

# Introductions to Sech

## Introduction to the hyperbolic functions

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### General

The six well-known hyperbolic functions are the hyperbolic sine  $\sinh(z)$ , hyperbolic cosine  $\cosh(z)$ , hyperbolic tangent  $\tanh(z)$ , hyperbolic cotangent  $\coth(z)$ , hyperbolic cosecant  $\operatorname{csch}(z)$ , and hyperbolic secant  $\operatorname{sech}(z)$ . They are among the most used elementary functions. The hyperbolic functions share many common properties and they have many properties and formulas that are similar to those of the trigonometric functions.

### Definitions of the hyperbolic functions

All hyperbolic functions can be defined as simple rational functions of the exponential function of  $z$ :

$$\sinh(z) = \frac{e^z - e^{-z}}{2}$$

$$\cosh(z) = \frac{e^z + e^{-z}}{2}$$

$$\tanh(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

$$\coth(z) = \frac{e^z + e^{-z}}{e^z - e^{-z}}$$

$$\operatorname{csch}(z) = \frac{2}{e^z - e^{-z}}$$

$$\operatorname{sech}(z) = \frac{2}{e^z + e^{-z}}.$$

The functions  $\tanh(z)$ ,  $\coth(z)$ ,  $\operatorname{csch}(z)$ , and  $\operatorname{sech}(z)$  can also be defined through the functions  $\sinh(z)$  and  $\cosh(z)$  using the following formulas:

$$\tanh(z) = \frac{\sinh(z)}{\cosh(z)}$$

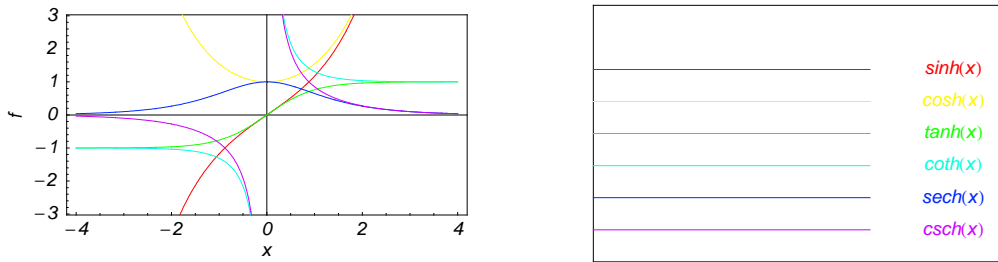
$$\coth(z) = \frac{\cosh(z)}{\sinh(z)}$$

$$\operatorname{csch}(z) = \frac{1}{\sinh(z)}$$

$$\operatorname{sech}(z) = \frac{1}{\cosh(z)}.$$

### A quick look at the hyperbolic functions

Here is a quick look at the graphics of the six hyperbolic functions along the real axis.



## Connections within the group of hyperbolic functions and with other function groups

### Representations through more general functions

The hyperbolic functions are particular cases of more general functions. Among these more general functions, four classes of special functions are of special relevance: Bessel, Jacobi, Mathieu, and hypergeometric functions.

For example,  $\sinh(z)$  and  $\cosh(z)$  have the following representations through Bessel, Mathieu, and hypergeometric functions:

$$\sinh(z) = -i \sqrt{\frac{\pi iz}{2}} J_{1/2}(iz) \quad \sinh(z) = \sqrt{\frac{\pi z}{2}} I_{1/2}(z) \quad \sinh(z) = -i \sqrt{\frac{\pi iz}{2}} Y_{-1/2}(iz) \quad \sinh(z) = \frac{1}{\sqrt{2\pi}} (\sqrt{-z} K_{1/2}(-z) - \sqrt{z})$$

$$\cosh(z) = \sqrt{\frac{\pi iz}{2}} J_{-1/2}(iz) \quad \cosh(z) = \sqrt{\frac{\pi z}{2}} I_{-1/2}(z) \quad \cosh(z) = -\sqrt{\frac{\pi iz}{2}} Y_{1/2}(iz) \quad \cosh(z) = \frac{1}{\sqrt{2\pi}} (\sqrt{-z} K_{1/2}(-z) + \sqrt{z})$$

$$\sinh(z) = -i \operatorname{Se}(1, 0, iz) \quad \cosh(z) = \operatorname{Ce}(1, 0, iz)$$

$$\sinh(z) = {}_zF_1\left(\frac{3}{2}; \frac{z^2}{4}\right) \quad \cosh(z) = {}_0F_1\left(\frac{1}{2}; \frac{z^2}{4}\right).$$

All hyperbolic functions can be represented as degenerate cases of the corresponding doubly periodic Jacobi elliptic functions when their second parameter is equal to 0 or 1:

$$\begin{aligned} \sinh(z) &= -i \operatorname{sd}(iz | 0) = -i \operatorname{sn}(iz | 0) & \sinh(z) &= \operatorname{sc}(z | 1) = \operatorname{sd}(z | 1) \\ \cosh(z) &= \operatorname{cd}(iz | 0) = \operatorname{cn}(iz | 0) & \cosh(z) &= \operatorname{nc}(z | 1) = \operatorname{nd}(z | 1) \\ \tanh(z) &= -i \operatorname{sc}(iz | 0) & \tanh(z) &= \operatorname{sn}(z | 1) \\ \operatorname{coth}(z) &= i \operatorname{cs}(iz | 0) & \operatorname{coth}(z) &= \operatorname{ns}(z | 1) \\ \operatorname{csch}(z) &= i \operatorname{ds}(iz | 0) = i \operatorname{ns}(iz | 0) & \operatorname{csch}(z) &= \operatorname{cs}(z | 1) = \operatorname{ds}(z | 1) \\ \operatorname{sech}(z) &= \operatorname{dc}(iz | 0) = \operatorname{nc}(iz | 0) & \operatorname{sech}(z) &= \operatorname{cn}(z | 1) = \operatorname{dn}(z | 1). \end{aligned}$$

### Representations through related equivalent functions

Each of the six hyperbolic functions can be represented through the corresponding trigonometric function:

$$\begin{aligned} \sinh(z) &= -i \sin(iz) & \sinh(iz) &= i \sin(z) \\ \cosh(z) &= \cos(iz) & \cosh(iz) &= \cos(z) \\ \tanh(z) &= -i \tan(iz) & \tanh(iz) &= i \tan(z) \\ \operatorname{coth}(z) &= i \cot(iz) & \operatorname{coth}(iz) &= -i \cot(z) \\ \operatorname{csch}(z) &= i \operatorname{csc}(iz) & \operatorname{csch}(iz) &= -i \operatorname{csc}(z) \\ \operatorname{sech}(z) &= \operatorname{sec}(iz) & \operatorname{sech}(iz) &= \operatorname{sec}(z). \end{aligned}$$

### Relations to inverse functions

Each of the six hyperbolic functions is connected with a corresponding inverse hyperbolic function by two formulas. One direction can be expressed through a simple formula, but the other direction is much more complicated because of the multivalued nature of the inverse function:

$$\sinh(\sinh^{-1}(z)) = z \quad \sinh^{-1}(\sinh(z)) = z /; -\frac{\pi}{2} < \text{Im}(z) < \frac{\pi}{2} \bigvee \left( \text{Im}(z) = -\frac{\pi}{2} \wedge \text{Re}(z) \leq 0 \right) \bigvee \left( \text{Im}(z) = \frac{\pi}{2} \wedge \text{Re}(z) \geq 0 \right)$$

$$\cosh(\cosh^{-1}(z)) = z \quad \cosh^{-1}(\cosh(z)) = z /; \text{Re}(z) > 0 \wedge -\pi < \text{Im}(z) \leq \pi \vee \text{Re}(z) = 0 \wedge \text{Im}(z) \geq 0$$

$$\tanh(\tanh^{-1}(z)) = z \quad \tanh^{-1}(\tanh(z)) = z /; -\frac{\pi}{2} < \text{Im}(z) < \frac{\pi}{2} \bigvee \left( \text{Im}(z) = -\frac{\pi}{2} \wedge \text{Re}(z) > 0 \right) \bigvee \left( \text{Im}(z) = \frac{\pi}{2} \wedge \text{Re}(z) < 0 \right)$$

$$\coth(\coth^{-1}(z)) = z \quad \coth^{-1}(\coth(z)) = z /; -\frac{\pi}{2} < \text{Im}(z) < \frac{\pi}{2} \bigvee \left( \text{Im}(z) = -\frac{\pi}{2} \wedge \text{Re}(z) > 0 \right) \bigvee \left( \text{Im}(z) = \frac{\pi}{2} \wedge \text{Re}(z) \leq 0 \right)$$

$$\text{csch}(\text{csch}^{-1}(z)) = z \quad \text{csch}^{-1}(\text{csch}(z)) = z /; -\frac{\pi}{2} < \text{Im}(z) < \frac{\pi}{2} \bigvee \left( \text{Im}(z) = -\frac{\pi}{2} \wedge \text{Re}(z) \leq 0 \right) \bigvee \left( \text{Im}(z) = \frac{\pi}{2} \wedge \text{Re}(z) \geq 0 \right)$$

$$\text{sech}(\text{sech}^{-1}(z)) = z \quad \text{sech}^{-1}(\text{sech}(z)) = z /; -\pi < \text{Im}(z) \leq \pi \wedge \text{Re}(z) > 0 \vee \text{Re}(z) = 0 \wedge \text{Im}(z) \geq 0.$$

### Representations through other hyperbolic functions

Each of the six hyperbolic functions can be represented through any other function as a rational function of that function with a linear argument. For example, the hyperbolic sine can be representative as a group-defining function because the other five functions can be expressed as:

$$\cosh(z) = -i \sinh\left(\frac{\pi i}{2} - z\right) \quad \cosh^2(z) = 1 + \sinh^2(z)$$

$$\tanh(z) = \frac{\sinh(z)}{\cosh(z)} = \frac{i \sinh(z)}{\sinh\left(\frac{\pi i}{2} - z\right)} \quad \tanh^2(z) = \frac{\sinh^2(z)}{1 + \sinh^2(z)}$$

$$\coth(z) = \frac{\cosh(z)}{\sinh(z)} = -\frac{i \sinh\left(\frac{\pi i}{2} - z\right)}{\sinh(z)} \quad \coth^2(z) = \frac{1 + \sinh^2(z)}{\sinh^2(z)}$$

$$\text{csch}(z) = \frac{1}{\sinh(z)} \quad \text{csch}^2(z) = \frac{1}{\sinh^2(z)}$$

$$\text{sech}(z) = \frac{1}{\cosh(z)} = \frac{i}{\sinh\left(\frac{\pi i}{2} - z\right)} \quad \text{sech}^2(z) = \frac{1}{1 + \sinh^2(z)}.$$

All six hyperbolic functions can be transformed into any other function of the group of hyperbolic functions if the argument  $z$  is replaced by  $p \pi i / 2 + q z$  with  $q^2 = 1 \wedge p \in \mathbb{Z}$ :

$$\sinh(-z - 2 \pi i) = -\sinh(z) \quad \sinh(z - 2 \pi i) = \sinh(z)$$

$$\sinh\left(-z - \frac{3 \pi i}{2}\right) = i \cosh(z) \quad \sinh\left(z - \frac{3 \pi i}{2}\right) = i \cosh(z)$$

$$\sinh(-z - \pi i) = \sinh(z) \quad \sinh(z - \pi i) = -\sinh(z)$$

$$\sinh\left(-z - \frac{\pi i}{2}\right) = -i \cosh(z) \quad \sinh\left(z - \frac{\pi i}{2}\right) = -i \cosh(z)$$

$$\sinh\left(z + \frac{\pi i}{2}\right) = i \cosh(z) \quad \sinh\left(\frac{\pi i}{2} - z\right) = i \cosh(z)$$

$$\sinh(z + \pi i) = -\sinh(z) \quad \sinh(\pi i - z) = \sinh(z)$$

$$\sinh\left(z + \frac{3 \pi i}{2}\right) = -i \cosh(z) \quad \sinh\left(\frac{3 \pi i}{2} - z\right) = -i \cosh(z)$$

$$\sinh(z + 2 \pi i) = \sinh(z) \quad \sinh(2 \pi i - z) = -\sinh(z)$$

$$\begin{aligned} \cosh(-z-2\pi i) &= \cosh(z) & \cosh(z-2\pi i) &= \cosh(z) \\ \cosh\left(-z-\frac{3\pi i}{2}\right) &= -i \sinh(z) & \cosh\left(z-\frac{3\pi i}{2}\right) &= i \sinh(z) \\ \cosh(-z-\pi i) &= -\cosh(z) & \cosh(z-\pi i) &= -\cosh(z) \\ \cosh\left(-z-\frac{\pi i}{2}\right) &= i \sinh(z) & \cosh\left(z-\frac{\pi i}{2}\right) &= -i \sinh(z) \\ \cosh\left(z+\frac{\pi i}{2}\right) &= i \sinh(z) & \cosh\left(\frac{\pi i}{2}-z\right) &= -i \sinh(z) \\ \cosh(z+\pi i) &= -\cosh(z) & \cosh(\pi i-z) &= -\cosh(z) \\ \cosh\left(z+\frac{3\pi i}{2}\right) &= -i \sinh(z) & \cosh\left(\frac{3\pi i}{2}-z\right) &= i \sinh(z) \\ \cosh(z+2\pi i) &= \cosh(z) & \cosh(2\pi i-z) &= \cosh(z) \end{aligned}$$

$$\begin{aligned} \tanh(-z-\pi i) &= -\tanh(z) & \tanh(z-\pi i) &= \tanh(z) \\ \tanh\left(-z-\frac{\pi i}{2}\right) &= -\coth(z) & \tanh\left(z-\frac{\pi i}{2}\right) &= \coth(z) \\ \tanh\left(z+\frac{\pi i}{2}\right) &= \coth(z) & \tanh\left(\frac{\pi i}{2}-z\right) &= -\coth(z) \\ \tanh(z+\pi i) &= \tanh(z) & \tanh(\pi i-z) &= -\tanh(z) \end{aligned}$$

$$\begin{aligned} \coth(-z-\pi i) &= -\coth(z) & \coth(z-\pi i) &= \coth(z) \\ \coth\left(-z-\frac{\pi i}{2}\right) &= -\tanh(z) & \coth\left(z-\frac{\pi i}{2}\right) &= \tanh(z) \\ \coth\left(z+\frac{\pi i}{2}\right) &= \tanh(z) & \coth\left(\frac{\pi i}{2}-z\right) &= -\tanh(z) \\ \coth(z+\pi i) &= \coth(z) & \coth(\pi i-z) &= -\coth(z) \end{aligned}$$

$$\begin{aligned} \operatorname{csch}(-z-2\pi i) &= -\operatorname{csch}(z) & \operatorname{csch}(z-2\pi i) &= \operatorname{csch}(z) \\ \operatorname{csch}\left(-z-\frac{3\pi i}{2}\right) &= -i \operatorname{sech}(z) & \operatorname{csch}\left(z-\frac{3\pi i}{2}\right) &= -i \operatorname{sech}(z) \\ \operatorname{csch}(-z-\pi i) &= \operatorname{csch}(z) & \operatorname{csch}(z-\pi i) &= -\operatorname{csch}(z) \\ \operatorname{csch}\left(-z-\frac{\pi i}{2}\right) &= i \operatorname{sech}(z) & \operatorname{csch}\left(z-\frac{\pi i}{2}\right) &= i \operatorname{sech}(z) \\ \operatorname{csch}\left(z+\frac{\pi i}{2}\right) &= -i \operatorname{sech}(z) & \operatorname{csch}\left(\frac{\pi i}{2}-z\right) &= -i \operatorname{sech}(z) \\ \operatorname{csch}(z+\pi i) &= -\operatorname{csch}(z) & \operatorname{csch}(\pi i-z) &= \operatorname{csch}(z) \\ \operatorname{csch}\left(z+\frac{3\pi i}{2}\right) &= i \operatorname{sech}(z) & \operatorname{csch}\left(\frac{3\pi i}{2}-z\right) &= i \operatorname{sech}(z) \\ \operatorname{csch}(z+2\pi i) &= \operatorname{csch}(z) & \operatorname{csch}(2\pi i-z) &= -\operatorname{csch}(z) \end{aligned}$$

$$\begin{aligned} \operatorname{sech}(-z-2\pi i) &= \operatorname{sech}(z) & \operatorname{sech}(z-2\pi i) &= \operatorname{sech}(z) \\ \operatorname{sech}\left(-z-\frac{3\pi i}{2}\right) &= i \operatorname{csch}(z) & \operatorname{sech}\left(z-\frac{3\pi i}{2}\right) &= -i \operatorname{csch}(z) \\ \operatorname{sech}(-z-\pi i) &= -\operatorname{sech}(z) & \operatorname{sech}(z-\pi i) &= -\operatorname{sech}(z) \\ \operatorname{sech}\left(-z-\frac{\pi i}{2}\right) &= -i \operatorname{csch}(z) & \operatorname{sech}\left(z-\frac{\pi i}{2}\right) &= i \operatorname{csch}(z) \\ \operatorname{sech}\left(z+\frac{\pi i}{2}\right) &= -i \operatorname{csch}(z) & \operatorname{sech}\left(\frac{\pi i}{2}-z\right) &= i \operatorname{csch}(z) \\ \operatorname{sech}(z+\pi i) &= -\operatorname{sech}(z) & \operatorname{sech}(\pi i-z) &= -\operatorname{sech}(z) \\ \operatorname{sech}\left(z+\frac{3\pi i}{2}\right) &= i \operatorname{csch}(z) & \operatorname{sech}\left(\frac{3\pi i}{2}-z\right) &= -i \operatorname{csch}(z) \\ \operatorname{sech}(z+2\pi i) &= \operatorname{sech}(z) & \operatorname{sech}(2\pi i-z) &= \operatorname{sech}(z). \end{aligned}$$

## The best-known properties and formulas for hyperbolic functions

### Real values for real arguments

For real values of argument  $z$ , the values of all the hyperbolic functions are real (or infinity).

In the points  $z = 2\pi n i / m$ ;  $n \in \mathbb{Z} \wedge m \in \mathbb{Z}$ , the values of the hyperbolic functions are algebraic. In several cases, they can even be rational numbers, 1, or  $i$  (e.g.  $\sinh(\pi i / 2) = i$ ,  $\operatorname{sech}(0) = 1$ , or  $\cosh(\pi i / 3) = 1/2$ ). They can be expressed using only square roots if  $n \in \mathbb{Z}$  and  $m$  is a product of a power of 2 and distinct Fermat primes  $\{3, 5, 17, 257, \dots\}$ .

### Simple values at zero

All hyperbolic functions has rather simple values for arguments  $z = 0$  and  $z = \pi i / 2$ :

$$\begin{aligned} \sinh(0) &= 0 & \sinh\left(\frac{\pi i}{2}\right) &= i \\ \cosh(0) &= 1 & \cosh\left(\frac{\pi i}{2}\right) &= 0 \\ \tanh(0) &= 0 & \tanh\left(\frac{\pi i}{2}\right) &= \infty \\ \coth(0) &= \infty & \coth\left(\frac{\pi i}{2}\right) &= 0 \\ \operatorname{csch}(0) &= \infty & \operatorname{csch}\left(\frac{\pi i}{2}\right) &= -i \\ \operatorname{sech}(0) &= 1 & \operatorname{sech}\left(\frac{\pi i}{2}\right) &= \infty. \end{aligned}$$

### Analyticity

All hyperbolic functions are defined for all complex values of  $z$ , and they are analytical functions of  $z$  over the whole complex  $z$ -plane and do not have branch cuts or branch points. The two functions  $\sinh(z)$  and  $\cosh(z)$  are entire functions with an essential singular point at  $z = \infty$ . All other hyperbolic functions are meromorphic functions with simple poles at points  $z = \pi k i$ ;  $k \in \mathbb{Z}$  (for  $\operatorname{csch}(z)$  and  $\coth(z)$ ) and at points  $z = \pi i / 2 + \pi k i$ ;  $k \in \mathbb{Z}$  (for  $\operatorname{sech}(z)$  and  $\tanh(z)$ ).

### Periodicity

All hyperbolic functions are periodic functions with a real period ( $2\pi i$  or  $\pi i$ ):

$$\begin{aligned} \sinh(z) &= \sinh(z + 2\pi i) & \sinh(z + 2\pi i k) &= \sinh(z) /; k \in \mathbb{Z} \\ \cosh(z) &= \cosh(z + 2\pi i) & \cosh(z + 2\pi i k) &= \cosh(z) /; k \in \mathbb{Z} \\ \tanh(z) &= \tanh(z + \pi i) & \tanh(z + \pi i k) &= \tanh(z) /; k \in \mathbb{Z} \\ \coth(z) &= \coth(z + \pi i) & \coth(z + \pi i k) &= \coth(z) /; k \in \mathbb{Z} \\ \operatorname{csch}(z) &= \operatorname{csch}(z + 2\pi i) & \operatorname{csch}(z + 2\pi i k) &= \operatorname{csch}(z) /; k \in \mathbb{Z} \\ \operatorname{sech}(z) &= \operatorname{sech}(z + 2\pi i) & \operatorname{sech}(z + 2\pi i k) &= \operatorname{sech}(z) /; k \in \mathbb{Z}. \end{aligned}$$

### Parity and symmetry

All hyperbolic functions have parity (either odd or even) and mirror symmetry:

$$\begin{aligned} \sinh(-z) &= -\sinh(z) & \sinh(\bar{z}) &= \overline{\sinh(z)} \\ \cosh(-z) &= \cosh(z) & \cosh(\bar{z}) &= \overline{\cosh(z)} \\ \tanh(-z) &= -\tanh(z) & \tanh(\bar{z}) &= \overline{\tanh(z)} \\ \coth(-z) &= -\coth(z) & \coth(\bar{z}) &= \overline{\coth(z)} \\ \operatorname{csch}(-z) &= -\operatorname{csch}(z) & \operatorname{csch}(\bar{z}) &= \overline{\operatorname{csch}(z)} \\ \operatorname{sech}(-z) &= \operatorname{sech}(z) & \operatorname{sech}(\bar{z}) &= \overline{\operatorname{sech}(z)}. \end{aligned}$$

### Simple representations of derivatives

The derivatives of all hyperbolic functions have simple representations that can be expressed through other hyperbolic functions:

$$\begin{aligned} \frac{\partial \sinh(z)}{\partial z} &= \cosh(z) & \frac{\partial \cosh(z)}{\partial z} &= \sinh(z) & \frac{\partial \tanh(z)}{\partial z} &= \operatorname{sech}^2(z) \\ \frac{\partial \coth(z)}{\partial z} &= -\operatorname{csch}^2(z) & \frac{\partial \operatorname{csch}(z)}{\partial z} &= -\coth(z) \operatorname{csch}(z) & \frac{\partial \operatorname{sech}(z)}{\partial z} &= -\operatorname{sech}(z) \tanh(z). \end{aligned}$$

### Simple differential equations

The solutions of the simplest second-order linear ordinary differential equation with constant coefficients can be represented through  $\sinh(z)$  and  $\cosh(z)$ . The other hyperbolic functions satisfy first-order nonlinear differential equations:

$$\begin{aligned} w''(z) - w(z) &= 0 ; w(z) = \cosh(z) \wedge w(0) = 1 \wedge w'(0) = 0 \\ w''(z) - w(z) &= 0 ; w(z) = \sinh(z) \wedge w(0) = 0 \wedge w'(0) = 1 \\ w''(z) - w(z) &= 0 ; w(z) = c_1 \cosh(z) + c_2 \sinh(z). \end{aligned}$$

All six hyperbolic functions satisfy first-order nonlinear differential equations:

$$\begin{aligned} w'(z) - \sqrt{1 + (w(z))^2} &= 0 ; w(z) = \sinh(z) \wedge w(0) = 0 \wedge |\operatorname{Im}(z)| < \frac{\pi}{2} \\ w'(z) - \sqrt{-1 + (w(z))^2} &= 0 ; w(z) = \cosh(z) \wedge w(0) = 1 \wedge |\operatorname{Im}(z)| < \frac{\pi}{2} \\ w'(z) + w(z)^2 - 1 &= 0 ; w(z) = \tanh(z) \wedge w(0) = 0 \\ w'(z) + w(z)^2 - 1 &= 0 ; w(z) = \coth(z) \wedge w\left(\frac{\pi i}{2}\right) = 0 \\ w'(z)^2 - w(z)^4 - w(z)^2 &= 0 ; w(z) = \operatorname{csch}(z) \\ w'(z)^2 + w(z)^4 - w(z)^2 &= 0 ; w(z) = \operatorname{sech}(z). \end{aligned}$$

### Applications of hyperbolic functions

Trigonometric functions are intimately related to triangle geometry. Functions like sine and cosine are often introduced as edge lengths of right-angled triangles. Hyperbolic functions occur in the theory of triangles in hyperbolic spaces.

Lobachevsky (1829) and J. Bolyai (1832) independently recognized that Euclid's fifth postulate—saying that for a given line and a point not on the line, there is exactly one line parallel to the first—might be changed and still be a consistent geometry. In the hyperbolic geometry it is allowable for more than one line to be parallel to the first (meaning that the parallel lines will never meet the first, however far they are extended). Translated into triangles, this means that the sum of the three angles is always less than  $\pi$ .

A particularly nice representation of the hyperbolic geometry can be realized in the unit disk of complex numbers (the Poincaré disk model). In this model, points are complex numbers in the unit disk, and the lines are either arcs of circles that meet the boundary of the unit circle orthogonal or diameters of the unit circle.

The distance  $d$  between two points (meaning complex numbers)  $A$  and  $B$  in the Poincaré disk is:

$$d(A, B) = 2 \tanh^{-1} \left( \left| \frac{A - B}{1 - \bar{B}A} \right| \right).$$

The attractive feature of the Poincaré disk model is that the hyperbolic angles agree with the Euclidean angles. Formally, the angle  $\alpha$  at a point  $A$  of two hyperbolic lines  $\overline{AB}$  and  $\overline{AC}$  is described by the formula:

$$\cos(\alpha) = \frac{\frac{-A+B}{1-A} \frac{-A+C}{1-A}}{\left| \frac{-A+B}{1-A} \right| \left| \frac{-A+C}{1-A} \right|}.$$

In the following, the values of the three angles of an hyperbolic triangle at the vertices  $A$ ,  $B$ , and  $C$  are denoted through  $\alpha$ ,  $\beta$ , and  $\gamma$ . The hyperbolic length of the three edges opposite to the angles are denoted  $a$ ,  $b$ , and  $c$ .

The cosine rule and the second cosine rule for hyperbolic triangles are:

$$\sinh(b) \sinh(c) \cos(\alpha) = \cosh(b) \cosh(c) - \cosh(a)$$

$$\sinh(a) \sinh(c) \cos(\beta) = \cosh(a) \cosh(c) - \cosh(b)$$

$$\sinh(a) \sinh(b) \cos(\gamma) = \cosh(a) \cosh(b) - \cosh(c)$$

$$\sin(\beta) \sin(\gamma) \cosh(a) = \cos(\beta) \cos(\gamma) + \cos(\alpha)$$

$$\sin(\alpha) \sin(\gamma) \cosh(b) = \cos(\alpha) \cos(\gamma) + \cos(\beta)$$

$$\sin(\alpha) \sin(\beta) \cosh(c) = \cos(\alpha) \cos(\beta) + \cos(\gamma).$$

The sine rule for hyperbolic triangles is:

$$\frac{\sin(\alpha)}{\sinh(a)} = \frac{\sin(\beta)}{\sinh(b)} = \frac{\sin(\gamma)}{\sinh(c)}.$$

For a right-angle triangle, the hyperbolic version of the Pythagorean theorem follows from the preceding formulas (the right angle is taken at vertex  $A$ ):

$$\cosh(a) = \cosh(b) \cosh(c).$$

Using the series expansion  $\cosh(x) \approx 1 + x^2/2$  at small scales the hyperbolic geometry is approximated by the familiar Euclidean geometry. The cosine formulas and the sine formulas for hyperbolic triangles with a right angle at vertex  $A$  become:

$$\cos(\beta) = \frac{\tanh(c)}{\tanh(a)}, \quad \sin(\beta) = \frac{\sinh(b)}{\sinh(a)}$$

$$\cos(\gamma) = \frac{\tanh(b)}{\tanh(a)}, \quad \sin(\gamma) = \frac{\sinh(c)}{\sinh(a)}.$$

The inscribed circle has the radius:

$$\rho = \sqrt{\tanh^{-1} \left( \frac{\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) + 2 \cos(\alpha) \cos(\beta) \cos(\gamma) - 1}{2(1 + \cos(\alpha))(1 + \cos(\beta))(1 + \cos(\gamma))} \right)}.$$

The circumscribed circle has the radius:

$$\rho = \tanh^{-1} \left( \frac{4 \sinh\left(\frac{a}{2}\right) \sinh\left(\frac{b}{2}\right) \sinh\left(\frac{c}{2}\right)}{\sin(\gamma) \sinh(a) \sinh(b)} \right).$$

**Other applications**

As rational functions of the exponential function, the hyperbolic functions appear virtually everywhere in quantitative sciences. It is impossible to list their numerous applications in teaching, science, engineering, and art.

**Introduction to the Hyperbolic Secant Function**

**Defining the hyperbolic secant function**

The hyperbolic secant function is an old mathematical function.

