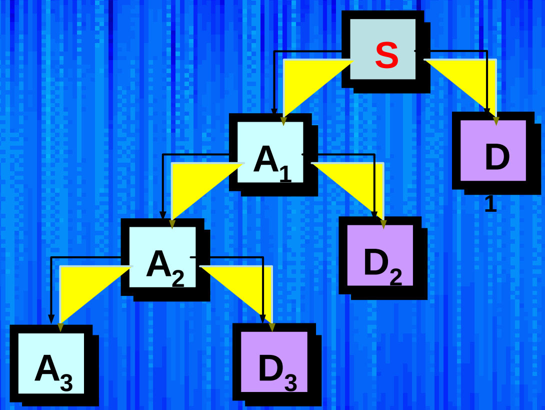
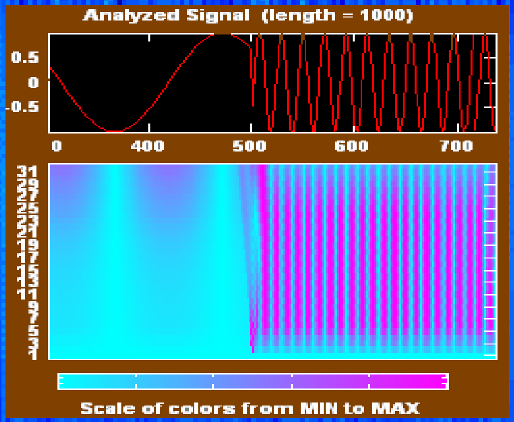
Workshop 118 on Wavelet Application in Transportation Engineering, Sunday, January 09, 2005

# Introduction to Wavelet

## — A Tutorial




Fengxiang Qiao, Ph.D.
Texas Southern University

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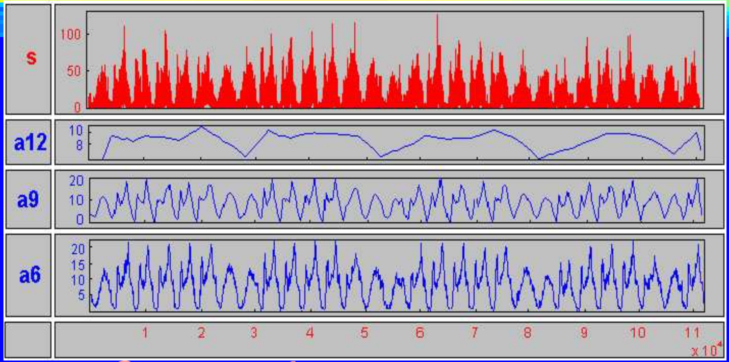
# OVERVIEW




- **Wavelet**
  - A small wave
- **Wavelet Transforms**
  - Convert a signal into a series of wavelets
  - Provide a way for analyzing waveforms, bounded in both frequency and duration
  - Allow signals to be stored more efficiently than by Fourier transform
  - Be able to better approximate real-world signals
  - Well-suited for approximating data with sharp discontinuities
- **“The Forest & the Trees”**
  - Notice gross features with a large "window“
  - Notice small features with a small "window”

# DEVELOPMENT IN HISTORY

- **Pre-1930**
  - Joseph Fourier (1807) with his theories of frequency analysis
- **The 1930s**
  - Using scale-varying basis functions; computing the energy of a function
- **1960-1980**
  - Guido Weiss and Ronald R. Coifman; Grossman and Morlet
- **Post-1980**
  - Stephane Mallat; Y. Meyer; Ingrid Daubechies; wavelet applications today

# PRE-1930

- **Fourier Synthesis**
  - Main branch leading to wavelets
  - By Joseph Fourier (born in France, 1768-1830) with frequency analysis theories (1807)
- **From the Notion of Frequency Analysis to Scale Analysis**
  - Analyzing *f*(*x*) by creating mathematical structures that vary in scale
    - Construct a function, shift it by some amount, change its scale, apply that structure in approximating a signal
    - Repeat the procedure. Take that basic structure, shift it, and scale it again. Apply it to the same signal to get a new approximation
- **Haar Wavelet**
  - The first mention of wavelets appeared in an appendix to the thesis of A. Haar (1909)
  - With *compact support*, vanishes outside of a finite interval
  - Not continuously differentiable

For any $2\pi$ periodical function $f(x)$:

$$f(x) = a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

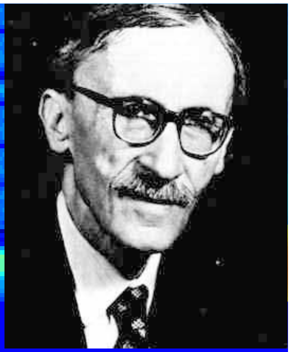
$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(kx)dx$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(kx)dx$$

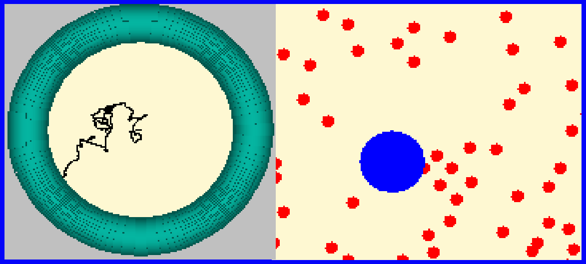
Jean-Baptiste-Joseph Fourier (1768-1830)

# THE 1930$^{s}$



- **Finding by the 1930s Physicist Paul Levy**
  - ♥ Haar basis function is superior to the Fourier basis functions for studying small complicated details in the Brownian motion
- **Energy of a Function by Littlewood, Paley, and Stein**
  - ♥ Different results were produced if the energy was concentrated around a few points or distributed over a larger interval

$$Energy = \frac{1}{2}\int_0^{2\pi} \left| f(x) \right|^2 dx$$



# 1960-1980

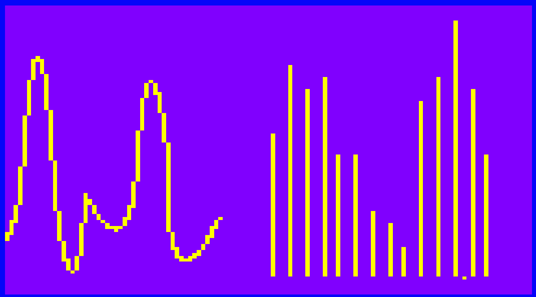
- **Created a Simplest Elements of a Function Space, Called *Atoms***
  - By the mathematicians Guido Weiss and Ronald R. Coifman
  - With the goal of finding the atoms for a common function
- **Using Wavelets for Numerical Image Processing**
  - David Marr developed an effective algorithm using a function varying in scale in the early 1980s
- **Defined Wavelets in the Context of Quantum Physics**
  - By Grossman and Morlet in 1980

# POST-1980

- An Additional Jump-start By Mallat
  - In 1985, Stephane Mallat discovered some relationships between quadrature mirror filters, pyramid algorithms, and orthonormal wavelet bases
- Y. Meyer's First Non-trivial Wavelets
  - Be continuously differentiable
  - Do not have compact support
- Ingrid Daubechies' Orthonormal Basis Functions
  - Based on Mallat's work
  - Perhaps the most elegant, and the cornerstone of wavelet applications today

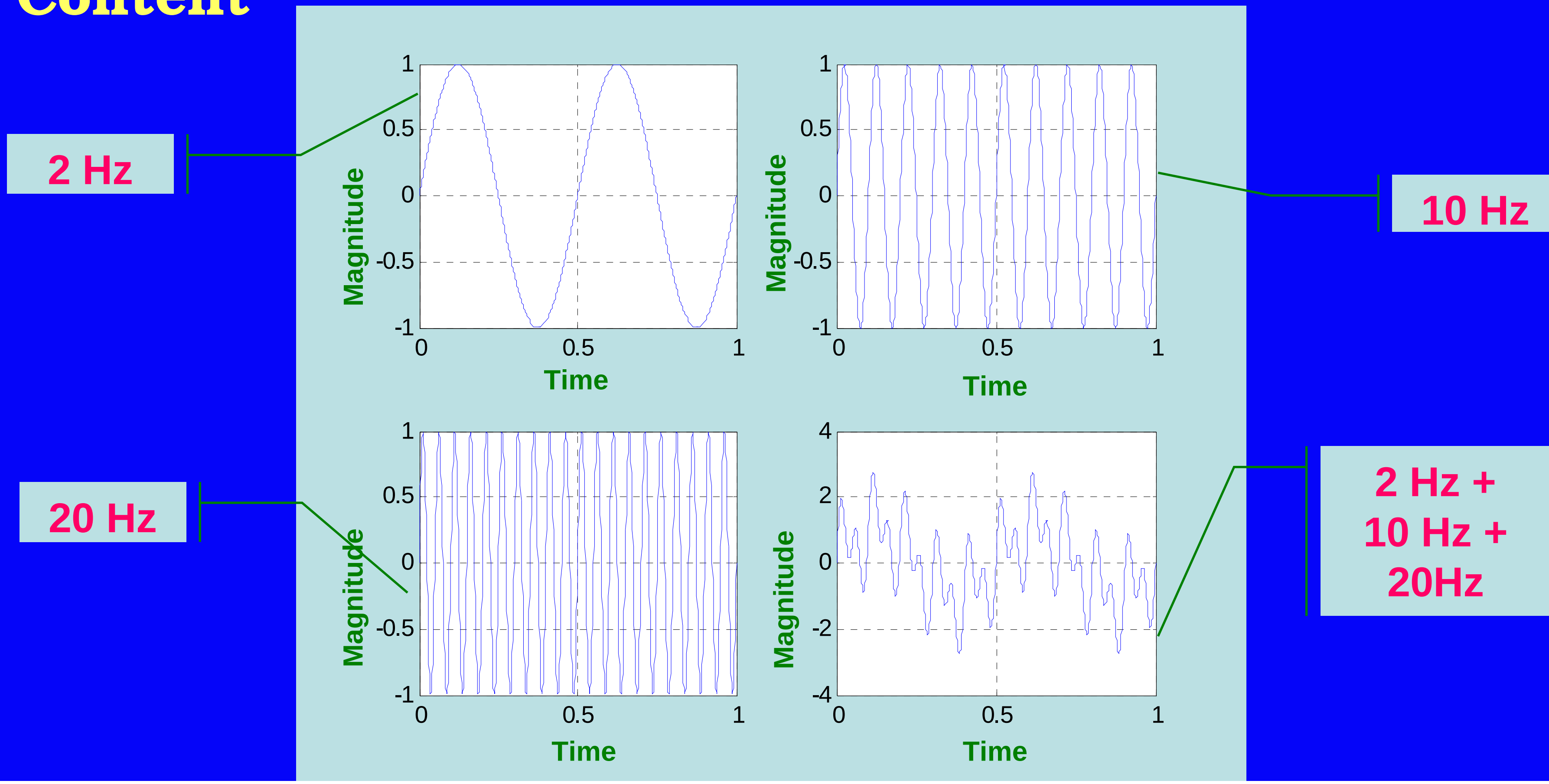
# MATHEMATICAL TRANSFORMATION

- **Why**
  - ♥ To obtain a further information from the signal that is not readily available in the raw signal.
- **Raw Signal**
  - ♥ Normally the time-domain signal
- **Processed Signal**
  - ♥ A signal that has been "transformed" by any of the available mathematical transformations
- **Fourier Transformation**
  - ♥ The most popular transformation



# TIME-DOMAIN SIGNAL

- The Independent Variable is Time
- The Dependent Variable is the Amplitude
- Most of the Information is Hidden in the Frequency Content



# FREQUENCY TRANSFORMS

- **Why Frequency Information is Needed**
  - Be able to see any information that is not obvious in time-domain
- **Types of Frequency Transformation**
  - Fourier Transform, Hilbert Transform, Short-time Fourier Transform, Wigner Distributions, the Radon Transform, the Wavelet Transform ...

