Investigating Graphs of Polynomial Functions

Warm Up

Lesson Presentation

Lesson Quiz

Warm Up

Identify all the real roots of each equation.

1. $x^3 - 7x^2 + 8x + 16 = 0$ $-1, 4$

2. $2x^3 - 14x - 12 = 0$ $-1, -2, 3$

3. $x^4 + x^3 - 25x^2 - 27x = 0$ 0

4. $x^4 - 26x^2 + 25 = 0$ $1, -1, 5, -5$

Objectives

Use properties of end behavior to analyze, describe, and graph polynomial functions.

Identify and use maxima and minima of polynomial functions to solve problems.

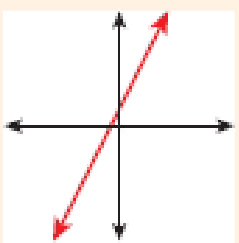
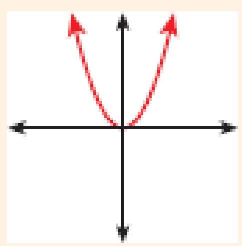
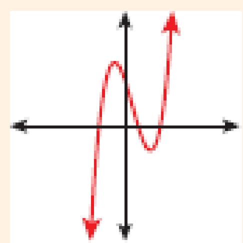
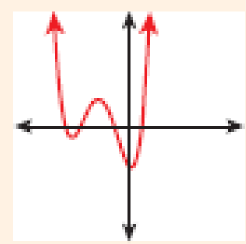
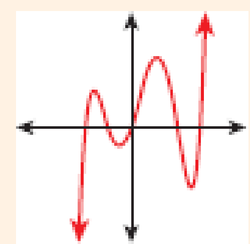
Vocabulary

end behavior
turning point
local maximum
local minimum

Investigating Graphs of Polynomial Functions

Polynomial functions are classified by their degree. The graphs of polynomial functions are classified by the degree of the polynomial. Each graph, based on the degree, has a distinctive shape and characteristics.

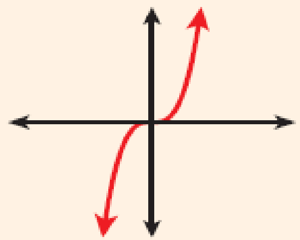
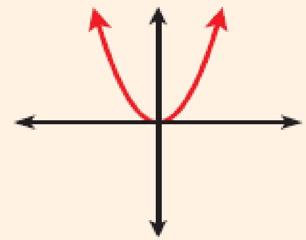
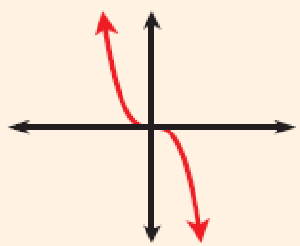
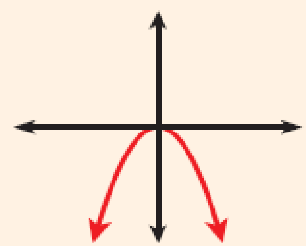
Graphs of Polynomial Functions

Linear function Degree 1	Quadratic function Degree 2	Cubic function Degree 3	Quartic function Degree 4	Quintic function Degree 5
				

End behavior is a description of the values of the function as x approaches infinity ($x \longrightarrow +\infty$) or negative infinity ($x \longrightarrow -\infty$). The degree and leading coefficient of a polynomial function determine its end behavior. It is helpful when you are graphing a polynomial function to know about the end behavior of the function.

Investigating Graphs of Polynomial Functions

Polynomial End Behavior

***P*(*x*) has...**	**Odd Degree**	**Even Degree**
Leading coefficient $a > 0$	As $x \to +\infty$, $P(x) \to +\infty$  As $x \to -\infty$, $P(x) \to -\infty$	As $x \to -\infty$, $P(x) \to +\infty$  As $x \to +\infty$, $P(x) \to +\infty$
Leading coefficient $a < 0$	As $x \to -\infty$, $P(x) \to +\infty$  As $x \to +\infty$, $P(x) \to -\infty$	As $x \to -\infty$, $P(x) \to -\infty$  As $x \to +\infty$, $P(x) \to -\infty$

Example 1: Determining End Behavior of Polynomial Functions

Identify the leading coefficient, degree, and end behavior.