This function is easily defined as the ratio of one and hyperbolic cosine functions:

$$\operatorname{sech}(z) = \frac{1}{\cosh(z)} = \frac{2}{e^z + e^{-z}}.$$

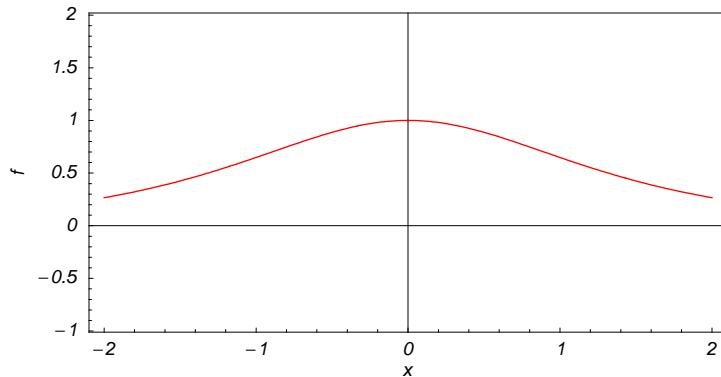
After comparison with the famous Euler formula for the cosine function  $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$ , it is easy to derive the following representation of the hyperbolic secant through the circular secant:

$$\operatorname{sech}(z) = \sec(iz).$$

The previous formula allows establishment of all the properties and formulas for the hyperbolic secant from corresponding properties and formulas for the circular secant.

**A quick look at the hyperbolic secant function**

Here is a graphic of the hyperbolic secant function  $f(x) = \operatorname{sech}(x)$  for real values of its argument  $x$ .



**Representation through more general functions**

The hyperbolic secant function  $\operatorname{sech}(z)$  can be represented using more general mathematical functions. As the reciprocal to the hyperbolic cosine function that is a particular case of the generalized hypergeometric, Bessel, Struve, and Mathieu functions, the hyperbolic secant function can also be represented as reciprocal to those special functions. Here are some examples:

$$\operatorname{sech}(z) = \frac{1}{{}_0F_1\left(\frac{1}{2}; \frac{z^2}{4}\right)} \quad \operatorname{sech}(z) = \frac{1}{\sqrt{\frac{\pi iz}{2}} J_{-\frac{1}{2}}(iz)} \quad \operatorname{sech}(z) = \frac{1}{\sqrt{\frac{\pi z}{2}} I_{-\frac{1}{2}}(z)}$$



$$\operatorname{sech}(z) = -\frac{1}{\sqrt{\frac{\pi iz}{2}} Y_{\frac{1}{2}}(iz)} \quad \operatorname{sech}(z) = \frac{1}{1 + \sqrt{\frac{\pi z}{2}} L_{\frac{1}{2}}(z)} \quad \operatorname{sech}(z) = \frac{1}{\operatorname{Ce}(1, 0, iz)}$$

But these representations are not very useful because they include complicated special functions in the denominators.

It is more useful to write the secant function as particular cases of one special function. That can be done using doubly periodic Jacobi elliptic functions that degenerate into the hyperbolic secant function when their second parameter is equal to 0 or 1:

$$\operatorname{sech}(z) = \operatorname{dc}(iz | 0) = \operatorname{nc}(iz | 0) = \operatorname{ds}\left(\frac{\pi}{2} - iz \mid 0\right) = \operatorname{ns}\left(\frac{\pi}{2} - iz \mid 0\right) = \operatorname{cn}(z | 1) = \operatorname{dn}(z | 1) = i \operatorname{cs}\left(\frac{\pi i}{2} - z \mid 1\right) = i \operatorname{ds}\left(\frac{\pi i}{2} - z \mid 1\right).$$

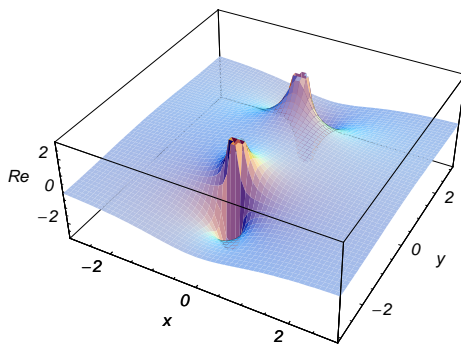
### Definition of the hyperbolic secant function for a complex argument

In the complex  $z$ -plane, the function  $\operatorname{sech}(z)$  is defined by the same formula used for real values:

$$\operatorname{sech}(z) = \frac{1}{\cosh(z)} = \frac{2}{e^z + e^{-z}}.$$

In the points  $z = \pi i / 2 + \pi k i$ ;  $k \in \mathbb{Z}$ , where  $\cosh(z)$  has zeros, the denominator of the last formula equals zero and  $\operatorname{sech}(z)$  has singularities (poles of the first order).

Here are two graphics showing the real and imaginary parts of the hyperbolic secant function over the complex plane.



### The best-known properties and formulas for the hyperbolic secant function

#### Values in points

The values of the hyperbolic secant function for special values of its argument can be easily derived from the corresponding values of the circular secant function in special points of the circle:

$$\begin{aligned} \operatorname{sech}(0) &= 1 & \operatorname{sech}\left(\frac{\pi i}{6}\right) &= \frac{2}{\sqrt{3}} & \operatorname{sech}\left(\frac{\pi i}{4}\right) &= \sqrt{2} & \operatorname{sech}\left(\frac{\pi i}{3}\right) &= 2 \\ \operatorname{sech}\left(\frac{\pi i}{2}\right) &= \infty & \operatorname{sech}\left(\frac{2\pi i}{3}\right) &= -2 & \operatorname{sech}\left(\frac{3\pi i}{4}\right) &= -\sqrt{2} & \operatorname{sech}\left(\frac{5\pi i}{6}\right) &= -\frac{2}{\sqrt{3}} \\ \operatorname{sech}(\pi i) &= -1 & \operatorname{sech}\left(\frac{7\pi i}{6}\right) &= -\frac{2}{\sqrt{3}} & \operatorname{sech}\left(\frac{5\pi i}{4}\right) &= -\sqrt{2} & \operatorname{sech}\left(\frac{4\pi i}{3}\right) &= -2 \\ \operatorname{sech}\left(\frac{3\pi i}{2}\right) &= \infty & \operatorname{sech}\left(\frac{5\pi i}{3}\right) &= 2 & \operatorname{sech}\left(\frac{7\pi i}{4}\right) &= \sqrt{2} & \operatorname{sech}\left(\frac{11\pi i}{6}\right) &= \frac{2}{\sqrt{3}} \end{aligned}$$

$$\operatorname{sech}(2\pi i) = 1 \quad \operatorname{sech}(\pi i m) = (-1)^m \quad ; \quad m \in \mathbb{Z} \quad \operatorname{sech}\left(\pi i \left(\frac{1}{2} + m\right)\right) = \infty \quad ; \quad m \in \mathbb{Z}.$$

The values at infinity can be expressed by the following formulas:

$$\operatorname{sech}(\infty) = 0 \quad \operatorname{sech}(-\infty) = 0.$$

### General characteristics

For real values of argument  $z$ , the values of  $\operatorname{sech}(z)$  are real.

In the points  $z = 2\pi n i / m$ ;  $n \in \mathbb{Z} \wedge m \in \mathbb{Z}$ , the values of  $\operatorname{sech}(z)$  are algebraic. In several cases, they can be integers  $-2, -1, 1$ , or  $2$ :

$$\operatorname{sech}(0) = 1 \quad \operatorname{sech}\left(\frac{\pi i}{3}\right) = 2 \quad \operatorname{sech}\left(\frac{2\pi i}{3}\right) = -2 \quad \operatorname{sech}(\pi i) = -1.$$

The values of  $\operatorname{sech}\left(\frac{\pi i \pi}{m}\right)$  can be expressed using only square roots if  $n \in \mathbb{Z}$  and  $m$  is a product of a power of 2 and distinct Fermat primes  $\{3, 5, 17, 257, \dots\}$ .

The function  $\operatorname{sech}(z)$  is an analytical function of  $z$  that is defined over the whole complex  $z$ -plane and does not have branch cuts and branch points. It has an infinite set of singular points:

- (a)  $z = \pi i / 2 + \pi i k$ ;  $k \in \mathbb{Z}$  are the simple poles with residues  $(-1)^{k-1} i$ .
- (b)  $z = \infty$  is an essential singular point.

It is a periodic function with period  $2\pi i$ :

$$\operatorname{sech}(z + 2\pi i) = \operatorname{sech}(z)$$

$$\operatorname{sech}(z) = \operatorname{sech}(z + 2\pi i k) \quad ; \quad k \in \mathbb{Z} \quad \operatorname{sech}(z) = (-1)^k \operatorname{sech}(z + \pi i k) \quad ; \quad k \in \mathbb{Z}.$$

The function  $\operatorname{sech}(z)$  is an even function with mirror symmetry:

$$\operatorname{sech}(-z) = \operatorname{sech}(z) \quad \operatorname{sech}(\bar{z}) = \overline{\operatorname{sech}(z)}.$$

### Differentiation

The first derivative of  $\operatorname{sech}(z)$  has simple representations using either the  $\operatorname{tanh}(z)$  function or the  $\operatorname{sech}(z)$  function:

$$\frac{\partial \operatorname{sech}(z)}{\partial z} = -\operatorname{tanh}(z) \operatorname{sech}(z).$$

The  $n^{\text{th}}$  derivative of  $\operatorname{sech}(z)$  has much more complicated representations than the symbolic  $n^{\text{th}}$  derivatives for  $\sinh(z)$  and  $\cosh(z)$ :

$$\frac{\partial^n \operatorname{sech}(z)}{\partial z^n} = \operatorname{sech}(z) \left( \delta_n + i^n (n+1)! \sum_{k=0}^n \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-1)^k 2^{1-k} (k-2j)^n \operatorname{sech}^k(z) \cosh\left(\frac{in\pi}{2} - (k-2j)z\right)}{(k+1)j!(k-j)!(n-k)!} \right); n \in \mathbb{N},$$

where  $\delta_n$  is the Kronecker delta symbol:  $\delta_0 = 1$  and  $\delta_n = 0$ ;  $n \neq 0$ .

**Ordinary differential equation**

The functions  $\operatorname{sech}(z)$  satisfies the following first-order nonlinear differential equation:

$$w'(z)^2 + w(z)^4 - w(z)^2 = 0 /; w(z) = \operatorname{sech}(z).$$

**Series representation**

The function  $\operatorname{sech}(z)$  has the following series expansion at the origin that converges for all finite values  $z$  with  $|z| < \frac{\pi}{2}$ :

$$\operatorname{sech}(z) = 1 - \frac{z^2}{2} + \frac{5z^4}{24} + \dots = \sum_{k=0}^{\infty} \frac{E_{2k} z^{2k}}{(2k)!},$$

where  $E_{2k}$  are the Euler numbers.

The hyperbolic secant function  $\operatorname{sech}(z)$  can also be presented using other kinds of series with the following formulas:

$$\operatorname{sech}(z) = \pi \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)}{\pi^2 \left(k + \frac{1}{2}\right)^2 + z^2} /; \frac{iz}{\pi} - \frac{1}{2} \notin \mathbb{Z}$$

$$\operatorname{sech}^2(z) = - \sum_{k=-\infty}^{\infty} \frac{1}{\left(z + \pi i \left(k + \frac{1}{2}\right)\right)^2} /; \frac{iz}{\pi} - \frac{1}{2} \notin \mathbb{Z}.$$

**Integral representation**

The function  $\operatorname{sech}(z)$  has a well-known integral representation through the following definite integral along the positive part of the real axis:

$$\operatorname{sech}(z) = \frac{2}{\pi} \int_0^{\infty} \frac{1}{t^2 + 1} t^{\frac{2iz}{\pi}} dt /; |\operatorname{Im}(z)| < \frac{\pi}{2}.$$

**Product representation**

The famous infinite product representation for  $\cosh(z)$  can be easily rewritten as the following product representation for the hyperbolic secant function:

$$\operatorname{sech}(z) = \prod_{k=1}^{\infty} \frac{\pi^2 (2k-1)^2}{\pi^2 (2k-1)^2 + 4z^2}.$$

### Limit representation

The hyperbolic secant function has the following limit representation:

$$\operatorname{sech}(z) = \lim_{n \rightarrow \infty} \left( \sum_{k=-n}^n \frac{(-1)^k}{\pi \left(k + \frac{1}{2}\right) - iz} \right) /; \frac{iz}{\pi} - \frac{1}{2} \notin \mathbb{Z}.$$

### Indefinite integration

Indefinite integrals of expressions that contain the hyperbolic secant function can sometimes be expressed using elementary functions. However, special functions are frequently needed to express the results even when the integrands have a simple form (if they can be evaluated in closed form). Here are some examples:

$$\int \operatorname{sech}(z) dz = 2 \tan^{-1} \left( \tanh \left( \frac{z}{2} \right) \right)$$

$$\int \sqrt{\operatorname{sech}(z)} dz = -2i \cosh^{\frac{1}{2}}(z) F \left( \frac{iz}{2} \middle| 2 \right) \operatorname{sech}^{\frac{1}{2}}(z)$$

$$\int \operatorname{sech}^v(az) dz = -\frac{\operatorname{sech}^{v-1}(az) \sinh(az)}{a(1-v) \sqrt{-\sinh^2(az)}} {}_2F_1 \left( \frac{1-v}{2}, \frac{1}{2}; \frac{3-v}{2}; \cosh^2(az) \right).$$

### Definite integration

Definite integrals that contain the hyperbolic secant function are sometimes simple and their values can be expressed through elementary functions. Here is one example:

$$\int_0^{\infty} t^2 \operatorname{sech}(t) dt = \frac{\pi^3}{8}.$$

Some special functions can be used to evaluate more complicated definite integrals. For example, gamma functions, incomplete beta functions, and the Catalan constant are needed to express the following integrals:

$$\int_0^{\frac{\pi}{2}} \operatorname{sech}^a(t) dt = \frac{1}{2} i \left( B_{\cosh^2 \left( \frac{\pi}{2} \right)} \left( \frac{1-a}{2}, \frac{1}{2} \right) - \frac{\sqrt{\pi} \Gamma \left( \frac{1}{2} - \frac{a}{2} \right)}{\Gamma \left( 1 - \frac{a}{2} \right)} \right) /; \operatorname{Re}(a) < 1$$

$$\int_0^{\infty} t \operatorname{sech}(t) dt = 2C.$$

### Finite summation

The following finite sum that contain the hyperbolic secant has a simple value:

$$\sum_{k=1}^n \frac{1}{2^{2k}} \operatorname{sech}^2 \left( \frac{z}{2^k} \right) = \frac{1}{2^{2n}} \operatorname{csch}^2 \left( \frac{z}{2^n} \right) - \operatorname{csch}^2(z) /; n \in \mathbb{N}.$$

### Infinite summation

The evaluation limit of the last formula in the previous subsection for  $n \rightarrow \infty$  gives the following value for the corresponding infinite sum:

$$\sum_{k=1}^{\infty} \frac{1}{2^{2k}} \operatorname{sech}^2\left(\frac{z}{2^k}\right) = \frac{1}{z^2} - \operatorname{csch}^2(z).$$

### Finite products

The following finite product from the hyperbolic secant can be represented through the hyperbolic cosecant function:

$$\prod_{k=0}^{n-1} \operatorname{sech}\left(z + \frac{\pi k}{n}\right) = (-1)^{n-1} 2^{n-1} \operatorname{csc}\left(\frac{n(\pi - 2z)}{2}\right); n \in \mathbb{N}^+.$$

### Infinite products

The following infinite product from the hyperbolic secant can be represented through the hyperbolic cosecant function:

$$\prod_{k=1}^{\infty} \operatorname{sech}\left(\frac{z}{2^k}\right) = z \operatorname{csch}(z).$$

### Addition formulas

The hyperbolic secant of a sum or difference can be represented in terms of hyperbolic sine and cosine as shown in the following formulas:

$$\operatorname{sech}(a + b) = \frac{1}{\cosh(a) \cosh(b) + \sinh(a) \sinh(b)}$$

$$\operatorname{sech}(a - b) = \frac{1}{\cosh(b) \cosh(a) - \sinh(a) \sinh(b)}.$$

### Multiple arguments

In the case of multiple arguments  $2z, 3z, \dots$ , the function  $\operatorname{sech}(nz)$  can be represented as a rational function that contains powers of the hyperbolic secant. Here are two examples:

$$\operatorname{sech}(2z) = \frac{\operatorname{sech}^2(z)}{2 - \operatorname{sech}^2(z)}$$

$$\operatorname{sech}(3z) = \frac{\operatorname{sech}^3(z)}{4 - 3 \operatorname{sech}^2(z)}$$

### Half-angle formulas

The hyperbolic secant of a half-angle can be represented by the following simple formula that is valid in a horizontal strip:

$$\operatorname{sech}\left(\frac{z}{2}\right) = \frac{\sqrt{2}}{\sqrt{\cosh(z) + 1}}; |\operatorname{Im}(z)| < \pi \vee (\operatorname{Im}(z) = \pi \wedge \operatorname{Re}(z) > 0) \vee (\operatorname{Im}(z) = -\pi \wedge \operatorname{Re}(z) < 0).$$

To make this formula correct for all complex  $z$ , a complicated prefactor is needed:

$$\operatorname{sech}\left(\frac{z}{2}\right) = c(z) \frac{\sqrt{2}}{\sqrt{\cosh(z) + 1}} ; c(z) = (-1)^{\lfloor \frac{\pi - \operatorname{Im}(z)}{2\pi} \rfloor} \left( 1 - \left( 1 + (-1)^{\lfloor \frac{\pi - \operatorname{Im}(z)}{2\pi} \rfloor + \lfloor -\frac{\pi - \operatorname{Im}(z)}{2\pi} \rfloor} \right) \theta(-\operatorname{Re}(z)) \right),$$

where  $c(z)$  contains the unit step, real part, imaginary part, and the floor functions.

### Sums of two direct functions

The sum and difference of two hyperbolic secant functions can be described by the following formulas:

$$\operatorname{sech}(a) + \operatorname{sech}(b) = 2 \cosh\left(\frac{a}{2} - \frac{b}{2}\right) \cosh\left(\frac{a}{2} + \frac{b}{2}\right) \operatorname{sech}(a) \operatorname{sech}(b)$$

$$\operatorname{sech}(a) - \operatorname{sech}(b) = -2 \sinh\left(\frac{a}{2} - \frac{b}{2}\right) \sinh\left(\frac{a}{2} + \frac{b}{2}\right) \operatorname{sech}(a) \operatorname{sech}(b).$$

### Products involving the direct function

The product of two hyperbolic secants and the product of hyperbolic secant and cosecant have the following representations:

$$\operatorname{sech}(a) \operatorname{sech}(b) = \frac{2}{\cosh(a - b) + \cosh(a + b)}$$

$$\operatorname{sech}(a) \operatorname{csch}(b) = \frac{2}{\sinh(a + b) - \sinh(a - b)}.$$

### Inequalities

One of the most famous inequalities for the hyperbolic secant function is the following:

$$\operatorname{sech}(x) < x \operatorname{csch}(x) ; x > 0 \wedge x \in \mathbb{R}.$$

### Relations with its inverse function

There are simple relations between the function  $\operatorname{sech}(z)$  and its inverse function  $\operatorname{sech}^{-1}(z)$ :

$$\operatorname{sech}\left(\operatorname{sech}^{-1}(z)\right) = z \quad \operatorname{sech}^{-1}\left(\operatorname{sech}(z)\right) = z ; -\pi < \operatorname{Im}(z) \leq \pi \wedge \operatorname{Re}(z) > 0 \vee \operatorname{Re}(z) = 0 \wedge \operatorname{Im}(z) \geq 0.$$

The second formula is valid at least in the horizontal half-strip  $-\pi < \operatorname{Im}(z) \leq \pi \wedge \operatorname{Re}(z) > 0$ . This can be expanded to a full horizontal strip:

$$\operatorname{sech}^{-1}\left(\operatorname{sech}(z)\right) = \sqrt{z^2} ; -\pi < \operatorname{Im}(z) < \pi \vee \operatorname{Im}(z) = -\pi \wedge \operatorname{Re}(z) \leq 0 \vee \operatorname{Im}(z) = \pi \wedge \operatorname{Re}(z) \geq 0.$$

Outside of this strip a much more complicated relation (that contains the unit step, real part, and the floor functions) holds:

$$\operatorname{sech}^{-1}\left(\operatorname{sech}(z)\right) = \sqrt{z^2} \left( 1 - \frac{\pi i}{2z} \left( \left( 1 - (-1)^{\lfloor \frac{\operatorname{Im}(z)}{\pi} \rfloor} \right) + 2 \left\lfloor \frac{\operatorname{Im}(z)}{\pi} \right\rfloor \right) \right) + \frac{\pi i}{2} \left( \frac{\sqrt{z^2}}{z} + 1 \right) \left( 1 + (-1)^{\lfloor \frac{\operatorname{Im}(z)}{2\pi} + \frac{1}{2} \rfloor + \lfloor -\frac{\operatorname{Im}(z)}{2\pi} - \frac{1}{2} \rfloor} \right) \theta(\operatorname{Re}(z)) ; z \neq 0.$$

### Representations through other hyperbolic functions

The hyperbolic secant and cosecant functions are connected by a very simple formula including the linear function in the argument:

$$\operatorname{sech}(z) = i \operatorname{csch}\left(\frac{\pi i}{2} - z\right).$$

The hyperbolic secant function can also be represented using other hyperbolic functions by the following formulas:

$$\operatorname{sech}(z) = \frac{i}{\sinh\left(\frac{\pi i}{2} - z\right)} \quad \operatorname{sech}(z) = \frac{1 - \tanh^2\left(\frac{z}{2}\right)}{1 + \tanh^2\left(\frac{z}{2}\right)} \quad \operatorname{sech}(z) = \frac{\operatorname{coth}^2\left(\frac{z}{2}\right) - 1}{\operatorname{coth}^2\left(\frac{z}{2}\right) + 1}.$$

### Representations through trigonometric functions

The hyperbolic secant function has the following representations using the trigonometric functions:

$$\operatorname{sech}(z) = \frac{1}{\sin\left(\frac{\pi}{2} - iz\right)} \quad \operatorname{sech}(z) = \frac{1}{\cos(iz)} \quad \operatorname{sech}(z) = \frac{1 + \tan^2\left(\frac{iz}{2}\right)}{1 - \tan^2\left(\frac{iz}{2}\right)} \quad \operatorname{sech}(z) = \frac{\cot^2\left(\frac{iz}{2}\right) + 1}{\cot^2\left(\frac{iz}{2}\right) - 1}$$

$$\operatorname{sech}(z) = \operatorname{csc}\left(\frac{\pi}{2} - iz\right) \quad \operatorname{sech}(z) = \operatorname{sec}(iz) \quad \operatorname{sech}(iz) = \operatorname{sec}(z).$$

### Applications

The hyperbolic secant function is used throughout mathematics, the exact sciences, and engineering.

## Introduction to the Hyperbolic Functions in *Mathematica*

### Overview

The following shows how the six hyperbolic functions are realized in *Mathematica*. Examples of evaluating *Mathematica* functions applied to various numeric and exact expressions that involve the hyperbolic functions or return them are shown. These involve numeric and symbolic calculations and plots.

### Notations

#### *Mathematica* forms of notations

All six hyperbolic functions are represented as built-in functions in *Mathematica*. Following *Mathematica*'s general naming convention, the `StandardForm` function names are simply capitalized versions of the traditional mathematics names. Here is a list `hypFunctions` of the six hyperbolic functions in `StandardForm`.

```
hypFunctions = {Sinh[z], Cosh[z], Tanh[z], Coth[z], Sech[z], Cosh[z]}
```

```
{Sinh[z], Cosh[z], Tanh[z], Coth[z], Sech[z], Cosh[z]}
```

Here is a list `hypFunctions` of the six trigonometric functions in `TraditionalForm`.

```
hypFunctions // TraditionalForm
```

```
{sinh(z), cosh(z), tanh(z), coth(z), sech(z), cosh(z)}
```

#### Additional forms of notations

*Mathematica* also knows the most popular forms of notations for the hyperbolic functions that are used in other programming languages. Here are three examples: CForm, TeXForm, and FortranForm.

```
hypFunctions /. {z -> 2 Pi z} // CForm
```

```
List(Sinh(2*Pi*z), Cosh(2*Pi*z), Tanh(2*Pi*z), Coth(2*Pi*z), Sech(2*Pi*z), Cosh(2*Pi*z))
```

```
hypFunctions /. {z -> 2 Pi z} // TeXForm
```

```
\{ \sinh (2\,\pi \,z), \cosh (2\,\pi \,z), \tanh (2\,\pi \,z), \coth (2\,\pi \,z),  
  \Mfunction{Sech}(2\,\pi \,z), \cosh (2\,\pi \,z) \}
```

```
hypFunctions /. {z -> 2 Pi z} // FortranForm
```

```
List(Sinh(2*Pi*z), Cosh(2*Pi*z), Tanh(2*Pi*z), Coth(2*Pi*z), Sech(2*Pi*z), Cosh(2*Pi*z))
```

## Automatic evaluations and transformations

### Evaluation for exact, machine-number, and high-precision arguments

For a simple exact argument, *Mathematica* returns an exact result. For instance, for the argument  $\pi i/6$ , the Sinh function evaluates to  $i/2$ .

```
Sinh[ $\frac{\pi i}{6}$ ]
```

```
 $\frac{i}{2}$ 
```

```
{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]} /. z ->  $\frac{\pi i}{6}$ 
```

```
{ $\frac{i}{2}$ ,  $\frac{\sqrt{3}}{2}$ ,  $\frac{i}{\sqrt{3}}$ ,  $-i\sqrt{3}$ ,  $-2i$ ,  $\frac{2}{\sqrt{3}}$ }
```

For a generic machine-number argument (a numerical argument with a decimal point and not too many digits), a machine number is returned.

```
Cosh[3.]
```

```
10.0677
```

```
{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]} /. z -> 2.
```

```
{3.62686, 3.7622, 0.964028, 1.03731, 0.275721, 0.265802}
```

The next inputs calculate 100-digit approximations of the six hyperbolic functions at  $z = 1$ .

```
N[Tanh[1], 40]
```

```
0.7615941559557648881194582826047935904128
```

```
Coth[1] // N[#, 50] &
```

```
1.3130352854993313036361612469308478329120139412405
```



```
N[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]} /. z -> 1, 100]
```

```
{1.175201193643801456882381850595600815155717981334095870229565413013307567304323895\
607117452089623392,
1.543080634815243778477905620757061682601529112365863704737402214710769063049223698\
964264726435543036,
0.761594155955764888119458282604793590412768597257936551596810500121953244576638483\
4589475216736767144,
1.313035285499331303636161246930847832912013941240452655543152967567084270461874382\
674679241480856303,
0.850918128239321545133842763287175284181724660910339616990421151729003364321465103\
8997301773288938124,
0.648054273663885399574977353226150323108489312071942023037865337318717595646712830\
2808547853078928924}
```

Within a second, it is possible to calculate thousands of digits for the hyperbolic functions. The next input calculates 10000 digits for  $\sinh(1)$ ,  $\cosh(1)$ ,  $\tanh(1)$ ,  $\coth(1)$ ,  $\operatorname{sech}(1)$ , and  $\operatorname{csch}(1)$  and analyzes the frequency of the occurrence of the digit  $k$  in the resulting decimal number.

```
Map[Function[w, {First[#], Length[#]} & /@ Split[Sort[First[RealDigits[w]]]],
N[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]} /. z -> 1, 10 000]]
```

```
{{{0, 980}, {1, 994}, {2, 996}, {3, 1014}, {4, 986}, {5, 1001},
{6, 1017}, {7, 1020}, {8, 981}, {9, 1011}}, {{0, 1015}, {1, 960}, {2, 997},
{3, 1037}, {4, 1070}, {5, 1018}, {6, 973}, {7, 997}, {8, 963}, {9, 970}},
{{0, 971}, {1, 1023}, {2, 1016}, {3, 970}, {4, 949}, {5, 1052}, {6, 981},
{7, 1056}, {8, 1010}, {9, 972}}, {{0, 975}, {1, 986}, {2, 1023},
{3, 1004}, {4, 1008}, {5, 977}, {6, 977}, {7, 1036}, {8, 1035}, {9, 979}},
{{0, 979}, {1, 1030}, {2, 987}, {3, 992}, {4, 1016}, {5, 1030}, {6, 1021},
{7, 969}, {8, 974}, {9, 1002}}, {{0, 1009}, {1, 971}, {2, 1018},
{3, 994}, {4, 1011}, {5, 1018}, {6, 958}, {7, 1019}, {8, 1016}, {9, 986}}}
```

Here are 50-digit approximations to the six hyperbolic functions at the complex argument  $z = 3 + 5i$ .

```
N[Csch[3 + 5 i], 100]
```

```
0.0280585164230800759963159842602743697051540123887285931631736730964453318082730911\
1484269546408531396 +
0.095323634674178402851915930706256451645442166878775479803879772793331583262276221\
38939784445056701747 i
```

```
N[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]} /. z -> 3 + 5 i, 50]
```

```
{2.8416922956063519438168753953062364359281841632360 -
 9.6541254768548391365515436340301659921919691213853 i,
2.8558150042273872913639018630946098374643609536732 -
9.6063834484325811198111562160434163877218590394033 i,
1.0041647106948152119205166259313184311852454735738 -
0.0027082358362240721322640353684331035927960259125751 i,
0.99584531857585412976042001587164841711026557204102 +
0.0026857984057585256446537711012814749378977439361108 i,
0.028058516423080075996315984260274369705154012388729 +
0.095323634674178402851915930706256451645442166878775 i,
0.028433530909971667358833684958646399417265586614624 +
0.095644640955286344684316595933099452259073530811833 i}
```

*Mathematica* always evaluates mathematical functions with machine precision, if the arguments are machine numbers. In this case, only six digits after the decimal point are shown in the results. The remaining digits are suppressed, but can be displayed using the function `InputForm`.

```
{Sinh[2.], N[Sinh[2]], N[Sinh[2], 16], N[Sinh[2], 5], N[Sinh[2], 20]}
{3.62686, 3.62686, 3.62686, 3.62686, 3.6268604078470187677}

% // InputForm
{3.6268604078470186, 3.6268604078470186, 3.6268604078470186, 3.6268604078470186,
3.62686040784701876766821398280126201644`20}
```

`Precision[%]`

16

### Simplification of the argument

*Mathematica* uses symmetries and periodicities of all the hyperbolic functions to simplify expressions. Here are some examples.

