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2016 Nonlinearity 29 915

(http://iopscience.iop.org/0951-7715/29/3/915)

IOP Publishing | London Mathematical Society Nonlinearity

Nonlinearity **29** (2016) 915–946 doi:10.1088/0951-7715/29/3/915

Inverse scattering transform for the integrable nonlocal nonlinear Schrödinger equation

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Received 11 June 2015, revised 1 December 2015
Accepted for publication 30 December 2015
Published 4 February 2016

Recommended by Dr Beatrice Pelloni

Abstract

A nonlocal nonlinear Schrödinger (NLS) equation was recently introduced and shown to be an integrable infinite dimensional Hamiltonian evolution equation. In this paper a detailed study of the inverse scattering transform of this nonlocal NLS equation is carried out. The direct and inverse scattering problems are analyzed. Key symmetries of the eigenfunctions and scattering data and conserved quantities are obtained. The inverse scattering theory is developed by using a novel left–right Riemann–Hilbert problem. The Cauchy problem for the nonlocal NLS equation is formulated and methods to find pure soliton solutions are presented; this leads to explicit time-periodic one and two soliton solutions. A detailed comparison with the classical NLS equation is given and brief remarks about nonlocal versions of the modified Korteweg–de Vries and sine-Gordon equations are made.

Keywords: integrable nonlocal NLS equation, left-right Riemann-Hilbert problem, PT symmetry
Mathematics Subject Classification numbers: 37K15, 37Q55, 35Q51

1. Introduction

Exactly solvable models and integrable evolution equations are ubiquitous in nonlinear science and play an essential role in many branches of physics. Generally speaking, many of these systems can be derived from basic principles and arise as universal models in diverse physical phenomena. For example, the Korteweg–de Vries (KdV) and modified Korteweg–de

Vries (mKdV) equations describe the evolution of weakly dispersive and small amplitude waves in quadratic and cubic nonlinear media respectively. Physically speaking, the KdV equation is well known in terms of its application to shallow water waves [5, 19].

The integrable cubic nonlinear Schrödinger (NLS) equation [28] is also a universal model. It describes the evolution of weakly nonlinear and quasi-monochromatic wave trains in media with cubic nonlinearities. In the context of nonlinear optics, the NLS equation is a key model describing optical wave propagation in Kerr media [9, 18]. Moreover, in the limit of deep water, the NLS equation can be derived from the classical irrotational and inviscid water waves equations [5]. The KdV, mKdV and NLS equations share the mathematical property that all are integrable and exactly solvable evolution equations.

There are also numerous continuous and discrete exactly integrable evolution equations that are physically relevant and apply to diverse problems in fluid mechanics, electromagnetics, gravitational waves, elasticity, fundamental physics and lattice dynamics, to name but a few [3, 5, 6]. Recently integrable continuous and discrete nonlocal nonlinear Schrödinger equations describing wave propagation in nonlinear *PT* symmetric media were also found [1, 2].

Interestingly there are multi-dimensional extensions of many of these equations. The best known integrable multi-dimsional equation is the Kadomtsev–Petviashvili (KP) equation [17]. It is too a universal model describing the evolution of weakly dispersive and small amplitude waves with additional weak transverse variation. It arises in water waves, plasma physics and internal waves [3, 5]. Remarkably, multi-line soliton solutions of the KP equation are observed virtually daily on shallow flat beaches [8].

Generally speaking integrability is established once an infinite number of constants of motion or an infinite number of conservation laws are obtained. However considerably more information about the solution can be obtained if the inverse scattering transform (IST) can be carried out [4]. Corresponding to rapidly decaying initial data, IST provides a linearization and a class of explicit solutions—e.g. solitons.

The method associates a compatible pair of linear equations (i.e. a Lax pair) with the integrable nonlinear equation. One of the equations, termed the scattering problem, is used to determine suitably analytic eigenfunctions and provides a mechanism to transform the initial data to appropriate scattering data. The other linear equation serves to determine the evolution of the scattering data. Using the analytic behavior of the eigenfunctions an inverse scattering problem is developed. It is convenient to transform the inverse scattering system to a generalized linear Riemann–Hilbert (RH) problem to solve for the underlying meromorphic functions. With the time dependence of the scattering data one can find the solution of the nonlinear evolution equation from the inverse or RH problem. The IST method is viewed as an extension of Fourier transforms to a class of nonlinear systems [3, 5, 13, 24].

Here we investigate in detail the following nonlocal nonlinear Schrödinger (NLS) equation

$$\mathrm{i}q_t(x,t) = q_{xx}(x,t) \pm 2q^2(x,t)q^*(-x,t), \tag{1.1}$$

where $*$ denotes complex conjugate. This equation was first introduced in [1] but due to size limitations many details had to be omitted. In the above equation we see that the nonlocality occurs in a remarkably simple way; namely one of the nonlinear terms has the dependent variable evaluated at $-x$, i.e. note the term $q^*(-x, t)$. We also note that integrable multidimensional extensions of the nonlocal NLS equation have been found and studied [15]. Another way to write the above equation is

$$\mathrm{i}q_t(x,t) = q_{xx}(x,t) + V(x,t)q(x,t), \tag{1.2}$$

where $V(x,t;q) = \pm 2q(x,t)q^*(-x,t)$. In this latter form the equation is viewed as a Schrödinger equation with a nonlinear '*PT*' symmetric potential [11]: $V(x, t) = V^*(-x, t)$. It is remarkable that such a simple nonlocal nonlinear Schrödinger equation turns out to be integrable. We also note that this equation is Galilean invariant. *PT*-symmetric equations have been extensively studied in recent years [14, 16, 20–23, 25–27]. This is discussed further in section 2.

There are other nonlocal integrable systems which have been analyzed; two that are well-known are the Benjamin-Ono and intermediate long wave equations [3, 5]. These equations were found in the study of long internal waves with a deep bottom layer. The IST for these systems has been known since the 1980s. We also remark that the following nonlocal evolution equations

$$q_t(x,t) + q_{xxx}(x,t) \pm 6q(x,t)q^*(-x,-t)q_x(x,t) = 0, \; q \in \mathbb{C}: \text{ complex nonlocal mKdV} \tag{1.3}$$

$$q_t(x,t) + q_{xxx}(x,t) \pm 6q(x,t)q(-x,-t)q_x(x,t) = 0, \; q \in \mathbb{R}: \text{ real nonlocal mKdV} \tag{1.4}$$

and

$$q_{xt}(x,t) + 2s(x,t)q(x,t) = 0, \; s_x(x,t) = (q(x,t)q(-x,-t))_t, \; q \in \mathbb{R}: \text{ real nonlocal sine–Gordon} \tag{1.5}$$

arise from a reduction of the AKNS scattering problem and are integrable. It is interesting that for this case (and 'odd' flows in the 2×2 hierarchy) the nonlocality shows up as sign inversions in both x and t. When $q(-x, -t) = q(x, t)$ all these equations reduce to their standard counterparts: mKdV and sine-Gordon. The IST formulation will be discussed in a future communication.

In this paper we develop the IST associated with nonlocal nonlinear Schrödinger (NLS) equation (1.1). It turns out that the general method of AKNS [5, 6] can be applied but with a new symmetry relationship imposed. This crucial symmetry relation has major consequences; it leads to new symmetries amongst the eigenfunctions, scattering data and a novel inverse problem which can be related to a Riemann–Hilbert problem formulated via eigenfunctions defined at both plus and minus infinity; here we refer to the method as a left/right Riemann–Hilbert approach. Explicitly this new symmetry condition leads to relationships between eigenfunctions defined at both $+\infty$ and $-\infty$ (hence the term left and right). In the standard NLS system the Riemann–Hilbert problem leads to a closed set of uncoupled integral equations at *either* $+\infty$ *or* $-\infty$. In the nonlocal NLS equation discussed in this paper a closed set of integral equations can again be determined. But to do this requires using both sets of eigenfunctions at $+\infty$ and $-\infty$ which, in turn, are coupled. The symmetry conditions relating the eigenfunctions at both $+\infty$ and $-\infty$ are fundamental. To our knowledge this is the first time that such symmetries between left and right Riemann–Hilbert problems have been employed in the IST associated with nonlinear wave problems.

The paper is organized as follows. In section 2 we employ the AKNS procedure to find a coupled NLS type system in terms of two potentials: $q(x, t)$ and $r(x, t)$. With the symmetry

$$r(x,t) = \sigma q^*(-x,t), \quad \sigma = \mp 1 \tag{1.6}$$

the nonlocal nonlinear Schrödinger (NLS) equation (1.1) results. In section 3 we provide a method to derive an infinite number of local and global conservation laws associated with the nonlocal nonlinear Schrödinger equation (1.1). This establishes its integrability as an infinite dimensional Hamiltonian dynamical system. In section 4 certain key asymptotic properties of the eigenfunctions and scattering data are discussed; this is followed by section 5 where the symmetries of the eigenfunctions as well as of the scattering data are established. The basic inverse scattering problem is developed in section 6 along with a detailed analysis of the left–right Riemann–Hilbert problem. The reconstruction formula of the potentials is presented in section 7. The time periodic one and two soliton solutions and some of their properties are given in section 9. Comparisons with the classical NLS equation is in section 10; the calculation for the scattering data associated with equation (1.1) for special box initial condition is given in section 11. We conclude in section 12.

2. Linear pair and compatibility condition

2.1. The nonlocal nonlinear Schrödinger equation

We begin our discussion by considering the AKNS scattering problem [5, 6]

$$\mathbf{v}_x = \mathsf{X}\mathbf{v}, \tag{2.1}$$

where $\mathbf{v} = \mathbf{v}(x, t)$ is a two-component vector, i.e. $\mathbf{v}(x, t) = (v_1(x, t), v_2(x, t))^T$ and $q(x, t)$, $r(x, t)$ are (in general) complex valued functions that vanish rapidly as $|x| \to \infty$ and k is a complex spectral parameter. The matrix X depends on the functions q and r as well as on the spectral parameter k

$$\mathsf{X} = \begin{pmatrix} -\mathrm{i}k & q(x,t) \\ r(x,t) & \mathrm{i}k \end{pmatrix}. \tag{2.2}$$

The time evolution of the eigenfunctions v_j, $j = 1, 2$ is given by

$$\mathbf{v}_t = \mathsf{T}\mathbf{v}, \tag{2.3}$$

where

$$\mathsf{T} = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & -\mathsf{A} \end{pmatrix}, \tag{2.4}$$

and A, B and C are scalar functions of $q(x, t)$, $r(x, t)$ given by

$$\mathsf{A} = 2\mathrm{i}k^2 + \mathrm{i}q(x,t)r(x,t), \tag{2.5}$$

$$\mathsf{B} = -2kq(x,t) - \mathrm{i}q_x(x,t), \tag{2.6}$$

$$\mathsf{C} = -2kr(x,t) + \mathrm{i}r_x(x,t). \tag{2.7}$$

The compatibility condition of system (2.1) and (2.3), i.e. $v_{jxt} = v_{jtx}$, $j = 1, 2$ yields

$$\mathrm{i}q_t(x,t) = q_{xx}(x,t) - 2r(x,t)q^2(x,t), \tag{2.8}$$

$$-\mathrm{i}r_t(x,t) = r_{xx}(x,t) - 2q(x,t)r^2(x,t). \tag{2.9}$$

Under the symmetry reduction

$$r(x,t) = \sigma q^*(-x,t), \quad \sigma = \mp 1, \tag{2.10}$$

system (2.8) and (2.9) are compatible and leads to the nonlocal nonlinear Schrödinger equation first introduced in [1] and mentioned above in the introduction. We write the resulting equation (1.1) again for the convenience of the reader:

$$\mathrm{i}q_t(x,t) = q_{xx}(x,t) \pm 2q^2(x,t)q^*(-x,t),$$

where again $*$ denotes complex conjugation and $q(x, t)$ is a complex valued function of the real variables x and t with corresponding Lax pairs given by

$$\mathsf{X} = \begin{pmatrix} -\mathrm{i}k & q(x,t) \\ \sigma q^*(-x,t) & \mathrm{i}k \end{pmatrix}, \tag{2.11}$$

$$\mathsf{T} = \begin{pmatrix} 2\mathrm{i}k^2 + \mathrm{i}\sigma q(x,t)q^*(-x,t) & -2kq(x,t) - \mathrm{i}q_x(x,t) \\ -2\sigma k q^*(-x,t) - \mathrm{i}\sigma q_x^*(-x,t) & -2\mathrm{i}k^2 - \mathrm{i}\sigma q(x,t)q^*(-x,t) \end{pmatrix}. \tag{2.12}$$

The crucial symmetry reduction (2.10) was first noted in [1] and leads to a novel class of nonlocal integrable nonlinear evolution equations including a nonlocal NLS hierarchy. This is a special and remarkably simple reduction of the more general AKNS system [4] which has not been previously found. A list of few important properties of equation (1.1) are shown below:

- Time-reversal symmetry: If $q(x, t)$ is a solution so is $q^*(x,-t)$.
- Invariance under the transformation $x \to -x$: If $q(x, t)$ is a solution so is $q(-x, t)$.
- Gauge invariance: If $q(x, t)$ is a solution so is $\mathrm{e}^{\mathrm{i}\theta_0}q(x,t)$ with real and constant θ_0.
- Complex translation invariance: If $q(x, t)$ is a solution so is $q(x + \mathrm{i}x_0, t)$ for any constant and real x_0.
- Equation (1.1) is a Hamiltonian dynamical system and is obtained using the variational formulation

$$\mathrm{i}q_t(x,t) = \frac{\delta H}{\delta q^*(-x,t)}, \tag{2.13}$$

where $\frac{\delta H}{\delta q^*(-x,t)}$ is the variational derivative of the Hamiltonian with respect to $q^*(-x, t)$ and is given by

$$H = \int_{-\infty}^{+\infty} [-q_x(x,t)q_x^*(-x,t) - \sigma q^2(x,t)q^{*2}(-x,t)]\,\mathrm{d}x, \quad \sigma = \mp 1. \tag{2.14}$$

- *PT* symmetry: If $q(x, t)$ is a solution so is $q^*(-x,-t)$. As mentioned in the Introduction, calling the quantity

$$V(x,t) = \pm 2q(x,t)q^*(-x,t), \tag{2.15}$$

which, in classical optics is referred to as a self-induced potential, implies that $V^*(-x, t) = V(x, t)$.; and with this at hand, equation (1.1) takes the form

$$\mathrm{i}q_t(x,t) = q_{xx}(x,t) + V(x,t)q(x,t). \tag{2.16}$$

This equivalent formulation allows one to connect equation (1.1) with *PT* symmetric optics for which $V(x, t)$ represents a 'waveguide' and the resulting equation remains invariant under the joint transformation of $x \to -x, t \to -t$ and a complex conjugate. Thus, the nonlocal equation (1.1) is *PT* symmetric [11]. We remark that wave propagation in *PT* symmetric coupled waveguides or photonic lattices has been observed in experiments in classical optics [14, 16, 20–23, 25–27].
- Galilean invariance: If $q(x, t)$ solves equation (1.1) with initial condition $q(x, 0)$ then

$$\tilde{q}(x,t) = q(x + 2\mathrm{i}\xi t, t)\mathrm{e}^{-\xi x}\mathrm{e}^{-\mathrm{i}\xi^2 t}, \tag{2.17}$$

also solves to equation (1.1) with corresponding initial condition $\tilde{q}(x, 0) = q(x, 0)\mathrm{e}^{-\xi x}$ for any real constant ξ. If one restricts the class of solutions to equation (1.1) to rapidly decaying solutions at $|x| \to \infty$ then this would put some restriction on the parameter ξ to guarantee that $\tilde{q}(x, t)$ itself remains within that class. We point out that the general q, r system (2.8) and (2.9) is also Galilean invariant: If $q(x, t)$, $r(x, t)$ solve system (2.8) and (2.9) so does

$$\tilde{q}(x, t) = q(x - \mathrm{i}Vt, t)\mathrm{e}^{-K_1 x}\mathrm{e}^{-\mathrm{i}\omega_1 t}, \tag{2.18}$$

$$\tilde{r}(x, t) = r(x - \mathrm{i}Vt, t)\mathrm{e}^{-K_2 x}\mathrm{e}^{-\mathrm{i}\omega_2 t}, \tag{2.19}$$

with $K_1 = -K_2$ and $\omega_2 = -\omega_1$ all being real parameters satisfying the relations $V = -2K_1$ and $\omega_1 = K_1^2$.