A. $Q(x) = -x^4 + 6x^3 - x + 9$

The leading coefficient is –1, which is negative.

The degree is 4, which is even.

As $x \to -\infty$, $P(x) \to -\infty$, and as $x \to +\infty$, $P(x) \to -\infty$.

B. $P(x) = 2x^5 + 6x^4 - x + 4$

The leading coefficient is 2, which is positive.

The degree is 5, which is odd.

As $x \to -\infty$, $P(x) \to -\infty$, and as $x \to +\infty$, $P(x) \to +\infty$.

Check It Out! Example 1

Identify the leading coefficient, degree, and end behavior.

a. $P(x) = 2x^5 + 3x^2 - 4x - 1$

The leading coefficient is 2, which is positive.

The degree is 5, which is odd.

As $x \rightarrow -\infty$, $P(x) \rightarrow -\infty$, and as $x \rightarrow +\infty$, $P(x) \rightarrow +\infty$.

b. $S(x) = -3x^2 + x + 1$

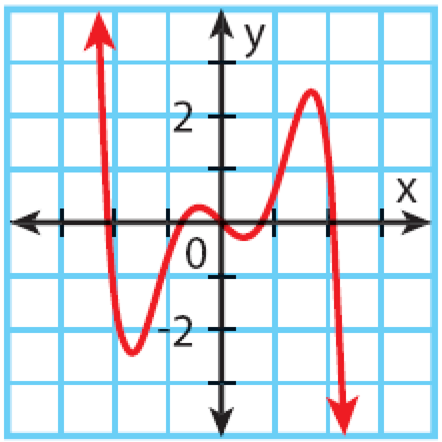
The leading coefficient is –3, which is negative.

The degree is 2, which is even.

As $x \rightarrow -\infty$, $P(x) \rightarrow -\infty$, and as $x \rightarrow +\infty$, $P(x) \rightarrow -\infty$.

Example 2A: Using Graphs to Analyze Polynomial Functions

Identify whether the function graphed has an odd or even degree and a positive or negative leading coefficient.

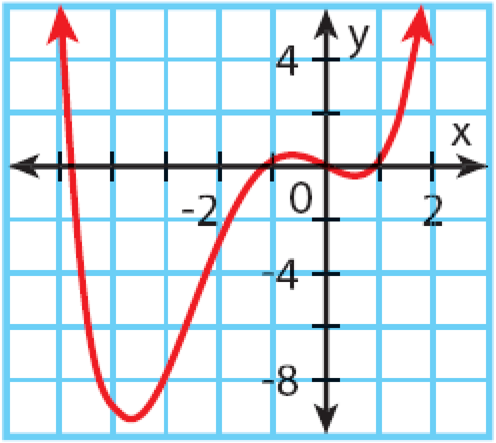


As $x \to -\infty$, $P(x) \to +\infty$, and as $x \to +\infty$, $P(x) \to -\infty$.

$P(x)$ is of odd degree with a negative leading coefficient.

Example 2B: Using Graphs to Analyze Polynomial Functions

Identify whether the function graphed has an odd or even degree and a positive or negative leading coefficient.

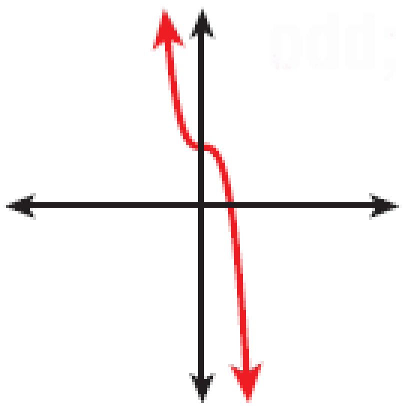


As $x \rightarrow -\infty$, $P(x) \rightarrow +\infty$, and as $x \rightarrow +\infty$, $P(x) \rightarrow +\infty$.

$P(x)$ is of even degree with a positive leading coefficient.