`Sinh[-z]`

`-Sinh[z]`

`Sinh[z +  $\pi i$ ]`

`-Sinh[z]`

`Sinh[z +  $2 \pi i$ ]`

`Sinh[z]`

`Sinh[z +  $34 \pi i$ ]`

`Sinh[z]`

`{Sinh[-z], Cosh[-z], Tanh[-z], Coth[-z], Csch[-z], Sech[-z]}`

`{-Sinh[z], Cosh[z], -Tanh[z], -Coth[z], -Csch[z], Sech[z]}`

**{Sinh[z + π i], Cosh[z + π i], Tanh[z + π i], Coth[z + π i], Csch[z + π i], Sech[z + π i]}**

**{-Sinh[z], -Cosh[z], Tanh[z], Coth[z], -Csch[z], -Sech[z]}**

**{Sinh[z + 2 π i], Cosh[z + 2 π i], Tanh[z + 2 π i],  
Coth[z + 2 π i], Csch[z + 2 π i], Sech[z + 2 π i]}**

**{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}**

**{Sinh[z + 342 π i], Cosh[z + 342 π i], Tanh[z + 342 π i],  
Coth[z + 342 π i], Csch[z + 342 π i], Sech[z + 342 π i]}**

**{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}**

*Mathematica* automatically simplifies the composition of the direct and the inverse hyperbolic functions into the argument.

**{Sinh[ArcSinh[z]], Cosh[ArcCosh[z]], Tanh[ArcTanh[z]],  
Coth[ArcCoth[z]], Csch[ArcCsch[z]], Sech[ArcSech[z]]}**

**{z, z, z, z, z, z}**

*Mathematica* also automatically simplifies the composition of the direct and any of the inverse hyperbolic functions into algebraic functions of the argument.

**{Sinh[ArcSinh[z]], Sinh[ArcCosh[z]], Sinh[ArcTanh[z]],  
Sinh[ArcCoth[z]], Sinh[ArcCsch[z]], Sinh[ArcSech[z]]}**

$$\left\{ z, \sqrt{\frac{-1+z}{1+z}} (1+z), \frac{z}{\sqrt{1-z^2}}, \frac{1}{\sqrt{1-\frac{1}{z^2}} z}, \frac{1}{z}, \sqrt{\frac{1-z}{1+z}} (1+z) \right\}$$

**{Cosh[ArcSinh[z]], Cosh[ArcCosh[z]], Cosh[ArcTanh[z]],  
Cosh[ArcCoth[z]], Cosh[ArcCsch[z]], Cosh[ArcSech[z]]}**

$$\left\{ \sqrt{1+z^2}, z, \frac{1}{\sqrt{1-z^2}}, \frac{1}{\sqrt{1-\frac{1}{z^2}}}, \sqrt{1+\frac{1}{z^2}}, \frac{1}{z} \right\}$$

**{Tanh[ArcSinh[z]], Tanh[ArcCosh[z]], Tanh[ArcTanh[z]],  
Tanh[ArcCoth[z]], Tanh[ArcCsch[z]], Tanh[ArcSech[z]]}**

$$\left\{ \frac{z}{\sqrt{1+z^2}}, \frac{\sqrt{\frac{-1+z}{1+z}} (1+z)}{z}, z, \frac{1}{z}, \frac{1}{\sqrt{1+\frac{1}{z^2}} z}, \sqrt{\frac{1-z}{1+z}} (1+z) \right\}$$

**{Coth[ArcSinh[z]], Coth[ArcCosh[z]], Coth[ArcTanh[z]],  
Coth[ArcCoth[z]], Coth[ArcCsch[z]], Coth[ArcSech[z]]}**

$$\left\{ \frac{\sqrt{1+z^2}}{z}, \frac{z}{\sqrt{\frac{-1+z}{1+z}}(1+z)}, \frac{1}{z}, z, \sqrt{1+\frac{1}{z^2}}z, \frac{1}{\sqrt{\frac{1-z}{1+z}}(1+z)} \right\}$$

$$\{\text{Csch}[\text{ArcSinh}[z]], \text{Csch}[\text{ArcCosh}[z]], \text{Csch}[\text{ArcTanh}[z]], \text{Csch}[\text{ArcCoth}[z]], \text{Csch}[\text{ArcCsch}[z]], \text{Csch}[\text{ArcSech}[z]]\}$$

$$\left\{ \frac{1}{z}, \frac{1}{\sqrt{\frac{-1+z}{1+z}}(1+z)}, \frac{\sqrt{1-z^2}}{z}, \sqrt{1-\frac{1}{z^2}}z, z, \frac{z}{\sqrt{\frac{1-z}{1+z}}(1+z)} \right\}$$

$$\{\text{Sech}[\text{ArcSinh}[z]], \text{Sech}[\text{ArcCosh}[z]], \text{Sech}[\text{ArcTanh}[z]], \text{Sech}[\text{ArcCoth}[z]], \text{Sech}[\text{ArcCsch}[z]], \text{Sech}[\text{ArcSech}[z]]\}$$

$$\left\{ \frac{1}{\sqrt{1+z^2}}, \frac{1}{z}, \sqrt{1-z^2}, \sqrt{1-\frac{1}{z^2}}, \frac{1}{\sqrt{1+\frac{1}{z^2}}}, z \right\}$$

In cases where the argument has the structure  $\pi k/2 + z$  or  $\pi k/2 - z$ ,  $e^{\pi k/2 + iz}$  or  $\pi k/2 - iz$  with integer  $k$ , trigonometric functions can be automatically transformed into other trigonometric or hyperbolic functions. Here are some examples.

$$\text{Tanh}\left[\frac{\pi i}{2} - z\right]$$

$$-\text{Coth}[z]$$

$$\text{Csch}[i z]$$

$$-i \text{Csc}[z]$$

$$\left\{ \text{Sinh}\left[\frac{\pi i}{2} - z\right], \text{Cosh}\left[\frac{\pi i}{2} - z\right], \text{Tanh}\left[\frac{\pi i}{2} - z\right], \text{Coth}\left[\frac{\pi i}{2} - z\right], \text{Csch}\left[\frac{\pi i}{2} - z\right], \text{Sech}\left[\frac{\pi i}{2} - z\right] \right\}$$

$$\{i \text{Cosh}[z], -i \text{Sinh}[z], -\text{Coth}[z], -\text{Tanh}[z], -i \text{Sech}[z], i \text{Csch}[z]\}$$

$$\{\text{Sinh}[i z], \text{Cosh}[i z], \text{Tanh}[i z], \text{Coth}[i z], \text{Csch}[i z], \text{Sech}[i z]\}$$

$$\{i \text{Sin}[z], \text{Cos}[z], i \text{Tan}[z], -i \text{Cot}[z], -i \text{Csc}[z], \text{Sec}[z]\}$$

### Simplification of simple expressions containing hyperbolic functions

Sometimes simple arithmetic operations containing hyperbolic functions can automatically produce other hyperbolic functions.

$$1 / \text{Sech}[z]$$

$$\text{Cosh}[z]$$

$$\{1 / \text{Sinh}[z], 1 / \text{Cosh}[z], 1 / \text{Tanh}[z], 1 / \text{Coth}[z], 1 / \text{Csch}[z], 1 / \text{Sech}[z], \text{Sinh}[z] / \text{Cosh}[z], \text{Cosh}[z] / \text{Sinh}[z], \text{Sinh}[z] / \text{Sinh}[\pi i / 2 - z], \text{Cosh}[z] / \text{Sinh}[z]^2\}$$

{Csch[z], Sech[z], Coth[z], Tanh[z], Sinh[z],  
Cosh[z], Tanh[z], Coth[z], -i Tanh[z], Coth[z] Csch[z]}

### Hyperbolic functions as special cases of more general functions

All hyperbolic functions can be treated as particular cases of some more advanced special functions. For example,  $\sinh(z)$  and  $\cosh(z)$  are sometimes the results of auto-simplifications from Bessel, Mathieu, Jacobi, hypergeometric, and Meijer functions (for appropriate values of their parameters).

**BesselI** $\left[\frac{1}{2}, z\right]$

$$\frac{\sqrt{\frac{2}{\pi}} \operatorname{Sinh}[z]}{\sqrt{z}}$$

**MathieuC** $[1, 0, i z]$

**Cosh** $[z]$

**JacobiSN** $[z, 1]$

**Tanh** $[z]$

{**BesselI** $\left[\frac{1}{2}, z\right]$ , **MathieuS** $[1, 0, i z]$ , **JacobiSD** $[i z, 0]$ ,

**HypergeometricPFQ** $\left[\{\}, \left\{\frac{3}{2}\right\}, \frac{z^2}{4}\right]$ , **MeijerG** $\left[\{\{\}, \{\}\}, \left\{\{0\}, \left\{-\frac{1}{2}\right\}\right\}, -\frac{z^2}{4}\right]$ }

$$\left\{\frac{\sqrt{\frac{2}{\pi}} \operatorname{Sinh}[z]}{\sqrt{z}}, i \operatorname{Sinh}[z], i \operatorname{Sinh}[z], \frac{\operatorname{Sinh}\left[\sqrt{z^2}\right]}{\sqrt{z^2}}, \frac{2 \operatorname{Sinh}[z]}{\sqrt{\pi} z}\right\}$$

{**BesselI** $\left[-\frac{1}{2}, z\right]$ , **MathieuC** $[1, 0, i z]$ , **JacobiCD** $[i z, 0]$ ,

**Hypergeometric0F1** $\left[\frac{1}{2}, \frac{z^2}{4}\right]$ , **MeijerG** $\left[\{\{\}, \{\}\}, \left\{\{0\}, \left\{\frac{1}{2}\right\}\right\}, -\frac{z^2}{4}\right]$ }

$$\left\{\frac{\sqrt{\frac{2}{\pi}} \operatorname{Cosh}[z]}{\sqrt{z}}, \operatorname{Cosh}[z], \operatorname{Cosh}[z], \operatorname{Cosh}\left[\sqrt{z^2}\right], \frac{\operatorname{Cosh}[z]}{\sqrt{\pi}}\right\}$$

{**JacobiSC** $[i z, 0]$ , **JacobiNS** $[z, 1]$ , **JacobiNS** $[i z, 0]$ , **JacobiDC** $[i z, 0]$ }

{ $i \operatorname{Tanh}[z]$ ,  $\operatorname{Coth}[z]$ ,  $-i \operatorname{Csch}[z]$ ,  $\operatorname{Sech}[z]$ }

### Equivalence transformations carried out by specialized *Mathematica* functions

#### General remarks

Automatic evaluation and transformations can sometimes be inconvenient: They act in only one chosen direction and the result can be overly complicated. For example, the expression  $i \cosh(z)/2$  is generally preferable to the more complicated  $\sinh(\pi i/2 - z) \cosh(\pi i/3)$ . *Mathematica* provides automatic transformation of the second expression into the first one. But compact expressions like  $\sinh(2z) \cosh(\pi i/16)$  should not be automatically expanded into the more complicated expression  $\sinh(z) \cosh(z) \left(2 + (2 + 2^{1/2})^{1/2}\right)^{1/2}$ . *Mathematica* has special functions that produce these types of expansions. Some of them are demonstrated in the next section.

### TrigExpand

The function `TrigExpand` expands out trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then expands out the products of the trigonometric and hyperbolic functions into sums of powers, using the trigonometric and hyperbolic identities where possible. Here are some examples.

```
TrigExpand[Sinh[x - y]]
```

$$\text{Cosh}[y] \text{Sinh}[x] - \text{Cosh}[x] \text{Sinh}[y]$$

```
Cosh[4 z] // TrigExpand
```

$$\text{Cosh}[z]^4 + 6 \text{Cosh}[z]^2 \text{Sinh}[z]^2 + \text{Sinh}[z]^4$$

```
TrigExpand[{{Sinh[x + y], Sinh[3 z]},
             {Cosh[x + y], Cosh[3 z]},
             {Tanh[x + y], Tanh[3 z]},
             {Coth[x + y], Coth[3 z]},
             {Csch[x + y], Csch[3 z]},
             {Sech[x + y], Sech[3 z]}}]
```

$$\left\{ \left\{ \text{Cosh}[y] \text{Sinh}[x] + \text{Cosh}[x] \text{Sinh}[y], 3 \text{Cosh}[z]^2 \text{Sinh}[z] + \text{Sinh}[z]^3 \right\}, \right. \\ \left. \left\{ \text{Cosh}[x] \text{Cosh}[y] + \text{Sinh}[x] \text{Sinh}[y], \text{Cosh}[z]^3 + 3 \text{Cosh}[z] \text{Sinh}[z]^2 \right\}, \right. \\ \left. \left\{ \frac{\text{Cosh}[y] \text{Sinh}[x]}{\text{Cosh}[x] \text{Cosh}[y] + \text{Sinh}[x] \text{Sinh}[y]} + \frac{\text{Cosh}[x] \text{Sinh}[y]}{\text{Cosh}[x] \text{Cosh}[y] + \text{Sinh}[x] \text{Sinh}[y]}, \right. \right. \\ \left. \left. \frac{3 \text{Cosh}[z]^2 \text{Sinh}[z]}{\text{Cosh}[z]^3 + 3 \text{Cosh}[z] \text{Sinh}[z]^2} + \frac{\text{Sinh}[z]^3}{\text{Cosh}[z]^3 + 3 \text{Cosh}[z] \text{Sinh}[z]^2} \right\}, \right. \\ \left. \left\{ \frac{\text{Cosh}[x] \text{Cosh}[y]}{\text{Cosh}[y] \text{Sinh}[x] + \text{Cosh}[x] \text{Sinh}[y]} + \frac{\text{Sinh}[x] \text{Sinh}[y]}{\text{Cosh}[y] \text{Sinh}[x] + \text{Cosh}[x] \text{Sinh}[y]}, \right. \right. \\ \left. \left. \frac{\text{Cosh}[z]^3}{3 \text{Cosh}[z]^2 \text{Sinh}[z] + \text{Sinh}[z]^3} + \frac{3 \text{Cosh}[z] \text{Sinh}[z]^2}{3 \text{Cosh}[z]^2 \text{Sinh}[z] + \text{Sinh}[z]^3} \right\}, \right. \\ \left. \left\{ \frac{1}{\text{Cosh}[y] \text{Sinh}[x] + \text{Cosh}[x] \text{Sinh}[y]}, \frac{1}{3 \text{Cosh}[z]^2 \text{Sinh}[z] + \text{Sinh}[z]^3} \right\}, \right. \\ \left. \left\{ \frac{1}{\text{Cosh}[x] \text{Cosh}[y] + \text{Sinh}[x] \text{Sinh}[y]}, \frac{1}{\text{Cosh}[z]^3 + 3 \text{Cosh}[z] \text{Sinh}[z]^2} \right\} \right\}$$

```
TableForm[ (# == TrigExpand[#]) & /@
```

```
  Flatten[{{Sinh[x + y], Sinh[3 z]}, {Cosh[x + y], Cosh[3 z]}, {Tanh[x + y], Tanh[3 z]},
           {Coth[x + y], Coth[3 z]}, {Csch[x + y], Csch[3 z]}, {Sech[x + y], Sech[3 z]}}]
```

$$\begin{aligned} \text{Sinh}[x + y] &== \text{Cosh}[y] \text{Sinh}[x] + \text{Cosh}[x] \text{Sinh}[y] \\ \text{Sinh}[3 z] &== 3 \text{Cosh}[z]^2 \text{Sinh}[z] + \text{Sinh}[z]^3 \\ \text{Cosh}[x + y] &== \text{Cosh}[x] \text{Cosh}[y] + \text{Sinh}[x] \text{Sinh}[y] \\ \text{Cosh}[3 z] &== \text{Cosh}[z]^3 + 3 \text{Cosh}[z] \text{Sinh}[z]^2 \\ \text{Tanh}[x + y] &== \frac{\text{Cosh}[y] \text{Sinh}[x]}{\text{Cosh}[x] \text{Cosh}[y] + \text{Sinh}[x] \text{Sinh}[y]} + \frac{\text{Cosh}[x] \text{Sinh}[y]}{\text{Cosh}[x] \text{Cosh}[y] + \text{Sinh}[x] \text{Sinh}[y]} \\ \text{Tanh}[3 z] &== \frac{3 \text{Cosh}[z]^2 \text{Sinh}[z]}{\text{Cosh}[z]^3 + 3 \text{Cosh}[z] \text{Sinh}[z]^2} + \frac{\text{Sinh}[z]^3}{\text{Cosh}[z]^3 + 3 \text{Cosh}[z] \text{Sinh}[z]^2} \\ \text{Coth}[x + y] &== \frac{\text{Cosh}[x] \text{Cosh}[y]}{\text{Cosh}[y] \text{Sinh}[x] + \text{Cosh}[x] \text{Sinh}[y]} + \frac{\text{Sinh}[x] \text{Sinh}[y]}{\text{Cosh}[y] \text{Sinh}[x] + \text{Cosh}[x] \text{Sinh}[y]} \\ \text{Coth}[3 z] &== \frac{\text{Cosh}[z]^3}{3 \text{Cosh}[z]^2 \text{Sinh}[z] + \text{Sinh}[z]^3} + \frac{3 \text{Cosh}[z] \text{Sinh}[z]^2}{3 \text{Cosh}[z]^2 \text{Sinh}[z] + \text{Sinh}[z]^3} \\ \text{Csch}[x + y] &== \frac{1}{\text{Cosh}[y] \text{Sinh}[x] + \text{Cosh}[x] \text{Sinh}[y]} \\ \text{Csch}[3 z] &== \frac{1}{3 \text{Cosh}[z]^2 \text{Sinh}[z] + \text{Sinh}[z]^3} \\ \text{Sech}[x + y] &== \frac{1}{\text{Cosh}[x] \text{Cosh}[y] + \text{Sinh}[x] \text{Sinh}[y]} \\ \text{Sech}[3 z] &== \frac{1}{\text{Cosh}[z]^3 + 3 \text{Cosh}[z] \text{Sinh}[z]^2} \end{aligned}$$

### TrigFactor

The command `TrigFactor` factors trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then factors the resulting polynomials in the trigonometric and hyperbolic functions, using the corresponding identities where possible. Here are some examples.

`TrigFactor[Sinh[x] + i Cosh[y]]`

$$\left( i \text{Cosh}\left[\frac{x}{2} - \frac{y}{2}\right] + \text{Sinh}\left[\frac{x}{2} - \frac{y}{2}\right] \right) \left( \text{Cosh}\left[\frac{x}{2} + \frac{y}{2}\right] - i \text{Sinh}\left[\frac{x}{2} + \frac{y}{2}\right] \right)$$

`Tanh[x] - Coth[y] // TrigFactor`

$$-\text{Cosh}[x - y] \text{Csch}[y] \text{Sech}[x]$$

`TrigFactor[{Sinh[x] + Sinh[y],  
Cosh[x] + Cosh[y],  
Tanh[x] + Tanh[y],  
Coth[x] + Coth[y],  
Csch[x] + Csch[y],  
Sech[x] + Sech[y]}]`

$$\left\{ 2 \text{Cosh}\left[\frac{x}{2} - \frac{y}{2}\right] \text{Sinh}\left[\frac{x}{2} + \frac{y}{2}\right], 2 \text{Cosh}\left[\frac{x}{2} - \frac{y}{2}\right] \text{Cosh}\left[\frac{x}{2} + \frac{y}{2}\right], \text{Sech}[x] \text{Sech}[y] \text{Sinh}[x + y], \right. \\ \left. \text{Csch}[x] \text{Csch}[y] \text{Sinh}[x + y], \frac{1}{2} \text{Cosh}\left[\frac{x}{2} - \frac{y}{2}\right] \text{Csch}\left[\frac{x}{2}\right] \text{Csch}\left[\frac{y}{2}\right] \text{Sech}\left[\frac{x}{2}\right] \text{Sech}\left[\frac{y}{2}\right] \text{Sinh}\left[\frac{x}{2} + \frac{y}{2}\right], \right. \\ \left. \frac{2 \text{Cosh}\left[\frac{x}{2} - \frac{y}{2}\right] \text{Cosh}\left[\frac{x}{2} + \frac{y}{2}\right]}{\left( \text{Cosh}\left[\frac{x}{2}\right] - i \text{Sinh}\left[\frac{x}{2}\right] \right) \left( \text{Cosh}\left[\frac{x}{2}\right] + i \text{Sinh}\left[\frac{x}{2}\right] \right) \left( \text{Cosh}\left[\frac{y}{2}\right] - i \text{Sinh}\left[\frac{y}{2}\right] \right) \left( \text{Cosh}\left[\frac{y}{2}\right] + i \text{Sinh}\left[\frac{y}{2}\right] \right)} \right\}$$

### TrigReduce

The command `TrigReduce` rewrites products and powers of trigonometric and hyperbolic functions in terms of those functions with combined arguments. In more detail, it typically yields a linear expression involving trigonometric and hyperbolic functions with more complicated arguments. `TrigReduce` is approximately opposite to `TrigExpand` and `TrigFactor`. Here are some examples.

**TrigReduce[Sinh[z]^3]**

$$\frac{1}{4} (-3 \operatorname{Sinh}[z] + \operatorname{Sinh}[3 z])$$

**Sinh[x] Cosh[y] // TrigReduce**

$$\frac{1}{2} (\operatorname{Sinh}[x - y] + \operatorname{Sinh}[x + y])$$

**TrigReduce[{Sinh[z]^2, Cosh[z]^2, Tanh[z]^2, Coth[z]^2, Csch[z]^2, Sech[z]^2}]**

$$\left\{ \frac{1}{2} (-1 + \operatorname{Cosh}[2 z]), \frac{1}{2} (1 + \operatorname{Cosh}[2 z]), \right. \\ \left. \frac{-1 + \operatorname{Cosh}[2 z]}{1 + \operatorname{Cosh}[2 z]}, \frac{1 + \operatorname{Cosh}[2 z]}{-1 + \operatorname{Cosh}[2 z]}, \frac{2}{-1 + \operatorname{Cosh}[2 z]}, \frac{2}{1 + \operatorname{Cosh}[2 z]} \right\}$$

**TrigReduce[TrigExpand[{{Sinh[x+y], Sinh[3 z], Sinh[x] Sinh[y]},  
{Cosh[x+y], Cosh[3 z], Cosh[x] Cosh[y]},  
{Tanh[x+y], Tanh[3 z], Tanh[x] Tanh[y]},  
{Coth[x+y], Coth[3 z], Coth[x] Coth[y]},  
{Csch[x+y], Csch[3 z], Csch[x] Csch[y]},  
{Sech[x+y], Sech[3 z], Sech[x] Sech[y]}]}]]**

$$\left\{ \left\{ \operatorname{Sinh}[x + y], \operatorname{Sinh}[3 z], \frac{1}{2} (-\operatorname{Cosh}[x - y] + \operatorname{Cosh}[x + y]) \right\}, \right. \\ \left\{ \operatorname{Cosh}[x + y], \operatorname{Cosh}[3 z], \frac{1}{2} (\operatorname{Cosh}[x - y] + \operatorname{Cosh}[x + y]) \right\}, \\ \left\{ \operatorname{Tanh}[x + y], \operatorname{Tanh}[3 z], \frac{-\operatorname{Cosh}[x - y] + \operatorname{Cosh}[x + y]}{\operatorname{Cosh}[x - y] + \operatorname{Cosh}[x + y]} \right\}, \\ \left\{ \operatorname{Coth}[x + y], \operatorname{Coth}[3 z], \frac{-\operatorname{Cosh}[x - y] - \operatorname{Cosh}[x + y]}{\operatorname{Cosh}[x - y] - \operatorname{Cosh}[x + y]} \right\}, \\ \left\{ \operatorname{Csch}[x + y], \operatorname{Csch}[3 z], -\frac{2}{\operatorname{Cosh}[x - y] - \operatorname{Cosh}[x + y]} \right\}, \\ \left. \left\{ \operatorname{Sech}[x + y], \operatorname{Sech}[3 z], \frac{2}{\operatorname{Cosh}[x - y] + \operatorname{Cosh}[x + y]} \right\} \right\}$$

**TrigReduce[TrigFactor[{Sinh[x]+Sinh[y], Cosh[x]+Cosh[y],  
Tanh[x]+Tanh[y], Coth[x]+Coth[y], Csch[x]+Csch[y], Sech[x]+Sech[y]}]]]**

$$\left\{ \operatorname{Sinh}[x] + \operatorname{Sinh}[y], \operatorname{Cosh}[x] + \operatorname{Cosh}[y], \frac{2 \operatorname{Sinh}[x + y]}{\operatorname{Cosh}[x - y] + \operatorname{Cosh}[x + y]}, \right. \\ \left. -\frac{2 \operatorname{Sinh}[x + y]}{\operatorname{Cosh}[x - y] - \operatorname{Cosh}[x + y]}, -\frac{2 (\operatorname{Sinh}[x] + \operatorname{Sinh}[y])}{\operatorname{Cosh}[x - y] - \operatorname{Cosh}[x + y]}, \frac{2 (\operatorname{Cosh}[x] + \operatorname{Cosh}[y])}{\operatorname{Cosh}[x - y] + \operatorname{Cosh}[x + y]} \right\}$$

**TrigToExp**



The command `TrigToExp` converts direct and inverse trigonometric and hyperbolic functions to exponentials or logarithmic functions. It tries, where possible, to give results that do not involve explicit complex numbers. Here are some examples.

`TrigToExp[Sinh[2 z]]`

$$-\frac{1}{2} e^{-2z} + \frac{e^{2z}}{2}$$

`Sinh[z] Tanh[2 z] // TrigToExp`

$$\frac{(-e^{-z} + e^z) (-e^{-2z} + e^{2z})}{2 (e^{-2z} + e^{2z})}$$

`TrigToExp[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}]`

$$\left\{ -\frac{e^{-z}}{2} + \frac{e^z}{2}, \frac{e^{-z}}{2} + \frac{e^z}{2}, \frac{-e^{-z} + e^z}{e^{-z} + e^z}, \frac{e^{-z} + e^z}{-e^{-z} + e^z}, \frac{2}{-e^{-z} + e^z}, \frac{2}{e^{-z} + e^z} \right\}$$

### ExpToTrig

The command `ExpToTrig` converts exponentials to trigonometric or hyperbolic functions. It tries, where possible, to give results that do not involve explicit complex numbers. It is approximately opposite to `TrigToExp`. Here are some examples.

`ExpToTrig[e^{x\beta}]`

$$\text{Cosh}[x\beta] + \text{Sinh}[x\beta]$$

$$\frac{e^{x\alpha} - e^{x\beta}}{e^{x\gamma} + e^{x\delta}} // \text{ExpToTrig}$$

$$\frac{\text{Cosh}[x\alpha] - \text{Cosh}[x\beta] + \text{Sinh}[x\alpha] - \text{Sinh}[x\beta]}{\text{Cosh}[x\gamma] + \text{Cosh}[x\delta] + \text{Sinh}[x\gamma] + \text{Sinh}[x\delta]}$$

`ExpToTrig[TrigToExp[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}]]`

$$\{\text{Sinh}[z], \text{Cosh}[z], \text{Tanh}[z], \text{Coth}[z], \text{Csch}[z], \text{Sech}[z]\}$$

`ExpToTrig[{α e^{-xβ} + α e^{xβ}, α e^{-xβ} + γ e^{ixβ}]`

$$\{2\alpha \text{Cosh}[x\beta], \gamma \text{Cos}[x\beta] + \alpha \text{Cosh}[x\beta] + i\gamma \text{Sin}[x\beta] - \alpha \text{Sinh}[x\beta]\}$$

### ComplexExpand

The function `ComplexExpand` expands expressions assuming that all the occurring variables are real. The value option `TargetFunctions` is a list of functions from the set `{Re, Im, Abs, Arg, Conjugate, Sign}`. `ComplexExpand` tries to give results in terms of the specified functions. Here are some examples.

