Chapter 2

Iterated, Line, and Surface Integrals

2.1 Iterated Integrals

• Iterated Double Integrals • Double Integrals over General Regions • Changing the Order of Integration • Triple Integrals • Triple Integrals over General Regions

We must confess in all humility that, while number is a product of our mind alone, space has a reality beyond the mind whose rules we cannot completely prescribe.
-Carl Gauss

Iterated Double Integrals

Let $\boldsymbol{f}$ be a real-valued function of two variables x, y defined on a rectangular region

$$R = \{(x, y) : a \leq x \leq b,\ c \leq y \leq d\}$$

where a, b, c, d are real numbers. We partition $[a, b]$ and $[c, d]$ as follows.

Let m, n be positive integers, let $a = x_0, x_1, \ldots, x_m = b$ be a partition of $[a, b]$, and let $c = y_0, y_1, \ldots, y_n = d$ be a partition of $[c, d]$. For $1 \leq i \leq m$ and $1 \leq j \leq n$, consider the rectangular subregions

$$R_{i,j} = \{(x, y) : x_{i-1} \leq x \leq x_i,\ y_{j-1} \leq y \leq y_j\}\,.$$

The collection $\Delta = \{R_{i,j} : i = 1, \ldots, m \text{ and } j = 1, \ldots, n\}$ is called a partition of R into sub-rectangles. Let $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$, and let $\|\Delta\|$ be the norm of Δ which is defined as the maximum of all the Δx_i's and Δy_j's. We denote the area of rectangle $R_{i,j}$ by

$$\Delta A_{i,j} = \Delta x_i \Delta y_j.$$

Let L be a real number. We symbolically write

$$L = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sum_{j=1}^{m} \boldsymbol{f}\left(x_{i,j}, y_{i,j}\right) \Delta A_{i,j}$$

if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left| \sum_{i=1}^{n} \sum_{j=1}^{m} \boldsymbol{f}\left(x_{i,j}, y_{i,j}\right) \Delta A_{i,j} - L \right| < \varepsilon$$

for all $(x_{i,j}, y_{i,j})$ in $R_{i,j}$ whenever $\|\Delta\| < \delta$. If such a limit L exists, we say $\boldsymbol{f}$ is **Riemann integrable** or **integrable** on R, and we write

$$\int\int_R \boldsymbol{f}(x, y) dA = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sum_{j=1}^{m} \boldsymbol{f}\left(x_{i,j}, y_{i,j}\right) \Delta A_{i,j}.$$

We quote a theorem from advanced calculus about Riemann integrable functions. We omit the proof of the theorem since it is beyond the scope of this book.[1]

Theorem 2.1 Riemann Integrable Functions

Let $z = \boldsymbol{f}(x, y)$ be a bounded real-valued function on a rectangular region

$$R = \{(x, y) : a \leq x \leq b,\ c \leq y \leq d\}.$$

If the points of discontinuities of $\boldsymbol{f}$ in R lie on a finite union of graphs of continuous functions of one independent variable, then $\boldsymbol{f}$ is Riemann integrable on R.

In particular, every continuous real-valued function on a rectangular region R is integrable on R. However, in order to evaluate $\int\int_R \boldsymbol{f}(x, y)dA$, we introduce the concept of an **iterated integral**.

If we fix a value of y, then $\boldsymbol{f}(x, y)$ is a function of x only, and we have definite integral $\int_a^b \boldsymbol{f}(x, y)dx$. To illustrate, we assume y is constant in the integration below:

$$\begin{aligned}
\int_0^4 x \sin y \ dx &= \left.\frac{1}{2}x^2 \sin y\right|_{x=2}^{x=4} \\
&= \left(\frac{1}{2}(4)^2 \sin y\right) - \left(\frac{1}{2}(2)^2 \sin y\right) \\
&= 6 \sin y.
\end{aligned}$$

Likewise, $\int_c^d \boldsymbol{f}(x, y)dy$ is a definite integral if x is held constant. The next theorem implies that a **double integral** $\int\int_R \boldsymbol{f}(x, y)dA$ may be evaluated as an iterated integral if $\boldsymbol{f}$ is continuous on R. Also, we omit the proof of the theorem that is usually discussed in advanced calculus texts.

Theorem 2.2 Fubini's Theorem and Iterated Integrals

If $z = \boldsymbol{f}(x, y)$ is the function in Theorem 2.1, then the identity below holds:

$$\int\int_R \boldsymbol{f}(x, y)dA = \int_a^b \int_c^d \boldsymbol{f}(x, y)dydx = \int_c^d \int_a^b \boldsymbol{f}(x, y)dxdy$$

In particular, if $\boldsymbol{f}$ is continuous on R, the above identity applies.

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[1] A bounded real-valued function on a rectangular region $R \subseteq \mathbb{R}^2$ whose set of discontinuities is of measure zero is Riemann integrable. Also, if $y = f(x)$, $a \leq x \leq b$, is a continuous function, then $\{(x, f(x)) : a \leq x \leq b\}$ is a set of measure zero in $\mathbb{R}^2$.

Example 1 Evaluating an Iterated Integral

Evaluate the integral $\int_1^3 \int_0^\pi x^2 \cos(xy) dy dx$.

Solution For the inner integral, we have

$$\int \cos(xy) dy = \frac{1}{x} \sin xy + C.$$

Consequently, we evaluate the iterated integral as follows:

$$\begin{aligned} \int_1^3 \int_0^\pi x^2 \cos(xy) dy dx &= \int_1^3 x \sin(xy)\Big|_{y=0}^{y=\pi} dx \\ &= \int_1^3 x \sin(\pi x) dx. \end{aligned}$$

Then we integrate by parts using the substitutions:

u	$=$	x	dv	$=$	$\sin(\pi x)dx$
du	$=$	dx	v	$=$	$-\frac{1}{\pi}\cos(\pi x)$

Since $\int u dv = uv - \int v du$, we obtain

$$\begin{aligned} \int_1^3 x \sin(\pi x) dx &= -\frac{x}{\pi} \cos(\pi x)\Big|_{x=1}^{x=3} + \frac{1}{\pi} \int_1^3 \cos(\pi x) dx \\ &= \frac{2}{\pi} + \frac{1}{\pi^2} \sin(\pi x)\Big|_{x=1}^{x=3} = \frac{2}{\pi}. \end{aligned}$$

Thus, we find $\int_1^3 \int_0^\pi x^2 \cos(xy) dy dx = \frac{2}{\pi}$.

$\square$

Try This 1

Evaluate $\int_0^4 \int_0^9 \sqrt{x} dy dx$.

Double Integrals over General Regions

Let $\phi_1, \phi_2, \psi_1, \psi_2$ be continuous real-valued functions satisfying $\phi_1(x) \leq \phi_2(x)$ and $\psi_1(y) \leq \psi_2(y)$ for all x in $[a, b]$, and y in $[c, d]$. We evaluate double integrals of a real-valued continuous function $z = \boldsymbol{f}(x, y)$ over **elementary regions** in the xy-plane, see Figures 1-2. We classify these regions as either of type R_x or R_y, or possibly of both types, depending on whether the boundary of the region are graphs of functions of x or y, respectively:

A) $R_x = \{(x, y) : a \leq x \leq b,\ \phi_1(x) \leq y \leq \phi_2(x)\}$

B) $R_y = \{(x, y) : c \leq y \leq d,\ \psi_1(y) \leq x \leq \psi_2(y)\}$

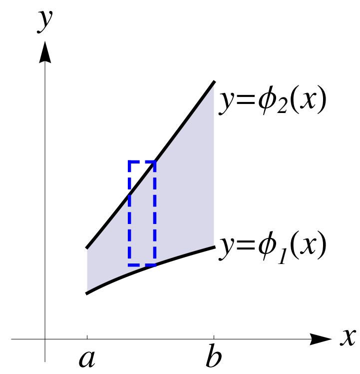


Figure 1 Region of type R_x

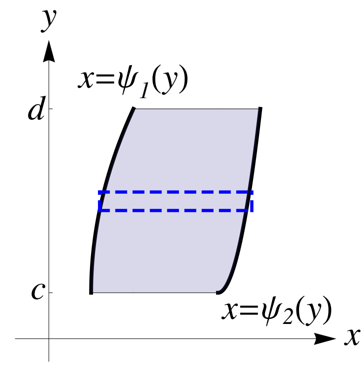


Figure 2 Region of type R_y

To evaluate $\int\int_{Rx} \boldsymbol{f}(x,y)dA$, let R be a rectangular region that contains R_x. We extend $\boldsymbol{f}$ to a function $\boldsymbol{F}$ on R:

$$\boldsymbol{F}(x,y) = \begin{cases} \boldsymbol{f}(x,y) & \text{if } (x,y) \in R_x \\ 0 & \text{if } (x,y) \notin R_x \end{cases}$$

Since $\boldsymbol{f}$ is bounded and continuous on R_x, the function $\boldsymbol{F}$ is bounded and possibly discontinuous only on points in the graph of ϕ_1 or ϕ_2. Then $\int\int_R \boldsymbol{F}(x,y)dA$ exists by Theorem 2.1. Suppose we choose R such that is consists of points (x,y) satisfying $a \leq x \leq b$ and $p \leq y \leq q$. Notice, for any x in $[a,b]$, we find $\boldsymbol{F}(x,y) = 0$ if y does not belong to the interval $[\phi_1(x), \phi_2(x)]$. Thus,

$$\int_p^q \boldsymbol{F}(x,y)dy = \int_{\phi_1(x)}^{\phi_2(x)} \boldsymbol{F}(x,y)dy = \int_{\phi_1(x)}^{\phi_2(x)} \boldsymbol{f}(x,y)dy.$$

and the latter integral exists by the continuity of $\boldsymbol{f}$ on R_x. By Fubini's Theorem, we obtain

$$\int\int_{Rx} \boldsymbol{F}(x,y)dA = \int_a^b \int_p^q \boldsymbol{F}(x,y)dydx = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \boldsymbol{f}(x,y)dydx.$$

We have a similar identity for double integrals over a type R_y region.

Theorem 2.3 Integrating over Regions of Type R_x or R_y

If $z = \boldsymbol{f}(x,y)$ is a continuous real-valued function on a region of type R_x or R_y, then

a) $\displaystyle\int\int_{Rx} \boldsymbol{f}(x,y)dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \boldsymbol{f}(x,y)dydx$

b) $\displaystyle\int\int_{Ry} \boldsymbol{f}(x,y)dA = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} \boldsymbol{f}(x,y)dxdy$

In Theorem 2.3, if $\boldsymbol{f}(x,y) = 1$ for all x, y, then $\displaystyle\int\int_R dA$ represents the area of region R where $R = R_x$ or $R = R_y$.

Also, we may apply double integrals to define the volume of a solid.

Definition 1 Volume of a Solid and Double Integrals

Let R be a region in the xy-plane of either type R_x or R_y. Let $z = \boldsymbol{f}(x, y)$ be a nonnegative real-valued continuous function defined on R. Let S be the solid consisting of the points $(x, y, \boldsymbol{f}(x, y))$ in 3-space where $(x, y) \in R$. The volume of S is defined as

$$\text{Volume} = \int\int_R \boldsymbol{f}(x, y)dA.$$

Example 2 Evaluating the Volume of a Solid

Find the volume of the solid that lies below the plane $\boldsymbol{f}(x, y) = \dfrac{1}{3}(6 - 2x - 2y)$ and above the region

$$R = \{(x, y) : 0 \leq y \leq 2,\ 0 \leq x \leq y/2\}$$

in the xy-plane, see Figure 3.

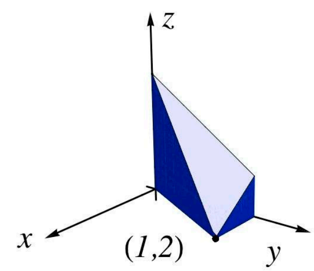


Figure 3
A solid bounded above by a plane, and with a triangular base in the xy-plane

Solution The region R is both type R_x and R_y, see Figure 4. We choose to evaluate the integral for the volume as a type R_y region.

$$\begin{aligned}
\text{Volume} &= \int\int_R \boldsymbol{f}(x, y)dA \\
&= \int_0^2 \int_0^{y/2} \frac{1}{3}(6 - 2y - 2x)dxdy \\
&= \frac{1}{3}\int_0^2 (6 - 2y)x - x^2 \Big|_{x=0}^{x=y/2} dy \\
&= \frac{1}{3}\int_0^2 \left(3y - y^2 - \frac{y^2}{4}\right) dy \\
\text{Volume} &= \frac{8}{9}\ \text{units}^3
\end{aligned}$$

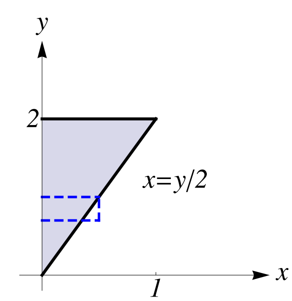


Figure 4
The base of the above solid in the xy-plane.

□

Try This 2

A solid S is bounded above by the plane $z = y$ and bounded below by the region

$$R = \{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1 - x^2\}.$$

Find the volume of S.

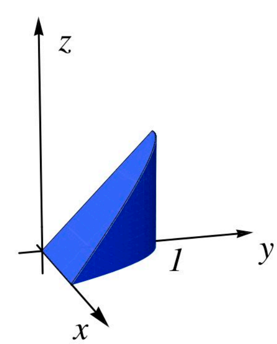


Figure 5
The solid for Try This 2.

Changing the Order of Integration

To be able to integrate $\int\int_R \boldsymbol{f}(x,y)dA$, there may be an advantage to evaluating one iterated integral over another. The reason is the order of integration, i.e., integrating with respect to x then integrating with respect to y, may be easier than integrating first with respect to y, then with respect to x secondly.

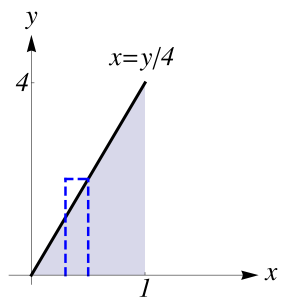


Figure 6
Region of integration

Example 3 Switching the Order of Integration

Sketch the region R of integration of

$$\int_0^4 \int_{y/4}^1 \sin(\pi x^2)dxdy.$$

Then switch the order of integration, and evaluate the resulting integral.

Solution In Figure 6, we see the triangular region R. To switch the order of integration to $dydx$, we partition the interval $[0,1]$ in the x-axis into smaller subintervals. If the order of integration is $dxdy$, partition $[0,4]$ in the y-axis.

Then draw a typical rectangle such that its base is a subinterval in the partition, and its length extends from the lower boundary to the upper boundary of the region. Since the order of integration is $dydx$, each point (x,y) in a typical rectangle satisfies $0 \le x \le 1$ and $0 \le y \le 4x$.

Then we integrate as follows:

$$\begin{aligned}
\int_0^4 \int_{y/4}^1 \sin(\pi x^2)dxdy &= \int_0^1 \int_0^{4x} \sin(\pi x^2)dydx \\
&= \int_0^1 y\sin(\pi x^2)\Big|_{y=0}^{y=4x} dx \\
&= \int_0^1 4x\sin(\pi x^2)dx \\
&= -\frac{2}{\pi}\cos(\pi x^2)\Big|_{x=0}^{x=1} \\
\int_0^4 \int_{y/4}^1 \sin(\pi x^2)dxdy &= \frac{4}{\pi}
\end{aligned}$$

□

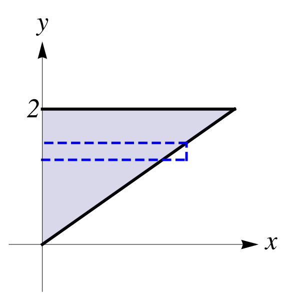


Figure 7
Region of integration
for Try This 3.

Try This 3

Sketch the region R of integration of

$$\int_0^4 \int_{x/2}^2 \frac{x}{\sqrt{y^3+1}}dydx.$$

Then switch the order of integration, and evaluate the resulting integral.

Triple Integrals

Let $w = \boldsymbol{g}(x, y, z)$ be a real-valued function that is defined on a Cartesian product

$$\begin{aligned} S &= [a, b] \times [c, d] \times [e, f] \\ &= \{(x, y, z) : a \leq x \leq b,\ c \leq y \leq d,\ e \leq z \leq f\} \end{aligned}$$

We partition each of $[a, b]$, $[c, d]$, and $[e, f]$ into finitely many subintervals. Let $[x_i, x_{i+1}]$, $[y_j, y_{j+1}]$, and $[z_k, z_{k+1}]$ be subintervals in the partitions, and we denote their lengths by Δx_i, Δy_j, Δz_k, respectively. We consider a box

$$S_{ijk} = [x_i, x_{i+1}] \times [y_j, y_{j+1}] \times [z_k, z_{k+1}]$$

whose volume is denoted by $\Delta V_{ijk} = \Delta x_i \Delta y_j \Delta z_k$. Let $\boldsymbol{p}_{ijk}$ be a point in S_{ijk}, and let $\|\Delta\|$ be the maximum of the norms of the partitions of $[a, b]$, $[c, d]$, and $[e, f]$.

We define $\boldsymbol{g}$ to be **Riemann integrable** or **integrable** on the box S if there is a number $L \in \mathbb{R}$ such that for each $\varepsilon > 0$ there exists a positive number $\delta > 0$ satisfying

$$\left| \sum_i \sum_j \sum_k \boldsymbol{g}\left(\boldsymbol{p}_{ijk}\right) \Delta V_{ijk} - L \right| < \varepsilon$$

for all $\boldsymbol{p}_{ijk}$ in S_{ijk} whenever $\|\Delta\| < \delta$. In the above sums, we evaluate over all the values of i, j, k. In such a case, we write

$$\int\int\int_S \boldsymbol{g}(x, y, z) dV = \lim_{\|\Delta\| \to 0} \sum_i \sum_j \sum_k \boldsymbol{g}\left(\boldsymbol{p}_{ijk}\right) \Delta V_{ijk} = L.$$

We state an integrability condition for $\boldsymbol{g}$ similar to Theorem 2.1. Also, we state a Fubini's theorem for **triple integrals** $\int\int\int_S \boldsymbol{g}(x, y, z) dV$. We omit the proofs since they are usually discussed in advanced calculus texts.

Theorem 2.4 Riemann Integrable Functions

Let $w = \boldsymbol{g}(x, y, z)$ be a bounded real-valued function on a box

$$S = [a, b] \times [c, d] \times [e, f].$$

If the points of discontinuities of $\boldsymbol{g}$ in S lie on a finite union of graphs of continuous functions of two independent variables, then $\boldsymbol{g}$ is integrable on S.

Theorem 2.5 Fubini's Theorem for Triple Integrals

If $w = \boldsymbol{g}(x, y, z)$ is the function in Theorem 2.4, then

$$\int\int\int_S \boldsymbol{g}(x, y, z) dV = \int_a^b \int_c^d \int_e^f \boldsymbol{g}(x, y, z) dz dy dx$$

Also, the six iterated triple integrals exist and are equal.

Example 4 Evaluating a Triple Integral

Evaluate the integral $\int_1^2 \int_0^3 \int_0^2 \left(x + z^2 - y^2\right) dxdydz$.

Solution We integrate as follows:

$$\begin{aligned}
\int_1^2 \int_0^3 \int_0^2 \left(x + z^2 - y^2\right) dxdydz &= \int_1^2 \int_0^3 \left(\frac{x^2}{2} + x\left(z^2 - y^2\right) \Big|_{x=0}^{x=2} dydz \right. \\
&= \int_1^2 \int_0^3 \left(2 + 2\left(z^2 - y^2\right)\right) dydz \\
&= 2\int_1^2 \int_0^3 \left(1 + z^2 - y^2\right) dydz \\
&= 2\int_1^2 \left(y\left(1 + z^2\right) - \frac{y^3}{3} \Big|_{y=0}^{y=3} dz \right. \\
&= 2\int_1^2 \left(3\left(1 + z^2\right) - 9\right) dz \\
&= 6\int_1^2 \left(z^2 - 2\right) dz \\
\int_1^2 \int_0^3 \int_0^2 \left(x + z^2 - y^2\right) dxdydz &= 2.
\end{aligned}$$

□

Try This 4

Evaluate the triple integral $\int_0^2 \int_0^1 \int_0^1 yze^{xy} dxdydz$.

Triple Integrals over General Regions

We wish to extend and evaluate triple integrals $\int\int\int_S \boldsymbol{g}(x, y, z)dV$ over **elementary regions** $S \subseteq \mathbb{R}^3$ that we are about to describe. We consider an elementary region in the xy-plane:

$$R_x = \{(x, y) : a \le x \le b, \phi_1(x) \le y \le \phi_2(x)\}$$

where ϕ_1, ϕ_2 are continuous functions of x such that $\phi_1(x) \le \phi_2(x)$. Then we associate an elementary region in 3-space such as

$$S_{yx} = \{(x, y, z) : (x, y) \in R_x,\ \Phi_1(x, y) \le z \le \Phi_2(x, y)\}.$$

where Φ_1, Φ_2 are continuous functions on R_x satisfying $\Phi_1(x, y) \le \Phi_2(x, y)$. Using similar ideas leading to Theorem 2.3, we have

$$\int\int\int_{S_{yx}} \boldsymbol{g}(x, y, z)dV = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\Phi_1(x,y)}^{\Phi_2(x,y)} \boldsymbol{g}(x, y, z)dzdydx$$

provided $\boldsymbol{g}$ is continuous on S_{yx}.

Similarly, if an elementary region in the xy-plane is given by

$$R_y = \{(x, y) : c \le y \le d, \psi_1(y) \le x \le \psi_2(y)\}$$

where ψ_1, ψ_2 are continuous functions of y such that $\psi_1(y) \le \psi_2(y)$, we associate an elementary region

$$S_{xy} = \{(x, y, z) : (x, y) \in R_y,\ \Psi_1(x, y) \le z \le \Psi_2(x, y)\}.$$

where Ψ_1, Ψ_2 are continuous functions on R_y satisfying $\Psi_1(x, y) \le \Psi_2(x, y)$. Likewise, we obtain

$$\int\int\int_{Sxy} \boldsymbol{g}(x, y, z)dV = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} \int_{\Psi_1(x,y)}^{\Psi_2(x,y)} \boldsymbol{g}(x, y, z)dzdxdy$$

whenever $\boldsymbol{g}$ is continuous on S_{xy}. There are six possible iterated triple integrals, and another one of these is

$$\int\int\int_S \boldsymbol{g}(x, y, z)dV = \int_e^f \int_{\gamma_1(z)}^{\gamma_2(z)} \int_{\Gamma_1(y,z)}^{\Gamma_2(y,z)} \boldsymbol{g}(x, y, z)dxdydz$$

where γ_1, γ_2 are continuous functions of z satisfying $\gamma_1(z) \le \gamma_2(z)$, and Γ_1, Γ_2 are continuous functions of (y, z) in some elementary region in the yz-plane such that $\Gamma_1(y, z) \le \Gamma_2(y, z)$.

We summarize and define an **elementary region** $S \subseteq \mathbb{R}^3$ in 3-space. For any point in S, two of its coordinates lie in an elementary region in a plane such as the xy-, yz-, or xz-plane, and the third coordinate lies between two continuous functions of the first two variables. Specifically, let x_1, x_2, x_3 be a permutation of x, y, z, and let $p < q$ be real constants. Let f_1, f_2, F_1, F_2 be continuous functions satisfying $f_1(x_3) \le f_2(x_3)$ whenever $p \le x_3 \le q$, and further $F_1(x_2, x_3) \le F_2(x_2, x_3)$ if $f_1(x_3) \le x_2 \le f_2(x_3)$. A point lies in S if its coordinates satisfy $p \le x_3 \le q$, $f_1(x_3) \le x_2 \le f_2(x_3)$, and $F_1(x_2, x_3) \le x_1 \le F_2(x_2, x_3)$. Following the idea of Theorem 2.3, we have the following theorem:

Theorem 2.6 Triple Integrals over Elementary Regions

If $\boldsymbol{g}$ is a real-valued continuous function on an elementary region $S \subseteq \mathbb{R}^3$, then

$$\int\int\int_S \boldsymbol{g}\ dV = \int_p^q \int_{f_1(x_3)}^{f_2(x_3)} \int_{F_1(x_2,x_3)}^{F_2(x_2,x_3)} \boldsymbol{g}\ dx_1dx_2dx_3$$

In Theorem 2.6, if $\boldsymbol{g}(x, y, z) = 1$ for all x, y, then $\displaystyle\int\int\int_S dV$ is the volume of S.