An infinite number of local and global conservation laws associated with equation (1.1) can be derived hence it is an integrable evolution equation. Using a left–right Riemann–Hilbert formulation, the inverse scattering transform is carried out and general solution to equation (1.1) corresponding to rapidly decaying initial data, is obtained including pure one and two solitons solutions. Key important properties of equation (1.1) are also contrasted with the classical NLS equation where the nonlocal nonlinear term $q^*(-x, t)$ is replaced by $q^*(x, t)$. In particular, the symmetries of the eigenfunctions and scattering data associated with the classical NLS equations with even initial data are shown to coincide with symmetries of (2.1) under the symmetry reduction (2.10).

2.2. The nonlocal real and complex mKdV equations

In the case where the functions A, B and C are third order polynomial in the spectral parameter k [3, 4], the compatibility condition of system (2.1) and (2.3) yields

$$q_t(x, t) + q_{xxx}(x, t) - 6q(x, t)r(x, t)q_x(x, t) = 0, \tag{2.20}$$

$$r_t(x, t) + r_{xxx}(x, t) - 6q(x, t)r(x, t)r_x(x, t) = 0, \tag{2.21}$$

Under the symmetry reduction

$$r(x, t) = \sigma q^*(-x, -t), \quad \sigma = \mp 1, \tag{2.22}$$

system (2.20) and (2.21) are compatible and leads to the nonlocal complex mKdV equation

$$q_t(x, t) + q_{xxx}(x, t) - 6\sigma q(x, t)q^*(-x, -t)q_x(x, t) = 0, \tag{2.23}$$

where again $*$ denotes complex conjugation and $q(x, t)$ is a complex valued function of the real variables x and t. When the function $q(x, t)$ is assumed to be real valued then equation (2.23) reduces back to the real mKdV equation (1.4). We point out that under space and time even initial conditions, the nonlocal mKdV equation reduces back to its classical (local) counterpart.

2.3. The nonlocal sine-Gordon equation

If in equation (2.4) we take $\mathsf{A} = \mathsf{A}_l/k, \mathsf{B} = \mathsf{B}_l/k$ and $\mathsf{C} = \mathsf{C}_l/k$ then the compatibility condition $v_{jxt} = v_{jtx}, j = 1, 2$ results in the following set of equation for the potents q and r

$$q_{xt}(x, t) + 2s(x, t)q(x, t) = 0, \tag{2.24}$$

$$r_{xt}(x,t) + 2s(x,t)r(x,t) = 0, \tag{2.25}$$

$$s_x(x,t) + (q(x,t)r(x,t))_t = 0, \tag{2.26}$$

where we have defined $A_1 = -is/2$. Under the reduction

$$r(x,t) = -q(-x,-t), \tag{2.27}$$

the system (2.24) and (2.25) are compatible and give rise to the nonlocal sine-Gordon equation

$$q_{xt}(x,t) + 2s(x,t)q(x,t) = 0, \qquad s(-x,-t) = s(x,t). \tag{2.28}$$

3. Infinite number of conserved quantities and conservation laws

3.1. Global conservation laws

The infinite number of conserved quantities of (1.1) is derived as follows. We assume that $q(x, t)$ decays rapidly at infinity. As a result, solutions of the scattering problem (2.1) can be defined subject to the following boundary conditions

$$\lim_{x\to-\infty} \phi(x,k) = \begin{pmatrix}1\\0\end{pmatrix} e^{-ikx}, \qquad \lim_{x\to-\infty} \overline{\phi}(x,k) = \begin{pmatrix}0\\1\end{pmatrix} e^{ikx}, \tag{3.1}$$

$$\lim_{x\to+\infty} \psi(x,k) = \begin{pmatrix}0\\1\end{pmatrix} e^{ikx}, \qquad \lim_{x\to+\infty} \overline{\psi}(x,k) = \begin{pmatrix}1\\0\end{pmatrix} e^{-ikx}. \tag{3.2}$$

Throughout the paper, $\overline{\phi}$ is *not* the complex conjugate of ϕ. We shall instead use ϕ^* to denote the complex conjugate of ϕ. If $\phi(x,t) = (\phi_1(x,t), \phi_2(x,t))^T$ is the solution to (2.1) that satisfies the boundary conditions (3.1) then, for $\mathrm{Im}\, k \geqslant 0$, the scattering data $a(k) \equiv \phi_1(x,t)e^{ikx}$ is analytic in the upper half complex k plane and approaches 1 as $x \to \pm\infty$. Substituting $\phi_1(x,t) = \exp[-ikx + \varphi(x,t)]$ into (2.1) we find (after eliminating ϕ_2) that the function $\mu(x,t) \equiv \varphi_x(x,t)$ satisfies the Riccati equation

$$q\frac{\partial}{\partial x}\left(\frac{\mu}{q}\right) + \mu^2 - qr - 2ik\mu = 0. \tag{3.3}$$

Since for $\mathrm{Im} k > 0$, $\lim_{|k|\to\infty} \varphi(x,k) = 0$ we substitute the asymptotic expansion

$$\mu(x,k) = \sum_{n=0}^{\infty} \frac{\mu_n(x,t)}{(2ik)^{n+1}}, \tag{3.4}$$

in equation (3.3) and equate powers of k to find

$$\mu_0(x,t) = -q(x,t)r(x,t), \tag{3.5}$$

$$\mu_1(x,t) = -q(x,t)r_x(x,t), \tag{3.6}$$

and for any integer $n \geqslant 1$

$$\mu_{n+1} = q\frac{\partial}{\partial x}\left(\frac{\mu_n}{q}\right) + \sum_{m=0}^{n-1} \mu_m \mu_{n-m-1}. \tag{3.7}$$

From the boundary conditions (3.1) it follows $\lim_{x\to-\infty} \phi_1(x,k)e^{ikx} = 1$ and $\lim_{x\to-\infty} \varphi(x,k) = 0$. Combining this result together with the definition of μ we have

$$\ln a(k) = \sum_{n=0}^{\infty} \frac{\mathcal{C}_n}{(2\mathrm{i}k)^{n+1}}, \tag{3.8}$$

where we have defined

$$\mathcal{C}_n = \int_{-\infty}^{+\infty} \mu_n(x,t)\mathrm{d}x. \tag{3.9}$$

Since for all k with $\operatorname{Im} k > 0$, $a(k)$ is time independent then it follows that $\mathcal{C}_n$ is also time independent. Below is a list of the first few conserved quantities under the symmetry reduction $r(x,t) = \sigma q^*(-x,t)$, $\sigma = \mp 1$:

$$\mathcal{C}_0 = -\sigma \int_{-\infty}^{+\infty} q(x,t)q^*(-x,t)\mathrm{d}x, \tag{3.10}$$

$$\mathcal{C}_1 = \frac{\sigma}{2} \int_{-\infty}^{+\infty} \left[q_x(x,t)q^*(-x,t) + q(x,t)q_x^*(-x,t)\right] \mathrm{d}x, \tag{3.11}$$

$$\mathcal{C}_2 = -\sigma \int_{-\infty}^{+\infty} \left[q_x(x,t)q_x^*(-x,t) - \sigma q^2(x,t)q^{*2}(-x,t)\right] \mathrm{d}x, \tag{3.12}$$

$$\begin{aligned}\mathcal{C}_3 = \frac{\sigma}{2} \int_{-\infty}^{+\infty} \quad & \{q_{xxx}(x,t)q^*(-x,t) + q(x,t)q_{xxx}^*(-x,t) \\ & - 2\sigma q(x,t)q^*(-x,t)\left[4q(x,t)q_x^*(-x,t) - q_x(x,t)q^*(-x,t)\right]\}\mathrm{d}x,\end{aligned} \tag{3.13}$$

$$\begin{aligned}\mathcal{C}_4 = -\sigma \int_{-\infty}^{+\infty} \quad & \{q_{xx}(x,t)q_{xx}^*(-x,t) - 6\sigma q(x,t)q^*(-x,t)q_x(x,t)q_x^*(-x,t) \\ & + \sigma[q^*(-x,t)q_x(x,t) - q(x,t)q_x^*(-x,t)]^2 + 2q^3(x,t)q^{*3}(-x,t)\}\mathrm{d}x,\end{aligned} \tag{3.14}$$

3.2. Local conservation laws

In this section we explain how to derive an infinite number of local conservation laws. We start with the time-dependent problem (2.3)

$$\phi_{1t} = \mathsf{A}\phi_1 + \mathsf{B}\phi_2. \tag{3.15}$$

Substituting the expression for μ and φ into (3.15) and taking the x derivative of the resulting equations we find

$$\partial_t \mu(x,t) = \partial_x \left(\mathsf{A}_{\mathrm{nonloc}} + \frac{\mu(x,t)\mathsf{B}_{\mathrm{nonloc}}}{q(x,t)} \right). \tag{3.16}$$

Recall that the nonlocal nonlinear Schrödinger equation (1.1) is obtained when

$$\mathsf{A}_{\mathrm{nonloc}} = 2\mathrm{i}k^2 + \mathrm{i}\sigma q(x,t)q^*(-x,t), \tag{3.17}$$

$$\mathsf{B}_{\mathrm{nonloc}} = -2kq(x,t) - \mathrm{i}q_x(x,t), \tag{3.18}$$

$$\mathsf{C}_{\mathrm{nonloc}} = -2k\sigma q^*(-x,t) - \mathrm{i}\sigma q_x^*(-x,t). \tag{3.19}$$

Substituting the power series expansion for μ from (3.4) and (3.17)–(3.18) into (3.16) we obtain

$$\partial_t\left(\sum_{n=0}^{\infty}\frac{\mu_n(x,t)}{(2\mathrm{i}k)^{n+1}}\right)=\mathrm{i}\partial_x\left[\sigma q(x,t)q^*(-x,t)+\left(2\mathrm{i}k-\frac{q_x(x,t)}{q(x,t)}\right)\sum_{n=0}^{\infty}\frac{\mu_n(x,t)}{(2\mathrm{i}k)^{n+1}}\right]. \tag{3.20}$$

The coefficients of $(2\mathrm{i}k)^{-n}$ are trivial for $n\leqslant -1$. For $n\geqslant 0$ we find

$$\partial_t\mu_n(x,t)+\mathrm{i}\partial_x\left[\frac{q_x(x,t)}{q(x,t)}\mu_n(x,t)-\mu_{n+1}(x,t)\right]=0,\quad n=0,1,2,3,\cdots. \tag{3.21}$$

We can write the conservation laws in the form

$$\frac{\partial\mathcal{T}}{\partial t}=-\mathrm{i}\frac{\partial\mathcal{X}}{\partial x}, \tag{3.22}$$

where $\mathcal{T}$ and $\mathcal{X}$ are the so-called densities and fluxes respectively. The first few conservation laws are given by

1. $\mathcal{T}=q(x,t)q^*(-x,t),\quad \mathcal{X}=q(x,t)q_x^*(-x,t)+q^*(-x,t)q_x(x,t).$
2. $\mathcal{T}=q(x,t)q_x^*(-x,t),\ \mathcal{X}=q_x^*(-x,t)q_x(x,t)+q(x,t)q_{xx}^*(-x,t)-\sigma q^2(x,t)q^{*2}(-x,t),$
3. $\mathcal{T}=q(x,t)q_{xx}^*(-x,t)-\sigma q^2(x,t)q^{*2}(-x,t),\ \mathcal{X}=q_x(x,t)q_{xx}^*(-x,t)+q(x,t)q_{xxx}^*(-x,t)$
   $-4\sigma q^2(x,t)q^*(-x,t)q_x^*(-x,t).$

4. Direct scattering problem

In this section we study the scattering problem (2.1) subject to the boundary conditions (3.1). It is expedient to reformulate the scattering problem in terms of eigenfunctions having constant boundary conditions (the so-called 'Jost functions') defined by (note: hereafter, for simplicity of notation, we often suppress the time dependence)

$$M(x,k)=\mathrm{e}^{\mathrm{i}kx}\phi(x,k),\qquad \overline{M}(x,k)=\mathrm{e}^{-\mathrm{i}kx}\overline{\phi}(x,k), \tag{4.1}$$

$$N(x,k)=\mathrm{e}^{-\mathrm{i}kx}\psi(x,k),\qquad \overline{N}(x,k)=\mathrm{e}^{\mathrm{i}kx}\overline{\psi}(x,k). \tag{4.2}$$