`ComplexExpand[Sinh[x + i y] Cosh[x - i y]]`

$$\text{Cos}[y]^2 \text{Cosh}[x] \text{Sinh}[x] + \text{Cosh}[x] \text{Sin}[y]^2 \text{Sinh}[x] + i (\text{Cos}[y] \text{Cosh}[x]^2 \text{Sin}[y] - \text{Cos}[y] \text{Sin}[y] \text{Sinh}[x]^2)$$

**Csch[x + i y] Sech[x - i y] // ComplexExpand**

$$-\frac{4 \cos [y]^2 \cosh [x] \sinh [x]}{(\cos [2 y]-\cosh [2 x])(\cos [2 y]+\cosh [2 x])}-\frac{4 \cosh [x] \sin [y]^2 \sinh [x]}{(\cos [2 y]-\cosh [2 x])(\cos [2 y]+\cosh [2 x])}+i\left(\frac{4 \cos [y] \cosh [x]^2 \sin [y]}{(\cos [2 y]-\cosh [2 x])(\cos [2 y]+\cosh [2 x])}-\frac{4 \cos [y] \sin [y] \sinh [x]^2}{(\cos [2 y]-\cosh [2 x])(\cos [2 y]+\cosh [2 x])}\right)$$

**li1 = {Sinh[x + i y], Cosh[x + i y], Tanh[x + i y], Coth[x + i y], Csch[x + i y], Sech[x + i y]}**

{Sinh[x + i y], Cosh[x + i y], Tanh[x + i y], Coth[x + i y], Csch[x + i y], Sech[x + i y]}

**ComplexExpand[li1]**

$$\left\{i \cosh [x] \sin [y]+\cos [y] \sinh [x], \cos [y] \cosh [x]+i \sin [y] \sinh [x], \frac{i \sin [2 y]}{\cos [2 y]+\cosh [2 x]}+\frac{\sinh [2 x]}{\cos [2 y]+\cosh [2 x]}, \frac{i \sin [2 y]}{\cos [2 y]-\cosh [2 x]}-\frac{\sinh [2 x]}{\cos [2 y]-\cosh [2 x]}, \frac{2 i \cosh [x] \sin [y]}{\cos [2 y]-\cosh [2 x]}-\frac{2 \cos [y] \sinh [x]}{\cos [2 y]-\cosh [2 x]}, \frac{2 \cos [y] \cosh [x]}{\cos [2 y]+\cosh [2 x]}-\frac{2 i \sin [y] \sinh [x]}{\cos [2 y]+\cosh [2 x]}\right\}$$

**ComplexExpand[Re[#] & /@ li1, TargetFunctions -> {Re, Im}]**

$$\left\{\cos [y] \sinh [x], \cos [y] \cosh [x], \frac{\sinh [2 x]}{\cos [2 y]+\cosh [2 x]}, -\frac{\sinh [2 x]}{\cos [2 y]-\cosh [2 x]}, -\frac{2 \cos [y] \sinh [x]}{\cos [2 y]-\cosh [2 x]}, \frac{2 \cos [y] \cosh [x]}{\cos [2 y]+\cosh [2 x]}\right\}$$

**ComplexExpand[Im[#] & /@ li1, TargetFunctions -> {Re, Im}]**

$$\left\{\cosh [x] \sin [y], \sin [y] \sinh [x], \frac{\sin [2 y]}{\cos [2 y]+\cosh [2 x]}, \frac{\sin [2 y]}{\cos [2 y]-\cosh [2 x]}, \frac{2 \cosh [x] \sin [y]}{\cos [2 y]-\cosh [2 x]}, -\frac{2 \sin [y] \sinh [x]}{\cos [2 y]+\cosh [2 x]}\right\}$$

**ComplexExpand[Abs[#] & /@ li1, TargetFunctions -> {Re, Im}]**

$$\left\{ \sqrt{\cosh[x]^2 \sin[y]^2 + \cos[y]^2 \sinh[x]^2}, \sqrt{\cos[y]^2 \cosh[x]^2 + \sin[y]^2 \sinh[x]^2}, \right. \\ \sqrt{\frac{\sin[2y]^2}{(\cos[2y] + \cosh[2x])^2} + \frac{\sinh[2x]^2}{(\cos[2y] + \cosh[2x])^2}}, \\ \sqrt{\frac{\sin[2y]^2}{(\cos[2y] - \cosh[2x])^2} + \frac{\sinh[2x]^2}{(\cos[2y] - \cosh[2x])^2}}, \\ \sqrt{\frac{4 \cosh[x]^2 \sin[y]^2}{(\cos[2y] - \cosh[2x])^2} + \frac{4 \cos[y]^2 \sinh[x]^2}{(\cos[2y] - \cosh[2x])^2}}, \\ \left. \sqrt{\frac{4 \cos[y]^2 \cosh[x]^2}{(\cos[2y] + \cosh[2x])^2} + \frac{4 \sin[y]^2 \sinh[x]^2}{(\cos[2y] + \cosh[2x])^2}} \right\}$$

**% // Simplify[#, {x, y} ∈ Reals] &**

$$\left\{ \frac{\sqrt{-\cos[2y] + \cosh[2x]}}{\sqrt{2}}, \frac{\sqrt{\cos[2y] + \cosh[2x]}}{\sqrt{2}}, \frac{\sqrt{\sin[2y]^2 + \sinh[2x]^2}}{\cos[2y] + \cosh[2x]}, \right. \\ \left. \sqrt{-\frac{\cos[2y] + \cosh[2x]}{\cos[2y] - \cosh[2x]}}, \frac{\sqrt{2}}{\sqrt{-\cos[2y] + \cosh[2x]}}, \frac{\sqrt{2}}{\sqrt{\cos[2y] + \cosh[2x]}} \right\}$$

**ComplexExpand[Arg[#] & /@ l11, TargetFunctions → {Re, Im}]**

$$\left\{ \text{ArcTan}[\cos[y] \sinh[x], \cosh[x] \sin[y]], \text{ArcTan}[\cos[y] \cosh[x], \sin[y] \sinh[x]], \right. \\ \text{ArcTan}\left[\frac{\sinh[2x]}{\cos[2y] + \cosh[2x]}, \frac{\sin[2y]}{\cos[2y] + \cosh[2x]}\right], \\ \text{ArcTan}\left[-\frac{\sinh[2x]}{\cos[2y] - \cosh[2x]}, \frac{\sin[2y]}{\cos[2y] - \cosh[2x]}\right], \\ \text{ArcTan}\left[-\frac{2 \cos[y] \sinh[x]}{\cos[2y] - \cosh[2x]}, \frac{2 \cosh[x] \sin[y]}{\cos[2y] - \cosh[2x]}\right], \\ \left. \text{ArcTan}\left[\frac{2 \cos[y] \cosh[x]}{\cos[2y] + \cosh[2x]}, -\frac{2 \sin[y] \sinh[x]}{\cos[2y] + \cosh[2x]}\right] \right\}$$

**% // Simplify[#, {x, y} ∈ Reals] &**

$$\left\{ \text{ArcTan}[\cos[y] \sinh[x], \cosh[x] \sin[y]], \text{ArcTan}[\cos[y] \cosh[x], \sin[y] \sinh[x]], \right. \\ \text{ArcTan}[\sinh[2x], \sin[2y]], \text{ArcTan}[\cosh[x] \sinh[x], -\cos[y] \sin[y]], \\ \left. \text{ArcTan}[\cos[y] \sinh[x], -\cosh[x] \sin[y]], \text{ArcTan}[\cos[y] \cosh[x], -\sin[y] \sinh[x]] \right\}$$

**ComplexExpand[Conjugate[#] & /@ l11, TargetFunctions → {Re, Im}] // Simplify**

$$\left\{ -i \operatorname{Cosh}[x] \operatorname{Sin}[y] + \operatorname{Cos}[y] \operatorname{Sinh}[x], \operatorname{Cos}[y] \operatorname{Cosh}[x] - i \operatorname{Sin}[y] \operatorname{Sinh}[x], \right. \\ \left. \frac{-i \operatorname{Sin}[2y] + \operatorname{Sinh}[2x]}{\operatorname{Cos}[2y] + \operatorname{Cosh}[2x]}, -\frac{i \operatorname{Sin}[2y] + \operatorname{Sinh}[2x]}{\operatorname{Cos}[2y] - \operatorname{Cosh}[2x]}, \right. \\ \left. \frac{1}{-i \operatorname{Cosh}[x] \operatorname{Sin}[y] + \operatorname{Cos}[y] \operatorname{Sinh}[x]}, \frac{1}{\operatorname{Cos}[y] \operatorname{Cosh}[x] - i \operatorname{Sin}[y] \operatorname{Sinh}[x]} \right\}$$

### Simplify

The command `Simplify` performs a sequence of algebraic transformations on an expression, and returns the simplest form it finds. Here are two examples.

```
Simplify[Sinh[2 z] / Sinh[z]]
```

```
2 Cosh[z]
```

```
Sinh[2 z] / Cosh[z] // Simplify
```

```
2 Sinh[z]
```

Here is a large collection of hyperbolic identities. All are written as one large logical conjunction.

$$\text{Simplify}[\#] \ \& \ /@ \left( \begin{aligned} & \text{Cosh}[z]^2 - \text{Sinh}[z]^2 == 1 \wedge \\ & \text{Sinh}[z]^2 == \frac{\text{Cosh}[2z] - 1}{2} \wedge \text{Cosh}[z]^2 == \frac{1 + \text{Cosh}[2z]}{2} \wedge \\ & \text{Tanh}[z]^2 == \frac{\text{Cosh}[2z] - 1}{\text{Cosh}[2z] + 1} \wedge \text{Coth}[z]^2 == \frac{\text{Cosh}[2z] + 1}{\text{Cosh}[2z] - 1} \wedge \\ & \text{Sinh}[2z] == 2 \text{Sinh}[z] \text{Cosh}[z] \wedge \text{Cosh}[2z] == \text{Cosh}[z]^2 + \text{Sinh}[z]^2 == 2 \text{Cosh}[z]^2 - 1 \wedge \\ & \text{Sinh}[a + b] == \text{Sinh}[a] \text{Cosh}[b] + \text{Cosh}[a] \text{Sinh}[b] \wedge \\ & \quad \text{Sinh}[a - b] == \text{Sinh}[a] \text{Cosh}[b] - \text{Cosh}[a] \text{Sinh}[b] \wedge \\ & \text{Cosh}[a + b] == \text{Cosh}[a] \text{Cosh}[b] + \text{Sinh}[a] \text{Sinh}[b] \wedge \\ & \quad \text{Cosh}[a - b] == \text{Cosh}[a] \text{Cosh}[b] - \text{Sinh}[a] \text{Sinh}[b] \wedge \\ & \text{Sinh}[a] + \text{Sinh}[b] == 2 \text{Sinh}\left[\frac{a+b}{2}\right] \text{Cosh}\left[\frac{a-b}{2}\right] \wedge \\ & \quad \text{Sinh}[a] - \text{Sinh}[b] == 2 \text{Cosh}\left[\frac{a+b}{2}\right] \text{Sinh}\left[\frac{a-b}{2}\right] \wedge \\ & \text{Cosh}[a] + \text{Cosh}[b] == 2 \text{Cosh}\left[\frac{a+b}{2}\right] \text{Cosh}\left[\frac{a-b}{2}\right] \wedge \\ & \quad \text{Cosh}[a] - \text{Cosh}[b] == -2 \text{Sinh}\left[\frac{a+b}{2}\right] \text{Sinh}\left[\frac{b-a}{2}\right] \wedge \\ & \text{Tanh}[a] + \text{Tanh}[b] == \frac{\text{Sinh}[a+b]}{\text{Cosh}[a] \text{Cosh}[b]} \wedge \text{Tanh}[a] - \text{Tanh}[b] == \frac{\text{Sinh}[a-b]}{\text{Cosh}[a] \text{Cosh}[b]} \wedge \\ & A \text{Sinh}[z] + B \text{Cosh}[z] == A \sqrt{1 - \frac{B^2}{A^2}} \text{Sinh}\left[z + \text{ArcTanh}\left[\frac{B}{A}\right]\right] \wedge \\ & \text{Sinh}[a] \text{Sinh}[b] == \frac{\text{Cosh}[a+b] - \text{Cosh}[a-b]}{2} \wedge \text{Cosh}[a] \text{Cosh}[b] == \\ & \quad \frac{\text{Cosh}[a-b] + \text{Cosh}[a+b]}{2} \wedge \text{Sinh}[a] \text{Cosh}[b] == \frac{\text{Sinh}[a+b] + \text{Sinh}[a-b]}{2} \wedge \\ & \text{Sinh}\left[\frac{z}{2}\right]^2 == \frac{\text{Cosh}[z] - 1}{2} \wedge \text{Cosh}\left[\frac{z}{2}\right]^2 == \frac{1 + \text{Cosh}[z]}{2} \wedge \\ & \text{Tanh}\left[\frac{z}{2}\right] == \frac{\text{Cosh}[z] - 1}{\text{Sinh}[z]} == \frac{\text{Sinh}[z]}{1 + \text{Cosh}[z]} \wedge \text{Coth}\left[\frac{z}{2}\right] == \frac{\text{Sinh}[z]}{\text{Cosh}[z] - 1} == \frac{1 + \text{Cosh}[z]}{\text{Sinh}[z]} \end{aligned} \right)$$

True

The command `Simplify` has the `Assumption` option. For example, *Mathematica* knows that  $\sinh(x) > 0$  for all real positive  $x$ , and uses the periodicity of hyperbolic functions for the symbolic integer coefficient  $k$  of  $k\pi i$ .

`Simplify[Abs[Sinh[x]] > 0, x > 0]`

True

`Abs[Sinh[x]] > 0 // Simplify[#, x > 0] &`

True

```
Simplify[{Sinh[z + 2 k π i], Cosh[z + 2 k π i], Tanh[z + k π i],
  Coth[z + k π i], Csch[z + 2 k π i], Sech[z + 2 k π i]}, k ∈ Integers]
```

```
{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}
```

```
Simplify[{Sinh[z + k π i] / Sinh[z], Cosh[z + k π i] / Cosh[z], Tanh[z + k π i] / Tanh[z],
  Coth[z + k π i] / Coth[z], Csch[z + k π i] / Csch[z], Sech[z + k π i] / Sech[z]}, k ∈ Integers]
```

```
{(-1)k, (-1)k, 1, 1, (-1)k, (-1)k}
```

*Mathematica* also knows that the composition of inverse and direct hyperbolic functions produces the value of the inner argument under the appropriate restriction. Here are some examples.

```
Simplify[{ArcSinh[Sinh[z]], ArcTanh[Tanh[z]],
  ArcCoth[Coth[z]], ArcCsch[Csch[z]]}, -π / 2 < Im[z] < π / 2]
```

```
{z, z, z, z}
```

```
Simplify[{ArcCosh[Cosh[z]], ArcSech[Sech[z]]}, -π < Im[z] < π ∧ Re[z] > 0]
```

```
{z, z}
```

### FunctionExpand (and Together)

While the hyperbolic functions auto-evaluate for simple fractions of  $\pi i$ , for more complicated cases they stay as hyperbolic functions to avoid the build up of large expressions. Using the function `FunctionExpand`, such expressions can be transformed into explicit radicals.

$$\text{Cosh}\left[\frac{\pi i}{32}\right]$$

$$\text{Cos}\left[\frac{\pi}{32}\right]$$

$$\text{FunctionExpand}\left[\text{Cosh}\left[\frac{\pi i}{32}\right]\right]$$

$$\frac{1}{2} \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}$$

$$\text{Coth}\left[\frac{\pi i}{24}\right] // \text{FunctionExpand}$$

$$\frac{i \left( \frac{\sqrt{2-\sqrt{2}}}{4} + \frac{1}{4} \sqrt{3(2+\sqrt{2})} \right)}{-\frac{1}{4} \sqrt{3(2-\sqrt{2})} + \frac{\sqrt{2+\sqrt{2}}}{4}}$$

$$\left\{ \operatorname{Sinh}\left[\frac{\pi i}{16}\right], \operatorname{Cosh}\left[\frac{\pi i}{16}\right], \operatorname{Tanh}\left[\frac{\pi i}{16}\right], \operatorname{Coth}\left[\frac{\pi i}{16}\right], \operatorname{Csch}\left[\frac{\pi i}{16}\right], \operatorname{Sech}\left[\frac{\pi i}{16}\right] \right\}$$

$$\left\{ i \operatorname{Sin}\left[\frac{\pi}{16}\right], \operatorname{Cos}\left[\frac{\pi}{16}\right], i \operatorname{Tan}\left[\frac{\pi}{16}\right], -i \operatorname{Cot}\left[\frac{\pi}{16}\right], -i \operatorname{Csc}\left[\frac{\pi}{16}\right], \operatorname{Sec}\left[\frac{\pi}{16}\right] \right\}$$

**FunctionExpand[%]**

$$\left\{ \frac{1}{2} i \sqrt{2 - \sqrt{2 + \sqrt{2}}}, \frac{1}{2} \sqrt{2 + \sqrt{2 + \sqrt{2}}}, i \sqrt{\frac{2 - \sqrt{2 + \sqrt{2}}}{2 + \sqrt{2 + \sqrt{2}}}}, \right.$$

$$\left. -i \sqrt{\frac{2 + \sqrt{2 + \sqrt{2}}}{2 - \sqrt{2 + \sqrt{2}}}}, -\frac{2i}{\sqrt{2 - \sqrt{2 + \sqrt{2}}}}, \frac{2}{\sqrt{2 + \sqrt{2 + \sqrt{2}}}} \right\}$$

$$\left\{ \operatorname{Sinh}\left[\frac{\pi i}{60}\right], \operatorname{Cosh}\left[\frac{\pi i}{60}\right], \operatorname{Tanh}\left[\frac{\pi i}{60}\right], \operatorname{Coth}\left[\frac{\pi i}{60}\right], \operatorname{Csch}\left[\frac{\pi i}{60}\right], \operatorname{Sech}\left[\frac{\pi i}{60}\right] \right\}$$

$$\left\{ i \operatorname{Sin}\left[\frac{\pi}{60}\right], \operatorname{Cos}\left[\frac{\pi}{60}\right], i \operatorname{Tan}\left[\frac{\pi}{60}\right], -i \operatorname{Cot}\left[\frac{\pi}{60}\right], -i \operatorname{Csc}\left[\frac{\pi}{60}\right], \operatorname{Sec}\left[\frac{\pi}{60}\right] \right\}$$

**Together[FunctionExpand[%]]**

$$\left\{ \frac{1}{16} i \left( -\sqrt{2} - \sqrt{6} + \sqrt{10} + \sqrt{30} + 2\sqrt{5 + \sqrt{5}} - 2\sqrt{3(5 + \sqrt{5})} \right), \right.$$

$$\frac{1}{16} \left( \sqrt{2} - \sqrt{6} - \sqrt{10} + \sqrt{30} + 2\sqrt{5 + \sqrt{5}} + 2\sqrt{3(5 + \sqrt{5})} \right),$$

$$- \frac{i \left( 1 + \sqrt{3} - \sqrt{5} - \sqrt{15} - \sqrt{2(5 + \sqrt{5})} + \sqrt{6(5 + \sqrt{5})} \right)}{1 - \sqrt{3} - \sqrt{5} + \sqrt{15} + \sqrt{2(5 + \sqrt{5})} + \sqrt{6(5 + \sqrt{5})}},$$

$$i \frac{\left( 1 - \sqrt{3} - \sqrt{5} + \sqrt{15} + \sqrt{2(5 + \sqrt{5})} + \sqrt{6(5 + \sqrt{5})} \right)}{1 + \sqrt{3} - \sqrt{5} - \sqrt{15} - \sqrt{2(5 + \sqrt{5})} + \sqrt{6(5 + \sqrt{5})}},$$

$$- \frac{16i}{-\sqrt{2} - \sqrt{6} + \sqrt{10} + \sqrt{30} + 2\sqrt{5 + \sqrt{5}} - 2\sqrt{3(5 + \sqrt{5})}},$$

$$\left. \frac{16}{\sqrt{2} - \sqrt{6} - \sqrt{10} + \sqrt{30} + 2\sqrt{5 + \sqrt{5}} + 2\sqrt{3(5 + \sqrt{5})}} \right\}$$

If the denominator contains squares of integers other than 2, the results always contain complex numbers (meaning that the imaginary number  $i = \sqrt{-1}$  appears unavoidably).

$$\left\{ \text{Sinh}\left[\frac{\pi i}{9}\right], \text{Cosh}\left[\frac{\pi i}{9}\right], \text{Tanh}\left[\frac{\pi i}{9}\right], \text{Coth}\left[\frac{\pi i}{9}\right], \text{Csch}\left[\frac{\pi i}{9}\right], \text{Sech}\left[\frac{\pi i}{9}\right] \right\}$$

$$\left\{ i \text{Sin}\left[\frac{\pi}{9}\right], \text{Cos}\left[\frac{\pi}{9}\right], i \text{Tan}\left[\frac{\pi}{9}\right], -i \text{Cot}\left[\frac{\pi}{9}\right], -i \text{Csc}\left[\frac{\pi}{9}\right], \text{Sec}\left[\frac{\pi}{9}\right] \right\}$$

**FunctionExpand[%] // Together**

$$\left\{ \frac{1}{8} \left( 2^{2/3} (-1 - i \sqrt{3})^{1/3} + i 2^{2/3} \sqrt{3} (-1 - i \sqrt{3})^{1/3} - 2^{2/3} (-1 + i \sqrt{3})^{1/3} + i 2^{2/3} \sqrt{3} (-1 + i \sqrt{3})^{1/3} \right), \right.$$

$$\frac{1}{8} \left( 2^{2/3} (-1 - i \sqrt{3})^{1/3} + i 2^{2/3} \sqrt{3} (-1 - i \sqrt{3})^{1/3} + 2^{2/3} (-1 + i \sqrt{3})^{1/3} - i 2^{2/3} \sqrt{3} (-1 + i \sqrt{3})^{1/3} \right),$$

$$\frac{-i (-1 - i \sqrt{3})^{1/3} + \sqrt{3} (-1 - i \sqrt{3})^{1/3} + i (-1 + i \sqrt{3})^{1/3} + \sqrt{3} (-1 + i \sqrt{3})^{1/3}}{-i (-1 - i \sqrt{3})^{1/3} + \sqrt{3} (-1 - i \sqrt{3})^{1/3} - i (-1 + i \sqrt{3})^{1/3} - \sqrt{3} (-1 + i \sqrt{3})^{1/3}},$$

$$\frac{-i (-1 - i \sqrt{3})^{1/3} + \sqrt{3} (-1 - i \sqrt{3})^{1/3} - i (-1 + i \sqrt{3})^{1/3} - \sqrt{3} (-1 + i \sqrt{3})^{1/3}}{-i (-1 - i \sqrt{3})^{1/3} + \sqrt{3} (-1 - i \sqrt{3})^{1/3} + i (-1 + i \sqrt{3})^{1/3} + \sqrt{3} (-1 + i \sqrt{3})^{1/3}},$$

$$- (8 i) / \left( -i 2^{2/3} (-1 - i \sqrt{3})^{1/3} + 2^{2/3} \sqrt{3} (-1 - i \sqrt{3})^{1/3} + i 2^{2/3} (-1 + i \sqrt{3})^{1/3} + 2^{2/3} \sqrt{3} (-1 + i \sqrt{3})^{1/3} \right),$$

$$- (8 i) / \left( -i 2^{2/3} (-1 - i \sqrt{3})^{1/3} + 2^{2/3} \sqrt{3} (-1 - i \sqrt{3})^{1/3} - i 2^{2/3} (-1 + i \sqrt{3})^{1/3} - 2^{2/3} \sqrt{3} (-1 + i \sqrt{3})^{1/3} \right) \left. \right\}$$

Here the function `RootReduce` is used to express the previous algebraic numbers as numbered roots of polynomial equations.

**RootReduce[Simplify[%]]**

$$\left\{ \text{Root}\left[3 + 36 \#1^2 + 96 \#1^4 + 64 \#1^6 \ \&, 4\right], \text{Root}\left[-1 - 6 \#1 + 8 \#1^3 \ \&, 3\right], \right.$$

$$\text{Root}\left[3 + 27 \#1^2 + 33 \#1^4 + \#1^6 \ \&, 4\right], \text{Root}\left[1 + 33 \#1^2 + 27 \#1^4 + 3 \#1^6 \ \&, 3\right],$$

$$\text{Root}\left[64 + 96 \#1^2 + 36 \#1^4 + 3 \#1^6 \ \&, 5\right], \text{Root}\left[-8 + 6 \#1^2 + \#1^3 \ \&, 3\right] \left. \right\}$$

The function `FunctionExpand` also reduces hyperbolic expressions with compound arguments or compositions, including hyperbolic functions, to simpler forms. Here are some examples.

**FunctionExpand[Coth[ $\sqrt{-z^2}$ ]]**



$$\frac{\sqrt{-z} \operatorname{Cot}[z]}{\sqrt{z}}$$

**Tanh** $\left[\sqrt{i z^2}\right]$  // **FunctionExpand**

$$\frac{(-1)^{3/4} \sqrt{-(-1)^{3/4} z} \sqrt{(-1)^{3/4} z} \operatorname{Tanh}\left[(-1)^{1/4} z\right]}{z}$$

**{Sinh** $\left[\sqrt{z^2}\right]$ , **Cosh** $\left[\sqrt{z^2}\right]$ , **Tanh** $\left[\sqrt{z^2}\right]$ ,  
**Coth** $\left[\sqrt{z^2}\right]$ , **Csch** $\left[\sqrt{z^2}\right]$ , **Sech** $\left[\sqrt{z^2}\right]$ } // **FunctionExpand**

$$\left\{ \frac{\sqrt{-i z} \sqrt{i z} \operatorname{Sinh}[z]}{z}, \operatorname{Cosh}[z], \frac{\sqrt{-i z} \sqrt{i z} \operatorname{Tanh}[z]}{z}, \right. \\ \left. \frac{\sqrt{-i z} \sqrt{i z} \operatorname{Coth}[z]}{z}, \frac{\sqrt{-i z} \sqrt{i z} \operatorname{Csch}[z]}{z}, \operatorname{Sech}[z] \right\}$$

Applying **Simplify** to the last expression gives a more compact result.

**Simplify** $[\%]$

$$\left\{ \frac{\sqrt{z^2} \operatorname{Sinh}[z]}{z}, \operatorname{Cosh}[z], \frac{\sqrt{z^2} \operatorname{Tanh}[z]}{z}, \frac{\sqrt{z^2} \operatorname{Coth}[z]}{z}, \frac{\sqrt{z^2} \operatorname{Csch}[z]}{z}, \operatorname{Sech}[z] \right\}$$

Here are some similar examples.

**Sinh** $[2 \operatorname{ArcTanh}[z]]$  // **FunctionExpand**

$$\frac{2 z}{1 - z^2}$$

**Cosh** $\left[\frac{\operatorname{ArcCoth}[z]}{2}\right]$  // **FunctionExpand**

$$\frac{\sqrt{1 + \frac{\sqrt{-i z} \sqrt{i z}}{\sqrt{(-1+z)(1+z)}}}}{\sqrt{2}}$$

**{Sinh** $[2 \operatorname{ArcSinh}[z]]$ , **Cosh** $[2 \operatorname{ArcCosh}[z]]$ , **Tanh** $[2 \operatorname{ArcTanh}[z]]$ ,  
**Coth** $[2 \operatorname{ArcCoth}[z]]$ , **Csch** $[2 \operatorname{ArcCsch}[z]]$ , **Sech** $[2 \operatorname{ArcSech}[z]]$ } // **FunctionExpand**

$$\left\{ 2 z \sqrt{i(-i+z)} \sqrt{-i(i+z)}, z^2 + (-1+z)(1+z), -\frac{2(-1+z)z(1+z)}{(1-z^2)(1+z^2)}, \right. \\ \left. \frac{1}{2} \left(1 - \frac{1}{z^2}\right) z \left(\frac{1}{(-1+z)(1+z)} + \frac{z^2}{(-1+z)(1+z)}\right), \frac{\sqrt{-z} z^{3/2}}{2 \sqrt{-1-z^2}}, \frac{z^2}{2-z^2} \right\}$$

$$\left\{ \text{Sinh}\left[\frac{\text{ArcSinh}[z]}{2}\right], \text{Cosh}\left[\frac{\text{ArcCosh}[z]}{2}\right], \text{Tanh}\left[\frac{\text{ArcTanh}[z]}{2}\right], \right. \\ \left. \text{Coth}\left[\frac{\text{ArcCoth}[z]}{2}\right], \text{Csch}\left[\frac{\text{ArcCsch}[z]}{2}\right], \text{Sech}\left[\frac{\text{ArcSech}[z]}{2}\right] \right\} // \text{FunctionExpand}$$

$$\left\{ \frac{z \sqrt{-1 + \sqrt{i(-i+z)} \sqrt{-i(i+z)}}}{\sqrt{2} \sqrt{-i z} \sqrt{i z}}, \frac{\sqrt{1+z}}{\sqrt{2}}, \frac{z}{1 + \sqrt{1-z} \sqrt{1+z}}, \right. \\ \left. z \left( 1 + \frac{\sqrt{(-1+z)(1+z)}}{\sqrt{-i z} \sqrt{i z}} \right), \frac{\sqrt{2} \sqrt{-\frac{i}{z}} \sqrt{\frac{i}{z}} z}{\sqrt{-1 + \frac{\sqrt{-1-z^2}}{\sqrt{-z} \sqrt{z}}}}, \frac{\sqrt{2} \sqrt{-z}}{\sqrt{-1-z}} \right\}$$

**Simplify[%]**

$$\left\{ \frac{z \sqrt{-1 + \sqrt{1+z^2}}}{\sqrt{2} \sqrt{z^2}}, \frac{\sqrt{1+z}}{\sqrt{2}}, \frac{z}{1 + \sqrt{1-z^2}}, z + \frac{\sqrt{z^2} \sqrt{-1+z^2}}{z}, \frac{\sqrt{2} \sqrt{\frac{1}{z^2}} z}{\sqrt{-1 + \sqrt{1 + \frac{1}{z^2}}}}, \frac{\sqrt{2}}{\sqrt{1 + \frac{1}{z}}} \right\}$$

### FullSimplify

The function FullSimplify tries a wider range of transformations than the function Simplify and returns the simplest form it finds. Here are some examples that contrast the results of applying these functions to the same expressions.

$$\text{Cosh}\left[\frac{1}{2} \text{Log}[1 - i z] - \frac{1}{2} \text{Log}[1 + i z]\right] // \text{Simplify}$$

$$\text{Cosh}\left[\frac{1}{2} (\text{Log}[1 - i z] - \text{Log}[1 + i z])\right]$$

**% // FullSimplify**

$$\frac{1}{\sqrt{1+z^2}}$$

$$\left\{ \text{Sinh}\left[-\text{Log}\left[i z + \sqrt{1-z^2}\right]\right], \text{Cosh}\left[-\text{Log}\left[i z + \sqrt{1-z^2}\right]\right], \right. \\ \text{Tanh}\left[-\text{Log}\left[i z + \sqrt{1-z^2}\right]\right], \text{Coth}\left[-\text{Log}\left[i z + \sqrt{1-z^2}\right]\right], \\ \left. \text{Csch}\left[-\text{Log}\left[i z + \sqrt{1-z^2}\right]\right], \text{Sech}\left[-\text{Log}\left[i z + \sqrt{1-z^2}\right]\right] \right\} // \text{Simplify}$$

$$\left\{ -i z, \frac{1 - z^2 + i z \sqrt{1 - z^2}}{i z + \sqrt{1 - z^2}}, -\frac{-1 + \left( i z + \sqrt{1 - z^2} \right)^2}{1 + \left( i z + \sqrt{1 - z^2} \right)^2}, \right.$$

$$\left. -\frac{1 + \left( i z + \sqrt{1 - z^2} \right)^2}{-1 + \left( i z + \sqrt{1 - z^2} \right)^2}, \frac{i}{z}, \frac{2 \left( i z + \sqrt{1 - z^2} \right)}{1 + \left( i z + \sqrt{1 - z^2} \right)^2} \right\}$$

$$\left\{ \text{Sinh}\left[-\text{Log}\left[i z + \sqrt{1 - z^2}\right]\right], \text{Cosh}\left[-\text{Log}\left[i z + \sqrt{1 - z^2}\right]\right], \right.$$

$$\text{Tanh}\left[-\text{Log}\left[i z + \sqrt{1 - z^2}\right]\right], \text{Coth}\left[-\text{Log}\left[i z + \sqrt{1 - z^2}\right]\right],$$

$$\left. \text{Csch}\left[-\text{Log}\left[i z + \sqrt{1 - z^2}\right]\right], \text{Sech}\left[-\text{Log}\left[i z + \sqrt{1 - z^2}\right]\right] \right\} // \text{FullSimplify}$$

$$\left\{ -i z, \sqrt{1 - z^2}, -\frac{i z}{\sqrt{1 - z^2}}, \frac{i \sqrt{1 - z^2}}{z}, \frac{i}{z}, \frac{1}{\sqrt{1 - z^2}} \right\}$$

### Operations performed by specialized *Mathematica* functions

#### Series expansions

Calculating the series expansion of hyperbolic functions to hundreds of terms can be done in seconds. Here are some examples.