Check It Out! Example 2a

Identify whether the function graphed has an odd or even degree and a positive or negative leading coefficient.

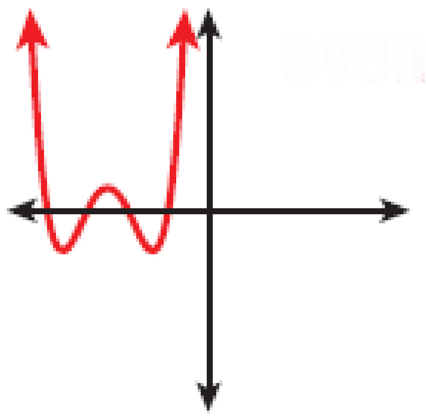


As $x \to -\infty$, $P(x) \to +\infty$, and as $x \to +\infty$, $P(x) \to -\infty$.

$P(x)$ is of odd degree with a negative leading coefficient.

Check It Out! Example 2b

Identify whether the function graphed has an odd or even degree and a positive or negative leading coefficient.



As $x \to -\infty$, $P(x) \to +\infty$, and as $x \to +\infty$, $P(x) \to +\infty$.

$P(x)$ is of even degree with a positive leading coefficient.

Now that you have studied factoring, solving polynomial equations, and end behavior, you can graph a polynomial function.

Steps for Graphing a Polynomial Function
1. Find the real zeros and *y*-intercept of the function.
2. Plot the *x*- and *y*-intercepts.
3. Make a table for several *x*-values that lie between the real zeros.
4. Plot the points from your table.
5. Determine the end behavior of the graph.
6. Sketch the graph.

Example 3: Graphing Polynomial Functions

Graph the function. $f(x) = x^3 + 4x^2 + x - 6$**.**

Step 1 Identify the possible rational roots by using the Rational Root Theorem.

±1, ±2, ±3, ±6 *p = –6, and q = 1.*

Step 2 Test all possible rational zeros until a zero is identified.

Test $x = -1$.

$$\begin{array}{r|rrrr} -1 & 1 & 4 & 1 & -6 \\ & & -1 & -3 & 2 \\ \hline & 1 & 3 & -2 & \boxed{-4} \end{array}$$

Test $x = 1$.

$$\begin{array}{r|rrrr} 1 & 1 & 4 & 1 & -6 \\ & & 1 & 5 & 6 \\ \hline & 1 & 5 & 6 & \boxed{0} \end{array}$$

$x = 1$ is a zero, and $f(x) = (x - 1)(x^2 + 5x + 6)$.

Example 3 Continued

Step 3 Write the equation in factored form.

Factor: $f(x) = (x - 1)(x + 2)(x + 3)$

The zeros are 1, –2, and –3.

Step 4 Plot other points as guidelines.

$f(0) = -6$, so the y-intercept is –6. Plot points between the zeros. Choose $x = -\frac{5}{2}$, and $x = -1$ for simple calculations.

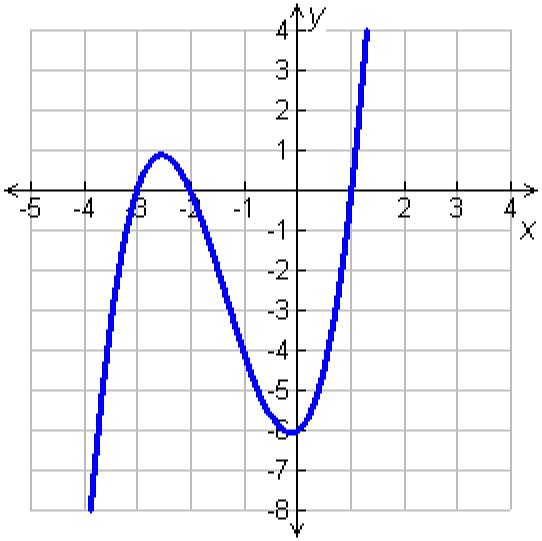
$f(\frac{5}{2}) = 0.875$, and $f(-1) = -4$.

Example 3 Continued

Step 5 Identify end behavior.