With this in mind, one can show that the Jost functions satisfy a linear implicit integral equation that in turn is used to establish the following important results: In the space of absolutely integrable functions $L^1(\mathbb{R})$ defined by $\int_{\mathbb{R}}|f|\mathrm{d}x<\infty$, one can show that $M(x,k)$, $N(x,k)$ are analytic functions in the upper half complex k plane whereas $\overline{M}(x,k),\overline{N}(x,k)$ are analytic functions in the lower half complex k plane [6]. From the integral representation one can also derive the large k asymptotics of the Jost functions used in the inverse problem, (c.f. [6])

$$M(x,k)=\begin{pmatrix}1-\dfrac{1}{2\mathrm{i}k}\displaystyle\int_{-\infty}^{x}r(z)q(z)\mathrm{d}z\\ -\dfrac{r(x)}{2\mathrm{i}k}\end{pmatrix}+O(k^{-2}), \tag{4.3}$$

$$N(x,k)=\begin{pmatrix}\dfrac{q(x)}{2\mathrm{i}k}\\ 1-\dfrac{1}{2\mathrm{i}k}\displaystyle\int_x^{+\infty}r(z)q(z)\mathrm{d}z\end{pmatrix}+O(k^{-2}),\tag{4.4}$$

$$\overline{M}(x,k)=\begin{pmatrix}\dfrac{q(x)}{2\mathrm{i}k}\\ 1+\dfrac{1}{2\mathrm{i}k}\displaystyle\int_{-\infty}^{x}r(z)q(z)\mathrm{d}z\end{pmatrix}+O(k^{-2}),\tag{4.5}$$

$$\overline{N}(x,k)=\begin{pmatrix}1+\dfrac{1}{2\mathrm{i}k}\displaystyle\int_x^{+\infty}r(z)q(z)\mathrm{d}z\\ -\dfrac{r(x)}{2\mathrm{i}k}\end{pmatrix}+O(k^{-2}).\tag{4.6}$$

The solutions $\phi(x,k)$ and $\overline{\phi}(x,k)$ of the scattering problem (2.1) with the boundary conditions (3.1) are linearly independent for all k satisfying $a(k)\neq 0$—see below. This follows from the fact that the Wronskian $W(u, v)$ is given by

$$W(u,v)\equiv u_1v_2-u_2v_1,\tag{4.7}$$

of any two solutions u and v to (2.1) is independent of x. Similar arguments hold for $\psi(x,k)$ and $\overline{\psi}(x,k)$. Therefore because the scattering problem (2.1) is a second order linear ODE, the pairs $\{\phi,\overline{\phi}\}$ and $\{\psi,\overline{\psi}\}$ are linearly dependent and one can express one basis set in terms of the other:

$$\phi(x,k)=a(k)\overline{\psi}(x,k)+b(k)\psi(x,k),\tag{4.8}$$

$$\overline{\phi}(x,k)=\overline{a}(k)\psi(x,k)+\overline{b}(k)\overline{\psi}(x,k).\tag{4.9}$$

With this result, the scattering coefficients are thus given by

$$a(k)=W(\phi,\psi),\tag{4.10}$$

$$\overline{a}(k)=W(\overline{\psi},\overline{\phi}).\tag{4.11}$$

$$b(k)=W(\overline{\psi},\phi),\tag{4.12}$$

$$\overline{b}(k)=W(\overline{\phi},\psi).\tag{4.13}$$

Note that the scattering data satisfy the relation

$$a(k)\overline{a}(k)-b(k)\overline{b}(k)=1.\tag{4.14}$$

If the potentials $q,r\to 0$ as $|x|\to\infty$ then from the analyticity properties of the Jost functions, it can be shown that $a(k)$ is analytic in the upper half complex k plane whereas $\overline{a}(k)$ is analytic in the lower half complex k plane [6]. In general, $b(k)$ and $\overline{b}(k)$ cannot be extended off the real k axis. Therefore, equation (4.14) is defined only for $\mathrm{Im}k=0$. The large k asymptotics of the scattering data $a(k)$ and $\overline{a}(k)$ is given by [6]

$$a(k) \sim 1 - \frac{1}{2\mathrm{i}k}\int_{-\infty}^{+\infty} r(z)q(z)\mathrm{d}z + O(k^{-2}), \tag{4.15}$$

$$\bar{a}(k) \sim 1 + \frac{1}{2\mathrm{i}k}\int_{-\infty}^{+\infty} r(z)q(z)\mathrm{d}z + O(k^{-2}). \tag{4.16}$$

5. Symmetry reduction $r(x,t) = \sigma q^*(-x,t),\ \sigma = \mp 1$: eigenfunctions and scattering data

5.1. Symmetry of the eigenfunctions

In this section we establish important symmetry properties of the eigenfunctions of the eigenvalue problem (2.1) valid under the symmetry reduction $r(x,t) = \sigma q^*(-x,t)$ where $\sigma = \mp 1$. To do so, let $v(x,k) \equiv (v_1(x,k),\ v_2(x,k))^{\mathrm{T}}$ be a solution to system (2.1) with $r(x,t) = \sigma q^*(-x,t)$. If we take the complex conjugation of (2.1) and let $x \to -x,\ k^* \to -k^*$ one reaches the following conclusion:

$$\text{If } \begin{pmatrix} v_1(x,k) \\ v_2(x,k) \end{pmatrix} \text{solves (2.1) then } \begin{pmatrix} v_2^*(-x,-k^*) \\ -\sigma v_1^*(-x,-k^*) \end{pmatrix} \text{solves (2.1).} \tag{5.1}$$

Therefore, because the solutions of the scattering problem (2.1) are uniquely determined by their respective boundary conditions (3.1) we obtain the important symmetry relations

$$\psi(x,k) = \begin{pmatrix} 0 & -\sigma \\ 1 & 0 \end{pmatrix}\phi^*(-x,-k^*), \tag{5.2}$$

$$\bar{\psi}(x,k) = \begin{pmatrix} 0 & 1 \\ -\sigma & 0 \end{pmatrix}\bar{\phi}^*(-x,-k^*). \tag{5.3}$$

With the definitions (4.1) and (4.2) one can readily write down the corresponding symmetry conditions of the Jost functions:

$$N(x,k) = \begin{pmatrix} 0 & -\sigma \\ 1 & 0 \end{pmatrix}M^*(-x,-k^*), \tag{5.4}$$

$$\bar{N}(x,k) = \begin{pmatrix} 0 & 1 \\ -\sigma & 0 \end{pmatrix}\bar{M}^*(-x,-k^*). \tag{5.5}$$

Thus we see that the symmetry $r(x,t) = \sigma q^*(-x,t)$ leads to symmetry between eigenfunctions defined at $\pm\infty$.

5.2. Symmetry of the scattering data

The symmetry in the eigenfunctions in turn imposes a very important symmetry in the scattering data $a(k),\ \bar{a}(k)$ and $b(k),\ \bar{b}(k)$. From the Wronskian representations for the scattering data and the above symmetry relations together with the fact that the Wronskian does not depend on x it follows for both signs $\sigma = \mp 1$

$$a(k) = a^*(-k^*), \tag{5.6}$$

$$\overline{a}(k) = \overline{a}^*(-k^*), \tag{5.7}$$

$$\overline{b}(k) = \sigma b^*(-k^*). \tag{5.8}$$

It therefore follows from equation (5.6) that if $k_j = \xi_j + \mathrm{i}\eta_j$ is a zero (eigenvalue) of $a(k)$ in the upper half k plane then $-k_j^* = -\xi_j + \mathrm{i}\eta_j$ is a zero of $a(k)$ in the upper half k plane. Similarly, if $\overline{k}_j$ is a zero of $\overline{a}(k)$ in the lower half k plane so is $-\overline{k}_j^*$ in the upper half k plane.

6. Inverse scattering: a left–right Riemann–Hilbert approach

The inverse problem consists of constructing the potential functions $r(x, t)$ and $q(x, t)$ from the scattering data, i.e. reflection coefficients $\rho(k,t)$ and $\overline{\rho}(k,t)$ defined on $\mathrm{Im}\, k = 0$ as well as the eigenvalues $k_j, \overline{k}_j$ and norming constants (in x) $C_j(t), \overline{C}_j(t)$. Our approach is based on solving two Riemann–Hilbert problems associated to what we refer as left and right scattering problems and use the symmetry conditions established in section 5 to relate between the two pieces. In doing so we will make frequent use of the so-called projection operators defined as follows. For any integrable function $f(k)$, $k \in \mathbb{C}$ that rapidly decays to zero as $|k| \to \infty$ we define the projection operators $P_\pm$ as

$$P_\pm f = \lim_{\epsilon \to 0} \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - (k \pm \mathrm{i}\epsilon)}. \tag{6.1}$$

One of the most important properties of the projection operators are

$$P_\pm f_\pm = \pm f_\pm, \qquad P_\pm f_\mp = 0, \tag{6.2}$$

where $f_+(z)$ and $f_-(z)$ are analytic functions in the upper and lower complex half plane respectively satisfying $f_\pm(z) \to 0$ as $|z| \to \infty$.

6.1. Left scattering problem

Our starting point is the 'left' scattering problem (4.8) and (4.9). Divide equation (4.8) by $a(k)$; (4.9) by $\overline{a}(k)$ and use the definition of the Jost functions given in (4.1) and (4.2) gives the equivalent formulation

$$\frac{M(x,k)}{a(k)} = \overline{N}(x,k) + \rho(k)\mathrm{e}^{2\mathrm{i}kx} N(x,k), \tag{6.3}$$

$$\frac{\overline{M}(x,k)}{\overline{a}(k)} = N(x,k) + \overline{\rho}(k)\mathrm{e}^{-2\mathrm{i}kx} \overline{N}(x,k), \tag{6.4}$$

where we have defined the left reflection coefficients

$$\rho(k) \equiv \frac{b(k)}{a(k)}, \qquad \overline{\rho}(k) \equiv \frac{\overline{b}(k)}{\overline{a}(k)}. \tag{6.5}$$

Taking into account the corresponding boundary conditions as well as the asymptotic behavior of $a(k)$ at large k we find

$$\lim_{|k| \to \infty} \left[\frac{M(x,k)}{a(k)} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{6.6}$$

The functions $M(x, k)$ and $a(k)$ are both analytic functions in the upper half complex k plane. Since $a(k)$ has simple zeros at $k = k_\ell$ then $\frac{M(x,k)}{a(k)}$ is not analytic and has simple poles at $k = k_\ell$ (note that $M(x, k)$ has no zeros in the upper half plane). Let $k = k_0$ be a simple zero of $a(k)$. Then around k_0 the Laurent series expansion for the function $\frac{M(x,k)}{a(k)}$ is given by

$$\frac{M(x,k)}{a(k)} = \frac{M(x,k_0)}{a'(k_0)(k-k_0)} + \text{analytic function.} \tag{6.7}$$

Let k_ℓ be an eigenvalue of $a(k)$ in the upper half complex k plane, i.e. $a(k_\ell) = 0$. Then equation (4.8) gives $\phi(x, k_\ell) = b(k_\ell)\psi(x, k_\ell)$ which, in terms of the Jost functions, reads

$$M(x,k_\ell) = b(k_\ell)N(x,k_\ell)\mathrm{e}^{2\mathrm{i}k_\ell x}. \tag{6.8}$$

We subtract from both sides of equation (6.3) the contributions from all poles and use (6.8) to find

$$\frac{M(x,k)}{a(k)} - \begin{pmatrix}1\\0\end{pmatrix} - \sum_{\ell=1}^{J}\frac{C_\ell N_\ell(x)\mathrm{e}^{2\mathrm{i}k_\ell x}}{k-k_\ell} = \overline{N}(x,k) - \begin{pmatrix}1\\0\end{pmatrix} - \sum_{\ell=1}^{J}\frac{C_\ell N_\ell(x)\mathrm{e}^{2\mathrm{i}k_\ell x}}{k-k_\ell} + \rho(k)\mathrm{e}^{2\mathrm{i}kx}N(x,k), \tag{6.9}$$

where we have defined the left norming constant C_ℓ as

$$C_\ell \equiv \frac{b(k_\ell)}{a'(k_\ell)}. \tag{6.10}$$

The left hand side of equation (6.9) is an analytic function in the upper half plane and goes to zero as $|k| \to \infty$ hence it forms a '+' function. Also, the function $\overline{N}(x,k) - \begin{pmatrix}1\\0\end{pmatrix}$ is an analytic function in the lower half plane and goes to zero as $|k| \to \infty$ therefore, it forms an '$-$' function. Apply P_- on (6.9) to find

$$\overline{N}(x,k) = \begin{pmatrix}1\\0\end{pmatrix} + \sum_{\ell=1}^{J}\frac{C_\ell N(x,k_\ell)\mathrm{e}^{2\mathrm{i}k_\ell x}}{k-k_\ell} + \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\rho(\xi)\mathrm{e}^{2\mathrm{i}\xi x}N(x,\xi)}{\xi-(k-\mathrm{i}0)}\mathrm{d}\xi. \tag{6.11}$$

Similarly, the functions $\overline{M}(x,k)$ and $\overline{a}(k)$ are both analytic functions in k in the lower half plane. Since $\overline{a}(k)$ has simple zeros at $k = \overline{k}_\ell$ then $\frac{\overline{M}(x,k)}{\overline{a}(k)}$ is not analytic and has simple poles at $k = \overline{k}_\ell$ (the function $\overline{M}(x,k)$ has no zeros in the lower half plane). Let $k = \overline{k}_0$ be a simple zero for $\overline{a}(k)$. Then around $\overline{k}_0$ the Laurent series expansion for the function $\frac{\overline{M}(x,k)}{\overline{a}(k)}$ is given by

$$\frac{\overline{M}(x,k)}{\overline{a}(k)} = \frac{\overline{M}(x,\overline{k}_0)}{\overline{a}'(\overline{k}_0)(k-\overline{k}_0)} + \text{analytic function.} \tag{6.12}$$

At an eigenvalue $\overline{k}_\ell$ we have $\overline{a}(\overline{k}_\ell) = 0$. Then equation (4.9) gives $\overline{\phi}(x,\overline{k}_\ell) = \overline{b}(\overline{k}_\ell)\overline{\psi}(x,\overline{k}_\ell)$ which imply

$$\overline{M}(x,\overline{k}_\ell) = \overline{b}(\overline{k}_\ell)\overline{N}(x,\overline{k}_\ell)\mathrm{e}^{-2\mathrm{i}\overline{k}_\ell x}. \tag{6.13}$$

Subtract all the poles from (6.4) and use the relation (6.13) to find

$$\frac{\overline{M}(x,k)}{\overline{a}(k)} - \begin{pmatrix}0\\1\end{pmatrix} - \sum_{\ell=1}^{\overline{J}}\frac{\overline{C}_\ell\overline{N}_\ell(x)\mathrm{e}^{-2\mathrm{i}\overline{k}_\ell x}}{k-\overline{k}_\ell} = N(x,k) - \begin{pmatrix}0\\1\end{pmatrix} - \sum_{\ell=1}^{\overline{J}}\frac{\overline{C}_\ell\overline{N}_\ell(x)\mathrm{e}^{-2\mathrm{i}\overline{k}_\ell x}}{k-\overline{k}_\ell} + \overline{\rho}(k)\mathrm{e}^{-2\mathrm{i}kx}\overline{N}(x,k), \tag{6.14}$$

where as before the left norming constant is given by

$$\overline{C}_\ell \equiv \frac{\overline{b}(\overline{k}_\ell)}{\overline{a}'(\overline{k}_\ell)}. \tag{6.15}$$