`Series[Sinh[z], {z, 0, 5}]`

$$z + \frac{z^3}{6} + \frac{z^5}{120} + O[z]^6$$

`Normal[%]`

$$z + \frac{z^3}{6} + \frac{z^5}{120}$$

`Series[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}, {z, 0, 3}]`

$$\left\{ z + \frac{z^3}{6} + O[z]^4, 1 + \frac{z^2}{2} + O[z]^4, z - \frac{z^3}{3} + O[z]^4, \right.$$

$$\left. \frac{1}{z} + \frac{z}{3} - \frac{z^3}{45} + O[z]^4, \frac{1}{z} - \frac{z}{6} + \frac{7 z^3}{360} + O[z]^4, 1 - \frac{z^2}{2} + O[z]^4 \right\}$$

`Series[Coth[z], {z, 0, 100}] // Timing`

$$\left\{ 0.79 \text{ Second}, \frac{1}{z} + \frac{z}{3} - \frac{z^3}{45} + \frac{2 z^5}{945} - \frac{z^7}{4725} + \frac{2 z^9}{93555} - \frac{1382 z^{11}}{638512875} + \right.$$

$$\left. \frac{4 z^{13}}{18243225} - \frac{3617 z^{15}}{162820783125} + \frac{87734 z^{17}}{38979295480125} - \frac{349222 z^{19}}{1531329465290625} + \right.$$

$$\begin{aligned}
 & \frac{310\,732\,z^{21}}{13\,447\,856\,940\,643\,125} - \frac{472\,728\,182\,z^{23}}{201\,919\,571\,963\,756\,521\,875} + \frac{2\,631\,724\,z^{25}}{11\,094\,481\,976\,030\,578\,125} - \\
 & \frac{13\,571\,120\,588\,z^{27}}{564\,653\,660\,170\,076\,273\,671\,875} + \frac{13\,785\,346\,041\,608\,z^{29}}{5\,660\,878\,804\,669\,082\,674\,070\,015\,625} - \\
 & \frac{7\,709\,321\,041\,217\,z^{31}}{31\,245\,110\,285\,511\,170\,603\,633\,203\,125} + \frac{303\,257\,395\,102\,z^{33}}{12\,130\,454\,581\,433\,748\,587\,292\,890\,625} - \\
 & \frac{52\,630\,543\,106\,106\,954\,746\,z^{35}}{20\,777\,977\,561\,866\,588\,586\,487\,628\,662\,044\,921\,875} + \frac{616\,840\,823\,966\,644\,z^{37}}{2\,403\,467\,618\,492\,375\,776\,343\,276\,883\,984\,375} - \\
 & \frac{522\,165\,436\,992\,898\,244\,102\,z^{39}}{20\,080\,431\,172\,289\,638\,826\,798\,401\,128\,390\,556\,640\,625} + \\
 & \frac{6\,080\,390\,575\,672\,283\,210\,764\,z^{41}}{2\,307\,789\,189\,818\,960\,127\,712\,594\,427\,864\,667\,427\,734\,375} - \\
 & \frac{10\,121\,188\,937\,927\,645\,176\,372\,z^{43}}{37\,913\,679\,547\,025\,773\,526\,706\,908\,457\,776\,679\,169\,921\,875} + \\
 & \frac{207\,461\,256\,206\,578\,143\,748\,856\,z^{45}}{7\,670\,102\,214\,448\,301\,053\,033\,358\,480\,610\,212\,529\,462\,890\,625} - \\
 & \frac{11\,218\,806\,737\,995\,635\,372\,498\,255\,094\,z^{47}}{4\,093\,648\,603\,384\,274\,996\,519\,698\,921\,478\,879\,580\,162\,286\,669\,921\,875} + \\
 & \frac{79\,209\,152\,838\,572\,743\,713\,996\,404\,z^{49}}{285\,258\,771\,457\,546\,764\,463\,363\,635\,252\,374\,414\,183\,254\,365\,234\,375} - \\
 & \frac{246\,512\,528\,657\,073\,833\,030\,130\,766\,724\,z^{51}}{8\,761\,982\,491\,474\,419\,367\,550\,817\,114\,626\,909\,562\,924\,278\,968\,505\,859\,375} + \\
 & \frac{233\,199\,709\,079\,078\,899\,371\,344\,990\,501\,528\,z^{53}}{81\,807\,125\,729\,900\,063\,867\,074\,959\,072\,425\,603\,825\,198\,823\,017\,351\,806\,640\,625} - \\
 & \frac{1\,416\,795\,959\,607\,558\,144\,963\,094\,708\,378\,988\,z^{55}}{4\,905\,352\,087\,939\,496\,310\,826\,487\,207\,538\,302\,184\,255\,342\,959\,123\,162\,841\,796\,875} + \\
 & \frac{23\,305\,824\,372\,104\,839\,134\,357\,731\,308\,699\,592\,z^{57}}{796\,392\,368\,980\,577\,121\,745\,974\,726\,570\,063\,253\,238\,310\,542\,073\,919\,837\,646\,484\,375} - \\
 & \frac{9\,721\,865\,123\,870\,044\,576\,322\,439\,952\,638\,561\,968\,331\,928\,z^{59}}{3\,278\,777\,586\,273\,629\,598\,615\,520\,165\,380\,455\,583\,231\,003\,564\,645\,636\,125\,000\,418\,914\,794\,921\,875} + \\
 & \frac{6\,348\,689\,256\,302\,894\,731\,330\,601\,216\,724\,328\,336\,z^{61}}{21\,132\,271\,510\,899\,613\,925\,529\,439\,369\,536\,628\,424\,678\,570\,233\,931\,462\,891\,949\,462\,890\,625} - \\
 & \frac{106\,783\,830\,147\,866\,529\,886\,385\,444\,979\,142\,647\,942\,017\,z^{63}}{3\,508\,062\,732\,166\,890\,409\,707\,514\,582\,539\,928\,001\,638\,766\,051\,683\,792\,497\,378\,070\,587\,158\,203\,125} + \\
 & \frac{(267\,745\,458\,568\,424\,664\,373\,021\,714\,282\,169\,516\,771\,254\,382\,z^{65})}{86\,812\,790\,293\,146\,213\,360\,651\,966\,604\,262\,937\,105\,495\,141\,563\,588\,806\,888\,204\,273\,501\,373\,291\,015} : \\
 & \frac{625 - (250\,471\,004\,320\,250\,327\,955\,196\,022\,920\,428\,000\,776\,938\,z^{67})}{801\,528\,196\,428\,242\,695\,121\,010\,267\,455\,843\,804\,062\,822\,357\,897\,831\,858\,125\,102\,407\,684\,326\,171\,875} : \\
 & + \frac{(172\,043\,582\,552\,384\,800\,434\,637\,321\,986\,040\,823\,829\,878\,646\,884\,z^{69})}{5\,433\,748\,964\,547\,053\,581\,149\,916\,185\,708\,338\,218\,048\,392\,402\,830\,337\,634\,114\,958\,370\,880\,742\,156} : \\
 & \frac{982\,421\,875 - (11\,655\,909\,923\,339\,888\,220\,876\,554\,489\,282\,134\,730\,564\,976\,603\,688\,520\,858\,z^{71})}{3\,633\,348\,205\,269\,879\,230\,856\,840\,004\,304\,821\,536\,968\,049\,780\,112\,803\,650\,817\,771\,432\,558\,560\,793} :
 \end{aligned}$$

$$\begin{aligned}
& 458\ 452\ 606\ 201\ 171\ 875 + \\
& (3\ 692\ 153\ 220\ 456\ 342\ 488\ 035\ 683\ 646\ 645\ 690\ 290\ 452\ 790\ 030\ 604\ z^{73}) / \\
& 11\ 359\ 005\ 221\ 796\ 317\ 918\ 049\ 302\ 062\ 760\ 294\ 302\ 183\ 889\ 391\ 189\ 419\ 445\ 133\ 951\ 612\ 582\ 060\ 536 \cdot \\
& 346\ 435\ 546\ 875 - (5\ 190\ 545\ 015\ 986\ 394\ 254\ 249\ 936\ 008\ 544\ 252\ 611\ 445\ 319\ 542\ 919\ 116\ z^{75}) / \\
& 157\ 606\ 197\ 452\ 423\ 911\ 112\ 934\ 066\ 120\ 799\ 083\ 442\ 801\ 465\ 302\ 753\ 194\ 801\ 233\ 578\ 624\ 576\ 089 \cdot \\
& 941\ 806\ 793\ 212\ 890\ 625 + \\
& (255\ 290\ 071\ 123\ 323\ 586\ 643\ 187\ 098\ 799\ 718\ 199\ 072\ 122\ 692\ 536\ 861\ 835\ 992\ z^{77}) / \\
& 76\ 505\ 736\ 228\ 426\ 953\ 173\ 738\ 238\ 352\ 183\ 101\ 801\ 688\ 392\ 812\ 244\ 485\ 181\ 277\ 127\ 930\ 109\ 049\ 138 \cdot \\
& 257\ 655\ 704\ 498\ 291\ 015\ 625 - \\
& (9\ 207\ 568\ 598\ 958\ 915\ 293\ 871\ 149\ 938\ 038\ 093\ 699\ 588\ 515\ 745\ 502\ 577\ 839\ 313\ 734\ z^{79}) / \\
& 27\ 233\ 582\ 984\ 369\ 795\ 892\ 070\ 228\ 410\ 001\ 578\ 355\ 986\ 013\ 571\ 390\ 071\ 723\ 225\ 259\ 349\ 721\ 067\ 988 \cdot \\
& 068\ 852\ 863\ 296\ 604\ 156\ 494\ 140\ 625 + \\
& (163\ 611\ 136\ 505\ 867\ 886\ 519\ 332\ 147\ 296\ 221\ 453\ 678\ 803\ 514\ 884\ 902\ 772\ 183\ 572\ z^{81}) / \\
& 4\ 776\ 089\ 171\ 877\ 348\ 057\ 451\ 105\ 924\ 101\ 750\ 653\ 118\ 402\ 745\ 283\ 825\ 543\ 113\ 171\ 217\ 116\ 857\ 704 \cdot \\
& 024\ 700\ 607\ 798\ 175\ 811\ 767\ 578\ 125 - \\
& (8\ 098\ 304\ 783\ 741\ 161\ 440\ 924\ 524\ 640\ 446\ 924\ 039\ 959\ 669\ 564\ 792\ 363\ 509\ 124\ 335\ 729\ 908\ z^{83}) / \\
& 2\ 333\ 207\ 846\ 470\ 426\ 678\ 843\ 707\ 227\ 616\ 712\ 214\ 909\ 162\ 634\ 745\ 895\ 349\ 325\ 948\ 586\ 531\ 533\ 393 \cdot \\
& 530\ 725\ 143\ 500\ 144\ 033\ 328\ 342\ 437\ 744\ 140\ 625 + \\
& (122\ 923\ 650\ 124\ 219\ 284\ 385\ 832\ 157\ 660\ 699\ 813\ 260\ 991\ 755\ 656\ 444\ 452\ 420\ 836\ 648\ z^{85}) / \\
& 349\ 538\ 086\ 043\ 843\ 717\ 584\ 559\ 187\ 055\ 386\ 621\ 548\ 470\ 304\ 913\ 596\ 772\ 372\ 737\ 435\ 524\ 697\ 231 \cdot \\
& 069\ 047\ 713\ 981\ 709\ 496\ 784\ 210\ 205\ 078\ 125 - \\
& (476\ 882\ 359\ 517\ 824\ 548\ 362\ 004\ 154\ 188\ 840\ 670\ 307\ 545\ 554\ 753\ 464\ 961\ 562\ 516\ 323\ 845\ 108\ z^{87}) / \\
& 13\ 383\ 510\ 964\ 174\ 348\ 021\ 497\ 060\ 628\ 653\ 950\ 829\ 663\ 288\ 548\ 327\ 870\ 152\ 944\ 013\ 988\ 358\ 928\ 114 \cdot \\
& 528\ 962\ 242\ 087\ 062\ 453\ 152\ 690\ 410\ 614\ 013\ 671\ 875 + \\
& (1\ 886\ 491\ 646\ 433\ 732\ 479\ 814\ 597\ 361\ 998\ 744\ 134\ 040\ 407\ 919\ 471\ 435\ 385\ 970\ 472\ 345\ 164\ 676\ 056 \\
& z^{89}) / \\
& 522\ 532\ 651\ 330\ 971\ 490\ 226\ 753\ 590\ 247\ 329\ 744\ 050\ 384\ 290\ 675\ 644\ 135\ 735\ 656\ 667\ 608\ 610\ 471 \cdot \\
& 400\ 391\ 047\ 234\ 539\ 824\ 350\ 830\ 981\ 313\ 610\ 076\ 904\ 296\ 875 - \\
& (450\ 638\ 590\ 680\ 882\ 618\ 431\ 105\ 331\ 665\ 591\ 912\ 924\ 988\ 342\ 163\ 281\ 788\ 877\ 675\ 244\ 114\ 763\ 912 \\
& z^{91}) / \\
& 1\ 231\ 931\ 818\ 039\ 911\ 948\ 327\ 467\ 370\ 123\ 161\ 265\ 684\ 460\ 571\ 086\ 659\ 079\ 080\ 437\ 659\ 781\ 065\ 743 \cdot \\
& 269\ 173\ 212\ 919\ 832\ 661\ 978\ 537\ 311\ 246\ 395\ 111\ 083\ 984\ 375 + \\
& (415\ 596\ 189\ 473\ 955\ 564\ 121\ 634\ 614\ 268\ 323\ 814\ 113\ 534\ 779\ 643\ 471\ 190\ 276\ 158\ 333\ 713\ 923\ 216 \\
& z^{93}) / \\
& 11\ 213\ 200\ 675\ 690\ 943\ 223\ 287\ 032\ 785\ 929\ 540\ 201\ 272\ 600\ 687\ 465\ 377\ 745\ 332\ 153\ 847\ 964\ 679\ 254 \cdot \\
& 692\ 602\ 138\ 023\ 498\ 144\ 562\ 090\ 675\ 557\ 613\ 372\ 802\ 734\ 375 - \\
& (423\ 200\ 899\ 194\ 533\ 026\ 195\ 195\ 456\ 219\ 648\ 467\ 346\ 087\ 908\ 778\ 120\ 468\ 301\ 277\ 466\ 840\ 101\ 336 \cdot \\
& 699\ 974\ 518\ z^{95}) / \\
& 112\ 694\ 926\ 530\ 960\ 148\ 011\ 367\ 752\ 417\ 874\ 063\ 473\ 378\ 698\ 369\ 880\ 587\ 800\ 838\ 274\ 234\ 349\ 237 \cdot \\
& 591\ 647\ 453\ 413\ 782\ 021\ 538\ 312\ 594\ 164\ 677\ 406\ 144\ 702\ 434\ 539\ 794\ 921\ 875 + \\
& (5\ 543\ 531\ 483\ 502\ 489\ 438\ 698\ 050\ 411\ 951\ 314\ 743\ 456\ 505\ 773\ 755\ 468\ 368\ 087\ 670\ 306\ 121\ 873\ 229 \cdot \\
& 244\ z^{97}) / \\
& 14\ 569\ 479\ 835\ 935\ 377\ 894\ 165\ 191\ 004\ 250\ 040\ 526\ 616\ 509\ 162\ 234\ 077\ 285\ 176\ 247\ 476\ 968\ 227\ 225 \cdot \\
& 810\ 918\ 346\ 966\ 001\ 491\ 701\ 692\ 846\ 112\ 140\ 419\ 483\ 184\ 814\ 453\ 125 - \\
& (378\ 392\ 151\ 276\ 488\ 501\ 180\ 909\ 732\ 277\ 974\ 887\ 490\ 811\ 366\ 132\ 267\ 744\ 533\ 542\ 784\ 817\ 245\ 581 \cdot \\
& 660\ 788\ 990\ 844\ z^{99}) / \\
& 9\ 815\ 205\ 420\ 757\ 514\ 710\ 108\ 178\ 059\ 369\ 553\ 458\ 327\ 392\ 260\ 750\ 404\ 049\ 930\ 407\ 987\ 933\ 582\ 359 \cdot \\
& 080\ 767\ 225\ 644\ 716\ 670\ 692\ 512\ 152\ 512\ 547\ 992\ 166\ 922\ 998\ 160\ 812\ 180\ 452\ 125 \cdot \dots \cdot (-101)
\end{aligned}$$

*Mathematica* comes with the add-on package `DiscreteMath`RSolve`` that allows finding the general terms of series for many functions. After loading this package, and using the package function `SeriesTerm`, the following  $n^{\text{th}}$  term for odd hyperbolic functions can be evaluated.

```
<< DiscreteMath`RSolve`
```

```
SeriesTerm[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}, {z, 0, n}] z^n
```

$$\left\{ z^n \text{If} \left[ \text{Odd}[n], \frac{1}{n!}, 0 \right], z^n \text{If} \left[ \text{Even}[n], \frac{1}{n!}, 0 \right], \right. \\ z^n \text{If} \left[ \text{Odd}[n], \frac{2^{1+n} (-1 + 2^{1+n}) \text{BernoulliB}[1 + n]}{(1 + n)!}, 0 \right], \\ \left. \frac{2^{1+n} z^n \text{BernoulliB}[1 + n]}{(1 + n)!}, \frac{2^{1+n} z^n \text{BernoulliB} \left[ 1 + n, \frac{1}{2} \right]}{(1 + n)!}, \frac{z^n \text{EulerE}[n]}{n!} \right\}$$

Here is a quick check of the last result.

This series should be evaluated to  $\{\sinh(z), \cosh(z), \tanh(z), \coth(z), \operatorname{csch}(z), \operatorname{sech}(z)\}$ , which can be concluded from the following relation.

```
Sum[#, {n, 0, 100}] & /@% -
```

```
Series[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}, {z, 0, 100}]
```

$$\left\{ O[z]^{101}, O[z]^{101}, O[z]^{101}, -\frac{1}{z} - 1 + O[z]^{101}, -\frac{1}{z} + O[z]^{101}, O[z]^{101} \right\}$$

### Differentiation

*Mathematica* can evaluate derivatives of hyperbolic functions of an arbitrary positive integer order.

```
D[Sinh[z], z]
```

```
Cosh[z]
```

```
Sinh[z] // D[#, z] &
```

```
Cosh[z]
```

```
∂z{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}
```

$$\{\text{Cosh}[z], \text{Sinh}[z], \text{Sech}[z]^2, -\text{Csch}[z]^2, -\text{Coth}[z] \text{Csch}[z], -\text{Sech}[z] \text{Tanh}[z]\}$$

```
∂{z,2}{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}
```

$$\{\text{Sinh}[z], \text{Cosh}[z], -2 \text{Sech}[z]^2 \text{Tanh}[z], 2 \text{Coth}[z] \text{Csch}[z]^2, \\ \text{Coth}[z]^2 \text{Csch}[z] + \text{Csch}[z]^3, -\text{Sech}[z]^3 + \text{Sech}[z] \text{Tanh}[z]^2\}$$

```
Table[D[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}, {z, n}], {n, 4}]
```

$$\left\{ \left\{ \text{Cosh}[z], \text{Sinh}[z], \text{Sech}[z]^2, -\text{Csch}[z]^2, -\text{Coth}[z] \text{Csch}[z], -\text{Sech}[z] \text{Tanh}[z] \right\}, \right. \\ \left\{ \text{Sinh}[z], \text{Cosh}[z], -2 \text{Sech}[z]^2 \text{Tanh}[z], 2 \text{Coth}[z] \text{Csch}[z]^2, \right. \\ \left. \text{Coth}[z]^2 \text{Csch}[z] + \text{Csch}[z]^3, -\text{Sech}[z]^3 + \text{Sech}[z] \text{Tanh}[z]^2 \right\}, \\ \left\{ \text{Cosh}[z], \text{Sinh}[z], -2 \text{Sech}[z]^4 + 4 \text{Sech}[z]^2 \text{Tanh}[z]^2, -4 \text{Coth}[z]^2 \text{Csch}[z]^2 - 2 \text{Csch}[z]^4, \right. \\ \left. -\text{Coth}[z]^3 \text{Csch}[z] - 5 \text{Coth}[z] \text{Csch}[z]^3, 5 \text{Sech}[z]^3 \text{Tanh}[z] - \text{Sech}[z] \text{Tanh}[z]^3 \right\}, \\ \left\{ \text{Sinh}[z], \text{Cosh}[z], 16 \text{Sech}[z]^4 \text{Tanh}[z] - 8 \text{Sech}[z]^2 \text{Tanh}[z]^3, \right. \\ \left. 8 \text{Coth}[z]^3 \text{Csch}[z]^2 + 16 \text{Coth}[z] \text{Csch}[z]^4, \text{Coth}[z]^4 \text{Csch}[z] + 18 \text{Coth}[z]^2 \text{Csch}[z]^3 + \right. \\ \left. 5 \text{Csch}[z]^5, 5 \text{Sech}[z]^5 - 18 \text{Sech}[z]^3 \text{Tanh}[z]^2 + \text{Sech}[z] \text{Tanh}[z]^4 \right\} \left. \right\}$$

**Finite summation**

Mathematica can calculate finite sums that contain hyperbolic functions. Here are two examples.

$$\text{Sum}[\text{Sinh}[a k], \{k, 0, n\}]$$

$$\frac{-1 + e^{a+a n}}{2 (-1 + e^a)} - \frac{e^{-a n} (-1 + e^{a+a n})}{2 (-1 + e^a)}$$

$$\sum_{k=0}^n (-1)^k \text{sinh}[a k]$$

$$-\frac{e^a + (-e^{-a})^n}{2 (1 + e^a)} + \frac{1 + e^a (-e^a)^n}{2 (1 + e^a)}$$

**Infinite summation**

Mathematica can calculate infinite sums that contain hyperbolic functions. Here are some examples.

$$\sum_{k=1}^{\infty} z^k \text{sinh}[k x]$$

$$-\frac{z}{2 (e^x - z)} - \frac{e^x z}{2 (-1 + e^x z)}$$

$$\sum_{k=1}^{\infty} \frac{\text{sinh}[k x]}{k!}$$

$$\frac{1}{2} (1 - e^{e^{-x}}) + \frac{1}{2} (-1 + e^{e^x})$$

$$\sum_{k=1}^{\infty} \frac{\text{Cosh}[k x]}{k}$$

$$-\frac{1}{2} \text{Log}[1 - e^{-x}] - \frac{1}{2} \text{Log}[1 - e^x]$$

**Finite products**

Mathematica can calculate some finite symbolic products that contain the hyperbolic functions. Here are two examples.

$$\prod_{k=1}^{n-1} \text{Sinh}\left[\frac{\pi k i}{n}\right]$$

$$\left(\frac{i}{2}\right)^{-1+n} n$$

$$\prod_{k=1}^{n-1} \text{Cosh}\left[z + \frac{\pi k i}{n}\right]$$

$$-(-1)^n 2^{1-n} \text{Sech}[z] \text{Sin}\left[\frac{1}{2} n (\pi + 2 i z)\right]$$

### Infinite products

*Mathematica* can calculate infinite products that contain hyperbolic functions. Here are some examples.

$$\prod_{k=1}^{\infty} \text{Exp}\left[z^k \text{Sinh}[k x]\right]$$

$$e^{-\frac{(-1+e^{2x})z}{2(e^x-z)(-1+e^x z)}}$$

$$\prod_{k=1}^{\infty} \text{Exp}\left[\frac{\text{Cosh}[k x]}{k!}\right]$$

$$e^{\frac{1}{2}(-2+e^{-x}+e^x)}$$

### Indefinite integration

*Mathematica* can calculate a huge set of doable indefinite integrals that contain hyperbolic functions. Here are some examples.

$$\int \text{sinh}[7 z] dz$$

$$\frac{1}{7} \text{Cosh}[7 z]$$

$$\int \left\{ \{\text{Sinh}[z], \text{Sinh}[z]^a\}, \{\text{Cosh}[z], \text{Cosh}[z]^a\}, \{\text{Tanh}[z], \text{Tanh}[z]^a\}, \right. \\ \left. \{\text{Coth}[z], \text{Coth}[z]^a\}, \{\text{Csch}[z], \text{Csch}[z]^a\}, \{\text{Sech}[z], \text{Sech}[z]^a\} \right\} dz$$



$$\left\{ \left\{ \text{Cosh}[z], \right. \right.$$

$$\left. \left. -\text{Cosh}[z] \text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{1-a}{2}, \frac{3}{2}, \text{Cosh}[z]^2\right] \text{Sinh}[z]^{1+a} \left(-\text{Sinh}[z]^2\right)^{\frac{1}{2}(-1-a)}\right\}, \right.$$

$$\left. \left\{ \text{Sinh}[z], -\frac{\text{Cosh}[z]^{1+a} \text{Hypergeometric2F1}\left[\frac{1+a}{2}, \frac{1}{2}, \frac{3+a}{2}, \text{Cosh}[z]^2\right] \text{Sinh}[z]}{(1+a) \sqrt{-\text{Sinh}[z]^2}} \right\}, \right.$$

$$\left. \left\{ \text{Log}[\text{Cosh}[z]], \frac{\text{Hypergeometric2F1}\left[\frac{1+a}{2}, 1, 1 + \frac{1+a}{2}, \text{Tanh}[z]^2\right] \text{Tanh}[z]^{1+a}}{1+a} \right\}, \right.$$

$$\left. \left\{ \text{Log}[\text{Sinh}[z]], \frac{\text{Coth}[z]^{1+a} \text{Hypergeometric2F1}\left[\frac{1+a}{2}, 1, 1 + \frac{1+a}{2}, \text{Coth}[z]^2\right]}{1+a} \right\}, \right.$$

$$\left. \left\{ -\text{Log}\left[\text{Cosh}\left[\frac{z}{2}\right]\right] + \text{Log}\left[\text{Sinh}\left[\frac{z}{2}\right]\right], \right. \right.$$

$$\left. \left. -\text{Cosh}[z] \text{Csch}[z]^{-1+a} \text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{1+a}{2}, \frac{3}{2}, \text{Cosh}[z]^2\right] \left(-\text{Sinh}[z]^2\right)^{\frac{1}{2}(-1+a)}\right\}, \right.$$

$$\left. \left\{ 2 \text{ArcTan}\left[\text{Tanh}\left[\frac{z}{2}\right]\right], -\frac{\text{Hypergeometric2F1}\left[\frac{1-a}{2}, \frac{1}{2}, \frac{3-a}{2}, \text{Cosh}[z]^2\right] \text{Sech}[z]^{-1+a} \text{Sinh}[z]}{(1-a) \sqrt{-\text{Sinh}[z]^2}} \right\} \right\}$$

### Definite integration

*Mathematica* can calculate wide classes of definite integrals that contain hyperbolic functions. Here are some examples.