The degree is odd and the leading coefficient is positive so as $x \to -\infty$, $P(x) \to -\infty$, and as $x \to +\infty$, $P(x) \to +\infty$.

Step 6 Sketch the graph of $f(x) = x^3 + 4x^2 + x - 6$ by using all of the information about $f(x)$.



Check It Out! Example 3a

Graph the function. $f(x) = x^3 - 2x^2 - 5x + 6$.

Step 1 Identify the possible rational roots by using the Rational Root Theorem.

$\pm 1, \pm 2, \pm 3, \pm 6$ *p = 6, and q = 1.*

Step 2 Test all possible rational zeros until a zero is identified.

Test $x = -1$.

$$\begin{array}{r|rrrr} -1 & 1 & -2 & -5 & 6 \\ & & -1 & 3 & 2 \\ \hline & 1 & -3 & -2 & \boxed{8} \end{array}$$

Test $x = 1$.

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -5 & 6 \\ & & 1 & -1 & -6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array}$$

$x = 1$ is a zero, and $f(x) = (x - 1)(x^2 - x - 6)$.

Check It Out! Example 3a Continued

Step 3 Write the equation in factored form.

Factor: $f(x) = (x - 1)(x + 2)(x - 3)$

The zeros are 1, –2, and 3.

Step 4 Plot other points as guidelines.

$f(0) = 6$, so the y-intercept is 6. Plot points between the zeros. Choose $x = -1$, and $x = 2$ for simple calculations.

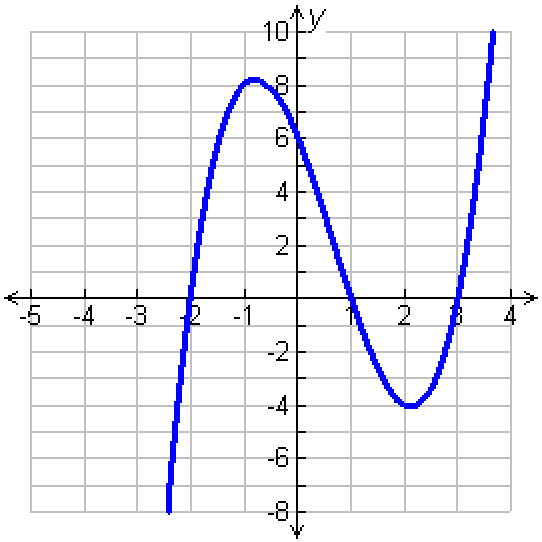
$f(-1) = 8$, and $f(2) = -4$.

Check It Out! Example 3a Continued

Step 5 Identify end behavior.

The degree is odd and the leading coefficient is positive so as $x \to -\infty$, $P(x) \to -\infty$, and as $x \to +\infty$, $P(x) \to +\infty$.

Step 6 Sketch the graph of $f(x) = x^3 - 2x^2 - 5x + 6$ by using all of the information about $f(x)$.



Check It Out! Example 3b

Graph the function. $f(x) = -2x^2 - x + 6$.

Step 1 Identify the possible rational roots by using the Rational Root Theorem.

±1, ±2, ±3, ±6 *p = 6, and q = −2.*

Step 2 Test all possible rational zeros until a zero is identified.

Test $x = -2$.

$$\begin{array}{r|rrr} -2 & -2 & -1 & 6 \\ & & 4 & -6 \\ \hline & -2 & 3 & \boxed{0} \end{array}$$

$x = -2$ is a zero, and $f(x) = (x + 2)(-2x + 3)$.

Check It Out! Example 3b Continued

Step 3 The equation is in factored form.

Factor: $f(x) = (x + 2)(-2x + 3)$.

The zeros are –2, and $\frac{3}{2}$.

Step 4 Plot other points as guidelines.

$f(0) = 6$, so the y-intercept is 6. Plot points between the zeros. Choose $x = -1$, and $x = 1$ for simple calculations.