# FREQUENCY ANALYSIS

- **Frequency Spectrum**
  - Be basically the frequency components (spectral components) of that signal
  - Show what frequencies exists in the signal
- **Fourier Transform (FT)**
  - One way to find the frequency content
  - Tells how much of each frequency exists in a signal

$$X(k+1) = \sum_{n=0}^{N-1} x(n+1) \cdot W_N^{kn}$$

$$x(n+1) = \frac{1}{N} \sum_{k=0}^{N-1} X(k+1) W_N^{-kn}$$

$$w_N = e^{-j\left(\frac{2\pi}{N}\right)}$$

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-2j\pi ft} dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f) \cdot e^{2j\pi ft} df$$

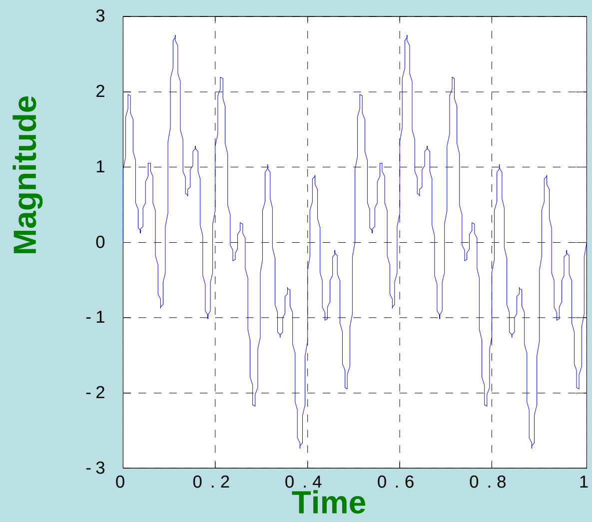
# STATIONARITY OF SIGNAL (1)

- Stationary Signal
  - ♥ Signals with frequency content unchanged in time
  - ♥ All frequency components exist at all times
- Non-stationary Signal
  - ♥ Frequency changes in time
  - ♥ One example: the "Chirp Signal"

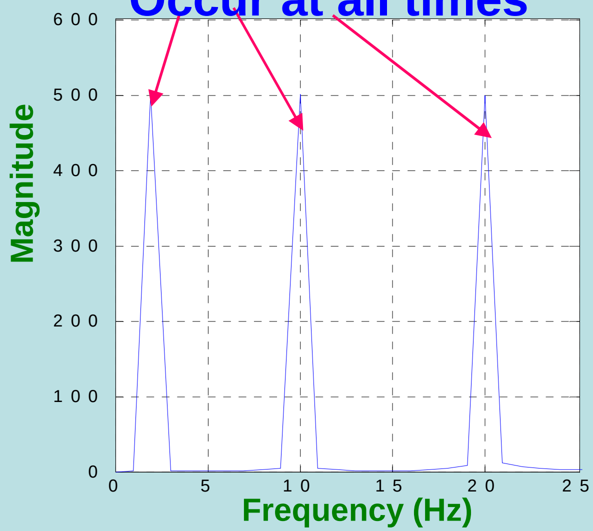
# STATIONARITY OF SIGNAL (2)

2 Hz + 10 Hz + 20Hz

Stationary



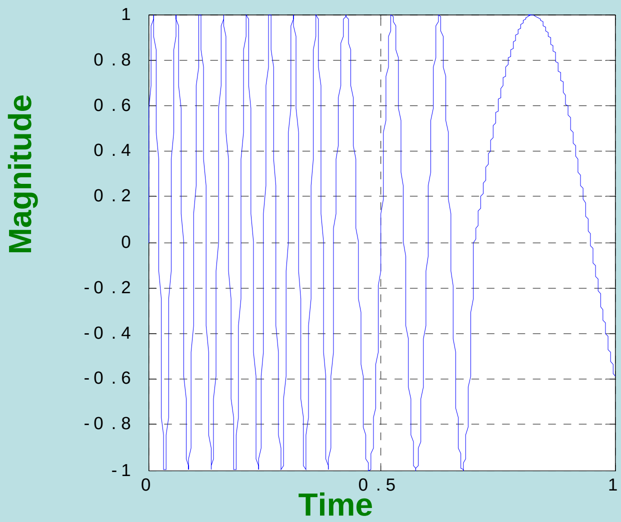

Occur at all times

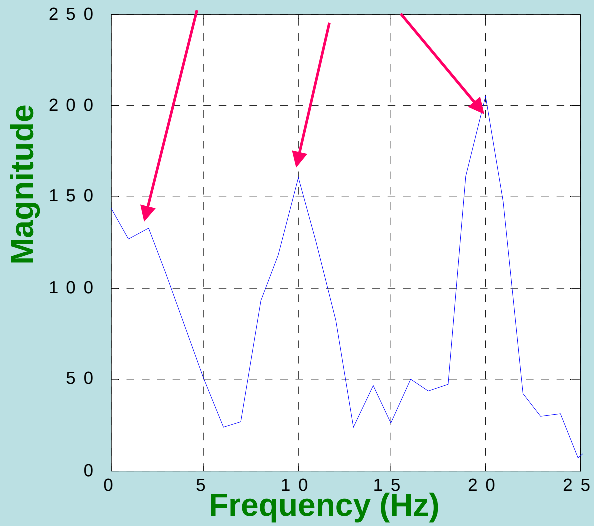



Do not appear at all times

0.0-0.4: 2 Hz +
0.4-0.7: 10 Hz +
0.7-1.0: 20Hz

Non-Stationary

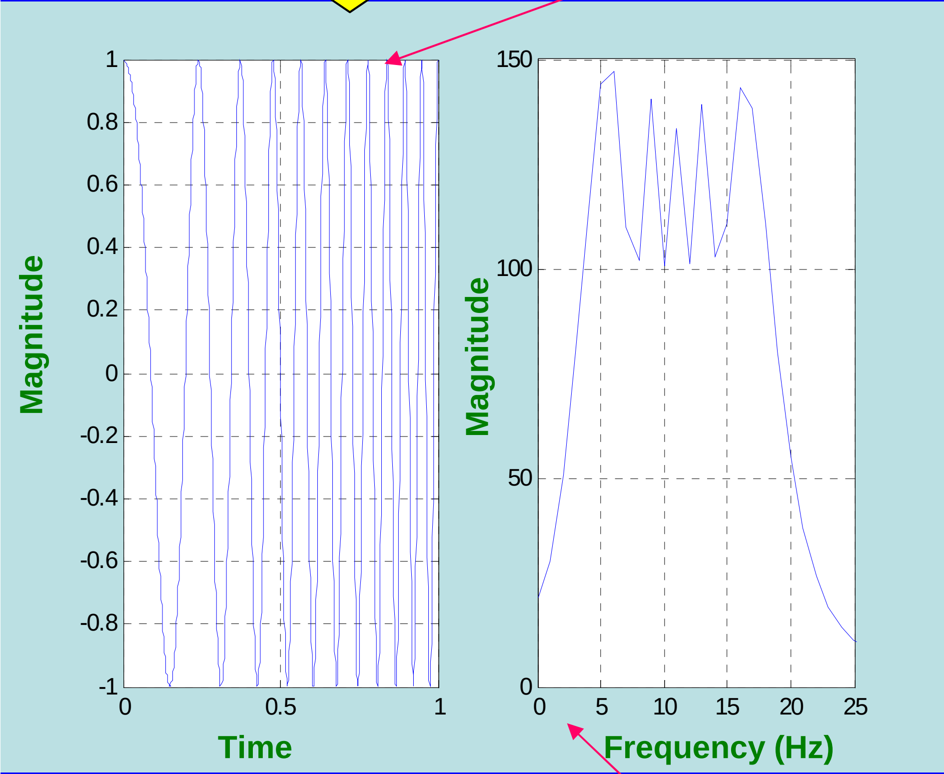


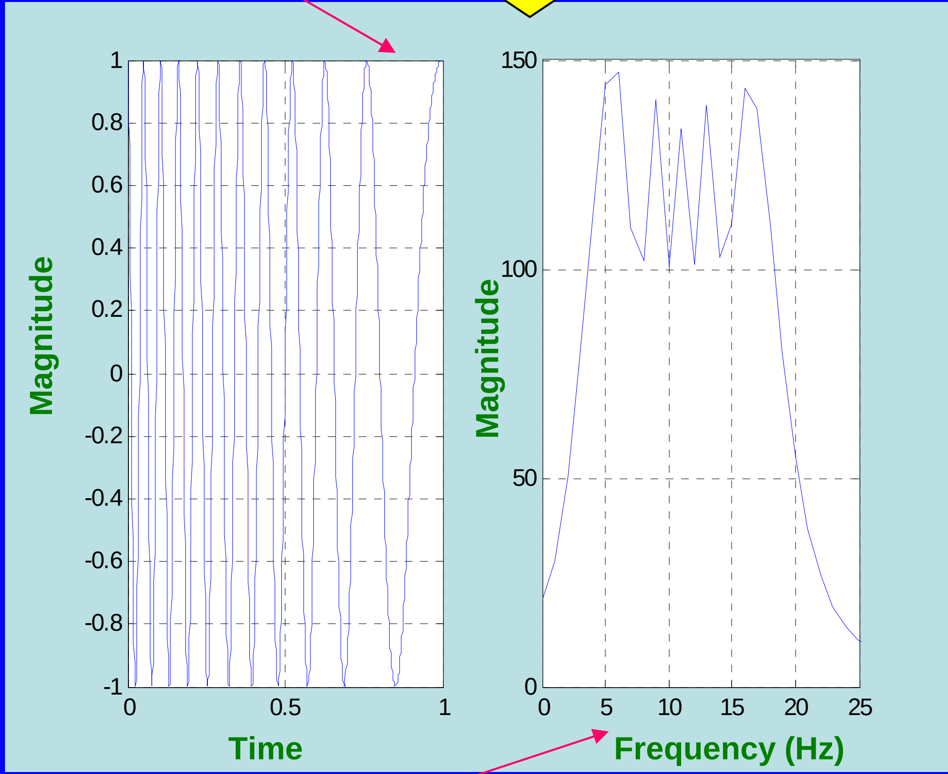



# CHIRP SIGNALS

- **Frequency: 2 Hz to 20 Hz**
- **Frequency: 20 Hz to 2 Hz**

Different in Time Domain




Same in Frequency Domain

**At what time the frequency components occur? FT can not tell!**

# NOTHING MORE, NOTHING LESS

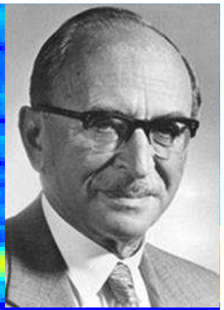
- **FT Only Gives what Frequency Components Exist in the Signal**
- **The Time and Frequency Information can not be Seen at the Same Time**
- **Time-frequency Representation of the Signal is Needed**