Example 5 Triple Integral

Let $S \subseteq \mathbb{R}^3$ be a region in the first octant that is bounded above by the plane $2x + y + z = 2$, and bounded below by the xy-plane. Then evaluate $\displaystyle\int\int\int_S 6xdV$.

Solution In Figure 8, the base of solid S in the xy-plane may be expressed as

$$R_x = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le 2 - 2x\}.$$

Then region S is given by

$$S_{yx} = \{(x, y, z) : (x, y) \in R_x,\ 0 \le z \le 2 - 2x - y\}.$$

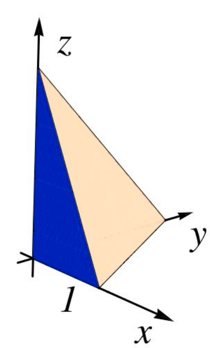


Figure 8
Solid of integration
for Example 5.

We apply Theorem 2.6, and evaluate the inner-most integral.

$$\begin{aligned}
\int\int\int_S 6x dV &= \int_0^1 \int_0^{2-2x} \int_0^{2-2x-y} 6x\ dzdydx \\
&= \int_0^1 \int_0^{2-2x} 6x(2-2x-y)dydx \\
&= \int_0^1 \int_0^{2-2x} \left(12x(1-x) - 6xy\right) dydx
\end{aligned}$$

Then we evaluate the inner-integral with respect to y:

$$\begin{aligned}
\int\int\int_S 6x dV &= \int_0^1 \left(12x(1-x)y - 3xy^2 \Big|_{y=0}^{y=2-2x} dx \right. \\
&= \int_0^1 \left(24x(1-x)^2 - 12x(1-x)^2\right) dx \\
&= \int_0^1 12x(1-x)^2 dx \\
\int\int\int_S 6x dV &= 1.
\end{aligned}$$

□

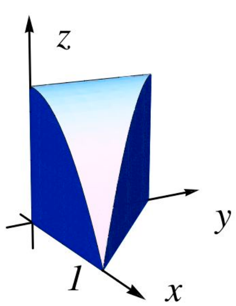


Figure 9
The solid of integration for Try This 5.

Try This 5

A solid $S \subseteq \mathbb{R}^3$ lies in the first octant, bounded by the surfaces $z = 1 - x^2$ and $x + y = 1$, and the coordinate planes. Then evaluate $\int\int\int_S 10x dV$.

2.1 Check-It Out

Evaluate the iterated integral.

1. $\displaystyle\int\int 10(2x+y)^3 dxdy$ 2. $\displaystyle\int_0^3 \int_0^1 xy^2 dydx$ 3. $\displaystyle\int_0^2 \int_0^x \int_0^y (x-y)dzdydx$

Rewrite the integral by switching the order of integration.

4. $\displaystyle\int_0^1 \int_0^x f(x,y)dydx$ 5. $\displaystyle\int_0^2 \int_0^{y/2} f(x,y)dxdy$

True or False. *If false, revise the statement to make it true or explain.*

1. $\displaystyle\int_a^b \int_c^d \boldsymbol{f}(x,y)dydx = \int_a^b \int_c^d \boldsymbol{f}(x,y)dxdy$

2. $\displaystyle\int_0^2 \int_0^2 (x+y)dydx = \int_0^2 xdx + \int_0^2 ydy$

3. $\displaystyle\int_0^1 \int_0^1 \int_0^1 xyz\ dzdydx = \frac{1}{8}$

4. If $R = \{(x,y) : x^2 + y^2 \le r^2\}$, then $\displaystyle\int\int_R dA = \pi r^2$.

5. The volume of the solid bounded by the hemisphere $z = \sqrt{4 - x^2 - y^2}$ and the plane $z = 0$ is given by $\displaystyle\int_{-2}^{2}\int_{-2}^{2} \sqrt{4 - x^2 - y^2}dydx$.

Exercises for Section 2.1

In Exercises 1-8, evaluate the iterated integrals.

1. $\displaystyle\int_{\pi/6}^{\pi/2} \int_0^1 6x\sin(xy)dydx$

2. $\displaystyle\int_{1/4}^{1/3} \int_0^1 \pi y \sec^2(\pi xy)dxdy$

3. $\displaystyle\int_1^2 \int_0^1 \int_1^3 (4xy + 2z)dzdxdy$

4. $\displaystyle\int_0^2 \int_0^z \int_0^y 12xdxdydz$

5. $\displaystyle\int_0^1 \int_{-x}^{x} 6e^{x+y}dydx$

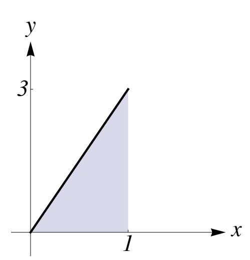
6. $\displaystyle\int_1^e \int_0^y \frac{dxdy}{x^2+y^2}$

7. $\displaystyle\int_1^2 \int_0^z \int_0^{y/2} \frac{4}{\sqrt{y^2-x^2}}dxdydz$

8. $\displaystyle\int_1^e \int_2^4 \int_x^{2x} \frac{y}{z\sqrt{z^2-x^2}}dzdydx$

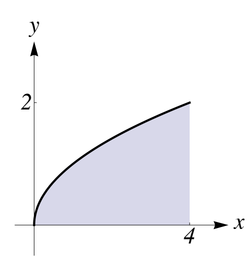
In Exercises 9-14, we see a region R of integration that is bounded by the graph of the indicated equation. Express the double integral of $\boldsymbol{f}(x,y)$ over R using $dydx$. Then switch the order of integration to $dxdy$.

9. $y = 3x$



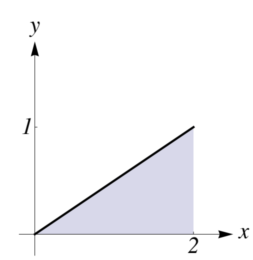
For No. 9

10. $y = \sqrt{x}$



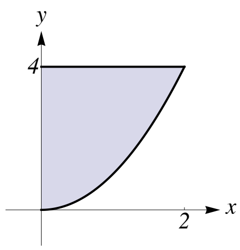
For No. 10

11. $y = x/2$



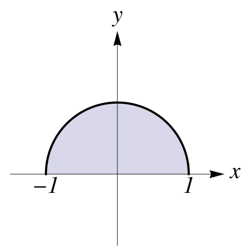
For No. 11

12. $y = x^2$



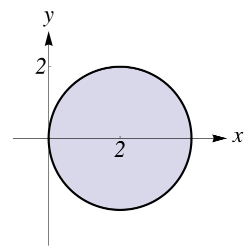
For No. 12

13. $y = \sqrt{1 - x^2}$



For No. 13

14. $(x-2)^2 + y^2 = 4$



For No. 14

In Exercises 15-26, switch the order of integration, and evaluate the integral with the new limits.

15. $\displaystyle\int_0^1 \int_0^{2x} 2xy\,dy\,dx$

16. $\displaystyle\int_0^4 \int_{x^2}^{8\sqrt{x}} dy\,dx$

17. $\displaystyle\int_0^2 \int_{y/2}^1 8xy\,dx\,dy$

18. $\displaystyle\int_0^4 \int_0^{\sqrt{x}} dy\,dx$

19. $\displaystyle\int_0^4 \int_{\sqrt{x}}^2 dy\,dx$

20. $\displaystyle\int_0^2 \int_0^{x^2} 2x\,dy\,dx$

21. $\displaystyle\int_0^2 \int_{x^2}^4 2x\,dy\,dx$

22. $\displaystyle\int_{-1}^1 \int_{|x|}^1 dy\,dx$

23. $\displaystyle\int_0^{\pi/2} \int_{\sin x}^1 \cos(x)\,dy\,dx$

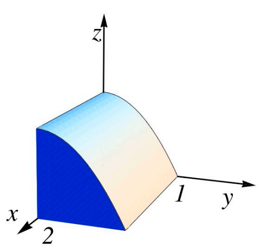
24. $\displaystyle\int_0^4 \int_{x^2}^{8\sqrt{x}} \frac{7\sqrt{y}}{192}\,dy\,dx$

25. $\displaystyle\int_0^1 \int_{2x}^2 4e^{y^2}\,dy\,dx$

26. $\displaystyle\int_0^1 \int_y^1 \cos\left(\frac{\pi x^2}{4}\right) dx\,dy$

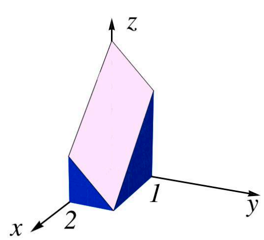
In Exercises 27-34, find the volume of the solid that is bounded by the coordinate planes, and the graph of the equations in 3-space.

27. $z = 1 - y^2$, $x = 2$, $y = 1$



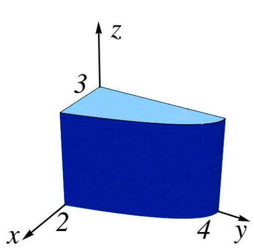
For No. 27

28. $x + y + z = 3$, $x = 2$, $y = 1$



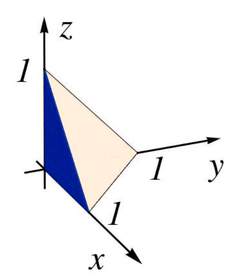
For No. 28

29. $z = 3$, $y = 4 - x^2$



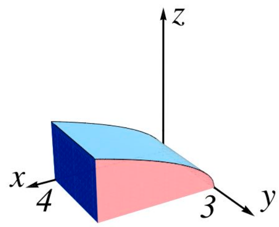
For No. 29

30. $x + y + z = 1$



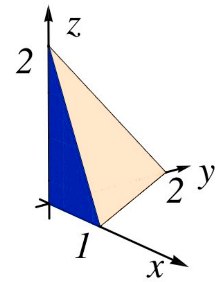
For No. 30

31. $z=\sqrt{x},\ x=4,\ y=3$



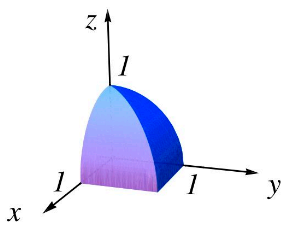
For No. 31

32. $2x+y+z=2$



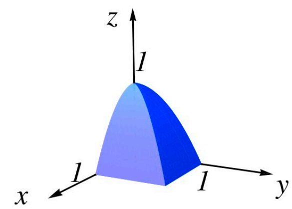
For No. 32

33. $x^2+z^2=1,\ y^2+z^2=1$



For No. 33

34. $z=1-y^2,\ z=1-x^2$



For No. 34

Miscellaneous Problems

35. Find the volume of the solid that is bounded by the plane $x+4y+16z=8$, and the coordinate planes.

36. Find the volume of the solid bounded by the plane $2x+y+z=2$, and the coordinate planes.

37. Find the volume of the solid in the first octant bounded by the plane $x+4y+z=4$, and the coordinate planes.

38. Find the volume of the solid in the first octant bounded by the plane $2x+y+z=4$, and the coordinate planes.

39. Find the volume of a solid in the first octant that is bounded by the surfaces $z=1-y^2$ and $x=2$.

40. Find the volume of a solid in the first octant that is bounded by the graphs of $z=1-y^2$, $y=2x$, $x=0$, and $z=0$.

41. Evaluate $\displaystyle\int\int_R \frac{x}{\sqrt{1-x^2}}dA$ where $R=\left\{(x,y): 0\le x\le \frac{1}{2},\ \arcsin x\le y\le \frac{\pi}{6}\right\}$.

42. Evaluate $\displaystyle\int\int_R \frac{\sin x}{x} dA$ where $R = \{(x, y) : 0 \leq y \leq 1,\ y \leq x \leq 1\}$.

43. Evaluate $\displaystyle\int\int\int_S (xy - z)dV$ where S is the solid in 3-space that is bounded by the graphs of $x = 1$, $y = 4$, $z = 1$, and the coordinate planes.

44. Evaluate $\displaystyle\int\int\int_S \sqrt{xyz}dV$ where S is the solid in 3-space that is bounded by the graphs of $x = 1$, $y = 1$, $z = 1$, and the coordinate planes.

45. Find the volume of a solid that is bounded by the circular cylinder $x^2 + y^2 = 1$, the plane $z + 2y = 2$, and the xy-plane.

46. Evaluate $\displaystyle\int\int\int_S 16dV$ where S is the solid in the first octant bounded by the paraboloid $z = x^2 + y^2$, the plane $z = 1$, and the coordinate planes.

47. A solid S consists of the points (x, y, z) satisfying $0 \leq x, y, z \leq 1$ and $2\sqrt{xy} \leq z$, see Figure for No. 47. Then find the volume of S. In terms of probability, if x, y, z are independent uniform random variables on $[0, 1]$, then the volume of S is the probability that the solutions t of the quadratic equation $xt^2 + zt + y = 0$ are real numbers..

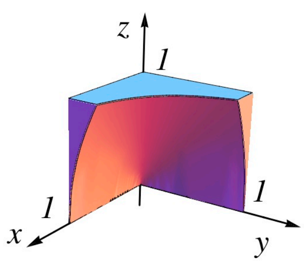


Figure for No. 47

2.2 Change of Variables in Integration

• Jacobian Determinant • Cylindrical Coordinates • Spherical Coordinates

Jacobian Determinant

The substitution method is an important technique in the integration of a function of one variable, namely,

$$\int_{x(a)}^{x(b)} f(x)dx = \int_a^b f(x(u))x'(u)du.$$

We extend the above method to the integration of multivariable functions.

Let $D = \{(u,v) \in \mathbb{R}^2 : a \le u \le b, c \le u \le d\}$ be a rectangular region in the uv-plane, see Figure 1. Let $x = x(u,v)$ and $y = y(u,v)$ be real-valued functions on D with continuous first partial derivatives. Let T be a transformation from D into the xy-plane defined by $T(x,y) = (x(u,v), y(u,v))$. Further, suppose T is a one-to-one function on D, i.e., T maps distinct points in D to distinct points in the xy-plane.

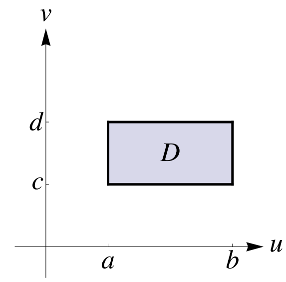


Figure 1
Region S in the uv-plane

Let C be the image of D under the transformation T, i.e,

$$C = T(D) = \{(x,y) \in \mathbb{R}^2 : x = x(u,v), y = y(u,v), (u,v) \in D\}.$$

If the sides of rectangle D are small, we approximate the area of C. The idea[2] is to use the linear approximation of T near (a,c) as described in (36), page 39. Moreover, we use a result from linear algebra claiming that a one-to-one linear transformation maps a rectangle to a parallelogram.

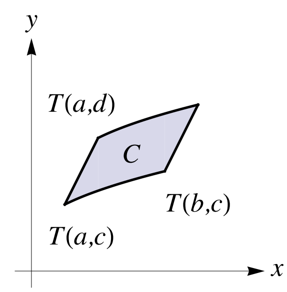


Figure 2
Region $T(R)$ in the xy-plane

In Figure 2, let $\boldsymbol{v}$ be the vector from point $T(a,c)$ to $T(b,c)$, and let $\boldsymbol{w}$ be the vector from $T(a,c)$ to $T(a,d)$. We may approximate $\boldsymbol{v}$ and $\boldsymbol{w}$ by the tangent vectors along the boundary of C, respectively, i.e.,

$$\boldsymbol{v} \approx \Delta u \left.\frac{\partial T}{\partial u}\right|_{(a,c)} \text{ and } \boldsymbol{w} \approx \Delta v \left.\frac{\partial T}{\partial v}\right|_{(a,c)}$$

if $\Delta u = b - a$ and $\Delta v = d - c$ are small. The area of D is $\Delta u \Delta v$, and the area of C is approximately the area of the parallelogram defined by $\boldsymbol{v}$ and $\boldsymbol{w}$. Recall, the area of a parallelogram is the magnitude of the cross product of the vectors defining the sides of the parallelogram. Then

$$\begin{aligned}
\text{Area of } C &\approx \|\boldsymbol{v} \times \boldsymbol{w}\| \\
&\approx \left\| \left.\frac{\partial T}{\partial u}\right|_{(a,c)} \times \left.\frac{\partial T}{\partial v}\right|_{(a,c)} \right\| (\text{Area of } D) \\
&= \left| \text{Det} \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & 0 \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & 0 \end{bmatrix} \right| (\text{Area of } D)
\end{aligned}$$

[2]The linear approximation of T near $\boldsymbol{x_0} = (a,c)$ is defined by $T'(\boldsymbol{x}) = A(\boldsymbol{x} - \boldsymbol{x_0}) + T(\boldsymbol{x_0})$ where A is a 2 by 2 matrix, and $\boldsymbol{x}, \boldsymbol{x_0} \in \mathbb{R}^2$ are realized as column vectors.

where the partial derivatives are evaluated at (a, c). Consequently, we find

$$\text{Area of } C \approx \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \text{ (Area of } D)$$

where

$$\frac{\partial(x,y)}{\partial(u,v)} = \text{Det}\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{bmatrix}.$$

The factor $\dfrac{\partial(x,y)}{\partial(u,v)}$ is called the **Jacobian determinant** of the transformation T.

Let $z = \boldsymbol{f}(x, y)$ be a real-valued continuous function on an elementary region R in the xy-plane. We *sketch* a proof of the change of variables theorem. For a detailed proof, consult an advanced calculus textbook.

$$\begin{aligned}
\int\int_R f(x,y)dxdy &\approx \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i, y_j)\Delta x_i \Delta y_j \\
&\approx \sum_{i=1}^{n}\sum_{j=1}^{m} f(x(u_i, v_j), y(u_i, v_j)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u \Delta v \\
&= \int\int_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv
\end{aligned}$$

We summarize the discussion into a theorem.

Theorem 2.7 Change of Variables

Let $z = \boldsymbol{f}(x, y)$ be a continuous real-valued function defined on an elementary region R in the xy-plane. Let $(x, y) = T(u, v)$ be a one-to-one function defined on an elementary region S in the uv-plane. Suppose the components of T, namely, $x = x(u, v)$ and $y = y(u, v)$ have continuous partial derivatives and $T(S) = R$. Then

$$\int\int_R f(x,y)dxdy = \int\int_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.$$

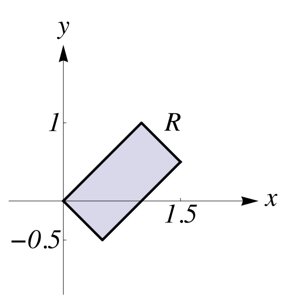


Figure 3
Region in xy-plane

Example 1 Applying a Change of Variables

Evaluate $\displaystyle\int\int_R \sqrt{x^2 - y^2}\, dA$ where R is the rectangular region with vertices at $(0, 0)$, $(1, 1)$, $(\frac{3}{2}, \frac{1}{2})$, and $(\frac{1}{2}, -\frac{1}{2})$. Let $x = \dfrac{1}{2}(u + v)$ and $y = \dfrac{1}{2}(u - v)$.

Solution Solving for u and v, we find $u = x + y$ and $v = x - y$. The four vertices of R correspond to four points in the uv-plane:

(x, y)	(u, v)
$(0, 0)$	$(0, 0)$
$(1, 1)$	$(2, 0)$
$(\frac{3}{2}, \frac{1}{2})$	$(2, 1)$
$(\frac{1}{2}, -\frac{1}{2})$	$(0, 1)$

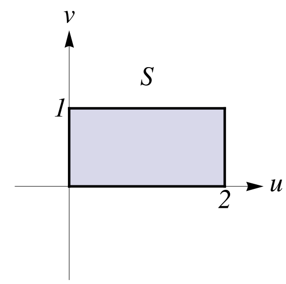


Figure 4
Region in uv-plane

The region R is mapped to region S in the uv-plane, see Figure 2. Since $x = (u+v)/2$ and $y = (u - v)/2$. The Jacobian determinant is given by

$$\frac{\partial(x, y)}{\partial(u, v)} = \text{Det}\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \text{Det}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = -\frac{1}{2}$$

Notice, $x^2 - y^2 = uv$. Applying Theorem 2.7, we obtain

$$\begin{aligned} \int\int_R \sqrt{x^2 - y^2}\, dA &= \int_0^2 \int_0^1 \sqrt{uv} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dvdu \\ &= \frac{1}{2} \left(\int_0^2 \sqrt{u} du \right) \left(\int_0^1 \sqrt{v} dv \right) \\ &= \frac{4\sqrt{2}}{9} \end{aligned}$$

□

Try This 1

Evaluate the same integral in Example 1, but let $x = u + v$ and $y = u - v$.

Example 2 Applying Polar Coordinates

Evaluate $\displaystyle\int\int_R e^{-(x^2+y^2)} dA$ where R is the region between the circles $x^2 + y^2 = 9$ and $x^2 + y^2 = 4$, where $y \geq 0$. See Figure 5.

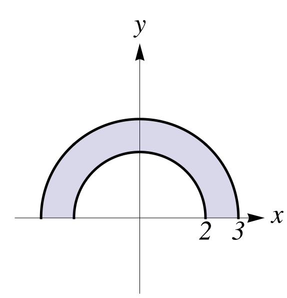


Figure 5
A region S between two concentric circles.

Solution We apply the change of variables $x = r\cos\theta$ and $y = r\sin\theta$, where $r \geq 0$. These equations arise from trigonometry, as shown below.

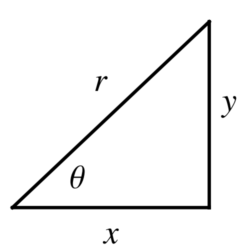


Figure 6 Right triangle trigonometry

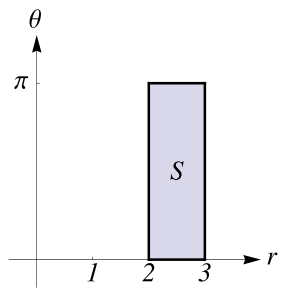


Figure 7
Rectangular region S
in the $r\theta$-plane.

Solving for r and θ, we have $r = \sqrt{x^2+y^2}$ and $\tan\theta = y/x$ if $x \neq 0$. In particular, r is the distance between point (x, y) and the origin. In Figure 5, we deduce that r satisfies $2 \le r \le 3$. Also, θ is the standard angle that the line through the origin and point (x, y) makes with the positive x-axis. From Figure 5, we see θ satisfies $0 \le \theta \le \pi$. The region S in the $r\theta$-plane corresponding to region R is shown in Figure 7.

Since $x = r\cos\theta$ and $y = r\sin\theta$, the Jacobian determinant is given by

$$\begin{aligned}\frac{\partial(x,y)}{\partial(r,\theta)} &= \operatorname{Det}\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{bmatrix} = \operatorname{Det}\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \\ &= r\left(\cos^2\theta + \sin^2\theta\right) = r.\end{aligned}$$

Applying $r^2 = x^2 + y^2$ and the change of variables, we obtain

$$\begin{aligned}\int\int_R e^{-(x^2+y^2)} dA &= \int\int_S e^{-r^2}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dr d\theta \\ &= \int_0^{\pi}\int_2^3 re^{-r^2} dr d\theta \\ &= -\frac{1}{2}\int_0^{\pi} e^{-r^2}\Big|_{r=2}^{r=3} d\theta \\ \int\int_R e^{-(x^2+y^2)} dA &= \frac{\pi}{2}\left(e^{-4} - e^{-9}\right).\end{aligned}$$

□

Try This 2

Evaluate $\int\int_R \sqrt{x^2+y^2} dA$ where R is the circular region bounded by $x^2 + y^2 = 1$.

Apply the transformation defined by $x = r\cos\theta$ and $y = r\sin\theta$.

Next, we discuss the change of variables for triple integrals. Let T be a one-to-one transformation from a rectangular box

$$B = \{(u,v,w) : a_1 \le u \le b_1, a_2 \le v \le b_2, a_3 \le w \le b_3\}$$

into $\mathbb{R}^3$ where $a_i < b_i$ are real constants, $i = 1, 2, 3$. We investigate the effect of T on the volume of B.