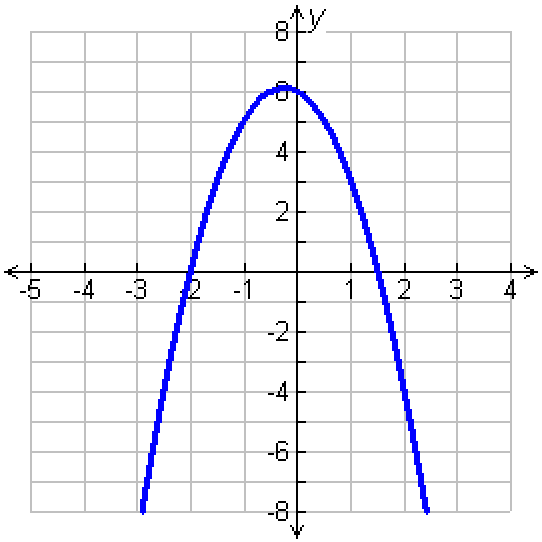
$f(-1) = 5$, and $f(1) = 3$.

Check It Out! Example 3b Continued

Step 5 Identify end behavior.

The degree is even and the leading coefficient is negative so as $x \rightarrow -\infty$, $P(x) \rightarrow -\infty$, and as $x \rightarrow +\infty$, $P(x) \rightarrow -\infty$.

Step 6 Sketch the graph of $f(x) = -2x^2 - x + 6$ by using all of the information about $f(x)$.



A **turning point** is where a graph changes from increasing to decreasing or from decreasing to increasing. A turning point corresponds to a *local maximum* or *minimum*.

Local Maxima and Minima

For a function $f(x)$, $f(a)$ is a **local maximum** if there is an interval around a such that $f(x) < f(a)$ for every x-value in the interval except a.

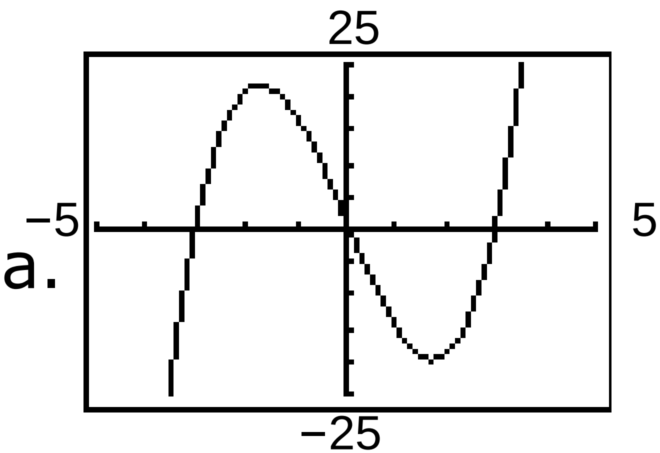
For a function $f(x)$, $f(a)$ is a **local minimum** if there is an interval around a such that $f(x) > f(a)$ for every x-value in the interval except a.

A polynomial function of degree n has at most $n – 1$ turning points and at most n x-intercepts. If the function has n distinct roots, then it has exactly $n – 1$ turning points and exactly n x-intercepts. You can use a graphing calculator to graph and estimate maximum and minimum values.

Example 4: Determine Maxima and Minima with a Calculator

Graph $f(x) = 2x^3 - 18x + 1$ on a calculator, and estimate the local maxima and minima.

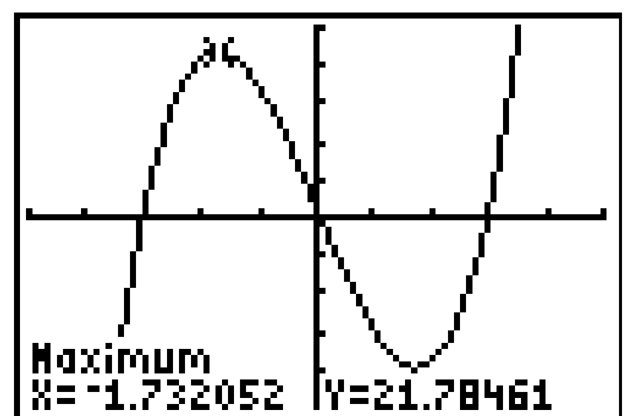
Step 1 Graph.



