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This programed text is an introduction to the algebra of complex numbers for engineering students, particularly because of its relevance to important problems of applications in electrical engineering. It is designed for a person who is well experienced with the algebra of real numbers and calculus, but who has no experience with complex number algebra. The main ideas in this programed text are (1) the origin of complex numbers, (2) graphical interpretation of the quadratic function whose solutions are complex numbers, (3) fundamental operations with complex numbers, (4) complex numbers of trigonometric functions of sin x and cos x, (5) exponential form of a complex number, (6) geometrical interpretation of a complex number using vectors in the plane, (7) differentiation of six ax and cos ax, and (8) finding the nth root of a complex number. (RP)

No. 8

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Complex numbers have important applications in many fields of engineering, particularly electrical engineering. Paralleling the case of real numbers, there is an algebra of complex numbers, and also a calculus. This programmed text is an introduction to the former, being designed for a person who is well experienced with the algebra of real numbers, and calculus to the extent of differentiation of e^x, but who has no experience with complex number algebra. However, it will also be useful as a review, and will provide an extension of skills, for a person who has had a slight introduction to complex numbers in a course in high school "advanced algebra", or the equivalent.

From algebra, it is recognized that the equation

$$x^2 = 1$$

has $x = +1$ and $x = -1$ as its solutions. The corresponding equation

$$x^2 = -1$$

has no solutions in the realm of real numbers. However, since this equation does occur, it is convenient, in effect, to invent a solution. This means that a suitable symbol must be invented, and an interpretation provided. Accordingly, we write

$$x = j \quad \text{and} \quad x = -j$$

as the solutions, and then recognize that the symbol j has the property

$$j^2 = \_\_\_\_\_\_\_\_$$

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Answer:

$$j^2 = -1$$

In electrical literature it is customary to use j, rather than the universally used i of mathematics literature. This is done because i has become so well established as the symbol of electric current.

Due to an unfortunate historical accident, j is called the imaginary unit (another name would have been better), and numbers like 2j (or j2) 3j, etc. are called imaginary numbers. Imaginary numbers are imaginary only in the sense that they cannot be combined in the ordinary way with real numbers. Now, using the idea of the imaginary unit (j), the solutions of

$$x^2 = -9$$

$$u^2 = -6$$

$$y^2 = -100$$

are

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<u>Answer</u>:

$x = j3$
$u = j\sqrt{6}$
$y = j10$

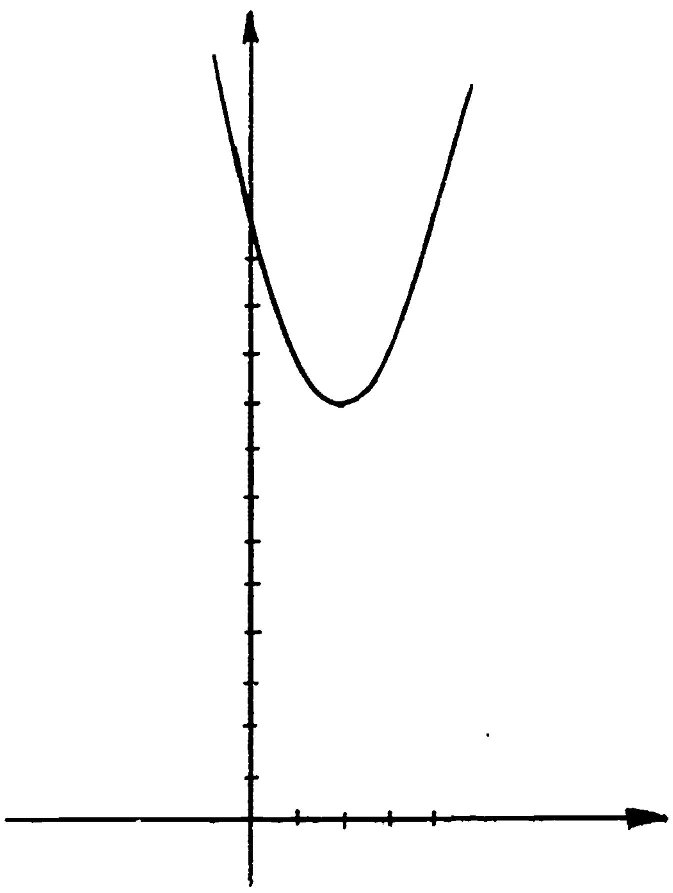

Figure 1.

Now consider the function of x

$$f(x) = x^2 - 4x + 13$$

which is plotted in Fig. 1. Observe that the graph does not cross the horizontal axis (i.e., the function never becomes zero). Thus, there are no real values of x for which

$$x^2 - 4x + 13 = 0$$

In other words, the equation has no real solutions. Now let us write the equation in modified form by "completing the square", as

$$x^2 - 4x + 4 = -9$$

or

$$(\qquad)^2 = -9$$

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Answer:

$$(x-2)^2 = -9$$

Thus, if

$$x - 2 = \pm j3$$

it will satisfy the equation $(x-2)^2 = -9$. However, we want values of x (not x - 2) that will satisfy $x^2 - 4x + 13 = 0$. It would seem that

$$x = 2 + j3 \quad \text{and} \quad x = 2 - j3$$

are these values. However, a combination like 2 + j3 is a new entity. How shall we deal with it in substituting it for x in $x^2 - 4x + 13$? That is, what are $(2 + j3)^2$ and $-4(2 + j3)$?

This is a matter for <u>definition</u>. We define addition and multiplication of these quantities to be the same as for binomials of <u>real</u> numbers. For example, $4(a + b) = 4a + 4b$, and $(a + b)^2 = a^2 + 2ab + b^2$. Thus, by definition,

$$-4(2 + j3) = \underline{\qquad\qquad\qquad}$$

$$(2 + j3)^2 = \underline{\qquad\qquad\qquad}$$

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Answer:

$-4(2+j3) = -8-j12$

$(2+j3)^2 = 4+j12-9 = -5+j12$

Thus,

$(2+j3)^2-4(2+j3)+13 = -5+j12-8-j12+13 = 0$

In a similar way, check that

$$x = 2 - j3$$

is also a solution of $x^2 - 4x + 13 = 0$. In doing so, you get

$-4(2 - j3) =$ ________________

$(2 - j3)^2 =$ ________________

Therefore, $x^2 - 4x + 13 =$ ________________________

10

<u>Answer</u>:

$$-4x = -8+j12$$

$$x^2 = -5-j12$$

$$x^2-4x+13 = 0$$

It was pointed out earlier that numbers like j3, 2j, etc. are called <u>imaginary numbers</u>, and that j is the imaginary unit. In the case of <u>real</u> numbers, there is a corresponding unit which, however, we do not write. To make the point, fill in the parentheses below with what you believe to be the "real unit" symbol to represent what is usually written merely as 2.

$$2 = (\quad) 2$$

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Answer:

$$2 = (1)2$$

Thus $2 + j3$ is really

$$(1)2 + (j)3$$

meaning 2 units of real number and 3 units of imaginary number.

Continuing to use the example $2 + j3 = (1)2 + (j)3$, the multiplier of (1) is called the real part, and the multiplier of (j) the imaginary part. (Note that the "imaginary part" is a real number.)

What are the real and imaginary parts of

Number	Real part	Imaginary part
$2 - j3$		
$j(2 + j3)$		
j^2	0	

Answer:

Number	Real Part	Imaginary Part
2-j3	2	-3
j(2+j3)	-3	2
j^2	-1	0

If you got any of these wrong, go to page 15.

If you got them all correct, go to page 17.

If you said 3 is the imaginary part of $2 - j3$, note that $2 - j3 = 2 + j(-3)$. Hence, (-3) is the imaginary part.

If you got the second example wrong, you might have neglected to observe that

$$j(2 + j3) = -3 + j2$$

Now find the real and imaginary parts of

j^3

$j + j^2$

$(1 + j)^2$

Answer:

Real	Imag.	
0,	-1 ;	$j^3 = j(j^2) = -j$
-1,	1 ;	$j + j^2 = -1 + j$
0,	2 ;	$(1+j)^2 = 1 + 2j + j^2 = 1 - 1 + j2 = j2$

If you got any of these wrong, seek help, or at least be sure you understand your error and are certain you will not make the same mistake again. Then proceed with page 17.

You are now beginning to accept the notion of a pair of numbers (a real number and an imaginary number) being a single entity; a complex number. It is customary to use a single symbol to represent a complex number, like $\bar{A} = 2 + j3$, $\bar{B} = 3 + j4$, etc. In this text, a bar is placed above the symbol to indicate that it represents a complex number. If a complex number $\bar{A}$ has a real part A_1 and an imaginary part A_2, write the number

$$\bar{A} =$$

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<u>Answer</u>:

$$A = A_1+jA_2$$

If you wrote A_1+A_2, you forgot that A_2 is real; it is the <u>multiplier</u> of j.

The general form we have been using ($\bar{A} = A_1 + jA_2$) is said to be the <u>rectangular form</u>. We have also said that algebraic manipulations are <u>defined</u> to be those that follow from treating $A_1 + jA_2$ as a binomial.

What do you get if you multiply the following, according to this rule?

$(1+j2)(2+j3) =$ ______________________

$(1-j2)(-2+j3) =$ ______________________

$(-1+j2)(2+j3) =$ ______________________

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<u>Answer</u>:

$$(1+j2)(2+j3) = -4+j7$$

$$(1-j2)(-2+j3) = 4+j7$$

$$(-1+j2)(2+j3) = -8+j$$

If we consider the literal case of two general complex numbers

$$\bar{A} = A_1 + jA_2, \ \bar{B} = B_1 + jB_2$$

what are the real and imaginary parts of $\overline{AB}$?

