Inverse Trigonometric Functions - Trigonometric Equations

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Abstract

This handout defines the inverse of the sine, cosine and tangent functions. It then shows how these inverse functions can be used to solve trigonometric equations.

1 Inverse Trigonometric Functions

1.1 Quick Review

It is assumed that the student is familiar with the concept of inverse functions. We will review this concept very briefly. For further review, please visit section 2.7 or the handout given in class on inverse functions.

The student will recall that if f is a one-to-one function with domain A and range B, then the inverse of f is the function denoted f^{-1}, with domain B and range A such that

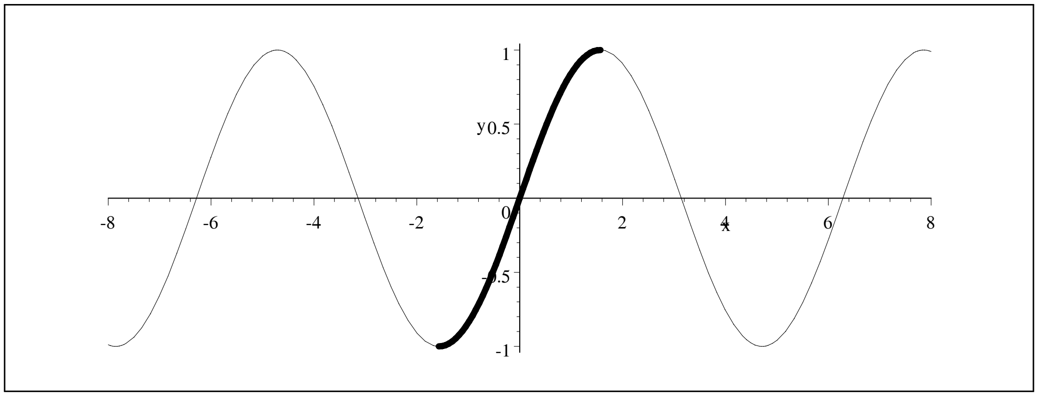
$$y = f^{-1}(x) \Longleftrightarrow x = f(y)$$

For a function to have an inverse, it must be one-to-one. Sometimes, when a function is not one-to-one, one can restrict its domain to make it one-to-one. For example, the function $f(x) = x^2$, with domain $(-\infty, \infty)$ is not one-to-one; it does not pass the horizontal line test. However, if we restrict its domain to $[0, \infty)$, it is one-to-one. It has an inverse, it is $f^{-1}(x) = \sqrt{x}$.

Because of their periodic nature, the trigonometric functions are not one-to-one.

1.2 Inverse of the Sine Function

Obviously, the graph of $\sin x$ does not pass the horizontal line test. Thus, $\sin x$ is not one-to-one, hence not invertible. We would like to restrict its domain to make it one-to-one. We would like to do it in such a way that we do not lose too much information. In other words, we would like its range to be the same, that is $[-1, 1]$. There are many ways this can be done. The way it is usually done is to restrict the domain of $\sin x$ to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The graph below shows $y = \sin x$. In bold, it shows $\sin x$ restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. It is easy to see that the restriction of $\sin x$ passes the horizontal line test.



Remembering that the domain of a function and the range of its inverse are the same, we have:

Definition 1 *The inverse sine function, denoted* $\sin^{-1}$ *is the function with domain* $[-1, 1]$*, range* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ *defined by*

$$y = \sin^{-1} x \Longleftrightarrow x = \sin y$$

The inverse sine function is also called arcsine, it is denoted by arcsin.

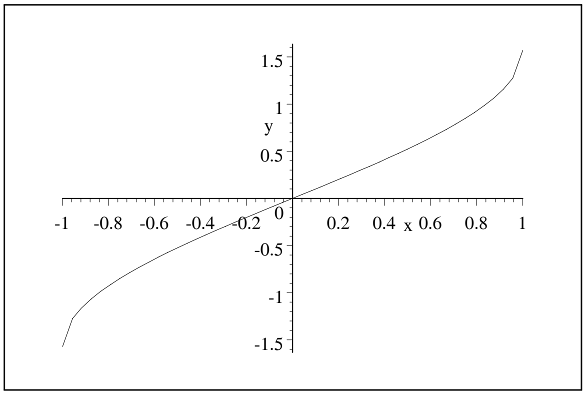
There are several important remarks to make at this point.

