > 0\ ,\ d\sin\gamma > 0$$
$$dr > 0 \qquad \rightarrow d\gamma < 0\ ,\ d\sin\gamma < 0$$

i.e., $$\frac{d\sin\gamma}{dr} = 0$$

Since (eq. 8.78) $\sin\gamma = \frac{vp}{r}$

$$\frac{d}{dr}\left(\frac{vp}{r}\right) < 0 \quad \Rightarrow \quad \frac{d}{dr}\left(\frac{v}{r}\right) < 0 \tag{8.82}$$

or

$$\frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2} < 0 \quad \Rightarrow \quad \frac{dv}{dr} < \frac{v}{r} \tag{8.83}$$

It follows from eq. (8.78) that at a point of the minimum M when $\sin\gamma = 1$

$$p = \frac{r_M}{v(r_M)} \tag{8.84}$$

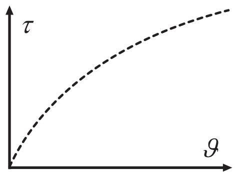
Eq. (8.84) means that for a given $v(r)$ the depth of M is determined by the ray parameter. In other words, different rays have different points of minimum.

Let source and receiver be at the Earth's surface.
Then eqs. (8.80) and (8.81) give

$$\vartheta = 2p \int_{r_M}^{R} \frac{v dr}{r(r^2 - v^2p^2)^{\frac{1}{2}}} \tag{8.85}$$

$$\tau = 2 \int_{r_M}^{R} \frac{r dr}{v\sqrt{r^2 - v^2 p^2}} \tag{8.86}$$

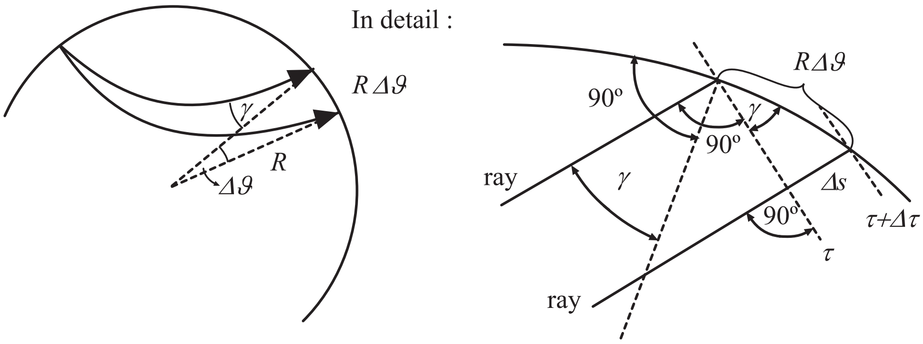
where R is the Earth's radius.



Eqs. (8.85) and (8.86) are parametric equations of a hodochrone - travel-time curve.
Thus, given $v(r)$, we can calculate a hodochrone using eqs. (8.85) and (8.86). We get one point of the hodochrone for one value of p or r_M.

8.22 Benndorf's equation

Consider a hodochrone, i.e., $\tau(\vartheta)$. Choose one point of the hodochrone. Find the corresponding ray. Consider two neighboring rays - see the figures



$$\begin{aligned}
\sin\gamma &= \frac{\Delta s}{R\Delta\vartheta}\,, \qquad \Delta s = v\Delta\tau \\
\sin\gamma &= \frac{v\Delta\tau}{R\Delta\vartheta}\,, \qquad p = \frac{R\sin\gamma}{v} \\
\frac{pv}{R} &= \frac{v\Delta\tau}{R\Delta\vartheta} \\
p &= \frac{\Delta\tau}{\Delta\vartheta}
\end{aligned}$$

In the limit we get

$$p = \frac{d\tau}{d\vartheta} \tag{8.87}$$

Eq. (8.87) is Benndorf's equation. Thus, if we choose one point of the hodochrone, $[\vartheta_1, \tau_1]$, then this point corresponds to the ray whose ray parameter, p_1, is equal to

$$\left.\frac{d\tau}{d\vartheta}\right|_{\vartheta_1}$$

If we compute values of p for large number of values of ϑ, we will get $p = p(\vartheta)$. Benndorf's equation thus gives the relation between rays and the hodochrone.