**Most of Transportation Signals are Non-stationary.**

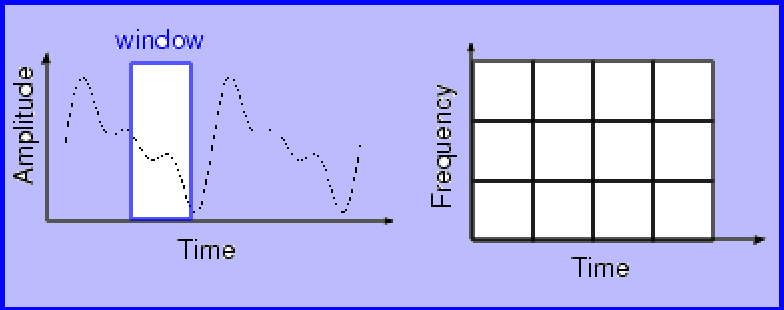
**(We need to know whether and also when an incident was happened.)**

**ONE EARLIER SOLUTION: SHORT-TIME FOURIER TRANSFORM (STFT)**

# SFORT TIME FOURIER TRANSFORM (STFT)



- **Dennis Gabor (1946) Used STFT**
  - **To analyze only a small section of the signal at a time -- a technique called *Windowing the Signal*.**
- **The Segment of Signal is Assumed *Stationary***
- **A 3D transform**




$$\mathrm{STFT}_X^{(\omega)}(t', f) = \int_t \left[x(t) \bullet \omega^*(t-t')\right] \bullet e^{-j2\pi ft}\,dt$$

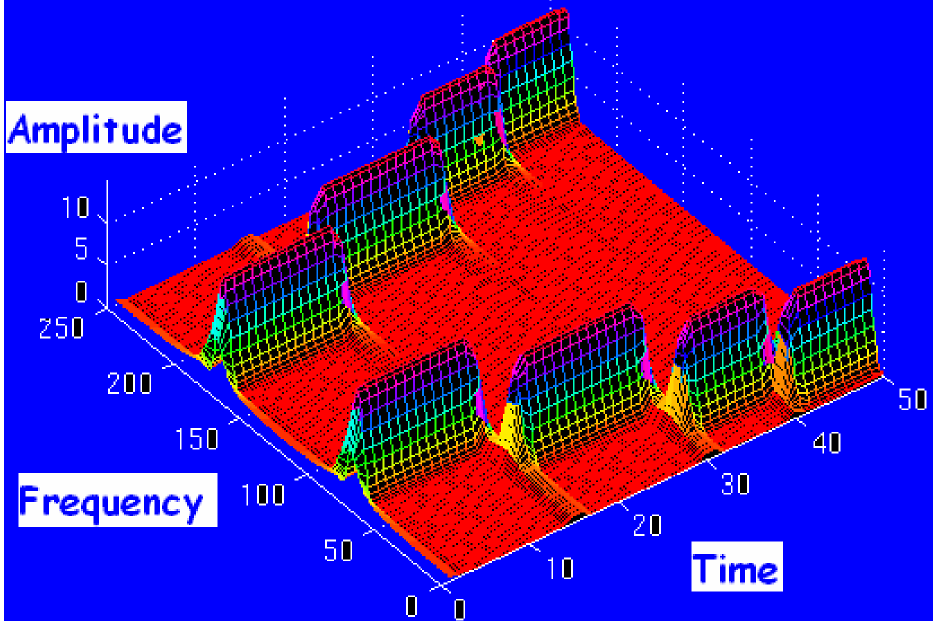
$\omega(t)$: the window function

**A function of time and frequency**

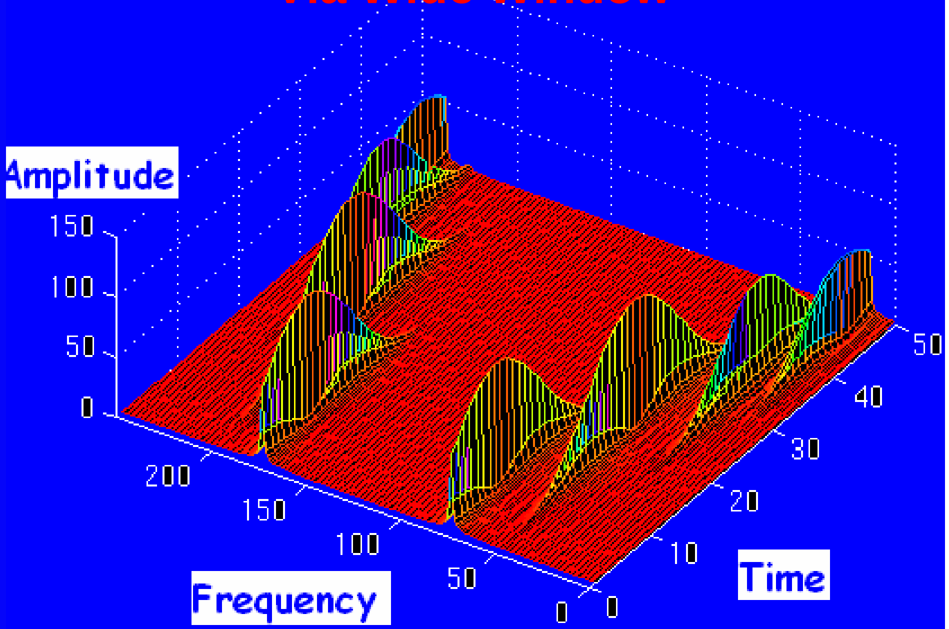
# DRAWBACKS OF STFT

- Unchanged Window
- Dilemma of Resolution
  - Narrow window -> poor frequency resolution
  - Wide window -> poor time resolution
- Heisenberg Uncertainty Principle
  - Cannot know what frequency exists at what time intervals

Via Narrow Window




Via Wide Window




The two figures were from Robi Poliker, 1994

# MULTIRESOLUTION ANALYSIS (MRA)

- **Wavelet Transform**
  - An alternative approach to the short time Fourier transform to overcome the resolution problem
  - Similar to STFT: signal is multiplied with a function
- **Multiresolution Analysis**
  - Analyze the signal at different frequencies with different resolutions
  - Good time resolution and poor frequency resolution at high frequencies
  - Good frequency resolution and poor time resolution at low frequencies
  - More suitable for short duration of higher frequency; and longer duration of lower frequency components

# ADVANTAGES OF WT OVER STFT

- Width of the Window is Changed as the Transform is Computed for Every Spectral Components
- Altered Resolutions are Placed

# PRINCIPLES OF WAELET TRANSFORM

- Split Up the Signal into a Bunch of Signals
- Representing the Same Signal, but all Corresponding to Different Frequency Bands
- Only Providing What Frequency Bands Exists at What Time Intervals

# DEFINITION OF CONTINUOUS WAVELET TRANSFORM

$$CWT_x^{\psi}(\tau, s) = \Psi_x^{\psi}(\tau, s) = \frac{1}{\sqrt{|s|}} \int x(t) \bullet \psi^{*}\left(\frac{t-\tau}{s}\right) dt$$

Translation

(The location of the window)

Scale

Mother Wavelet

- **Wavelet**
  - **Small wave**
  - **Means the window function is of finite length**
- **Mother Wavelet**
  - **A prototype for generating the other window functions**
  - **All the used windows are its dilated or compressed and shifted versions**

# SCALE

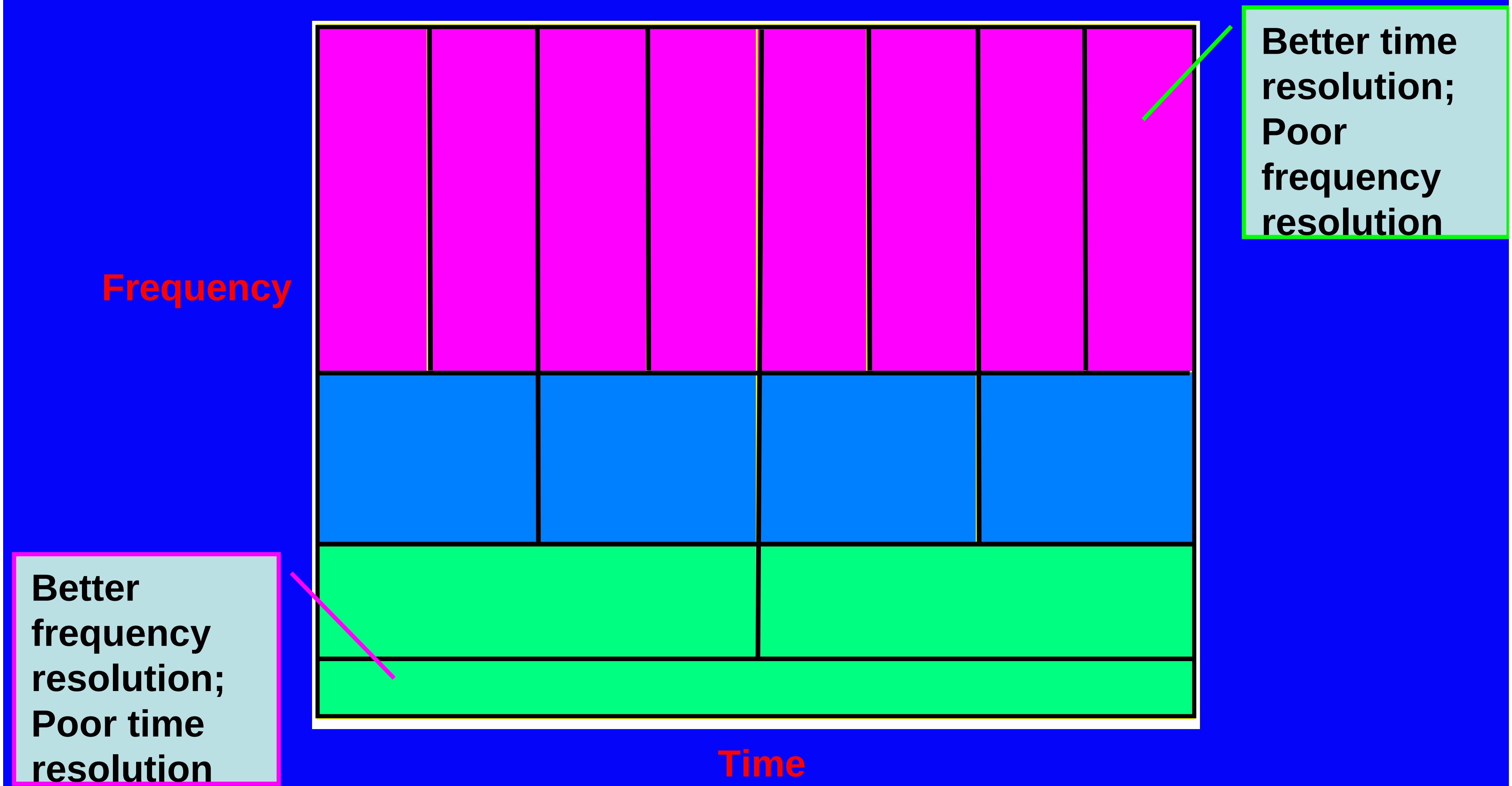
- Scale
  - S>1: dilate the signal
  - S<1: compress the signal
- Low Frequency -> High Scale -> Non-detailed Global View of Signal -> Span Entire Signal
- High Frequency -> Low Scale -> Detailed View  Last in Short Time
- Only Limited Interval of Scales is Necessary