Remark 2 $\sin^{-1} x$ *is the number* y *in the interval* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ *such that* $\sin y = x$. *As you know, since* $\sin x$ *is periodic, there is more than one number* y *such that* $\sin y = x$. *There is an infinite amount. We only want the one in the interval* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. *For example,* $\sin^{-1} 1 = \frac{\pi}{2}$ *because* $\sin \frac{\pi}{2} = 1$. *There are other numbers* a *such that* $\sin a = 1$. $a = \frac{\pi}{2} + 2n\pi$ *will work. Out of all these numbers, the only one we want is* $\frac{\pi}{2}$ *because it is the only one in interval* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Remark 3 *From the general properties of inverse functions, we have*

$$\begin{aligned} \sin\left(\sin^{-1} x\right) &= x \text{ for every } x \text{ in } [-1, 1] \\ \sin^{-1}(\sin x) &= x \text{ for every } x \text{ in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \end{aligned}$$

From the general properties of inverse functions, we deduce that the graph of $\sin^{-1}$ is a reflection of the graph of sin (restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$) about the line $y = x$. The graph of $\sin^{-1}$ is shown below.



Example 4 *Find* $\sin^{-1}\frac{1}{2}$. *Since* $\sin\frac{\pi}{6} = \frac{1}{2}$, *and* $\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, *it follows that* $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$.

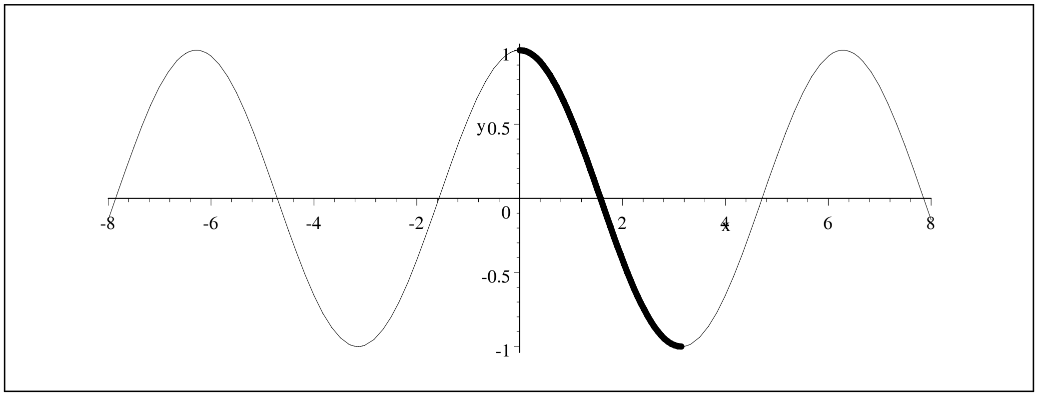
Example 5 *Find* $\sin^{-1} .2$. *Here, we need to use our calculator.* $\sin^{-1} .2 = 0.20136$.

Example 6 *Find* $\sin^{-1} 5$.
Since the domain of $\sin^{-1}$ *is* $[-1, 1]$, $\sin^{-1} 5$ *is not defined.*

Remark 7 *Given* a, $\sin^{-1} a$ *is the angle* θ *such that* $\sin\theta = a$. *Thus,* $\sin^{-1} a$ *is an angle. When you use your calculator to perform this type of operations, you must be aware of whether your calculator is set in degrees or radians; the answer you get will be different, though it will represent the same angle. For example, if your calculator is set in degrees, then* $\sin^{-1} .2 = 11.53696$, *yet above we found* $.20136$. *These two numbers are different, yet they represent the same angle (do the conversion). So, when writing the answer to such a question, you should specify the units. For example,* $\sin^{-1} .2 = 0.20136$ *rad, or* $\sin^{-1} .2 = 11.53696°$.