Real part of $\overline{AB}$ = ________________

Imaginary part of $\overline{AB}$ = ________________

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Answer:

Real part = $A_1B_1 - A_2B_2$

Imaginary part = $A_2B_1 + A_1B_2$

That is, $\overline{AB} = (A_1+jA_2)(B_1+jB_2) = (A_1B_1-A_2B_2) + j(A_2B_1+A_1B_2)$

Let us investigate whether this definition of multiplication is consistent with the theory of polynomials which says that if x_1 and x_2 are the two roots of $x^2 + ax + b = 0$, then

$$x^2 + ax + b = (x - x_1)(x - x_2)$$

Earlier, we found that $x_1 = 2 + j3$ and $x_2 = 2 - j3$ were the two complex roots of the numerical example $x^2 - 4x + 13 = 0$. Let us use the expression above on this example, for which

$$(x - x_1)(x - x_2) = [x - (2 + j3)][x - (2 - j3)]$$

Expand the right-hand side to get coefficients a and b.

a =

b =

Answer:

$a = -4$

$b = 13$

To get this, write

$$[x-(2+j3)][x-(2-j3)] = x^2-x(2-j3+2+j3) + (2+j3)(2-j3)$$

$$= x^2-4x+4-j^2 9+j(6-6) = x^2-4x+13$$

Observe that this was the polynomial from which we originally started.

Division is defined as the inverse of multiplication. That is, if $A_1 + jA_2$ is a symbol for the quotient

$$A_1 + jA_2 = \frac{1 + j2}{2 + j3}$$

we mean that

$$(A_1 + jA_2)(2 + j3) = 1 + j2$$

Finding the quotient consists of finding A_1 and A_2 such that the product on the left yields $1 + j2$. This is most easily done by multiplying both sides of the equation by $(2 - j3)$, because $(2 + j3)(2 - j3) = 4 + 9 = 13$, a real number. Thus, we get

$$(A_1 + jA_2)(2 + j3)(2 - j3) = (1 + j2)(2 - j3)$$

or

$$A_1 + jA_2 = \underline{\qquad\qquad\qquad\qquad}$$

<u>Answer</u>:

$$A_1+jA_2 = \frac{8}{13} + j\,\frac{1}{13}$$

An alternative sequence of steps is to multiply numerator and denominator of the original fraction by (2-j3). Then, since (2+j3)(2-j3) = 13 is a real number, the desired quotient is obtained in the rectangular form. Thus,

$$A_1+jA_2 = \frac{1+j2}{2+j3} = \frac{(1+j2)(2-j3)}{(2+j3)(2-j3)} = \frac{8+j}{4+9} = \frac{8}{13} + j\,\frac{1}{13}$$

This process is called <u>rationalizing</u>.

Division is an important algebraic manipulation. You had better try the following examples:

$$\frac{1}{3+j4} =$$

$$\frac{2+j2}{2-j2} =$$

$$\frac{1}{j} =$$

Answer:

$\frac{3}{25} - j\,\frac{4}{25}$

j

-j

The number $2 - j3$, used as the multiplier of $2 + j3$ to obtain the real number 13, is called the <u>complex conjugate</u> of $2 + j3$. In general, given a complex number $\overline{B} = B_1 + jB_2$, where B_1 and B_2 are real, the complex conjugate is

$$\overline{B}^* = B_1 - jB_2$$

where the * symbol means the complex conjugate of $\overline{B}$. The conjugate is useful because

$$\overline{B}\,\overline{B}^*$$

is <u>always</u> a real number (has zero imaginary part). Prove this by forming the product

$$\overline{B}\,\overline{B}^* = \underline{\hspace{4cm}}$$

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<u>Answer</u>:

$$(B_1+jB_2)(B_1-jB_2) = B_1^2 + j(B_2B_1-B_1B_2) + B_2^2 = B_1^2 + B_2^2$$

B_1 and B_2 are real, hence $B_1^2 + B_2^2$ is real.

Let us summarize the rules of algebra, in literal form for the two complex numbers

$$\overline{A} = A_1 + jA_2$$

$$\overline{B} = B_1 + jB_2$$

and the real number C. These are

$C\overline{A}$ = ______________

$\overline{A} + \overline{B}$ = ______________

$\overline{AB}$ = ________________

(Write out the answers in rectangular form.)

Answer:

Check your answers, carefully, for signs!

$C\overline{A} = CA_1 + jCA_2$

$\overline{A} + \overline{B} = A_1 + B_1 + j(A_2 + B_2)$

$\overline{AB} = A_1B_1 - A_2B_2 + j(A_2B_1 + A_1B_2)$

If you did not get the answer for $\overline{AB}$, go to page 33.

If you got it, go to page 35.

Perhaps you got $A_1B_1 + A_2B_2$ for the real part of $\overline{AB}$. If so, you forgot that $j^2 = -1$. Writing out the work in detail,

$$\overline{AB} = (A_1+jA_2)(B_1+jB_2) = A_1B_1 + j^2A_2B_2 + jA_2B_1 + jA_1B_2$$

$$= A_1B_1 - A_2B_2 + j(A_2B_1 + A_1B_2)$$

As another check, do the case

$(c + jd)(e - jf) =$

<u>Answer</u>:

$$ce + df + j(de-cf)$$

If you don't get this, go back to the other side, or seek help.

Then go to page 35.

$$\frac{1+j2}{2+j3} = \frac{(1+j2)(2-j3)}{(2+j3)(2-j3)}$$

$$= \frac{2+6+j(4-3)}{4+9}$$

$$= \frac{8}{13} + j\,\frac{1}{13}$$

Now that you can multiply two numbers in literal form, let's continue the summary by trying an example of division. The earlier numerical example is repeated on the right of page 34, for reference. Using the same principle, if

$$\overline{A} = A_1 + jA_2 \quad , \quad \overline{B} = B_1 + jB_2$$

what is $\frac{\overline{A}}{\overline{B}}$?

$$\frac{\overline{A}}{\overline{B}} =$$

Answer:

$$\frac{A_1B_1+A_2B_2+j(A_2B_1-A_1B_2)}{B_1^2 + B_2^2}$$

If you did not get this, go to page 37.

If you got it, go to page 39.

The most probable errors are in signs, or perhaps the denominator still has a j in it. Let's work it out in detail.

$$\frac{A_1+jA_2}{B_1+jB_2} = \frac{(A_1+jA_2)(B_1-jB_2)}{(B_1+jB_2)(B_1-jB_2)} = \frac{A_1B_1 + A_2B_2 + j(A_2B_1-A_1B_2)}{B_1{}^2 + \cancel{jB_2B_1} - \cancel{jB_2B_1} + B_2{}^2}$$

Now do

$$\frac{c + jd}{e - jf} =$$

and state what is the complex conjugate of e - jf.

Answer:

$$\frac{ce - df + j(de+cf)}{e^2 + f^2}$$

conjugate of e-jf is e+jf.

If you had trouble with this, go back to the other side, or seek help.

Then go to page 39.

For further practice with the algebra of complex numbers, find the following, clearly identifying the real and imaginary parts. Use the notation $\overline{A} = A_1+jA_2$ and $\overline{B} = B_1+jB_2$.

1) $j\overline{A}$

2) $\overline{A} + j\overline{B}$

3) $(\overline{A}+j\overline{B})(\overline{A}-j\overline{B})$

Answer:

1) $j\bar{A} = -A_2 + jA_1$

2) $\bar{A}+j\bar{B} = A_1-B_2+j(A_2+B_1)$

3) $(\bar{A}+j\bar{B})(\bar{A}-j\bar{B}) = [A_1-B_2+j(A_2+B_1)][A_1+B_2+j(A_2-B_2)] = A_1^2 - A_2^2 + B_1^2 - B_2^2 + j2(A_1A_2+B_1B_2)$

If you did not get all of these, go to page 41.
If you did, go to page 43.

If you missed (1) or (2), you forgot what we mean by the real and imaginary parts. They must be real numbers.

In (3) you may have written

$$(\bar{A}+j\bar{B})(\bar{A}-j\bar{B}) = \bar{A}^2 + \bar{B}^2$$

This is correct as far as you have gone, but the real and imaginary parts have not been identified, because $\bar{A}^2$ and $\bar{B}^2$ are complex. What are they?

$$\bar{A}^2 =$$

$$\bar{B}^2 =$$

<u>Answer</u>:

$$\bar{A}^2 = (A_1+jA_2)^2$$

$$= A_1^2-A_2^2+j2A_1A_2$$

$$\bar{B}^2 = (B_1+jB_2)^2$$

$$= B_1^2-B_2^2+j2B_1B_2$$

Therefore, $\bar{A}^2+\bar{B}^2 = A_1^2-A_2^2+B_1^2-B_2^2+j2(A_1A_2+B_1B_2)$ as shown on page 40.

Go to page 43.

As one more exercise, this time on division. If $\overline{A}$ and $\overline{B}$ are the same as before, and $\overline{C} = C_1 + jC_2$, what is

$$\frac{\overline{C}}{\overline{A}+j\overline{B}} = \qquad ?$$

Answer:

$$\frac{C_1(A_1+B_1)+C_2(A_2+B_2)+j[C_2(A_1+B_1)-C_1(A_2+B_2)]}{(A_1+B_1)^2+(A_2+B_2)^2}$$

If you did not get this, go to page 45.

If you did, go to page 47.

