INTRODUCTORY MATHEMATICS FOR ECONOMICS MSCS.
LECTURE 3: MULTIVARIABLE FUNCTIONS AND CONSTRAINED OPTIMIZATION.

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SEPTEMBER 2009.

Functions of more than one variable.

Production Function: $Y = F(K,L)$

Utility Function: $U = U(x_1, x_2)$

How do we differentiate these? This is called partial differentiation. If you differentiate a function with respect to one of its two (or more) variables, then you treat the other values as constants.

$$Y = K^{\alpha} L^{1-\alpha}$$

$$\frac{\partial Y}{\partial K} = \alpha K^{\alpha-1} L^{1-\alpha} \qquad \frac{\partial Y}{\partial K} = (1-\alpha) K^{\alpha} L^{-\alpha}$$

Some more maths examples:

$$Y = ze^x \Rightarrow \frac{\partial Y}{\partial z} = e^x \quad \text{and} \quad \frac{\partial Y}{\partial x} = ze^x$$

$$Y = (z+x)e^{x+z} \Rightarrow \frac{\partial Y}{\partial z} = (1+z+x)e^{x+z} \quad \text{and} \quad \frac{\partial Y}{\partial x} = (1+z+x)e^{x+z}$$

The rules of differentiation all apply to the partial derivates (product/quotient/chain rule etc.).

Second order partial derivates: simply do it twice!

$$Y = ze^x \Rightarrow \frac{\partial Y}{\partial z} = e^x \text{ and } \frac{\partial Y}{\partial x} = ze^x$$

$$\frac{\partial^2 Y}{\partial z^2} = 0 \text{ and } \frac{\partial^2 Y}{\partial x^2} = ze^x \text{ ; } \frac{\partial^2 Y}{\partial x \partial z} = e^x$$

The last item is called a *cross-partial derivative*: you differentiate first with x and then with z (or the other way around: you get the same result – Young's Theorem).

Total Differential.

Consider $y = f(x,z)$. How much does the dependant variable (y) change if there is a small change in the independent variables (x,z).

$$dy = f_x dx + f_z dz$$

Where f_z, f_x are the partial derivatives of f with respect to x and z (equivalent to f'). This expression is called the Total Differential.

Economic Application: Indifference curves: Combinations of (x,z) that keep u constant.

$$U = U(x,z)$$ Utility depends on x,y.
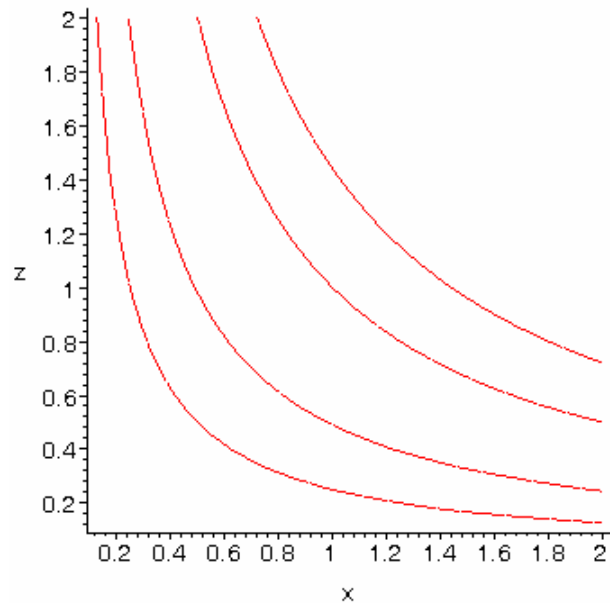
$$dU = U_x dx + U_z dz$$
Let x and y change by dx and dy: the change in u is dU

$$0 = U_x dx + U_z dz$$ For the indifference curve, we only allow changes in x,y that leave utility unchanged

$$\left. \frac{dx}{dz} \right|_{U=\bar{U}} = -\frac{U_z}{U_x} = MRS.$$ The slope of the indifference curve in (z,x) space is the MRS

For example: Cobb-Douglas preferences $U = x^{0.5}z^{0.5}$

$$U = z^{0.5}x^{0.5}$$

$$dU = 0.5\frac{z^{0.5}}{x^{0.5}}dx + 0.5\frac{x^{0.5}}{z^{0.5}}dz = 0$$

$$dx0.5\frac{z^{0.5}}{x^{0.5}} = -0.5\frac{x^{0.5}}{z^{0.5}}dz$$

$$\left.\frac{dz}{dx}\right|_{U=\bar{U}} = -\frac{z}{x} = MRS$$

Indifference curves: U= 0.5, 0.7, 1, 1.2.

Note: we can treat dx and dz just like any number/variable and do algebra with them….

Implicit Differentiation.