$$\int_0^{\pi/2} \sqrt[3]{\text{Sinh}[z]} \, dz$$

$$-\frac{(-1)^{1/3} \sqrt{\pi} \text{Gamma}\left[\frac{2}{3}\right]}{2 \text{Gamma}\left[\frac{7}{6}\right]} + (-1)^{1/3} \text{Cosh}\left[\frac{\pi}{2}\right] \text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{1}{3}, \frac{3}{2}, \text{Cosh}\left[\frac{\pi}{2}\right]^2\right]$$

$$\int_1^{\pi/2} \left\{ \sqrt{\text{Sinh}[z]}, \sqrt{\text{Cosh}[z]}, \sqrt{\text{Tanh}[z]}, \sqrt{\text{Coth}[z]}, \sqrt{\text{Csch}[z]}, \sqrt{\text{Sech}[z]} \right\} dz$$

$$\begin{aligned}
 & \left\{ 2 (-1)^{1/4} \text{EllipticE} \left[ \left( \frac{1}{4} - \frac{i}{4} \right) \pi, 2 \right] - 2 (-1)^{1/4} \text{EllipticE} \left[ \frac{1}{4} (-2i + \pi), 2 \right], \right. \\
 & 2i \text{EllipticE} \left[ \frac{i}{2}, 2 \right] - 2i \text{EllipticE} \left[ \frac{i\pi}{4}, 2 \right], \\
 & \frac{1}{2} \left( i \text{Log} \left[ 1 - i \sqrt{\frac{-1 + e^2}{1 + e^2}} \right] - i \text{Log} \left[ 1 + i \sqrt{\frac{-1 + e^2}{1 + e^2}} \right] - \right. \\
 & \quad i \text{Log} \left[ 1 - i \sqrt{\frac{-1 + e^\pi}{1 + e^\pi}} \right] + i \text{Log} \left[ 1 + i \sqrt{\frac{-1 + e^\pi}{1 + e^\pi}} \right] + \text{Log} \left[ 1 - \sqrt{\text{Tanh}[1]} \right] - \\
 & \quad \left. \left. \text{Log} \left[ 1 + \sqrt{\text{Tanh}[1]} \right] - \text{Log} \left[ 1 - \sqrt{\text{Tanh} \left[ \frac{\pi}{2} \right]} \right] + \text{Log} \left[ 1 + \sqrt{\text{Tanh} \left[ \frac{\pi}{2} \right]} \right] \right) \right\}, \\
 & \frac{1}{2} i \left( \text{Log} \left[ 1 - i \sqrt{\frac{1 + e^2}{-1 + e^2}} \right] - \text{Log} \left[ 1 + i \sqrt{\frac{1 + e^2}{-1 + e^2}} \right] - \text{Log} \left[ 1 - i \sqrt{\frac{1 + e^\pi}{-1 + e^\pi}} \right] + \right. \\
 & \quad \left. \text{Log} \left[ 1 + i \sqrt{\frac{1 + e^\pi}{-1 + e^\pi}} \right] - i \text{Log} \left[ -1 + \sqrt{\text{Coth}[1]} \right] + i \text{Log} \left[ 1 + \sqrt{\text{Coth}[1]} \right] + \right. \\
 & \quad \left. i \text{Log} \left[ -1 + \sqrt{\text{Coth} \left[ \frac{\pi}{2} \right]} \right] - i \text{Log} \left[ 1 + \sqrt{\text{Coth} \left[ \frac{\pi}{2} \right]} \right] \right) \right\}, \\
 & 2 (-1)^{3/4} \text{EllipticF} \left[ \left( \frac{1}{4} - \frac{i}{4} \right) \pi, 2 \right] - 2 (-1)^{3/4} \text{EllipticF} \left[ \frac{1}{4} (-2i + \pi), 2 \right], \\
 & 2i \text{EllipticF} \left[ \frac{i}{2}, 2 \right] - 2i \text{EllipticF} \left[ \frac{i\pi}{4}, 2 \right] \} \\
 & \int_1^{\frac{\pi}{4}} \{ \{ \text{Sinh}[z], \text{Sinh}[z]^a \}, \{ \text{Cosh}[z], \text{Cosh}[z]^a \}, \{ \text{Tanh}[z], \text{Tanh}[z]^a \}, \\
 & \quad \{ \text{Coth}[z], \text{Coth}[z]^a \}, \{ \text{Csch}[z], \text{Csch}[z]^a \}, \{ \text{Sech}[z], \text{Sech}[z]^a \} \} dz
 \end{aligned}$$

$$\begin{aligned}
 & \left\{ \left\{ -\text{Cosh}[1] + \text{Cosh}\left[\frac{\pi}{4}\right], \right. \right. \\
 & \quad (-1)^{-\frac{1}{2}-\frac{a}{2}} \text{Cosh}[1] \text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{1-a}{2}, \frac{3}{2}, \text{Cosh}[1]^2\right] \text{Sinh}[1]^{1+2\left(-\frac{1}{2}-\frac{a}{2}\right)+a} - \\
 & \quad (-1)^{-\frac{1}{2}-\frac{a}{2}} \text{Cosh}\left[\frac{\pi}{4}\right] \text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{1-a}{2}, \frac{3}{2}, \text{Cosh}\left[\frac{\pi}{4}\right]^2\right] \text{Sinh}\left[\frac{\pi}{4}\right]^{1+2\left(-\frac{1}{2}-\frac{a}{2}\right)+a}\right\}, \\
 & \left\{ -\text{Sinh}[1] + \text{Sinh}\left[\frac{\pi}{4}\right], -\frac{i \text{Cosh}[1]^{1+a} \text{Hypergeometric2F1}\left[\frac{1+a}{2}, \frac{1}{2}, \frac{3+a}{2}, \text{Cosh}[1]^2\right]}{1+a} + \right. \\
 & \quad \left. \frac{i \text{Cosh}\left[\frac{\pi}{4}\right]^{1+a} \text{Hypergeometric2F1}\left[\frac{1+a}{2}, \frac{1}{2}, \frac{3+a}{2}, \text{Cosh}\left[\frac{\pi}{4}\right]^2\right]}{1+a} \right\}, \\
 & \left\{ -\text{Log}[\text{Cosh}[1]] + \text{Log}\left[\text{Cosh}\left[\frac{\pi}{4}\right]\right], -\frac{\text{Hypergeometric2F1}\left[\frac{1+a}{2}, 1, 1 + \frac{1+a}{2}, \text{Tanh}[1]^2\right] \text{Tanh}[1]^{1+a}}{1+a} + \right. \\
 & \quad \left. \frac{\text{Hypergeometric2F1}\left[\frac{1+a}{2}, 1, 1 + \frac{1+a}{2}, \text{Tanh}\left[\frac{\pi}{4}\right]^2\right] \text{Tanh}\left[\frac{\pi}{4}\right]^{1+a}}{1+a} \right\}, \\
 & \left\{ -\text{Log}[\text{Sinh}[1]] + \text{Log}\left[\text{Sinh}\left[\frac{\pi}{4}\right]\right], -\frac{\text{Coth}[1]^{1+a} \text{Hypergeometric2F1}\left[\frac{1+a}{2}, 1, 1 + \frac{1+a}{2}, \text{Coth}[1]^2\right]}{1+a} + \right. \\
 & \quad \left. \frac{\text{Coth}\left[\frac{\pi}{4}\right]^{1+a} \text{Hypergeometric2F1}\left[\frac{1+a}{2}, 1, 1 + \frac{1+a}{2}, \text{Coth}\left[\frac{\pi}{4}\right]^2\right]}{1+a} \right\}, \\
 & \left\{ \text{Log}\left[\text{Cosh}\left[\frac{1}{2}\right]\right] - \text{Log}\left[\text{Cosh}\left[\frac{\pi}{8}\right]\right] - \text{Log}\left[\text{Sinh}\left[\frac{1}{2}\right]\right] + \text{Log}\left[\text{Sinh}\left[\frac{\pi}{8}\right]\right], \right. \\
 & \quad (-1)^{-\frac{1}{2}+\frac{a}{2}} \text{Cosh}[1] \text{Csch}[1]^{-1+a} \text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{1+a}{2}, \frac{3}{2}, \text{Cosh}[1]^2\right] \text{Sinh}[1]^{2\left(-\frac{1}{2}+\frac{a}{2}\right)} - \\
 & \quad (-1)^{-\frac{1}{2}+\frac{a}{2}} \text{Cosh}\left[\frac{\pi}{4}\right] \text{Csch}\left[\frac{\pi}{4}\right]^{-1+a} \text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{1+a}{2}, \frac{3}{2}, \text{Cosh}\left[\frac{\pi}{4}\right]^2\right] \text{Sinh}\left[\frac{\pi}{4}\right]^{2\left(-\frac{1}{2}+\frac{a}{2}\right)}\right\}, \\
 & \left\{ -2 \text{ArcTan}\left[\text{Tanh}\left[\frac{1}{2}\right]\right] + 2 \text{ArcTan}\left[\text{Tanh}\left[\frac{\pi}{8}\right]\right], \right. \\
 & \quad \frac{i \text{Hypergeometric2F1}\left[\frac{1-a}{2}, \frac{1}{2}, \frac{3-a}{2}, \text{Cosh}[1]^2\right] \text{Sech}[1]^{-1+a}}{-1+a} - \\
 & \quad \left. \frac{i \text{Hypergeometric2F1}\left[\frac{1-a}{2}, \frac{1}{2}, \frac{3-a}{2}, \text{Cosh}\left[\frac{\pi}{4}\right]^2\right] \text{Sech}\left[\frac{\pi}{4}\right]^{-1+a}}{-1+a} \right\} \\
 & \int_0^\infty \left\{ \frac{1}{a+b \text{Sinh}[z]}, \frac{1}{a+b \text{Cosh}[z]}, \right. \\
 & \quad \left. \frac{1}{a+b \text{Tanh}[z]}, \frac{1}{a+b \text{Coth}[z]}, \frac{1}{a+b \text{Csch}[z]}, \frac{1}{a+b \text{Sech}[z]} \right\} dz
 \end{aligned}$$

$$\left\{ -\frac{1}{\sqrt{-a^2 - b^2}} \left( i \left( \operatorname{Log} \left[ 1 - \frac{ia}{\sqrt{-a^2 - b^2}} \right] - \operatorname{Log} \left[ 1 + \frac{ia}{\sqrt{-a^2 - b^2}} \right] + \right. \right. \right. \\ \left. \left. \left. \operatorname{Log} \left[ \frac{ia - ib + \sqrt{-a^2 - b^2}}{\sqrt{-a^2 - b^2}} \right] - \operatorname{Log} \left[ \frac{-ia + ib + \sqrt{-a^2 - b^2}}{\sqrt{-a^2 - b^2}} \right] \right) \right) , \\ -\frac{1}{\sqrt{-a^2 + b^2}} \left( i \left( \operatorname{Log} \left[ 1 - \frac{ia}{\sqrt{-a^2 + b^2}} \right] - \operatorname{Log} \left[ 1 + \frac{ia}{\sqrt{-a^2 + b^2}} \right] - \right. \right. \\ \left. \left. \left. \operatorname{Log} \left[ \frac{-ia - ib + \sqrt{-a^2 + b^2}}{\sqrt{-a^2 + b^2}} \right] + \operatorname{Log} \left[ \frac{ia + ib + \sqrt{-a^2 + b^2}}{\sqrt{-a^2 + b^2}} \right] \right) \right) , \\ \frac{b (\operatorname{Log} [2a] - \operatorname{Log} [a + b])}{a^2 - b^2} , \frac{b (\operatorname{Log} [-a - b] - \operatorname{Log} [-2b])}{-a^2 + b^2} , \\ \frac{1}{a \sqrt{-a^2 - b^2}} \\ \left( i b \left( \operatorname{Log} \left[ \frac{-ib + \sqrt{-a^2 - b^2}}{\sqrt{-a^2 - b^2}} \right] - \operatorname{Log} \left[ \frac{ia - ib + \sqrt{-a^2 - b^2}}{\sqrt{-a^2 - b^2}} \right] - \right. \right. \\ \left. \left. \left. \operatorname{Log} \left[ \frac{ib + \sqrt{-a^2 - b^2}}{\sqrt{-a^2 - b^2}} \right] + \operatorname{Log} \left[ \frac{-ia + ib + \sqrt{-a^2 - b^2}}{\sqrt{-a^2 - b^2}} \right] \right) \right) , \\ \frac{1}{a \sqrt{a^2 - b^2}} \left( i b \left( \operatorname{Log} \left[ 1 - \frac{ib}{\sqrt{a^2 - b^2}} \right] - \operatorname{Log} \left[ 1 + \frac{ib}{\sqrt{a^2 - b^2}} \right] - \right. \right. \\ \left. \left. \left. \operatorname{Log} \left[ \frac{-ia - ib + \sqrt{a^2 - b^2}}{\sqrt{a^2 - b^2}} \right] + \operatorname{Log} \left[ \frac{ia + ib + \sqrt{a^2 - b^2}}{\sqrt{a^2 - b^2}} \right] \right) \right) \right\}$$

### Limit operation

*Mathematica* can calculate limits that contain hyperbolic functions. Here are some examples.

$$\text{Limit} \left[ \frac{\operatorname{Sinh}[z]}{z} + \operatorname{Cosh}[z]^3, z \rightarrow 0 \right]$$

2

$$\text{Limit} \left[ \left( \frac{\operatorname{Tanh}[x]}{x} \right)^{\frac{1}{x^2}}, x \rightarrow 0 \right]$$

$$\frac{1}{e^{1/3}}$$

$$\text{Limit}\left[\frac{\text{Sinh}\left[\sqrt{z^2}\right]}{z}, z \rightarrow 0, \text{Direction} \rightarrow 1\right]$$

-1

$$\text{Limit}\left[\frac{\text{Sinh}\left[\sqrt{z^2}\right]}{z}, z \rightarrow 0, \text{Direction} \rightarrow -1\right]$$

1

### Solving equations

The next input solves equations that contain hyperbolic functions. The message indicates that the multivalued functions are used to express the result and that some solutions might be absent.

$$\text{Solve}\left[\text{Tanh}[z]^2 + 3 \text{Sinh}\left[z + \frac{\pi}{6}\right] = 4, z\right]$$

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.

$$\left\{ \left\{ z \rightarrow -\text{ArcSech}\left[ 3 \left( -2 \text{Root}\left[1 + 6 \#1^2 - 6 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9 \#1^4 + 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9 \text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \&, 1\right] - 3 \text{Root}\left[1 + 6 \#1^2 - 6 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9 \#1^4 + 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9 \text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \&, 1\right]^3 - 3 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \text{Root}\left[1 + 6 \#1^2 - 6 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9 \#1^4 + 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9 \text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \&, 1\right]^3 + 3 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \text{Root}\left[1 + 6 \#1^2 - 6 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9 \#1^4 + 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9 \text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \&, 1\right]^5 + 2 \text{Root}\left[1 + 6 \#1^2 - 6 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9 \#1^4 + 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9 \text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \&, 1\right]^2 \text{Sinh}\left[\frac{\pi}{6}\right] + 6 \text{Root}\left[1 + 6 \#1^2 - 6 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9 \#1^4 + 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9 \text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \&, 1\right]^4 \text{Sinh}\left[\frac{\pi}{6}\right] - 3 \text{Root}\left[1 + 6 \#1^2 - 6 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9 \#1^4 + 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 -$$









$$\begin{aligned}
 & 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \&, 6\right]^3 - \\
 & 3 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \operatorname{Root}\left[1 + 6 \#1^2 - 6 \operatorname{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9 \#1^4 + 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - \right. \\
 & \quad \left. 18 \operatorname{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \&, 6\right]^3 + \\
 & 3 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \operatorname{Root}\left[1 + 6 \#1^2 - 6 \operatorname{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9 \#1^4 + 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - \right. \\
 & \quad \left. 18 \operatorname{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \&, 6\right]^5 + \\
 & 2 \operatorname{Root}\left[1 + 6 \#1^2 - 6 \operatorname{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9 \#1^4 + 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18 \operatorname{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - \right. \\
 & \quad \left. 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \&, 6\right]^2 \operatorname{Sinh}\left[\frac{\pi}{6}\right] + \\
 & 6 \operatorname{Root}\left[1 + 6 \#1^2 - 6 \operatorname{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9 \#1^4 + 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18 \operatorname{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - \right. \\
 & \quad \left. 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \&, 6\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right] - \\
 & 3 \operatorname{Root}\left[1 + 6 \#1^2 - 6 \operatorname{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9 \#1^4 + 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18 \operatorname{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - \right. \\
 & \quad \left. 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \&, 6\right]^5 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 \left. \right\} \left. \right\} \left. \right\}
 \end{aligned}$$

Complete solutions can be obtained by using the function Reduce.

**Reduce[Sinh[x] == a, x] // InputForm**

```
// InputForm = C[1] ∈ Integers &&
(x == I * Pi - ArcSinh[a] + (2 * I) * Pi * C[1] || x == ArcSinh[a] + (2 * I) * Pi * C[1])
```

**Reduce[Cosh[x] == a, x] // InputForm**

```
// InputForm =
C[1] ∈ Integers && (x == -ArcCosh[a] + (2 * I) * Pi * C[1] || x == ArcCosh[a] + (2 * I) * Pi * C[1])
```

**Reduce[Tanh[x] == a, x] // InputForm**

```
// InputForm = C[1] ∈ Integers && -1 + a^2 ≠ 0 && x == ArcTanh[a] + I * Pi * C[1]
```

**Reduce[Coth[x] == a, x] // InputForm**

```
// InputForm = C[1] ∈ Integers && -1 + a^2 ≠ 0 && x == ArcCoth[a] + I * Pi * C[1]
```

**Reduce[Csch[x] == a, x] // InputForm**

```
// InputForm = C[1] ∈ Integers && a ≠ 0 &&
(x == I * Pi - ArcSinh[a^(-1)] + (2 * I) * Pi * C[1] || x == ArcSinh[a^(-1)] + (2 * I) * Pi * C[1])
```

**Reduce[Sech[x] == a, x] // InputForm**

```
// InputForm = C[1] ∈ Integers && a ≠ 0 &&
(x == -ArcCosh[a^(-1)] + (2 * I) * Pi * C[1] || x == ArcCosh[a^(-1)] + (2 * I) * Pi * C[1])
```

### Solving differential equations

Here are differential equations whose linear-independent solutions are hyperbolic functions. The solutions of the simplest second-order linear ordinary differential equation with constant coefficients can be represented through  $\sinh(z)$  and  $\cosh(z)$ .

```
{DSolve[w' [z] - w[z] == 0, w[z], z],
 DSolve[w' [z] + w[z]^2 - 1 == 0, w[z], z]} // (ExpToTrig // @ #) &
{{w[z] → C[1] Cosh[z] + C[2] Cosh[z] + C[1] Sinh[z] - C[2] Sinh[z]},
 {{w[z] →  $\frac{\text{Cosh}[2 z] + \text{Cosh}[2 C[1]] + \text{Sinh}[2 z] + \text{Sinh}[2 C[1]]}{\text{Cosh}[2 z] - \text{Cosh}[2 C[1]] + \text{Sinh}[2 z] - \text{Sinh}[2 C[1]]}$ }}}
```

All hyperbolic functions satisfy first-order nonlinear differential equations. In carrying out the algorithm to solve the nonlinear differential equation, *Mathematica* has to solve a transcendental equation. In doing so, the generically multivariate inverse of a function is encountered, and a message is issued that a solution branch is potentially missed.

```
DSolve[{w' [z] ==  $\sqrt{1 + w[z]^2}$ , w[0] == 0}, w[z], z]
```

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.

```
{{w[z] → Sinh[z]}}
```

```
DSolve[{w' [z] ==  $\sqrt{-1 + w[z]^2}$ , w[0] == 1}, w[z], z] // FullSimplify
```

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. More...

```
{{w[z] → Cosh[z]}}
```

```
DSolve[{w' [z] + w[z]^2 - 1 == 0, w[0] == 0}, w[z], z] // FullSimplify
```

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. More...

```
{{w[z] → Tanh[z]}}
```

```
DSolve[{w' [z] - w[z]^2 + 1 == 0, w[ $\frac{\pi i}{2}$ ] == 0}, w[z], z] // FullSimplify
```

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. More...

```
{{w[z] → -Coth[z]}}
```

### Integral transforms

*Mathematica* supports the main integral transforms like direct and inverse Fourier, Laplace, and Z transforms that can give results containing classical or generalized functions. Here are some transforms of hyperbolic functions.

`LaplaceTransform[Sinh[t], t, s]`

$$\frac{1}{-1 + s^2}$$

`LaplaceTransform[Cosh[t], t, s]`

$$\frac{s}{-1 + s^2}$$

`FourierTransform[Csch[t], t, s]`

$$i \sqrt{\frac{\pi}{2}} \operatorname{Tanh}\left[\frac{\pi s}{2}\right]$$

`FourierTransform[Sech[t], t, s]`

$$\sqrt{\frac{\pi}{2}} \operatorname{Sech}\left[\frac{\pi s}{2}\right]$$

### Plotting

*Mathematica* has built-in functions for 2D and 3D graphics. Here are some examples.

```
Plot[Sin[Sinh[Sum[z^k, {k, 0, 5}]]], {z, -3/2, 4/5}, PlotRange -> All, PlotPoints -> 120];
```

```
Plot3D[Re[Tanh[x + i y]], {x, -2, 2}, {y, -2, 2},
  PlotPoints -> 240, PlotRange -> {-5, 5},
  ClipFill -> None, Mesh -> False, AxesLabel -> {"x", "y", None}];
```

```
ContourPlot[Arg[Sech[1/(x + i y)]]], {x, -1/4, 1/4}, {y, -1/3, 1/3},
  PlotPoints -> 400, PlotRange -> {-pi, pi}, FrameLabel -> {"x", "y", None, None},
  ColorFunction -> Hue, ContourLines -> False, Contours -> 200];
```

## Introduction to the Hyperbolic Secant Function in *Mathematica*

### Overview

The following shows how the hyperbolic secant function is realized in *Mathematica*. Examples of evaluating *Mathematica* functions applied to various numeric and exact expressions that involve the hyperbolic secant function or return it are shown. These involve numeric and symbolic calculations and plots.

### Notations

*Mathematica* forms of notations

Following *Mathematica*'s general naming convention, function names in `StandardForm` are just the capitalized versions of their traditional mathematics names. This shows the hyperbolic secant function in `StandardForm`.

```
Sech[z]
```

```
Sech[z]
```

This shows the hyperbolic secant function in `TraditionalForm`.

```
% // TraditionalForm
```

```
sech(z)
```

### Additional forms of notations

*Mathematica* also knows the most popular forms of notations for the hyperbolic secant function that are used in other programming languages. Here are three examples: `CForm`, `TeXForm`, and `FortranForm`.

```
{CForm[Sech[2 π z]], TeXForm[Sech[2 π z]], FortranForm[Sech[2 π z]]}
```

```
{Sech (2 * Pi * z), \Mfunction {Sech} (2 \, \pi \, z), Sech (2 * Pi * z)}
```

## Automatic evaluations and transformations

### Evaluation for exact and machine-number values of arguments

For the exact argument  $z = \pi/4$ , *Mathematica* returns an exact result.

```
Sech $\left[\frac{\pi i}{4}\right]$ 
```

```
 $\sqrt{2}$ 
```

```
Sech[z] /. z →  $\frac{\pi i}{4}$ 
```

```
 $\sqrt{2}$ 
```

For a machine-number argument (numerical argument with a decimal point), a machine number is also returned.

```
Sech[4.]
```

```
0.036619
```

```
Sech[z] /. z → 2.
```

```
0.265802
```

The next inputs calculate 100-digit approximations at  $z = 1$  and  $z = 2$ .

```
N[Sech[z] /. z → 1, 100]
```

```
0.6480542736638853995749773532261503231084893120719420230378653373187175956467128302\
808547853078928924
```

```
N[Sech[2], 100]
```

```
0.2658022288340796921208627398198889715307826544322680697146411474667245461885868155\
190997054051077860
```

```
Sech[2] // N[#, 100] &
```

```
0.2658022288340796921208627398198889715307826544322680697146411474667245461885868155\
190997054051077860
```

It is possible to calculate thousands of digits for the hyperbolic secant function in less than a second. The next input calculates 10000 digits for  $\text{sech}(1)$  and analyzes the frequency of the digit  $k$  in the resulting decimal number.

```
Map[Function[w, {First[#], Length[#]} & /@ Split[Sort[First[RealDigits[w]]]],  
N[{Sech[z]} /. z -> 1, 10 000]]
```

```
{{{0, 1009}, {1, 971}, {2, 1018}, {3, 994},  
{4, 1011}, {5, 1018}, {6, 958}, {7, 1019}, {8, 1016}, {9, 986}}}
```

Here is a 50-digit approximation to the hyperbolic secant function at the complex argument  $z = 3 - 2i$ .

```
N[Sech[3 - 2 i], 50]
```

```
-0.041674964411144270048349908380328896071871370342714 +  
0.090611137196237596529661198337272798422607482677162 i
```

```
{N[Sech[z] /. z -> 3 - 2 i, 50], Sech[3 - 2 i] // N[#, 50] &}
```

```
{-0.041674964411144270048349908380328896071871370342714 +  
0.090611137196237596529661198337272798422607482677162 i,  
-0.041674964411144270048349908380328896071871370342714 +  
0.090611137196237596529661198337272798422607482677162 i}
```

*Mathematica* automatically evaluates mathematical functions with machine precision, if the arguments of the function are numerical values and include machine-number elements. In this case, only six digits after the decimal point are shown in the results. The remaining digits are suppressed, but can be displayed using the function `InputForm`.

```
{Sech[3.], N[Sech[3]], N[Sech[3], 16], N[Sech[3], 5], N[Sech[3], 20]}
```

```
{0.0993279, 0.0993279, 0.0993279, 0.0993279, 0.099327927419433207829}
```

```
% // InputForm
```

```
{0.0993279274194332, 0.0993279274194332, 0.0993279274194332, 0.0993279274194332,  
0.09932792741943320782901244809925379905`20}
```

### Simplification of the argument

*Mathematica* knows the symmetry and periodicity of the hyperbolic secant function. Here are some examples.

```
Sech[-3]
```

```
Sech[3]
```

$$\{\text{Sech}[-z], \text{Sech}[z + \pi i], \text{Sech}[z + 2\pi i], \text{Sech}[-z + 21\pi i]\}$$

$$\{\text{Sech}[z], -\text{Sech}[z], \text{Sech}[z], -\text{Sech}[z]\}$$

*Mathematica* automatically simplifies the composition of the direct and the inverse hyperbolic secant functions to the inner argument.

$$\text{Sech}[\text{ArcSech}[z]]$$

$$z$$

*Mathematica* also automatically simplifies the composition of the direct and any of the inverse hyperbolic functions into algebraic functions of the argument.

$$\{\text{Sech}[\text{ArcSinh}[z]], \text{Sech}[\text{ArcCosh}[z]], \text{Sech}[\text{ArcTanh}[z]],$$

$$\text{Sech}[\text{ArcCoth}[z]], \text{Sech}[\text{ArcCsch}[z]], \text{Sech}[\text{ArcSech}[z]]\}$$

$$\left\{ \frac{1}{\sqrt{1+z^2}}, \frac{1}{z}, \sqrt{1-z^2}, \sqrt{1-\frac{1}{z^2}}, \frac{1}{\sqrt{1+\frac{1}{z^2}}}, z \right\}$$

In cases where the argument has the structure  $\pi ik/2 + z$  or  $\pi ik/2 - z$ , and  $\pi ik/2 + iz$  or  $\pi ik/2 - iz$  with integer  $k$ , the hyperbolic secant function can be automatically transformed into hyperbolic or trigonometric secant or cosecant functions.

$$\text{Sech}\left[\frac{\pi i}{2} - 4\right]$$

$$i \text{Csch}[4]$$

$$\left\{ \text{Sech}\left[\frac{\pi i}{2} - z\right], \text{Sech}\left[\frac{\pi i}{2} + z\right], \text{Sech}\left[-\frac{\pi i}{2} - z\right], \text{Sech}\left[-\frac{\pi i}{2} + z\right], \text{Sech}[\pi i - z], \text{Sech}[\pi i + z] \right\}$$

$$\{i \text{Csch}[z], -i \text{Csch}[z], -i \text{Csch}[z], i \text{Csch}[z], -\text{Sech}[z], -\text{Sech}[z]\}$$

$$\text{Sech}[i 5]$$

$$\text{Sec}[5]$$

$$\left\{ \text{Sech}[i z], \text{Sech}\left[\frac{\pi i}{2} - i z\right], \text{Sech}\left[\frac{\pi i}{2} + i z\right], \text{Sech}[\pi i - i z], \text{Sech}[\pi i + i z] \right\}$$

$$\{\text{Sec}[z], \text{Csc}[z], -\text{Csc}[z], -\text{Sec}[z], -\text{Sec}[z]\}$$

### Simplification of combination of hyperbolic secant functions

Sometimes simple arithmetic operations containing the hyperbolic secant function can automatically generate other equal hyperbolic functions.

$$1 / \text{Sech}[4]$$

$$\text{Cosh}[4]$$

```
{1 / Sech[z], 1 / Sech[π i / 2 - z], Sech[π i / 2 - z] / Sech[z],
  Sech[z] / Sech[π i / 2 - z], 1 / Sech[π i / 2 - z], Sech[π i / 2 - z] / Sech[z]^2}
{Cosh[z], -i Sinh[z], i Coth[z], -i Tanh[z], -i Sinh[z], i Cosh[z] Coth[z]}
```

**The hyperbolic secant function arising as special cases from more general functions**

The hyperbolic secant function can be treated as a particular case of other, more general special functions. For example,  $\operatorname{sech}(z)$  appears automatically from Bessel, Struve, Mathieu, Jacobi, hypergeometric, and Meijer functions or their reciprocals for appropriate parameters.

```
{√(2/π) / (√z BesselI[-1/2, z]), 1 / (1 + √(πz/2) StruveL[1/2, z]), 1 / MathieuC[1, 0, i z],
  JacobiDC[i z, 0], JacobiNC[i z, 0], JacobiDS[π/2 - i z, 0], JacobiNS[π/2 - i z, 0],
  JacobiCN[z, 1], JacobiDN[z, 1], i JacobiCS[π i/2 - z, 1], i JacobiDS[π i/2 - z, 1],
  1 / HypergeometricPFQ[{}, {1/2}, z^2/4], 1 / MeijerG[{{}, {}], {{-1/2}, {0}}, -z^2/4]}
{Sech[z], 1 / (1 + √(π/2) √z (-√(2/π)/√z + √(2/π) Cosh[z]/√z)), Sech[z], Sech[z], Sech[z], Sech[z],
  Sech[z], Sech[z], Sech[z], Sech[z], Sech[z], Sech[√z^2], 1/2 √π √-z^2 Sech[z]}
% // Simplify
{Sech[z], Sech[z], Sech[z], Sech[z], Sech[z], Sech[z], Sech[z],
  Sech[z], Sech[z], Sech[z], Sech[z], Sech[√z^2], 1/2 √π √-z^2 Sech[z]}
```

**Equivalence transformations using specialized *Mathematica* functions**

**General remarks**

Almost everybody prefers using  $1 - \operatorname{sech}(z)$  instead of  $\operatorname{sech}(\pi i - z) + \operatorname{sech}(0)$ . *Mathematica* automatically transforms the second expression into the first one. The automatic application of transformation rules to mathematical expressions can give overly complicated results. Compact expressions like  $\operatorname{sech}(\pi i / 16)$  should not be automatically expanded into the more complicated expression  $2 / (2 + (2 + 2^{1/2}))$ . *Mathematica* has special functions that produce such expansions. Some are demonstrated in the next section.