Appendix

Convolution

Let us introduce concept of convolution by an intuitive physical consideration.
Consider some physical system. Denote an input (input signal) to the system by $x(x)$ and system's response to the input by $y(t)$.

$$x(t) \to \boxed{\text{SYSTEM}} \to y(t)$$

Let us assume the following properties of the system :

Linearity

Let $y(t)$ be the system's response to the input $x(t)$. For brevity we will use the symbolic notation

$$x(t) \to y(t) \qquad \Leftrightarrow \qquad a\, x(t) \to a\, y(t)$$

Let $x_1(t) \to y_1(t)$ and $x_2(t) \to y_2(t)$. Then

$$x_1(t) + x_2(t) \to y_1(t) + y_2(t)$$

Consequently

$$a_1\, x_1(t) + a_2\, x_2(t) \to a_1\, y_1(t) + a_2\, y_2(t)$$

Invariability with respect to time

Let $x(t) \to y(t)$, then $x(t-\tau) \to y(t-\tau)$

Causality

Let $x(t) = 0$ for $t < t_0$, then $y(t) = 0$ for $t < t_0$.

Consider now Dirac delta function (or Dirac delta impulse) $\delta(t)$ as input to the system. Denote the system's response $h(t)$, i.e.,

$$\delta(t) \to h(t)$$

Since $\mathcal{F}[\delta(t)] = 1$, i.e., all frequencies are equally present in the Dirac delta impulse, the system's response $h(t)$ is called the impulse response of the system and characterizes transfer properties of the system in the time domain.

Due to invariability of the system with respect to time

$$\delta(t-\tau) \to h(t-\tau)$$

Consider again some general input signal $x(t)$. It can be represented using the Dirac Delta function :

$$x(t) = \int_0^t x(\tau)\delta(t-\tau)d\tau \tag{A.1}$$

Relation (A.1) means, that the general signal $x(t)$ is an 'integral linear combination' of the Dirac delta impulses $\delta(t-\tau)$. Relation (A.1) and linearity of the system imply, that the response to the input signal to the input signal $x(t)$ is the same integral combination of the impulse responses $h(t-\tau)$:

$$y(t) = \int_{0}^{t} x(\tau)h(t-\tau)d\tau \tag{A.2}$$

Relation (A.2) is symbolically denoted as

$$y(t) = x(t) * h(t) \tag{A.3}$$

The integral in relation (A.2) is called the convolutory integral, or simply, the convolution. Relation (A.2) means that once we know the impulse response of a system we can compute the output of the system for an arbitrary input using the convolution.

Now we define the convolution of two functions $x_1(t)$ and $x_2(t)$ as

$$\rho^{x_1,x_2}(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau \tag{A.4}$$

and symbolically denote

$$\rho^{x_1,x_2}(t) = x_1(t) * x_2(t) \tag{A.5}$$

Relation (A.4) is valid for general functions $x_1(t)$ and $x_2(t)$ defined in the interval $t \in (-\infty, \infty)$. If one or both of the functions are causal, relation (A.4) can be modified.