Take the Total differential. Now, suppose we want to see what happens if we hold one of the variables constant:

$$U = U(x,z)$$

$$dU = U_x dx + U_z dz$$

$$dz = 0 \Rightarrow dU = U_x dx$$

$$\left. \frac{dU}{dx} \right|_z = U_x$$

Implicit differentiation says that if we hold z constant, there is an implicit relationship (function) between y and x.

This is obvious: the partial differential equals the implicit derivative when we hold z constant.

However, we can do this operation even when we do not know the exact function. For example, we might not know the Utility function U, but just that (y,z,x) satisfy a general relationship F(x,y,z)=0. We can then use the Total differential to solve for any of the total differentials:

$$F(x, y, z) = 0$$

$$F_x dx + F_y dy + F_z dz = 0$$

If $dz = 0$

$$\Rightarrow F_x dx + F_y dy = 0$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

This is almost the same algebra as the indifference curve. Note, at the last step: this can only be done if $F_y \neq 0.$ (cannot divide by zero).

Total Derivative.

Suppose that we have two functions of the form:

$$y = f(x,z)$$
$$x = g(z)$$

We can substitute the second function into the first and then differentiate using the function of a function rule:

$$y = f(g(z),z)$$
$$\frac{dy}{dz} = \frac{\partial f}{\partial x}\frac{dg}{dz} + \frac{\partial f}{\partial z}$$

The Total effect of z on y takes two forms: the *direct effect* represented by the partial derivative $\frac{\partial f}{\partial z}$, and the *indirect effect* via g on x: $\frac{\partial f}{\partial x}\frac{dg}{dz}$.

Example:

$$y = x^{0.5} z^{0.5}$$
$$x = 1 - z$$

We can substitute the second relation into the first and differentiate w.r.t. z:

$$y = (1-z)^{0.5} z^{0.5} \Rightarrow \frac{dy}{dz} = -0.5(1-z)^{-0.5} z^{0.5} + 0.5(1-z)^{0.5} z^{-0.5}$$

The first expression is the indirect effect: the second the direct.

Unconstrained Dynamic Optimization with two or more variables.

Similar to one variable, but have more dimensions! In this course, I stick to two dimensions (can look in books for n dimensional case).

Maximize $y = f(x, z)$

Step 1: First order conditions

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial z} = 0$$

A necessary condition for a local or global maximum is that both (all if more than two variables) are zero.

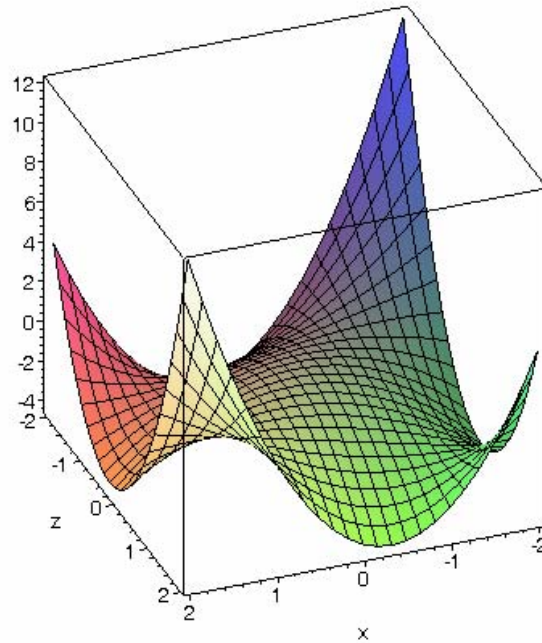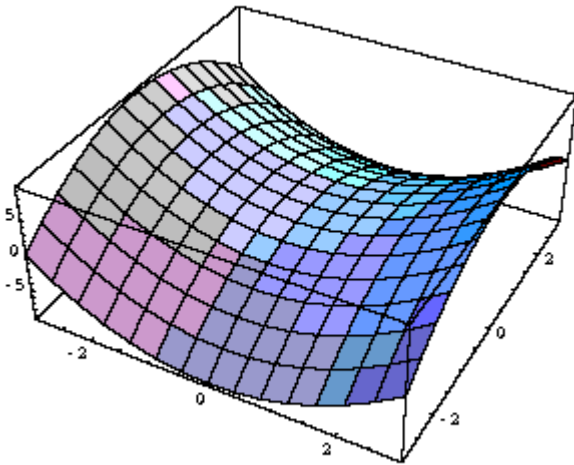Step 2: Second order conditions.

A local maximum requires that the function is strictly concave at the point where the first order conditions are met.

There are three second order conditions are

$$\frac{\partial^2 y}{\partial x^2} < 0 \quad \text{and} \quad \frac{\partial^2 y}{\partial z^2} < 0 \quad \text{and} \quad \left(\frac{\partial^2 y}{\partial x^2}\right)\left(\frac{\partial^2 y}{\partial z^2}\right) - \left(\frac{\partial^2 y}{\partial z \partial x}\right)^2 > 0$$

These three second order conditions ensure strict concavity. First two are obvious: third arises because of the extra dimension.