A common error on this problem is to think $\bar{A}-j\bar{B}$ is the complex conjugate of $\bar{A}+j\bar{B}$, leading to an inccrrect result.

What is the complex conjugate of $\bar{A}+j\bar{B}$?

$$(\bar{A}+j\bar{B})^* = \underline{\hspace{6cm}}?$$

<u>Answer</u>:

$$A_1 + B_1 - j(A_2 + B_2)$$

The complex conjugate is obtained by changing the sign on the <u>imaginary part</u>, which is not necessarily the quantity multiplying j. It <u>is</u> this coefficient only if the coefficient is real.

If you did not get this, go to page 47.
If you did, go to page 49.

What are the following complex conjugates, if $\overline{A} = A_1 + jA_2$,

$(\overline{A}^2)^* =$ ____________________

$(\overline{A}*)^* =$ ____________________ ?

<u>Answer</u>:

$$(\bar{A}^2)^* = A_1^2 - A_2^2 - j2A_1A_2$$

$$(\bar{A}^*)^* = A_1 + jA_2 = \bar{A}$$

The special complex number

$$\bar{U} = \cos\theta + j\sin\theta$$

where θ is a real quantity, has particular significance.

If θ is regarded as a real variable, the derivative of $\bar{U}$ with respect to θ is

$$\frac{d\bar{U}}{d\theta} = -\sin\theta + j\cos\theta$$

By bringing out j as a factor, this can also be written as

$$-\sin\theta + j\cos\theta = j(\qquad\qquad)$$

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Answer:

$$-\sin\theta + j\cos\theta = j(\cos\theta + j\sin\theta)$$

$$= j(\overline{U})$$

That is, $\frac{d}{d\theta}(\overline{U}) = j\overline{U}$

The derivative of this function is j times the function itself.

For real numbers a and x,

$$\frac{d}{dx}(e^{ax}) = a(e^{ax})$$

This result bears a strong resemblance to the property of the derivative of a real exponential function, e^{ax}, as shown in the REMINDER on page 50.

For the function $e^{a\theta}$, since θ is real, state in words the characteristics of its derivative with respect to θ:

____

____

____.

Answer:

the derivative of $e^{a\theta}$ with respect to θ is a times the function, or $ae^{a\theta}$.

Reminder

For real θ, and $\overline{U} = \cos\theta + j\sin\theta$, we had

$$\frac{d\overline{U}}{d\theta} = j\overline{U}$$

and, for real a and θ,

$$\frac{d(e^{a\theta})}{d\theta} = a(e^{a\theta})$$

Our important results are summarized on the right of page 52. We see that so far as differentiation is concerned, $\overline{U}$ (which is $\cos\,\theta + j\,\sin\,\theta$) has the same properties as $e^{a\theta}$, if <u>a</u> is replaced by j. This leads us to think of the symbol $e^{j\theta}$ as a representation for U. But what does it mean to have a number (e) raised to an <u>imaginary power</u>? Algebra provides <u>no</u> answer to this question. However, what we have done is to <u>define</u> $e^{j\theta}$ as $\overline{U}$. Thus, <u>by definition</u>

$$e^{j\theta} = \cos\,\theta + j\,\sin\,\theta$$

In accordance with this definition, evaluate the following:

$$e^{j0} =$$
$$e^{j\pi/3} =$$
$$e^{j\pi} =$$
$$e^{j3\pi/2} =$$

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<u>Answer</u>:

$$e^{j0} = 1 + j0$$

$$e^{j\pi/3} = \cos\frac{\pi}{3} + j\sin\frac{\pi}{3} = .5 + j\frac{\sqrt{3}}{2}$$

$$e^{j\pi} = -1 + j0$$

$$e^{j3\pi/2} = 0 - j$$

<u>Reminder</u>

$$e^{ax_1} e^{ax_2} = e^{a(x_1+x_2)}$$

for real numbers a, x_1, and x_2.

On the right of page 54 is shown the "law of exponents", whereby upon multiplication of two exponentials, the exponents add. It is important to know whether $e^{j\theta}$ has this same property, so that it may be used in the usual way. To check this, consider the two complex numbers

$$e^{j\theta_1} = \cos \theta_1 + j \sin \theta_1$$

$$e^{j\theta_2} = \cos \theta_2 + j \sin \theta_2$$

If the law of exponentials is obeyed, we will be able to prove that

$$(e^{j\theta_1})(e^{j\theta_2}) = \underline{\hspace{6cm}}$$

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Answer:

$e^{j(\theta_1+\theta_2)}$

In other words, it is required to prove

$$(e^{j\theta_1})(e^{j\theta_2}) = \cos(\theta_1+\theta_2) + j\sin(\theta_1+\theta_2)$$

The product required is

$$e^{j\theta_1} e^{j\theta_2} = (\cos \theta_1 + j \sin \theta_1)(\cos \theta_2 + j \sin \theta_2)$$

$$= \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + j(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)$$

But, we have two trigonometric identities for the sine and cosine of the sum of two angles:

$$\cos(\theta_1 + \theta_2) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$\sin(\theta_1 + \theta_2) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Therefore, you can simplify the above product to yield

$$e^{j\theta_1} e^{j\theta_2} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

<u>Answer</u>:

$$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2$$

$$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2$$

Thus,

$$e^{j\theta_1}e^{j\theta_2} = \cos(\theta_1+\theta_2) + j\,\sin(\theta_1+\theta_2) = e^{j(\theta_1+\theta_2)}$$

and <u>this</u> is what we set out to prove!

Since $e^{j\theta} = \cos\theta + j \sin\theta$, it follows that

$$Ae^{j\theta} = A\cos\theta + jA\sin\theta$$

where A is a real number. The next question to be considered is, when given a complex number

$$\overline{A} = A_1 + jA_2$$

whether it can also be written $Ae^{j\theta}$. This is possible if A and θ can be found, such that

$A\cos\theta =$ ________________ and $A\sin\theta =$ ________________

Answer:

$$A \cos \theta = A_1$$

$$A \sin \theta = A_2$$

Thus, if in $\bar{A} = A_1 + jA_2$, A_1 and A_2 are given, solution of these equations for A and θ will permit expressing the same number in the form $Ae^{j\theta}$.

Reminder

The two equations on the left give:

$$A^2\cos^2\theta = A_1{}^2$$

$$A^2\sin^2\theta = A_2{}^2$$

and

$$\tan\theta = \frac{A_2}{A_1}$$

These equations are shown in an alternate form on the right of page 60. Adding the first pair of equations gives

$$A^2(\cos^2\theta + \sin^2\theta) = A_1^2 + A_2^2$$

or, since $\cos^2\theta + \sin^2\theta = 1$ we have

$$A^2 = A_1^2 + A_2^2, \text{ or } A = \sqrt{A_1^2 + A_2^2}$$

From the last of these equations, we can write

$$\theta = \arctan \frac{A_2}{A_1}$$

$Ae^{j\theta}$ is called the <u>exponential</u> <u>form</u> of a complex number. As numerical examples, write the following in the exponential forms (example $1-j = \sqrt{2}\ e^{-j45^\circ}$).

$2 + j2 =$

$3 - j3 =$

$-\sqrt{3} + j1 =$

62

Answer:

$$2\sqrt{2}\, e^{j45^{\circ}}$$

$$3\sqrt{2}\, e^{j225^{\circ}}$$

$$2e^{j150^{\circ}}$$

More precise notation would designate angles in radians. However, in numerical cases there is no possibility of error in writing angles in degrees.

Let us summarize what we have done:

1) In order to obtain solutions for certain algebraic equations, the <u>imaginary</u> unit j was invented.
2) Other equations lead to the necessity of inventing <u>complex</u> numbers. A complex number is written (1) times real part + (j) times imaginary part, or more briefly $A_1 + jA_2$, where A_1 and A_2 are real numbers.
3) Addition and multiplication of complex numbers are <u>defined</u> as those results obtained from ordinary algebra, treating $A_1 + jA_2$ as a binomial. Division is defined as the inverse of multiplication.

4) The conjugate of $\bar{A} = A_1 + jA_2$ is written $\bar{A}^* = A_1 - jA_2$.

5) Any complex number $\bar{A} = A_1 + jA_2$ can also be written in the exponential form $\bar{A} = Ae^{j\theta}$, where

$$A = \sqrt{A_1^2 + A_2^2} \qquad \text{and} \qquad \theta = \arctan(A_2/A_1).$$

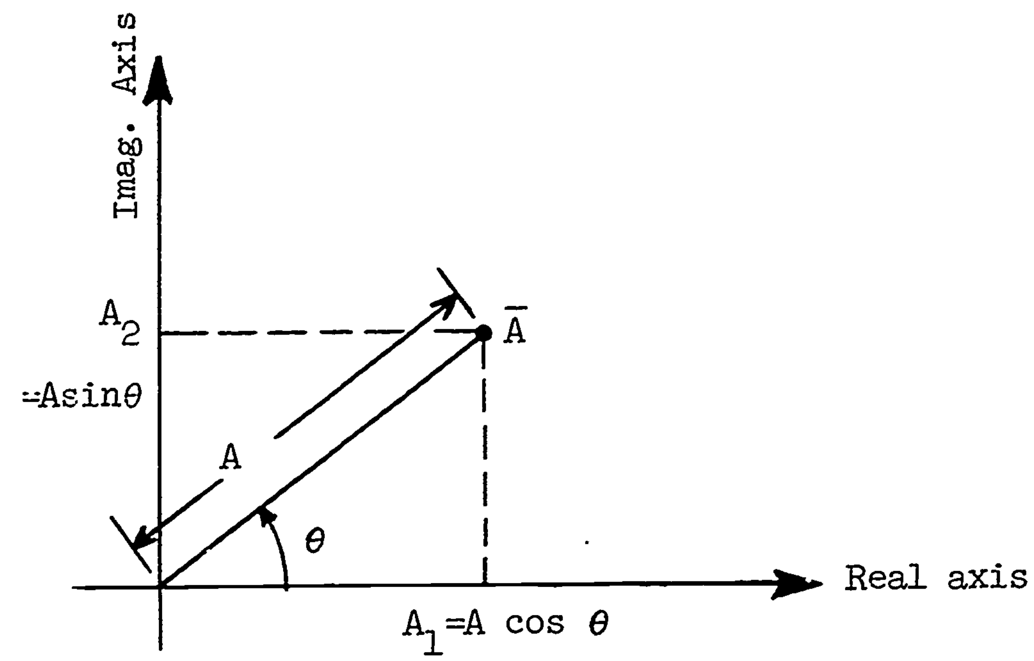


Figure 2.