**TrigExpand**

The function `TrigExpand` expands out trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then expands out products of the trigonometric and hyperbolic functions into sums of powers, using trigonometric and hyperbolic identities where possible. Here are some examples.

**TrigExpand[Sech[x - y]]**

$$\frac{1}{\text{Cosh}[x] \text{Cosh}[y] - \text{Sinh}[x] \text{Sinh}[y]}$$

**Sech[4 z] // TrigExpand**

$$\frac{1}{\text{Cosh}[z]^4 + 6 \text{Cosh}[z]^2 \text{Sinh}[z]^2 + \text{Sinh}[z]^4}$$

**Sech[2 z]^2 // TrigExpand**

$$\frac{1}{\text{Cosh}[z]^4 + 2 \text{Cosh}[z]^2 \text{Sinh}[z]^2 + \text{Sinh}[z]^4}$$

**TrigExpand[{Sech[x + y + z], Sech[3 z]}]**

$$\left\{ \frac{1}{\text{Cosh}[x] \text{Cosh}[y] \text{Cosh}[z] + \text{Cosh}[z] \text{Sinh}[x] \text{Sinh}[y] + \text{Cosh}[y] \text{Sinh}[x] \text{Sinh}[z] + \text{Cosh}[x] \text{Sinh}[y] \text{Sinh}[z]}, \frac{1}{\text{Cosh}[z]^3 + 3 \text{Cosh}[z] \text{Sinh}[z]^2} \right\}$$

### TrigFactor

The function `TrigFactor` factors trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then factors the resulting polynomials into trigonometric and hyperbolic functions, using trigonometric and hyperbolic identities where possible. Here are some examples.

**TrigFactor[Sech[x] + Sech[y]]**

$$\frac{2 \text{Cosh}\left[\frac{x}{2} - \frac{y}{2}\right] \text{Cosh}\left[\frac{x}{2} + \frac{y}{2}\right]}{\left(\text{Cosh}\left[\frac{x}{2}\right] - i \text{Sinh}\left[\frac{x}{2}\right]\right) \left(\text{Cosh}\left[\frac{x}{2}\right] + i \text{Sinh}\left[\frac{x}{2}\right]\right) \left(\text{Cosh}\left[\frac{y}{2}\right] - i \text{Sinh}\left[\frac{y}{2}\right]\right) \left(\text{Cosh}\left[\frac{y}{2}\right] + i \text{Sinh}\left[\frac{y}{2}\right]\right)}$$

**Sech[x] - Csch[y] // TrigFactor**

$$\frac{\text{Csch}\left[\frac{y}{2}\right] \text{Sech}\left[\frac{y}{2}\right] (-\text{Cosh}[x] + \text{Sinh}[y])}{2 \left(\text{Cosh}\left[\frac{x}{2}\right] - i \text{Sinh}\left[\frac{x}{2}\right]\right) \left(\text{Cosh}\left[\frac{x}{2}\right] + i \text{Sinh}\left[\frac{x}{2}\right]\right)}$$

### TrigReduce

The function `TrigReduce` rewrites the products and powers of trigonometric and hyperbolic functions in terms of trigonometric and hyperbolic functions with combined arguments. In more detail, it typically yields a linear expression involving trigonometric and hyperbolic functions with more complicated arguments. `TrigReduce` is approximately opposite to `TrigExpand` and `TrigFactor`. Here are some examples.



**TrigReduce[Sech[x] Sech[y]]**

$$\frac{2}{\text{Cosh}[x - y] + \text{Cosh}[x + y]}$$

**Sech[x] Csch[y] // TrigReduce**

$$-\frac{2}{\text{Sinh}[x - y] - \text{Sinh}[x + y]}$$

**Table[TrigReduce[Sech[z]^n], {n, 2, 5}]**

$$\left\{ \frac{2}{1 + \text{Cosh}[2z]}, \frac{4}{3 \text{Cosh}[z] + \text{Cosh}[3z]}, \frac{8}{3 + 4 \text{Cosh}[2z] + \text{Cosh}[4z]}, \frac{16}{10 \text{Cosh}[z] + 5 \text{Cosh}[3z] + \text{Cosh}[5z]} \right\}$$

**TrigReduce[TrigExpand[{Sech[x + y + z], Sech[3z], Sech[x] Sech[y]}]]**

$$\left\{ \text{Sech}[x + y + z], \text{Sech}[3z], \frac{2}{\text{Cosh}[x - y] + \text{Cosh}[x + y]} \right\}$$

**TrigFactor[Sech[x] + Sech[y]] // TrigReduce**

$$\frac{2 (\text{Cosh}[x] + \text{Cosh}[y])}{\text{Cosh}[x - y] + \text{Cosh}[x + y]}$$

### TrigToExp

The function `TrigToExp` converts trigonometric and hyperbolic functions to exponentials. It tries, where possible, to give results that do not involve explicit complex numbers. Here are some examples.

**TrigToExp[Sech[z]]**

$$\frac{2}{e^{-z} + e^z}$$

**Sech[a z] + Sech[b z] // TrigToExp**

$$\frac{2}{e^{-az} + e^{az}} + \frac{2}{e^{-bz} + e^{bz}}$$

### ExpToTrig

The function `ExpToTrig` converts exponentials to trigonometric and hyperbolic functions. It is approximately opposite to `TrigToExp`. Here are some examples.

**ExpToTrig[TrigToExp[Sech[z]]]**

Sech[z]

$$\left\{ \alpha e^{-x\beta} + \alpha e^{x\beta} / (\alpha e^{-x\beta} + \gamma e^{x\beta}) \right\} // \text{ExpToTrig}$$

$$\left\{ \alpha \operatorname{Cosh}[x \beta] - \alpha \operatorname{Sinh}[x \beta] + \frac{\alpha (\operatorname{Cosh}[x \beta] + \operatorname{Sinh}[x \beta])}{\alpha \operatorname{Cosh}[x \beta] + \gamma \operatorname{Cosh}[x \beta] - \alpha \operatorname{Sinh}[x \beta] + \gamma \operatorname{Sinh}[x \beta]} \right\}$$

### ComplexExpand

The function `ComplexExpand` expands expressions assuming that all the variables are real. The option `TargetFunctions` can be given as a list of functions from the set `{Re, Im, Abs, Arg, Conjugate, Sign}`. `ComplexExpand` will try to give results in terms of the functions specified. Here are some examples.

`ComplexExpand[Sech[x + i y]]`

$$\frac{2 \operatorname{Cos}[y] \operatorname{Cosh}[x]}{\operatorname{Cos}[2 y] + \operatorname{Cosh}[2 x]} - \frac{2 i \operatorname{Sin}[y] \operatorname{Sinh}[x]}{\operatorname{Cos}[2 y] + \operatorname{Cosh}[2 x]}$$

`Sech[x + i y] + Sech[x - i y] // ComplexExpand`

$$\frac{4 \operatorname{Cos}[y] \operatorname{Cosh}[x]}{\operatorname{Cos}[2 y] + \operatorname{Cosh}[2 x]}$$

`ComplexExpand[Re[Sech[x + i y]], TargetFunctions -> {Re, Im}]`

$$\frac{2 \operatorname{Cos}[y] \operatorname{Cosh}[x]}{\operatorname{Cos}[2 y] + \operatorname{Cosh}[2 x]}$$

`ComplexExpand[Im[Sech[x + i y]], TargetFunctions -> {Re, Im}]`

$$\frac{2 \operatorname{Sin}[y] \operatorname{Sinh}[x]}{\operatorname{Cos}[2 y] + \operatorname{Cosh}[2 x]}$$

`ComplexExpand[Abs[Sech[x + i y]], TargetFunctions -> {Re, Im}]`

$$\sqrt{\frac{4 \operatorname{Cos}[y]^2 \operatorname{Cosh}[x]^2}{(\operatorname{Cos}[2 y] + \operatorname{Cosh}[2 x])^2} + \frac{4 \operatorname{Sin}[y]^2 \operatorname{Sinh}[x]^2}{(\operatorname{Cos}[2 y] + \operatorname{Cosh}[2 x])^2}}$$

`ComplexExpand[Abs[Sech[x + i y]], TargetFunctions -> {Re, Im}] // Simplify[#, {x, y} ∈ Reals] &`

$$\frac{\sqrt{2}}{\sqrt{\operatorname{Cos}[2 y] + \operatorname{Cosh}[2 x]}}$$

`ComplexExpand[Re[Sech[x + i y]] + Im[Sech[x + i y]], TargetFunctions -> {Re, Im}]`

$$\frac{2 \operatorname{Cos}[y] \operatorname{Cosh}[x]}{\operatorname{Cos}[2 y] + \operatorname{Cosh}[2 x]} - \frac{2 \operatorname{Sin}[y] \operatorname{Sinh}[x]}{\operatorname{Cos}[2 y] + \operatorname{Cosh}[2 x]}$$

`ComplexExpand[Arg[Sech[x + i y]], TargetFunctions -> {Re, Im}]`

$$\operatorname{ArcTan}\left[\frac{2 \operatorname{Cos}[y] \operatorname{Cosh}[x]}{\operatorname{Cos}[2 y] + \operatorname{Cosh}[2 x]}, -\frac{2 \operatorname{Sin}[y] \operatorname{Sinh}[x]}{\operatorname{Cos}[2 y] + \operatorname{Cosh}[2 x]}\right]$$

`ComplexExpand[Arg[Sech[x + i y]], TargetFunctions -> {Re, Im}] // Simplify[#, {x, y} ∈ Reals] &`

ArcTan[Cos[y] Cosh[x], -Sin[y] Sinh[x]]

ComplexExpand[Conjugate[Sech[x + i y]], TargetFunctions -> {Re, Im}] // Simplify

$$\frac{1}{\cos[y] \cosh[x] - i \sin[y] \sinh[x]}$$

**Simplify**

The function Simplify performs a sequence of algebraic transformations on the expression, and returns the simplest form it finds. Here are some examples.

$$\left( \operatorname{sech}[z_1]^2 \operatorname{sech}[z_2]^2 - \operatorname{sech}[z_1]^2 - \operatorname{sech}[z_2]^2 \right) \operatorname{sech}[z_1 + z_2]^2 + 2 \operatorname{sech}[z_1] \operatorname{sech}[z_2] \operatorname{sech}[z_1 + z_2] // \text{Simplify}$$

$$\operatorname{sech}[z_1]^2 \operatorname{sech}[z_2]^2$$

$$\text{Simplify} \left[ \operatorname{sech} \left[ z - \frac{\pi i}{3} \right] \operatorname{sech} \left[ \frac{\pi i}{3} + z \right] + \operatorname{sech} \left[ z - \frac{\pi i}{3} \right] \operatorname{sech}[z] + \operatorname{sech}[z] \operatorname{sech} \left[ \frac{\pi i}{3} + z \right] \right]$$

$$\frac{8}{-1 + 2 \cosh[2 z]}$$

Here is a collection of hyperbolic identities. Each is written as a logical conjunction.

$$\text{Simplify}[\#] \ \& \ /@ \left( \operatorname{sech}[2 z] (2 - \operatorname{sech}[z]^2) == \operatorname{sech}[z]^2 \wedge \right.$$

$$\operatorname{sech}[4 z] \left( \cosh[z]^4 + 6 \cosh[z]^2 \sinh[z]^2 + \sinh[z]^4 \right) == 1 \wedge$$

$$\operatorname{sech}[a + i b] == \frac{2 \cos[b] \cosh[a] - 2 i \sinh[a] \sin[b]}{\cosh[2 a] + \cos[2 b]} \wedge \operatorname{sech}[a]^2 - \operatorname{sech}[b]^2 ==$$

$$\operatorname{sech}[a]^2 \operatorname{sech}[b]^2 \sinh[b - a] \sinh[a + b] \wedge \operatorname{sech}[\operatorname{ArcCsch}[z]] == \frac{1}{\sqrt{1 + \frac{1}{z^2}}} \wedge$$

$$\operatorname{sech} \left[ \frac{z}{2} \right]^2 == \frac{2}{1 + \cosh[z]} \wedge \operatorname{sech}[x] \operatorname{sech}[y] == \frac{2}{\cosh[x - y] + \cosh[x + y]} \wedge$$

$$\left. \operatorname{Tanh}[a]^2 - \operatorname{Tanh}[b]^2 == \operatorname{sech}[a]^2 \operatorname{sech}[b]^2 \sinh[a - b] \sinh[a + b] \right)$$

True

The command Simplify has the Assumption option. For example, Mathematica recognizes the periodicity of the hyperbolic function for the symbolic integer coefficient k of k pi i.

**Simplify**[{**Sech**[ $z + 2 k \pi i$ ], **Sech**[ $z + k \pi i$ ] / **Sech**[ $z$ ]},  $k \in \text{Integers}$ ]

$$\{\text{Sech}[z], (-1)^k\}$$

*Mathematica* also knows that the composition of inverse and direct hyperbolic functions produces the value of the internal argument under the corresponding restriction.

**ArcSech**[**Sech**[ $z$ ]]

ArcSech[Sech[ $z$ ]]

**Simplify**[**ArcSech**[**Sech**[ $z$ ]],  $0 < \text{Im}[z] < \pi \wedge \text{Re}[z] > 0$ ]

$z$

**FunctionExpand (and Together)**

While the hyperbolic secant function auto-evaluates for simple fractions of  $\pi$ , for more complicated cases it stays as a hyperbolic secant function to avoid the build up of large expressions. Using the function `FunctionExpand`, the hyperbolic secant function can sometimes be transformed into explicit radicals. Here are some examples.

$$\left\{ \text{Sech}\left[\frac{\pi i}{16}\right], \text{Sech}\left[\frac{\pi i}{60}\right] \right\}$$

$$\left\{ \text{Sec}\left[\frac{\pi}{16}\right], \text{Sec}\left[\frac{\pi}{60}\right] \right\}$$

**FunctionExpand**[%]

$$\left\{ \frac{2}{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}, \frac{1}{-\frac{\frac{1}{8}\sqrt{3}(-1+\sqrt{5})-\frac{1}{4}\sqrt{\frac{1}{2}(5+\sqrt{5})}}{\sqrt{2}} - \frac{\frac{1}{8}(-1+\sqrt{5})-\frac{1}{4}\sqrt{\frac{3}{2}(5+\sqrt{5})}}{\sqrt{2}}}} \right\}$$

**Together**[%]

$$\left\{ \frac{2}{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}, \frac{16}{\sqrt{2} - \sqrt{6} - \sqrt{10} + \sqrt{30} + 2\sqrt{5 + \sqrt{5}} + 2\sqrt{3(5 + \sqrt{5})}} \right\}$$

If the denominator contains squares of integers other than 2, the results always contain complex numbers deeply inside of the expression (meaning that the imaginary number  $i = \sqrt{-1}$  appears unavoidably).

$$\left\{ \text{Sech}\left[\frac{\pi i}{9}\right] \right\}$$

$$\left\{ \text{Sec}\left[\frac{\pi}{9}\right] \right\}$$

**FunctionExpand**[%]**//Together**

$$\left\{ - (8 i) / \left( -i 2^{2/3} (-1 - i \sqrt{3})^{1/3} + 2^{2/3} \sqrt{3} (-1 - i \sqrt{3})^{1/3} - i 2^{2/3} (-1 + i \sqrt{3})^{1/3} - 2^{2/3} \sqrt{3} (-1 + i \sqrt{3})^{1/3} \right) \right\}$$

The function `RootReduce` allows for writing the previous algebraic numbers as the roots of polynomial equations.

`RootReduce[Simplify[%]]`

$$\{ \text{Root}[-8 + 6 \#1^2 + \#1^3 \&, 3] \}$$

The function `FunctionExpand` also reduces hyperbolic expressions with compound arguments or compositions, including hyperbolic functions, to simpler ones. Here are some examples.

`{Sech[Sqrt[z^2]], Sech[ArcSech[z]/2], Sech[2 ArcSech[z]], Sech[3 ArcSinh[z]]} // FunctionExpand`

$$\left\{ \text{Sech}[z], \frac{\sqrt{2} \sqrt{-z}}{\sqrt{-1-z}}, \frac{z^2}{2-z^2}, \frac{1}{3 z^2 \sqrt{i(-i+z)} \sqrt{-i(i+z)} + (i(-i+z))^{3/2} (-i(i+z))^{3/2}} \right\}$$

Applying `Simplify` to the previous expression gives a more compact result.

`Simplify[%]`

$$\left\{ \text{Sech}[z], \frac{\sqrt{2}}{\sqrt{1+\frac{1}{z}}}, \frac{z^2}{2-z^2}, \frac{1}{\sqrt{1+z^2} (1+4z^2)} \right\}$$

### FullSimplify

The function `FullSimplify` tries a wider range of transformations than `Simplify` and returns the simplest form it finds. Here are some examples that compare the results of applying these functions to the same expressions.

$$\begin{aligned} \text{set1} = & \left\{ \text{Sech}\left[\text{Log}\left[z + \sqrt{1+z^2}\right]\right], \text{Sech}\left[\frac{\pi i}{2} + \text{Log}\left[z + \sqrt{1+z^2}\right]\right], \right. \\ & \text{Sech}\left[\frac{1}{2} \text{Log}[1-z] - \frac{1}{2} \text{Log}[1+z]\right], \text{Sech}\left[\frac{1}{2} \text{Log}\left[1 - \frac{1}{z}\right] - \frac{1}{2} \text{Log}\left[1 + \frac{1}{z}\right]\right], \\ & \left. \text{Sech}\left[\text{Log}\left[\sqrt{1 + \frac{1}{z^2}} + \frac{1}{z}\right]\right], \text{Sech}\left[\frac{\pi i}{2} + \text{Log}\left[\sqrt{1 + \frac{1}{z^2}} + \frac{1}{z}\right]\right] \right\} \end{aligned}$$

$$\left\{ \frac{2 \left( z + \sqrt{1 + z^2} \right)}{1 + \left( z + \sqrt{1 + z^2} \right)^2}, -\frac{2 i \left( z + \sqrt{1 + z^2} \right)}{-1 + \left( z + \sqrt{1 + z^2} \right)^2}, \operatorname{Sech} \left[ \frac{1}{2} \operatorname{Log} [1 - z] - \frac{1}{2} \operatorname{Log} [1 + z] \right], \right.$$

$$\left. \operatorname{Sech} \left[ \frac{1}{2} \operatorname{Log} \left[ 1 - \frac{1}{z} \right] - \frac{1}{2} \operatorname{Log} \left[ 1 + \frac{1}{z} \right] \right], \frac{2 \left( \sqrt{1 + \frac{1}{z^2}} + \frac{1}{z} \right)}{1 + \left( \sqrt{1 + \frac{1}{z^2}} + \frac{1}{z} \right)^2}, -\frac{2 i \left( \sqrt{1 + \frac{1}{z^2}} + \frac{1}{z} \right)}{-1 + \left( \sqrt{1 + \frac{1}{z^2}} + \frac{1}{z} \right)^2} \right\}$$

**set1 // Simplify**

$$\left\{ \frac{z + \sqrt{1 + z^2}}{1 + z^2 + z \sqrt{1 + z^2}}, -\frac{i}{z}, \operatorname{Sech} \left[ \frac{1}{2} (\operatorname{Log} [1 - z] - \operatorname{Log} [1 + z]) \right], \right.$$

$$\left. \operatorname{Sech} \left[ \frac{1}{2} \left( -\operatorname{Log} \left[ 1 + \frac{1}{z} \right] + \operatorname{Log} \left[ \frac{-1 + z}{z} \right] \right) \right], \frac{z \left( 1 + \sqrt{1 + \frac{1}{z^2}} z \right)}{1 + \sqrt{1 + \frac{1}{z^2}} z + z^2}, -i z \right\}$$

**set1 // FullSimplify**

$$\left\{ \frac{1}{\sqrt{1 + z^2}}, -\frac{i}{z}, \sqrt{1 - z^2}, \sqrt{1 - \frac{1}{z^2}}, \frac{1}{\sqrt{1 + \frac{1}{z^2}}}, -i z \right\}$$

## Operations under special *Mathematica* functions

### Series expansions

Calculating the series expansion of a hyperbolic secant function to hundreds of terms can be done in seconds.

**Series[Sech[z], {z, 0, 3}]**

$$1 - \frac{z^2}{2} + O[z]^4$$

**Normal[%]**

$$1 - \frac{z^2}{2}$$

**Series[Sech[z], {z, 0, 100}] // Timing**

$$\left\{ 0.4 \text{ Second}, 1 - \frac{z^2}{2} + \frac{5 z^4}{24} - \frac{61 z^6}{720} + \frac{277 z^8}{8064} - \frac{50 521 z^{10}}{3 628 800} + \right.$$

$$\left. \frac{540 553 z^{12}}{95 800 320} - \frac{199 360 981 z^{14}}{87 178 291 200} + \frac{3 878 302 429 z^{16}}{4 184 557 977 600} - \frac{2 404 879 675 441 z^{18}}{6 402 373 705 728 000} + \right.$$

$$\begin{aligned}
 & \frac{14\,814\,847\,529\,501\,z^{20}}{97\,316\,080\,327\,065\,600} - \frac{69\,348\,874\,393\,137\,901\,z^{22}}{1\,124\,000\,727\,777\,607\,680\,000} + \frac{238\,685\,140\,977\,801\,337\,z^{24}}{9\,545\,360\,026\,665\,222\,144\,000} - \\
 & \frac{4\,087\,072\,509\,293\,123\,892\,361\,z^{26}}{403\,291\,461\,126\,605\,635\,584\,000\,000} + \frac{13\,181\,680\,435\,827\,682\,794\,403\,z^{28}}{3\,209\,350\,995\,912\,777\,478\,963\,200\,000} - \\
 & \frac{441\,543\,893\,249\,023\,104\,553\,682\,821\,z^{30}}{265\,252\,859\,812\,191\,058\,636\,308\,480\,000\,000} + \frac{2\,088\,463\,430\,347\,521\,052\,196\,056\,349\,z^{32}}{3\,095\,656\,905\,102\,276\,825\,496\,682\,496\,000\,000} - \\
 & \frac{80\,723\,299\,235\,887\,898\,062\,168\,247\,453\,281\,z^{34}}{295\,232\,799\,039\,604\,140\,847\,618\,609\,643\,520\,000\,000} + \\
 & \frac{634\,185\,543\,684\,887\,724\,959\,185\,841\,096\,293\,z^{36}}{5\,722\,974\,258\,306\,172\,576\,430\,760\,740\,782\,080\,000\,000} - \\
 & \frac{23\,489\,580\,527\,043\,108\,252\,017\,828\,576\,198\,947\,741\,z^{38}}{523\,022\,617\,466\,601\,111\,760\,007\,224\,100\,074\,291\,200\,000\,000} + \\
 & \frac{594\,046\,028\,724\,599\,200\,715\,086\,271\,256\,233\,067\,377\,z^{40}}{32\,636\,611\,329\,915\,909\,373\,824\,450\,783\,844\,635\,770\,880\,000\,000} - \\
 & \frac{10\,364\,622\,733\,519\,612\,119\,397\,957\,304\,745\,185\,976\,310\,201\,z^{42}}{1\,405\,006\,117\,752\,879\,898\,543\,142\,606\,244\,511\,569\,936\,384\,000\,000\,000} + \\
 & \frac{1\,589\,515\,884\,519\,518\,540\,721\,608\,102\,017\,614\,123\,903\,854\,761\,z^{44}}{531\,654\,314\,957\,689\,753\,608\,725\,162\,202\,923\,178\,063\,927\,705\,600\,000\,000} - \\
 & \frac{350\,923\,027\,193\,976\,051\,443\,948\,867\,093\,355\,168\,290\,742\,509\,719\,z^{46}}{289\,611\,692\,621\,688\,892\,097\,384\,496\,252\,644\,994\,366\,402\,723\,840\,000\,000\,000} + \\
 & \frac{5\,516\,994\,249\,383\,296\,071\,214\,195\,242\,422\,482\,492\,286\,460\,673\,697\,z^{48}}{11\,234\,312\,753\,426\,310\,109\,377\,637\,146\,944\,230\,804\,091\,842\,854\,912\,000\,000\,000} - \\
 & \frac{6\,053\,285\,248\,188\,621\,896\,314\,383\,785\,111\,649\,088\,103\,498\,225\,146\,815\,121\,z^{50}}{30\,414\,093\,201\,713\,378\,043\,612\,608\,166\,064\,768\,844\,377\,641\,568\,960\,512\,000\,000\,000\,000} + \\
 & \frac{41\,975\,241\,849\,578\,121\,598\,166\,907\,316\,650\,470\,861\,184\,801\,527\,518\,616\,623\,z^{52}}{520\,375\,323\,683\,508\,894\,010\,713\,786\,170\,346\,883\,711\,545\,196\,392\,795\,340\,800\,000\,000\,000} - \\
 & \frac{175\,503\,719\,511\,831\,141\,815\,382\,152\,695\,110\,156\,842\,145\,116\,853\,760\,461\,828\,967\,z^{54}}{5\,368\,458\,077\,656\,776\,290\,048\,668\,434\,489\,013\,513\,564\,617\,780\,739\,719\,561\,216\,000\,000\,000\,000} + \\
 & \frac{64\,967\,737\,216\,691\,318\,649\,670\,939\,570\,279\,884\,325\,581\,992\,086\,894\,110\,409\,620\,041\,z^{56}}{4\,903\,438\,536\,585\,265\,185\,200\,314\,810\,094\,654\,825\,768\,951\,578\,490\,816\,956\,465\,152\,000\,000\,000\,000} \\
 & \left( \frac{12\,622\,019\,251\,806\,218\,719\,903\,409\,237\,287\,489\,255\,482\,341\,061\,191\,825\,594\,069\,964\,920\,041\,z^{58}}{2\,350\,561\,331\,282\,878\,571\,829\,474\,910\,515\,074\,683\,828\,862\,318\,181\,142\,924\,420\,699\,914\,240\,000\,000\,000} \right) + \\
 & \left( \frac{1\,185\,526\,120\,888\,983\,993\,227\,221\,456\,081\,974\,382\,241\,144\,834\,647\,221\,872\,733\,754\,203\,223\,z^{60}}{544\,745\,473\,829\,223\,577\,366\,699\,913\,795\,310\,270\,425\,805\,080\,612\,847\,525\,528\,594\,284\,544\,000\,000\,000} \right) - \\
 & \left( \frac{27\,757\,101\,702\,071\,580\,597\,366\,980\,908\,371\,527\,449\,233\,019\,594\,800\,917\,578\,033\,782\,766\,889\,782\,501\,z^{62}}{31\,469\,973\,260\,387\,937\,525\,653\,122\,354\,950\,764\,088\,012\,280\,797\,258\,232\,192\,163\,168\,247\,821\,107\,200\,000\,000\,000} \right) + \\
 & \left( \frac{28\,085\,512\,897\,843\,894\,225\,849\,466\,174\,345\,363\,694\,335\,517\,968\,712\,302\,712\,983\,804\,234\,006\,003\,567\,z^{64}}{78\,567\,759\,867\,420\,535\,048\,565\,566\,151\,802\,774\,490\,938\,400\,108\,077\,518\,389\,351\,018\,188\,987\,432\,960} \right)
 \end{aligned}$$