Let $x = x(u,v,w)$, $y = y(u,v,w)$, $z = z(u,v,w)$ be real-valued functions on B with continuous partial derivatives such that $(x, y, z) = T(u, v, w)$. The derivative T' of T at (a_1, a_2, a_3) provides the best linear approximation[3] of T near (a_1, a_2, a_3).

[3]The linear approximation of T near $\boldsymbol{x_0} = (a_1, a_2, a_3)$ is defined by $T'(\boldsymbol{x}) = A(\boldsymbol{x} - \boldsymbol{x_0}) + T(\boldsymbol{x_0})$ where A is a 3 by 3 matrix, and $\boldsymbol{x}, \boldsymbol{x_0} \in \mathbb{R}^3$ are realized as column vectors. See (36), page 39.

We apply a result from linear algebra stating that an injective linear transformation in 3-space should map B onto a parallelepiped C. If all the sides of B are small, then the volume of $T(B)$ is approximately the volume of the parallelepiped $T'(B)$ whose three defining edges are vectors

$$\boldsymbol{v}_1 = \Delta u \frac{\partial T}{\partial u}, \quad \boldsymbol{v}_2 = \Delta v \frac{\partial T}{\partial v}, \quad \boldsymbol{v}_3 = \Delta w \frac{\partial T}{\partial w}.$$

where the partial derivatives are evaluated at (a,b,c), $\Delta u = b_1 - a_1$, $\Delta v = b_2 - a_2$, and $\Delta w = b_3 - a_3$. By Theorem 1.2, the volume of $T'(B)$ is the absolute value of the triple scalar product of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$. Applying identity (1) in page 152, the triple scalar product is the determinant of the 3 by 3 matrix whose rows are the vectors. That is,

$$(\boldsymbol{v}_1 \times \boldsymbol{v}_2) \cdot \boldsymbol{v}_3 \quad = \quad \frac{\partial(x,y,w)}{\partial(u,v,w)} \mathrm{Vol}(B)$$

where the volume of B is $\mathrm{Vol}(B) = \Delta u \Delta v \Delta w$, and the Jacobian determinant is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \mathrm{Det} \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

and the partial derivatives are evaluated at (a,b,c). Then an effect of transformation T on the volume of B is given by

$$\mathrm{Vol}(T(B)) \approx \mathrm{Vol}(T'(B)) = \left| \frac{\partial(x,y,w)}{\partial(u,v,w)} \right| \mathrm{Vol}(B).$$

Furthermore, let $A = \boldsymbol{F}(x,y,z)$ be a real-valued function. We sketch a proof of the change of variables for triple integrals.

$$\begin{aligned} \int\int\int_M \boldsymbol{F}(x,y,z)dV \quad &\approx \quad \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{p} \boldsymbol{F}(x_i,y_j,z_k)\Delta x_i \Delta y_j \Delta z_k \\ &\approx \quad \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{p} (\boldsymbol{F}\circ T)(u_i,v_j,w_k) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| \Delta u \Delta v \Delta w \\ &= \quad \int\int\int_N (\boldsymbol{F}\circ T)(u,v,w) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dV \end{aligned}$$

A detailed proof of the change of variables is found in advanced calculus books.

Theorem 2.8 Change of Variables II

Let $A = \boldsymbol{F}(x,y,z)$ be a continuous real-valued function defined on an elementary region M in the xyz-space. Let $(x,y,z) = T(u,v,w)$ be a one-to-one function defined on an elementary region N in the uvw-space. Suppose the components of T, namely, $x = x(u,v,w)$, $y = y(u,v,w)$ and $z = y(u,v,w)$ have continuous partial derivatives and $T(N) = M$. Then

$$\int\int\int_M \boldsymbol{F}(x,y,z)dV = \int\int\int_N (\boldsymbol{F}\circ T)(u,v,w) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dV.$$

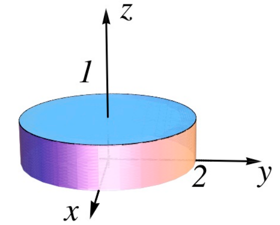


Figure 8
Cylindrical solid of radius 2

Example 3 Triple Integral and Change of Variables

Evaluate $\int\int\int_M \sqrt{x^2+y^2}\, dV$ where M is the solid bounded by the circular cylinder $x^2+y^2=4$, and the planes $z=0$ and $z=1$, see Figure 8.

Apply the transformation $x=r\cos\theta$, $y=r\sin\theta$, and $z=z$.

Solution The base of the solid M lies in the xy-plane. Notice, $x^2+y^2=r^2$. Then each point $(x,y,0)$ in the base satisfies $x^2+y^2\le 4$, or equivalently $0\le r\le 2$ and $0\le\theta\le 2\pi$. In addition, the z-component of any point in M satisfies $0\le z\le 1$.

Then the solid defined by

$$N=\{(r,\theta,z): 0\le r\le 2, 0\le\theta\le 2\pi, 0\le z\le 1\}$$

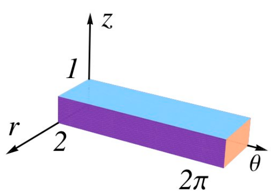


Figure 9
Rectangular solid N

is mapped into M by the transformation, see Figure 9. The Jacobian determinant of the transformation is given by

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \text{Det}\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = r$$

Applying the change of variables theorem, we obtain

$$\begin{aligned} \int\int\int_M \sqrt{x^2+y^2}\, dV &= \int\int\int_N \sqrt{r^2}\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| dV \\ &= \int_0^1\int_0^{2\pi}\int_0^2 r^2 dr d\theta dz \\ &= \frac{16\pi}{3}. \end{aligned}$$

□

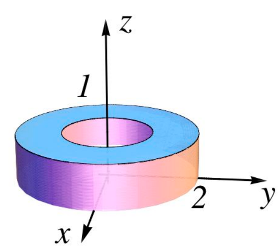


Figure 10
Washer M between two cylindrical cylinders

Try This 3

Evaluate $\int\int\int_M e^z dV$ where M is the solid between two circular cylinders $x^2+y^2=1$ and $x^2+y^2=4$, and the planes $z=0$ and $z=1$. See Figure 10, and apply the same transformations used in Example 3.

Example 4 Changing the Variables in a Triple Integral

Evaluate $\displaystyle\iiint_S \frac{dV}{1+x^2+y^2+z^2}$ where the solid S is the unit ball bounded by $x^2+y^2+z^2=1$. Apply the transformation $x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, and $z=\rho\cos\phi$ where $0\le\rho\le1$, $0\le\phi\le\pi$, and $0\le\theta\le2\pi$.

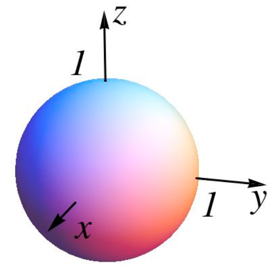


Figure 11
Solid ball S of radius 1.

Solution We find

$$\begin{aligned} x^2+y^2+z^2 &= (\rho\sin\phi\cos\theta)^2+(\rho\sin\phi\sin\theta)^2+(\rho\cos\phi)^2 \\ &= (\rho\sin\phi)^2(\cos^2\theta+\sin^2\theta)+(\rho\cos\phi)^2 \\ &= (\rho\sin\phi)^2+(\rho\cos\phi)^2 \\ &= \rho^2. \end{aligned}$$

Observe, ρ is the distance between (x,y,z) and the origin. Also, θ is the angle between the x-axis and the vector from the origin to $(x,y,0)$ for $\tan\theta=y/x$. The dot product of (x,y,z) with $\boldsymbol{k}$ satisfies

$$(\rho\sin\phi\cos\theta,\rho\sin\phi\sin\theta,\rho\cos\phi)\cdot\boldsymbol{k}=\rho\cos\phi.$$

In particular, ϕ is the angle between vector (x,y,z) and $\boldsymbol{k}$.

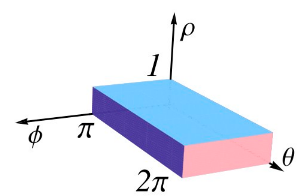


Figure 12
Rectangular solid T

The solid ball S is mapped into a rectangular solid T, see Figure 12. The Jacobian determinant of the transformation is given below but we postpone the proof to page 83 of the section:

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \operatorname{Det}\begin{bmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial\theta} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\phi} & \frac{\partial z}{\partial\theta} \end{bmatrix} = \rho^2\sin\phi.$$

Applying the change of variables, we obtain

$$\begin{aligned} \iiint_S \frac{dV}{1+x^2+y^2+z^2} &= \iiint_T \frac{1}{1+\rho^2}\left|\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right| d\rho d\phi d\theta \\ &= \int_0^{2\pi}\int_0^{\pi}\int_0^1 \frac{\rho^2\sin\phi}{1+\rho^2}d\rho d\phi d\theta \\ &= 4\pi\int_0^1 \frac{\rho^2}{1+\rho^2}d\rho \\ &= 4\pi\int_0^1\left(1-\frac{1}{1+\rho^2}\right)d\rho \\ &= 4\pi(\rho-\arctan\rho\Big|_{\rho=0}^{\rho=1} \\ \iiint_S \frac{dV}{1+x^2+y^2+z^2} &= \pi(4-\pi) \end{aligned}$$

□

Try This 4

Evaluate $\displaystyle\int\int\int_S dV$ where S is the unit ball that is bounded by the sphere $x^2 + y^2 + z^2 = 1$. Apply the transformation $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho \cos\phi$ where $0 \le \rho \le 1$, $0 \le \phi \le \pi$, and $0 \le \theta \le 2\pi$.

Cylindrical Coordinates

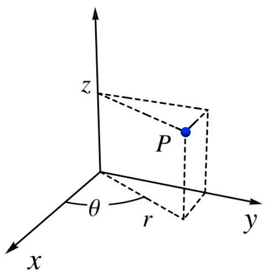


Figure 13
Cylindrical coordinates (r, θ, z) of point P.

The cylindrical coordinates of a point $(x, y, z) \in \mathbb{R}^3$ in Cartesian coordinates are (r, θ, z) where

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x} \text{ if } x \ne 0.$$

In particular, θ is the angle between the positive x-axis and the line segment joining the origin to $(x, y, 0)$. Usually, we require $0 \le \theta < 2\pi$. Also, (r, θ) represent the polar coordinates of (x, y). Note, r is a real number representing the directed distance from the origin to (x, y). The cylindrical coordinates of a point are not unique. For instance, the cylindrical coordinates $(2, \pi/6, 3)$, $(2, 13\pi/6, 3)$, and $(-2, 7\pi/6, 3)$ represent the same point in 3-space. The identities below

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

are helpful when converting to Cartesian coordinates from cylindrical coordinates.

Example 5 Switching between Cartesian and Cylindrical Coordinates

Find the cylindrical coordinates of the point $P(x, y, z) = (2, -2\sqrt{3}, 4)$ in Cartesian coordinates. Then find the Cartesian coordinates of the point $Q(r, \theta, z) = \left(4, \frac{5\pi}{6}, 1\right)$ given in cylindrical coordinates.

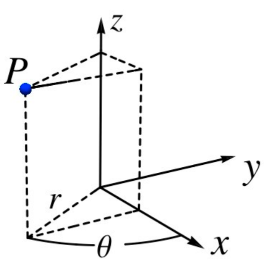


Figure 14
Cylindrical point
$(r, \theta, z) = \left(4, -\frac{\pi}{3}, 4\right)$.

Solution Since $P(x, y, z) = (2, -2\sqrt{3}, 4)$, we find

$$r = \sqrt{x^2 + y^2} = \sqrt{4 + 12} = 4.$$

From the identity $\tan\theta = y/x = -\sqrt{3}$, we may choose $\theta = -\frac{\pi}{3}$, see Figure 14. Then the cylindrical coordinates of point P are

$$(r, \theta, z) = \left(4, -\frac{\pi}{3} + 2k\pi, 4\right), \ \left(-4, \frac{2\pi}{3} + 2k\pi, 4\right)$$

where k is an integer.

Using the cylindrical coordinates $Q(r, \theta, z) = \left(4, \frac{5\pi}{6}, 1\right)$, we find

$$\begin{aligned} x &= r\cos\theta = 4\cos\frac{5\pi}{6} = -2\sqrt{3} \\ y &= r\sin\theta = 4\sin\frac{5\pi}{6} = 2. \end{aligned}$$

Thus, the Cartesian coordinates of Q are

$$Q(x, y, z) = (-2\sqrt{3}, 2, 1)$$

since $z = 1$.

□

Try This 5

Find the cylindrical coordinates of point $P(x, y, z) = (-1, 1, 2)$. Also, find the Cartesian coordinates of point $\left(3, \frac{3\pi}{2}, 1\right)$ given in cylindrical coordinates.

Example 6 Finding the Volume of a Solid

Find the volume of the solid S bounded by the cone $x^2 + y^2 = z^2$ and the sphere $x^2 + y^2 + z^2 = 2$ where $z \geq 0$, see Figure 15.

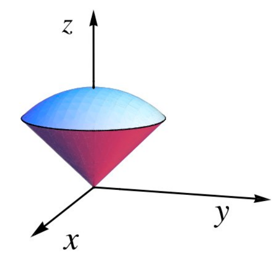


Figure 15
Solid S bounded between a cone and a sphere.

Solution The volume of S is $\int\int\int_S dV$. In cylindrical coordinates, we have the identity $x^2 + y^2 = r^2$. Then an equation of the cone $x^2 + y^2 = z^2$ in cylindrical coordinates is $z = r$. The cylindrical coordinates for the sphere is $r^2 + z^2 = 2$.

To find where the surfaces intersect, substitute the former equation into the later one. Then $2z^2 = 2$ and $z = 1$ for $z \geq 0$. Thus, every point (x, y, z) in S satisfies $x^2 + y^2 \leq 1$ and $\sqrt{x^2 + y^2} \leq z \leq \sqrt{2 - x^2 - y^2}$.

Since $x^2 + y^2 \leq 1$, we have $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$, see the base of the solid in Figure 16. The solid T corresponds to solid S under the change in coordinates from rectangular to cylindrical coordinates. In Example 3, the Jacobian determinant associated to the change of variables is

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = r.$$

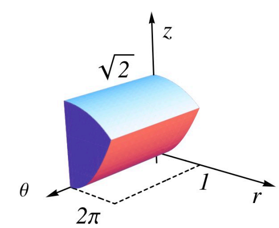


Figure 16
Solid T between $z = r^2$, $r^2 + z^2 = 2$, and $\theta = 2\pi$ in the first octant.

From $\sqrt{x^2 + y^2} \leq z \leq \sqrt{2 - x^2 - y^2}$, we obtain $r \leq z \leq \sqrt{2 - r^2}$. Applying the change of variables theorem for integration, we find

$$\begin{aligned}
\int\int\int_S dV &= \int\int\int_T \left|\frac{\partial(x, y, z)}{\partial(r, \theta, z)}\right| dV \\
&= \int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} r dz dr d\theta \\
&= \int_0^{2\pi}\int_0^1 \left(r\sqrt{2 - r^2} - r^2\right) dr d\theta \\
\text{Volume} &= \frac{4\pi}{3}(\sqrt{2} - 1).
\end{aligned}$$

□

Try This 6

Find the volume of the solid bounded by the ellipsoid $x^2 + y^2 + 4z^2 = 4$.

Express the volume as a triple integral that uses cylindrical coordinates.

See Figure 17.

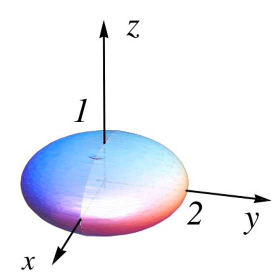


Figure 17
The ellipsoid

$$x^2 + y^2 + 4z^2 = 4.$$

Spherical Coordinates

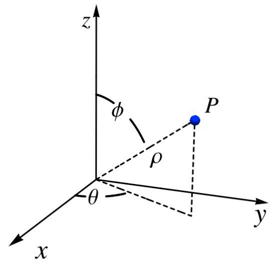


Figure 18
Spherical coordinates (ρ, θ, ϕ) of point P.

The spherical coordinates of a point $(x, y, z) \in \mathbb{R}^3$ in Cartesian coordinates are (ρ, θ, ϕ) where

$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \tan\theta = \frac{y}{x} \text{ if } x \neq 0, \text{ and } \rho\cos\phi = (x, y, z) \cdot \boldsymbol{k}.$$

The coordinate $0 \leq \theta < 2\pi$ is the angle between the positive x-axis and a vector from the origin to point $(x, y, 0)$, see Figure 18. The coordinate $\rho \geq 0$ is the distance between (x, y, z) and the origin. Further, we require $0 \leq \phi \leq \pi$ and ϕ is the angle between the positive z-axis and the line segment joining the origin to (x, y, z). The identities below

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$$

are useful when converting to Cartesian coordinates from spherical coordinates.

Example 7 Switching between Cartesian and Spherical Coordinates

Find the spherical coordinates of point $A(x, y, z) = \left(\frac{3}{2}, \frac{\sqrt{3}}{2}, 1\right)$.

Then find the Cartesian coordinates of the point $B(\rho, \theta, \phi) = \left(8, \frac{7\pi}{4}, \frac{\pi}{6}\right)$.

Solution Since $A(x, y, z) = \left(\frac{3}{2}, \frac{\sqrt{3}}{2}, 1\right)$, we find

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{\frac{9}{4} + \frac{3}{4} + 1} = 2.$$

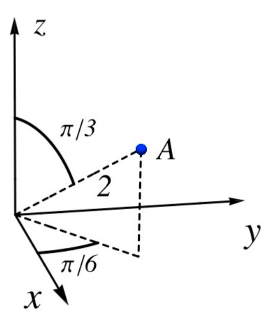


Figure 19
Spherical coordinates of point $A\left(2, \frac{\pi}{6}, \frac{\pi}{3}\right)$.

Using the identity $\tan\theta = \frac{y}{x} = \frac{1}{\sqrt{3}}$, we choose $\theta = \frac{\pi}{6}$, see Figure 19. Also, we find

$$\begin{aligned} \rho\cos\phi &= (x, y, z) \cdot \boldsymbol{k} \\ 2\cos\phi &= \left(\frac{3}{2}, \frac{\sqrt{3}}{2}, 1\right) \cdot \boldsymbol{k} = 1 \\ \phi &= \frac{\pi}{3}. \end{aligned}$$

Then the spherical coordinates of point A are

$$A(\rho, \theta, \phi) = \left(2, \frac{\pi}{6}, \frac{\pi}{3}\right).$$

Next, we determine the Cartesian coordinates of $B(\rho, \theta, \phi) = \left(8, \frac{7\pi}{4}, \frac{\pi}{6}\right)$.

$$\begin{aligned} x &= \rho\sin\phi\cos\theta = 8\sin\frac{\pi}{6}\cos\frac{7\pi}{4} = 2\sqrt{2} \\ y &= \rho\sin\phi\sin\theta = 8\sin\frac{\pi}{6}\sin\frac{7\pi}{4} = -2\sqrt{2} \\ z &= \rho\cos\phi = 8\cos\frac{\pi}{6} = 4\sqrt{3} \end{aligned}$$

Thus, the Cartesian coordinates are $B(x, y, z) = (2\sqrt{2}, -2\sqrt{2}, 4\sqrt{3})$.

□

Try This 7

Find the spherical coordinates of point $C(x, y, z) = (3, 3\sqrt{3}, 6)$. Also, find the rectangular (or Cartesian) coordinates of $D(\rho, \theta, \phi) = \left(2\sqrt{2}, \frac{3\pi}{4}, \frac{\pi}{3}\right)$.

For the purpose of evaluating triple integrals, we evaluate the Jacobian determinant of the transformation to Cartesian from spherical coordinates. The spherical coordinates (ρ, ϕ, θ) of a point (x, y, z) satisfy

$$x = \rho \sin\phi \cos\theta, \; y = \rho \sin\phi \sin\theta, \; z = \rho \cos\phi.$$

We compute the determinant below by expanding the minors in the third row, i.e.,

$$\begin{aligned}
\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} &= \text{Det} \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{bmatrix} \\
&= \text{Det}[A_{3,1}]\cos\phi - \text{Det}\,[A_{3,2}]\,(-\rho\sin\phi)
\end{aligned}$$

where the 2 by 2 matrices $A_{3,j}$ are obtained from the above 3 by 3 matrix by crossing out the third row and jth column, $j = 1, 2$. Namely, the matrices $A_{3,j}$ are

$$A_{3,1} = \begin{bmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{bmatrix}, \quad A_{3,2} = \begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{bmatrix}$$

Then

$$\begin{aligned}
\text{Det}[A_{3,1}] &= \rho^2 \sin\phi\cos\phi\cos^2\theta + \rho^2\sin\phi\cos\phi\sin^2\theta \\
&= \rho^2\sin\phi\cos\phi(\cos^2\theta + \sin^2\theta) \;=\; \rho^2\sin\phi\cos\phi.
\end{aligned}$$

Using a similar calculation, we find

$$\text{Det}[A_{3,2}] \;=\; \rho\sin^2\phi.$$

Consequently, the Jacobian determinant for the change of variables to Cartesian coordinates from spherical coordinates is given by

$$\begin{aligned}
\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} &= \text{Det}[A_{3,1}]\cos\phi - \text{Det}\,[A_{3,2}]\,(-\rho\sin\phi) \\
&= (\rho^2\sin\phi\cos\phi)\cos\phi - (\rho\sin^2\phi)(-\rho\sin\phi) \\
&= \rho^2\sin\phi. \qquad (1)
\end{aligned}$$

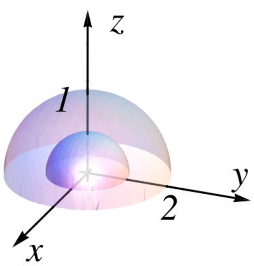


Figure 20
A solid S bounded by two hemispheres.

Example 8 Triple Integral and Spherical Coordinates

Evaluate $\int\int\int_S zdV$ where S is the solid between the two spheres of radii 1 and 2, and centered at the origin. See Figure 20.

Solution In spherical coordinates, we have $z = \rho\cos\phi$. Also, the Jacobian determinant for spherical coordinates is $\rho^2\sin\phi$, see (1).

Analyzing Figure 20, the spherical coordinate ρ satisfies $1 \le \rho \le 2$ since the distance between a point in S and the origin is a value between 1 and 2, inclusive. The coordinate ϕ satisfies $0 \le \phi \le \pi/2$ for the angle between vector $\boldsymbol{k}$ and a vector from the origin to a point in S is a value from 0 to $\pi/2$ radians. Also, the polar angle of any point is S satisfies $0 \le \theta < 2\pi$. In Figure 21, we see that the spherical coordinates of S describe a rectangular box.

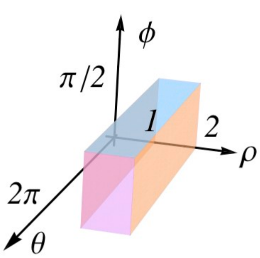


Figure 21
The solid S in spherical coordinates.

Then by the change of variables theorem, we obtain

$$\begin{aligned}
\int\int\int_S zdV &= \int_0^{2\pi}\int_0^{\pi/2}\int_1^2 \rho^3\sin\phi\cos\phi\; d\rho\; d\phi\; d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/2} \left.\frac{\rho^4}{4}\right|_{\rho=1}^{\rho=2} \sin\phi\cos\phi\; d\phi\; d\theta \\
&= \frac{15}{4}\int_0^{2\pi}\int_0^{\pi/2} \sin\phi\cos\phi\; d\phi\; d\theta \\
&= \frac{15}{4}\int_0^{2\pi} \left.\frac{1}{2}\sin^2(\phi)\right|_{\phi=0}^{\phi=\pi/2} d\theta \\
&= \frac{15}{8}\int_0^{2\pi} d\theta \\
\int\int\int_S zdV &= \frac{15\pi}{4}.
\end{aligned}$$

□

Try This 8

Set up a triple integral in spherical coordinates for the volume of the part of solid S in Figure 20 that lies in the first octant. Then evaluate the integral.

2.2 Check-It Out

Evaluate the integral by applying the indicated transformation.

1. $\displaystyle\int\int_R e^{x^2+y^2}dA$ where R is the circular region of radius 1 and centered at the origin.

   Apply the transformations $x = r\cos\theta$ and $y = r\sin\theta$.