All reference to a geometrical interpretation of a complex number was purposely omitted, up to this point. However, in the notation $\bar{A} = Ae^{j\theta} = A_1 + jA_2$, the fact that

$$A_1 = \underline{\hspace{2cm}} \quad \text{and} \quad A_2 = \underline{\hspace{3cm}}$$

is reminiscent of the diagram in Fig. 2, where A_1 and A_2 are plotted along perpendicular axes. They determine a point $\bar{A}$, and this point in turn determines a line of length A making an angle θ with the horizontal. A is called the magnitude of $\bar{A}$, and θ its angle. Thus, we may look upon $\bar{A}$ as a point in a plane, called the complex plane.

68

Answer:

$A_1 = A \cos \theta$

$A_2 = A \sin \theta$

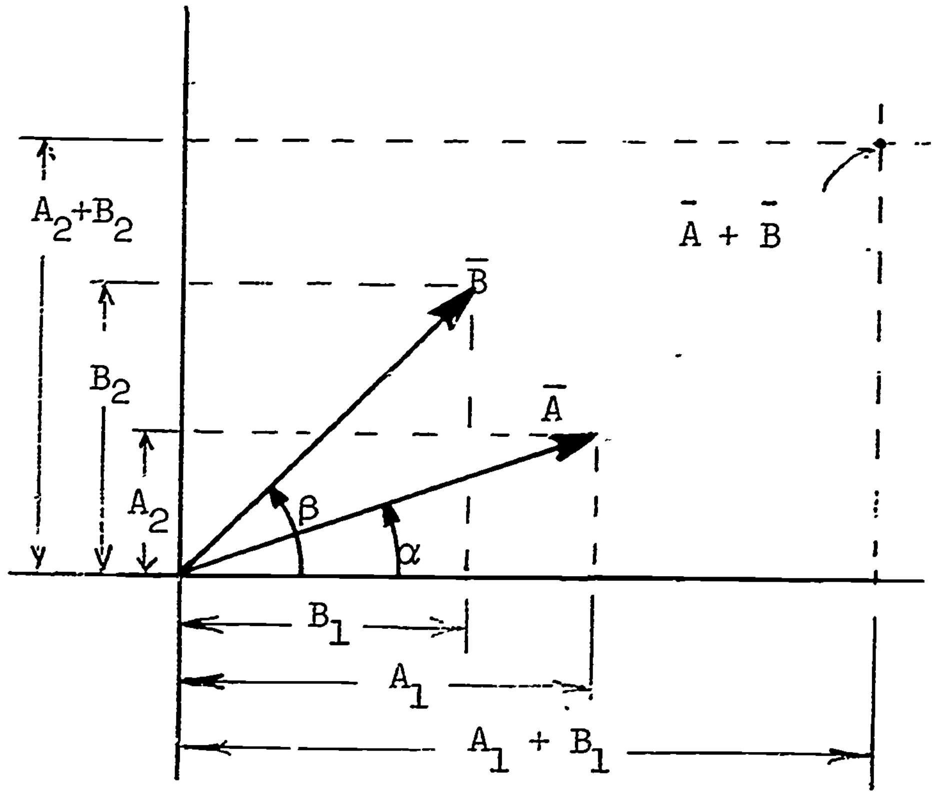


Figure 3.

This geometrical portrayal yields convenient interpretations for addition and multiplication of complex numbers. Referring to Fig. 3,

$$\bar{A} + \bar{B} = (A_1+B_1) + j(A_2+B_2)$$

This sum can be obtained by adding components. By putting arrowheads on the lines, as in Fig. 3, you can get the idea that the sum $\bar{A} + \bar{B}$ can be obtained graphically by adding line $\bar{A}$ and $\bar{B}$ like ________________.

Answer:

vectors.

In other words, as in

Fig. 4

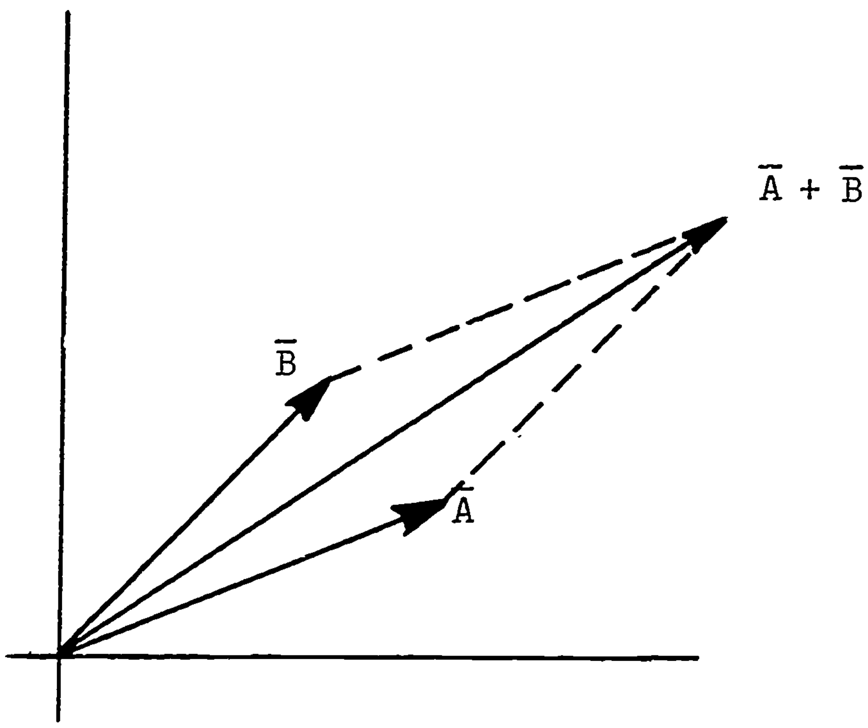


Figure 4.

Using the exponential forms, the product is

$$\overline{AB} = Ae^{j\alpha}\ Be^{j\beta} = \underline{\qquad\qquad\qquad}$$

In other words, the product has a magnitude equal to ______________________

____

and an angle equal to the ____.

Answer:

$$ABe^{j(\alpha+\beta)}$$

The product has a magnitude equal to the product of the magnitudes of the numbers multiplied, and an angle equal to the sum of the angles of the numbers multiplied, as shown in Fig. 5.

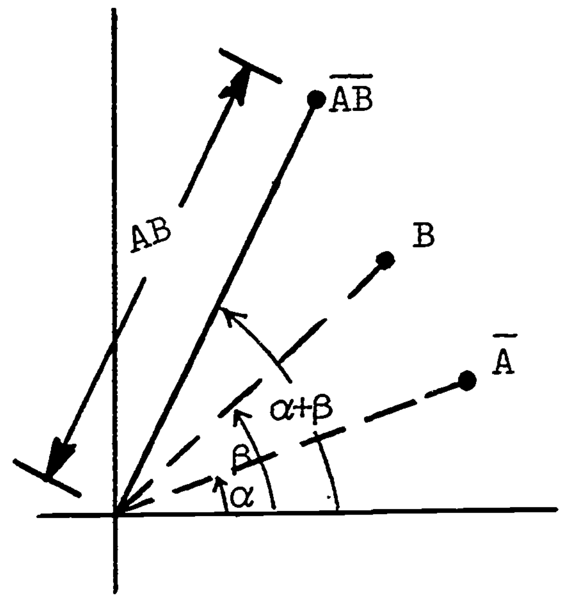


Figure 5.

The notation $Ae^{j\alpha}$ for the exponential form is convenient because it is similar to real exponentials and hence the rules of algebra and calculus apply. However, when the only purpose is to specify a complex number (say in the answer to a problem) by giving its magnitude and angle, an alternate form

$$\bar{A} = A\underline{/\alpha} \quad \text{(for example, } \bar{A} = 4\underline{/45^\circ}\text{)}$$

is often used. This is called the polar form, because A and α are the polar coordinates of the point $\bar{A}$ in the complex plane.

Specify the following in polar coordinates:

$20 + j40 =$

$-40 + j20 =$

$-40 - j20 =$

74

Answer:

$$\sqrt{(20)^2+(40)^2} = 20\sqrt{5}$$

$$\arctan \frac{40}{20} = 63.5^o$$

$$20+j40 = 20\sqrt{5}\ \underline{/63.5^o}$$

$$-40+j20 = 20\sqrt{5}\ \underline{/153.5^o}$$

$$-40-j20 = 20\sqrt{5}\ \underline{/-153.5^o}$$

Check your own answers and make any corrections needed.