# COMPUTATION OF CWT

$$\mathrm{CWT}_x^{\psi}(\tau, s) = \Psi_x^{\psi}(\tau, s) = \frac{1}{\sqrt{|s|}} \int x(t) \bullet \psi^*\left(\frac{t-\tau}{s}\right) dt$$

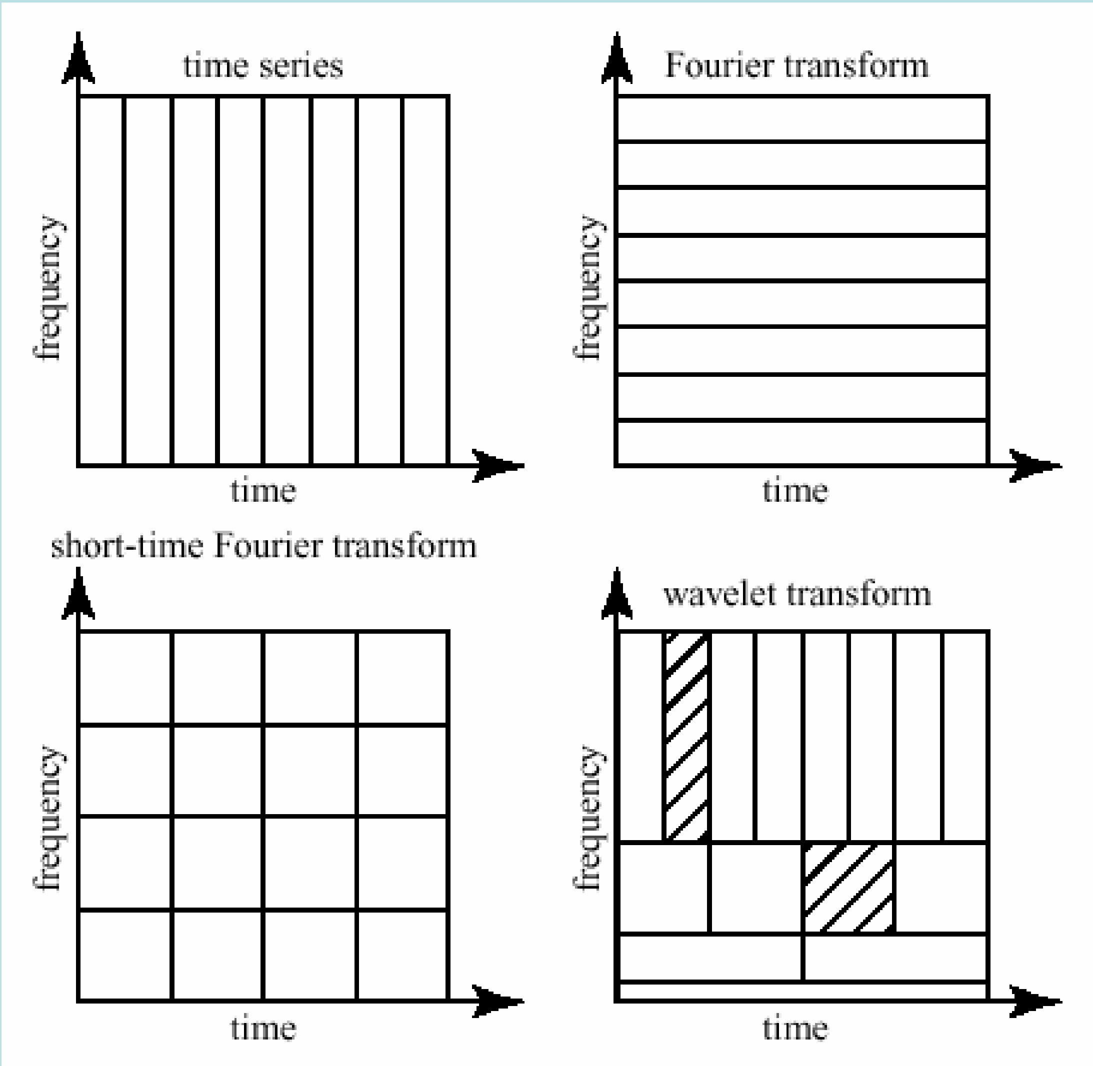
- **Step 1:** The wavelet is placed at the beginning of the signal, and set s=1 (the most compressed wavelet);
- **Step 2:** The wavelet function at scale "1" is multiplied by the signal, and integrated over all times; then multiplied by $1/\sqrt{s}$ ;
- **Step 3:** Shift the wavelet to $t=\tau$ , and get the transform value at $t=\tau$ and $s=1$;
- **Step 4:** Repeat the procedure until the wavelet reaches the end of the signal;
- **Step 5:** Scale s is increased by a sufficiently small value, the above procedure is repeated for all *s;*
- **Step 6:** Each computation for a given *s* fills the single row of the time-scale plane;
- **Step 7:** CWT is obtained if all s are calculated.

# RESOLUTION OF TIME & FREQUENCY




- Each box represents a equal portion
- Resolution in STFT is selected once for entire analysis

# COMPARSION OF TRANSFORMATIONS




From http://www.cerm.unifi.it/EUcourse2001/Gunther_lecturenotes.pdf, p.10

# MATHEMATICAL EXPLAINATION

$$\mathrm{CWT}_x^{\psi}(\tau, s) = \Psi_x^{\psi}(\tau, s) = \frac{1}{\sqrt{|s|}} \int x(t) \bullet \psi^*\left(\frac{t-\tau}{s}\right) dt$$
$$= \int X(T) \bullet \psi_{\tau,s}^*(t) dt$$

$$\psi_{\tau,s}^*(t) = \frac{1}{\sqrt{s}} \psi\left(\frac{t-\tau}{s}\right)$$

**CWT can be regarded as the inner product of the signal with a basis function $\psi_{\tau,s}^*(t)$**

# DISCRETIZATION OF CWT

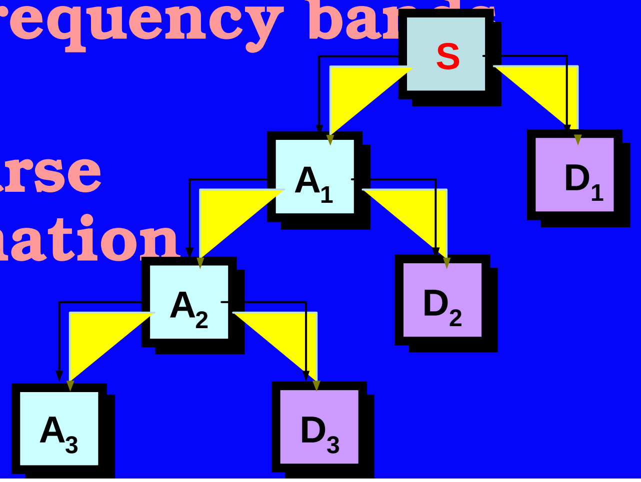
- It is Necessary to Sample the Time-Frequency (scale) Plane.
- At High Scale $s$ (Lower Frequency $f$), the Sampling Rate $N$ can be Decreased.
- The Scale Parameter $s$ is Normally Discretized on a Logarithmic Grid.
- The most Common Value is 2.

$$N_2 = s_1 / s_2 \cdot N_1 = f_1 / f_2 \cdot N_1$$

| | | | | |
|---|---|---|---|---|
| S | 2 | 4 | 8 | … |
| N | 32 | 16 | 8 | … |

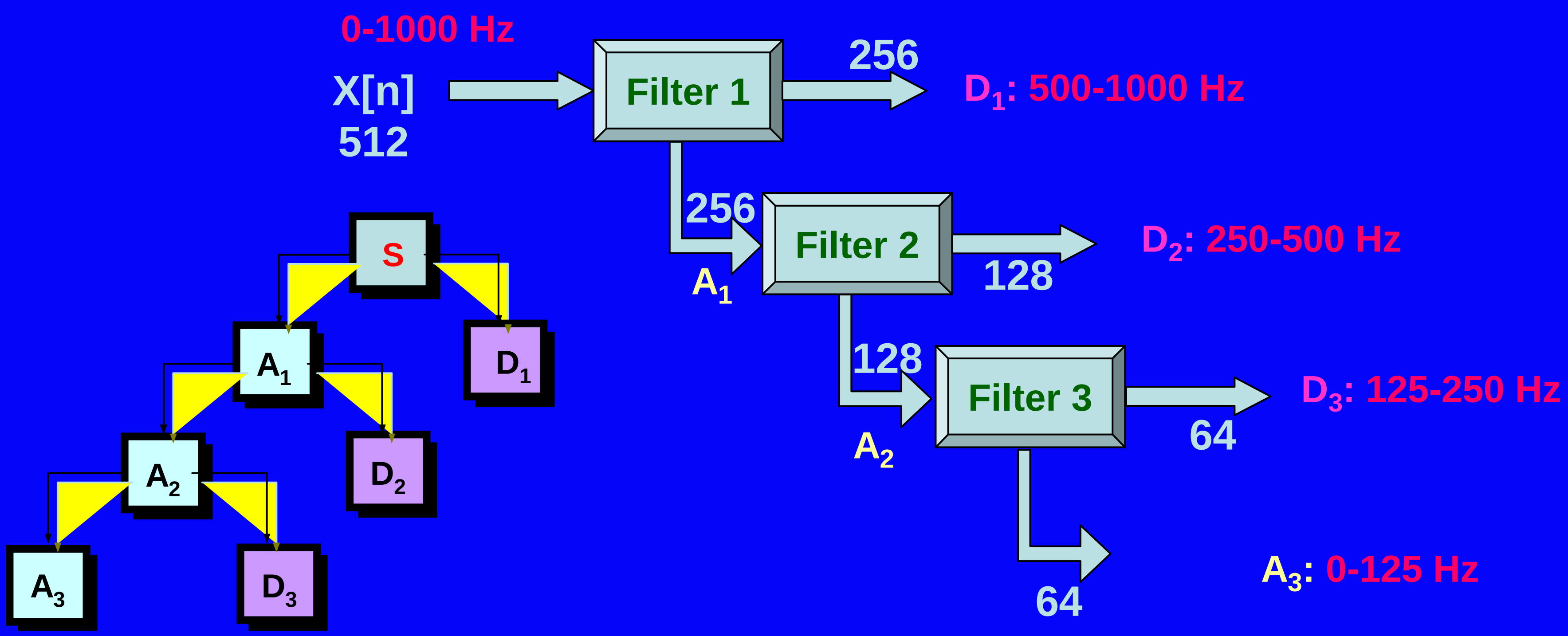
# EFFECTIVE & FAST DWT

- The Discretized CWT is not a True Discrete Transform
- Discrete Wavelet Transform (DWT)
  - Provides sufficient information both for analysis and synthesis
  - Reduce the computation time sufficiently
  - Easier to implement
  - Analyze the signal at different frequency bands with different resolutions
  - Decompose the signal into a coarse approximation and detail information