2. $\displaystyle\int\int\int_S \sqrt{x^2+y^2+z^2}dV$ where S is the unit ball, i.e., the solid bounded by the unit sphere $x^2+y^2+z^2 = 1$. Apply a change of variables by using spherical coordinates, see Example 8.

True or False. *If false, revise the statement to make it true or explain.*

1. If $x = u - v$ and $y = u + v$, the Jacobian determinant satisfies $\dfrac{\partial(x,y)}{\partial(u,v)} = 2$.

2. If $x = r\cos\theta$ and $y = r\sin\theta$, the Jacobian determinant satisfies $\dfrac{\partial(x,y)}{\partial(r,\theta)} = r^2$.

3. If $R = \{(x,y) : x^2+y^2 \le 25\}$, then $\displaystyle\int\int_R e^{\sqrt{x^2+y^2}}dA = \int_0^{2\pi}\int_0^5 re^r drd\theta$.

4. The Jacobian determinant for spherical coordinates satisfies $\dfrac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \rho\sin\phi$.

5. If $x = 2u$ and $y = 3v$, then by the change of variables we have
   $\displaystyle\int_0^3\int_0^2 \boldsymbol{f}(x,y)dxdy = \int_0^1\int_0^1 \boldsymbol{f}(2u,3v)dudv.$

Exercises for Section 2.2

In Exercises 1-8, evaluate the integral by applying the indicated change of variables. The region of integration R in Cartesian coordinates is shown. The region S to where R is mapped by the change of variables is shown, too. In addition, state the Jacobian determinant of the change of variables.

1. $\displaystyle\int\int_R ydA$ where R is the quadrilateral region with vertices at $(0,0)$, $(4,0)$, $(1,2)$, and $(5,2)$.

   Let $x = \dfrac{v-u}{2}$ and $y = v$.

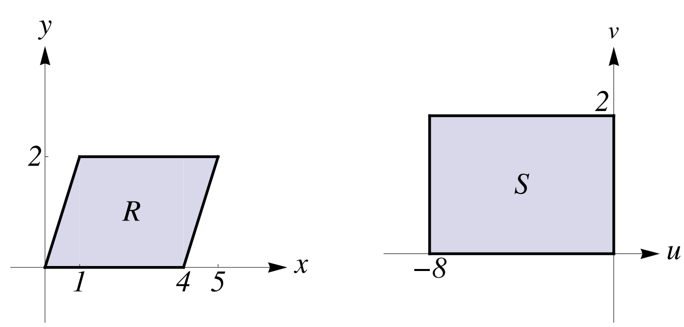


Figure for 1

2. $\displaystyle\int\int_R (y-x)\,dA$ where R is the quadrilateral region bounded by the lines $y = x+1$, $y = x+2$, $y = 2x-1$, and $y = 2x+2$. Let $x = u - v$ and $y = 2u - v$.

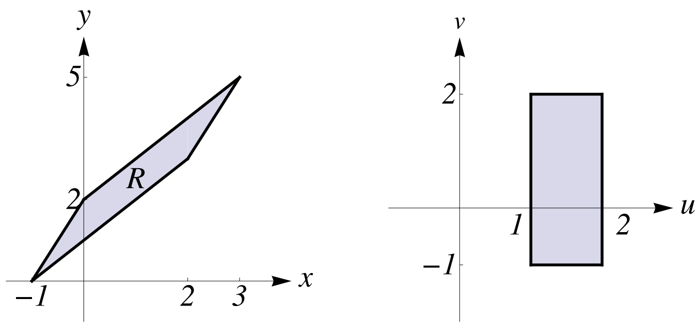


Figure for 2

3. $\displaystyle\int\int_R \frac{x-y}{x+y}\,dA$ where R is the quadrilateral region with vertices at $(1,0)$, $(2,0)$, $(1,1)$, and $(\frac{1}{2},\frac{1}{2})$.

Let $x = \dfrac{u+v}{2}$ and $y = \dfrac{u-v}{2}$.

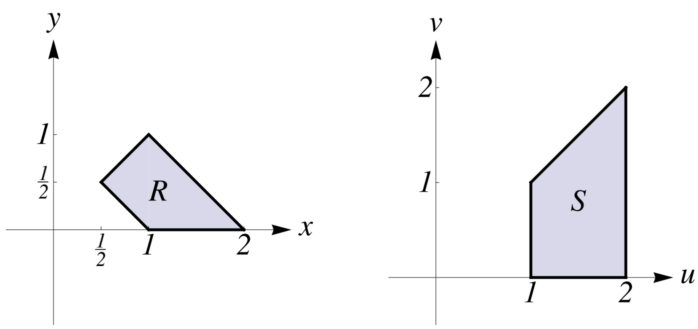


Figure for 3

4. $\displaystyle\int\int_R \left(x^2+y^2\right)^2 dA$ where R is the circular unit circle centered at the origin.

Let $x = r\cos\theta$ and $y = r\sin\theta$.

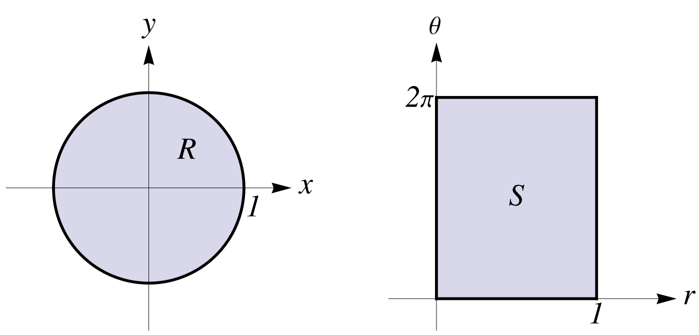


Figure for 4

5. $\displaystyle\int\int\int_R z dV$ where R is the solid in the first octant bounded by the surfaces $z = x^2 + y^2$, $x^2 + y^2 = 1$, and the coordinate planes. Let $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$.

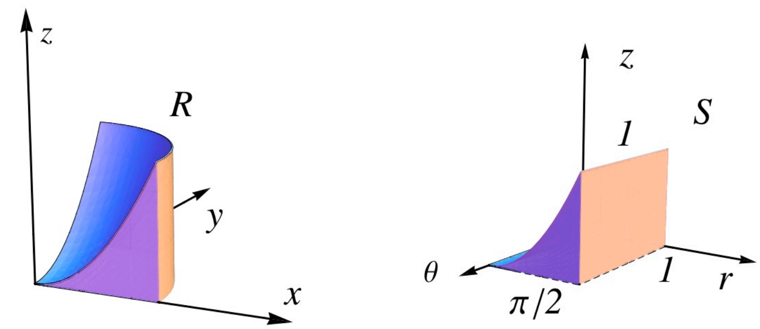


Figure for 5

6. $\displaystyle\int\int\int_R e^{(x^2+y^2+z^2)^{3/2}} dV$ where R is the solid bounded by the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$. Change the variables to spherical coordinates.

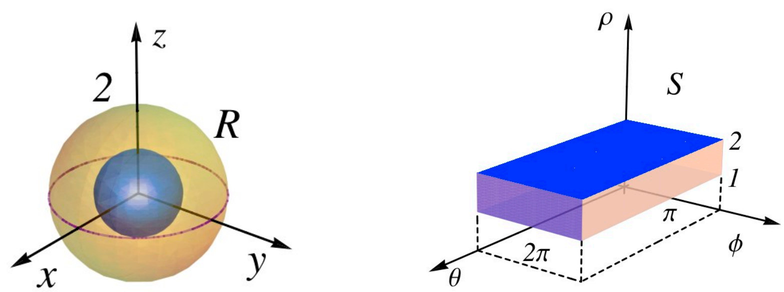


Figure for 6

7. $\displaystyle\int\int\int_R 2z dV$ where R is the solid bounded by the surfaces $z = \sqrt{4 - x^2 - y^2}$, $z = \sqrt{2 - x^2 - y^2}$, $x = 1$, $y = x$, and $y = 0$. Change the variables to cylindrical coordinates.

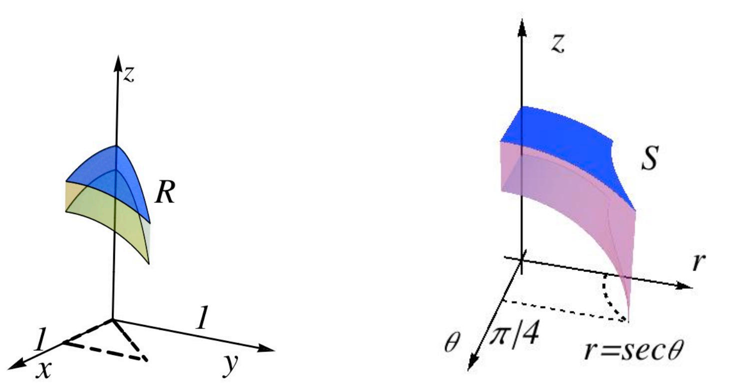


Figure for 7

8. $\displaystyle\iiint_R z\,dV$ where R is the solid bounded by the surfaces $z = \sqrt{x^2+y^2}$, $x^2+y^2 = 1$, and $z = 0$. Change the variables to spherical coordinates.

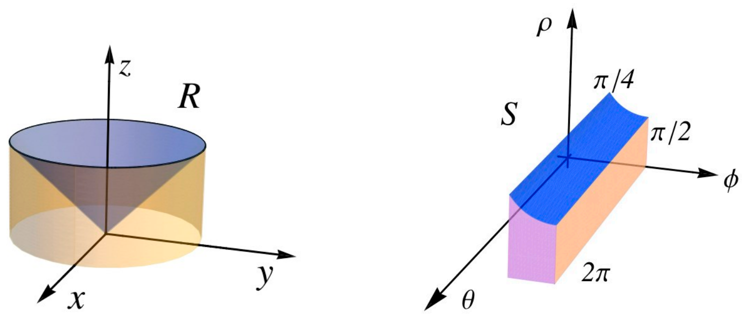


Figure for 8

In Exercises 9-14, if a point is defined in Cartesian coordinates, express the point in cylindrical coordinates. If the point is given in cylindrical coordinates, change to Cartesian coordinates.

9. $(x, y, z) = (2\sqrt{3}, 2, 1)$

10. $(r, \theta, z) = \left(2, \dfrac{5\pi}{3}, 3\right)$

11. $(x, y, z) = (-6, 2\sqrt{3}, 2)$

12. $(r, \theta, z) = \left(2, \dfrac{3\pi}{2}, 4\right)$

13. $(r, \theta, z) = \left(\sqrt{2}, \dfrac{2\pi}{3}, -2\right)$

14. $(x, y, z) = \left(-\dfrac{\sqrt{3}}{2}, -\dfrac{3}{2}, -3\right)$

In Exercises 15-20, if a point is given in Cartesian coordinates, express the point in spherical coordinates. If the point is expressed in spherical coordinates, convert to Cartesian coordinates.

15. $(x, y, z) = (1, \sqrt{3}, 2\sqrt{3})$

16. $(\rho, \theta, \phi) = \left(4, \dfrac{2\pi}{3}, \dfrac{5\pi}{6}\right)$

17. $(x, y, z) = \left(\dfrac{3}{2}, -\dfrac{\sqrt{3}}{2}, -1\right)$

18. $(\rho, \theta, \phi) = \left(2, \dfrac{7\pi}{6}, \dfrac{2\pi}{3}\right)$

19. $(\rho, \theta, \phi) = \left(2, \pi, \dfrac{3\pi}{4}\right)$

20. $(x, y, z) = \left(6\sqrt{3}, -6, 4\sqrt{3}\right)$

In Exercises 21-24, evaluate the integral by applying the indicated change of variables. Sketch the indicated region R. Also, sketch the image S of the transformation defined by the change of variables.

21. $\displaystyle\iint_R (x-y)\,dA$ where R is a quadrilateral region with vertices $(0, 0)$, $(1, 1)$, $(-5, 1)$, and $(-6, 0)$.

Apply the change of variables $x = 2u + v$ and $y = v$.

22. $\int\int_R (y-x)\,dA$ where R is a quadrilateral region with vertices $(0, 0)$, $(2, 0)$, $(3, 1)$, and $(1, 1)$.

Apply the change of variables $x = -u + v$ and $y = v$.

23. $\displaystyle\iint_R dA$ where R is the region bounded by the lines $y = x$, $y = x + 1$, $y = -x$, and $y = -x + 1$.

Let $x = \dfrac{-u+v}{2}$ and $y = \dfrac{u+v}{2}$.

24. $\displaystyle\int\int_R \sqrt{y+2x}\, dA$ where R is the region bounded by the lines $y = 4x+2$, $y = 4x+5$, $y = -2x+3$, and $y = -2x+1$. Let $x = \dfrac{-u+v}{6}$ and $y = \dfrac{u+2v}{3}$.

In Exercises 25-28, solve for x and y in the given substitution. Then evaluate the integral by applying a change of variables. Sketch the region R in the Cartesian plane, and the image S of the transformation defined by the change of variables.

25. $\displaystyle\int\int_R \sqrt{x-y}\, dA$ where R is a trapezoid with vertices $(1,0)$, $(2,0)$, $(0,-2)$ and $(0,-1)$.

    Substitute $u = x+y$ and $v = x-y$.

26. $\displaystyle\int\int_R xy\, dA$ where R is the region in the first quadrant that is bounded by the graphs of $y = x$, $y = 3x$, $xy = 1$, and $xy = 3$. Substitute $u = xy$ and $v = y$.

27. $\displaystyle\int\int_R (x+y)\, dA$ where R is the triangular region with vertices $(0,0)$, $(2,1)$, and $(1,2)$.

    Substitute $u = -x+2y$ and $v = x+y$.

28. $\displaystyle\int\int_R (x-y)(1+x+y)dA$ where R is the triangular region with vertices $(0,0)$, $(1,0)$, and $(1,1)$.

    Substitute $u = x+y$ and $v = x-y$.

29. $\displaystyle\int\int_R (x^2-y^2)dA$ where R is a quadrilateral region bounded by the lines $y = x$, $y = x-2$, $y = -x$, and $y = -x+3$. Substitute $x = \dfrac{u+v}{2}$ and $y = \dfrac{-u+v}{2}$.

30. Verify the identity $\displaystyle\int\int_{R_n} e^{-x^2-y^2} dA = \pi\left(1-e^{-n^2}\right)$ where $R_n = \{(x,y) : x^2+y^2 \le n^2\}$, $n \ge 1$.

31. We adopt the notation of Exercise 30. Applying the Monotone Convergence Theorem[4], we have

$$\int\int_{\mathbb{R}^2} e^{-x^2-y^2} dA = \lim_{n\to\infty} \int\int_{R_n} e^{-x^2-y^2} dA.$$

    Moreover, by Fubini's theorem, we have

$$\int\int_{\mathbb{R}^2} e^{-x^2-y^2} dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} dxdy.$$

    Then verify the identity $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

[4]The Monotone Convergence Theorem and a version of Fubini's Theorem used in Exercise 31 are proved in a course such as integration theory or real analysis. The proof is beyond the scope of this text.

32. A solid is bounded by the surfaces $x^2 + y^2 = 1$, $z = 0$, and $z = 2$. Sketch the solid. Then find the volume of the solid by evaluating a triple integral in cylindrical coordinates.

In Exercises 33-36, evaluate the integral by changing the variables to spherical coordinates. Include a graph of solid R.

33. $\int\int\int_R \left(1 - x^2 - y^2 - z^2\right) dV$ where R is the unit ball of radius 1 and centered at the origin.

34. $\int\int\int_R z\, dV$ where R is a solid in the first octant that lies inside the sphere $x^2 + y^2 + z^2 = 4$.

35. $\int\int\int_R dV$ which is the volume of the solid R inside the sphere $\rho = 1$ and above the cone $\phi = \frac{\pi}{3}$.

36. $\int\int\int_R dV$ which is the volume of the solid R inside the sphere $\rho = 2$, above the plane $z = 0$, and below the cone $\phi = \frac{\pi}{4}$.

In Exercises 37-38, evaluate the integral by changing the variables to cylindrical coordinates. Also, sketch the solid R.

37. $\int\int\int_R xy\, dV$ where R is the solid in the first octant bounded by $x^2 + y^2 = 4$, $z = 0$, and $z = 1$.

38. $\int\int\int_R \frac{z}{1 + x^2 + y^2}\, dV$ where R is the solid bounded by $z = \sqrt{x^2 + y^2}$ and $z = 1$.

2.3 Line Integrals and a Fundamental Theorem

• The Arc Length Function • Line Integral • Re-parametrizing a Curve • Fundamental Theorem for Line Integrals • Path Integrals

The Arc Length Function

Geometrically, a curve is an arc of a graph such as a parabola, circle, line, and others. In this section we study integrals of functions that are defined on curves. We begin with a definition of the parametrization of a curve.

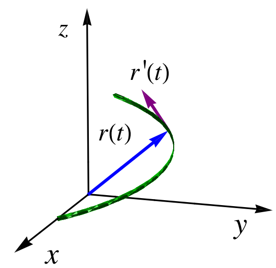


Figure 1
A position vector $r(t)$ and a tangent vector $r'(t)$

Definition 2 Parametrization of a Curve

Let $\boldsymbol{r}$ be a continuous function from a closed interval $[a, b]$ to $\mathbb{R}^n$ where $n = 2, 3$. Suppose $\boldsymbol{r}$ satisfies the two conditions below.

a) $\boldsymbol{r}$ is one-to-one on $[a, b]$, or $\boldsymbol{r}$ is one-to-one on $[a, b)$ and $\boldsymbol{r}(a) = \boldsymbol{r}(b)$.

b) $\boldsymbol{r'}(t)$ is continuous on (a, b) such that $\|\boldsymbol{r'}(t)\|$ and $\frac{1}{\|\boldsymbol{r'}(t)\|}$ are bounded.

In such a case, $\boldsymbol{r}$ parametrizes the **smooth curve** C defined by

$$C = \{\boldsymbol{r}(t) \mid a \le t \le b\}.$$

For simplicity, we may write that a smooth curve is a curve. If $\boldsymbol{r}$ is one-to-one on $[a, b]$, we say C is a **non-intersecting curve**. If $\boldsymbol{r}$ is one-to-one on $[a, b)$ and $\boldsymbol{r}(a) = \boldsymbol{r}(b)$, we say C is a **simple closed curve**.

Suppose $n = 3$ and $\boldsymbol{r}(t) = (x(t), y(t), z(t))$ is a vector in standard position, i.e., the initial point of $\boldsymbol{r}(t)$ is the origin, and the terminal point is $(x(t), y(t), z(t))$.

If $a < t < b$ and $\Delta t > 0$ is a small, the difference quotient

$$\frac{\boldsymbol{r}\,(t + \Delta t) - \boldsymbol{r}(t)}{\Delta t}$$

is a vector whose initial point is the terminal point of $\boldsymbol{r}(t)$. As $\Delta t \to 0$, the limit $\boldsymbol{r'}(t)$ is a vector that is tangent to the curve C at point $\boldsymbol{r}(t)$, see Figure 1.

To define the arc length of C, partition $[a, b]$ into m subintervals. Let $t_0 = a$, $\Delta t = (b - a)/m$, and $t_k = t_{k-1} + \Delta t$ for $k = 1, \ldots, m - 1$. Using the partition, subdivide C into m subarcs, and find the sum of the distances between the initial and terminal points of each subarc.

The distance between the endpoints of $\boldsymbol{r}(t_k)$ and $\boldsymbol{r}(t_{k-1})$ satisfies

$$\|\boldsymbol{r}\,(t_k) - \boldsymbol{r}(t_{k-1})\| \quad = \quad \sqrt{x'\,(p_k)^2 + y'\,(q_k)^2 + z'\,(v_k)^2}\;\Delta t$$

because of the Mean Value Theorem, where $t_{k-1} < p_k, q_k, v_k < t_k$. We recall

$$\|\boldsymbol{r'}(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}.$$

Since x', y', z' are continuous, we estimate each of p_k, q_k, v_k by a single value t_k^* in (t_{k-1}, t_k). Consequently,

$$\|\boldsymbol{r}\,(t_k) - \boldsymbol{r}(t_{k-1})\| \quad \approx \quad \|\boldsymbol{r'}(t_k^*)\|\;\Delta t.$$

Then a reasonable approximation for the arc length of C is the Riemann sum

$$\sum_{k=0}^{m-1} \|\boldsymbol{r}(t_k) - \boldsymbol{r}(t_{k-1})\| \approx \sum_{k=0}^{m-1} \|\boldsymbol{r}'(t_k^*)\| \ \Delta t.$$

By definition, $\|\boldsymbol{r}'\|$ is bounded and continuous, and consequently, integrable.
We let $m \to \infty$, and define the limit of the Riemann sums as the arc length of C.

$$\text{Length}(C) \ = \ \int_a^b \|\boldsymbol{r}'(t)\| \ dt.$$

The arc length of C is independent of the parametrization; this follows from a modification of Theorem 2.9 to be discussed later in the section. Moreover, the **arc length function** is defined by

$$s(t) = \int_a^t \|\boldsymbol{r}'(w)\| \ dw. \tag{2}$$

Example 1 Arc Length Function

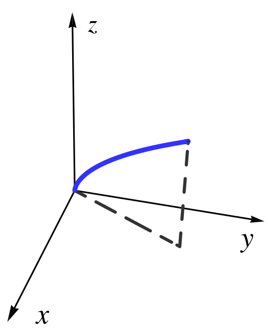


Figure 2
A curve emanating from the origin.

Find the arc length function for the curve parametrized by

$$\boldsymbol{r}(t) = \left(t^{3/2}, 2t^{3/2}, \frac{3t}{2}\right), \ 0 \le t \le \frac{3}{5}.$$

Then find the arc length from $\boldsymbol{r}(0)$ to $\boldsymbol{r}(3/5)$, see Figure 2.

Solution The derivative is

$$\boldsymbol{r}'(t) = \left(\frac{3}{2}\sqrt{t}, \ 3\sqrt{t}, \ \frac{3}{2}\right).$$

Then the magnitude of derivative is

$$\begin{aligned} \|\boldsymbol{r}'(t)\| &= \sqrt{\frac{9}{4}t + 9t + \frac{9}{4}} = \sqrt{\frac{45}{4}t + \frac{9}{4}} \\ &= \frac{3}{2}\sqrt{5t+1}. \end{aligned}$$

Thus, the arc length function satisfies

$$\begin{aligned} s(t) &= \int_0^t \|\boldsymbol{r}'(w)\| \ dw = \int_0^t \frac{3}{2}\sqrt{5w+1}dw \\ &= \frac{3}{10}(5w+1)^{3/2}\frac{2}{3}\Big|_{w=0}^{w=t} \\ &= \frac{1}{5}\left((5t+1)^{3/2} - 1\right) \end{aligned}$$

Hence, the arc length from $\boldsymbol{r}(0)$ to $\boldsymbol{r}(3/5)$ is given by

$$s\left(\frac{3}{5}\right) \ = \ \frac{1}{5}\left(4^{3/2} - 1\right) \ v = \ \frac{7}{5}.$$

□

Try This 1

Find the arc length function for

$$\boldsymbol{r}(\theta) = (\cos\theta, \sin\theta, \theta),\ 0 \le \theta \le 2\pi.$$

The graph of $\boldsymbol{r}$ is a helix, see Figure 3.

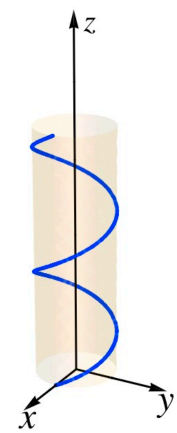


Figure 3
The helix $\boldsymbol{r}$ lies on a circular cylinder of radius 1.

The Line Integral

A line integral generalizes the concept of an integral $\int_a^b f(x)dx$ of a real-valued function f defined on $[a, b]$. In a line integral, we integrate a vector field $\boldsymbol{F}$ that is defined on a curve.

Definition 3 Line Integral

Let $C \subseteq \mathbb{R}^n$ be a curve parametrized by a function $\boldsymbol{r}$ defined on $I = [a, b]$ where $n = 2, 3$. Let $\boldsymbol{F}$ be a vector field defined on C such that $\boldsymbol{F} \circ \boldsymbol{r}$ is continuous from I to $\mathbb{R}^n$. The **line integral of $\boldsymbol{F}$ along** C is denoted by $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$, and defined by

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)dt$$

Later in the section, we show the line integral of a vector field along a curve is independent of the parametrization provided the parameterization is orientation preserving, see Theorem 2.10.