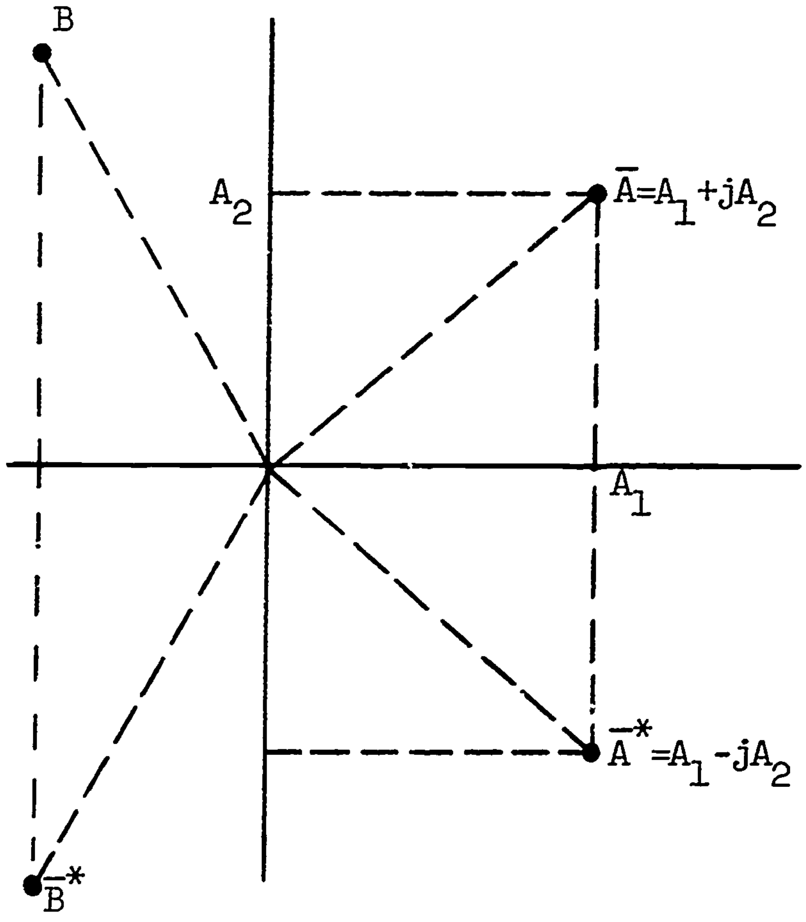
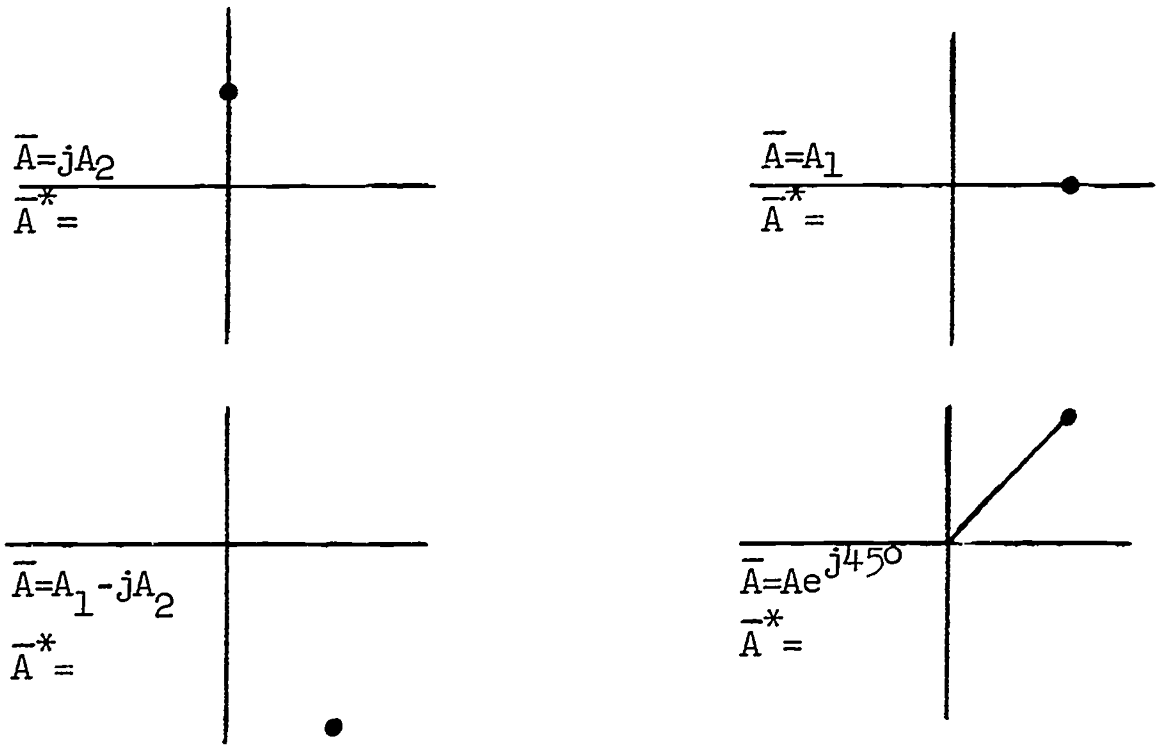


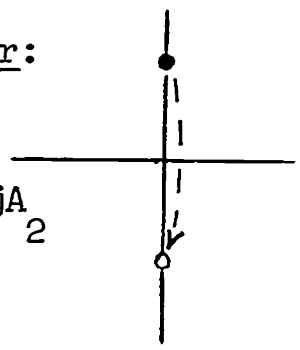
Figure 6.

Figure 6 shows a geometrical interpretation of the complex conjugate as the reflection of a point in the real axis.

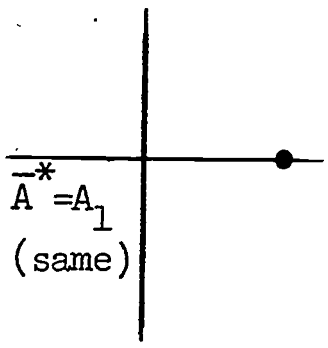
Show points on the following axes to illustrate the complex conjugates of the points indicated:



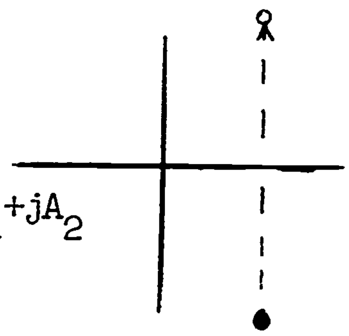
Answer:



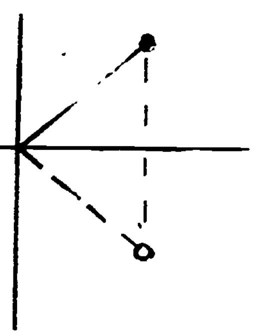
$\bar{A}^* = -jA_2$



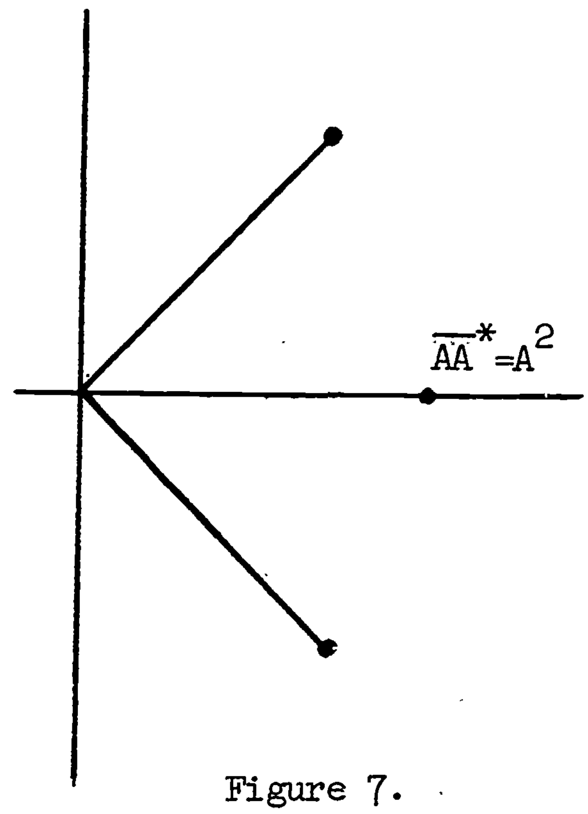
$\bar{A}^* = A_1$
(same)



$\bar{A}^* = A_1 + jA_2$



$\bar{A} = Ae^{j45^o}$
$\bar{A}^* = Ae^{-j45^o}$



$\bar{A}\bar{A}^* = A^2$

Figure 7.

When defining the complex conjugate, we used the example $(2+j3)(2-j3) = 13$ to show that multiplying a number by a conjugate yields a real number. Now we can see what that real number is. If $\overline{A} = Ae^{j\alpha}$, as in Fig. 7, then

$$\overline{A}\,\overline{A}^{*} = Ae^{j\alpha}\,Ae^{-j\alpha} = A^{2}$$

Thus, we have shown that the product of a complex number by its conjugate is

____________________________________.

Answer:

a real number which is the square of the magnitude, of the complex number.