The graph appears to have one local maxima and one local minima.

Step 2 Find the maximum.

Press 2nd CALC TRACE to access the **CALC** menu. Choose **4:maximum**.The local maximum is approximately 21.7846.

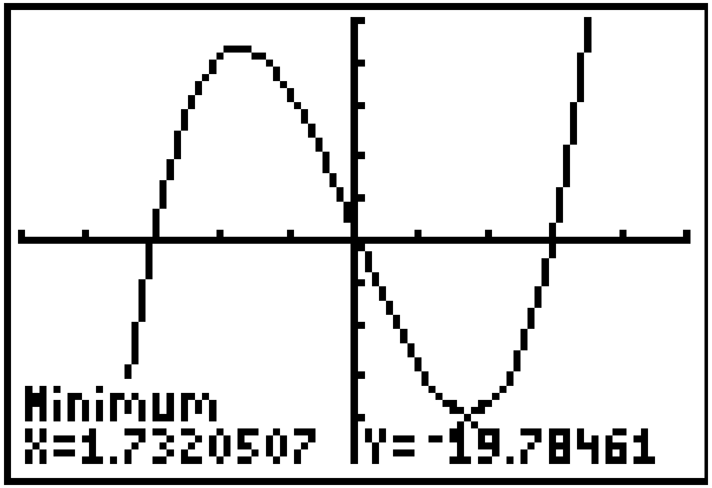


Example 4 Continued

Graph $f(x) = 2x^3 – 18x + 1$ on a calculator, and estimate the local maxima and minima.

Step 3 Find the minimum.

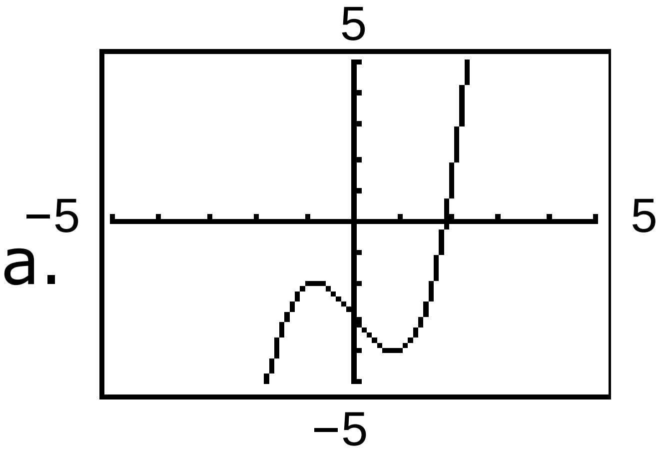
Press 2nd TRACE (CALC) to access the **CALC** menu. Choose **3:minimum**.The local minimum is approximately –19.7846.



Check It Out! Example 4a

Graph $g(x) = x^3 - 2x - 3$ on a calculator, and estimate the local maxima and minima.

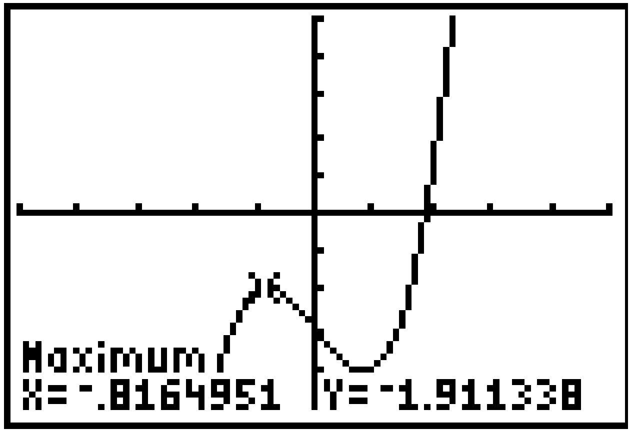
Step 1 Graph.



The graph appears to have one local maxima and one local minima.

Step 2 Find the maximum.

Press 2nd CALC TRACE to access the **CALC** menu. Choose **4:maximum**.The local maximum is approximately –1.9113.