We discuss an application of line integrals. Suppose a vector field $\boldsymbol{F}$ represents a force that causes a particle to move along a curve C parametrized by $\boldsymbol{r}$. That is, the particle is at position $\boldsymbol{r}(t)$ at time t. We show the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ represents the work done by the force on the particle.

The **unit tangent vector** to $\boldsymbol{r}(t)$ is defined by

$$\boldsymbol{T}(t) = \frac{1}{\|\boldsymbol{r}'(t)\|}\boldsymbol{r}'(t). \tag{3}$$

Since the differential of the arc length function is $ds = \|\boldsymbol{r}'(t)\|\ dt$, we write

$$\begin{aligned} \int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|}\|\boldsymbol{r}'(t)\|dt \\ &= \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{T}(t)\ ds. \end{aligned} \tag{4}$$

Recall, the projection of vector $\boldsymbol{v} = \boldsymbol{F}(\boldsymbol{r}(t))$ onto unit vector $\boldsymbol{w} = \boldsymbol{T}(t)$ satisfies

$$\mathrm{Proj}_{\boldsymbol{w}}(\boldsymbol{v}) = (\boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{T}(t))\,\boldsymbol{T}(t)$$

see identity (15) in page 15. The scalar $\boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{T}(t)\ ds$ in the integrand of (4) represents the work done by the force $\boldsymbol{F}$ along the tangential component $\boldsymbol{T}(t)$, as the particle moves through an arc length of ds. Then the line integral (4) is the work done by a force $\boldsymbol{F}$ acting on a particle that is moving along curve C.

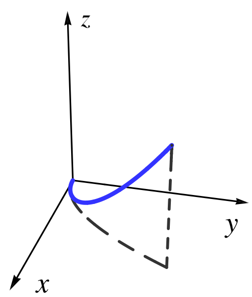


Figure 4
The curve $\boldsymbol{r}(t) = \left(t, t^2, t^3\right)$, $0 \leq t \leq 1$

Example 2 Evaluating a Line Integral

Let $\boldsymbol{F}(x, y, z) = (yz, x, 2x)$ be a vector field. Let C be the curve parametrized by $\boldsymbol{r}(t) = \left(t, t^2, t^3\right),\ \ 0 \leq t \leq 1$, see Figure 4. Then evaluate $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$.

Solution Let $\boldsymbol{r}(t) = (x(t), y(t), z(t))$ where $x(t) = t$, $y(t) = t^2$, and $z(t) = t^3$. Evaluating the vector field, we obtain

$$\begin{aligned}\boldsymbol{F}(\boldsymbol{r}(t)) &= \boldsymbol{F}(x(t), y(t), z(t)) \\ &= (y(t)z(t), x(t), 2x(t)) \\ &= \left(t^5, t, 2t\right).\end{aligned}$$

Also, the derivative of the curve is

$$\boldsymbol{r}'(t) = \left(1, 2t, 3t^2\right).$$

Applying Definition 3, the line integral is given by

$$\begin{aligned}\int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_0^1 \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)dt \\ &= \int_0^1 \left(t^5, t, 2t\right) \cdot \left(1, 2t, 3t^2\right) dt \\ &= \int_0^1 \left(t^5 + 2t^2 + 6t^3\right) dt \\ &= \left(\frac{t^6}{6} + \frac{2t^3}{3} + \frac{3t^4}{2}\right|_{t=0}^{t=1} \\ \int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= \frac{7}{3}.\end{aligned}$$

□

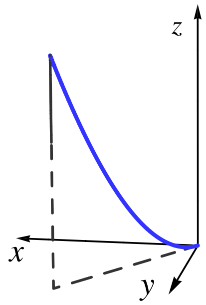


Figure 5
For **Try This 2**

Try This 2

Let $\boldsymbol{F}(x, y, z) = (x, z, y)$ be a vector field. Let $\boldsymbol{r}(t) = \left(t, 2t, t^2\right)$, $0 \leq t \leq 2$, be a parametrization of curve C. Then evaluate the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$.

Re-parametrizing a Curve

We study the effect of *re-parametrizing* a curve has on a line integral. Let C be a curve that is parametrized by a function $\boldsymbol{r}$ defined on $[a, b]$. The **orientation** of C with respect to $\boldsymbol{r}$ is the direction along C that the points $\boldsymbol{r}(t)$ trace as t increases. For instance, the orientation may be counter clockwise on a simple closed curve.

Let α be a one-to-one continuous function from $[c, d]$ onto $[a, b]$ such that the derivative α' and $\frac{1}{\alpha'}$ are bounded and continuous on (c, d). We say the composite function $\boldsymbol{r} \circ \alpha$ is a **re-parametrization** of $\boldsymbol{r}$.

Notice, $\boldsymbol{r} \circ \alpha$ parametrizes C. We say $\boldsymbol{r} \circ \alpha$ is **orientation preserving** if the orientations of C with respect to $\boldsymbol{r}$ and to $\boldsymbol{r} \circ \alpha$ are the same. Otherwise, we say $\boldsymbol{r} \circ \alpha$ is **orientation reversing**.

The line integral of vector field $\boldsymbol{F}$ using the parametrization $\boldsymbol{r_1} = \boldsymbol{r} \circ \alpha$ satisfies

$$\begin{aligned}
\int_c^d \boldsymbol{F}(\boldsymbol{r_1}(t)) \cdot \boldsymbol{r_1}'(t)\ dt &= \int_c^d \boldsymbol{F}(\boldsymbol{r}(\alpha(t)) \cdot \boldsymbol{r}'(\alpha(t))\ \alpha'(t)\ dt \qquad \text{Chain Rule} \\
&= \int_{\alpha(c)}^{\alpha(d)} \boldsymbol{F}(\boldsymbol{r}(w)) \cdot \boldsymbol{r}'(w)\ dw \quad \text{where } w = \alpha(t), dw = \alpha'(t)dt \\
&= \pm \int_a^b \boldsymbol{F}(\boldsymbol{r}(w)) \cdot \boldsymbol{r}'(w)\ dw \qquad (5)
\end{aligned}$$

for $\int_a^b f(x)dx = -\int_b^a f(x)dx$. The sign in (5) depends on whether $\alpha(c) = a$ or $\alpha(c) = b$. We choose the plus sign if $\alpha(c) = a$, i.e., $\boldsymbol{r_1}$ is orientation preserving. Otherwise, we choose the negative sign if $\boldsymbol{r_1}$ is orientation reversing.

Moreover, any two parametrizations of a curve C with the same initial points are re-parametrizations of each other, see theorem below.

Theorem 2.9 Parametrization of Curves

Let $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ be functions defined on $I_1 = [a, b]$ and $I_2 = [c, d]$, respectively, that parametrize a curve C. If $\boldsymbol{r_1}(a) = \boldsymbol{r_2}(c)$, then $\boldsymbol{r_1}$ is a re-parametrization of $\boldsymbol{r_2}$.

Proof Let $t \in (a, b)$, and choose $s \in (c, d)$ such that $\boldsymbol{r_1}(t) = \boldsymbol{r_2}(s)$. Then the function given by $s = f(t)$, $f(a) = c$, and $f(b) = d$ is a bijection from I_1 onto I_2. By definition, $\boldsymbol{r_1} = \boldsymbol{r_2} \circ f$. We show f is differentiable. If $\Delta t \neq 0$ is small enough, choose $\Delta s \neq 0$ such that $\boldsymbol{r_1}(t + \Delta t) = \boldsymbol{r_2}(s + \Delta s)$. Then

$$\boldsymbol{r_1}(t + \Delta t) - \boldsymbol{r_1}(t) = \boldsymbol{r_2}(s + \Delta s) - \boldsymbol{r_2}(s).$$

By definition, we have $f\,(t + \Delta t) = s + \Delta s$.

Then

$$f'(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\dfrac{\|\boldsymbol{r_1}(t+\Delta t) - \boldsymbol{r_1}(t)\|}{\Delta t}}{\dfrac{\|\boldsymbol{r_2}(s+\Delta s) - \boldsymbol{r_2}(s)\|}{\Delta s}}.$$

By the continuity of the curves, $\Delta t \to 0$ if and only if $\Delta s \to 0$. Consequently,

$$f'(t) = \frac{\|\boldsymbol{r_1}'(t)\|}{\|\boldsymbol{r_2}'(s)\|}. \tag{6}$$

By definition, $\boldsymbol{r_i}'$ and $\frac{1}{\boldsymbol{r_i}'}$ are bounded for $i = 1, 2$. Then f' and $\frac{1}{f'}$ are bounded on (a, b). Hence, $\boldsymbol{r_1} = \boldsymbol{r_2} \circ f$, i.e., $\boldsymbol{r_1}$ is a re-parametrization of $\boldsymbol{r_2}$.

□

A curve C has exactly two orientations. If we fix an orientation for C, the other orientation is called the opposite orientation and which we denote by $-C$.

Let C be a non-intersecting oriented curve, and let $\boldsymbol{F}$ be a continuous vector field on C. Applying identity (5) and Theorem 2.9, the line integral of $\boldsymbol{F}$ along C is independent of any orientation preserving parametrization $\boldsymbol{r}$ of C.

Next, we analyze the line integral on a simple closed curve C. Note, the initial points of two functions $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ that parametrize C may not be the same. We assume the orientations of C with respect to $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ are the same.

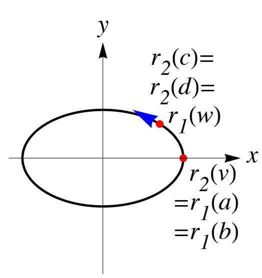


Figure 6
Parametrizations $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ with different initial points.

Suppose $\boldsymbol{r_1}$ is defined on $[a, b]$, and $\boldsymbol{r_2}$ is defined on $[c, d]$. Choose w, v such that $\boldsymbol{r_1}(w) = \boldsymbol{r_2}(c) = \boldsymbol{r_2}(d)$, and $\boldsymbol{r_2}(v) = \boldsymbol{r_1}(a) = \boldsymbol{r_1}(b)$, as in Figure 6. Then

$$\begin{aligned}
\int_a^b \boldsymbol{F}(\boldsymbol{r_1}(t)) \cdot \boldsymbol{r_1}'(t)dt &= \int_a^w \boldsymbol{F}(\boldsymbol{r_1}(t)) \cdot \boldsymbol{r_1}'(t)dt + \int_w^b \boldsymbol{F}(\boldsymbol{r_1}(t)) \cdot \boldsymbol{r_1}'(t)dt \\
&= \int_v^d \boldsymbol{F}(\boldsymbol{r_2}(t)) \cdot \boldsymbol{r_2}'(t)dt + \int_c^v \boldsymbol{F}(\boldsymbol{r_2}(t)) \cdot \boldsymbol{r_2}'(t)dt \\
&= \int_c^d \boldsymbol{F}(\boldsymbol{r_2}(t)) \cdot \boldsymbol{r_2}'(t)dt
\end{aligned}$$

where in the middle equation we applied identity (5) with the plus sign. Thus, the line integral of a vector field along an oriented simple closed curve is independent of the orientation preserving parametrization of the curve. We summarize below.

Theorem 2.10 Line Integrals and Re-parametrization of Curves

Let C be an oriented curve, and let $\boldsymbol{F}$ be a continuous vector field defined on C. Then the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ is independent of the orientation preserving parametrization of C. Also,

$$\int_{-C} \boldsymbol{F} \cdot d\boldsymbol{r} = -\int_C \boldsymbol{F} \cdot d\boldsymbol{r}.$$

Moreover, the lengths of curves C and $-C$ are the same.

Proof We only have to prove the last statement. Let $\boldsymbol{r}$ be a parametrization of C, and let $\boldsymbol{r_1} = \boldsymbol{r} \circ \alpha$ be a re-parametrization where α is a one-to-one function from $[c,d]$ onto $[a,b]$ such that α' and $1/\alpha'$ are continuous and bounded on (a,b). Since α is one-to-one, α' is entirely nonpositive or entirely nonnegative. Then

$$\begin{aligned}
\int_c^d \|\boldsymbol{r_1}'(t)\|\, dt &= \int_c^d \|\boldsymbol{r}'(\alpha(t))\|\, |\alpha'(t)|\, dt \\
&= \begin{cases} \displaystyle\int_c^d \|\boldsymbol{r}'(\alpha(t))\|\, \alpha'(t)\, dt & \text{if} \quad \alpha \text{ is increasing} \\ \displaystyle -\int_c^d \|\boldsymbol{r}'(\alpha(t))\|\, \alpha'(t)\, dt & \text{if} \quad \alpha \text{ is decreasing} \end{cases} \\
&= \begin{cases} \displaystyle\int_a^b \|\boldsymbol{r}'(w)\|\, dw & \text{if} \quad w = \alpha(t), \text{ and } \alpha \text{ is increasing} \\ \displaystyle -\int_b^a \|\boldsymbol{r}'(w)\|\, dw & \text{if} \quad \alpha \text{ is decreasing} \end{cases} \\
\int_c^d \|\boldsymbol{r_1}'(t)\|\, dt &= \int_a^b \|\boldsymbol{r}'(t)\|\, dt
\end{aligned}$$

Hence, the arc lengths of C and $-C$ are equal.

□

Given a parametrization $\boldsymbol{r}$ of a curve C defined on $[a,b]$, there is a natural orientation-reversing parametrization $\boldsymbol{r}^*$ of $-C$. Namely, $\boldsymbol{r}^*$ is defined on $[a,b]$ and

$$\boldsymbol{r}^*(t) = \boldsymbol{r}\,(a+b-t)\,. \tag{7}$$

Example 3 Evaluating a Line Integral

Let $\boldsymbol{F}(x,y,z) = (\sin \pi x, \cos \pi z, y)$ be a vector field. Let C be the oriented line segment from $A(1,0,0)$ to $B(0,2,1)$, see Figure 7. Then evaluate $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$.

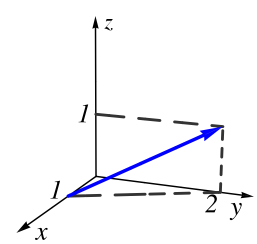


Figure 7
Line segment from $A(1,0,0)$ to $B(0,2,1)$.

Solution The vector from A to B is $(-1,2,1)$. Then a parametric equation of the line segment from A to B is

$$\boldsymbol{r}(t) = (-1,2,1)t + (1,0,0) = (1-t, 2t, t)\,,\;\; 0 \le t \le 1.$$

Since $\boldsymbol{r}'(t) = (-1,2,1)$, we obtain

$$\begin{aligned}
\int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_0^1 (\sin \pi(1-t), \cos \pi t, 2t) \cdot (-1,2,1) dt \\
&= \int_0^1 (-\sin \pi(1-t) + 2\cos \pi t + 2t)\, dt \\
&= -\frac{1}{\pi} \cos(\pi(1-t)) + \frac{2}{\pi} \sin(\pi t) + t^2 \,\Big|_0^1 \\
\int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= 1 - \frac{2}{\pi}.
\end{aligned}$$

□

Try This 3

Evaluate the line integral in Example 3 where C is the oriented line segment from $A(2,0,0)$ to $B(0,4,2)$.

Example 4 Evaluating a Line Integral

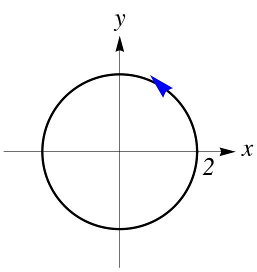


Figure 8
A counter-clockwise path around a circle of radius 2

Let $\boldsymbol{F}(x,y) = (-y, x)$ be a vector field. Let C be a circle of radius 2, centered at the origin, and oriented counter clockwise. Then evaluate the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$.

Solution The standard equation of circle C is $x^2 + y^2 = 4$, see Figure 8. Using the identity $\cos^2\theta + \sin^2\theta = 1$, the parametrization

$$\boldsymbol{r}(\theta) = (2\cos\theta, 2\sin\theta)\,,\ 0 \le \theta \le 2\pi$$

traces the circle in a counter clockwise motion. The derivative is

$$\boldsymbol{r}'(\theta) = (-2\sin\theta,\ 2\cos\theta)\,.$$

According to Theorem 2.10, the line integral is independent of the orientation preserving parametrization. Thus,

$$\begin{aligned}
\int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_0^{2\pi} \boldsymbol{F}\left(2\cos\theta, 2\sin\theta\right) \cdot (-2\sin\theta,\ 2\cos\theta)\, d\theta \\
&= \int_0^{2\pi} (-2\sin\theta, 2\cos\theta) \cdot (-2\sin\theta,\ 2\cos\theta)\, d\theta \\
&= \int_0^{2\pi} \left(4\sin^2\theta + 4\cos^2\theta\right) d\theta \ = \ \int_0^{2\pi} 4\ d\theta \\
\int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= 8\pi.
\end{aligned}$$

□

Try This 4

Let $\boldsymbol{F}(x,y) = (-y, 1)$ be a vector field. Let C be the unit circle oriented counter clockwise. Then evaluate the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$.

We extend the definition of a line integral of a vector field. Let C be a finite union of smooth curves C_i , $i = 1, \ldots, n$.

$$C = C_1 \cup \cdots \cup C_n$$

We denote the line integral of $\boldsymbol{F}$ along C by $\int_C \boldsymbol{F} \cdot d\boldsymbol{s}$, and define it by

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} + \cdots + \int_{C_n} \boldsymbol{F} \cdot d\boldsymbol{r}. \tag{8}$$

Using Theorem 2.10, it can be shown that the left side is independent of the finite union of smooth curves C_i. We say C is a **piecewise-smooth curve**.

Example 5 Line Integral Along a Triangular Image

Let $\boldsymbol{F}(x, y, z) = (xz, 2y, yz)$ be a vector field. Let C be a simple closed triangular path that bounds the plane $2x + 2y + z = 4$, and the coordinate planes. The orientation is counter clockwise as shown in Figure 9. Then evaluate $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$.

Solution We parametrize the segment C_1 from $(2, 0, 0)$ to $(0, 2, 0)$. Since y is increasing from $y = 0$ to $y = 2$, let $y = t$ where $0 \leq t \leq 2$. Notice, the plane $x + y = 2$ contains C_1. Then $x = 2 - t$, and we obtain a parametrization of C_1:

$$\boldsymbol{r_1}(t) = (2 - t, t, 0), \ 0 \leq t \leq 2.$$

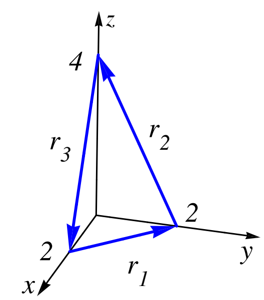


Figure 9
A triangular path in the plane $2x + 2y + z = 4$.

Applying Definition 3 and $\boldsymbol{F}(x, y, z) = (xz, 2y, yz)$, we find

$$\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} \ = \ \int_0^2 (0, 2t, 0) \cdot (-1, 1, 0)\, dt \ = \ 4.$$

Secondly, consider the segment C_2 from $(0, 2, 0)$ to $(0, 0, 4)$. Then z is increasing $z = 0$ to $z = 4$. Let $z = t$ for $0 \leq t \leq 4$. Note, the plane $2y + z = 4$ contains C_2. Then $y = (4 - t)/2$, and we have a parametrization for C_2:

$$\boldsymbol{r_2}(t) = \left(0, \frac{4 - t}{2}, t\right), \ 0 \leq t \leq 4.$$

The line integral along C_2 satisfies

$$\begin{aligned} \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_0^4 \left(0, 4 - t, \frac{4t - t^2}{2}\right) \cdot \left(0, -\frac{1}{2}, 1\right) dt \\ &= \frac{1}{2} \int_0^4 \left(-t^2 + 5t - 4\right) dt \ = \ \frac{4}{3}. \end{aligned}$$

Similarly, a parametrization of the segment C_3 from $(0, 0, 4)$ to $(2, 0, 0)$ is given by

$$\boldsymbol{r_3}(t) = (t, 0, 4 - 2t)\,, \ 0 \leq t \leq 2.$$

Then the line integral along C_3 satisfies

$$\begin{aligned} \int_{C_3} \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_0^2 \boldsymbol{F}\,(t, 0, 4 - 2t) \cdot (1, 0, -2)\, dt \\ &= \int_0^2 \left(4t - 2t^2, 0, 0\right) \cdot (1, 0, -2)\, dt \ = \ \frac{8}{3}. \end{aligned}$$

Finally, we obtain

$$\begin{aligned} \int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_3} \boldsymbol{F} \cdot d\boldsymbol{r} \\ &= 4 + \frac{4}{3} + \frac{8}{3} \\ &= 8. \end{aligned}$$

□

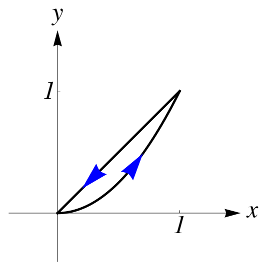


Figure 10
An oriented closed path along $y = x^2$ and $y = x$.

Try This 5

Let $\boldsymbol{F}(x, y) = (2y, 1)$ be a vector field. Let C be a counter clockwise closed curve bounded $y = x^2$ and $y = x$, see Figure 10. Then evaluate the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$.

Fundamental Theorem for Line Integrals

If a vector field $\boldsymbol{F}$ is conservative, the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ depends only on $\boldsymbol{F}$, and the initial and terminal points of C, as the next theorem shows.

Theorem 2.11 Fundamental Theorem for Line Integrals

Let $\boldsymbol{F}$ be a continuous vector field satisfying $\boldsymbol{F} = \nabla f$ for some real-valued function f. Let C be an oriented curve that is parametrized by a function $\boldsymbol{r}$ with initial point $\boldsymbol{r}(a)$ and terminal point $\boldsymbol{r}(b)$. Then

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = f\left(\boldsymbol{r}(b)\right) - f\left(\boldsymbol{r}(a)\right).$$

Proof For simplicity, let F be a function of two variables. Let $\boldsymbol{F} = \left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}\right)$. By the chain rule and the Fundamental Theorem of Calculus, we find

$$\begin{aligned}
\int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_a^b \left(\left.\frac{\partial f}{\partial x}\right|_{\boldsymbol{r}(t)}, \left.\frac{\partial f}{\partial y}\right|_{\boldsymbol{r}(t)} \right) \cdot \boldsymbol{r}'(t) dt \\
&= \int_a^b \left(f \circ \boldsymbol{r}\right)'(t)\ dt \\
&= f\left(\boldsymbol{r}(b)\right) - f\left(\boldsymbol{r}(a)\right).
\end{aligned}$$

□

We recall a characterization of conservative vector fields, see Theorem 1.10. Let $\boldsymbol{G}$ be a vector field defined on $\mathbb{R}^3$ with continuous first partial derivatives except for finitely many points. Then $\operatorname{curl}(\boldsymbol{G}) = \boldsymbol{0}$ if and only if $\boldsymbol{G} = \nabla g$ for some real-valued function g.

If $\boldsymbol{G} = (M, N, P)$ where M, N, P are real-valued functions of x, y, z, then the proof of Theorem 1.10 shows

$$\operatorname{curl} \boldsymbol{G} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\boldsymbol{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\boldsymbol{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\boldsymbol{k}.$$

Thus, $\operatorname{curl}(\boldsymbol{G}) = \mathbf{0}$ if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}, \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \tag{9}$$

Often, we may denote the line integral of $\boldsymbol{G} = (M, N, P)$ along C by

$$\int_C \boldsymbol{G} \cdot d\boldsymbol{r} = \int_C (M dx + N dy + P dz).$$

For vector fields $\boldsymbol{F} = (M, N)$ of two variables x and y, we may denote the line integral by

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C (M dx + N dy).$$

Example 6 Fundamental Theorem for Line Integrals

Let C be a curve that is parametrized by $\boldsymbol{r}(t) = (2\sin t,\ 2\cos t)$, $0 \leq t \leq \frac{\pi}{2}$. Then evaluate $\int_C (y dx + (x-1) dy)$ by the Fundamental Theorem for Line Integrals.