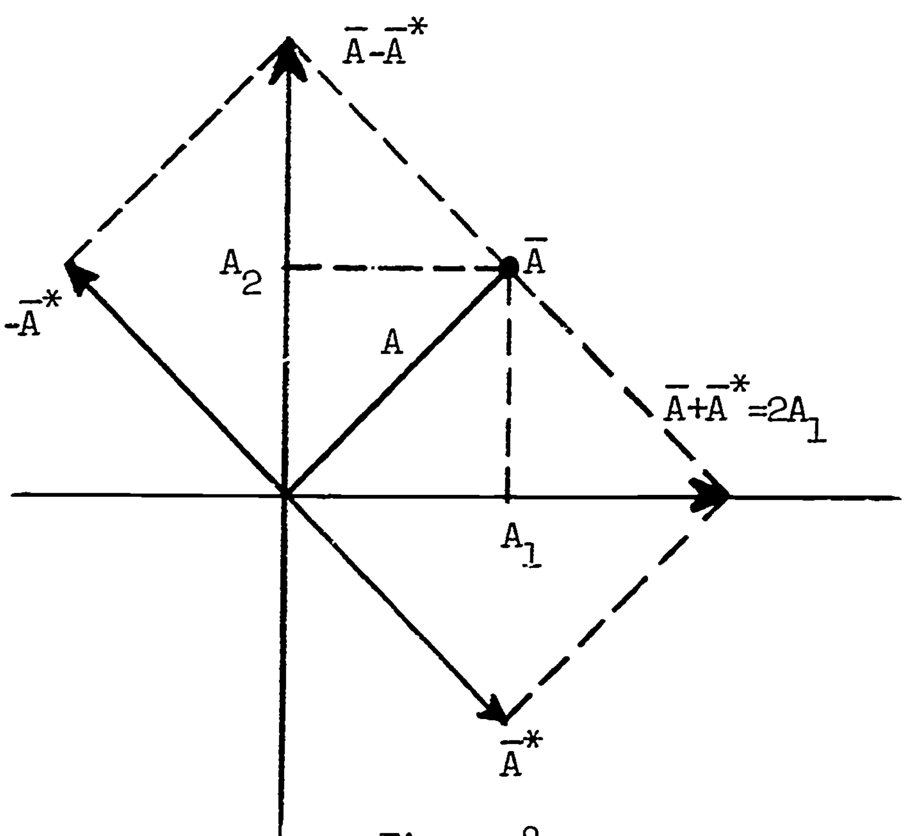


Figure 8.

The conjugate can also be used to give the real and imaginary parts of the complex number. Thus, if $\bar{A} = A_1 + jA_2$, then

$$\bar{A} + \bar{A}^* = A_1 + jA_2 + A_1 - jA_2 = 2A_1$$

$$\bar{A} - \bar{A}^* = A_1 + jA_2 - A_1 + jA_2 = 2jA_2$$

The geometrical equivalents of these equations are shown in Fig. 8. Thus, in terms of $\bar{A}$ and $\bar{A}^*$,

Real part of A =

Imaginary part of A =

<u>Answer</u>:

$$\text{Real part} = \frac{\bar{A}+\bar{A}^*}{2} = A_1$$

$$\text{Imaginary part} = \frac{\bar{A}-\bar{A}^*}{2j} = A_2$$

Be sure you have the j in the denominator of the second one. This was obtained by

$$\bar{A}-\bar{A}^* = 2jA_2$$

as shown in Fig. 8.

An important practical application of the last frame arises when $\bar{A}$ is expressed in the exponential form

$$\bar{A} = Ae^{j\theta} = \underline{\qquad\qquad} + \underline{\qquad\qquad}$$

Thus, in terms of $Ae^{j\theta}$ and its conjugate

$$A \cos \theta =$$

$$A \sin \theta =$$

82

<u>Answer</u>:

$$A \cos \theta = \frac{Ae^{j\theta} + Ae^{-j\theta}}{2} = \frac{\bar{A} + \bar{A}^*}{2}$$

$$A \sin \theta = \frac{Ae^{j\theta} - Ae^{-j\theta}}{2j} = \frac{\bar{A} - \bar{A}^*}{2j}$$

As an exercise with these formulas, write sin 2x in terms of exponentials:

$\sin 2x =$

84

<u>Answer</u>:

$$\sin 2x = \frac{e^{j2x} - e^{-j2x}}{2j}$$

The numerator of the answer shown on page 84 is of the form $\overline{A}^2 - \overline{B}^2$, if

$\overline{A}$ = _______ and $\overline{B}$ = __________

Thus, we can write

$$\frac{e^{j2x} - e^{-j2x}}{2j} = \frac{(\qquad)(\qquad)}{2j}$$

<u>Answer</u>:

$$\bar{A} = e^{jx}$$

$$\bar{B} = e^{-jx}$$

$$\sin 2x = \frac{e^{j2x} - e^{-j2x}}{2j} = \frac{(e^{jx} - e^{-jx})(e^{jx} + e^{-jx})}{2j}$$

Thus, recognizing that

$$\frac{e^{jx} - e^{-jx}}{2j} = \underline{\hspace{4cm}}$$

$$e^{jx} + e^{-jx} = \underline{2(\hspace{3cm})}$$

we have the result

$$\sin 2x =$$

<u>Answer</u>: $\frac{e^{jx} - e^{-jx}}{2j} = \sin x$ $\qquad e^{jx} + e^{-jx} = 2 \cos x$

$\sin 2x = 2 \sin x \cos x$

This example shows that complex numbers provide very simple proofs for trigonometric identities.

Let's try two other examples. Using exponentials,

$$\cos^2 x = (\qquad\qquad)(\qquad\qquad) = (\qquad\qquad) + (\qquad\qquad) + (\qquad\qquad)$$

$$\cos x \cos y = (\qquad\qquad)(\qquad\qquad) = (\qquad\qquad) + (\qquad\qquad) + (\qquad\qquad) + (\qquad\qquad)$$

90

<u>Answer</u>:

$$\cos^2 x = \frac{(e^{jx}+e^{-jx})(e^{jx}+e^{-jx})}{4} = \frac{e^{j2x}+2e^{j0}+e^{-j2x}}{4}$$

$$= \frac{1}{2} + \frac{e^{j2x}}{4} + \frac{e^{-j2x}}{4}$$

$$\cos x \cos y = \frac{(e^{jx}+e^{-jx})(e^{jy}+e^{-jy})}{4} = \frac{e^{j(x+y)}}{4} + \frac{e^{j(x-y)}}{4} + \frac{e^{-j(x-y)}}{4} + \frac{e^{-j(x+y)}}{4}$$

Finally, these partial answers, which are summarized for you on page 90, give the customary results

$\cos^2 x =$ ____________________

$\cos x \cos y =$ ____________________________

<u>Answer</u>:

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\cos x \cos y = \frac{\cos(x+y) + \cos(x-y)}{2}$$

If you have done these three examples, you are getting accustomed to the interchangeability of sin x and $(e^{jx}-e^{-jx})/(2j)$; and cos x and $(e^{jx}+e^{-jx})/2$. Now, using these same formulas, obtain formulas for the derivatives

$$\frac{d}{dx}(\cos ax) = \frac{d}{dx}\underset{\text{exponential}}{(\qquad\qquad)} = \underset{\text{exponential}}{\underline{\qquad\qquad}} = \underset{\text{trig. function}}{\underline{\qquad\qquad}}$$

$$\frac{d}{dx}(\sin ax) = \frac{d}{dx}\underset{\text{exponential}}{(\qquad\qquad)} = \underset{\text{exponential}}{\underline{\qquad\qquad}} = \underset{\text{trig. function}}{\underline{\qquad\qquad}}$$

Answer:

$$\frac{d}{dx}(\cos ax) = \frac{d}{dx}\left(\frac{e^{jax}+e^{-jax}}{2}\right) = \frac{jae^{jax}-jae^{-jax}}{2} = a\,\frac{e^{jax}-e^{-jax}}{(-2j)} = -a \sin ax$$

$$\frac{d}{dx}(\sin ax) = \frac{d}{dx}\left(\frac{e^{jax}-e^{-jax}}{2j}\right) = \frac{jae^{jax}+jae^{-jax}}{2j} = a\,\frac{e^{jax}+e^{-jax}}{2} = a \cos ax$$

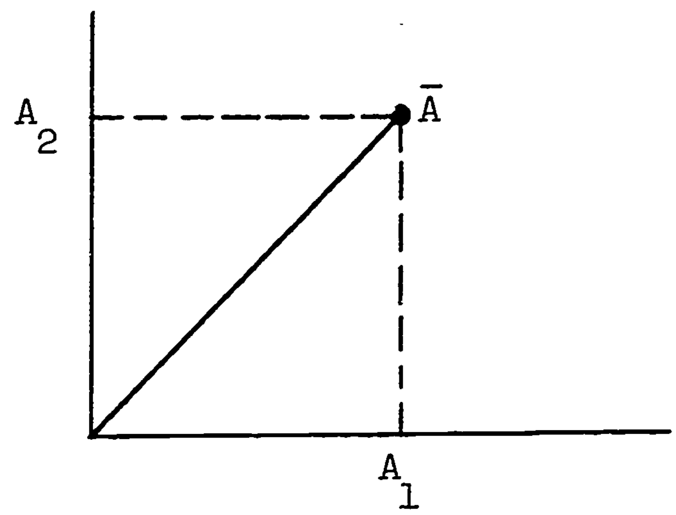


Figure 9.