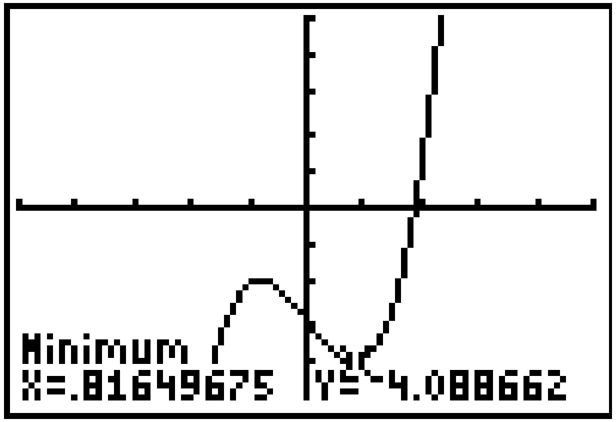


Check It Out! Example 4a Continued

Graph $g(x) = x^3 – 2x – 3$ on a calculator, and estimate the local maxima and minima.

Step 3 Find the minimum.

Press 2nd TRACE (CALC) to access the **CALC** menu. Choose **3:minimum**.The local minimum is approximately –4.0887.



Check It Out! Example 4b

Graph $h(x) = x^4 + 4x^2 - 6$ on a calculator, and estimate the local maxima and minima.

Step 1 Graph.

The graph appears to have one local maxima and one local minima.

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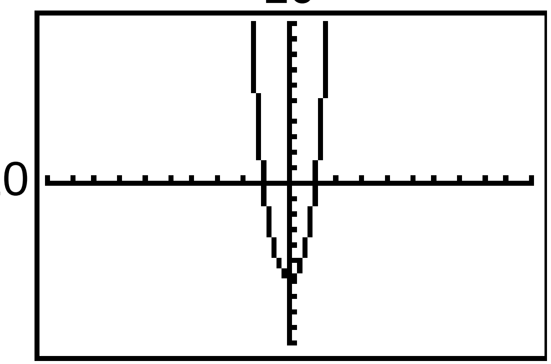
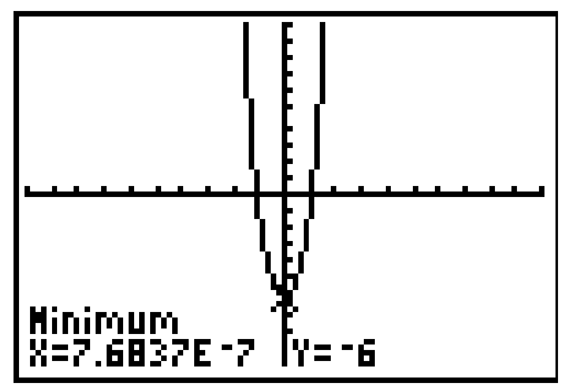
−10  10

−10

Step 2 There appears to be no maximum.

Step 3 Find the minimum.

Press 2nd CALC TRACE to access the **CALC** menu. Choose **3:minimum**. The local minimum is −6.



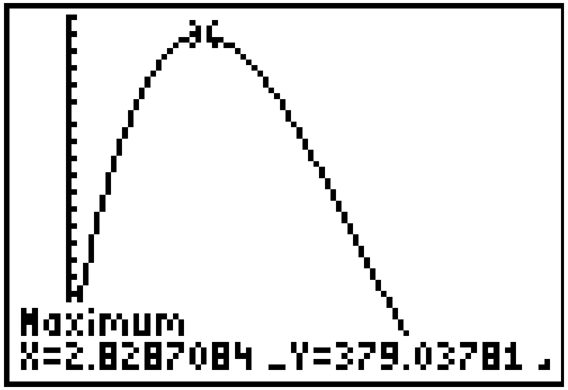
Example 5: Art Application

An artist plans to construct an open box from a 15 in. by 20 in. sheet of metal by cutting squares from the corners and folding up the sides. Find the maximum volume of the box and the corresponding dimensions.

Find a formula to represent the volume.