A) $x_1(t)$ is causal : $\rho^{x_1,x_2}(t) = \int_0^\infty x_1(\tau)x_2(t-\tau)d\tau$
B) $x_2(t)$ is causal : $\rho^{x_1,x_2}(t) = \int_{-\infty}^t x_1(\tau)x_2(t-\tau)d\tau$
C) both $x_1(t)$ and $x_2(t)$ are causal : $\rho^{x_1,x_2}(t) = \int_0^t x_1(\tau)x_2(t-\tau)d\tau$

Basic properties of the convolution

1. $x_1(t) * x_2(t) = x_2(t) * x_1(t)$
2. $x_1(t) * [x_2(t) + x_3(t)] = x_1(t) * x_2(t) + x_1(t) * x_3(t)$
3. $x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t)$
4. $x(t) * \delta(t) = x(t)$
   $x(t) * \delta(t-\tau) = x(t-\tau)$
   $x(t-t') * \delta(t-\tau) = x(t-t'-\tau)$
   $\delta(t-t') * \delta(t-\tau) = \delta(t-t'-\tau)$
5. $\rho^{x_1,x_2}(t-\tau) = x_1(t-\tau) * x_2(t) = x_1(t) * x_2(t-\tau)$
6. $\frac{d}{dt}[x_1(t) * x_2(t)] = \frac{dx_1(t)}{dt} * x_2(t) = x_1(t) * \frac{dx_2(t)}{dt}$
7. $\int_{-\infty}^t x_1(\eta) * x_2(\eta)d\eta = [\int_{-\infty}^t x_1(\eta)d\eta] * x_2(t) = x_1(t) * [\int_{-\infty}^t x_2(\eta)d\eta]$

Convolution theorem

Find the Fourier transform of the convolution (A.4).

$$
\begin{aligned}
\mathcal{F}[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_1(\eta)x_2(\tau-\eta)d\eta\right]e^{-i2\pi f\tau}d\tau &= \\
\int_{-\infty}^{\infty} x_1(\eta)\left[\int_{-\infty}^{\infty} x_2(\tau-\eta)e^{-i2\pi f\tau}d\tau\right]d\eta &= \\
\int_{-\infty}^{\infty} x_1(\eta)e^{-i2\pi f\eta}X_2(f)d\eta &= X_1(f)X_2(f)
\end{aligned}
$$

This allows us to formulate the convolution theorem Let $X_1(f) = \mathcal{F}\{x_1(t)\}$ and $X_2(f) = \mathcal{F}\{x_2(t)\}$. Then

$$\mathcal{F}[x_1(t) * x_2(t)] = X_1(f)X_2(f) \tag{A.6}$$

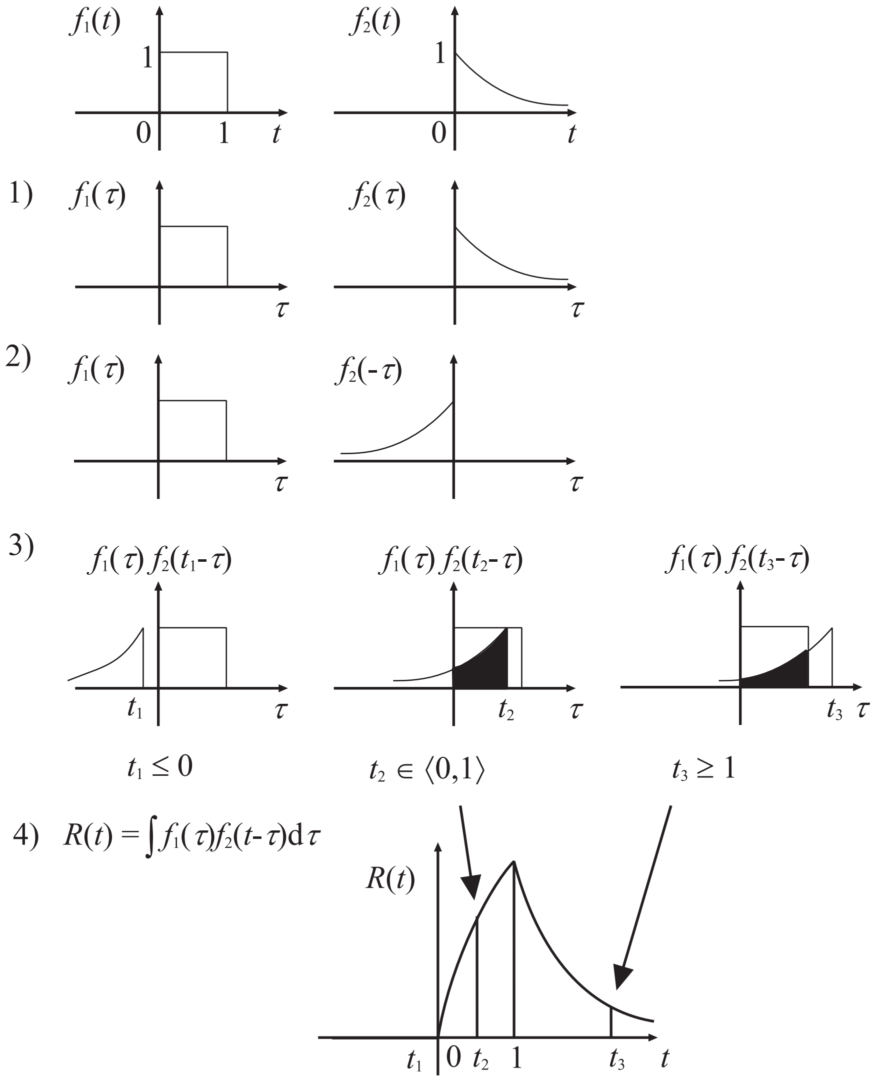
Relation (A.6) is one of the most important relations in the spectral analysis and is frequently used in mathematics, physics, geophysics and other scientific and engineering disciplines.