Solution We apply (9) to show that $\boldsymbol{F}$ is the gradient of a real-valued function. We write $\boldsymbol{F} = (M, N, P)$ where $M = y$, $N = x - 1$, and $P = 0$. Since M, N are functions of x and y only, we find $\operatorname{curl}(\boldsymbol{F}) = \mathbf{0}$ exactly when

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

Clearly, the above statement is true. Then $\operatorname{curl}(\boldsymbol{F}) = \mathbf{0}$. Now, we may eyeball and directly find a function f whose gradient is $\boldsymbol{F}$. In fact,

$$\nabla(xy - y) = (y, x - 1) = \boldsymbol{F}$$

and we let $f(x, y) = xy - y + C$ where C is a constant.

We present a different method for finding f. Notice, $\frac{\partial f}{\partial x} = y$ and $\frac{\partial f}{\partial y} = x - 1$. Integrating with respect to x, we find

$$f(x, y) = xy + \phi(y)$$

where ϕ is a function of y only. Since $\frac{\partial f}{\partial y} = x + \phi'(y)$, we obtain

$$\begin{aligned} x + \phi'(y) &= x - 1 \\ \phi'(y) &= -1 \\ \phi(y) &= -y + C. \end{aligned}$$

In any case, we have

$$f(x, y) = xy + \phi(y) = xy - y + C.$$

Hence, by the Fundamental Theorem for Line Integrals we find

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = f(\boldsymbol{r}(\pi/2)) - f(\boldsymbol{r}(0)) = f(2, 0) - f(0, 2) = 2.$$

□

Try This 6

Let C be an oriented curve with initial point $(1,0)$ and terminal point $(2,2)$. Then evaluate $\int_C \left(y^2dx + 2xydy\right)$ by the Fundamental Theorem for Line Integrals.

Example 7 Fundamental Theorem for Line Integrals

Let C be an oriented path from point $A(3,1,1)$ to $B(1,1,2)$ where $y \neq 0$. If

$$\boldsymbol{F}(x,y,z) = \left(\frac{z}{y},\ -\frac{xz}{y^2},\ \frac{x}{y} - 1\right)$$

is a vector field, evaluate $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ by the Fundamental Theorem for Line Integrals.

Solution We find a function f satisfying $\nabla f = \boldsymbol{F}$, but we omit the details in verifying (9). We may eyeball f and choose $f(x,y,z) = xz/y - z$.

Alternatively, since $\partial f/\partial x = z/y$, we integrate with respect to x. Then

$$f(x,y,z) = \frac{xz}{y} + \phi(y,z)$$

where ϕ is a function of y and z only. Note, $\frac{\partial f}{\partial z} = \frac{x}{y} + \frac{\partial \phi}{\partial z}$. Also, $\frac{\partial f}{\partial z} = \frac{x}{y} - 1$ for $\nabla f = \boldsymbol{F}$. Thus,

$$\begin{aligned} \frac{x}{y} + \frac{\partial \phi}{\partial z} &= \frac{x}{y} - 1 \\ \frac{\partial \phi}{\partial z} &= -1. \end{aligned}$$

Then $\phi(y,z) = -z + \psi(y)$ where ψ is a function of y only. Thus,

$$f(x,y,z) = \frac{xz}{y} - z + \psi(y). \tag{10}$$

Observe, $\frac{\partial f}{\partial y} = -\frac{xz}{y^2} + \psi'(y)$ and $\frac{\partial f}{\partial y} = -\frac{xz}{y^2}$ for $\nabla f = \boldsymbol{F}$. Combining, we obtain

$$\begin{aligned} -\frac{xz}{y^2} + \psi'(y) &= -\frac{xz}{y^2} \\ \psi(y) &= C \end{aligned}$$

where C is a constant. Then we rewrite (10) as follows:

$$f(x,y,z) = \frac{xz}{y} - z + C.$$

Hence, by the Fundamental Theorem for Line Integrals we find

$$\begin{aligned} \int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= f(1,1,2) - f(3,1,1) = 2 - 0 \\ &= 2. \end{aligned}$$

□

Try This 7

Apply the Fundamental Theorem for Line Integrals in evaluating

$$\int_C \left(dx + \frac{1}{z} dy - \frac{y}{z^2} dz \right)$$

where C is an oriented smooth path from $A(2, 2, 2)$ to $B(4, 4, 2)$ where $z \neq 0$.

Path Integrals

Let C be a curve parametrized by a function $\boldsymbol{r}$. Recall, the differential of the arc length function in (2) satisfies

$$ds = \|\boldsymbol{r}'(t)\| \; dt.$$

Next, we define the **path integral** of real-valued function f along C.

Definition 4 Path Integral

Let C be a smooth curve parametrized by a function $\boldsymbol{r}$ defined on $[a, b]$. Let f be a real-valued function such that $f \circ \boldsymbol{r}$ is continuous on $[a, b]$. The path integral of f along C is denoted by $\int_C f(\boldsymbol{x}) ds$, and defined by

$$\int_C f(\boldsymbol{x}) ds = \int_a^b f(\boldsymbol{r}(t)) \; \|\boldsymbol{r}'(t)\| \; dt.$$

The line integral of f along C is independent of the parametrization $\boldsymbol{r}$ of C. The proof of which is similar to that of Theorem 2.10. That is, if $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ are two functions that parametrize C, and defined on $[a_1, b_1]$, $[a_2, b_2]$, respectively, then

$$\int_{a_1}^{b_1} f(\boldsymbol{r_1}(t)) \; \|\boldsymbol{r_1}'(t)\| \; dt = \int_{a_2}^{b_2} f(\boldsymbol{r_2}(t)) \; \|\boldsymbol{r_2}'(t)\| \; dt.$$

The above identity holds even if $\boldsymbol{r_1}$ is orientation reversing of $\boldsymbol{r_2}$.

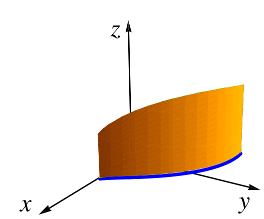


Figure 11
If $f(\boldsymbol{x}) \geq 0$, the path integral $\int_C f(\boldsymbol{x}) ds$ is the area of a fence built on a curve.

Geometrically, if $f(\boldsymbol{r}(t)) \geq 0$ represents the height of a fence at $\boldsymbol{r}(t)$, the path integral $\int_C f(\boldsymbol{x}) ds$ is the area of the fence, see Figure 11. The product $f(\boldsymbol{x}) ds$ is the area of a rectangle with base ds and height $f(\boldsymbol{x})$.

A parameter s for a parametrizing function $\boldsymbol{r}$ is called **arc length parameter** if $\|\boldsymbol{r}'(s)\| = 1$ for all s except possibly at the endpoints. Consequently, the arc length of $\boldsymbol{r}$ in subinterval $[c, d]$ is $d - c$. To re-parametrize $\boldsymbol{r}(t)$ by using the arc length parameter s, we suggest the following guidelines.

Parametrizing by the Arc Length Parameter

Let $\boldsymbol{r}$ be a smooth curve on $[a, b]$.

1. Evaluate the arc length function, $s = \phi(t) = \displaystyle\int_a^t \|\boldsymbol{r}'(t)\|\, dt$.

2. Since $s = \phi(t)$ is an increasing function, solve for the inverse function $t = \phi^{-1}(s)$.

3. The arc length parametrization is $\boldsymbol{r_1}(s) = \boldsymbol{r}\left(\phi^{-1}(s)\right)$.

Let C be a curve parametrized by $\boldsymbol{r}(s)$ where s is the arc length parameter. From definition (3), the unit tangent vector to the curve C at $\boldsymbol{r}(s)$ satisfies

$$\boldsymbol{T}(s) = \frac{\boldsymbol{r}'(s)}{\|\boldsymbol{r}'(s)\|} = \boldsymbol{r}'(s)$$

see (3). The **curvature** κ of C is a function that is defined on C where

$$\kappa(\boldsymbol{x}) = \|\boldsymbol{T'}(s)\| \tag{11}$$

and $\boldsymbol{x} = \boldsymbol{r}(s)$. The curvature κ is independent of the parametrization of C.

In particular, let $\boldsymbol{r_1}(s_1)$ and $\boldsymbol{r_2}(s_2)$ be two functions that parametrize C where s_1 and s_2 are the arc length parameters. If $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ have the same orientations, then $\boldsymbol{r_1} = \boldsymbol{r_2} \circ f$ where $f'(s_1) = 1$, see identity (6). By the chain rule, we find

$$\frac{d\boldsymbol{r_1}}{ds_1} = \frac{d\boldsymbol{r_2}}{ds_2} f'(s_1) = \frac{d\boldsymbol{r_2}}{ds_2}.$$

If $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ have opposite orientations, $\boldsymbol{r_1}$ and $\boldsymbol{r_2}^*$ have the same orientations where $\boldsymbol{r_2}^*(s_2) = \boldsymbol{r_2}(a_2 + b_2 - s_2)$, see (7). Similarly, $\boldsymbol{r_1} = \boldsymbol{r_2}^* \circ f$ and

$$\frac{d\boldsymbol{r_1}}{ds_1} = \left.\frac{d\boldsymbol{r_2}^*}{ds_2}\right|_{s_2 = f(s_1)} f'(s_1) = \left.\frac{d\boldsymbol{r_2}^*}{ds_2}\right|_{s_2 = f(s_1)} = -\left.\frac{d\boldsymbol{r_2}}{ds_2}\right|_{a_2 + b_2 - s_2}.$$

Thus, we find

$$\frac{d\boldsymbol{r_1}}{ds_1} = \pm\frac{d\boldsymbol{r_2}}{ds_2}.$$

where the sign depends on whether $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ are orientation preserving or reversing. In any case, the magnitudes of the second derivatives are equal, i.e.,

$$\left\|\frac{d^2\boldsymbol{r_1}}{ds_1^2}\right\| = \left\|\frac{d^2\boldsymbol{r_2}}{ds_2^2}\right\|.$$

Thus, the curvature κ is independent of the parametrization.

Example 8 Evaluating a Path Integral

Find the curvature function κ of the curve C parametrized by

$$\boldsymbol{r}(s) = \frac{1}{\sqrt{2}}(\cos s, \sin s, s)$$

where $0 \leq s \leq 2\pi$. Then evaluate the path integral $\int_C \kappa ds$.

Solution Indeed, s is the arc length parameter for

$$\boldsymbol{r}'(s) = \frac{1}{\sqrt{2}}(-\sin s, \cos s, 1)$$

and

$$\|\boldsymbol{r}'(s)\| = \frac{1}{\sqrt{2}}\|(-\sin s, \cos s, 1)\| = \frac{1}{\sqrt{2}}\left(\sqrt{2}\right) = 1.$$

Since $\boldsymbol{T}(s) = \boldsymbol{r}'(s)$, the curvature κ at $\boldsymbol{x} = \boldsymbol{r}(s)$ satisfies

$$\begin{aligned}\kappa(\boldsymbol{x}) &= \|\boldsymbol{T}'(s)\| = \|\boldsymbol{r}''(s)\| = \left\|\frac{1}{\sqrt{2}}(-\cos s, -\sin s, 0)\right\| \\ &= \frac{1}{\sqrt{2}}.\end{aligned}$$

Applying Definition 4 and since $\|\boldsymbol{r}'(s)\| = 1$, the path integral of κ along C is given by

$$\begin{aligned}\int_C \kappa(\boldsymbol{x})ds &= \int_0^{2\pi} \kappa(\boldsymbol{r}(t))\ \|\boldsymbol{r}'(t)\|\ dt = \int_0^{2\pi} \frac{1}{\sqrt{2}}dt \\ &= \sqrt{2}\pi.\end{aligned}$$

□

Try This 8

Find the curvature function κ for the circle C defined by

$$\boldsymbol{r}(s) = R\left(\cos\left(\frac{s}{R}\right), \sin\left(\frac{s}{R}\right)\right)$$

where $0 \leq s \leq 2\pi R$, and $R > 0$. Then evaluate the path integral $\int_C \kappa ds$.

2.3 Check-It Out

1. Find the arc length function for the curve $\boldsymbol{r}(t) = (t, mt + b)$.

2. Let $\boldsymbol{F}(x, y) = (-y, 3x)$ be a vector field. Let C be the curve parametrized by $\boldsymbol{r}(t) = (\cos t, \sin t)$, $0 \leq t \leq \frac{\pi}{2}$. Then evaluate $\int_C F \cdot d\boldsymbol{r}$.

3. Let $\boldsymbol{F}(x, y) = (y, x+1)$ be a vector field. Let C be the curve parametrized by $\boldsymbol{r}(t) = (3\cos t, 3\sin t)$, $0 \leq t \leq \frac{\pi}{3}$. Then evaluate $\int_C F \cdot d\boldsymbol{r}$ by the Fundamental Theorem for Line Integrals.

4. Let $\boldsymbol{G}(x, y, z) = (x+y, y, z)$ be a vector field, and let C be the line segment that joins point $A(2, 1, 0)$ to $B(1, 2, 2)$. Then evaluate $\int_C F \cdot d\boldsymbol{r}$.

True or False. *If false, revise the statement to make it true or explain.*

1. A parametrization of the unit circle that is oriented in the counter clockwise direction is given by $\boldsymbol{r}(\theta) = (\sin\theta, \cos\theta)$, $0 \leq \theta \leq 2\pi$.

2. The line segment from $(1, 0, 0)$ to $(0, 1, 1)$ is parametrized by $\boldsymbol{r}(t) = (1-t, t, t)$ where $0 \leq t \leq 1$.

3. The unit tangent vector to a curve parametrized by $\boldsymbol{r}(t)$, $a \leq t \leq b$, is given by $\boldsymbol{r}'(t)$.

4. If C is a curve parametrized by $\boldsymbol{r}(t) = (t, 0, 1)$ where $0 \leq t \leq 1$, then the line integral of a vector field $\boldsymbol{F}$ on C satisfies $\displaystyle\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^1 \boldsymbol{F}(t, 0, 1) \cdot \boldsymbol{i}\ dt$

5. A definite integral $\displaystyle\int_a^b f(x)dx$ is a line integral.

6. If C is a curve parametrized by $\boldsymbol{r}(t) = (t, 0, 1)$ where $0 \leq t \leq 1$, then the line integral of a vector field $\boldsymbol{F}$ on C satisfies $\displaystyle\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^1 \boldsymbol{F}(t, 0, 1) \cdot \boldsymbol{i}\ dt$

7. If $\boldsymbol{F}(x, y) = (y, 2x)$, the Fundamental Theorem for Line Integrals applies to $\displaystyle\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$.

8. Let $\boldsymbol{r}(t)$ be a parametrization of a curve where $a \leq t \leq b$. Then t is the arc length parameter if there is a constant $c > 0$ satisfying $\|r'(t)\| = c$ for all $a < t < b$.

9. Let $\boldsymbol{r}(t)$ be a parametrization of a curve where t is the arc length parameter. Then the curvature of the curve is $\|r''(t)\|$.

10. Let C be an oriented curve with initial point (a, b) and terminal point (c, d). If $\boldsymbol{F}(x, y) = (y, x)$, then $\displaystyle\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = cd - ab$.

Exercises for Section 2.3

In Exercises 1-4, evaluate the line integral of the vector field along the curve C parametrized by $\boldsymbol{r}$.

1. $\displaystyle\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ where $\boldsymbol{F}(x, y) = (2x, y)$, and $\boldsymbol{r}(t) = (\sin t, \cos t)$, $0 \leq t \leq \dfrac{\pi}{6}$.

2. $\displaystyle\int_C (xdx + xydy)$ where $\boldsymbol{r}(t) = (2\sin t, \cos t)$, $0 \leq t \leq \dfrac{\pi}{6}$.

3. $\int_C \boldsymbol{G}\cdot d\boldsymbol{r}$ where $\boldsymbol{G}(x,y,z)=(y,y+z,x)$, and $\boldsymbol{r}(t)=(2-t,t+1,t)$, $0\le t\le 2$.

4. $\int_C (\sin y\ dx+\cos z\ dy-\sin x\ dz)$ where $\boldsymbol{r}(t)=(t,2t,t)$, $0\le t\le \dfrac{\pi}{3}$.

In Exercises 5-12, parametrize the path C by a curve $\boldsymbol{r}$ that is one-to-one except possibly at the endpoints. Then evaluate the line integral of the indicated vector field $\boldsymbol{F}$ or $\boldsymbol{G}$ along the path C.

5. $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ where $\boldsymbol{F}(x,y)=(-y,x)$, and C is the directed line segment from $(1,0)$ to $(3,4)$.

6. $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ where $\boldsymbol{F}(x,y)=(y,x)$, and C is the directed line segment from $(1,1)$ to $(2,3)$.

7. $\int_C (dx+xdy)$ where C is the oriented line segment from the origin to $(2,4)$.

8. $\int_C (3xdx+(x+y)dy)$ where C is the line segment from the point $(2,2)$ to $(0,4)$.

9. $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ where $\boldsymbol{F}(x,y)=(-4y,4x)$, C is a path along the unit circle that joins $(1,0)$ to $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ in the counter clockwise direction.

10. $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ where $\boldsymbol{F}(x,y)=(x+y,x-y)$, and C is a path along a circle of radius 2, centered at the origin, and joining $(2,0)$ to $(1,\sqrt{3})$ in the counter clockwise direction.

11. $\int_C \boldsymbol{G}\cdot d\boldsymbol{r}$ where $\boldsymbol{G}(x,y,z)=(x+2y,y-2z,x+z)$, and C is the line segment from $(1,2,3)$ to $(3,6,5)$.

12. $\int_C (ydx+z^2dy+xdz)$ where C is the line segment from $(0,1,1)$ to $(\frac{1}{2},2,0)$.

In Exercises 13-26, evaluate the line integral by the Fundamental Theorem for Line Integrals. State the function f such that the gradient ∇f is the conservative vector field in the integral.

13. $\int_C (y^2dx+2xydy)$ where C is a path from the origin to $(-1,3)$.

14. $\int_C (x^2dx-\dfrac{y}{2}dy)$ where C is an oriented path from the $(1,2)$ to $(3,6)$.

15. $\int_C (\cos x\cos y\ dx-\sin x\sin y\ dy)$ where C is the path defined by $\boldsymbol{r}(t)=\left(t,\dfrac{t}{2}\right)$, $0\le t\le\frac{\pi}{3}$.

16. $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ where C is a path from point $A(1,0)$ to $B(0,1)$, and $\boldsymbol{F}(x,y)=(ye^{xy}+1,xe^{xy})$.

17. $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ where C is the image of the curve $\boldsymbol{r}(t)=(\arccos t,\arcsin t)$, $0\le t\le 1$, $\boldsymbol{F}(x,y)=(e^y\cos x,e^y\sin x)$.

18. $\displaystyle\int_C 6ye^{2x}dx + 3e^{2x}dy$ where C is the path defined by $r(t) = \big(t, t^2+2\big)$, $0 \le t \le 1$.

19. $\displaystyle\int_C 3\cos(2y)dx - 6x\sin(2y)dy$ where C is the path defined by $r(t) = (t, (t-1)\pi)$, $1 \le t \le 2$.

20. $\displaystyle\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ where $F(x,y) = (-2e^y\sin(x), 2e^y\cos x)$, and C is defined by $r(t) = (t, 6t/\pi)$, $0 \le t \le \dfrac{\pi}{3}$.

21. $\displaystyle\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ where $F(x,y) = \big(y\cos(x)e^{y\sin x}, \sin(x)e^{y\sin x}\big)$, and C is defined by $r(t) = (t, 2t)$, $0 \le t \le \dfrac{\pi}{3}$.

22. $\displaystyle\int_C \left(\arcsin\left(\frac{y}{2}\right)dx + \frac{x}{\sqrt{4-y^2}}dy\right)$ where C is a path joining point $(1,0)$ to $(\sqrt{3},1)$ with $y \ne \pm 2$.

23. $\displaystyle\int_C \big(yz^2dx + xz^2dy + 2xyzdz\big)$ where C is a path from point $P\,(2,2,1)$ to $Q\,(3,1,2)$.

24. $\displaystyle\int_C (e^y\cos x\ dx + e^y\sin x\ dy + dz)$ where C is a path from point $A\left(\frac{\pi}{2},0,1\right)$ to $B\left(\frac{\pi}{3},1,2\right)$.

25. $\displaystyle\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ where C is a path from point $P(0,2,1)$ to $Q(2,1,2)$, and $\boldsymbol{F}(x,y,z) = (ye^{xy-z}, xe^{xy-z}. -e^{xy-z})$.

26. $\displaystyle\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ where C is a path from point $P(1,0,1)$ to $Q(\sqrt{3},2,1)$, and $\boldsymbol{F}(x,y,z) = \left(\dfrac{z}{1+x^2}, 1, \arctan x\right)$.

In Exercises 27-34, evaluate the path integral of the given real-valued function along the smooth curve C parametrized by $\boldsymbol{r}$. See Definition 4.

27. $\displaystyle\int_C f(x,y)ds$ where $f(x,y) = xy - x^2$, $\boldsymbol{r}(t) = (3t, 4t)$, $0 \le t \le \dfrac{1}{3}$.

28. $\displaystyle\int_C f(x,y)ds$ where $f(x,y) = \dfrac{4}{13}(x-y)$, $\boldsymbol{r}(t) = (12t, 5t)$, $0 \le t \le \dfrac{1}{2}$.

29. $\displaystyle\int_C (x+y)ds$ where $\boldsymbol{r}(t) = (4\sin 2t, 4\cos 2t)$, $0 \le t \le \dfrac{\pi}{8}$.

30. $\displaystyle\int_C f(x,y,z)ds$ where $f(x,y,z) = x - y + z$, $\boldsymbol{r}(t) = (\cos t, \sin t, \sqrt{3}\ t)$, $0 \le t \le \dfrac{\pi}{2}$.

31. $\displaystyle\int_C f(x,y,z)ds$ where $f(x,y,z) = xy$, $\boldsymbol{r}(t) = (2\cos t, 2\sin t, \sqrt{5}\ t)$, $0 \le t \le \dfrac{\pi}{6}$.

32. $\displaystyle\int_C f(x,y,z)ds$ where $f(x,y,z) = \dfrac{2\cos x}{16+y}$, $\boldsymbol{r}(t) = \big(8t, 2t^2, t^3/3\big)$, $0 \le t \le \dfrac{\pi}{16}$.

33. $\displaystyle\int_C g(x,y,z)ds$ where $g(x,y,z) = yz - x^2$, $\boldsymbol{r}(t) = (t, 2t, 2t)$, $0 \le t \le 2$.

34. $\displaystyle\int_C g(x,y,z)ds$ where $g(x,y,z) = xyz$, $\boldsymbol{r}(t) = \big(\sqrt{t}, 2\sqrt{t}, 2\sqrt{t}\big)$, $0 \le t \le 3$.

In Exercises 35-40, find the unit tangent vector $\boldsymbol{T}(t)$ to the smooth curve parametrized by $\boldsymbol{r}$. See identity (3) in page 93.

35. $\boldsymbol{r}(t) = \left(t, t^2\right)$

36. $\boldsymbol{r}(t) = (8t, 4t, 8t)$

37. $\boldsymbol{r}(t) = (a\cos t, a\sin t),\ \ a > 0$

38. $\boldsymbol{r}(t) = (b\cos t, b\sin t, t),\ \ b > 0$

39. $\boldsymbol{r}(t) = \left(t, \sqrt{\tfrac{3}{2}}\, t^2, t^3\right)$

40. $\boldsymbol{r}(t) = \left(\sqrt{2}\, t, e^t, e^{-t}\right)$

In Exercises 41-44, find the curvature of the curve parametrized by $\boldsymbol{r}$ at the indicated point P. By the chain rule, the curvature κ satisfies

$$\kappa = \|T'(s)\| = \frac{\|dT/dt\|}{ds/dt} = \frac{\|T'(t)\|}{\|r'(t)\|}.$$

41. $\boldsymbol{r}(t) = \left(t, t^2\right),\ P(0,0)$

42. $\boldsymbol{r}(t) = (t, mt),\ P(t, mt)$

43. $\boldsymbol{r}(t) = (3\cos t, 4\sin t),\ \ P(0,4)$

44. $\boldsymbol{r}(t) = \left(t, 2\sqrt{t}\right),\ \ P(1,2)$

In Exercises 45-46, a fence is built on a curve C parametrized by $\boldsymbol{r}$. The height of the fence at $\boldsymbol{r}(t)$ is given by $f(\boldsymbol{r}(t))$ where f is the indicated real-valued function. Find the area of the fence.