1.3 Inverse of the Cosine Function

Things are similar for cosine. The main difference is the interval on which we restrict the cosine function to make it invertible. This interval is $[0, \pi]$. The graph below shows $\cos x$ and its restriction to $[0, \pi]$ in bold.



Remembering that the domain of a function and the range of its inverse are the same, we have:

Definition 8 *The inverse cosine function, denoted* $\cos^{-1}$ *is the function with domain* $[-1, 1]$*, range* $[0, \pi]$ *defined by*

$$y = \cos^{-1} x \Longleftrightarrow x = \cos y$$

The inverse cosine function is also called arccosine, it is denoted by arccos*.*

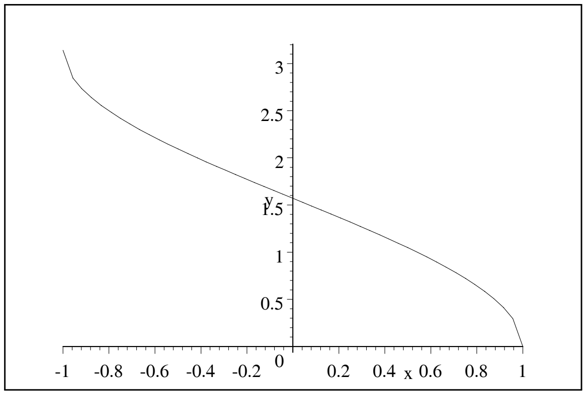
There are several important remarks to make at this point.

Remark 9 $\cos^{-1} x$ *is the number* y *in the interval* $[0, \pi]$ *such that* $\cos y = x$*. As you know, since* $\cos x$ *is periodic, there is more than one number* y *such that* $\cos y = x$*. There is an infinite amount. We only want the one in the interval* $[0, \pi]$*. For example,* $\cos^{-1} 0 = \dfrac{\pi}{2}$ *because* $\cos 0 = 1$*. There are other numbers* a *such that* $\cos a = 1$*.* $a = 2n\pi$ *will work. Out of all these numbers, the only one we want is* $\dfrac{\pi}{2}$ *because it is the only one in interval* $[0, \pi]$*.*

Remark 10 *From the general properties of inverse functions, we have*

$$\begin{aligned} \cos\left(\cos^{-1} x\right) &= x \text{ for every } x \text{ in } [-1, 1] \\ \cos^{-1}(\cos x) &= x \text{ for every } x \text{ in } [0, \pi] \end{aligned}$$

From the general properties of inverse functions, we deduce that the graph of $\cos^{-1}$ is a reflection of the graph of cos (restricted to $[0, \pi]$) about the line $y = x$. The graph of $\cos^{-1}$ is shown below.



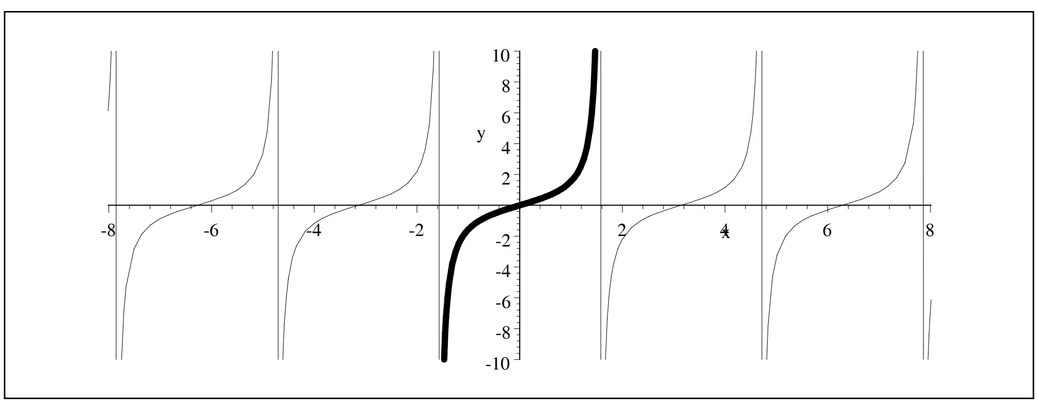
Example 11 *Find* $\cos^{-1}\frac{1}{2}$. *Since* $\cos\frac{\pi}{3} = \frac{1}{2}$, *and* $\frac{\pi}{3} \in [0, \pi]$, *it follows that* $\cos^{-1}\frac{1}{2} = \frac{\pi}{3}$ *rad.*