Can have a "saddlepoint" if first two are satisfied but not the third. Here are a couple of saddlepoints: the vertical axis is y, and in both cases the first order conditions are satisfied at x=z=0. But in neither case iis it a maximum!

Economic Example.

In economics we usually make assumptions that ensure that the multivariate function is strictly concave (when maximizing).

Multi-Product competitive firm.

$$\Pi = P^x x + P^z z - C(x, z)$$

$$c(x, z) = x^2 + z^2 - a.x.z$$

The first order conditions are:

$$\Pi_x = P^x - 2x + az$$

$$\Pi_z = P^z - 2z + ax$$

Second order conditions are:

$$\Pi_{xx} = -2 \text{ and } \Pi_{zz} = -2$$

$$\text{since } \Pi_{xz} = \Pi_{zx} = a$$

$$\Pi_{xx}\Pi_{zz} - \left(\Pi_{xz}\right)^2 = 4 - a^2$$

We have a (global) maximum if $-2 < a < 2$: global because $-2 < a < 2$ ensures profits are a strictly concave function everywhere.

To solve for the optimum: we need to express x and z as a function of the two prices.

$$z = \frac{P^z}{2} + \frac{a}{2}x$$

Use the first order condition for z to obtain an expression for z in terms of P^z and x.

$$P^x - 2x + a\left(\frac{P^z}{2} + \frac{a}{2}x\right) = 0 \Rightarrow P^x + \frac{a}{2}P^z = \left(\frac{4-a^2}{2}\right)x$$

Substitute into the first order condition for x

Hence:

$$x = \left(\frac{2}{4-a^2}\right)P^x + \frac{a}{4-a^2}P^z$$

Rearrange and solve for x

Likewise we can show that

$$z = \left(\frac{2}{4-a^2}\right)P^z + \frac{a}{4-a^2}P^x$$

The model is symmetric, so the solution for z should look like that for x: you simply swap the prices around.

Note: in this model, there is an interdependence in the marginal cost if $a \neq 0$. If a>0, then we have diseconomies of scope: production of z increases x marginal cost and vice versa. If a<0, we have economies of scope: production of x lowers the MC of z.

We can see that if $a < -2$ or $a > 2$ then the first order conditions can imply nonsense: negative outputs….

Constrained Optimization.

In economics, we mostly use *constrained optimisation*. This means maximization subject to some constraint.

For example: a household will maximize utility subject to a budget constraint: its total expenditure is less than income Y.

Budget constraint: two goods. $P_1 x_1 + P_2 x_2 \leq Y$. This is an *inequality* constraint total expenditure is less than or equal to Y. However, if we assume that both goods are liked (non-satiation, "more is better"), then we know that all the money will be spent! So then we have an equality constraint:
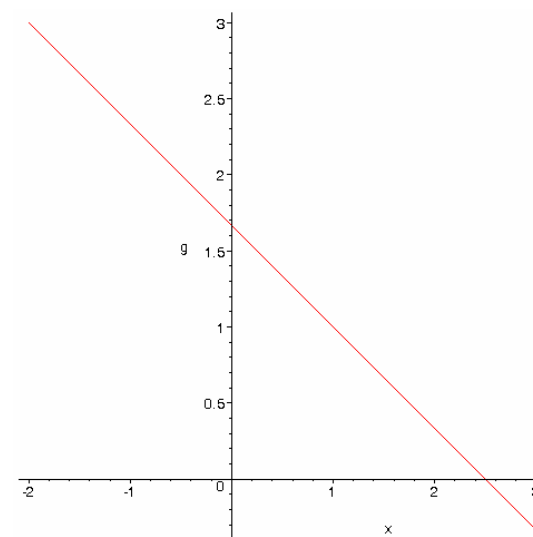
$$P_1 x_1 + P_2 x_2 = Y$$

This defines a straight line:

The Budget constraint can also
Be written as:

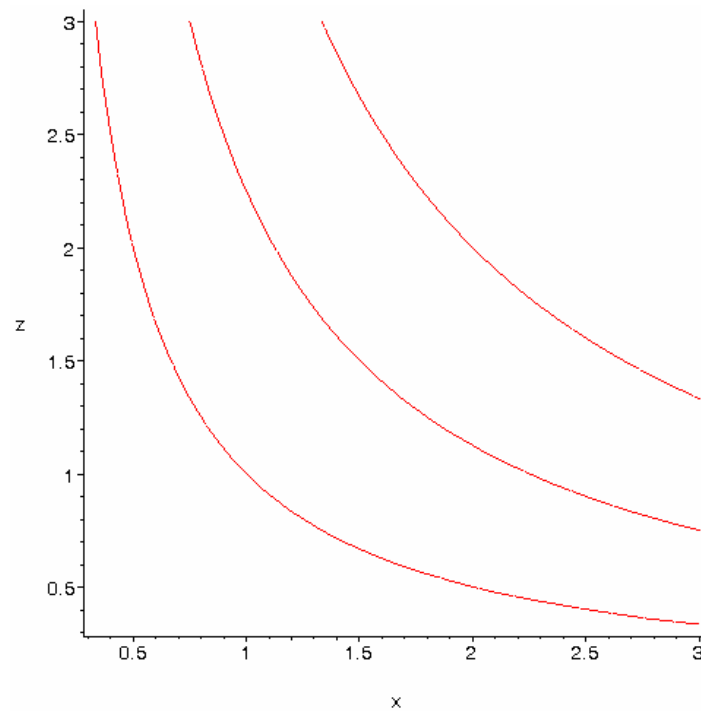$$x_2 = \frac{Y}{P_2} - \frac{P_1}{P_2} x_1$$

The *slope* of the budget line is the *price ratio*.

Of course, we also require the quantities consumed to be non-negative: so are only interested in the "positive orthant", where both $x_1 \geq 0, x_2 \geq 0$.

So the utility function can be represented by "indifference curves": these are like altitude contours on a map. For example

$$U(x_1, x_2) = x_1^{0.5} x_2^{0.5}$$

Lagrange Multipliers: Joseph-Louis Lagrange (1736-1813).

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0$$

Joseph-Louis Lagrange   1736 - 1813
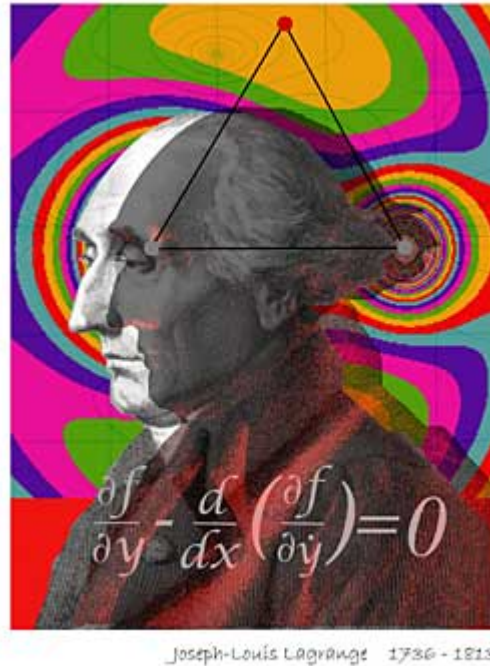
In Economics we use this method a lot: we can apply it when we have an equality constraint.

$$Max \quad U(x_1, x_2)$$
$$\text{s.t.} \quad P_1 x_1 + P_2 x_2 = Y$$

How to solve this: we invent a variable, the *lagrange multiplier* λ. Then we treat the constrained optimisation like an unconstrained one. We specify the *Lagrangean* $L(x_1, x_2, \lambda)$ where

$$L(x_1, x_2, \lambda) = U(x_1, x_2) + \lambda[Y - p_1 x_1 - p_2 x_2]$$

That is, we have the original objective function (utility function) and add on the invented variable λ times the constraint. Now, note a that:

- Since the constraint is $Y - P_1 x_1 - P_2 x_2 = 0$, the term in square brackets is zero when the constraint is satisfied, so that $L(x_1, x_2, \lambda) = U(x_1, x_2)$. The value of the lagrangean equals utility.

Now, having defined the lagrangean, we next take the first order conditions:

$$L_1 = \frac{\partial L}{\partial x_1} = U_1 - \lambda P_1 = 0$$

$$L_2 = \frac{\partial L}{\partial x_2} = U_2 - \lambda P_2 = 0$$

$$L_\lambda = \frac{\partial L}{\partial \lambda} = Y - P_1 x_1 - P_2 x_2 = 0$$

This gives us three equations and three unkowns $((x_1, x_2, \lambda)$. So, we can solve for the three variables. Lagrange: the solution to the these three equations give us the solution to the constrained optimization! (There are also some second order conditions: but, so long as U is concave, it is OK)!

Magic: invent a new variable and it lets you treat a constrained optimization like an unconstrained one.