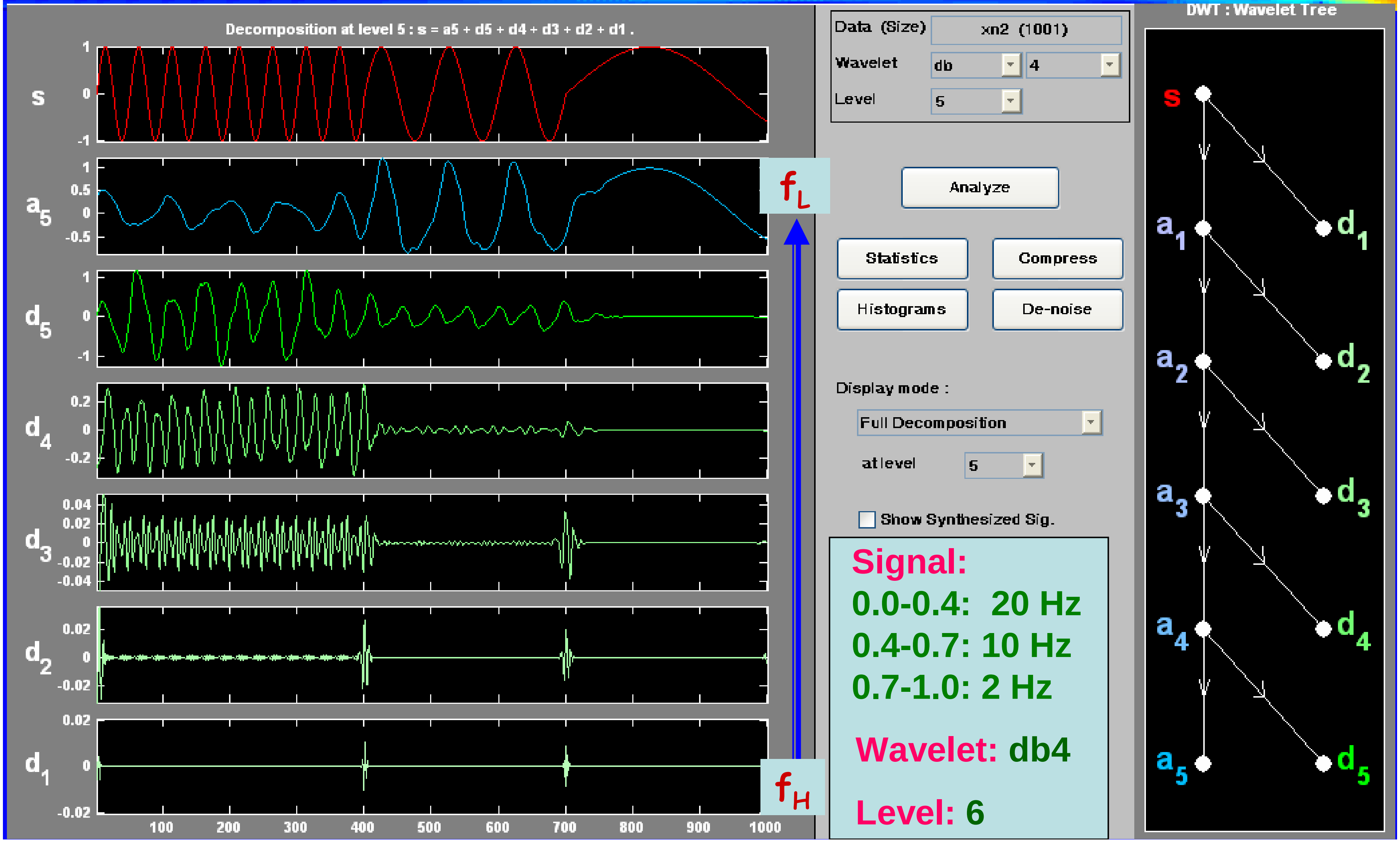


# SUBBABD CODING ALGORITHM

- **Halves the Time Resolution**
  - ♥ Only half number of samples resulted
- **Doubles the Frequency Resolution**
  - ♥ The spanned frequency band halved



# DECOMPOSING NON-STATIONARY SIGNALS (1)




**Signal:**

0.0-0.4: 20 Hz

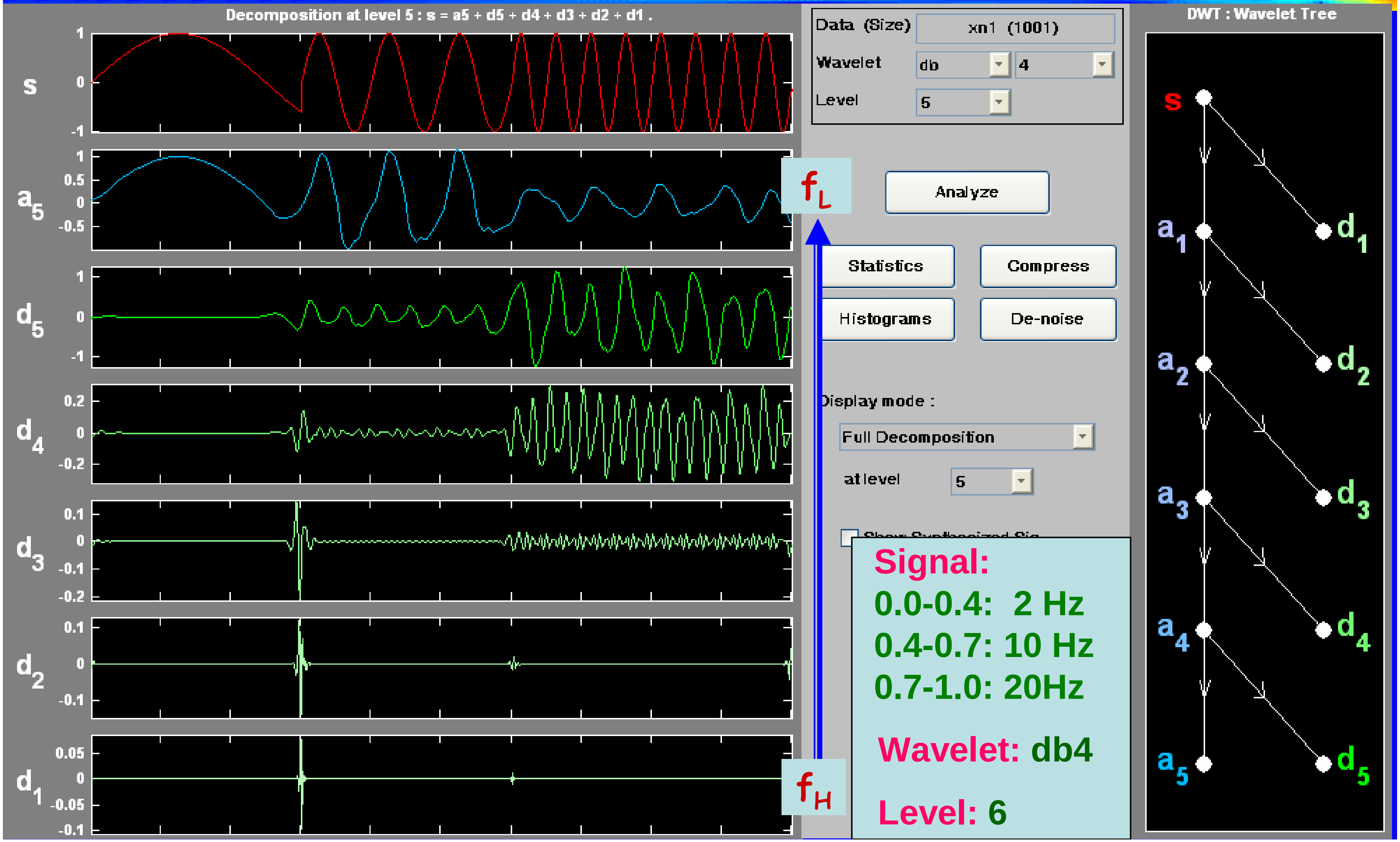
0.4-0.7: 10 Hz

0.7-1.0: 2 Hz

**Wavelet:** db4

**Level:** 6

# DECOMPOSING NON-STATIONARY SIGNALS (2)




**Signal:**
0.0-0.4: 2 Hz
0.4-0.7: 10 Hz
0.7-1.0: 20Hz

**Wavelet:** db4

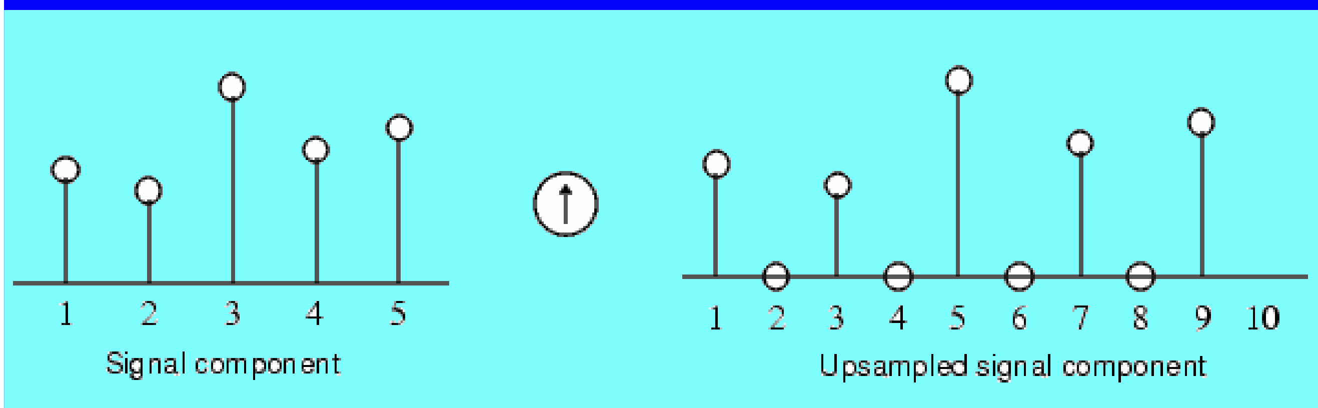
**Level:** 6

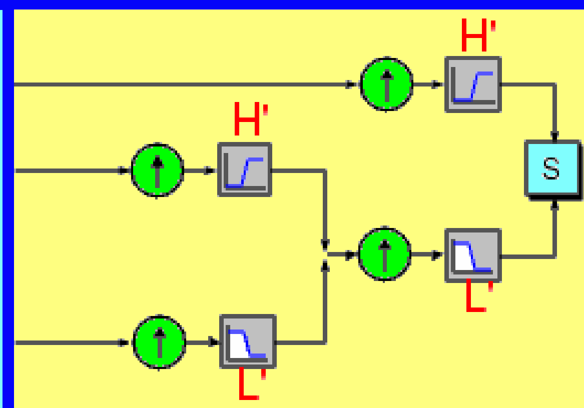
# RECONSTRUCTION (1)