45. $\boldsymbol{r}(t) = (\cos t, \sin t),\ 0 \le t \le 2\pi,\ f(x,y) = 2 + xy$

46. $\boldsymbol{r}(t) = (t, 3t),\ 0 \le t \le 1,\ f(x,y) = 1 + y - x$

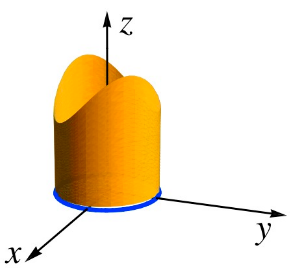


Figure for 45

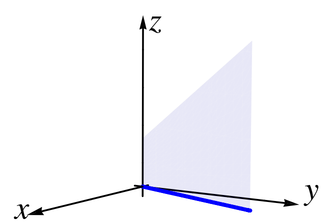


Figure for 46

In Exercises 47-48, a curve C is parametrized by a function $\boldsymbol{r}$. Find the work done by the given force $\boldsymbol{F}$ on a particle that is moving along C.

47. $\boldsymbol{r}(t) = \left(t^2, t\right),\ 0 \le t \le 1,\ \boldsymbol{F}(x,y) = (y, -x)$

48. $\boldsymbol{r}(t) = (e^t, t),\ 0 \le t \le 2,\ \boldsymbol{F}(x,y) = \left(\dfrac{\ln x}{y}, x + y^2\right)$

49. Let $\boldsymbol{r}$ be a smooth curve defined on $[a,b]$, and let $\boldsymbol{T}(t)$ be the unit tangent vector at $\boldsymbol{r}(t)$, see definition (3). Let s be the arc length parameter where $ds/dt = \|\boldsymbol{r}'(t)\|$.

   a) Verify the identities $\boldsymbol{r}'(t) = \dfrac{ds}{dt}\boldsymbol{T}(t)$, and

$$\boldsymbol{r}''(t) = \frac{d^2 s}{dt^2}\boldsymbol{T}(t) + \frac{ds}{dt}\boldsymbol{T}'(t).$$

b) Using the fact that $\boldsymbol{T}(t) \times \boldsymbol{T}(t) = \mathbf{0}$, prove

$$\boldsymbol{r}'(t) \times \boldsymbol{r}''(t) = \left(\frac{ds}{dt}\right)^2 \big(\boldsymbol{T}(t) \times \boldsymbol{T}'(t)\big).$$

c) Using the fact that $\boldsymbol{T}(t) \cdot \boldsymbol{T}(t) = 1$, prove $\boldsymbol{T}(t)$ is perpendicular to $\boldsymbol{T}'(t)$.

d) Show

$$\|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)\| = \left(\frac{ds}{dt}\right)^2 \|\boldsymbol{T}'(t)\|.$$

e) Using the fact that curvature satisfies $\kappa(t) = \dfrac{\|T'(t)\|}{\|r'(t)\|}$, prove the identity

$$\kappa(t) = \frac{\|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)\|}{\|\boldsymbol{r}'(t)\|^3}. \tag{12}$$

50. Let C be a parabolic curve parametrized by $\boldsymbol{r} = \big(0, t, t^2\big)$.

a) Applying the curvature identity (12), verify $\kappa(t) = \dfrac{2}{(1+4t^2)^{3/2}}$.

b) Evaluate $\displaystyle\int_{-\infty}^{\infty} \kappa(t)dt$.

c) Evaluate the path integral $\displaystyle\int_C \kappa ds$.

51. Let C be a hyperboloid curve parametrized by $\boldsymbol{r} = (\sinh t, \cosh t, 0)$.

a) Applying the curvature identity (12), verify $\kappa(t) = (\text{sech } 2t)^{3/2}$.

b) Evaluate the path integral $\displaystyle\int_C \kappa ds$.

52. Let $\boldsymbol{r}$ be a smooth curve defined on $[a, b]$, and let $\boldsymbol{T}(t)$ be the unit tangent vector at $\boldsymbol{r}(t)$.

a) The *normal vector to the curve* at $\boldsymbol{r}(t)$ is defined by

$$\boldsymbol{N}(t) = \frac{\boldsymbol{T}'(t)}{\|\boldsymbol{T}'(t)\|}. \tag{13}$$

b) Using the fact that $\boldsymbol{T}(t) \cdot \boldsymbol{T}(t) = 1$, prove $\boldsymbol{N}(t)$ is perpendicular to $\boldsymbol{T}(t)$.

c) The *binormal vector to the curve* at $\boldsymbol{r}(t)$ is defined by

$$\boldsymbol{B}(t) = \boldsymbol{T}(t) \times \boldsymbol{N}(t). \tag{14}$$

Using the fact that $\boldsymbol{B}(t) \cdot \boldsymbol{B}(t) = 1$, show $\boldsymbol{B}'(t) \cdot \boldsymbol{B}(t) = 0$.

d) Using the fact that $\boldsymbol{B}(t) \cdot \boldsymbol{T}(t) = 0 = \boldsymbol{B}(t) \cdot \boldsymbol{T}'(t)$, prove $\boldsymbol{B}'(t) \cdot \boldsymbol{T}(t) = 0$.

e) Applying c) and d), prove $\boldsymbol{B}'(t)$ is multiple of $\boldsymbol{N}(t)$.

53. If $\boldsymbol{r}(t) = (t, t^2, t^2)$, find the unit tangent vector $\boldsymbol{T}(t)$, the normal vector $\boldsymbol{N}(t)$, and the binormal vector $\boldsymbol{B}(t)$ at $\boldsymbol{r}(t)$.

2.4 Surface Integrals

• Area of a Surface • Surface Integrals of Vector Fields • Surface Integrals of Real-Valued Functions

Area of a Surface

In Section 2.3, we discussed line integrals along smooth curves. In this section, we study surface integrals, or integrals of vector fields defined on surfaces. When the surface is a subset of the xy-plane, the surface integral is a double integral as we will see. We begin with the definition of a parametrized surface.

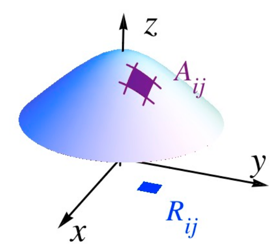


Figure 1
Surface parametrized by two independent variables.

Definition 5 Parametrization of a Surface

Let $R \subseteq \mathbb{R}^2$ be an elementary region[5]. Let $\boldsymbol{r}$ be a continuous function from R to $\mathbb{R}^3$ that satisfies the two conditions below except possibly on the boundary of R.

a) $\boldsymbol{r}$ is one-to-one with continuous partial derivatives, and

b) the functions
$$\phi(u,v) = \left\| \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right\| \tag{15}$$
and $\dfrac{1}{\phi(u,v)}$ are nonzero and bounded.

In such a case, we say $\boldsymbol{r}$ parametrizes the **surface** defined by
$$M = \{\boldsymbol{r}(u,v) \mid (u,v) \in R\}.$$

The function ϕ in (15) is integrable on R, and we omit its proof since it is similar to the application of Theorem 2.1 to Theorem 2.3 in pages 60 and 62.

Next, we define the **surface area** of M. For the moment, let R be a rectangular region. Partition R into sub-rectangles R_{ij} with uniform width Δu and uniform height Δv. Let (u_i, v_j) be the vertex of R_{ij} nearest the origin, as in Figure 1.

We approximate the image $\boldsymbol{r}(R_{ij})$ by a rectangular region A_{ij} whose sides are the vectors $\Delta u \frac{\partial \boldsymbol{r}}{\partial u}\big|_{(u_i,v_j)}$ and $\Delta v \frac{\partial \boldsymbol{r}}{\partial v}\big|_{(u_i,v_j)}$. The area of rectangle A_{ij} is the magnitude of the cross product of the vectors that define the rectangle. That is,

$$\begin{aligned} \text{Area}\,(A_{ij}) &= \left\| \left(\Delta u \frac{\partial \boldsymbol{r}}{\partial u}\Big|_{(u_i,v_j)} \right) \times \left(\Delta v \frac{\partial \boldsymbol{r}}{\partial v}\Big|_{(u_i,v_j)} \right) \right\| \\ &= \left\| \; \frac{\partial \boldsymbol{r}}{\partial u}\Big|_{(u_i,v_j)} \times \frac{\partial \boldsymbol{r}}{\partial v}\Big|_{(u_i,v_j)} \right\| \Delta u \Delta v \end{aligned}$$

[5] Elementary regions are discussed in page 61

The Riemann sum of the areas of rectangles A_{ij} approaches a limit as the norm of the partition of R approaches zero, due to the boundedness and continuity in (15). The limit which is an integral is a reasonable definition for the surface area of M. Notice, the integral of $\|\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\|$ over R exists even if R is an elementary region.

In general, let $\boldsymbol{r}$ be a function satisfying Definition 5. Let M be the corresponding surface parametrized by $\boldsymbol{r}$. We define the surface area of M as follows:

$$\text{(Surface Area of } M) = \int\int_R \left\| \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right\| dA \tag{16}$$

where $dA = dudv$ or $dA = dvdu$. The surface area of M is independent of the parametrization $\boldsymbol{r}$ in Definition 5; the proof is left as an exercise and depends on the change of variables theorem.

We consider a special case where a surface M is the graph of a real-valued function $z = f(x, y)$ with continuous partial derivatives. We parametrize M by the one-to-one function

$$\boldsymbol{r}(x, y) = (x, y, f(x, y)).$$

The cross product of the partial derivatives of $\boldsymbol{r}$ is given by

$$\begin{aligned} \frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} \\ &= \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right). \end{aligned}$$

Applying (16), the area of the surface defined by $z = f(x, y)$, $(x, y) \in R$, satisfies

$$\begin{aligned} \text{(Surface Area of } M) &= \int\int_R \left\| \frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y} \right\| dA \\ &= \int\int_R \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dA. \end{aligned} \tag{17}$$

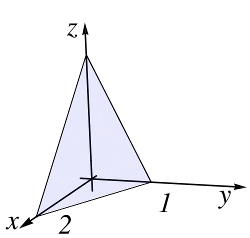


Figure 2
The plane $x + 2y + z = 2$ in the first octant.

Example 1 Evaluating a Surface Area

Find the surface area of the portion of the plane $x + 2y + z = 2$ in the first octant, see Figure 2.

Solution Solving for z, we find and write

$$z = f(x, y) = 2 - x - 2y.$$

Let M be the triangular region in the first octant defined by $z = f(x, y)$. The subset R of the xy-plane below the triangular region satisfies

$$R = \left\{ (x, y) \;\middle|\; 0 \le x \le 2,\ 0 \le y \le \frac{2-x}{2} \right\}.$$

We apply identity (17) to find the surface area of M. Notice,

$$\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}=\sqrt{1+(-1)^2+(-2)^2}=\sqrt{6}.$$

Then the surface area of M is given by

$$\begin{aligned}(\text{Surface Area of } M) &= \int\int_R \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dA \\ &= \sqrt{6}\int_0^2\int_0^{(2-x)/2} dydx \\ &= \sqrt{6}\,(\text{Area of } R) \\ &= \sqrt{6}\ \text{ sq. units}\end{aligned}$$

since the area of triangle R is 1 square unit. □

Try This 1

Find the surface area of the portion of the plane $2x+2y+z=4$ in the first octant above the triangular region in the xy-plane with vertices $(0,0,0)$, $(1,0,0)$, and $(1,1,0)$.

Example 2 Surface Area

Find the surface area of the part of the sphere

$$x^2+y^2+z^2=25$$

where $x^2+y^2 \le 16$ and $z \ge 0$, see Figure 3.

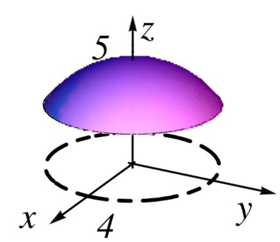


Figure 3
A dome above a circle of radius 4.

Solution We solve for z and write

$$z=f(x,y)=\sqrt{25-x^2-y^2}.$$

Evaluating the partial derivatives, we find

a) $\dfrac{\partial f}{\partial x}=\dfrac{-x}{\sqrt{25-x^2-y^2}}$ **b)** $\dfrac{\partial f}{\partial y}=\dfrac{-y}{\sqrt{25-x^2-y^2}}$

Then we obtain

$$\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}=\frac{5}{\sqrt{25-x^2-y^2}}.$$

Let M be the upper part of the sphere that lies above circular region

$$R = \left\{(x, y) \mid x^2 + y^2 \le 16\right\}.$$

We apply identity (17) to find the surface area of M.

$$\begin{aligned}
(\text{Surface Area of } M) &= \int\int_R \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA \\
&= \int\int_R \frac{5\, dA}{\sqrt{25 - x^2 - y^2}}.
\end{aligned}$$

We apply a change of variables to polar coordinates. Let

a) $x = r\cos\theta$ **b)** $y = r\sin\theta$, **c)** $dA = rdrd\theta$

where $0 \le r \le 4$ and $0 \le \theta \le 2\pi$. Since $x^2 + y^2 = r^2$, we obtain

$$\begin{aligned}
(\text{Surface Area of } M) &= \int_0^4 \int_0^{2\pi} \frac{5r\, d\theta dr}{\sqrt{25 - r^2}} \\
&= 10\pi \int_0^4 \frac{rdr}{\sqrt{25 - r^2}} \\
&= -10\pi\sqrt{25 - r^2}\Big|_{r=0}^{r=4} \\
(\text{Surface Area of } M) &= 20\pi \text{ sq. units}
\end{aligned}$$

□

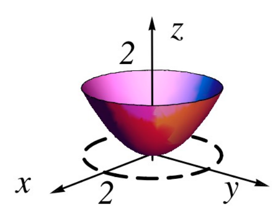


Figure 4
For Try This 2

Try This 2

Find the area of the surface $z = \frac{1}{2}\left(x^2 + y^2\right)$ that lies below the plane $z = 2$. See Figure 4.

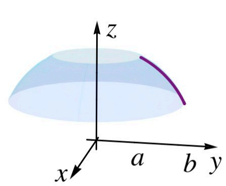


Figure 5
The curve $z = f(y)$ is revolved about the z-axis.

Next, we evaluate the area of a *surface of revolution*. Let C_1 be a curve in the yz-plane that is parametrized by

$$\boldsymbol{r_1}(t) = (0, y(t), z(t)),\ a \le t \le b \tag{18}$$

where $y(t) \ge 0$, as in Figure 5. A surface of revolution M is generated when C_1 is revolved about the z-axis. Then surface M is parametrized by

$$\boldsymbol{r}(\theta, t) = (y(t)\cos\theta, y(t)\sin\theta, z(t)) \tag{19}$$

where $0 \le \theta < 2\pi$, $t \in [a, b]$. We leave the verification of (19) as an exercise.

Moreover, the cross product of the partial derivatives of $\boldsymbol{r}$ satisfies

$$\begin{aligned}\frac{\partial \boldsymbol{r}}{\partial \theta} \times \frac{\partial \boldsymbol{r}}{\partial t} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -y(t)\sin\theta & y(t)\cos\theta & 0 \\ y'(t)\cos\theta & y'(t)\sin\theta & z'(t) \end{vmatrix} \\ &= (y(t)z'(t)\cos\theta,\ y(t)z'(t)\sin\theta,\ -y(t)y'(t)).\end{aligned}$$

Since $y(t) \geq 0$, we find

$$\begin{aligned}\left\| \frac{\partial \boldsymbol{r}}{\partial \theta} \times \frac{\partial \boldsymbol{r}}{\partial t} \right\| &= y(t)\sqrt{[y'(t)]^2 + [z'(t)]^2} \\ &= y(t)\ \|\boldsymbol{r_1}'(t)\| \qquad (20)\end{aligned}$$

Applying (16), the area of the surface of revolution M defined by (18) and (19) is

$$\begin{aligned}(\text{Area of Surface of Revolution}) &= \int_a^b \int_0^{2\pi} \left\| \frac{\partial \boldsymbol{r}}{\partial \theta} \times \frac{\partial \boldsymbol{r}}{\partial t} \right\| d\theta dt \\ &= 2\pi \int_a^b y(t)\ \|\boldsymbol{r_1}'(t)\|\ dt. \qquad (21)\end{aligned}$$

Example 3 Surface Area of a Sphere

Parametrize the sphere S of radius 2 that is centered at the origin.
Then find the surface area of S.

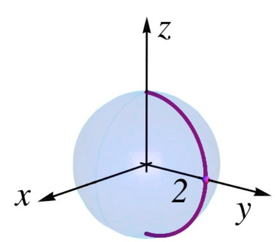


Figure 6
Semicircle generates
a sphere by a rotation.

Solution Let C_1 be the semicircle of radius 2 in the yz-plane given by

$$\boldsymbol{r_1}(\psi) = (0, 2\cos\psi, 2\sin\psi),\ -\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}.$$

The sphere S is generated when C_1 is revolved about the z-axis, see Figure 6. Applying (19) we parametrize S as follows

$$\boldsymbol{r}(\theta, \psi) = (2\cos\psi\cos\theta, 2\cos\psi\sin\theta, 2\sin\psi)$$

where $0 \leq \theta < 2\pi$. From the definition of $\boldsymbol{r_1}(\psi)$, we find $\|\boldsymbol{r_1}'(\psi)\| = 2$ and let $y(\psi) = 2\cos\psi$ be the y-component of $\boldsymbol{r_1}(\psi)$. Using (21), we obtain

$$\begin{aligned}(\text{Surface Area of } S) &= 2\pi \int_a^b y(\psi)\ \|\boldsymbol{r_1}'(\psi)\|\ d\psi \\ &= 2\pi \int_{-\pi/2}^{\pi/2} (2\cos\psi)(2)\ d\psi \\ &= 8\pi \sin\psi \Big|_{-\pi/2}^{\pi/2} \\ &= 16\pi \ \text{ sq. units}\end{aligned}$$

□

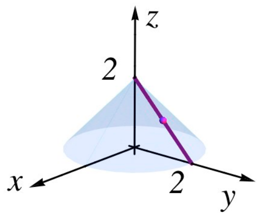


Figure 7
Right circular cone with radius 2 and height 2 units.

Try This 3

Parametrize a right circular cone S whose base is a circle of radius 2 and the height is 2 units. Assume the base lies on the xy-plane and is centered at the origin, see Figure 7. Then find the surface area of the cone S.

Surface Integrals of Vector Fields

Let $\boldsymbol{r}$ be a function from an elementary region $R \subseteq \mathbb{R}^2$ into $\mathbb{R}^3$ as in Definition 5. Let $M \subseteq \mathbb{R}^3$ be the surface that is parametrized by $\boldsymbol{r}$. Let $\boldsymbol{F}$ be a vector-valued function defined on M with values in $\mathbb{R}^3$ such that $F \circ r$ is continuous. In such a case, we say $\boldsymbol{F}$ is a *continuous vector field defined on M*.

Definition 6 Surface Integral of a Vector Field

Let $M \subseteq \mathbb{R}^3$ be a surface parametrized by a function $\boldsymbol{r}$ satisfying Definition 5. Let $\boldsymbol{F}$ be a continuous vector field defined on M, where the values of $\boldsymbol{F}$ lie in $\mathbb{R}^3$. The **surface integral** of $\boldsymbol{F}$ on M denoted by $\int\int_M \boldsymbol{F} \cdot d\boldsymbol{S}$ is defined by

$$\int\int_M \boldsymbol{F} \cdot d\boldsymbol{S} = \int\int_R \boldsymbol{F}(\boldsymbol{r}(u,v)) \cdot \left(\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right) dA$$

where $dA = dudv$ or $dA = dvdu$, and R is the domain of definition of $\boldsymbol{r}$.

An **orientation on a surface** M is a *continuous* function that assigns to each interior point p of M a unit vector $N(p)$ that is perpendicular to the tangent plane at p. We claim the surface integral $\int\int_M \boldsymbol{F} \cdot d\boldsymbol{S}$ is independent of the orientation preserving parametrization $\boldsymbol{r}$ of M, as we briefly explain below.

By Definition 5, $\boldsymbol{r}$ is defined on an elementary region R. Given $p = \boldsymbol{r}(u,v)$ where (u,v) lies in the interior of R, the unit vector below

$$N_{\boldsymbol{r}}(p) = \frac{1}{\left\| \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right\|} \left(\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right)$$

is normal to the tangent plane at p, where the partial derivatives are evaluated at (u,v). For the details, see Exercise 25. We say $\boldsymbol{r}$ is **orientation preserving** if

$$N_{\boldsymbol{r}}(p) = N(p)$$

for all interior points $p \in M$. While $\boldsymbol{r}$ is **orientation reversing** if

$$N_{\boldsymbol{r}}(p) = -N(p)$$

for all interior points $p \in M$.

Example 4 Evaluating a Surface Integral

Let $\boldsymbol{F}(x,y,z) = (0,0,y)$ be a vector field. Let R be a triangular region in the xy-plane with vertices at $(1,2)$, $(0,2)$, and the origin, see Figure 8. Let M be the surface that is oriented, and parametrized by

$$\boldsymbol{r}(x,y) = \left(x, y, \sqrt{x^2+y^2+1}\right),\ (x,y) \in R.$$

Then evaluate the surface integral $\int\int_M \boldsymbol{F}\cdot d\boldsymbol{S}$.

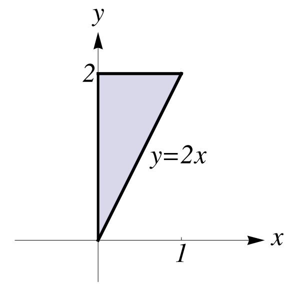


Figure 8
Triangular region R

Solution The surface M is a portion of the larger surface $z = \sqrt{x^2+y^2+1}$ that lies above the region R in the xy-plane, see Figure 9. The cross product of the partial derivatives of $\boldsymbol{r}$ satisfies

$$\begin{aligned}\frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & \frac{x}{\sqrt{x^2+y^2+1}} \\ 0 & 1 & \frac{y}{\sqrt{x^2+y^2+1}} \end{vmatrix} \\ &= \left(\frac{-x}{\sqrt{x^2+y^2+1}}, \frac{-y}{\sqrt{x^2+y^2+1}}, 1\right).\end{aligned}$$

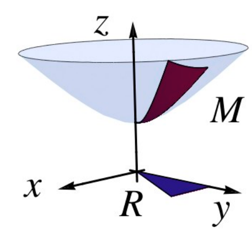


Figure 9
Surface M above region R in Figure 8

We apply Definition 6 to evaluate the surface integral. Notice, $\boldsymbol{F}(\boldsymbol{r}(x,y)) = (0,0,y)$.

$$\begin{aligned}\int\int_M \boldsymbol{F}\cdot d\boldsymbol{S} &= \int\int_R \boldsymbol{F}(\boldsymbol{r}(x,y))\cdot\left(\frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y}\right) dA \\ &= \int_0^1\int_{2x}^2 (0,0,y)\cdot\left(\frac{-x}{\sqrt{x^2+y^2+1}}, \frac{-y}{\sqrt{x^2+y^2+1}}, 1\right) dydx \\ &= \int_0^1\int_{2x}^2 ydydx = \int_0^1 \frac{y^2}{2}\bigg|_{y=2x}^{y=2} dx = \int_0^1 \left(2-2x^2\right) dx \\ \int\int_M \boldsymbol{F}\cdot d\boldsymbol{S} &= \frac{4}{3}.\end{aligned}$$

□

Try This 4

Let $\boldsymbol{F}(x,y,z) = (0,0,1)$ be a vector field. Let R be a triangular region in the xy-plane with vertices at $(0,\frac{1}{2})$, $(\frac{1}{2},0)$, and the origin. Let M be the surface parametrized by

$$\boldsymbol{r}(x,y) = \left(x, y, \sqrt{1-x^2-y^2}\right),\ (x,y) \in R.$$

Then evaluate the surface integral $\int\int_M \boldsymbol{F}\cdot d\boldsymbol{S}$.

Example 5 Evaluating a Surface Integral

Let $\boldsymbol{F}(x, y, z) = (x, y, z)$ be a vector field on the unit sphere centered at $(0, 0, 0)$. Assume the sphere is oriented by unit vectors that point away from the origin. Then evaluate the surface integral $\int\int_M \boldsymbol{F} \cdot d\boldsymbol{S}$.