From the first order conditions we can get the general properties of the optimum.

$$L_1 = U_1 - \lambda P_1 = 0 \Rightarrow \lambda = \frac{U_1}{P_1}$$ Take the first order conditions for both goods to get an expression for λ

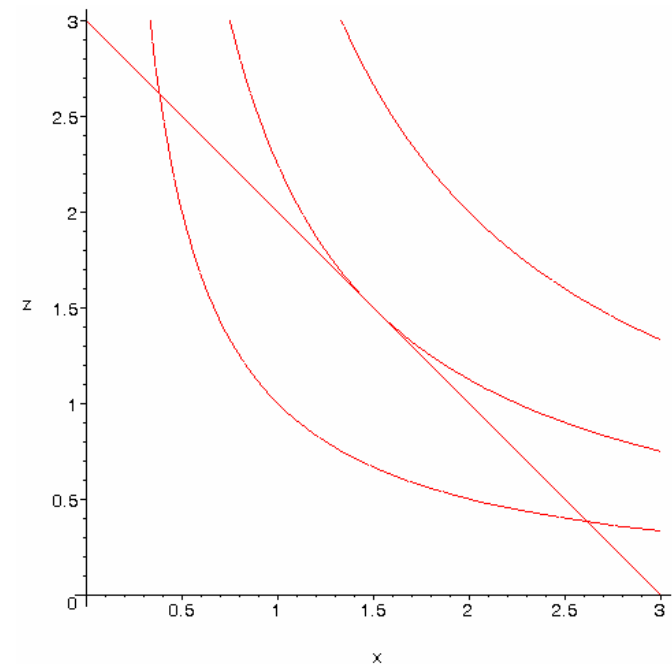$$L_2 = U_2 - \lambda P_2 = 0 \Rightarrow \lambda = \frac{U_2}{P_2}$$

Hence

$$\frac{U_1}{P_1} = \frac{U_2}{P_2} \Rightarrow \frac{U_1}{U_2} = \frac{P_1}{P_2}$$ Hence the MRS = slope of budget line (*sounds familiar?*)

Hence we have the *tangency* condition: at the optimum consumption bundle, the indifference curve is tangent to the budget constraint.

$$P_1 = P_2 = 1 : Y - x_1 - x_2 = 0$$

For example $U = x_1^{0.5} x_2^{0.5}$

$$x_1* = x_2* = 1.5.$$

3.15

We can also use this method to get the exact solution if we have explicit functional forms.

$$\max U = x_1^{0.5} x_2^{0.5}$$

$s.t$

$$3 - x_1 - x_2 = 0$$

First Step: form the Lagrangean: $\quad L = x_1^{0.5} x_2^{0.5} + \lambda[Y - x_1 - x_2]$

$$L_1 = 0.5 x_1^{-0.5} x_2^{0.5} - \lambda = 0$$

Second step: First order conditions. $\quad L_2 = 0.5 x_1^{0.5} x_2^{-0.5} - \lambda = 0$

$$L_\lambda = 3 - x_1 - x_2 = 0$$

Third Step: solve for (x_1, x_2, λ)

From $L_1 = L_2 = 0$, we have MRS=1: $\quad 0.5 x_1^{-0.5} x_2^{0.5} = 0.5 x_1^{0.5} x_2^{-0.5} \Rightarrow x_1 = x_2$. This means that the optimal solution lies on a ray from the origin where the x's are equal.

Now we need to find where on the ray: we use the budget constraint: $L_\lambda = 0 \Rightarrow x_1 + x_2 = 3$

$x_1 = x_2$ and $x_1 + x_2 = 3 \Rightarrow 2x_i = 3 \Rightarrow x_i^* = 1.5$.

To find the Lagrange multiplier: We can use the first or second condition:

$$L_1 = 0 \Rightarrow 0.5x*_1^{-0.5} \; x*_2^{0.5} = \lambda$$
$$\Rightarrow \lambda* = 0.5.$$

Finally: what is the maximum level of utility (which indifference curve are you on?):

$$U* = x*_1^{0.5} \; x*_2^{0.5} = 1.5$$

So, what is the meaning of the Lagrange multiplier?

The Lagrange multiplier tells you the increase in the maximum utility you can obtain if you get a little more income. This is often called the "shadow price". It is the derivative of maximum utility with respect to Y.

In this case: $dU* = \lambda.dY$. So, let us suppose that we had Y=4. The budget constrain moves out: the first order conditions for the x's are the same: so they will be chosen to be equal. If they are equal, then the optimal solution becomes 2 of each. In this case, $U* = 2$. An increase in income dY of 1 has given rise to an increase in maximum utility $dU*$ of 0.5. So, we can see exactly that $\lambda = 0.5$ gives the ratio (derivative) of these.

So, this is really magic: you introduce an extra number: not only does it let you solve the problem, but it also means something useful! COOL.....

Some more standard results.