Example 12 *Find* $\cos^{-1}.2$. *Here, we need to use our calculator.* $\cos^{-1}.2 =$ 1.3694 *rad.*

Remark 13 *Like for* $\sin^{-}$, *when computing* $\cos^{-1} a$ *for a given, it is important to know the units of the answer.*

1.4 Inverse of the Tangent Function

The graph of $\tan x$ is shown below.



The function $y = \tan x$ is periodic of period π. Its domain is all reals except numbers of the form $x = \frac{\pi}{2} + n\pi$. Its graph has vertical asymptotes at these values of x. The range of $y = \tan x$ is all reals. To make it invertible, we restrict it to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Definition 14 *The inverse tangent function, denoted* $\tan^{-1}$ *is the function with domain* $\mathbf{R}$*, range* $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *defined by*

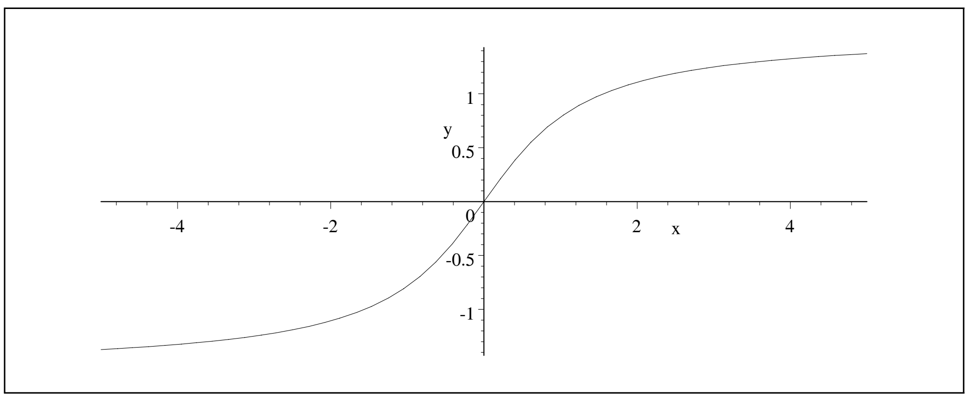
$$y = \tan^{-1} x \Longleftrightarrow x = \tan y$$

The inverse tangent function is also called arctangent, it is denoted by arctan*.*

Remark 15 *From the general properties of inverse functions, we have*

$$\begin{aligned} \tan\left(\tan^{-1} x\right) &= x \text{ for every } x \text{ in } \mathbf{R} \\ \tan^{-1}(\tan x) &= x \text{ for every } x \text{ in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{aligned}$$

From the general properties of inverse functions, we deduce that the graph of $\tan^{-1}$ is a reflection of the graph of tan (restricted to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$) about the line $y = x$. The graph of $\tan^{-1}$ is shown below.



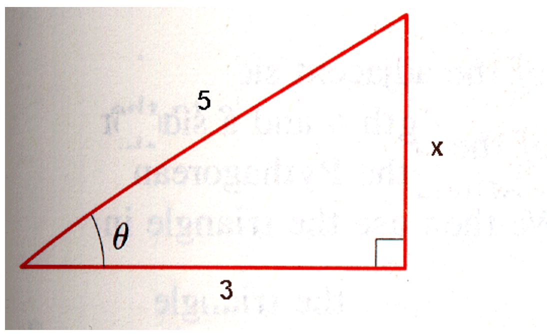
Example 16 *Find* $\tan^{-1} 1$*. Since* $\tan \frac{\pi}{4} = 1$*, and* $\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$*, it follows that* $\tan^{-1} 1 = \frac{\pi}{4}$*.*

1.5 Composing Trigonometric Functions and their Inverses

It is often important to be able to evaluate expressions of the form $\tan\left(\sin^{-1} a\right)$, $\sin\left(\cos^{-1} a\right)$, $\cos\left(\tan^{-1} a\right)$, ...for a given value of a. This is the case in particular in calculus. This can be done with a calculator. There is also an elegant way to do it using triangles. This way gives a very accurate answer (exact most of the time), while the calculator will only give an approximation. We illustrate it with an example.