$V(x) = x(15 - 2x)(20 - 2x)$ $V = lwh$

Graph $V(x)$. Note that values of x greater than 7.5 or less than 0 do not make sense for this problem.



The graph has a local maximum of about 379.04 when $x \approx 2.83$. So the largest open box will have dimensions of 2.83 in. by 9.34 in. by 14.34 in. and a volume of 379.04 in^3.

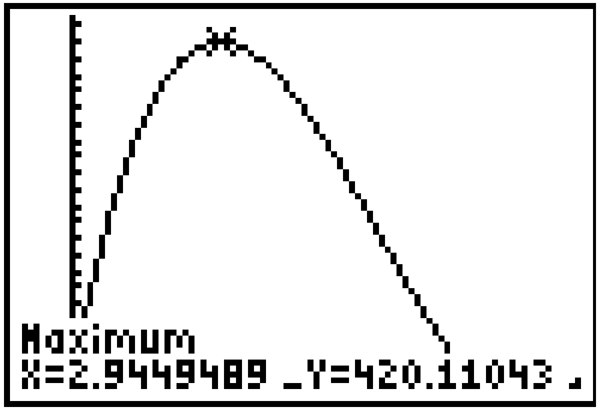
Check It Out! Example 5

A welder plans to construct an open box from a 16 ft. by 20 ft. sheet of metal by cutting squares from the corners and folding up the sides. Find the maximum volume of the box and the corresponding dimensions.

Find a formula to represent the volume.

$V(x) = x(16 - 2x)(20 - 2x)$ $V = lwh$

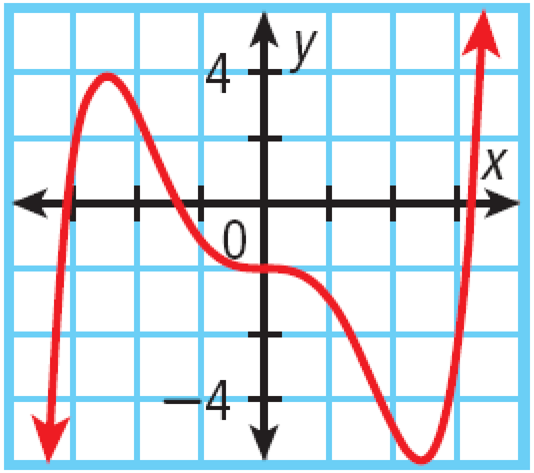
Graph $V(x)$. Note that values of x greater than 8 or less than 0 do not make sense for this problem.



The graph has a local maximum of about 420.11 when $x \approx 2.94$. So the largest open box will have dimensions of 2.94 ft by 10.12 ft by 14.12 ft and a volume of 420.11 ft^3.

Lesson Quiz: Part I

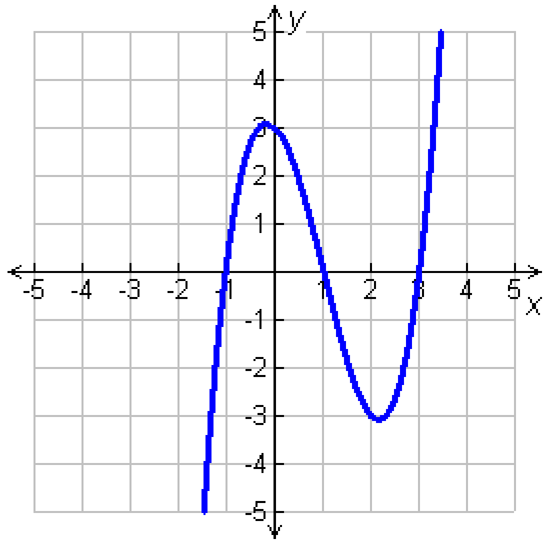
1. Identify whether the function graphed has an odd or even degree and a positive or negative leading coefficient.



odd; positive

Lesson Quiz: Part II

2. Graph the function $f(x) = x^3 - 3x^2 - x + 3$.



3. Estimate the local maxima and minima of $f(x) = x^3 - 15x - 2$. 20.3607; –24.3607