Note that the left hand side of equation (6.14) is an analytic function in the lower half plane and goes to zero as $|k| \to \infty$ hence it forms a '$-$' function. Also, the function $N(x,k) - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an analytic function in the upper half plane and goes to zero as $|k| \to \infty$ therefore, it forms an '$+$' function. Apply P_+ on (6.14) to find

$$N(x,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{\ell=1}^{\overline{J}} \frac{\overline{C}_\ell \overline{N}(x,\overline{k}_\ell) \mathrm{e}^{-2\mathrm{i}\overline{k}_\ell x}}{k - \overline{k}_\ell} - \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\overline{\rho}(\xi)\mathrm{e}^{-2\mathrm{i}\xi x}\overline{N}(x,\xi)}{\xi - (k + \mathrm{i}0)} \mathrm{d}\xi. \tag{6.16}$$

6.2. Time evolution of the scattering data: left scattering problem

The time evolution of the scattering data is derived from the evolution equation (2.3) and is given by see [3]

$$a(k,t) = a(k,0), \qquad \overline{a}(k,t) = \overline{a}(k,0) \tag{6.17}$$

$$b(k,t) = \mathrm{e}^{-4\mathrm{i}k^2 t} b(k,0), \qquad \overline{b}(k,t) = \mathrm{e}^{4\mathrm{i}k^2 t}\overline{b}(k,0). \tag{6.18}$$

Equation (6.17) implies that the zeros of the scattering data k_j and $\overline{k}_j$ (the soliton eigenvalues) are time independent. The time-evolution of the reflection coefficients and norming constants follows from (6.17) and (6.18) and are respectively given by

$$\rho(k,t) = \mathrm{e}^{-4\mathrm{i}k^2 t} b(k,0)/a(k,0), \quad \overline{\rho}(k,t) = \mathrm{e}^{4\mathrm{i}k^2 t}\overline{b}(k,0)/\overline{a}(k,0), \tag{6.19}$$

$$C_\ell = C_\ell(0)\mathrm{e}^{-4\mathrm{i}k_\ell^2 t}, \qquad \overline{C}_\ell = \overline{C}_\ell(0)\mathrm{e}^{4\mathrm{i}\overline{k}_\ell^2 t}, \tag{6.20}$$

6.3. Right scattering problem

To account for the symmetry conditions (5.5) and (5.4), we view the system (4.8) and (4.9) as a left scattering problem and supplement it with the right scattering problem

$$\psi(x,k) = \alpha(k)\overline{\phi}(x,k) + \beta(k)\phi(x,k), \tag{6.21}$$

$$\overline{\psi}(x,k) = \overline{\alpha}(k)\phi(x,k) + \overline{\beta}(k)\overline{\phi}(x,k). \tag{6.22}$$

where $\alpha(k)$, $\overline{\alpha}(k)$, $\beta(k)$ and $\overline{\beta}(k)$ are the right scattering data. We can also write system (6.21) and (6.22) in the matrix form

$$\Psi(x,k) = S_R(k)\Phi(x,k), \tag{6.23}$$

where the right scattering matrix is

$$S_R(k) = \begin{pmatrix} \overline{\alpha}(k) & \overline{\beta}(k) \\ \beta(k) & \alpha(k) \end{pmatrix}, \tag{6.24}$$

and $\Phi(x,k)\equiv(\phi(x,k),\ \overline{\phi}(x,k))^T$, $\Psi(x,k)\equiv(\overline{\psi}(x,k),\ \psi(x,k))^T$ where superscript T denotes matrix transpose. Following the same steps as for the left RH above, we can formulate the corresponding RH problem on the right and find the following linear integral equations which govern the functions $M(x,k), \overline{M}(x,k)$:

$$\overline{M}(x,k)=\begin{pmatrix}0\\1\end{pmatrix}+\sum_{\ell=1}^{J}\frac{B_\ell\, M(x,k_\ell)\mathrm{e}^{-2\mathrm{i}k_\ell x}}{k-k_\ell}+\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{R(\xi)\mathrm{e}^{-2\mathrm{i}\xi x}M(x,\xi)}{\xi-(k-\mathrm{i}0)}\mathrm{d}\xi, \tag{6.25}$$

$$M(x,k)=\begin{pmatrix}1\\0\end{pmatrix}+\sum_{\ell=1}^{\overline{J}}\frac{\overline{B}_\ell\, \overline{M}(x,\overline{k}_\ell)\mathrm{e}^{2\mathrm{i}\overline{k}_\ell x}}{k-\overline{k}_\ell}-\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\overline{R}(\xi)\mathrm{e}^{2\mathrm{i}\xi x}\overline{M}(x,\xi)}{\xi-(k+\mathrm{i}0)}\mathrm{d}\xi, \tag{6.26}$$

where $R\,(k)$ and $\overline{R}(k)$ are the right reflection coefficients defined by

$$R(k)=\frac{\beta(k)}{\alpha(k)},\qquad \overline{R}(k)=\frac{\overline{\beta}(k)}{\overline{\alpha}(k)}, \tag{6.27}$$

and $B_\ell, \overline{B}_\ell$ are the right norming constants defined by

$$B_\ell\ \equiv\ \frac{\beta(k_\ell)}{\alpha'(k_\ell)},\qquad \overline{B}_\ell\ \equiv\frac{\overline{\beta}(\overline{k}_\ell)}{\overline{\alpha}'(\overline{k}_\ell)}. \tag{6.28}$$

6.4. Time evolution of the scattering data: right scattering problem

The derivation of the time evolution of the right scattering data follows closely that of the left case and are given by

$$\alpha(k,t)=\alpha(k,0),\qquad \overline{\alpha}(k,t)=\overline{\alpha}(k,0) \tag{6.29}$$

$$\beta(k,t)=\mathrm{e}^{+4\mathrm{i}k^2t}\beta(k,0),\qquad \overline{\beta}(k,t)=\mathrm{e}^{-4\mathrm{i}k^2t}\overline{\beta}(k,0). \tag{6.30}$$

The time evolution of the norming constants and reflection coefficients follows from (6.29) and (6.30)

$$B_\ell\ =\ B_\ell(0)\mathrm{e}^{+4\mathrm{i}k_\ell^2t},\qquad \overline{B}_\ell\ =\ \overline{B}_\ell(0)\mathrm{e}^{-4\mathrm{i}\overline{k}_\ell^2t}. \tag{6.31}$$

6.5. Relation between the reflection coefficients

The left scattering problem (4.8) and (4.9) can be rewritten in the matrix form

$$\Phi(x,k)=S_L(k)\Psi(x,k), \tag{6.32}$$

where $\Phi(x,k)\equiv(\phi(x,k),\ \overline{\phi}(x,k))^T$, $\Psi(x,k)\equiv(\overline{\psi}(x,k),\ \psi(x,k))^T$ (here superscript T stands for matrix transpose) and $S_L(k)$ is the so-called left scattering matrix

$$S_L(k)=\begin{pmatrix}a(k) & b(k)\\ \overline{b}(k) & \overline{a}(k)\end{pmatrix}. \tag{6.33}$$

With this notation at hand, the scattering matrix (6.33) is related to the right scattering matrix (6.24) throughout the relation $S_R(k)=S_L^{-1}(k)$ which explicitly gives

$$\begin{aligned} &\overline{\alpha}(k) = \overline{a}(k), \qquad \alpha(k) = a(k), \\ &\overline{\beta}(k) = -b(k), \qquad \beta(k) = -\overline{b}(k). \end{aligned} \tag{6.34}$$

From the definition of the right reflection coefficient R given in (6.27) we have

$$R(k) = \frac{\beta(k)}{\alpha(k)} = -\frac{\overline{b}(k)}{a(k)} = -\sigma \frac{b^*(-k^*)}{a^*(-k^*)} = -\sigma \rho^*(-k^*). \tag{6.35}$$

In obtaining the result (6.35) we used the definition of the left reflection coefficient ρ given in (6.5) as well as the symmetry relation between the scattering data $a, \overline{a}$ and $\overline{b}$ given in (5.6)–(5.8) respectively. Similar symmetry result hold between $\overline{R}$ and $\overline{\rho}$, i.e.

$$\overline{R}(k) = -\sigma \overline{\rho}^*(-k^*). \tag{6.36}$$

With this at hand we have reduced the number of scattering data (reflection coefficients) in half. Next we aim at achieving the same goal for the norming constants.

6.6. Closing the system

To close the system we substitute $k = \overline{k}_j$ in equations (6.11), (6.25) and $k = k_j$ in equations (6.16), (6.26) then impose the symmetry condition $r(x) = \sigma q^*(-x)$ where $\sigma = \mp 1$ throughout the symmetry relation between the Jost functions (5.4) and (5.5) as well as the reflection coefficients to find

$$\begin{pmatrix} \overline{M}_2^*(-x, -\overline{k}_j^*) \\ -\sigma \overline{M}_1^*(-x, -\overline{k}_j^*) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{\ell=1}^{J} \frac{C_\ell e^{2ik_\ell x}}{\overline{k}_j - k_\ell} \begin{pmatrix} N_1(x, k_\ell) \\ N_2(x, k_\ell) \end{pmatrix} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\rho(\xi) e^{2i\xi x}}{\xi - (\overline{k}_j - i0)} \begin{pmatrix} N_1(x, \xi) \\ N_2(x, \xi) \end{pmatrix} d\xi, \tag{6.37}$$

$$\begin{pmatrix} \overline{M}_1(x, \overline{k}_j) \\ \overline{M}_2(x, \overline{k}_j) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{\ell=1}^{J} \frac{B_\ell e^{-2ik_\ell x}}{\overline{k}_j - k_\ell} \begin{pmatrix} N_2^*(-x, -k_\ell^*) \\ -\sigma N_1^*(-x, -k_\ell^*) \end{pmatrix} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\rho^*(-\xi) e^{-2i\xi x}}{\xi - (\overline{k}_j - i0)} \begin{pmatrix} -\sigma N_2^*(-x, -\xi) \\ N_1^*(-x, -\xi) \end{pmatrix} d\xi, \tag{6.38}$$

$$\begin{pmatrix} N_1(x, k_j) \\ N_2(x, k_j) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{\ell=1}^{J} \frac{\overline{C}_\ell e^{-2i\overline{k}_\ell x}}{k_j - \overline{k}_\ell} \begin{pmatrix} \overline{M}_2^*(-x, -\overline{k}_\ell^*) \\ -\sigma \overline{M}_1^*(-x, -\overline{k}_\ell^*) \end{pmatrix} - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\overline{\rho}(\xi) e^{-2i\xi x}}{\xi - (k_j + i0)} \begin{pmatrix} \overline{M}_2^*(-x, -\xi) \\ -\sigma \overline{M}_1^*(-x, -\xi) \end{pmatrix} d\xi, \tag{6.39}$$

$$\begin{pmatrix} N_2^*(-x, -k_j^*) \\ -\sigma N_1^*(-x, -k_j^*) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{\ell=1}^{J} \frac{\overline{B}_\ell e^{2i\overline{k}_\ell x}}{k_j - \overline{k}_\ell} \begin{pmatrix} \overline{M}_1(x, \overline{k}_\ell) \\ \overline{M}_2(x, \overline{k}_\ell) \end{pmatrix} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\sigma \overline{\rho}^*(-\xi) e^{2i\xi x}}{\xi - (k_j + i0)} \begin{pmatrix} \overline{M}_1(x, \xi) \\ \overline{M}_2(x, \xi) \end{pmatrix} d\xi, \tag{6.40}$$

Equations (6.37)–(6.40) are integro-algebraic equations that (can in principle) solve the inverse problem. As we shall see later, the above system will be used to construct breathing one soliton solution. Indeed the system can be reduced to two equations in two unknowns, e.g. for $N_1(x, k)$ and $N_2(x, k)$ by appropriately substituting $\bar{M}_1(x, k), \bar{M}_2(x, k)$ from equation (6.38) into equation (6.39) leaving only one vector equation for $N_1(x, k)$ and $N_2(x, k)$. The question of solving systems such as (6.37)–(6.40) or the reduced system (6.39) has been studied by Beals and Coifman in [10]. In this paper they give explicit conditions as to when the IST method yields a unique solution (possibly for finite time) to nonlinear evolution equations associated which contain the 2×2 scattering problems with general potentials q, r. However we note that soliton solutions to the nonlocal NLS equation can admit simple pole singularities in time. Hence, in general, one can only expect to find solutions for finite time. A full discussion of the solutions to the nonlocal NLS equation, obtained by IST, is outside the scope of this paper and remains to be studied.