Example 17 *Find* $\sin\left(\cos^{-1}\dfrac{3}{5}\right)$.

We begin by looking at the inverse trigonometric function. We say, let $\theta = \cos^{-1}\dfrac{3}{5}$. *Then,* $\sin\left(\cos^{-1}\dfrac{3}{5}\right) = \sin\theta$. *So, we must find* $\sin\theta$. *By definition, if* $\theta = \cos^{-1}\dfrac{3}{5}$, *then* $cos\theta = \dfrac{3}{5}$. *The problem is now to find* $\sin\theta$, *knowing* $\cos\theta$. *We know how to do this. This is the problem of finding all the trigonometric ratios, knowing one of them. We begin by drawing a triangle in which* $\cos\theta = \dfrac{3}{5}$. *Such a triangle is shown below:*



Then, $\sin\theta = \dfrac{x}{5}$, *so we need to find* x. *Using the Pythagorean theorem, we have*

$$\begin{aligned} x^2 + 9 &= 25 \\ x &= 4 \end{aligned}$$

Therefore, $\sin\theta = \dfrac{4}{5}$. *Hence,*

$$\sin\left(\cos^{-1}\frac{3}{5}\right) = \frac{4}{5}$$

1.6 Practice Problems

Do # 1, 5, 9, 21, 23, 45, 49, 55 on pages 548, 549.

2 Trigonometric Equations

A trigonometric equation is an equation involving one or more trigonometric functions.

2.1 Simple Equations Involving the Sine Function

2.1.1 Solving $\sin x = a$, a given.

One solution is $x = \sin^{-1} a$. However, this will only give us the solution in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. In other words, it will only give us a solution which is either in the first quadrant, or in the fourth. If the solution is in the first quadrant, there is also an angle in the second quadrant, call it x' for which the value of the sine function is the same. If the solution is in the fourth quadrant, there is also an angle in the third quadrant, call it x' for which the value of the sine function is the same. In both cases, the value of the second angle is $x' = \pi - x$. Once we have these two solutions x and x', because $\sin x$ is periodic, of period 2π, $x + 2n\pi$, and $x' + 2n\pi$ where n is any integer. are also solutions.

Proposition 18 *All the solutions of* $\sin x = a$ *are* $x + 2n\pi$ *and* $\pi - x + 2n\pi$*, where* n *is any integer, and* $x = \sin^{-1} a$*.*

Example 19 *Solve* $\sin x = \frac{1}{2}$.
One solution is $x = \sin^{-1} \frac{1}{2} = \frac{\pi}{6}$*. It follows that another solution is* $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$*. Therefore, all the solutions are* $\frac{\pi}{6} + 2n\pi$ *and* $\frac{5\pi}{6} + 2n\pi$*.*

2.1.2 Equations Involving the Sine Function

The general outline to solve such equations is:

1. Solve for $\sin x$
2. Solve for x as explained above.

Example 20 *Solve* $2 \sin x + 3 = 4$
First, we solve for $\sin x$

$$\begin{aligned} 2 \sin x + 3 &= 4 \\ 2 \sin x &= 1 \\ \sin x &= \frac{1}{2} \end{aligned}$$

Then, we solve for x*. Since we have already done that in the previous example, we know that the solutions are* $\frac{\pi}{6} + 2n\pi$ *and* $\frac{5\pi}{6} + 2n\pi$*.*

2.2 Simple Equations Involving the cosine Function

2.2.1 Solving $\cos x = a$, a given.

This is similar to the equation $\sin x = a$, however there are some differences. One solution is $x = \cos^{-1} a$. This will give us a value for x in the interval $[0, \pi]$.