When a complex number $\bar{A} = A_1 + jA_2$ is plotted in the complex plane, as in Fig. 9, the length of the line from the origin to the point $\bar{A}$ is called the absolute value of $\bar{A}$. A convenient notation is to place a symbol between vertical bars, to imply the absolute value. Thus, from Fig. 9,

$$|\bar{A}| = \sqrt{A_1^2 + A_2^2} \qquad \text{(positive square root)}$$

What are the absolute values of:

$|2 + j2| =$ ______________

$|\cos \alpha + j \sin \alpha| =$ ______________

<u>Answer</u>:

$$|2+j2| = \sqrt{4+4} = 2\sqrt{2}$$

$$|\cos\alpha + j\sin\alpha| = \sqrt{\cos^2\alpha+\sin^2\alpha} = 1$$

Observe that absolute values are always positive, or, preferably stated, non-negative (to include the possibility of zero).

The second of the previous examples, namely

$$|\cos \alpha + j \sin \alpha| = 1$$

is a reminder that for any real number α, the complex number $e^{j\alpha}$ has the absolute value

$$|e^{j\alpha}| = \underline{\qquad\qquad}$$

And, if A is a positive real number, the above is a special case of

$$|Ae^{j\alpha}| = \underline{\qquad\qquad\qquad}$$

98

Answer:

$$|e^{j\alpha}| = 1$$

$$|Ae^{j\alpha}| = A$$

If $\bar{A} = Ae^{j\alpha}$ and $\bar{B} = Be^{j\beta}$, with A and B greater than zero, then

$$\overline{AB} = ABe^{j(\alpha+\beta)}$$

and so the absolute value of the product $\overline{AB}$ is

$$|\overline{AB}| = |\quad||\quad|$$

or, in words, the absolute value of a product of two complex numbers is

the _______ of the _________ __________ of the numbers.

100

<u>Answer</u>:

$$|\overline{AB}| = |\overline{A}|\ |\overline{B}|$$

The absolute value of the product of two complex numbers is the <u>product</u> of the <u>absolute values</u> of the numbers.

By an induction proof, a similar statement may be made for the product of any number of complex numbers.

To make sure you are familiar with the idea of absolute values, find

$$|1+j| = \underline{\qquad\qquad\qquad}$$

$$|2+j3| = \underline{\qquad\qquad\qquad}$$

$$|3-j| = \underline{\qquad\qquad\qquad}$$

and also find

$$|(1+j)(2+j3)(3+j)| = \underline{\qquad\qquad\qquad}$$

<u>Answer</u>:

$$|1+j| = \sqrt{2}$$

$$|2+j3| = \sqrt{13}$$

$$|3-j| = \sqrt{10}$$

$$|(1+j)(2+j3)(3+j)| = \sqrt{260}$$

Observe that this can be done in two ways. One way is to multiply all three of the complex numbers together, and then find the absolute value. The second way is to use the result of the last page to get absolute value as the product of the absolute values of each complex number. Carry out whichever one of these you did not do.

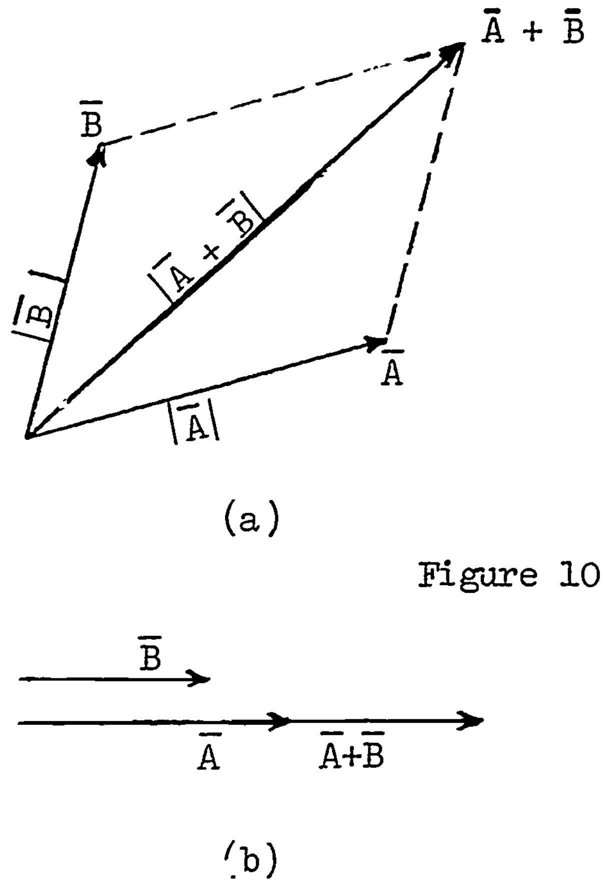


Figure 10

Let's consider what can be said of the absolute value of the sum of two complex numbers. This can be done by reference to Fig. 10, recalling that absolute values are the lengths of lines, as labeled. Two cases are shown. For (a), we see that

$$|\bar{A} + \bar{B}| < |\bar{A}| + |\bar{B}|$$

and for (b)

$$|\bar{A} + \bar{B}| = |\bar{A}| + |\bar{B}|$$

Both of these are included in the single statement

$$|\bar{A} + \bar{B}| \underset{?}{\underline{\quad\quad}} |\bar{A}| + |\bar{B}|$$

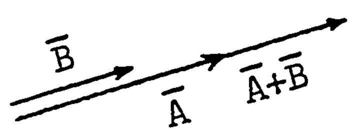
It can also be said that $|\bar{A} + \bar{B}| = |\bar{A}| + |\bar{B}|$ whenever the angles of $|\bar{A}|$ and $|\bar{B}|$ are ________________.

<u>Answer</u>:

$$|\bar{A} + \bar{B}| \leqq |\bar{A}| + |\bar{B}|$$

$|\bar{A} + \bar{B}| = |\bar{A}| + |\bar{B}|$ whenever the angles of $\bar{A}$ and $\bar{B}$ are <u>the same</u>.

If you answered <u>zero</u> for this, you were thinking about Fig. 10b. But the statement is still true for the case shown below:



For the following cases, determine whether the absolute value of the sum is equal to, or less than, the sum of the absolute values:

$$|(1+j) + 2e^{j\pi/4}| \underset{?}{\underline{\quad}} |1+j| + |2e^{j\pi/4}|$$

$$|(1-j) + 2e^{j\pi/4}| \underset{?}{\underline{\quad}} |1-j| + |2+e^{j\pi/4}|$$

Answer:

$$|(1+j) + (2e^{j\pi/4})| = |1+\sqrt{2} + j(1+\sqrt{2})| = (1+\sqrt{2})\sqrt{2} = \sqrt{2} + 2$$

$|1+j| + |2e^{j\pi/4}| = \sqrt{2} + 2$: These are equal. Observe that $1+j$ and $2e^{j\pi/4}$ both have angles of 45°.

$$|1-j+2e^{j\pi/4}| = |1+\ 2 + j(-1+\sqrt{2})| = \sqrt{(1+\sqrt{2})^2+(1-\sqrt{2})^2} = \sqrt{2+4} = \sqrt{6}$$

$|1-j| + |2e^{j\pi/4}| = \sqrt{2} + 2$: This is greater than $\sqrt{6}$.

Now we shall go on to another topic; finding roots (square root, etc.) of complex numbers. The exponential form is convenient for this. We already know that $\pm j = \sqrt{-1}$, but let us use the exponential form to write $\sqrt{-1}$. First, in exponential form we have

$$-1 = e^{j(\quad)}$$

or

$$-1 = e^{j(\quad +2\pi)}$$ (since an angle is not changed if 2π is added.)

108

<u>Answer</u>:

$$-1 = e^{j\pi}$$

$$= e^{j(\pi+2\pi)}.$$

Let the desired square root be written $Ae^{j\alpha}$. Then

$$(Ae^{j\alpha})^2 = e^{j\pi}$$

or

$$(Ae^{j\alpha})^2 = e^{j3\pi}$$

These yield the result

$$A^2 = \underline{\hspace{3cm}}$$

$$2\alpha = \underline{\hspace{3cm}} \text{ or } \underline{\hspace{3cm}}$$

Thus, the square root of -1 becomes, in exponential form

$$\underline{\hspace{3cm}} \text{ or } \underline{\hspace{3cm}}$$

110

Answer:

$$A^2 = 1$$

$$2\alpha = \pi \text{ or } 3\pi$$

$$e^{j\pi/2} \quad \text{or} \quad e^{j3\pi/2}$$

Observe that we have used $A = 1$ (not $A = -1$). This is in agreement with customary practice of using positive numbers for magnitudes. The number $e^{j3\pi/2}$ is equivalent to $e^{-j\pi/2}$.

These two results ($\sqrt{-1} = e^{j\pi/2}$ or $e^{-j\pi/2}$) are equivalent (using Euler's identity) to

$$e^{j\pi/2} = \cos(\quad) + j\sin(\quad) =$$

$$e^{-j\pi/2} = \cos(\quad) + j\sin(\quad) =$$

Answer:

$$e^{j\pi/2} = \cos \frac{\pi}{2} + j \sin \frac{\pi}{2} = 0 + j = j$$

$$e^{-j\pi/2} = \cos(-\frac{\pi}{2}) + j \sin(-\frac{\pi}{2}) = 0 - j = -j$$

This, of course, is what we expected. The importance of this example was to show how use of the exponential form yields a known result.