- **What**
  - ♥ How those components can be assembled back into the original signal without loss of information?
  - ♥ A Process After *decomposition* or *analysis*.
  - ♥ Also called *synthesis*
- **How**
  - ♥ Reconstruct the signal from the wavelet coefficients
  - ♥ Where wavelet analysis involves filtering and downsampling, the wavelet reconstruction process consists of upsampling and filtering

# RECONSTRUCTION (2)

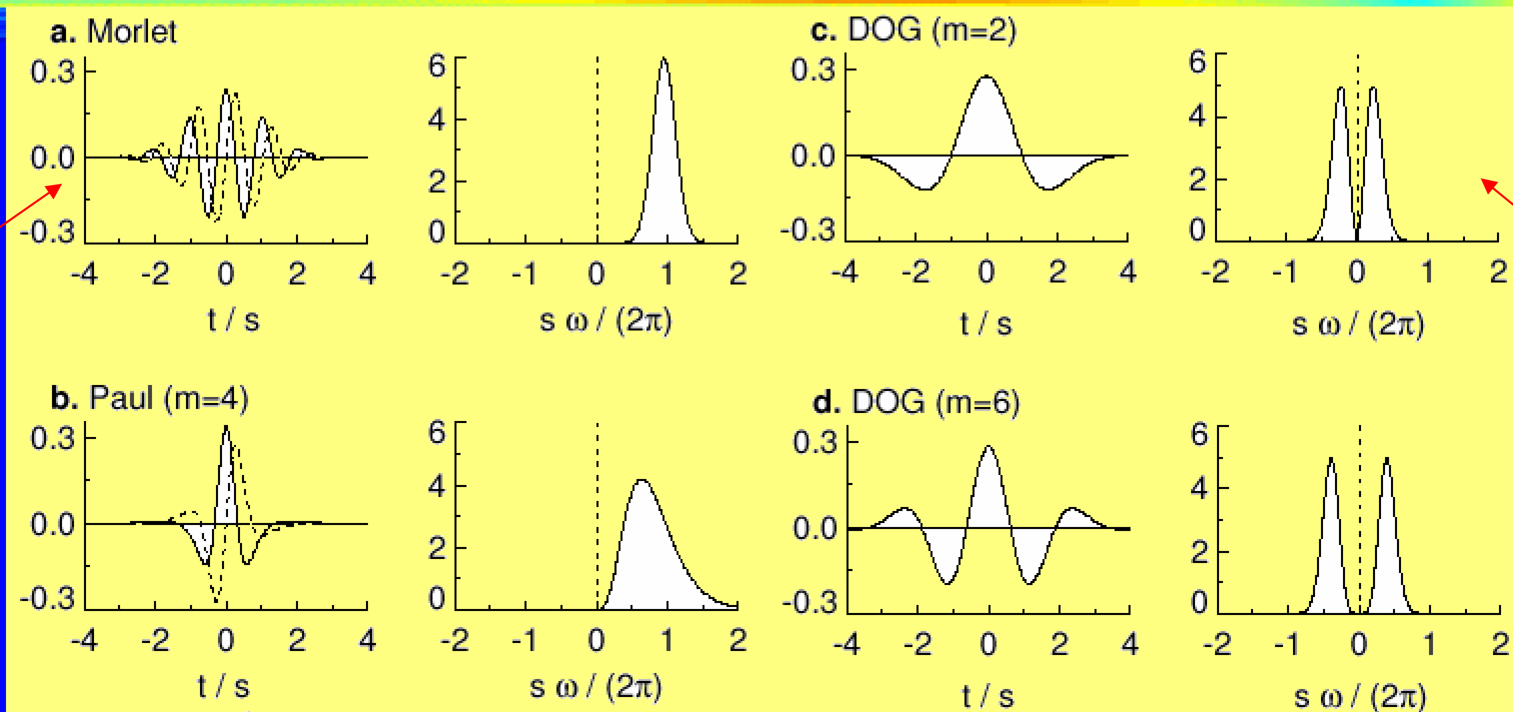
- Lengthening a signal component by inserting zeros between samples (upsampling)
- MATLAB Commands: idwt and waverec; idwt2 and waverec2.



# WAVELET BASES

Time domain




Frequency domain

Wavelet Basis Functions:

Morlet $(\omega_0 = \text{frequency})$: $\pi^{-1/4} e^{j\omega_0\eta} e^{-\eta^2/2}$

Paul $(m = \text{order})$: DOG $\dfrac{2^m i^m m!}{\sqrt{\pi(2m)!}}(1 - i\eta)^{-(m+1)}$

DOG $(m = \text{derivative})$: $\dfrac{(-1)^{m+1}}{\sqrt{\Gamma\left(m + \frac{1}{2}\right)}} \dfrac{d^m}{d\eta^m}\left(e^{-\eta^2/2}\right)$

Derivative Of a Gaussian
M=2 is the **Marr** or **Mexican hat** wavelet

# WAVELET FAMILY PROPERTIES

| Property | morl | mexh | meyr | haar | *db*N | symN | coifN | biorNr.Nd | rbioNr.Nd | gaus | dmey | cgau | cmor | fbsp | shan |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Crude | • | • | | | | | | | | • | | • | • | • | • |
| Infinitely regular | • | • | • | | | | | | | • | | • | • | • | • |
| Arbitrary regularity | | | | | • | • | • | • | • | | | | | | |
| Compactly supported orthogonal | | | | • | • | • | • | | | | | | | | |
| Compactly supported biothogonal | | | | | | | | • | • | | | | | | |
| Symmetry | • | • | • | • | | | | • | • | • | • | • | • | • | • |
| Asymmetry | | | | | • | | | | | | | | | | |
| Near symmetry | | | | | | • | • | | | | | | | | |
| Arbitrary number of vanishing moments | | | | | • | • | • | • | • | | | | | | |
| Vanishing moments for $\phi$ | | | | | | | • | | | | | | | | |
| Existence of $\phi$ | | | • | • | • | • | • | • | • | | | | | | |
| Orthogonal analysis | | | • | • | • | • | • | | | | | | | | |
| Biorthogonal analysis | | | • | • | • | • | • | • | • | | | | | | |
| Exact reconstruction | ≈ | • | • | • | • | • | • | • | • | • | ≈ | • | • | • | • |
| FIR filters | | | | • | • | • | • | • | • | | • | | | | |
| Continuous transform | • | • | • | • | • | • | • | • | • | • | | | | | |
| Discrete transform | | | • | • | • | • | • | • | • | | • | | | | |
| Fast algorithm | | | | • | • | • | • | • | • | | • | | | | |
| Explicit expression | • | • | | • | | | | For splines | For splines | • | | • | • | • | • |
| Complex valued | | | | | | | | | | | | • | • | • | • |
| Complex continuous transform | | | | | | | | | | | | • | • | • | • |
| FIR-based approximation | | | | | | | | | | | • | | | | |

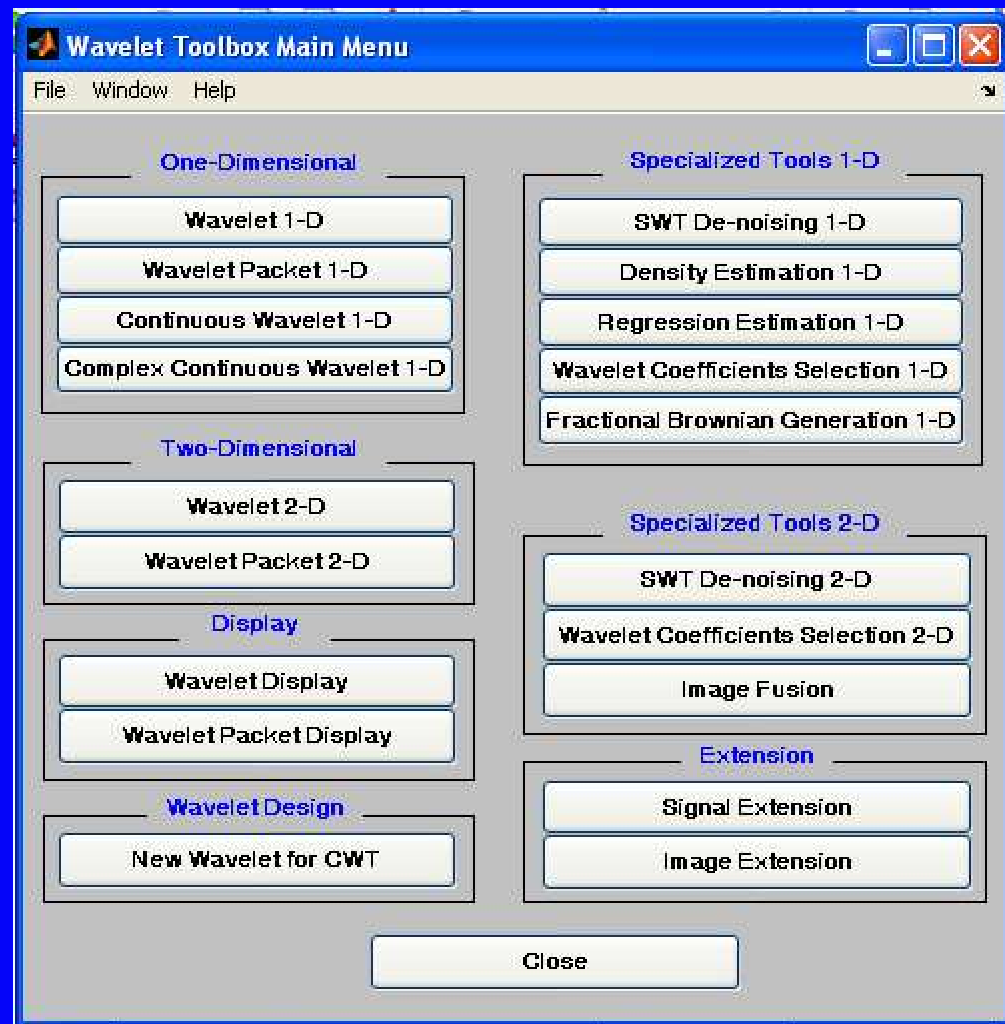
# WAVELET SOFTWARE

- A Lot of Toolboxes and Software have been Developed
- One of the Most Popular Ones is the MATLAB Wavelet Toolbox

http://www.mathworks.com/access/helpdesk/help/toolbox/wavelet/wavelet.html

# GUI VERSION IN MATLAB

- Graphical User Interfaces
- From the MATLAB prompt, type wavemenu, the Wavelet Toolbox Main Menu appears