In other words, x will be either in the first quadrant, or in the second. If x is in the first quadrant, then there is also a value in the fourth quadrant for which the value of the cosine function is the same. This value is $2\pi - x$. If x is in the second quadrant, then there is also a value in the third quadrant for which the value of the cosine function is the same. This value is also $2\pi - x$. Then, adding $2n\pi$ to either solution will also be a solution, because $\cos x$ has period 2π. Therefore, all the solutions of $\cos x = a$ are $x + 2n\pi$ and $2\pi - x + 2n\pi$ where n is any integer, and $x = \cos^{-1} a$. However, we notice that $2\pi - x + 2n\pi$ is the same thing as $-x + 2n\pi$. In summary, we have:

Proposition 21 *All the solutions of* $\cos x = a$ *are* $x + 2n\pi$ *and* $-x + 2n\pi$*, where* n *is any integer, and* $x = \cos^{-1} a$*.*

Example 22 *Solve* $2\cos x + \sqrt{3} = 0$
First, we solve for $\cos x$*.*

$$\begin{aligned} 2\cos x + \sqrt{3} &= 0 \\ \cos x &= -\frac{\sqrt{3}}{2} \end{aligned}$$

Next, we solve for x*.*

$$\begin{aligned} x &= \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) \\ x &= \frac{5\pi}{6} \end{aligned}$$

Therefore, all the solutions are $x = \pm\frac{5\pi}{6} + 2n\pi$

2.2.2 Equations Involving the cosine Function

The general outline to solve such equations is:

1. Solve for $\cos x$
2. Solve for x as explained above.

Example 23 *solve* $2\cos^2 x - 1 = 0$
First, we solve for $\cos x$*.*

$$\begin{aligned} 2\cos^2 x - 1 &= 0 \\ \cos^2 x &= \frac{1}{2} \\ \cos x &= \pm\frac{1}{\sqrt{2}} \\ &= \pm\frac{\sqrt{2}}{2} \end{aligned}$$

We now have two equations, we solve them separately.

$$\begin{aligned} \cos x &= \frac{\sqrt{2}}{2} \\ x &= \cos^{-1} \frac{\sqrt{2}}{2} \\ x &= \frac{\pi}{4} \end{aligned}$$

Therefore, all the solutions of the first equation are $x = \frac{\pi}{4} + 2n\pi$, $x = -\frac{\pi}{4} + 2n\pi$. *The second equation gives us*

$$\begin{aligned} \cos x &= -\frac{\sqrt{2}}{2} \\ x &= \cos^{-1} \left(-\frac{\sqrt{2}}{2}\right) \\ x &= \frac{3\pi}{4} \end{aligned}$$

Therefore, all the solutions of the first equation are $x = \frac{3\pi}{4} + 2n\pi$, $x = -\frac{3\pi}{4} + 2n\pi$. *Combining the two results, we get that the solutions of* $2\cos^2 x - 1 = 0$ *are* $x = \frac{\pi}{4} + 2n\pi$, $x = -\frac{\pi}{4} + 2n\pi$, $x = \frac{3\pi}{4} + 2n\pi$, $x = -\frac{3\pi}{4} + 2n$. *These can be written as* $\frac{\pi}{4} + n\frac{\pi}{2}$.

2.3 Simple Equations Involving the tangent Function

2.3.1 Solving $\tan x = a$, a given.

The function $y = \tan x$ is periodic, of period π. A solution of $\tan x = a$ is $x = \tan^{-1} a$, this is a solution in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Because this interval contains a full period of tangent, there are no other solutions, except the ones obtained by adding multiples of π (the period). Therefore, we have:

Proposition 24 *All the solutions of* $\tan x = a$ *are* $x + n\pi$, *where* n *is any integer, and* $x = \tan^{-1} a$.