With this example, the general case of $\sqrt[3]{\bar{B}}$, where $\bar{B} = Be^{j\beta}$ can be handled. Thus, if

$$Ae^{j\alpha} = \sqrt[3]{Be^{j\beta}}$$

we have

$$A^3 = \underline{\qquad\qquad}$$

$$3\alpha = \underline{\qquad\qquad}, \underline{\qquad\qquad + 2\pi}, \underline{\qquad\qquad + 4\pi}.$$

Thus, in terms of B and β,

$$\sqrt[3]{\bar{B}} = (\qquad)e^{j(\qquad)}$$

or

$$= (\qquad)e^{j(\qquad)}$$

or

$$= (\qquad)e^{j(\qquad)}$$

114

<u>Answer</u>: $A^3 = 1$; $3\alpha = \beta,\ \beta+2\pi,\ \beta+4\pi$

$$\sqrt[3]{\overline{B}} = \sqrt[3]{B}\ e^{j\beta/n}$$

or

$$= \sqrt[3]{B}\ e^{j(\beta+2\pi)/n}$$

or

$$= \sqrt[3]{B}\ e^{j(\beta+4\pi)/n}$$

Higher order roots are treated in a similar way.

As exercises, find

$$\sqrt[4]{-1} = \underline{\hspace{3cm}} = \underline{\hspace{4cm}}$$
$$\underline{\hspace{3cm}} = \underline{\hspace{4cm}}$$
$$\underline{\hspace{3cm}} = \underline{\hspace{4cm}}$$
$$\underline{\hspace{3cm}} = \underline{\hspace{4cm}}$$

Four separate answers

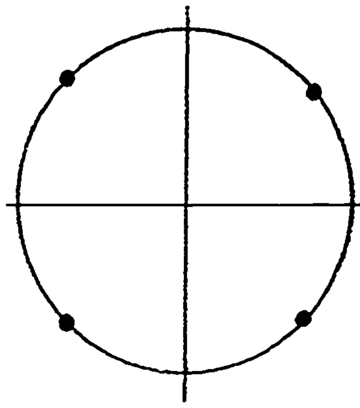
exponential rectangular

$$\sqrt{2+j2} = \underline{\hspace{3.5cm}} = \underline{\hspace{3cm}}$$
$$\underline{\hspace{3.5cm}} = \underline{\hspace{3cm}}$$

exponential rectangular

Answer:

$$\sqrt[4]{-1} = e^{j\pi/4} = \frac{1}{\sqrt{2}}(1+j)$$
$$= e^{j3\pi/4} = \frac{1}{\sqrt{2}}(-1+j)$$
$$= e^{j5\pi/4} = -\frac{1}{\sqrt{2}}(-1-j)$$
$$= e^{j7\pi/4} = \frac{1}{\sqrt{2}}(1-j)$$



The values of $\sqrt[4]{-1}$ are located as shown in this figure.

$$\sqrt{2+j2} = 2\sqrt{2}e^{j\pi/8} = 2\sqrt{2}\cos 22.5^{\circ} + j2\sqrt{2}\sin 22.5^{\circ} = 2\sqrt{2}(.927)+j2\sqrt{2}(.382)$$
$$= 2.62 + j1.08$$

$$\sqrt{2+j2} = 2\sqrt{2}e^{j9\pi/8} = 2\sqrt{2}\cos 202.5^{\circ} + j2\sqrt{2}\sin 202.5^{\circ} = -2.62 - j1.08$$

As a check that $\frac{1}{\sqrt{2}}(1+j)$ and $\frac{1}{\sqrt{2}}(-1+j)$ are two of the four values of $\sqrt[4]{-1}$, in rectangular form, compute

$$\left[\frac{1}{\sqrt{2}}(1+j)\right]^4 =$$

and

$$\left[\frac{1}{\sqrt{2}}(-1+j)\right]^4 =$$

<u>Answer</u>:

$$\left[\frac{1}{\sqrt{2}}(1+j)\right]^4 = \frac{1}{4}(1+j)^4 = \frac{1}{4}(1+j2-1)^2 = \frac{4}{4}j^2 = -1$$

$$\left[\frac{1}{\sqrt{2}}(-1+j)\right]^4 = \frac{1}{4}(-1+j)^4 = \frac{1}{4}(1-j2-1)^2 = \frac{4}{4}j^2 = -1$$

Observe that $\sqrt[4]{1}$ also involves complex numbers. Treating it as we did $\sqrt[4]{-1}$, we have

$\sqrt[4]{1}$ = __________ = __________

__________ = __________

__________ = __________

__________ = __________

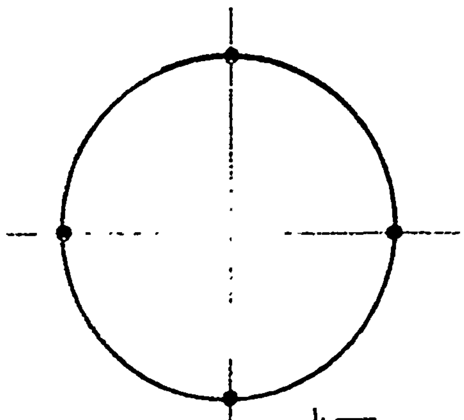
Answer:

$e^{j0} = 1+j0$

$e^{j\pi/2} = 0+j$

$e^{j\pi} = -1+j0$

$e^{j3\pi/2} = 0-j$



The values of $\sqrt[4]{1}$ are located as shown in this figure.

The two examples $\sqrt[4]{-1}$ and $\sqrt[4]{1}$ illustrate a general result for the nth root of a number. The absolute values of all the roots are _______________.

In the complex plan they therefore fall on a _________________________.

They are separated by ______________________ angles.

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Answer:

absolute values are equal

they fall on a circle

they are separated by equal angles

This completes our elementary treatment of the algebra of complex numbers. Let's see whether you can now do some sample problems.

Given, $\bar{A} = 3 + j4$, $\bar{B} = 6 - j2$

Find:

$$\frac{\bar{A}}{\bar{B}}$$

$$\left|\frac{\bar{A}}{\bar{B}}\right|$$

$$\frac{\bar{A}}{\bar{B}^*}$$

$$\left|\frac{\bar{A}}{\bar{B}^*}\right|$$

Answer:

$$\frac{\bar{A}}{\bar{B}} = \frac{3+j4}{6-j2} = \frac{(3+j4)(6+j2)}{36+4} = \frac{18-8+j(24+6)}{40} = .25+j.75$$

$$\left|\frac{\bar{A}}{\bar{B}}\right| = \frac{\sqrt{9+16}}{\sqrt{36+4}} = \frac{5}{\sqrt{40}} = \frac{5}{6.33} = .79 \quad ; \text{ or} \sqrt{(.25)^2+(.75)^2} = \sqrt{.625} = .79$$

$$\frac{\bar{A}}{\bar{B}^*} = \frac{3+j4}{6+j2} = \frac{(3+j4)(6-j2)}{36+4} = \frac{18+8+j(24-6)}{40} = .65 + j.45$$

$$\left|\frac{\bar{A}}{\bar{B}^*}\right| = \sqrt{(.65)^2+(.45)^2} = \sqrt{.625} = .79$$

Note that if $|\bar{A}/\bar{B}|$ is known, additional calculation of $|\bar{A}/\bar{B}^*|$ is not needed, because the absolute values of $\bar{B}$ and $\bar{B}^*$ are the same.

If a complex number $\bar{A}$ with absolute value A and angle α is squared, the absolute value and angle of the square will be

absolute value = ____________

angle = ______________

The sum of this number and its conjugate will be ____________________ (in terms of A and α).

Answer:

For square: absolute value = A^2

angle = 2α

The sum of $\bar{A}$ and $\bar{A}^*$ is $2A\cos\alpha$.

This last answer is twice the real part of A. If you forgot this, recall that

$$\bar{A} = A\cos\alpha + jA\sin\alpha$$

$$\bar{A}^* = A\cos\alpha - jA\sin\alpha$$

Given the complex number $\overline{A} = Ae^{j\alpha}$, which, if any, of the following are true?

Answer (true or false)

a) $|\overline{A} + \overline{A}^{*}| < 2A$ ________________

b) $|\overline{A} - \overline{A}^{*}| < 2A$ ________________

c) $|\overline{A} + \overline{A}^{*}| = |\overline{A} - \overline{A}^{*}|$ ________________

<u>Answer</u>:

(a) and (b) are true. $|\bar{A} + \bar{A}^*| = 2A|\cos\alpha| < 2A$

$|\bar{A} - \bar{A}^*| = 2A|\sin\alpha| < 2A$

However, since $\cos\alpha$ in general does not equal $\sin\alpha$, (c) is false.

If $\bar{A}$ and $\bar{B}$ are two complex numbers, which of the following, if any, are true:

	Answer (true or false)
1) $\lvert\bar{A}\rvert\ \lvert\bar{B}\rvert > \lvert\overline{AB}\rvert$	______________
2) $\lvert\bar{A} + \bar{B}^{*}\rvert < \lvert\bar{A}^{*}\rvert + \lvert\bar{B}\rvert$	______________
3) $\left\lvert\dfrac{\bar{A}}{\bar{B}}\right\rvert < \dfrac{\lvert\bar{A}\rvert}{\lvert\bar{B}\rvert}$	______________
4) $\lvert AB^{*}\rvert = \lvert A^{*}B\rvert$	______________

<u>Answer</u>:

(1) and (3) are false: an equality sign applies to these.

(2) is true: to see this, observe that $|\bar{A}^*| = |\bar{A}|$ and $|\bar{B}| = |\bar{B}^*|$ since conjugates have the same absolute values. But, $|\bar{A}+\bar{B}^*| < |\bar{A}| + |\bar{B}^*|$. Thus, $|\bar{A}+\bar{B}^*| < |\bar{A}^*| + |\bar{B}|$.

(4) is true because $\bar{A}^*\bar{B} = (\bar{A}\bar{B}^*)^*$, and conjugates have equal absolute values.