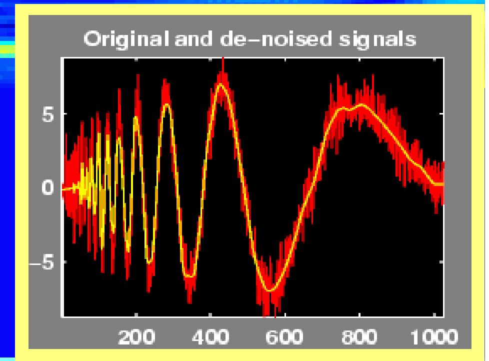
# OTHER SOFTWARE SOURCES

- WaveLib [http://www-sim.int-evry.fr/~bourges/WaveLib.html]
- EPIC [http://www.cis.upenn.edu/~eero/epic.html]
- Imager Wavelet Library [http://www.cs.ubc.ca/nest/imager/contributions/bobl/wvlt/download/]
- Mathematica wavelet programs [http://timna.Mines.EDU/wavelets/]
- Morletpackage [ftp://ftp.nosc.mil/pub/Shensa/]
- p-wavelets [ftp://pandemonium.physics.missouri.edu/pub/wavelets/]
- WaveLab [http://playfair.Stanford.EDU/~wavelab/]
- Rice Wavelet Tools [http://jazz.rice.edu/RWT/]
- Uvi_Wave Software [http://www.tsc.uvigo.es/~wavelets/uvi_wave.html]
- WAVBOX [ftp://simplicity.stanford.edu/pub/taswell/]
- Wavecompress [ftp://ftp.nosc.mil/pub/Shensa/]
- WaveThresh[http://www.stats.bris.ac.uk/pub/software/wavethresh/WaveThresh.html]
- WPLIB [ftp://pascal.math.yale.edu/pub/wavelets/software/wplib/]
- W-Transform Matlab Toolbox [ftp://info.mcs.anl.gov/pub/W-transform/]
- XWPL [ftp://pascal.math.yale.edu/pub/wavelets/software/xwpl/]
- …

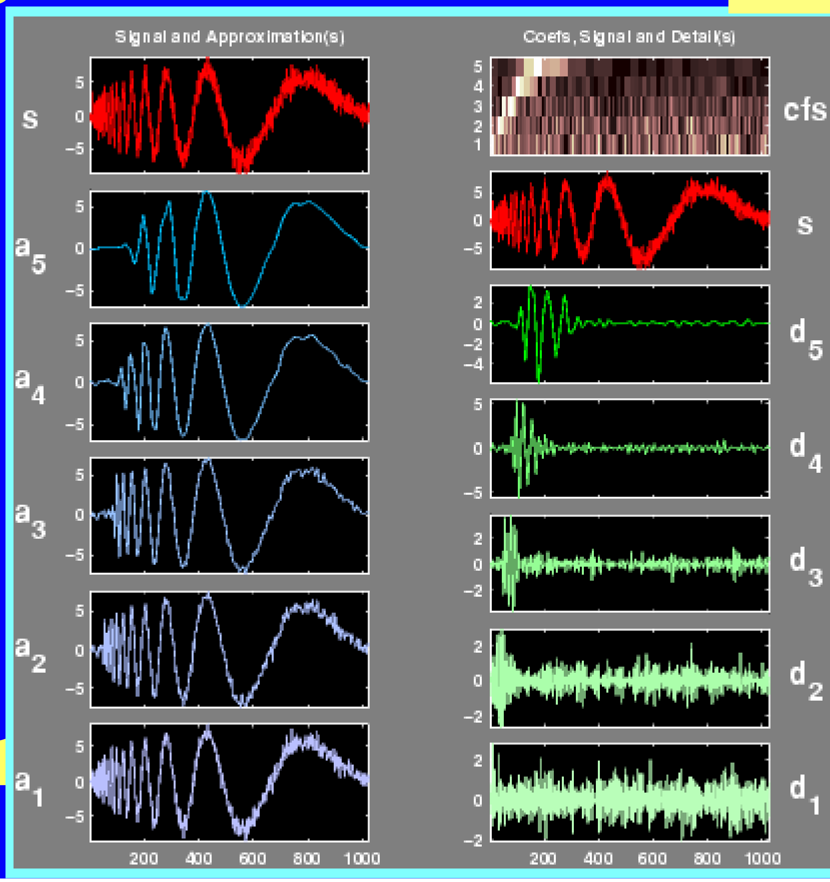
# WAVELET APPLICATIONS

- **Typical Application Fields**
  - ♥ Astronomy, acoustics, nuclear engineering, sub-band coding, signal and image processing, neurophysiology, music, magnetic resonance imaging, speech discrimination, optics, fractals, turbulence, earthquake-prediction, radar, human vision, and pure mathematics applications
- **Sample Applications**
  - ♥ Identifying pure frequencies
  - ♥ De-noising signals
  - ♥ Detecting discontinuities and breakdown points
  - ♥ Detecting self-similarity
  - ♥ Compressing images

# DE-NOISING SIGNALS




- **Highest Frequencies Appear at the Start of The Original Signal**
- **Approximations Appear Less and Less Noisy**
- **Also Lose Progressively More High-frequency Information.**
- **In $A_5$, About the First 20% of the Signal is Truncated**




**Example Analysis**
Noisy Doppler

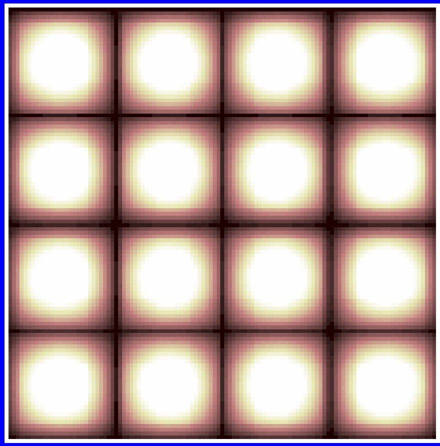
**MAT-file**
noisdopp.mat

**Wavelet**
sym4

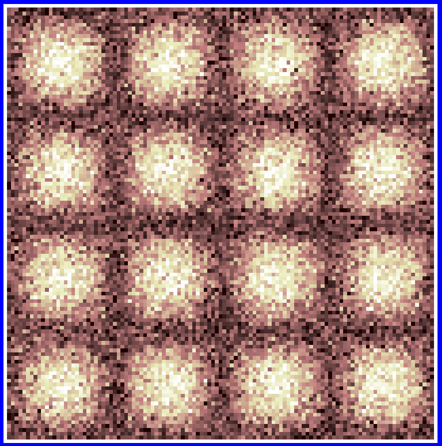
**Level**
5

# ANOTHER DE-NOISING

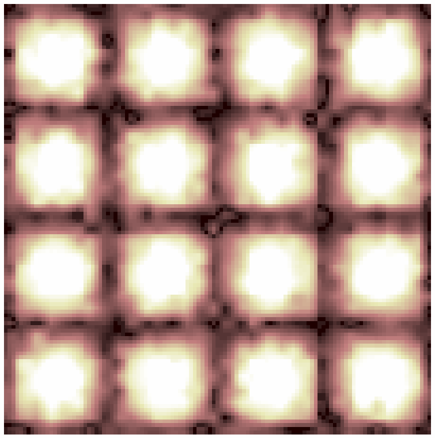
Original Image



Noisy Image



De-noised Image



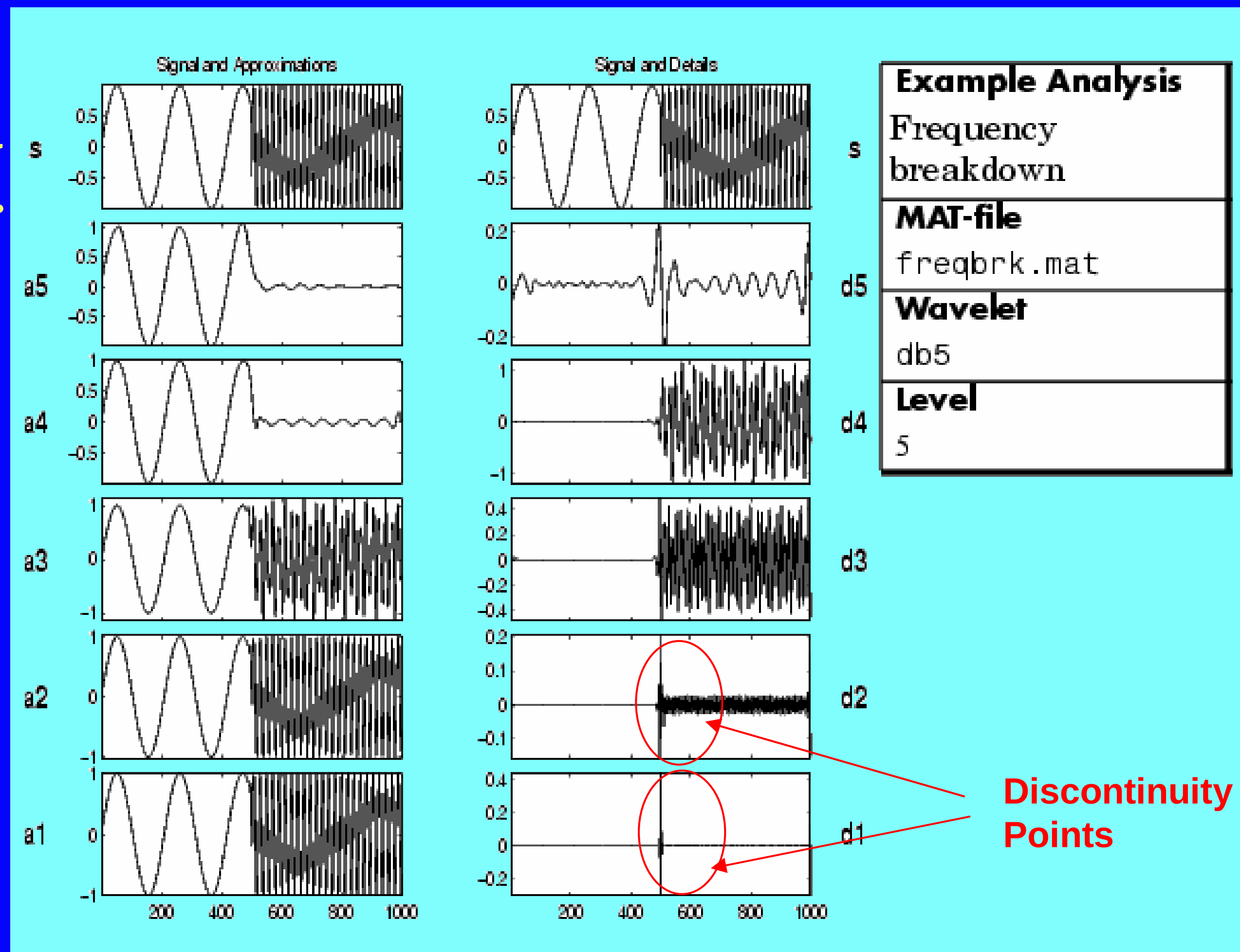
```
% Use wdencmp for image de-noising.

    % find default values (see ddencmp).
    [thr,sorh,keepapp] = ddencmp('den','wv',x);

    % denoise image using global thresholding option.
    xd = wdencmp('gbl',x,'sym4',2,thr,sorh,keepapp);
```