6.7. Eigenvalues on the imaginary axis and norming constants

When formulating the left and right RH problem for the Jost eigenfunctions we have subtracted only the contribution from all poles the are located at $k_j, \bar{k}_j$ the zeros of the scattering data $a(k)$ and $\bar{a}(k)$ respectively. However it follows from the symmetry of the scattering data (5.6) and (5.7) that there are two more contribution to the pole arising from $-k_j^*, \bar{k}_j^*$. Thus, in order for the above analysis to be consistent we restrict ourselves to the case for which all eigenvalues are located on the imaginary axis, i.e.

$$k_j = -k_j^*, \qquad \bar{k}_j = -\bar{k}_j^*. \tag{6.41}$$

To obtain the symmetry condition that relates the left norming constants C_j and $\bar{C}_j$ to the right norming constants B_j and $\bar{B}_j$ we restrict the equations (6.37)–(6.40) to eigenvalues satisfying (6.41) and obtain after some algebra the following set of equations

$$\begin{pmatrix} \bar{M}_1(x,\bar{k}_j) \\ \bar{M}_2(x,\bar{k}_j) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{\ell=1}^{J} \frac{C_\ell^* \mathrm{e}^{-2\mathrm{i}k_\ell x}}{-\bar{k}_j + k_\ell} \begin{pmatrix} -\sigma N_2^*(-x,k_\ell) \\ N_1^*(-x,k_\ell) \end{pmatrix} + \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\rho^*(-\xi)\mathrm{e}^{-2\mathrm{i}\xi x}}{\xi - (\bar{k}_j - \mathrm{i}0)} \begin{pmatrix} -\sigma N_2^*(-x,-\xi) \\ N_1^*(-x,-\xi) \end{pmatrix} \mathrm{d}\xi, \tag{6.42}$$

$$\begin{pmatrix} \bar{M}_1(x,\bar{k}_j) \\ \bar{M}_2(x,\bar{k}_j) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{\ell=1}^{J} \frac{B_\ell \mathrm{e}^{-2\mathrm{i}k_\ell x}}{\bar{k}_j - k_\ell} \begin{pmatrix} N_2^*(-x,k_\ell) \\ -\sigma N_1^*(-x,k_\ell) \end{pmatrix} + \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\rho^*(-\xi)\mathrm{e}^{-2\mathrm{i}\xi x}}{\xi - (\bar{k}_j - \mathrm{i}0)} \begin{pmatrix} -\sigma N_2^*(-x,-\xi) \\ N_1^*(-x,-\xi) \end{pmatrix} \mathrm{d}\xi, \tag{6.43}$$

$$\begin{pmatrix} N_1(x,k_j) \\ N_2(x,k_j) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{\ell=1}^{\bar{J}} \frac{\bar{C}_\ell \mathrm{e}^{-2\mathrm{i}\bar{k}_\ell x}}{k_j - \bar{k}_\ell} \begin{pmatrix} \bar{M}_2^*(-x,\bar{k}_\ell) \\ -\sigma \bar{M}_1^*(-x,\bar{k}_\ell) \end{pmatrix} - \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\bar{\rho}(\xi)\mathrm{e}^{-2\mathrm{i}\xi x}}{\xi - (k_j + \mathrm{i}0)} \begin{pmatrix} \bar{M}_2^*(-x,\xi) \\ -\sigma \bar{M}_1^*(-x,\xi) \end{pmatrix} \mathrm{d}\xi, \tag{6.44}$$

$$\begin{pmatrix} N_1(x,k_j) \\ N_2(x,k_j) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{\ell=1}^{\bar{J}} \frac{\bar{B}_\ell^* \mathrm{e}^{-2\mathrm{i}\bar{k}_\ell x}}{-k_j + \bar{k}_\ell} \begin{pmatrix} -\sigma \bar{M}_2^*(-x,\bar{k}_\ell) \\ \bar{M}_1^*(-x,\bar{k}_\ell) \end{pmatrix} - \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\bar{\rho}(\xi)\mathrm{e}^{-2\mathrm{i}\xi x}}{\xi - (k_j + \mathrm{i}0)} \begin{pmatrix} \bar{M}_2^*(-x,\xi) \\ -\sigma \bar{M}_1^*(-x,\xi) \end{pmatrix} \mathrm{d}\xi, \tag{6.45}$$

Comparing equation (6.42) with (6.43) one finds

$$\bar{B}_\ell = \sigma \bar{C}_\ell^*, \tag{6.46}$$

Similarly, one obtains from (6.44) and (6.45) the result

$$B_\ell = \sigma C_\ell^*. \tag{6.47}$$

It should be noted that both relations are valid under the symmetry reduction $r(x) = \sigma q^*(-x)$. Later in section 9 we will show that further restrictions on the norming constants only allow $\sigma = -1$.

6.8. Additional symmetry between the eigenfunctions

At an eigenvalue $k = k_\ell, \ell = 1, 2, \cdots, N$ (a zero of $a(k)$ in the upper half complex k plane), the eigen functions $\phi(x, k_\ell)$ and $\psi(x, k_\ell)$ are linearly dependent, i.e. $\phi(x, k_\ell) = b(k_\ell)\psi(x, k_\ell)$ which, in terms of the Jost functions, it reads (see equation (6.8))

$$M_1(x, k_\ell) = b_\ell e^{2ik_\ell x} N_1(x, k_\ell), \tag{6.48}$$

$$M_2(x, k_\ell) = b_\ell e^{2ik_\ell x} N_2(x, k_\ell). \tag{6.49}$$

Equations (6.48) and (6.49) yields the product relation

$$M_1(x, k_\ell) N_2(x, k_\ell) = M_2(x, k_\ell) N_1(x, k_\ell), \tag{6.50}$$

between the Jost functions N_j and $M_j, j = 1, 2$ valid at an eigenvalue $k_\ell, \ell = 1, 2 \cdots N$. Now use the symmetry condition (5.4) and (5.5) between the Jost functions in (6.50) to find

$$N_2(x, k_\ell) N_2^*(-x, k_\ell) = N_1(x, k_\ell) N_1^*(-x, k_\ell), \qquad \ell = 1, 2, \cdots, N. \tag{6.51}$$

Similar relation can be derived for the other set of the Jost functions $\overline{N}_j$ and $\overline{M}_j$, $j = 1, 2$ valid at an eigenvalue $\bar{k}_\ell, \ell = 1, 2 \cdots N$. Indeed, starting from the right scattering problem (6.23) one finds that at an eigenvalue $\bar{k}_\ell$ the Jost functions $\overline{N}_j$ and $\overline{M}_j$, $j = 1, 2$ are linearly dependent and satisfy

$$\overline{N}_1(x, \bar{k}_\ell) = \bar{\beta}_\ell \, e^{2i\bar{k}_\ell x} \, \overline{M}_1(x, \bar{k}_\ell), \tag{6.52}$$

$$\overline{N}_2(x, \bar{k}_\ell) = \bar{\beta}_\ell \, e^{2i\bar{k}_\ell x} \, \overline{M}_2(x, \bar{k}_\ell). \tag{6.53}$$

Eliminating $\bar{\beta}_\ell \, e^{2i\bar{k}_\ell x}$ from equations (6.52), (6.53) and make use of the symmetry condition (5.4), (5.5) between the Jost functions gives

$$\overline{M}_2(x, \bar{k}_\ell) \overline{M}_2^*(-x, \bar{k}_\ell) = \overline{M}_1(x, \bar{k}_\ell) \overline{M}_1^*(-x, \bar{k}_\ell), \qquad \ell = 1, 2, \cdots, N. \tag{6.54}$$

The above relations (6.51) and (6.54) are useful in finding explicit soliton solutions.

The above relations (6.51) and (6.54) are useful in order to find the norming constants C_j and $\overline{C}_j$ and their dependence on the soliton eigenvalues k_j and $\bar{k}_j$.

7. Recovery of the potentials

Recall from equation (6.11) that

$$\overline{N}(x, k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{\ell=1}^{J} \frac{C_\ell N_\ell(x) e^{2ik_\ell x}}{k - k_\ell} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\rho(\xi) e^{2i\xi x} N(x, \xi)}{\xi - (k - i0)} d\xi. \tag{7.1}$$

The large k behavior of $\overline{N}_2(x, k)$ is thus given by

$$\overline{N}_2(x, k) \underset{|k| \to \infty}{\sim} \frac{1}{k} \sum_{\ell=1}^{J} C_\ell N_2(x, k_\ell) e^{2ik_\ell x} - \frac{1}{k} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \rho(\xi) e^{2i\xi x} N_2(x, \xi) d\xi. \tag{7.2}$$

From (4.6) we have

$$\overline{N}_2(x, k) \sim -\frac{r(x)}{2ik}, \tag{7.3}$$

therefore, we have the result:

$$r(x) = -2\mathrm{i}\left(\sum_{\ell=1}^{J} C_\ell N_2(x,k_\ell)\mathrm{e}^{2\mathrm{i}k_\ell x} - \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\rho(\xi)\mathrm{e}^{2\mathrm{i}\xi x}N_2(x,\xi)\mathrm{d}\xi\right). \tag{7.4}$$

We use the symmetries from (5.5) to find for $r(x) = \mp q^*(-x)$

$$\overline{M}_1(x,k) = \pm\overline{N}_2^*(-x,-k^*) \tag{7.5}$$

together with the asymptotic relation (4.5)

$$\overline{M}_1(x,k) \sim \frac{q(x)}{2\mathrm{i}k}, \tag{7.6}$$

to find that

$$q(x) \sim \pm 2\mathrm{i}k\overline{N}_2^*(-x,-k^*). \tag{7.7}$$

From (7.2) we have

$$\overline{N}_2^*(x,k) \underset{|k|\to\infty}{\sim} \frac{1}{k^*}\sum_{\ell=1}^{J} C_\ell^* N_2^*(x,k_\ell)\mathrm{e}^{-2\mathrm{i}k_\ell^* x} + \frac{1}{k^*}\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\rho^*(\xi)\mathrm{e}^{-2\mathrm{i}\xi x}N_2^*(x,\xi)\mathrm{d}\xi. \tag{7.8}$$

$$\overline{N}_2^*(-x,-k^*) \underset{|k|\to\infty}{\sim} -\frac{1}{k}\sum_{\ell=1}^{J} C_\ell^* N_2^*(-x,k_\ell)\mathrm{e}^{2\mathrm{i}k_\ell^* x} - \frac{1}{k}\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\rho^*(\xi)\mathrm{e}^{2\mathrm{i}\xi x}N_2^*(-x,\xi)\mathrm{d}\xi. \tag{7.9}$$

Therefore, comparing (7.9) with (7.7) we find ($\sigma = \mp 1$)

$$q(x) = 2\mathrm{i}\sigma\sum_{\ell=1}^{J} C_\ell^* N_2^*(-x,k_\ell)\mathrm{e}^{2\mathrm{i}k_\ell^* x} + \frac{\sigma}{\pi}\int_{-\infty}^{\infty}\rho^*(\xi)\mathrm{e}^{2\mathrm{i}\xi x}N_2^*(-x,\xi)\mathrm{d}\xi. \tag{7.10}$$

8. Characterization of scattering data: general trace formulae

The functions $a(k)$ and $\bar{a}(k)$ are analytic in the upper and lower half complex k plane respectively and tend to unity as $k\to\infty$. We assume $a(k)$ and $\bar{a}(k)$ have the simple zeros $\{k_j,\ \mathrm{Im}\,k_j > 0\}_{j=1}^{J}$ and $\{\bar{k}_j,\ \mathrm{Im}\bar{k}_j < 0\}_{j=1}^{J}$, respectively, and define

$$a(k) = \prod_{m=1}^{J}\frac{k-k_m}{k-\bar{k}_m}\alpha(k), \qquad \bar{a}(k) = \prod_{m=1}^{J}\frac{k-\bar{k}_m}{k-k_m}\bar{\alpha}(k). \tag{8.1}$$

Thus $\alpha(k), \bar{\alpha}(k)$ are analytic in the upper and lower half complex plane respectively, tend to unity for $k\to\infty$ and have **no zeros** in their respective half planes. Therefore we have

$$\log\alpha(k) = \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{\log\alpha(\xi)}{\xi-k}\mathrm{d}\xi, \qquad \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{\log\bar{\alpha}(\xi)}{\xi-k}\mathrm{d}\xi = 0 \qquad \mathrm{Im}\,k > 0,$$

$$\log\bar{\alpha}(k) = -\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{\log\bar{\alpha}(\xi)}{\xi-k}\mathrm{d}\xi, \qquad \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{\log\alpha(\xi)}{\xi-k}\mathrm{d}\xi = 0 \qquad \mathrm{Im}\,k < 0.$$

Adding/subtracting the above equations in each half plane respectively, yields

$$\log \alpha(k) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{+\infty} \frac{\log \alpha(\xi)\bar{\alpha}(\xi)}{\xi - k} \mathrm{d}\xi \qquad \mathrm{Im}\, k > 0$$
$$\log \bar{\alpha}(k) = -\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{+\infty} \frac{\log \alpha(\xi)\bar{\alpha}(\xi)}{\xi - k} \mathrm{d}\xi \qquad \mathrm{Im}\, k < 0.$$

Then from equation (8.1) one finds

$$\log a(k) = \sum_{m=1}^{J} \log\left(\frac{k - k_m}{k - \bar{k}_m}\right) + \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{+\infty} \frac{\log(\alpha(\xi)\bar{\alpha}(\xi))}{\xi - k} \mathrm{d}\xi, \qquad \mathrm{Im}\, k > 0 \tag{8.2}$$

$$\log \bar{a}(k) = \sum_{m=1}^{J} \log\left(\frac{k - \bar{k}_m}{k - k_m}\right) - \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{+\infty} \frac{\log(\alpha(\xi)\bar{\alpha}(\xi))}{\xi - k} \mathrm{d}\xi, \qquad \mathrm{Im}\, k < 0. \tag{8.3}$$

Since from equation (8.1) we have $\alpha(k)\bar{\alpha}(k) = a(k)\bar{a}(k)$, equations (8.2)–(8.3) together with the unitarity condition $a(k)\bar{a}(k) - b(k)\bar{b}(k) = 1$ yield

$$\log a(k) = \sum_{m=1}^{J} \log\left(\frac{k - k_m}{k - \bar{k}_m}\right) + \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{+\infty} \frac{\log(1 + b(k)\bar{b}(k))}{\xi - k} \mathrm{d}\xi, \qquad \mathrm{Im}\, k > 0$$

$$\log \bar{a}(k) = \sum_{m=1}^{J} \log\left(\frac{k - \bar{k}_m}{k - k_m}\right) - \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{+\infty} \frac{\log(1 + b(k)\bar{b}(k))}{\xi - k} \mathrm{d}\xi, \qquad \mathrm{Im}\, k < 0.$$

From the symmetry conditions (5.8)

$$\bar{b}(k) = \mp b^{*}(-k),$$

we arrive at

$$\log a(k) = \sum_{m=1}^{J} \log\left(\frac{k - k_m}{k - \bar{k}_m}\right) + \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{+\infty} \frac{\log(1 \mp b(\xi)b^{*}(-\xi))))}{\xi - k} \mathrm{d}\xi, \qquad \mathrm{Im}\, k > 0 \tag{8.4}$$

$$\log \bar{a}(k) = \sum_{m=1}^{J} \log\left(\frac{k - \bar{k}_m}{k - k_m}\right) - \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{+\infty} \frac{\log(1 + \mp b(\xi))b^{*}(-\xi)))}{\xi - k} \mathrm{d}\xi, \qquad \mathrm{Im}\, k < 0. \tag{8.5}$$

Thus we can reconstruct $a(k), \bar{a}(k)$ in terms of the eigenvalues (zero's) and only **one function** $\boldsymbol{b(k)}$. And in the inverse problem we do not need

$$\rho(k) = \frac{b(k)}{a(k)}, \ \bar{\rho}(k) = \frac{\bar{b}(k)}{\bar{a}(k)}$$

independently. We only require **one function** $b(k)$, since $\bar{\rho}(k) = \frac{\mp b^{*}(-k)}{\bar{a}(k)}$ and we can determine $a(k), \bar{a}(k)$ from **only** $b(k)$ using equations (8.4)–(8.5). To find the norming constants for the solitons

$$C_j = \frac{b_j}{a_j'}, \qquad \overline{C}_j = \frac{\bar{b}_j}{\bar{a}_j'},$$

we need $a'_j, \bar{a}'_j$ when $r(x) = -q^*(-x)$. These derivatives in general position are found to be

$$a'(k_j) = \frac{\prod_{m\neq j}^{J}(k_j - k_m)}{\prod_{m=1}^{J}(k_j - \bar{k}_m)} \mathrm{e}^{\frac{1}{2\pi \mathrm{i}}\int \frac{\log(1-b(\xi)b^*(-\xi))}{\xi - k_j}\mathrm{d}\xi} \tag{8.6}$$

Similarly for $\bar{a}'(k)$:

$$\bar{a}'(\bar{k}_j) = \frac{\prod_{m\neq j}^{J}(\bar{k}_j - \bar{k}_m)}{\prod_{m=1}^{J}(\bar{k}_j - k_m)} \mathrm{e}^{\frac{1}{2\pi \mathrm{i}}\int \frac{\log(1-b(\xi)b^*(-\xi))}{\bar{k}_j - \xi}\mathrm{d}\xi} \tag{8.7}$$

Pure solitons have $\rho(k) = 0$:

$$a'(k_j) = \frac{\prod_{m\neq j}^{J}(k_j - k_m)}{\prod_{m=1}^{J}(k_j - \bar{k}_m)}$$

Similarly for $\bar{a}'(k)$:

$$\bar{a}'(\bar{k}_j) = \frac{\prod_{m\neq j}^{J}(\bar{k}_j - \bar{k}_m)}{\prod_{m=1}^{J}(\bar{k}_j - k_m)}$$

Thus we have:

$$C_j = \frac{\mathrm{e}^{\mathrm{i}\theta_j}}{a'(k_j)}, \qquad \bar{C}_j = \frac{\mathrm{e}^{\mathrm{i}\bar{\theta}_j}}{\bar{a}'(\bar{k}_j)}$$

So, in the pure soliton case $a'_j, \bar{a}'_j$ depend only on the zero's $k_j, \bar{k}_j$. But in the general case $a'_j, \bar{a}'_j$ depend on $k_j, \bar{k}_j$ and $b(k)$ (via $b(k)b^*(-k)$) as seen from the above formulae: (8.6)–(8.7).