Example 1. Two goods, but let us take one good as the numeraire: set its price equal to 1 (good 2), so P is the price of good 1 relative to good two.

max $x_1^\alpha x_2^{1-\alpha}$

s.t. $Y - px_1 - x_2 = 0$

Form Lagrangean: $\qquad\qquad L = x_1^\alpha x_2^{1-\alpha} + \lambda[Y - px_1 - x_2]$

$$L_1 = \alpha x_1^{\alpha-1} x_2^{(1-\alpha)} - \lambda p = 0$$

First order conditions: $\qquad L_2 = (1-\alpha)x_1^\alpha x_2^{-\alpha} - \lambda = 0$

$$L_\lambda = Y - px_1 - x_2 = 0$$

Note, since $\alpha x_1^{\alpha-1} x_2^{(1-\alpha)} = \alpha \dfrac{U}{x_1}$ and $\alpha x_1^{\alpha-1} x_2^{(1-\alpha)} = (1-\alpha)\dfrac{U}{x_2}$ we can rewrite the first two first order conditions

as $\alpha \dfrac{U}{x_1} = \lambda p$ and $(1-\alpha)\dfrac{U}{x_2} = \lambda$. Hence MRS=P becomes

$$\alpha \frac{U}{x_1 p} = (1-\alpha)\frac{U}{x_2} \Rightarrow \frac{\alpha}{1-\alpha}\frac{x_2}{x_1} = p \Rightarrow \frac{x_2}{x_1} = \frac{1-\alpha}{\alpha}p$$

This gives us the *ratio* of the x's as a function of p: this defines a slope of the ray from the origin. We then use the budget constraint to solve for the levels. From the previous expression:

$$x_2 = x_1 \frac{1-\alpha}{\alpha} p$$

So that the budget constraint becomes

$$Y - px_1 - x_1 \frac{1-\alpha}{\alpha} p = 0$$

$$Y = x_1 p[1 + \frac{1-\alpha}{\alpha}] = x_1 \frac{p}{\alpha}$$

$$x*_1 = \alpha \frac{Y}{p} \quad \text{or} \quad px*_1 = \alpha Y$$

The α gives the share of expenditure on good 1. This is constant, implying that the demand curve for a Cobb-Douglas utility is a rectangular hyperbola.

**Example 2: Cost Minimization.**

The firm wants to minimize cost (the dual; of maximizing output) of producing given output Y.

Cost: input costs $wL + rK$

Constraint: the choice of r,K must be sufficient to produce Y.

$$\min \quad rK + wL$$

$$s.t. \quad F(K,L) \geq Y.$$

Now: this is a minimisation problem. We know in (r,K)-space, that the iso-cost lines are given by $rK + wL = C_i \Rightarrow K = \dfrac{C_i}{r} - \dfrac{w}{r}L$. That is, negatively sloped lines with slope w/r and lines further away from origin mean higher cost. We consider iso-cost lines for $w=2$ and $r=3$: cost levels 2,3,5.
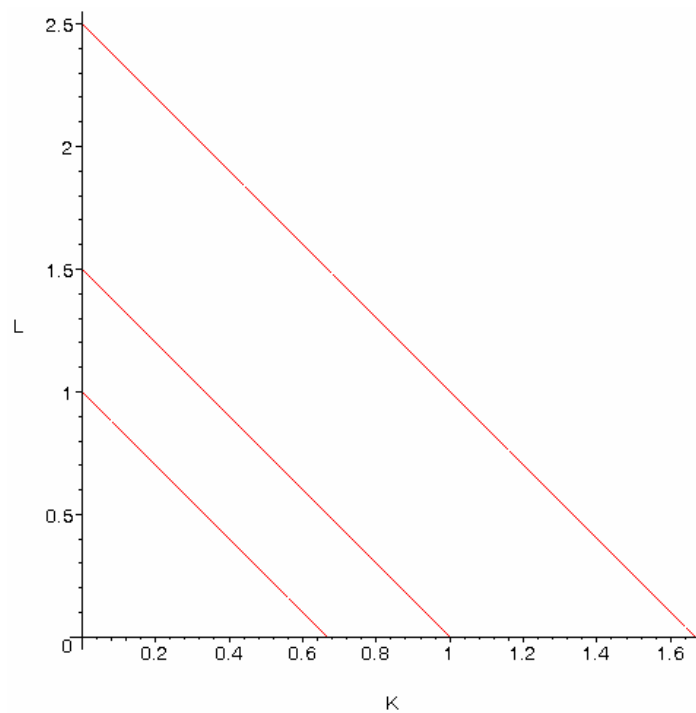
Higher levels of cost are represented by lines further out from origin: slope is -2/3,