$$\begin{aligned}
& 000\ 000\ 000\ 000 - \\
& (1\ 292\ 833\ 476\ 501\ 932\ 691\ 491\ 921\ 433\ 154\ 096\ 790\ 811\ 117\ 425\ 770\ 553\ 589\ 060\ 815\ 696\ 637\ 002\ 753 \dots \\
& \quad 202\ 901\ z^{66}) / \\
& 8\ 923\ 687\ 525\ 859\ 722\ 360\ 716\ 872\ 790\ 948\ 241\ 846\ 627\ 755\ 153\ 914\ 735\ 648\ 079\ 674\ 210\ 978\ 214\ 445 \dots \\
& \quad 056\ 000\ 000\ 000\ 000\ 000 + \\
& (29\ 123\ 688\ 760\ 279\ 263\ 001\ 430\ 094\ 018\ 988\ 465\ 332\ 372\ 162\ 571\ 662\ 986\ 597\ 289\ 539\ 553\ 612\ 919\ 097 \dots \\
& \quad 772\ 580\ 417\ z^{68}) / \\
& 496\ 007\ 108\ 487\ 366\ 119\ 920\ 198\ 083\ 713\ 834\ 316\ 209\ 479\ 840\ 271\ 073\ 534\ 474\ 342\ 147\ 603\ 644\ 289 \dots \\
& \quad 142\ 436\ 659\ 200\ 000\ 000\ 000\ 000 - \\
& (285\ 051\ 783\ 223\ 697\ 718\ 732\ 198\ 729\ 556\ 739\ 339\ 504\ 255\ 241\ 778\ 255\ 239\ 879\ 353\ 211\ 106\ 980\ 427 \dots \\
& \quad 546\ 235\ 397\ 447\ 421\ z^{70}) / \\
& 11\ 978\ 571\ 669\ 969\ 891\ 796\ 072\ 783\ 721\ 689\ 098\ 736\ 458\ 938\ 142\ 546\ 425\ 857\ 555\ 362\ 864\ 628\ 009\ 582 \dots \\
& \quad 789\ 845\ 319\ 680\ 000\ 000\ 000\ 000 + \\
& (245\ 561\ 214\ 460\ 517\ 407\ 673\ 652\ 820\ 081\ 679\ 655\ 456\ 146\ 784\ 275\ 186\ 259\ 846\ 056\ 537\ 278\ 029\ 024 \dots \\
& \quad 968\ 924\ 948\ 811\ 253\ z^{72}) / \\
& 25\ 461\ 313\ 254\ 422\ 489\ 339\ 511\ 047\ 977\ 245\ 186\ 170\ 801\ 701\ 365\ 778\ 515\ 169\ 988\ 779\ 610\ 801\ 823\ 279 \dots \\
& \quad 510\ 057\ 910\ 272\ 000\ 000\ 000\ 000 - \\
& (1\ 292\ 973\ 664\ 187\ 864\ 170\ 497\ 603\ 235\ 938\ 698\ 754\ 076\ 170\ 519\ 123\ 672\ 606\ 411\ 370\ 597\ 343\ 787\ 035 \dots \\
& \quad 331\ 808\ 195\ 731\ 850\ 937\ 881\ z^{74}) / \\
& 330\ 788\ 544\ 151\ 938\ 641\ 225\ 953\ 028\ 221\ 253\ 782\ 145\ 683\ 251\ 820\ 934\ 971\ 170\ 611\ 926\ 835\ 411\ 235 \dots \\
& \quad 700\ 971\ 565\ 459\ 250\ 872\ 320\ 000\ 000\ 000\ 000 + \\
& (597\ 385\ 636\ 656\ 915\ 390\ 186\ 148\ 730\ 443\ 428\ 121\ 138\ 584\ 473\ 874\ 136\ 140\ 562\ 762\ 566\ 693\ 379\ 607 \dots \\
& \quad 634\ 403\ 131\ 161\ 792\ 057\ 690\ 569\ z^{76}) / \\
& 377\ 098\ 940\ 333\ 210\ 050\ 997\ 586\ 452\ 172\ 229\ 311\ 646\ 078\ 907\ 075\ 865\ 867\ 134\ 497\ 596\ 592\ 368\ 808 \dots \\
& \quad 699\ 107\ 584\ 623\ 545\ 994\ 444\ 800\ 000\ 000\ 000\ 000 - \\
& (7\ 270\ 601\ 714\ 016\ 864\ 143\ 803\ 280\ 651\ 699\ 281\ 851\ 647\ 234\ 288\ 049\ 207\ 905\ 108\ 309\ 583\ 687\ 335\ 688 \dots \\
& \quad 017\ 641\ 546\ 191\ 095\ 009\ 395\ 592\ 341\ z^{78}) / \\
& 11\ 324\ 281\ 178\ 206\ 297\ 831\ 457\ 521\ 158\ 732\ 046\ 228\ 731\ 749\ 579\ 488\ 251\ 990\ 048\ 962\ 825\ 668\ 835\ 325 \dots \\
& \quad 234\ 200\ 766\ 245\ 086\ 213\ 177\ 344\ 000\ 000\ 000\ 000\ 000 + \\
& (1\ 068\ 746\ 958\ 875\ 908\ 008\ 291\ 678\ 180\ 949\ 390\ 301\ 975\ 323\ 397\ 102\ 930\ 046\ 690\ 221\ 009\ 795\ 234\ 210 \dots \\
& \quad 820\ 441\ 976\ 633\ 130\ 205\ 853\ 968\ 257\ z^{80}) / \\
& 4\ 107\ 285\ 913\ 702\ 370\ 289\ 515\ 726\ 469\ 049\ 442\ 305\ 055\ 073\ 592\ 101\ 334\ 437\ 710\ 728\ 554\ 274\ 148\ 594 \dots \\
& \quad 288\ 674\ 252\ 090\ 039\ 877\ 605\ 785\ 600\ 000\ 000\ 000\ 000 - \\
& (85\ 112\ 138\ 214\ 108\ 313\ 434\ 479\ 955\ 312\ 191\ 866\ 587\ 577\ 821\ 260\ 136\ 739\ 840\ 362\ 642\ 167\ 337\ 133\ 108 \dots \\
& \quad 921\ 391\ 785\ 317\ 917\ 129\ 176\ 247\ 795\ 509\ z^{82}) / \\
& 807\ 070\ 176\ 063\ 300\ 806\ 183\ 596\ 276\ 722\ 249\ 484\ 964\ 029\ 361\ 745\ 319\ 742\ 983\ 295\ 304\ 035\ 219\ 006 \dots \\
& \quad 341\ 085\ 141\ 957\ 567\ 286\ 601\ 832\ 202\ 240\ 000\ 000\ 000\ 000 + \\
& (75\ 147\ 245\ 410\ 003\ 320\ 910\ 727\ 888\ 281\ 178\ 604\ 989\ 642\ 358\ 351\ 638\ 149\ 809\ 389\ 998\ 326\ 458\ 656\ 760 \dots \\
& \quad 950\ 050\ 385\ 743\ 229\ 535\ 819\ 860\ 148\ 866\ 073\ z^{84}) / \\
& 1\ 758\ 217\ 578\ 018\ 755\ 048\ 806\ 041\ 156\ 037\ 204\ 927\ 321\ 308\ 374\ 664\ 217\ 234\ 545\ 806\ 913\ 731\ 065\ 852 \dots \\
& \quad 144\ 146\ 877\ 522\ 485\ 794\ 678\ 364\ 325\ 281\ 792\ 000\ 000\ 000\ 000\ 000 - \\
& (419\ 664\ 316\ 404\ 024\ 471\ 322\ 573\ 414\ 069\ 418\ 891\ 818\ 962\ 628\ 391\ 683\ 907\ 039\ 212\ 228\ 549\ 032\ 921 \dots \\
& \quad 853\ 217\ 838\ 146\ 608\ 053\ 808\ 786\ 365\ 440\ 570\ 254\ 969\ 261\ z^{86}) / \\
& 24\ 227\ 095\ 383\ 672\ 732\ 381\ 765\ 523\ 203\ 441\ 259\ 715\ 284\ 870\ 552\ 429\ 381\ 750\ 838\ 764\ 496\ 720\ 162\ 249 \dots \\
& \quad 742\ 450\ 276\ 789\ 464\ 634\ 901\ 319\ 465\ 571\ 660\ 595\ 200\ 000\ 000\ 000\ 000 + \\
& (260\ 431\ 918\ 104\ 809\ 279\ 625\ 171\ 738\ 266\ 163\ 736\ 271\ 351\ 522\ 797\ 322\ 006\ 135\ 619\ 151\ 648\ 480\ 857 \dots \\
& \quad 326\ 745\ 852\ 459\ 424\ 735\ 439\ 948\ 718\ 349\ 601\ 240\ 929\ 373\ 797\ z^{88}) /
\end{aligned}$$



$$\begin{aligned}
 & 37\ 096\ 528\ 451\ 479\ 687\ 822\ 959\ 369\ 129\ 109\ 256\ 876\ 044\ 193\ 789\ 879\ 869\ 336\ 884\ 316\ 197\ 377\ 912\ 436\ \vdots \\
 & 805\ 639\ 863\ 820\ 028\ 248\ 960\ 900\ 365\ 683\ 326\ 703\ 370\ 240\ 000\ 000\ 000\ 000\ 000\ 000\ 000 - \\
 & (4\ 227\ 240\ 686\ 139\ 909\ 064\ 705\ 589\ 929\ 214\ 593\ 102\ 933\ 845\ 388\ 672\ 369\ 082\ 676\ 644\ 542\ 650\ 248\ 228\ \vdots \\
 & 369\ 590\ 525\ 634\ 078\ 984\ 302\ 153\ 217\ 507\ 945\ 782\ 396\ 923\ 579\ 721\ z^{90}) / \\
 & 1\ 485\ 715\ 964\ 481\ 761\ 497\ 309\ 522\ 733\ 620\ 825\ 737\ 885\ 569\ 961\ 284\ 688\ 766\ 942\ 216\ 863\ 704\ 985\ 393\ \vdots \\
 & 094\ 065\ 876\ 545\ 992\ 131\ 370\ 884\ 059\ 645\ 617\ 234\ 469\ 978\ 112\ 000\ 000\ 000\ 000\ 000\ 000\ 000 + \\
 & (42\ 815\ 560\ 573\ 064\ 428\ 078\ 010\ 992\ 956\ 950\ 942\ 861\ 432\ 484\ 026\ 397\ 489\ 883\ 900\ 223\ 747\ 790\ 418\ 497\ \vdots \\
 & 948\ 114\ 193\ 578\ 680\ 162\ 571\ 793\ 178\ 004\ 310\ 552\ 004\ 531\ 066\ 419\ z^{92}) / \\
 & 37\ 129\ 594\ 192\ 959\ 126\ 135\ 747\ 236\ 793\ 652\ 397\ 246\ 501\ 467\ 808\ 583\ 326\ 438\ 329\ 073\ 381\ 904\ 888\ 689\ \vdots \\
 & 503\ 043\ 338\ 635\ 958\ 578\ 618\ 033\ 872\ 695\ 843\ 244\ 724\ 348\ 518\ 400\ 000\ 000\ 000\ 000\ 000\ 000 - \\
 & (50\ 817\ 990\ 724\ 580\ 425\ 164\ 559\ 757\ 643\ 090\ 736\ 003\ 482\ 435\ 671\ 513\ 413\ 926\ 813\ 239\ 886\ 828\ 210\ 876\ \vdots \\
 & 247\ 074\ 897\ 752\ 122\ 164\ 140\ 484\ 881\ 907\ 534\ 297\ 068\ 189\ 565\ 042\ 330\ 181\ z^{94}) / \\
 & 108\ 736\ 615\ 665\ 674\ 308\ 027\ 365\ 285\ 256\ 786\ 601\ 004\ 186\ 803\ 580\ 182\ 872\ 307\ 497\ 374\ 434\ 045\ 199\ \vdots \\
 & 869\ 417\ 927\ 630\ 229\ 109\ 214\ 583\ 415\ 458\ 560\ 865\ 651\ 202\ 385\ 340\ 530\ 688\ 000\ 000\ 000\ 000\ 000\ 000 + \\
 & 000 + \\
 & (50\ 039\ 705\ 954\ 360\ 322\ 297\ 158\ 013\ 840\ 847\ 032\ 326\ 634\ 079\ 577\ 003\ 760\ 741\ 000\ 075\ 956\ 513\ 037\ 893\ \vdots \\
 & 631\ 352\ 039\ 017\ 573\ 706\ 673\ 862\ 825\ 679\ 894\ 245\ 530\ 824\ 535\ 763\ 677\ z^{96}) / \\
 & 264\ 187\ 840\ 714\ 111\ 516\ 871\ 972\ 848\ 425\ 452\ 269\ 212\ 303\ 016\ 542\ 748\ 737\ 692\ 261\ 352\ 467\ 906\ 130\ \vdots \\
 & 325\ 025\ 008\ 754\ 780\ 828\ 981\ 920\ 632\ 387\ 127\ 607\ 867\ 726\ 182\ 219\ 776\ 000\ 000\ 000\ 000\ 000\ 000 - \\
 & (10\ 192\ 301\ 945\ 118\ 109\ 544\ 467\ 431\ 855\ 447\ 637\ 929\ 383\ 937\ 537\ 761\ 510\ 320\ 573\ 414\ 521\ 901\ 629\ 141\ \vdots \\
 & 453\ 868\ 275\ 940\ 735\ 720\ 302\ 151\ 703\ 532\ 547\ 073\ 126\ 053\ 721\ 100\ 491\ 517\ 027\ 671\ z^{98}) / \\
 & 132\ 773\ 104\ 913\ 848\ 559\ 797\ 551\ 911\ 874\ 045\ 668\ 645\ 206\ 954\ 423\ 647\ 206\ 528\ 087\ 870\ 102\ 742\ 353\ \vdots \\
 & 098\ 918\ 645\ 054\ 689\ 805\ 059\ 263\ 790\ 420\ 050\ 786\ 628\ 008\ 569\ 865\ 202\ 995\ 036\ 160\ 000\ 000\ 000\ 000\ \vdots \\
 & 000\ 000\ 000 + \\
 & (1\ 222\ 538\ 251\ 225\ 725\ 261\ 917\ 727\ 930\ 427\ 130\ 894\ 769\ 496\ 002\ 798\ 382\ 713\ 718\ 102\ 029\ 114\ 525\ 323\ \vdots \\
 & 524\ 473\ 694\ 269\ 201\ 451\ 423\ 144\ 149\ 126\ 167\ 703\ 052\ 004\ 443\ 591\ 028\ 403\ 213\ 886\ 579\ z^{100}) / \\
 & 39\ 295\ 248\ 607\ 976\ 485\ 339\ 662\ 837\ 413\ 164\ 926\ 522\ 406\ 723\ 479\ 739\ 630\ 092\ 039\ 142\ 692\ 723\ 199\ 997\ \vdots \\
 & 149\ 438\ 151\ 133\ 247\ 989\ 960\ 639\ 278\ 422\ 609\ 650\ 874\ 620\ 529\ 789\ 972\ 720\ 386\ 048\ 000\ 000\ 000\ 000\ \vdots \\
 & 000\ 000\ 000 + O[z]^{101}
 \end{aligned}$$

*Mathematica* comes with the add-on package `DiscreteMath`RSolve`` that allows finding the general terms of series for many functions. After loading this package, and using the package function `SeriesTerm`, the following  $n^{\text{th}}$  term of `sech(z)` can be evaluated.

```

<< DiscreteMath`RSolve`
SeriesTerm[Sech[z], {z, 0, n}] z^n
-----
    z^n EulerE[n]
      n!

```

Here is a quick check of the last result.

This series should be evaluated to `sech(z)`, which can be concluded from the following relation.

$$\text{Table}\left[\frac{z^n \text{EulerE}[n]}{n!} - \text{SeriesCoefficient}[\text{Series}[\text{Sech}[z], \{z, 0, 105\}], n] z^n, \{n, 100\}\right]$$

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

$$\text{Sum}\left[\frac{z^n \text{EulerE}[n]}{n!}, \{n, 0, 100\}\right] - \text{Series}[\text{Sech}[z], \{z, 0, 100\}]$$

$$O[z]^{101}$$

### Differentiation

*Mathematica* can evaluate derivatives of the hyperbolic secant function of an arbitrary positive integer order.

$\partial_z \text{Sech}[z]$

$$-\text{Sech}[z] \text{Tanh}[z]$$

$\partial_{\{z,2\}} \text{Sech}[z]$

$$-\text{Sech}[z]^3 + \text{Sech}[z] \text{Tanh}[z]^2$$

$\text{Table}[\text{D}[\text{Sech}[z], \{z, n\}], \{n, 10\}]$

$$\{-\text{Sech}[z] \text{Tanh}[z], -\text{Sech}[z]^3 + \text{Sech}[z] \text{Tanh}[z]^2, \\ 5 \text{Sech}[z]^3 \text{Tanh}[z] - \text{Sech}[z] \text{Tanh}[z]^3, 5 \text{Sech}[z]^5 - 18 \text{Sech}[z]^3 \text{Tanh}[z]^2 + \text{Sech}[z] \text{Tanh}[z]^4, \\ -61 \text{Sech}[z]^5 \text{Tanh}[z] + 58 \text{Sech}[z]^3 \text{Tanh}[z]^3 - \text{Sech}[z] \text{Tanh}[z]^5, \\ -61 \text{Sech}[z]^7 + 479 \text{Sech}[z]^5 \text{Tanh}[z]^2 - 179 \text{Sech}[z]^3 \text{Tanh}[z]^4 + \text{Sech}[z] \text{Tanh}[z]^6, \\ 1385 \text{Sech}[z]^7 \text{Tanh}[z] - 3111 \text{Sech}[z]^5 \text{Tanh}[z]^3 + 543 \text{Sech}[z]^3 \text{Tanh}[z]^5 - \text{Sech}[z] \text{Tanh}[z]^7, \\ 1385 \text{Sech}[z]^9 - 19028 \text{Sech}[z]^7 \text{Tanh}[z]^2 + 18270 \text{Sech}[z]^5 \text{Tanh}[z]^4 - 1636 \text{Sech}[z]^3 \text{Tanh}[z]^6 + \\ \text{Sech}[z] \text{Tanh}[z]^8, -50521 \text{Sech}[z]^9 \text{Tanh}[z] + 206276 \text{Sech}[z]^7 \text{Tanh}[z]^3 - \\ 101166 \text{Sech}[z]^5 \text{Tanh}[z]^5 + 4916 \text{Sech}[z]^3 \text{Tanh}[z]^7 - \text{Sech}[z] \text{Tanh}[z]^9, \\ -50521 \text{Sech}[z]^{11} + 1073517 \text{Sech}[z]^9 \text{Tanh}[z]^2 - 1949762 \text{Sech}[z]^7 \text{Tanh}[z]^4 + \\ 540242 \text{Sech}[z]^5 \text{Tanh}[z]^6 - 14757 \text{Sech}[z]^3 \text{Tanh}[z]^8 + \text{Sech}[z] \text{Tanh}[z]^{10}\}$$

### Indefinite integration

*Mathematica* can calculate a huge set of doable indefinite integrals that contain the hyperbolic secant function. The results can contain special functions. Here are some examples.

$$\int \text{sech}[z] \, dz$$

$$2 \text{ArcTan}\left[\text{Tanh}\left[\frac{z}{2}\right]\right]$$

$$\int \frac{1}{\sqrt{\text{Sech}[z]}} \, dz$$

$$-\frac{2 i \text{EllipticE}\left[\frac{i z}{2}, 2\right]}{\sqrt{\text{Cosh}[z]} \sqrt{\text{Sech}[z]}}$$

$$\int \operatorname{sech}[z]^a dz$$

$$\frac{\operatorname{Hypergeometric2F1}\left[\frac{1-a}{2}, \frac{1}{2}, \frac{3-a}{2}, \operatorname{Cosh}[z]^2\right] \operatorname{Sech}[z]^{-1+a} \operatorname{Sinh}[z]}{(1-a) \sqrt{-\operatorname{Sinh}[z]^2}}$$

$$\int \frac{1}{a + b \operatorname{Sech}[z]} dz // \text{Simplify}$$

$$z + \frac{2 b \operatorname{ArcTan}\left[\frac{(-a+b) \operatorname{Tanh}\left[\frac{z}{2}\right]}{\sqrt{a^2-b^2}}\right]}{\sqrt{a^2-b^2}}$$

$$\frac{\quad}{a}$$

**Definite integration**

Mathematica can calculate wide classes of definite integrals that contain the hyperbolic secant function. Here are some examples.

$$\int_0^\infty t^2 \operatorname{Sech}[t] dt$$

$$\frac{\pi^3}{8}$$

$$\int_0^{\frac{\pi}{2}} \frac{t e^{-a z}}{\operatorname{Sech}[t]} dt$$

$$e^{-a z} \left( 1 - \operatorname{Cosh}\left[\frac{\pi}{2}\right] + \frac{1}{2} \pi \operatorname{Sinh}\left[\frac{\pi}{2}\right] \right)$$

$$\int_0^\infty t \operatorname{Sech}[t] dt$$

$$2 \operatorname{Catalan}$$

$$\int_0^{\pi/2} \sqrt{\operatorname{Sech}[z]} dz$$

$$-2 i \operatorname{EllipticF}\left[\frac{i \pi}{4}, 2\right]$$

$$\int_0^\infty e^{-a z} \operatorname{Sech}[z] dz$$

$$\operatorname{If}\left[\operatorname{Re}[a] > 0, \frac{1}{2} \left( -\operatorname{PolyGamma}\left[0, \frac{1+a}{4}\right] + \operatorname{PolyGamma}\left[0, \frac{3+a}{4}\right] \right), \int_0^\infty e^{-a z} \operatorname{Sech}[z] dz \right]$$

**Limit operation**

Mathematica can calculate limits that contain the hyperbolic secant function. Here are some examples.

$$\text{Limit}\left[\frac{\text{Sech}[3z] - 1}{z^2}, z \rightarrow 0\right]$$

$$-\frac{9}{2}$$

$$\text{Limit}\left[z \text{Sech}\left[\pi i / 2 - 2\sqrt{z^2}\right], z \rightarrow 0, \text{Direction} \rightarrow 1\right]$$

$$-\frac{i}{2}$$

$$\text{Limit}\left[z \text{Sech}\left[\pi i / 2 - 2\sqrt{z^2}\right], z \rightarrow 0, \text{Direction} \rightarrow -1\right]$$

$$\frac{i}{2}$$

### Solving equations

The next inputs solve two equations that contain the hyperbolic secant function. Because of the multivalued nature of the inverse hyperbolic secant function, a message is printed indicating that only some of the possible solutions are returned.

$$\text{Solve}\left[\text{Sech}[z]^2 + 2 \text{Sech}\left[z + \text{Pi} / 6\right] = 4, z\right]$$

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.

$$\left\{ \left\{ z \rightarrow \text{ArcCosh} \left[ \text{Root} \left[ -\text{Sinh} \left[ \frac{\pi}{6} \right]^2 - \text{Cosh} \left[ \frac{\pi}{6} \right]^2 \#1^2 + 9 \text{Sinh} \left[ \frac{\pi}{6} \right]^2 \#1^2 - 4 \text{Cosh} \left[ \frac{\pi}{6} \right] \#1^3 - 4 \#1^4 + 8 \text{Cosh} \left[ \frac{\pi}{6} \right]^2 \#1^4 - 24 \text{Sinh} \left[ \frac{\pi}{6} \right]^2 \#1^4 + 16 \text{Cosh} \left[ \frac{\pi}{6} \right] \#1^5 - 16 \text{Cosh} \left[ \frac{\pi}{6} \right]^2 \#1^6 + 16 \text{Sinh} \left[ \frac{\pi}{6} \right]^2 \#1^6 \&, 1 \right] \right] \right\}, \right. \\ \left. \left\{ z \rightarrow \text{ArcCosh} \left[ \text{Root} \left[ -\text{Sinh} \left[ \frac{\pi}{6} \right]^2 - \text{Cosh} \left[ \frac{\pi}{6} \right]^2 \#1^2 + 9 \text{Sinh} \left[ \frac{\pi}{6} \right]^2 \#1^2 - 4 \text{Cosh} \left[ \frac{\pi}{6} \right] \#1^3 - 4 \#1^4 + 8 \text{Cosh} \left[ \frac{\pi}{6} \right]^2 \#1^4 - 24 \text{Sinh} \left[ \frac{\pi}{6} \right]^2 \#1^4 + 16 \text{Cosh} \left[ \frac{\pi}{6} \right] \#1^5 - 16 \text{Cosh} \left[ \frac{\pi}{6} \right]^2 \#1^6 + 16 \text{Sinh} \left[ \frac{\pi}{6} \right]^2 \#1^6 \&, 2 \right] \right] \right\}, \right. \\ \left. \left\{ z \rightarrow -\text{ArcCosh} \left[ \text{Root} \left[ -\text{Sinh} \left[ \frac{\pi}{6} \right]^2 - \text{Cosh} \left[ \frac{\pi}{6} \right]^2 \#1^2 + 9 \text{Sinh} \left[ \frac{\pi}{6} \right]^2 \#1^2 - 4 \text{Cosh} \left[ \frac{\pi}{6} \right] \#1^3 - 4 \#1^4 + 8 \text{Cosh} \left[ \frac{\pi}{6} \right]^2 \#1^4 - 24 \text{Sinh} \left[ \frac{\pi}{6} \right]^2 \#1^4 + 16 \text{Cosh} \left[ \frac{\pi}{6} \right] \#1^5 - 16 \text{Cosh} \left[ \frac{\pi}{6} \right]^2 \#1^6 + 16 \text{Sinh} \left[ \frac{\pi}{6} \right]^2 \#1^6 \&, 3 \right] \right] \right\}, \right. \\ \left. \left\{ z \rightarrow -\text{ArcCosh} \left[ \text{Root} \left[ -\text{Sinh} \left[ \frac{\pi}{6} \right]^2 - \text{Cosh} \left[ \frac{\pi}{6} \right]^2 \#1^2 + 9 \text{Sinh} \left[ \frac{\pi}{6} \right]^2 \#1^2 - 4 \text{Cosh} \left[ \frac{\pi}{6} \right] \#1^3 - 4 \#1^4 + 8 \text{Cosh} \left[ \frac{\pi}{6} \right]^2 \#1^4 - 24 \text{Sinh} \left[ \frac{\pi}{6} \right]^2 \#1^4 + 16 \text{Cosh} \left[ \frac{\pi}{6} \right] \#1^5 - 16 \text{Cosh} \left[ \frac{\pi}{6} \right]^2 \#1^6 + 16 \text{Sinh} \left[ \frac{\pi}{6} \right]^2 \#1^6 \&, 4 \right] \right] \right\}, \right. \\ \left. \left\{ z \rightarrow -\text{ArcCosh} \left[ \text{Root} \left[ -\text{Sinh} \left[ \frac{\pi}{6} \right]^2 - \text{Cosh} \left[ \frac{\pi}{6} \right]^2 \#1^2 + 9 \text{Sinh} \left[ \frac{\pi}{6} \right]^2 \#1^2 - 4 \text{Cosh} \left[ \frac{\pi}{6} \right] \#1^3 - 4 \#1^4 + 8 \text{Cosh} \left[ \frac{\pi}{6} \right]^2 \#1^4 - 24 \text{Sinh} \left[ \frac{\pi}{6} \right]^2 \#1^4 + 16 \text{Cosh} \left[ \frac{\pi}{6} \right] \#1^5 - 16 \text{Cosh} \left[ \frac{\pi}{6} \right]^2 \#1^6 + 16 \text{Sinh} \left[ \frac{\pi}{6} \right]^2 \#1^6 \&, 5 \right] \right] \right\}, \right. \\ \left. \left\{ z \rightarrow -\text{ArcCosh} \left[ \text{Root} \left[ -\text{Sinh} \left[ \frac{\pi}{6} \right]^2 - \text{Cosh} \left[ \frac{\pi}{6} \right]^2 \#1^2 + 9 \text{Sinh} \left[ \frac{\pi}{6} \right]^2 \#1^2 - 4 \text{Cosh} \left[ \frac{\pi}{6} \right] \#1^3 - 4 \#1^4 + 8 \text{Cosh} \left[ \frac{\pi}{6} \right]^2 \#1^4 - 24 \text{Sinh} \left[ \frac{\pi}{6} \right]^2 \#1^4 + 16 \text{Cosh} \left[ \frac{\pi}{6} \right] \#1^5 - 16 \text{Cosh} \left[ \frac{\pi}{6} \right]^2 \#1^6 + 16 \text{Sinh} \left[ \frac{\pi}{6} \right]^2 \#1^6 \&, 6 \right] \right] \right\} \right\}$$

**Solve[Sech[x] == a, x]**

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.

$$\{\{x \rightarrow -\text{ArcSech}[a]\}, \{x \rightarrow \text{ArcSech}[a]\}\}$$

A complete solution of the previous equation can be obtained using the function Reduce.

**Reduce[Sech[x] == a, x] // InputForm**

$$\text{C}[1] \in \text{Integers} \&\& a \neq 0 \&\& (x == -\text{ArcCosh}[a^{(-1)}] + (2*I)*\text{Pi}*C[1] \mid \mid x == \text{ArcCosh}[a^{(-1)}] + (2*I)*\text{Pi}*C[1])$$

### Solving differential equations

Here is a linear first-order differential equation that is obeyed by the hyperbolic secant function.

```
DSolve[w'[z] == -Tanh[z] Sech[z], w[z], z]
{{w[z] -> C[1] + Sech[z]}}
```

Here is a nonlinear second-order differential equation that is obeyed by the hyperbolic secant function. *Mathematica* solves the differential equation as a rational function of  $e^z$ . But it is straightforward to show that a hyperbolic secant function is also a solution.

```
DSolve[-w[z]^2 + 2 w'[z]^2 - w[z] w''[z] == 0, w[z], z]
{{w[z] -> \frac{e^z C[2]}{-1 + e^{2 z + 2 C[1]}}}}
```

```
-w[z]^2 + 2 w'[z]^2 - w[z] w''[z] == 0 /. w -> Function[z, e^{-C[1]} C[2] Sech[z + C[1]]] // Simplify
True
```

### Plotting

*Mathematica* has built-in functions for 2D and 3D graphics. Here are some examples.

```
Plot[Sech[z], {z, -4, 4}];
```

```
Plot3D[Re[Sech[x + i y]], {x, -π, π}, {y, 0, π},
  PlotPoints -> 240, PlotRange -> {-5, 5},
  ClipFill -> None, Mesh -> False, AxesLabel -> {"x", "y", None}];
```

```
ContourPlot[Arg[Sech[\frac{1}{x + i y}]], {x, -\frac{1}{3}, \frac{1}{3}}, {y, -\frac{1}{3}, \frac{1}{3}},
  PlotPoints -> 400, PlotRange -> {-π, π}, FrameLabel -> {"x", "y", None, None},
  ColorFunction -> (Hue[0.78 #] &), ContourLines -> False, Contours -> 200];
```

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