9. Soliton solutions

9.1. Norming constants

Pure soliton solutions correspond to zero reflection coefficients, i.e. $\rho(\xi) = 0$ and $\overline{\rho(\xi)} = 0$ for all real ξ. In this case the system (6.37)–(6.40) reduces to an algebraic equations that would determine the functional form of the soliton(s). Unlike the case of the classical NLS equation for which the two norming constants C_j and $\overline{C}_j$ are related through a symmetry condition (similarly the soliton eigenvalues), here, as we shall see, the norming constants are determined either using the symmetry condition (6.51) and (6.54) which is tedious to apply or via a trace formula [6, 12] that leads to a simple formula relating the norming constants to the soliton eigenvalues. We start the derivation from equation (5.4) whose first component satisfy (at an eigenvalue k_j)

$$N_1(x, k_j) = -\sigma M_2^*(-x, -k_j^*). \tag{9.1}$$

At an eigenvalue $k = k_j$, the eigenfunctions $M(x, k_j)$ and $N(x, k_j)$ are related via equation (6.8) which in our case give

$$M_2(x, k_j) = b_j \mathrm{e}^{2\mathrm{i}k_j x} N_2(x, k_j), \tag{9.2}$$

where $b_j \equiv b(k_j)$. Equation (9.2) combined with (9.1) gives

$$N_1(x, k_j) = -\sigma b_j^* \mathrm{e}^{-2\mathrm{i}k_j x} N_2^*(-x, -k_j^*). \tag{9.3}$$

On the other hand, the eigenfunctions $N_2(x,k_j)$ and $M_1(x,k_j)$ are related through the symmetry condition (5.4), i.e.

$$N_2(x,k_j) = M_1^*(-x,-k_j^*), \tag{9.4}$$

which together with (see equation (6.8))

$$M_1(x,k_j) = b_j e^{2ik_jx} N_1(x,k_j), \tag{9.5}$$

one finds

$$N_2(x,k_j) = b_j^* e^{-2ik_jx} N_1^*(-x,-k_j^*). \tag{9.6}$$

By complex conjugating equation (9.6); making the transformation $x \to -x, k_j \to -k_j^*$ and substituting the resulting expression into (9.3) we find the important relation

$$-\sigma|b_j|^2 = 1. \tag{9.7}$$

This result imply that the phase of b_j is arbitrary and $b_j = e^{i\theta_j}$. Moreover we see that these types of soliton solutions can only be obtained when $\sigma = -1$. Similar derivation holds for the functions $\overline{M}(x,\bar{k}_j)$ and $\overline{N}(x,\bar{k}_j)$. Indeed starting from the symmetry (5.5) and (6.13) one arrives after some calculations to a similar result as the one shown above

$$-\sigma|\bar{b}_j|^2 = 1, \tag{9.8}$$

where $\bar{b}_j \equiv \bar{b}(\bar{k}_j)$. The phase of the scattering data $\bar{b}_j$ is also free and we thus write $\bar{b}_j = e^{i\bar{\theta}_j}$. To find the norming constants

$$C_j = \frac{b_j}{a_j'}, \qquad \bar{C}_j = \frac{\bar{b}_j}{\bar{a}_j'}, \tag{9.9}$$

we need to compute a_j' and $\bar{a}_j'$. This is accomplished with the help of the trace formula [6, 12] which, for a general N eigenvalues $k_j, \bar{k}_j$, $j = 1, 2, \cdots N$ takes the form

$$a(k) = \prod_{j=1}^{N} \frac{k-k_j}{k-\bar{k}_j}. \tag{9.10}$$

By taking the derivative of $a(k)$ with respect to k we find

$$a'(k) = \prod_{j=1}^{N} \frac{k-k_j}{k-\bar{k}_j}\left[\sum_{\ell=1}^{N}\left(\frac{1}{k-k_\ell} - \frac{1}{k-\bar{k}_\ell}\right)\right]. \tag{9.11}$$

Evaluating equation (9.11) at the poles $k = k_n$ we find

$$a'(k_n) = \lim_{k\to k_n} \frac{\prod_{j=1}^{N}(k-k_j)}{\prod_{j=1}^{N}(k-\bar{k}_j)} \sum_{\ell=1}^{N} \frac{(k_\ell-\bar{k}_\ell)}{(k-k_\ell)(k-\bar{k}_\ell)}. \tag{9.12}$$

For the special case for which all the eigenvalues reside on the imaginary axis we let $k_j = i\eta_j, \bar{k}_j = -i\bar{\eta}_j$ for $j = 1, 2, \cdots N$ with all $\eta_j, \bar{\eta}_j$ being real positive numbers. In this case, equation (9.12) simplifies to

$$a'(k_n) = \lim_{\eta \to \eta_n} \frac{\prod_{j=1}^{N}(\eta - \eta_j)}{\prod_{j=1}^{N}(\eta + \bar{\eta}_j)} \sum_{\ell=1}^{N} \frac{(\eta_\ell + \bar{\eta}_\ell)}{\mathrm{i}(\eta - \eta_\ell)(\eta + \bar{\eta}_\ell)}, \tag{9.13}$$

where we defined $k \equiv \mathrm{i}\eta$ with $\eta > 0$. Similar results can be obtained for the scattering data $\bar{a}(k)$. Indeed, starting from

$$\bar{a}(k) = \prod_{j=1}^{N} \frac{k - \bar{k}_j}{k - k_j}, \tag{9.14}$$

one can derive the relation

$$\bar{a}'(k) = \prod_{j=1}^{N} \frac{k - \bar{k}_j}{k - k_j} \left[\sum_{\ell=1}^{N} \left(\frac{1}{k - \bar{k}_\ell} - \frac{1}{k - k_\ell} \right) \right]. \tag{9.15}$$

As before, evaluating equation (9.15) at the poles $k = \bar{k}_n$ we find

$$\bar{a}'(\bar{k}_n) = \lim_{k \to \bar{k}_n} \frac{\prod_{j=1}^{N}(k - \bar{k}_j)}{\prod_{j=1}^{N}(k - k_j)} \sum_{\ell=1}^{N} \frac{\bar{k}_\ell - k_\ell}{(k - \bar{k}_\ell)(k - k_\ell)}. \tag{9.16}$$

Again, when all the eigenvalues reside on the imaginary axis, equation (9.16) simplifies to

$$\bar{a}'(\bar{k}_n) = \lim_{\bar{\eta} \to \bar{\eta}_n} \frac{\prod_{j=1}^{N}(\bar{\eta} - \bar{\eta}_j)}{\prod_{j=1}^{N}(\bar{\eta} + \eta_j)} \sum_{\ell=1}^{N} \frac{\mathrm{i}(\bar{\eta}_\ell + \eta_\ell)}{(\bar{\eta} - \bar{\eta}_\ell)(\bar{\eta} + \eta_\ell)}. \tag{9.17}$$

where now $k \equiv -\mathrm{i}\bar{\eta}$ with $\bar{\eta} > 0$ and $k_j = \mathrm{i}\eta_j, \bar{k}_j = -\mathrm{i}\bar{\eta}_j$ for $j = 1, 2, \cdots N$ with all $\eta_j, \bar{\eta}_j$ being positive real numbers. Below we list the norming constants associated with the one and two soliton solutions:

- One soliton solution. Here we take $N = 1$ in equations (9.13), (9.17) and find

$$a'(k_1) = \frac{1}{\mathrm{i}(\eta_1 + \bar{\eta}_1)}, \qquad \bar{a}'(\bar{k}_1) = \frac{i}{\eta_1 + \bar{\eta}_1}. \tag{9.18}$$

  The norming constants are readily obtained from (9.9) and are given by

$$|C_1| = \eta_1 + \bar{\eta}_1, \qquad |\bar{C}_1| = \eta_1 + \bar{\eta}_1. \tag{9.19}$$

- Two soliton solution. Substituting $N = 2$ in equations (9.13), (9.17) we find

$$|C_1| = \frac{(\eta_1 + \bar{\eta}_1)(\eta_1 + \bar{\eta}_2)}{|\eta_2 - \eta_1|}, \qquad |C_2| = \frac{(\eta_2 + \bar{\eta}_2)(\eta_2 + \bar{\eta}_1)}{|\eta_2 - \eta_1|}, \tag{9.20}$$

$$|\bar{C}_1| = \frac{(\eta_1 + \bar{\eta}_1)(\eta_2 + \bar{\eta}_1)}{|\bar{\eta}_2 - \bar{\eta}_1|}, \qquad |\bar{C}_2| = \frac{(\eta_2 + \bar{\eta}_2)(\eta_1 + \bar{\eta}_2)}{|\bar{\eta}_2 - \bar{\eta}_1|}. \tag{9.21}$$

9.2. 1-Soliton solution

In this section we give an explicit form for the one soliton solution by setting $J = \bar{J} = 1$. In this case, system (6.38) and (6.40) give ($\sigma = -1$)

$$N_2^*(-x,k_1) = 1 - \frac{\overline{C}_1^* \mathrm{e}^{2\mathrm{i}\bar{k}_1 x}}{k_1 - \bar{k}_1} \overline{M}_1(x,\bar{k}_1), \tag{9.22}$$

$$N_1^*(-x,k_1) = -\frac{\overline{C}_1^* \mathrm{e}^{2\mathrm{i}\bar{k}_1 x}}{k_1 - \bar{k}_1} \overline{M}_2(x,\bar{k}_1), \tag{9.23}$$

$$\overline{M}_2^*(-x,\bar{k}_1) = 1 + \frac{C_1 \mathrm{e}^{2\mathrm{i}k_1 x}}{\bar{k}_1 - k_1} N_1(x,k_1), \tag{9.24}$$

$$\overline{M}_1^*(-x,\bar{k}_1) = \frac{C_1 \mathrm{e}^{2\mathrm{i}k_1 x}}{\bar{k}_1 - k_1} N_2(x,k_1). \tag{9.25}$$

Now let $k_1 = \mathrm{i}\eta_1$ and $\bar{k}_1 = -\mathrm{i}\bar{\eta}_1$ with $\eta_1, \bar{\eta}_1$ both being real and positive. Substituting these quantities into system (9.22)–(9.25) gives after lengthy algebra

$$N_1(x,\mathrm{i}\eta_1) = \frac{\overline{C}_1 \mathrm{e}^{-2\bar{\eta}_1 x}}{\mathrm{i}(\eta_1 + \bar{\eta}_1)} \Bigg/ \left(1 - \frac{C_1 \overline{C}_1 \mathrm{e}^{-2(\eta_1+\bar{\eta}_1)x}}{(\eta_1+\bar{\eta}_1)^2}\right), \tag{9.26}$$

$$N_2(x,\mathrm{i}\eta_1) = \frac{1}{1 - \frac{\overline{C}_1 C_1 \mathrm{e}^{-2(\eta_1+\bar{\eta}_1)x}}{(\eta_1+\bar{\eta}_1)^2}}, \tag{9.27}$$

$$\overline{M}_2(x,-\mathrm{i}\bar{\eta}_1) = 1 + \frac{C_1^* \mathrm{e}^{2\eta_1 x}}{\mathrm{i}(\eta_1 + \bar{\eta}_1)} N_1^*(-x,\mathrm{i}\eta_1), \tag{9.28}$$

$$\overline{M}_1(x,-\mathrm{i}\bar{\eta}_1) = \frac{C_1^* \mathrm{e}^{2\eta_1 x}}{\mathrm{i}(\eta_1 + \bar{\eta}_1)} N_2^*(-x,\mathrm{i}\eta_1). \tag{9.29}$$

The explicit form of the 1-soliton solution follows from (7.10) by setting the reflection coefficient ρ to zero,

$$q(x) = -2\mathrm{i}C_1^* N_2^*(-x,k_1)\mathrm{e}^{2\mathrm{i}k_1^* x}. \tag{9.30}$$

The norming constants C_1 and $\overline{C}_1$ are given in equation (9.19), i.e. $|\overline{C}_1| = (\eta_1 + \bar{\eta}_1)$ and $|\overline{C}_1| = (\eta_1 + \bar{\eta}_1)$ with arbitrary phases. Note that we would have arrived at the same results had we used the symmetry conditions (6.51) and (6.54) between the Jost functions. After some algebra we obtain

$$q(x) = -\frac{2\mathrm{i}C_1^*(\eta_1+\bar{\eta}_1)^2 \mathrm{e}^{2\eta_1 x}}{(\eta_1+\bar{\eta}_1)^2 - \overline{C}_1^* C_1^* \mathrm{e}^{2(\eta_1+\bar{\eta}_1)x}}. \tag{9.31}$$

The time evolution of the norming constants C_1 and $\overline{C}_1$ is found from equation (6.20) and is given by

$$C_1 = C_1(0)\mathrm{e}^{+4\mathrm{i}\eta_1^2 t} = |c|\mathrm{e}^{\mathrm{i}(\varphi+\pi/2)}\mathrm{e}^{+4\mathrm{i}\eta_1^2 t}, \qquad |c| = \eta_1 + \bar{\eta}_1, \tag{9.32}$$

$$\overline{C}_1 = \overline{C}_1(0)\mathrm{e}^{-4\mathrm{i}\bar{\eta}_1^2 t} = |\bar{c}|\mathrm{e}^{\mathrm{i}(\bar{\varphi}+\pi/2)}\mathrm{e}^{-4\mathrm{i}\bar{\eta}_1^2 t}, \qquad |\bar{c}| = \eta_1 + \bar{\eta}_1. \tag{9.33}$$