Now let us look at the constraint: $F(K,L) \geq Y$. Again, if we assume that the marginal products of capital and labour are both *strictly positive*, we know that to minimize cost, the firm will use as little as possible. Hence the inequality constrain becomes an equality constraint $F(K,L) = Y$. Suppose that

$$F(K,L) = K^{0.5}L^{0.5} = Y = 2$$
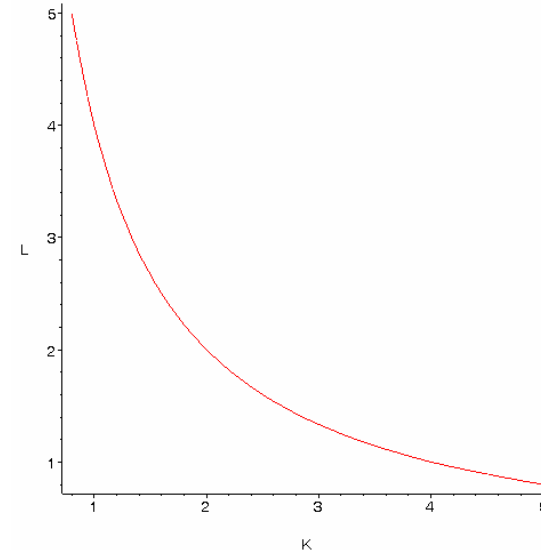
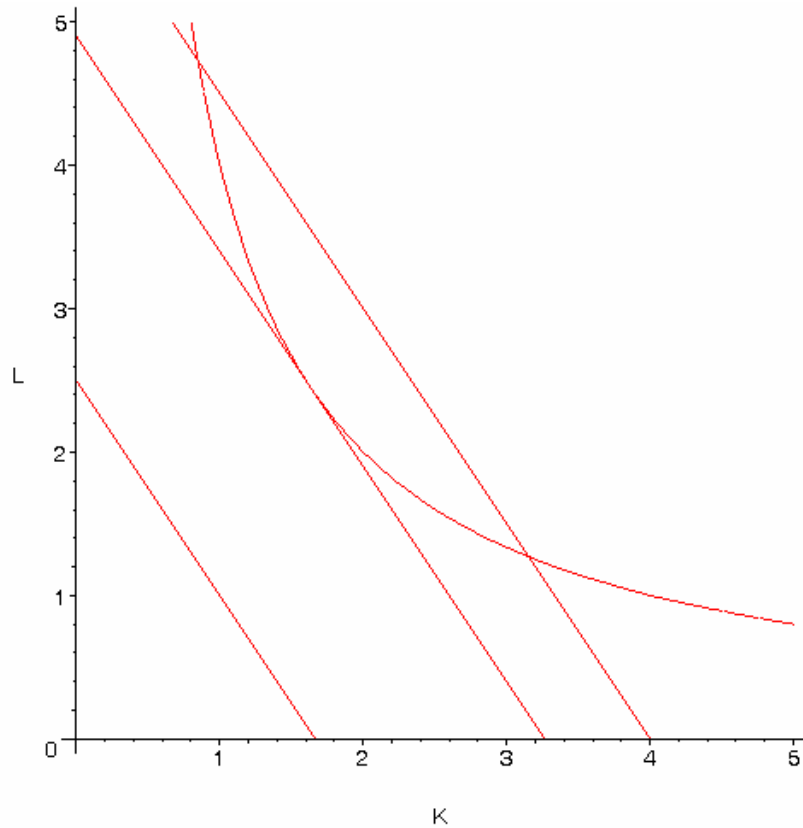Then the constraint is a particular isoquant line:

The slope of the isoquant:

$$dY = 0.5\frac{F}{K}dK + 0.5\frac{F}{L}dL = 0$$

$$\Rightarrow \left.\frac{dL}{dK}\right|_Y = -\frac{L}{K}$$

This is the ratio of the marginal products.

The solution to find the lowest iso-cost line that produces this output (cuts the isoquant) :

There are three isocost lines. The one nearest the origin is the lowest (C=5), but cannot produce Y=2.

The one furthest out is the highest cost level (C=12), and can produce Y=2.

The one that minimizes cost is the one where the isocost is tangential to the isoquant (C=9.8)

Now, let us see how this comes out of the LAGRANGEAN method.

Cost minimization step one. Specify the lagrangean.

$$L(K, L, \lambda) = wk + rK + \lambda[Y - F(K, L)]$$

F.O.C.

$$L_L = w - \lambda F_L = 0$$

$$L_K = r - \lambda F_K = 0$$

$$L_\lambda = F(K,L) - Y = 0.$$

Solve for (K,L,λ).

$$\lambda = \frac{w}{F_L} = \frac{r}{F_K} \Rightarrow \frac{w}{r} = \frac{F_L}{F_K}$$

Looks familiar? Slope of isocost = slope of isoquant (tangency condition).

Lastly, the third FOC tells us that we need a tangency AND it must produce Y.