It is easy to show (analogously with the proof of relation (A.6) that

$$\mathcal{F}[x_1(t)x_2(t)] = X_1(f) * X_2(f) \tag{A.7}$$

The r.h.s. of relation (A.7) is called the spectral or frequency convolution while that in relation (A.4) is called the time convolution.

Graphical interpretation of convolution



1. new variable τ
2. $f_2(\tau) \to f_2(-\tau)$ - horizontal mirror
3. $f_2(-\tau) \to f_2(t-\tau)$ - horizontal translation by t
4. multiply $f_2(\tau)f_2(t-\tau)$, area under the graph is the value of convolution in the time t

References / Recommended Reading

Aki, K., Richards, P. G.: Quantitative seismology: Theory and methods. *W. H. Freeman* 1980; *University Science Books* 2002

Carcione, J. M.: Wave fields in real media: wave propagation in anisotropic, anelastic and porous media. *Pergamon*, 2001.

Červený, V.: Seismic ray theory. *Cambridge University Press* 2001

Červený, V., Hron, F.: The ray series method and dynamic ray tracing systems for 3-D inhomogeneous media. *Bull. Seism. Soc. Am.* 70, 47-77.

Gubbins, D.: Seismology and plate tectonics. *Cambridge University Press* 1990

Kennett, B. L. N.: The seismic wavefield I, II. *Cambridge University Press* 2000, 2001

Lay, T., Wallace, T. C.: Modern global seismology. *Academic Press* 1995

Moczo, P., Kristek, J., Halada, L.: The Finite-Difference Method for Seismologists. An Introduction. *Comenius University*, Bratislava, 2004.

Moczo, P., Robertsson, J. O. A., Eisner, L.: The Finite-Difference Time-Domain Method for Modelling of Seismic Wave Propagation. In Advances in Wave Propagation in Heterogeneous Earth, R.-S. Wu, V. Maupin, and R. Dmowska, eds., in the series *Advances in Geophysics* , Vol. 48, *Elsevier*, 2006.

Novotný, O.: Seismic surface waves. *Universidad Federal da Bahia, Brasilia*, 1999

Pujol, J.: Elastic wave propagation and generation in seismology. *Cambridge University Press* 2003

Scholz, Ch. H.: The mechanics of earthquakes and faulting. *Cambridge University Press* 2002

Shearer, P. M.: Introduction to seismology. *Cambridge University Press* 1999

Udías, A.: Principles of seismology. *Cambridge University Press* 1999

Index