Substituting (9.32) and (9.33) back in (9.31) we find after some algebra the most general two parameter family of a breathing one soliton solution

$$q(x,t) = -\frac{2(\eta_1+\bar{\eta}_1)\mathrm{e}^{\mathrm{i}\bar{\varphi}}\mathrm{e}^{-4\mathrm{i}\bar{\eta}_1^2 t}\mathrm{e}^{-2\bar{\eta}_1 x}}{1+\mathrm{e}^{\mathrm{i}(\varphi+\bar{\varphi})}\mathrm{e}^{4\mathrm{i}(\eta_1^2-\bar{\eta}_1^2)t}\mathrm{e}^{-2(\eta_1+\bar{\eta}_1)x}}. \tag{9.34}$$

To obtain the one soliton solution for the corresponding local NLS equation we let $\bar{\eta}_1 = \eta_1$ and $\bar{C}_1 = -C_1^*$. This implies $\varphi + \bar{\varphi} = 0$. Next we show that the one soliton solution (9.34) is indeed Galilean invariant. To establish this result, we use the transformation (2.17) and obtain

$$\tilde{q}(x,t) = -\frac{2(\eta_1+\bar{\eta}_1)\mathrm{e}^{\mathrm{i}\bar{\varphi}}\mathrm{e}^{-4\mathrm{i}\bar{\eta}_1^2 t}\mathrm{e}^{-2\bar{\eta}_1(x+2\mathrm{i}\xi t)}\mathrm{e}^{-\xi x}\mathrm{e}^{-\mathrm{i}\xi^2 t}}{1+\mathrm{e}^{\mathrm{i}(\varphi+\bar{\varphi})}\mathrm{e}^{4\mathrm{i}(\eta_1^2-\bar{\eta}_1^2)t}\mathrm{e}^{-2(\eta_1+\bar{\eta}_1)(x+2\mathrm{i}\xi t)}}. \tag{9.35}$$

By defining the new soliton eigenvalues $\tilde{\eta}_1$ and $\tilde{\bar{\eta}}_1$ via the transformation

$$\tilde{\bar{\eta}}_1 = \bar{\eta}_1 + \xi/2, \tag{9.36}$$

$$\tilde{\eta}_1 = \eta_1 - \xi/2, \tag{9.37}$$

the one soliton solution then reads

$$\tilde{q}(x,t) = -\frac{2(\tilde{\eta}_1+\tilde{\bar{\eta}}_1)\mathrm{e}^{\mathrm{i}\bar{\varphi}}\mathrm{e}^{-4\mathrm{i}\tilde{\bar{\eta}}_1^2 t}\mathrm{e}^{-2\tilde{\bar{\eta}}_1 x}}{1+\mathrm{e}^{\mathrm{i}(\varphi+\bar{\varphi})}\mathrm{e}^{4\mathrm{i}(\tilde{\eta}_1^2-\tilde{\bar{\eta}}_1^2)t}\mathrm{e}^{-2(\tilde{\eta}_1+\tilde{\bar{\eta}}_1)x}}. \tag{9.38}$$

In order for the soliton $\tilde{q}(x,t)$ the remain within the class of rapidly decaying functions one requires

$$\xi > -2\bar{\eta}_1, \tag{9.39}$$

where $-2\bar{\eta}_1$ is exactly the asymptotic decay rate of the soliton as $x \to +\infty$. One interesting feature of the one soliton solution (9.34) is the formation of singularity at a finite time. Indeed, at the origin ($x = 0$) the solution (9.34) becomes singular when

$$t_n = \frac{(2n+1)\pi - (\varphi+\bar{\varphi})}{4(\eta_1^2-\bar{\eta}_1^2)}, \qquad n \in \mathbb{Z}. \tag{9.40}$$

It should be pointed out that not all members of the one-soliton family develop a singularity at finite time. For example, if one takes $\eta_1 = \bar{\eta}_1 \equiv \eta$ then the one soliton solution (9.34) reads

$$q(x,t) = -\frac{4\eta\mathrm{e}^{\mathrm{i}\bar{\varphi}}\mathrm{e}^{-4\mathrm{i}\eta^2 t}\mathrm{e}^{-2\eta x}}{1+\mathrm{e}^{\mathrm{i}(\varphi+\bar{\varphi})}\mathrm{e}^{-4\eta x}}. \tag{9.41}$$

Thus the soliton given in (9.41) develops no singularity so long $\varphi + \bar{\varphi} \neq (2n+1)\pi$ for any integer n.

9.3. 2-Soliton solution

In this section we construct a 2-soliton solution to the *PT* invariant nonlinear Schrödinger equation (1.1) with $\sigma = -1$. Such solution corresponds to soliton eigenvalues

$$\begin{aligned} &k_1 = \mathrm{i}\eta_1, \quad k_2 = \mathrm{i}\eta_2, \quad \eta_j > 0 \quad j = 1,2, \\ &\bar{k}_1 = -\mathrm{i}\bar{\eta}_1, \quad \bar{k}_2 = -\mathrm{i}\bar{\eta}_2, \quad \bar{\eta}_j > 0; \quad j = 1,2. \end{aligned} \tag{9.42}$$

Substituting $J=2$ in equation (7.10) and set the reflection coefficient ρ to zero we find

$$q(x) = 2\mathrm{i}\sigma C_1^* N_2^*(-x,k_1)\mathrm{e}^{2\eta_1 x} + C_2^* N_2^*(-x,k_2)\mathrm{e}^{2\eta_2 x}, \tag{9.43}$$

where $C_j, \overline{C}_j$, $j=1,2$ are the norming constants (in x) whose time evolution is determined by

$$C_1(t) = C_1(0)\mathrm{e}^{4\mathrm{i}\eta_1^2 t}, \qquad C_2(t) = C_2(0)\mathrm{e}^{4\mathrm{i}\eta_2^2 t}, \tag{9.44}$$

$$\overline{C}_1(t) = \overline{C}_1(0)\mathrm{e}^{-4\mathrm{i}\overline{\eta}_1^2 t}, \qquad \overline{C}_2(t) = \overline{C}_2(0)\mathrm{e}^{-4\mathrm{i}\overline{\eta}_2^2 t}, \tag{9.45}$$

and $|C_j|, |\overline{C}_j|$ are given in (9.20). To determine the functional form of the eigenfunctions $N_2^*(-x,k_1)$ and $N_2^*(-x,k_2)$ we solve the system of equations given in (6.37) and (6.40) with $J=\overline{J}=2$, and $\ell, j=1,2$. After some lengthy calculations we arrive at the following set of algebraic equations satisfied by the soliton eigenfunctions

$$N_2^*(-x,k_1) = 1 - \gamma_1 \mathrm{e}^{2\overline{\eta}_1 x}\, \overline{M}_1(x,\bar{k}_1) - \gamma_2 \mathrm{e}^{2\overline{\eta}_2 x}\, \overline{M}_1(x,\bar{k}_2), \tag{9.46}$$

$$N_2^*(-x,k_2) = 1 - \delta_1 \mathrm{e}^{2\overline{\eta}_1 x}\, \overline{M}_1(x,\bar{k}_1) - \delta_2 \mathrm{e}^{2\overline{\eta}_2 x}\, \overline{M}_1(x,\bar{k}_2), \tag{9.47}$$

$$N_1^*(-x,k_1) = -\gamma_1 \mathrm{e}^{2\overline{\eta}_1 x}\, \overline{M}_2(x,\bar{k}_1) - \gamma_2 \mathrm{e}^{2\overline{\eta}_2 x}\, \overline{M}_2(x,\bar{k}_2), \tag{9.48}$$

$$N_1^*(-x,k_2) = -\delta_1 \mathrm{e}^{2\overline{\eta}_1 x}\, \overline{M}_2(x,\bar{k}_1) - \delta_2 \mathrm{e}^{2\overline{\eta}_2 x}\, \overline{M}_2(x,\bar{k}_2), \tag{9.49}$$

$$\overline{M}_2^*(-x,\bar{k}_1) = 1 + \alpha_1 \mathrm{e}^{-2\eta_1 x}\, N_1(x,k_1) + \alpha_2 \mathrm{e}^{-2\eta_2 x}\, N_1(x,k_2), \tag{9.50}$$

$$\overline{M}_2^*(-x,\bar{k}_2) = 1 + \beta_1 \mathrm{e}^{-2\eta_1 x}\, N_1(x,k_1) + \beta_2 \mathrm{e}^{-2\eta_2 x}\, N_1(x,k_2), \tag{9.51}$$

$$\overline{M}_1^*(-x,\bar{k}_1) = \alpha_1 \mathrm{e}^{-2\eta_1 x}\, N_2(x,k_1) + \alpha_2 \mathrm{e}^{-2\eta_2 x}\, N_2(x,k_2), \tag{9.52}$$

$$\overline{M}_1^*(-x,\bar{k}_2) = \beta_1 \mathrm{e}^{-2\eta_1 x}\, N_2(x,k_1) + \beta_2 \mathrm{e}^{-2\eta_2 x}\, N_2(x,k_2), \tag{9.53}$$

where the quantities $\alpha_j, \beta_j, \gamma_j$ and $\delta_j, j=1,2$ are time dependent and defined as

$$\alpha_1 = \frac{\mathrm{i}C_1(0)\mathrm{e}^{4\mathrm{i}\eta_1^2 t}}{\overline{\eta}_1 + \eta_1}, \qquad \alpha_2 = \frac{\mathrm{i}C_2(0)\mathrm{e}^{4\mathrm{i}\eta_2^2 t}}{\overline{\eta}_1 + \eta_2} \tag{9.54}$$

$$\beta_1 = \frac{\mathrm{i}C_1(0)\mathrm{e}^{4\mathrm{i}\eta_1^2 t}}{\overline{\eta}_2 + \eta_1}, \qquad \beta_2 = \frac{\mathrm{i}C_2(0)\mathrm{e}^{4\mathrm{i}\eta_2^2 t}}{\overline{\eta}_2 + \eta_2} \tag{9.55}$$

$$\gamma_1 = \frac{\overline{C}_1^*(0)\mathrm{e}^{4\mathrm{i}\overline{\eta}_1^2 t}}{\mathrm{i}(\eta_1 + \overline{\eta}_1)}, \qquad \gamma_2 = \frac{\overline{C}_2^*(0)\mathrm{e}^{4\mathrm{i}\overline{\eta}_2^2 t}}{\mathrm{i}(\eta_1 + \overline{\eta}_2)} \tag{9.56}$$

$$\delta_1 = \frac{\overline{C}_1^*(0)\mathrm{e}^{4\mathrm{i}\overline{\eta}_1^2 t}}{\mathrm{i}(\eta_2 + \overline{\eta}_1)}, \qquad \delta_2 = \frac{\overline{C}_2^*(0)\mathrm{e}^{4\mathrm{i}\overline{\eta}_2^2 t}}{\mathrm{i}(\eta_2 + \overline{\eta}_2)} \tag{9.57}$$

Solving system (9.46)–(9.53) we get

$$N_2^*(-x,k_1)=\frac{\alpha_2^*(\delta_1-\gamma_1)\mathrm{e}^{2\overline{\eta}_1x}+\beta_2^*(\delta_2-\gamma_2)\mathrm{e}^{2\overline{\eta}_2x}+\mathrm{e}^{-2\eta_2x}}{\mathsf{G}_1(x)-\mathsf{G}_2(x)}, \tag{9.58}$$

where we defined the functions:

$$\mathsf{G}_1(x)=\left[\gamma_1\alpha_1^*\mathrm{e}^{2(\eta_1+\overline{\eta}_1)x}+\gamma_2\beta_1^*\mathrm{e}^{2(\eta_1+\overline{\eta}_2)x}+1\right]\left[\delta_1\alpha_2^*\mathrm{e}^{2\overline{\eta}_1x}+\delta_2\beta_2^*\mathrm{e}^{2\overline{\eta}_2x}+\mathrm{e}^{-2\eta_2x}\right], \tag{9.59}$$

and

$$\mathsf{G}_2(x)=\left[\delta_1\alpha_1^*\mathrm{e}^{2(\eta_1+\overline{\eta}_1)x}+\delta_2\beta_1^*\mathrm{e}^{2(\eta_1+\overline{\eta}_2)x}\right]\left[\gamma_1\alpha_2^*\mathrm{e}^{2\overline{\eta}_1x}+\gamma_2\beta_2^*\mathrm{e}^{2\overline{\eta}_2x}\right]. \tag{9.60}$$

The solution for $N_2^*(-x,k_2)$ is given by

$$N_2^*(-x,k_2)=\frac{\alpha_1^*(\gamma_1-\delta_1)\mathrm{e}^{2\overline{\eta}_1x}+\beta_1^*(\gamma_2-\delta_2)\mathrm{e}^{2\overline{\eta}_2x}+\mathrm{e}^{-2\eta_1x}}{\mathsf{H}_1(x)-\mathsf{H}_2(x)}, \tag{9.61}$$

where we defined the functions

$$\mathsf{H}_1(x)=\left[\gamma_1\alpha_1^*\mathrm{e}^{2\overline{\eta}_1x}+\gamma_2\beta_1^*\mathrm{e}^{2\overline{\eta}_2x}+\mathrm{e}^{-2\eta_1x}\right]\left[\delta_1\alpha_2^*\mathrm{e}^{2(\eta_2+\overline{\eta}_1)x}+\delta_2\beta_2^*\mathrm{e}^{2(\eta_2+\overline{\eta}_2)x}+1\right], \tag{9.62}$$

and

$$\mathsf{H}_2(x)=\left[\delta_1\alpha_1^*\mathrm{e}^{2\overline{\eta}_1x}+\delta_2\widetilde{\beta}^*{}_1\mathrm{e}^{2\overline{\eta}_2x}\right]\left[\gamma_1\alpha_2^*\mathrm{e}^{2(\eta_2+\overline{\eta}_1)x}+\gamma_2\beta_2^*\mathrm{e}^{2(\eta_2+\overline{\eta}_2)x}\right]. \tag{9.63}$$

Substituting equations (9.58)–(9.61) into equation (9.43) yields the 2-soliton solution.

10. Comparison with the classical NLS equation

In this part we briefly contrast the properties of the nonlocal NLS (1.1) with that of the classical (local) NLS equation:

$$\mathrm{i}q_t(x,t)=q_{xx}(x,t)\pm 2|q(x,t)|^2q(x,t), \tag{10.1}$$

obtained from equation (1.1) by replacing the term $q^*(-x,t)$ with $q^*(x,t)$. In table 1 we summarize and highlight the key differences between the classical and nonlocal NLS equations.