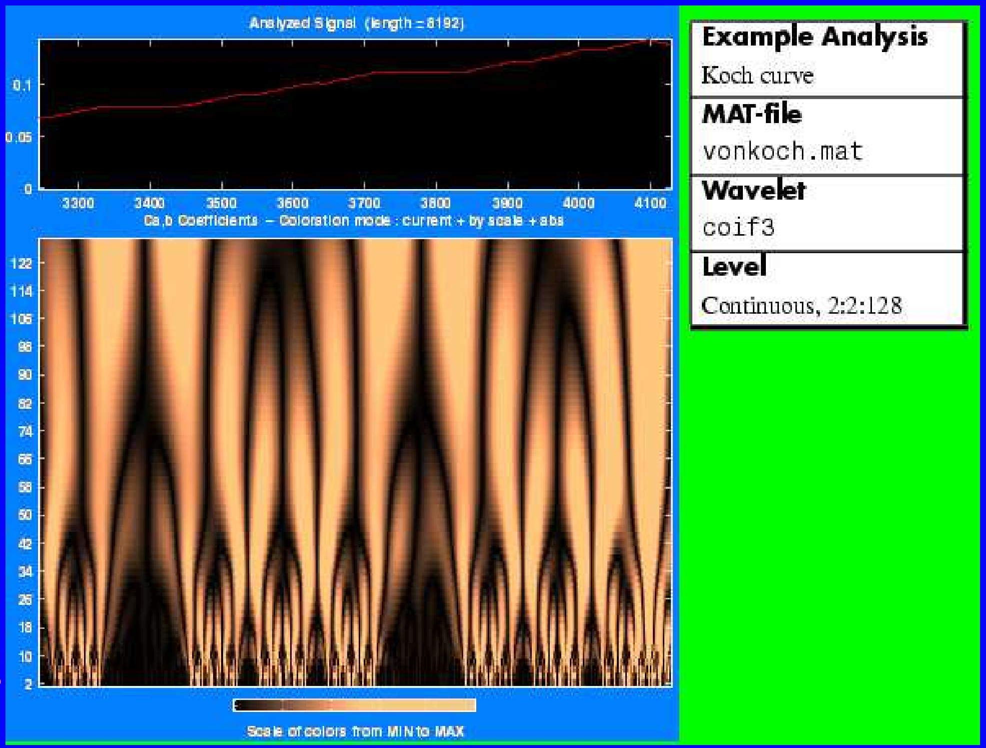
# DETECTING DISCONTINUITIES AND BREAKDOWN POINTS

- The Discontinuous Signal Consists of a Slow Sine Wave Abruptly Followed by a Medium Sine Wave.
- The 1st and 2nd Level Details ($D_1$ and $D_2$) Show the Discontinuity Most Clearly
- Things to be Detected
  - The site of the change
  - The type of change (a rupture of the signal, or an abrupt change in its first or second derivative)
  - The amplitude of the change



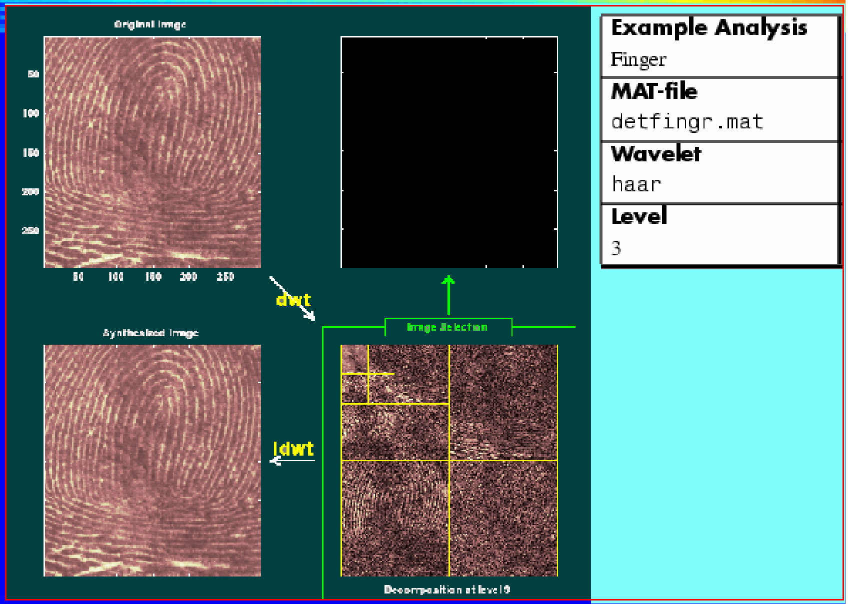
# DETECTING SELF-SIMILARITY

- **Purpose**
  - How analysis by wavelets can detect a self-similar, or fractal, signal.
  - The signal here is the Koch curve -- a synthetic signal that is built recursively
- **Analysis**
  - If a signal is similar to itself at different scales, then the "resemblance index" or wavelet coefficients also will be similar at different scales.
  - In the coefficients plot, which shows scale on the vertical axis, this self-similarity generates a characteristic pattern.

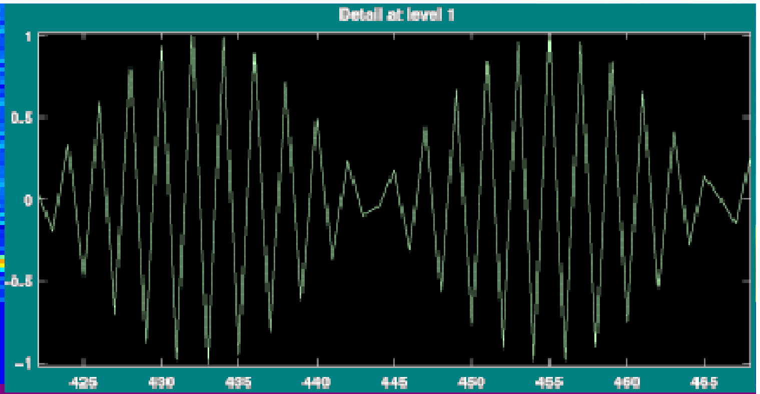


# COMPRESSING IMAGES

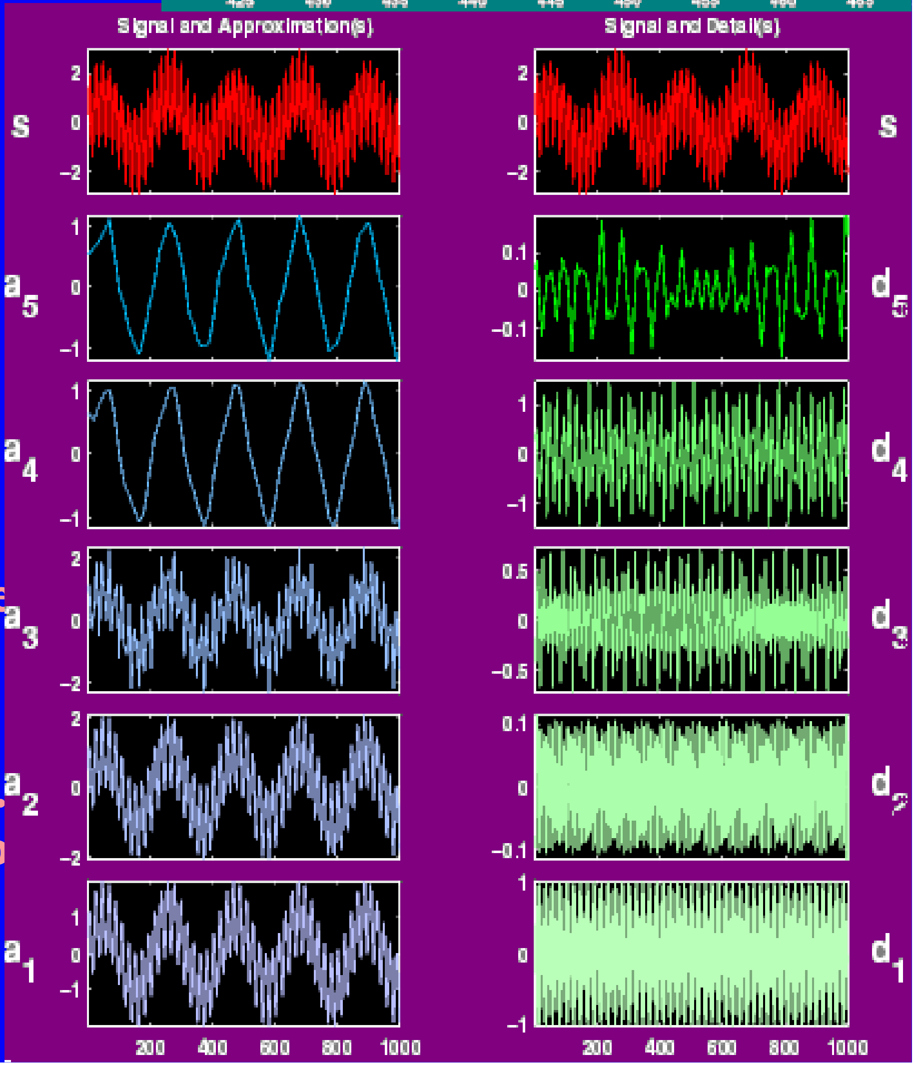
- **Fingerprints**
  - FBI maintains a large database of fingerprints — about 30 million sets of them.
  - The cost of storing all this data runs to hundreds of millions of dollars.
- **Results**
  - Values under the threshold are forced to zero, achieving about 42% zeros while retaining almost all (99.96%) the energy of the original image.
  - By turning to wavelets, the FBI has achieved a 15:1 compression ratio
  - better than the more traditional JPEG compression




# IDENTIFYING PURE FREQUENCIES




- **Purpose**
  - Resolving a signal into constituent sinusoids of different frequencies
  - The signal is a sum of three pure sine waves
- **Analysis**
  - D1 contains signal components whose period is between 1 and 2.
  - Zooming in on detail D1 reveals that each "belly" is composed of 10 oscillations.
  - D3 and D4 contain the medium sine frequencies.
  - There is a breakdown between approximations A3 and A4 -> The medium frequency been subtracted.
  - Approximations A1 to A3 be used to estimate the medium sine.
  - Zooming in on A1 reveals a period of around 20.



# SUMMARY

- Historical Background Introduced
- Frequency Domain Analysis Help to See any Information that is not Obvious in Time-domain
- Traditional Fourier Transform (FT) cannot Tell where a Frequency Starts and Ends
- Short-Term Fourier Transform (STFT) Uses Unchanged Windows, cannot Solve the Resolution Problem
- Continuous Wavelet Transform (CWT), Uses Wavelets as Windows with Altered Frequency and Time Resolutions
- Discrete Wavelet Transform (DWT) is more Effective and Faster
- Many Wavelet Families have been Developed with Different Properties
- A lot of Software are available, which Enable more Developments and Applications of Wavelet
- Wavelet Transform can be used in many Fields including Mathematics, Science, Engineering, Astronomy, …
- This Tutorial does not Cover all the Areas of Wavelet
- The theories and applications of wavelet is still in developing

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Questions ?