2.3.2 Equations Involving the cosine Function

The general outline to solve such equations is:

1. Solve for $\tan x$
2. Solve for x as explained above.

2.4 Trigonometric Equations

In general, a trigonometric equations may contain more than one trigonometric function. Solving it may require using all the algebra rules and trigonometric identities you know. We illustrate this with a few examples. You should keep in mind the following techniques to try:

- Try writing everything in terms of the sine and cosine functions
- Simplify the expression using identities.
- Sometimes, squaring both sides helps. In this case, you must check your solutions, some of them may not work.
- Try factoring common factors.

Example 25 *Solve* $\cos^2 x - \cos x = 0$

$$\cos^2 x - \cos x = 0$$

$$\cos x\,(\cos x - 1) = 0$$

A product is 0 *if one of the factors is zero. Thus, we get* $\cos x = 0$ *or* $\cos x - 1 = 0$*. We solve each equation separately. The first equation gives us:*

$$\begin{aligned} \cos x &= 0 \\ x &= \cos^{-1} 0 \\ x &= \frac{\pi}{2} \end{aligned}$$

Therefore all the solutions of this first equation are $x = \dfrac{\pi}{2} + 2n\pi$ *and* $x = \dfrac{-\pi}{2} + 2n\pi$ *which is the same as* $x = \dfrac{\pi}{2} + n\pi$*. The second equation gives us:*

$$\begin{aligned} \cos x - 1 &= 0 \\ \cos x &= 1 \\ x &= \cos^{-1} 0 \\ x &= 0 \end{aligned}$$

Therefore, all the solutions of the second equation are $x = 2n\pi$*. Combining the solution of both equations, we see that the solutions of* $\cos^2 x - \cos x = 0$ *are* $x = \dfrac{\pi}{2} + n\pi$ *and* $x = 2n\pi$*.*

Remark 26 *Sometimes, we only want the solutions in a given interval. In the above problem, we might have asked to find the solutions in* $[0, 2\pi]$*. Out of all the solutions we found, we would only keep the ones in the given interval. In this case, they would be* $\left\{0, \dfrac{\pi}{2}, \dfrac{3\pi}{2}, 2\pi\right\}$*.*

Example 27 *Solve* $\sin x = \cos x$.

$$\begin{aligned} \sin x &= \cos x \\ \frac{\sin x}{\cos x} &= 1 \\ \tan x &= 1 \\ x &= \tan^{-1} 1 \\ x &= \frac{\pi}{4} \end{aligned}$$

Therefore, all the solutions are $x = \frac{\pi}{4} + n\pi$.

Example 28 *Solve* $\sin^2 x - \sin x - 2 = 0$
Here, we notice that this is an equation involving $\sin x$ *only. So, we proceed in two steps. First, we solve for* $\sin x$. *Then, we solve for* x.

- *Solving for* $\sin x$. *To make things more readable, we let* $y = \sin x$. *The equation becomes* $y^2 - y - 2 = 0$. *This is a quadratic equation in* y *that we can solve easily.*

  $$\begin{aligned} y^2 - y - 2 &= 0 \\ (y-2)(y+1) &= 0 \\ y &= 2 \text{ or } y = -1 \end{aligned}$$

  We must solve $\sin x = 2$ *and* $\sin x = -1$.

- *Solution of* $\sin x = 2$. *Since the range of* $\sin x$ *is* $[-1, 1]$, $\sin x = 2$ *has no solutions.*

- *Solution of* $\sin x = -1$.

  $$\begin{aligned} \sin x &= -1 \\ x &= \sin^{-1}(-1) \\ x &= -\frac{\pi}{2} \end{aligned}$$

  Therefore, all the solutions are $x = -\frac{\pi}{2} + 2n\pi$ *and* $x = \pi - \left(-\frac{\pi}{2}\right) + 2n\pi$. *In other words,* $x = -\frac{\pi}{2} + 2n\pi$ *and* $x = \frac{3\pi}{2} + 2n\pi$. *These two formulas generate the same solutions. Therefore, the solutions to* $\sin^2 x - \sin x - 2 = 0$ *are* $x = -\frac{\pi}{2} + 2n\pi$.

2.5 Practice Problems

Do # 1, 3, 5, 7, 9, 11, 13 on page 561.