Three different scenarios will be addressed all of which concerning equation (10.1):

1. general initial conditions posed on the whole real line,
2. even initial conditions posed on the whole real line and,
3. general initial conditions on the semi-infinite interval ($x\geqslant 0$.)

In [28], it was shown that (10.1) is integrable on the whole real line. Furthermore, it was found that the symmetries of the eigenfunctions of the associated Zakharov–Shabat scattering problem are such that the eigenfunctions in the upper half complex plane are related to those in the lower half plane. This is in sharp contrast to the nonlocal case where the eigenfunctions at the upper/lower half plane are not related. On the other hand, if one restricts the class of initial conditions to be even (in x) then one obtains extra symmetry conditions on the scattering data that resembles the one we find. This leads us to the important conclusion that soliton solutions to (1.1) will have a classical NLS limit so long (10.1) admits an *even* soliton solution.

Table 1. Symmetry relations between various quantities related to the classical and nonlocal integrable NLS equations. Here $\sigma = \mp 1$.

Symmetry property	Classical NLS equation $\mathrm{i}q_t(x,t) = q_{xx}(x,t) \pm 2|q(x,t)|^2 q(x,t)$	Nonlocal NLS equation $\mathrm{i}q_t(x,t) = q_{xx}(x,t) \pm 2q^2(x,t)q^*(-x,t)$
Scattering data:	$\bar{a}(k) = a^*(k^*)$ $\bar{b}(k) = \sigma b^*(k^*)$	$a(k) = a^*(-k^*)$ $\bar{a}(\bar{k}) = \bar{a}^*(-\bar{k}^*)$ $\bar{b}(k) = \sigma b^*(k^*)$ a and $\bar{a}$ are not related
Eigenvalues:	$\bar{k}_j = k_j^*$	If $k_j, \bar{k}_j$ are eigenvalues then $-k_j^*, -\bar{k}_j^*$ are also eigenvalues. No relations between k_j and $\bar{k}_j$.
Eigenfunctions	$\bar{N}(x,k) = \begin{pmatrix} N^{(2)}(x,k^*) \\ \sigma N^{(1)}(x,k^*) \end{pmatrix}^*$ $\bar{M}(x,k) = \begin{pmatrix} \sigma M^{(2)}(x,k^*) \\ M^{(1)}(x,k^*) \end{pmatrix}^*$	$N(x,k) = \Lambda M^*(-x,-k^*)$ $\bar{N}(x,k) = \Lambda^{-1}\bar{M}^*(-x,-k^*)$ $\Lambda = \begin{pmatrix} 0 & -\sigma \\ 1 & 0 \end{pmatrix}$
Norming constants	$\bar{C}_j = \sigma C_j^*$ free parameters; independent of k_j and $\bar{k}_j$	$\bar{C}_j, C_j$ depend on k_j and $\bar{k}_j$ via equations (9.9), (9.13) and (9.17)
Galilean invariance	$\tilde{q}(x,t) = q(x - 2\zeta t, t)\mathrm{e}^{-\mathrm{i}\zeta x}\mathrm{e}^{\mathrm{i}\zeta^2 t}$	$\tilde{q}(x,t) = q(x + 2\mathrm{i}\xi t, t)\mathrm{e}^{-\xi x}\mathrm{e}^{-\mathrm{i}\xi^2 t}$

Finally, we point out that similar symmetry results were obtained in [7] for the classical NLS (10.1) on the semi-infinite interval.

11. Some special potentials: box initial conditions

In the previous sections we obtained pure soliton solutions for the nonlocal NLS equation corresponding to reflectionless potentials. However, more general choices of initial data $q(x, 0)$ give rise to non zero reflection coefficients in which case it is not (generally speaking) solvable by inverse scattering transform. As an example, we consider in this section few special cases of a box initial conditions and compute the eigenvalues and conserved quantities.

11.1. Single box function

The first example we consider the box initial condition

$$q(x,0) = \begin{cases} 0 & \text{for} \quad -\infty < x < 0 \\ h & \text{for} \quad 0 < x < L \\ 0 & \text{for} \quad x > 0 \end{cases} \tag{11.1}$$

where h, L are both real and positive. The initial condition for $r(x, 0)$ is obtained from (11.1) by imposing the symmetry condition $r(x, 0) = -q^*(-x, 0)$, i.e.

$$r(x,0) = \begin{cases} 0 & \text{for} \quad -\infty < x < -L \\ -h & \text{for} \quad -L < x < 0 \\ 0 & \text{for} \quad x > 0 \end{cases} \tag{11.2}$$

We use the scattering problem

$$v_{1x} = -\mathrm{i}kv_1 + q(x,t)v_2, \tag{11.3}$$
$$v_{2x} = \mathrm{i}kv_2 + r(x,t)v_1, \tag{11.4}$$

with the scattering functions satisfying the boundary conditions (3.1) and (3.2). Since this is a linear, constant coefficient, second order differential equation, solution to system (11.3) and (11.4) is thus given by

$$\begin{pmatrix} v_1(x,k) \\ v_2(x,k) \end{pmatrix} = \begin{pmatrix} \dfrac{h}{2\mathrm{i}k}\mathrm{c}_1\mathrm{e}^{\mathrm{i}kx} + \mathrm{c}_2\mathrm{e}^{-\mathrm{i}kx} \\ \mathrm{c}_1\mathrm{e}^{\mathrm{i}kx} \end{pmatrix} \quad \text{for} \quad 0 < x < L. \tag{11.5}$$

$$\begin{pmatrix} v_1(x,k) \\ v_2(x,k) \end{pmatrix} = \begin{pmatrix} \tilde{\mathrm{c}}_1\mathrm{e}^{-\mathrm{i}kx} \\ \tilde{\mathrm{c}}_2\mathrm{e}^{\mathrm{i}kx} + \dfrac{h}{2\mathrm{i}k}\tilde{\mathrm{c}}_1\mathrm{e}^{-\mathrm{i}kx} \end{pmatrix} \quad \text{for} \quad -L < x < 0. \tag{11.6}$$

Matching the eigenfunctions at $x = 0$ and $x = -L$ using the boundary conditions (3.1) and (3.2) gives

$$\tilde{\mathrm{c}}_1 = 1, \qquad \tilde{\mathrm{c}}_2 = -\frac{h}{2\mathrm{i}k}\mathrm{e}^{2\mathrm{i}kL}, \tag{11.7}$$

$$\mathrm{c}_1 = \frac{h}{2\mathrm{i}k}(1 - \mathrm{e}^{2\mathrm{i}kL}), \qquad \mathrm{c}_2 = 1 + \left(\frac{h}{2\mathrm{i}k}\right)^2(\mathrm{e}^{2\mathrm{i}kL} - 1). \tag{11.8}$$

On the other hand for $x > L$ we have from (3.2) and (4.8)

$$\phi(x,t) = \begin{pmatrix} a(k)\mathrm{e}^{-\mathrm{i}kx} \\ b(k)\mathrm{e}^{\mathrm{i}kx} \end{pmatrix} \quad \text{for} \quad x > L. \tag{11.9}$$

Matching the eigenfunctions at $x = L$ gives the formula for the scattering data

$$a(k) = 1 + \left(\frac{h}{2\mathrm{i}k}\right)^2(\mathrm{e}^{2\mathrm{i}kL} - 1) - \left(\frac{h}{2\mathrm{i}k}\right)^2(\mathrm{e}^{4\mathrm{i}kL} - \mathrm{e}^{2\mathrm{i}kL}). \tag{11.10}$$

$$b(k) = -h\mathrm{e}^{\mathrm{i}kL}\frac{\sin(kL)}{k}. \tag{11.11}$$

The asymptotic behavior of the scattering coefficient $a(k)$ for large and small k are readily obtained from (11.10) as

$$a(k) \sim 1 - \frac{h^2}{(2\mathrm{i}k)^2} \quad \text{as} \quad k \to \infty, \tag{11.12}$$

$$a(k) \sim 1 - h^2L^2 + O(k) \quad \text{as} \quad k \to 0. \tag{11.13}$$

The zeros of the scattering data $a(k)$ are implicitly given by solutions to

$$\mathrm{e}^{\mathrm{i}z} = 1 \pm \frac{iz}{hL}, \tag{11.14}$$

where $z = 2kL$. Note that $k = 0$ is an improper eigenvalue for ant h, L. If $\mathrm{Im}z > 0$ then we choose the negative sign whereas positive sign for $\mathrm{Im}z > 0$. The conserved quantities can be derived from (3.8). Using the large k asypmtotic of the scattering data $a(k)$ leads to $\mathcal{C}_{2n} = 0$

$$\mathcal{C}_1 = -h^2, \quad \mathcal{C}_3 = -h^4/2. \tag{11.15}$$

11.2. Separated boxes

As a second example we consider two separated boxes initial condition

$$q(x,0) = \begin{cases} 0 & \text{for} \quad -\infty < x < L_1 \\ H & \text{for} \quad L_1 < x < L_2 \\ 0 & \text{for} \quad x > L_2 \end{cases} \tag{11.16}$$

where H, L_j, $j = 1, 2$ are real and positive. The initial condition for $r(x, 0)$ is obtained from (11.1) by imposing the symmetry condition $r(x, 0) = -q^*(-x, 0)$, i.e.

$$r(x,0) = \begin{cases} 0 & \text{for} \quad -\infty < x < -L_2 \\ -H & \text{for} \quad -L_2 < x < -L_1 \\ 0 & \text{for} \quad x > -L_1 \end{cases} \tag{11.17}$$

Solution to the scattering problem (11.3) and (11.4) satisfying the boundary conditions (3.1) and (3.2) is thus given by

$$\begin{pmatrix} v_1(x,k) \\ v_2(x,k) \end{pmatrix} = \begin{pmatrix} \tilde{\mathrm{c}}_1 \mathrm{e}^{-\mathrm{i}kx} \\ \tilde{\mathrm{c}}_2 \mathrm{e}^{\mathrm{i}kx} + \dfrac{h}{2\mathrm{i}k}\tilde{\mathrm{c}}_1 \mathrm{e}^{-\mathrm{i}kx} \end{pmatrix} \quad \text{for} \quad -L_2 < x < -L_1. \tag{11.18}$$

$$\begin{pmatrix} v_1(x,k) \\ v_2(x,k) \end{pmatrix} = \begin{pmatrix} \tilde{\tilde{\mathrm{c}}}_1 \mathrm{e}^{-\mathrm{i}kx} \\ \tilde{\tilde{\mathrm{c}}}_2 \mathrm{e}^{\mathrm{i}kx} \end{pmatrix} \quad \text{for} \quad -L_1 < x < L_1. \tag{11.19}$$

$$\begin{pmatrix} v_1(x,k) \\ v_2(x,k) \end{pmatrix} = \begin{pmatrix} \dfrac{h}{2\mathrm{i}k}\mathrm{c}_1 \mathrm{e}^{\mathrm{i}kx} + \mathrm{c}_2 \mathrm{e}^{-\mathrm{i}kx} \\ \mathrm{c}_1 \mathrm{e}^{\mathrm{i}kx} \end{pmatrix} \quad \text{for} \quad L_1 < x < L_2. \tag{11.20}$$

Matching the eigenfunctions at $x = -L_2, -L_1$ and $x = L_1$ using the boundary conditions (3.1) and (3.2) gives

$$\tilde{\mathrm{c}}_1 = 1, \qquad \tilde{\mathrm{c}}_2 = -\frac{h}{2\mathrm{i}k}\mathrm{e}^{2\mathrm{i}kL_2}, \tag{11.21}$$

$$\mathrm{c}_1 = \frac{h}{2\mathrm{i}k}(1 - \mathrm{e}^{2\mathrm{i}kL_2}), \qquad \mathrm{c}_2 = 1 + \left(\frac{h}{2\mathrm{i}k}\right)^2 (\mathrm{e}^{2\mathrm{i}kL_2} - 1)\mathrm{e}^{2\mathrm{i}kL_1}. \tag{11.22}$$

$$\tilde{\tilde{\mathrm{c}}}_1 = 1, \qquad \tilde{\tilde{\mathrm{c}}}_2 = \frac{h}{2\mathrm{i}k}(1 - \mathrm{e}^{2\mathrm{i}kL_2}). \tag{11.23}$$

Now for $x > L_2$ the eigenfunctions are given by (11.9) Matching the eigenfunctions at $x = L$ gives the formula for the scattering data

$$a(k) = 1 + \left(\frac{h}{2ik}\right)^2 (e^{2ikL} - 1) - \left(\frac{h}{2ik}\right)^2 (e^{4ikL} - e^{2ikL}), \tag{11.24}$$

$$b(k) = -he^{ikL}\frac{\sin(kL)}{k}. \tag{11.25}$$

The asymptotic behavior of the scattering coefficient $a(k)$ for large and small k are readily obtained from (11.10) as

$$a(k) \sim 1 - \frac{h^2}{(2ik)^2} \quad \text{as} \quad k \to \infty, \tag{11.26}$$

$$a(k) \sim 1 - h^2L^2 + O(k) \quad \text{as} \quad k \to 0. \tag{11.27}$$

The zeros of the scattering data $a(k)$ are implicitly given by solutions to

$$e^{iz} = 1 \pm \frac{iz}{hL}, \tag{11.28}$$

where $z = 2kL$. Note that $k = 0$ is an improper eigenvalue for ant h, L. If $\mathrm{Im}z > 0$ then we choose the negative sign whereas positive sign for $\mathrm{Im}z > 0$. The conserved quantities can be derived from (3.8). Using the large k asypmtotic of the scattering data $a(k)$ leads to $\mathcal{C}_{2n} = 0$

$$\mathcal{C}_1 = -h^2, \quad \mathcal{C}_3 = -h^4/2. \tag{11.29}$$

12. Conclusion

A detailed study of the inverse scattering transform for the recently introduced integrable nonlocal nonlinear Schrödinger (NLS) equation is carried out. The direct and inverse scattering problems are analyzed and key symmetries of the eigenfunctions and scattering data and conserved quantities obtained. The inverse scattering theory is formulated using a new left–right Riemann–Hilbert problem and the Cauchy problem for the nonlocal NLS equation is formulated and methods to find pure soliton solutions presented. An explicit time-periodic one and two soliton solutions are given. A detailed comparison with the classical nonlinear Schrödinger (NLS) equation is presented as is scattering data for special box type initial conditions. Two nonlocal versions of the modified Korteweg–de Vries and sine-Gordon equations are given.

Acknowledgments

The authors appreciate very useful comments of G Biondini. The research of MJA was partially supported by NSF 253 under Grant No. DMS-1310200 and the U.S. Air Force Office 254 of Scientific Research, under Grant No. FA9550-12-1-0207.

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