Let us use our example: $w=2, r=3, Y=2$. with the production function $F(K,L) = K^{0.5} L^{0.5}$:

$$\frac{w}{r} = \frac{2}{3} = \frac{F_L}{F_k} = \frac{K}{L} \Rightarrow L = \frac{3}{2}K$$

The tangency condition implies that the solution must lie on a ray from the origin with slope 1.5. We can substitute this into the production function to solve for the optimal K,L

$$F(K,L) = K^{0.5} L^{0.5} = Y = 2 \Rightarrow K^{0.5}\left(\frac{3}{2}K\right)^{0.5} = 2 \Rightarrow K^* = \frac{2}{\sqrt{1.5}} \square \, 1.63$$

3.23

Hence: $\left(\dfrac{2}{\sqrt{1.5}}\right)^{0.5} L^{0.5} = 2 \Rightarrow L = 2\sqrt{1.5} \square\, 2.45.$

The minimum level of cost is thus: $C* = w.L* + rK* = 4\sqrt{1.5} + \dfrac{6}{\sqrt{1.5}} \square\, 4.90 + 4.90 \square\, 9.80$

Example 3. Cobb-Douglas cost minimization.

The Cobb-Douglas specification is used a lot in macroeconomics. Historically, the shares of labour and capital pretty constant not so true in last 10 years, but true in 1950-90.

Let us set r=1 (numeraire), so now w is the wage-rental ratio.

$\min \quad wL + K$

$s.t. \quad K^{\alpha} L^{1-\alpha} = Y$

Set up lagrangean

$$L(K, L, \lambda) = wL + K + \lambda[Y - K^{\alpha} L^{1-\alpha}]$$

FOC:

$$L_K = 1 - \lambda \alpha \frac{F}{K} = 0 \; ; \; L_L = w - \lambda(1-\alpha)\frac{F}{L} = 0 \; ; \; L_{\lambda} = Y - K^{\alpha} L^{1-\alpha} = 0$$

From FOC for K,L: $w = \dfrac{(1-\alpha)}{\alpha}\dfrac{K}{L} \Rightarrow L = \dfrac{(1-\alpha)}{\alpha}\dfrac{K}{w}$

Put this back into the technological constraint ($L_\lambda = 0$):

$$Y - K^\alpha \left(\frac{(1-\alpha)}{\alpha}\frac{K}{w}\right)^{1-\alpha} = 0 \Rightarrow Y = K\left(\frac{(1-\alpha)}{\alpha}\frac{1}{w}\right)^{1-\alpha} \Rightarrow K^* = Yw^{1-\alpha}\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha}$$

Hence we have $L^* = Yw^{-\alpha}\left(\dfrac{\alpha}{1-\alpha}\right)^{-\alpha}$

$$C^* = wL^* + K^* = Yw^{1-\alpha}\left[\left(\frac{\alpha}{1-\alpha}\right)^{-\alpha} + \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha}\right]$$

The minimum total cost is thus:

$$= Yw^{1-\alpha}\left(\frac{\alpha}{1-\alpha}\right)^{-\alpha}\left(\frac{1}{1-\alpha}\right)$$
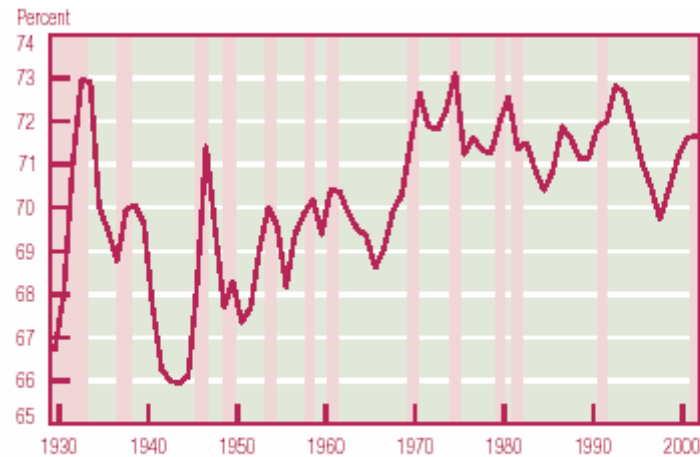
Now: note that the share of labour costs in total costs is

$$\frac{wL^*}{C^*} = \frac{Yw^{1-\alpha}\left(\dfrac{\alpha}{1-\alpha}\right)^{-\alpha}}{Yw^{1-\alpha}\left(\dfrac{\alpha}{1-\alpha}\right)^{-\alpha}\left(\dfrac{1}{1-\alpha}\right)} = 1-\alpha$$

Hence the share of capital in total costs is α.

Thus, if labours share of income is fairly constant, you can "calibrate" the Cobb-Douglas parameter α directly from the data!

In US/UK calibrates at around α=0.3.

Here is labour's share in US over 1930-2000+. Highest is 0.73 (α=0.27) and lowest 0.66 (α=0.34).

SOURCES: National Bureau of Economic Research; and authors' calculations based on NIPA data.

Conclusions

- Partial Differentiation: when you differentiate with one variable, treat others as fixed. Apply same rules.
- Second order: get "cross-partial" derivatives. The conditions for concavity and convexity need to be extended to allow for this cross effect.
- Constrained optimization: maximize subject to a constraint: budget constraint for consumer, technology for firm etc.
- Lagrange: magic method for solving constrained optimisation.