Solution In Example 3, we parametrized a sphere of radius 2 centered at the origin. Likewise, we obtain a parametrization $\boldsymbol{r}$ of the unit sphere, namely,

$$\boldsymbol{r}(\theta, \psi) = (\cos\psi\cos\theta, \cos\psi\sin\theta, \sin\psi)$$

where $0 \leq \theta < 2\pi$ and $-\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}$. For the cross product, we simplify and find

$$\begin{aligned} \frac{\partial \boldsymbol{r}}{\partial \theta} \times \frac{\partial \boldsymbol{r}}{\partial \psi} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -\cos\psi\sin\theta & \cos\psi\cos\theta & 0 \\ -\sin\psi\cos\theta & -\sin\psi\sin\theta & \cos\psi \end{vmatrix} \\ &= \left(\cos^2\psi\cos\theta,\ \cos^2\psi\sin\theta,\ \cos\psi\sin\psi\right). \end{aligned}$$

Analyzing the third component of $\frac{\partial \boldsymbol{r}}{\partial \theta} \times \frac{\partial \boldsymbol{r}}{\partial \psi}$, we observe the cross product vector points away from the origin. In particular, $\boldsymbol{r}$ is orientation preserving.

Moreover, we find

$$\begin{aligned} \boldsymbol{F}(\boldsymbol{r}(\theta, \psi)) \cdot \left(\frac{\partial \boldsymbol{r}}{\partial \theta} \times \frac{\partial \boldsymbol{r}}{\partial \psi}\right) &= \\ (\cos\psi\cos\theta, \cos\psi\sin\theta, \sin\psi) \cdot \left(\cos^2\psi\cos\theta,\ \cos^2\psi\sin\theta,\ \cos\psi\sin\psi\right) &= \\ \cos^3\psi(\cos^2\theta + \sin^2\theta) + \cos\psi\sin^2\psi &= \\ \cos^3\psi + \cos\psi\sin^2\psi &= \\ \cos\psi(\cos^2\theta + \sin^2\theta) &= \cos\psi. \end{aligned}$$

Applying Definition 6, we evaluate the surface integral as follows:

$$\begin{aligned} \int\int_M \boldsymbol{F} \cdot d\boldsymbol{S} &= \int\int_R \boldsymbol{F}(\boldsymbol{r}(\theta, \psi)) \cdot \left(\frac{\partial \boldsymbol{r}}{\partial \theta} \times \frac{\partial \boldsymbol{r}}{\partial \psi}\right) dA \\ &= \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \cos\psi \ d\theta d\psi \\ &= 2\pi \int_{-\pi/2}^{\pi/2} \cos\psi \ d\psi \\ \int\int_M \boldsymbol{F} \cdot d\boldsymbol{S} &= 4\pi. \end{aligned}$$

□

Try This 5

The orientation on the unit sphere M are unit vectors that point away from the origin. If $\boldsymbol{F}(x, y, z) = (0, 0, z)$, evaluate the surface integral $\int\int_M \boldsymbol{F} \cdot d\boldsymbol{S}$.

In the short sequel, we give a geometric interpretation of a surface integral $\int\int_M \boldsymbol{F} \cdot d\boldsymbol{S}$. Suppose a fluid is flowing through a surface M such that the fluid's direction and velocity at point p in S is given by vector $\boldsymbol{F}(p)$ ft/sec. The component of $\boldsymbol{F}(p)$ in the vector $\frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y}$ normal to S at p is given by the dot product

$$\boldsymbol{F}(p) \cdot \frac{1}{\|\frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y}\|} \left(\frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y} \right) \quad \text{ft/sec}$$

We partition M into rectangular-like subregions where $\|\frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y}\| dxdy$ ft^2 is the area of a subregion. When we multiply the above dot product to the area of a subregion, we obtain

$$\boldsymbol{F}(p) \cdot \left(\frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y} \right) dxdy \quad \text{ft}^3/\text{sec}$$

that describes the volume of fluid flowing through the subregion per second. Summing up and taking the limit as the norm of the partition approaches zero, we obtain the surface integral $\int\int_M \boldsymbol{F} \cdot d\boldsymbol{S}$ that is the volume of fluid flowing through surface M per second.

Surface Integrals of Real-Valued Functions

Definition 7 The Integral of a Real-Valued Function Defined on a Surface

Let f be a real-valued function on a surface M. Suppose M is parametrized by a function $\boldsymbol{r}$, as in Definition 5, such that $f \circ r$ is continuous. We denote the integral of f on M by $\int\int_M f dS$, and define it by

$$\int\int_M f dS = \int\int_R f(\boldsymbol{r}(x, y)) \left\| \frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y} \right\| dA \tag{22}$$

where R is an elementary region in the xy-plane on which $\boldsymbol{r}$ is defined.

The surface integral of a real-valued function f on M is independent of the parametrization $\boldsymbol{r}$ of M. That is, whether $\boldsymbol{r}$ is orientation preserving or reversing, the integral (22) is invariant. See Exercise 25 for a proof.

In particular, if f is a nonnegative function, the surface integral of f describes the volume of a solid. Moreover, the base of the solid is the surface M and the height of the solid at point $\boldsymbol{r}(x, y)$ is $f(\boldsymbol{r}(x, y))$. The integrand in (22) represents the volume of a rectangular box whose base has area $\left\| \frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y} \right\| dA$ and the height of the box is $f(\boldsymbol{r}(x, y))$.

Example 6 Evaluating a Surface Integral

Let M be the rectangular region in the plane $z = 1 - y$ where $0 \leq x, y \leq 1$.
Let $f(x, y, z) = x$ be a real-valued function that is defined on M.
Then evaluate the surface integral $\int\int_M f dS$.

Solution To parametrize M where $z = 1 - y$, we let

$$\boldsymbol{r}(x, y) = (x, y, 1 - y)$$

where (x, y) lies in a rectangular region R in the xy-plane defined by $0 \leq x, y \leq 1$. For the cross product of the partial derivatives of $\boldsymbol{r}$, we find

$$\begin{aligned} \frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix} \\ &= (0, 1, 1)\,. \end{aligned}$$

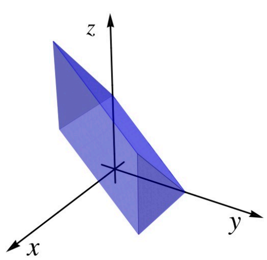


Figure 10
The base of a solid is the surface M in the plane $z = 1 - y$.

Using the given function f, we obtain

$$f(\boldsymbol{r}(x, y)) = f\,(x, y, 1 - y) = x.$$

Applying Definition 7, the surface integral satisfies

$$\begin{aligned} \int\int_M f dS &= \int\int_R f(\boldsymbol{r}(x, y)) \left\| \frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y} \right\| dA \\ &= \int_0^1 \int_0^1 x\sqrt{2}\; dxdy \\ &= \frac{\sqrt{2}}{2}. \end{aligned}$$

The surface integral $\int\int_M f dS$ represents the volume of a solid in Figure 10. The base of the solid is the surface M, and the height at $(x, y, z) \in M$ of the solid is $f(x, y, z)$.

□

Try This 6

Let M be the rectangular region in the plane $z = 1 - y$ where $0 \leq x, y \leq 1$, i.e., M is the base of the solid in Figure 10. If $f(x, y, z) = y + z$ is defined on M, evaluate the surface integral $\int\int_M f dS$.

2.4 Check-It Out

In Exercises 1-3, let M be the triangular part of the plane $x+y+z=1$ in the first octant. Suppose M is oriented by unit vectors normal to M that point away from the origin.

1. Find the surface area of M.
2. If $\boldsymbol{F}(x,y,z)=(0,0,y+z)$, evaluate the surface integral $\int\int_M \boldsymbol{F}\cdot d\boldsymbol{S}$
3. If $f(x,y,z)=y+z$, then evaluate the surface integral $\int\int_M f dS$

True or False. *If false, revise the statement to make it true or explain.*

1. The plane $x+y+z=4$ in the first octant is parametrized by $\boldsymbol{r}(x,y)=(x,y,4-x-y)$ where $0\le x,y\le 4$.
2. Let M be a surface parametrized by $\boldsymbol{r}(s,t)=(s,t,1)$ where $0\le s,t\le 1$. If $\boldsymbol{F}$ is vector field on M, then $\displaystyle\int\int_M \boldsymbol{F}\cdot d\boldsymbol{S}=\int_0^1\int_0^1 \boldsymbol{F}(s,t,1)\cdot\boldsymbol{k}\ dsdt$
3. The surface area of $z=f(x,y)$ is given by $\displaystyle\int\int_R \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\ dA$ for some region R in the xy-plane.
4. The surface area of M is given by $\displaystyle\int\int_R \|\boldsymbol{r}(s,t)\|\ dA$ where $\boldsymbol{r}$ is a function that parametrizes M, and $\boldsymbol{r}$ is defined on some region R in the st-plane.
5. If $\boldsymbol{F}(x,y,z)=(x,y,z)$, and M is parametrized by $\boldsymbol{r}(x,y)=(x,y,1-x-y)$, then $\displaystyle\int\int_M \boldsymbol{F}\cdot d\boldsymbol{S}=\int\int_R (1+x+y)dA$ for some region R in the xy-plane.
6. If $f(x,y,z)=x^2+y^2+z$, and M is parametrized by $\boldsymbol{r}(s,t)=(s,t,s^2+t^2)$, then $\displaystyle\int\int_M f dS=\int\int_R (s^2+t^2)\sqrt{1+4s^2+4t^2}\ dA$ for some region R in the st-plane.
7. If $\boldsymbol{F}$ is a vector field on a surface M defined by $z=f(x,y)$, then $\displaystyle\int\int_M \boldsymbol{F}\cdot d\boldsymbol{S}=\int\int_R \boldsymbol{F}(x,y,f(x,y))\cdot\left(-\frac{\partial f}{\partial x},-\frac{\partial f}{\partial y},1\right)\ dA$ for some region R in the xy-plane.

Exercises for Section 2.4

In Exercises 1-6, find a parametrization of the indicated surface. Then find the area of the surface.

1. The quadrilateral region defined by $2x + y + z = 4$ where $0 \leq x \leq 1$ and $0 \leq y \leq 2$.

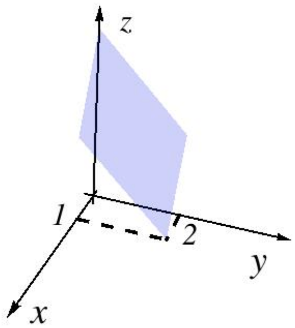


Figure for 1

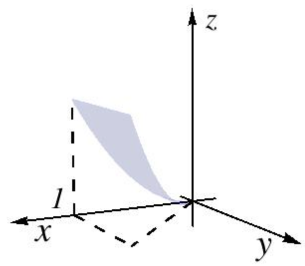


Figure for 2

2. The surface in the parabolic cylinder $z = \frac{\sqrt{3}}{2}x^2$ where $0 \leq x \leq 1$ and $0 \leq y \leq x$.

3. The elliptical-shaped surface $x + y + z = 3$ such that $x^2 + y^2 \leq 1$.

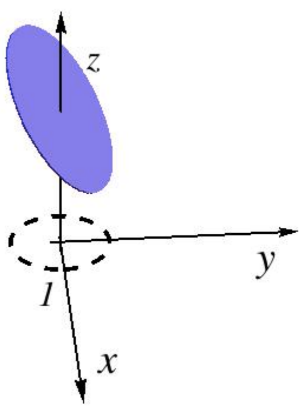


Figure for 3

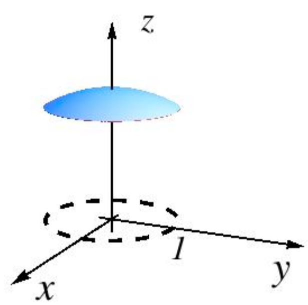


Figure for 4

4. The dome-shaped surface $x^2 + y^2 + z^2 = 4$ such that $x^2 + y^2 \leq 1$ and $z > 0$.

5. The surface consisting of points (x, y, z) satisfying $x^2 + y^2 = 1$ and $0 \leq z \leq 1 - y$.

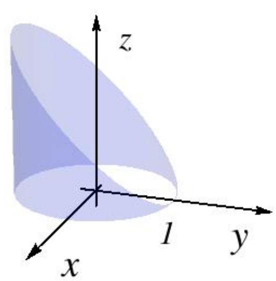


Figure for 5

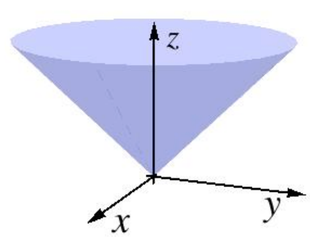


Figure for 6

6. The cone defined by $z = \sqrt{x^2 + y^2}$ where $x^2 + y^2 \leq 16$. The top is not included.

In Exercises 7-16, evaluate the surface integral $\int\int_M \boldsymbol{F} \cdot d\boldsymbol{S}$ for the indicated vector field $\boldsymbol{F}$ defined on the oriented surface M. Assume M is oriented by unit vectors pointing upward.

7. Let $\boldsymbol{F}(x, y, z) = (y, 0, z)$, and let M be parametrized by $\boldsymbol{r}(t, \theta) = (2t\cos\theta, 2t\sin\theta, 1)$ where $0 \le \theta \le 2\pi$ and $0 \le t \le 1$.

8. Let $\boldsymbol{F}(x, y, z) = (0, y + z, x)$, and let M be parametrized by $\boldsymbol{r}(t, \theta) = (t\cos\theta, t\sin\theta, 1 - t\sin\theta)$ where $0 \le \theta \le 2\pi$ and $0 \le t \le 1$.

9. Let $\boldsymbol{F}(x, y, z) = (y, -x, z)$, and let M be parametrized by $\boldsymbol{r}(\psi, \theta) = (\sin\psi\cos\theta, \sin\psi\sin\theta, \cos\psi)$ where $0 \le \psi \le \pi$ and $0 \le \theta \le 2\pi$. Note, M is the unit sphere.

10. Let $\boldsymbol{F}(x, y, z) = \left(-x/z, -y/z, 1/z^2\right)$, and let M be parametrized by $\boldsymbol{r}(x, y) = \left(x, y, \sqrt{1 + x^2 + y^2}\right)$ where $0 \le x \le a$, $0 \le y \le b$.

11. Let $\boldsymbol{F}(x, y, z) = (0, -x, 2y)$, and let M be the triangular planar region defined by $x + y + 2z = 2$ in the first octant.

12. Let $\boldsymbol{F}(x, y, z) = (0, 0, 2)$, and let M be upper unit hemisphere $z = \sqrt{1 - x^2 - y^2}$.

13. Let $\boldsymbol{F}(x, y, z) = (2y - 2z, -1, 2x)$, and let M be the triangular planar region defined by $x - y + z = 1$ in the octant where $x, z > 0$ and $y < 0$.

14. Let $\boldsymbol{F} = (0, z, 0)$, and let M be the portion of the plane $2x + 2y + z = 4$ in the first octant.

15. Let $\boldsymbol{F} = \mathrm{curl}(0, 0, y)$, and let M be the portion of the sphere $x^2 + y^2 + z^2 = 25$ in the first octant.

16. Let $\boldsymbol{F} = \mathrm{curl}(0, x^2, 0)$, and let M be the hemisphere $z = \sqrt{4 - x^2 - y^2}$.

In Exercises 17-24, evaluate the surface integral $\int\int_M f dS$ for the indicated real-valued function f defined on the given surface M.

17. Let $f(x, y, z) = 3z$, and let M be the surface parametrized by $\boldsymbol{r}(\theta, t) = (t\cos\theta, t\sin\theta, t)$ where $0 \le t \le \sqrt{2}$ and $0 \le \theta \le 2\pi$.

18. Let $f(x, y, z) = x^2$, and let M be the surface parametrized by $\boldsymbol{r}(\theta, t) = (t\cos\theta, t\sin\theta, t + 1)$ where $0 \le t \le 2$ and $0 \le \theta \le 2\pi$.

19. Let $f(x, y, z) = x^2 + y^2$, and let M be the surface parametrized by $\boldsymbol{r}(x, y) = (x, y, \sqrt{x^2 + y^2})$ such that $x^2 + y^2 \le 1$.

20. Let $f(x, y, z) = z\sqrt{x^2 + y^2}$, and let M be the disk of radius 2 defined by $x^2 + y^2 \le 4$ and $z = 3$.

21. Let $f(x, y, z) = 2x + y + z$, and let M be the plane $2x + y + z = 4$ in the first octant.

22. Let $f(x, y, z) = 2y$, and let M be the plane $x + y + z = 3$ in the first octant.

23. Let $f(x, y, z) = 6xy$, and let M be the triangular region with vertices $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$.

24. Let $f(x, y, z) = 6z$, and let M be the triangular region with vertices $(2, 0, 0)$, $(0, 0, 4)$, and $(1, 1, 1)$.

The Effects of Re-parametrizations on Surface Integrals

25. Let $\boldsymbol{r_1}, \boldsymbol{r_2}$ be two functions that parametrize a surface $M \subseteq \mathbb{R}^3$, as in Definition 5. Suppose $\boldsymbol{r_1}, \boldsymbol{r_2}$ are defined on elementary regions $R_1, R_2 \subseteq \mathbb{R}^2$, respectively. To each interior point (s, t) in R_2, choose an interior point (u, v) in R_1 such that

$$\boldsymbol{r_1}(u, v) = \boldsymbol{r_2}(s, t).$$

   a) Prove there exists a function h that is one-to-one and differentiable in the interior of R_2 such that $(u, v) = h(s, t)$. We sketch a proof below.

   Let $\boldsymbol{r_2}(s_0, t_0) = \boldsymbol{r_1}(u_0, v_0) = (x_0, y_0, z_0) \in M$. Since the cross product $\frac{\partial \boldsymbol{r_2}}{\partial s}\big|_{(s_0,t_0)} \times \frac{\partial \boldsymbol{r_2}}{\partial t}\big|_{(s_0,t_0)}$ is nonzero, without loss of generality we assume the third component of the cross product is nonzero,

$$\text{Det}\begin{bmatrix} \frac{\partial x_2}{\partial s} & \frac{\partial x_2}{\partial t} \\ \frac{\partial y_2}{\partial s} & \frac{\partial y_2}{\partial t} \end{bmatrix} = \frac{\partial x_2}{\partial s}\frac{\partial y_2}{\partial t} - \frac{\partial x_2}{\partial t}\frac{\partial y_2}{\partial s} \neq 0$$

   where $\boldsymbol{r_2}(s, t) = (x_2, y_2, z_2)$. We apply the Inverse Function Theorem to the system of equations:

$$\begin{cases} x_2(s, t) = x \\ y_2(s, t) = y \end{cases}$$

   Since the above determinant is nonzero, it is possible to express (s, t) as a differentiable function of (x, y) near (x_0, y_0). That is, $(s, t) = f(x, y)$ for some invertible function f defined on an open disk containing (x_0, y_0) with continuous partial derivatives.

   Similarly, since $\frac{\partial \boldsymbol{r_1}}{\partial u}\big|_{(u_0,v_0)} \times \frac{\partial \boldsymbol{r_1}}{\partial v}\big|_{(u_0,v_0)}$ is nonzero, we write $(u, v) = g(x, y)$, $(u, v) = g(x, z)$, or $(u, v) = g(y, z)$ for some function g with continuous partial derivatives in an open disk containing (x_0, y_0), (x_0, z_0), or (y_0, z_0), respectively. In any case, we obtain $(s, t) = f \circ g^{-1}(u, v)$.

   b) Apply the Chain Rule to verify

$$\begin{aligned} \frac{\partial \boldsymbol{r_2}}{\partial s} \times \frac{\partial \boldsymbol{r_2}}{\partial t} &= \left(\frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}\right)\left(\frac{\partial \boldsymbol{r_1}}{\partial x} \times \frac{\partial \boldsymbol{r_1}}{\partial y}\right) \\ &= \frac{\partial(x, y)}{\partial(s, t)}\left(\frac{\partial \boldsymbol{r_1}}{\partial x} \times \frac{\partial \boldsymbol{r_1}}{\partial y}\right) \end{aligned}$$

   where $\frac{\partial(x,y)}{\partial(s,t)}$ is the Jacobian determinant of the transformation from $(s, t) \in R_1$ to $(x, y) \in R_2$. Thus, the tangent plane to the surface M is well-defined since the cross-products $\frac{\partial \boldsymbol{r_2}}{\partial s} \times \frac{\partial \boldsymbol{r_2}}{\partial t}$ and $\frac{\partial \boldsymbol{r_1}}{\partial x} \times \frac{\partial \boldsymbol{r_1}}{\partial y}$ are parallel.

c) Let $\boldsymbol{F}$ be a continuous vector field defined on M and with values in $\mathbb{R}^3$. Applying part a) and the change of variables theorem, we find

$$\begin{aligned}
\int\int_{R_2} \boldsymbol{F}\left(\boldsymbol{r_2}(s,t)\right)\cdot\left(\frac{\partial \boldsymbol{r_2}}{\partial s}\times\frac{\partial \boldsymbol{r_2}}{\partial t}\right) dsdt &= \int\int_{R_1} \boldsymbol{F}\left(\boldsymbol{r_1}(x,y)\right)\cdot\left(\frac{\partial(x,y)}{\partial(s,t)}\left(\frac{\partial \boldsymbol{r_1}}{\partial x}\times\frac{\partial \boldsymbol{r_1}}{\partial y}\right)\right)\left|\frac{\partial(s,t)}{\partial(x,y)}\right| dxdy \\
&= \int\int_{R_1} \boldsymbol{F}\left(\boldsymbol{r_1}(x,y)\right)\cdot\left(\frac{\partial \boldsymbol{r_1}}{\partial x}\times\frac{\partial \boldsymbol{r_1}}{\partial y}\right)\frac{\partial(x,y)}{\partial(s,t)}\left|\frac{\partial(s,t)}{\partial(x,y)}\right| dxdy \\
&= \pm\int\int_{R_1} \boldsymbol{F}\left(\boldsymbol{r_1}(x,y)\right)\cdot\left(\frac{\partial \boldsymbol{r_1}}{\partial x}\times\frac{\partial \boldsymbol{r_1}}{\partial y}\right) dxdy
\end{aligned}$$

where the sign is positive if the Jacobian determinant $\frac{\partial(x,y)}{\partial(s,t)}$ is positive, and the sign is negative if the Jacobian determinant is negative.

The Jacobian determinant is positive when $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ are orientation preserving. Then the integral of $\boldsymbol{F}$ on surface M is independent of the orientation-preserving parametrization $\boldsymbol{r}$ of M.

d) Applying the change of variable theorem, we find

$$\begin{aligned}
\int\int_{R_2} f\left(\boldsymbol{r_2}(s,t)\right)\left\|\frac{\partial \boldsymbol{r_2}}{\partial s}\times\frac{\partial \boldsymbol{r_2}}{\partial t}\right\| dsdt &= \int\int_{R_1} f\left(\boldsymbol{r_1}(x,y)\right)\left\|\frac{\partial(x,y)}{\partial(s,t)}\left(\frac{\partial \boldsymbol{r_1}}{\partial x}\times\frac{\partial \boldsymbol{r_1}}{\partial y}\right)\right\|\left|\frac{\partial(s,t)}{\partial(x,y)}\right| dxdy \\
&= \int\int_{R_1} f\left(\boldsymbol{r_1}(x,y)\right)\left\|\left(\frac{\partial \boldsymbol{r_1}}{\partial x}\times\frac{\partial \boldsymbol{r_1}}{\partial y}\right)\right\| dxdy.
\end{aligned}$$

In particular, the integral of the scalar-valued function f over M, namely, $\int\int_M f dS$, is independent of the parametrization $\boldsymbol{r}$ of M. Moreover, the path integral is independent of the orientation of $\boldsymbol{r}$, i.e., the orientation may be preserving or reversing.

2.5 Integral Theorems of Green, Stokes, and Gauss

• Stokes' Theorem • Stokes' Theorem Applied to Surfaces that are Graphs of Functions • Green's Theorem • Gauss' Divergence Theorem • Proofs of the Integral Theorems

Stokes' Theorem

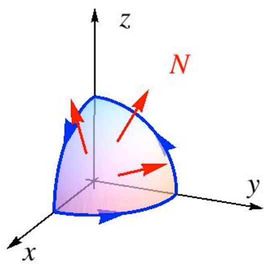


Figure 1
An oriented surface M, and its boundary ∂M is oriented positively.

Stokes' Theorem relates a surface integral to a line integral along the boundary of the surface. In the short sequel, we develop the relationship.

Let M be an oriented surface parametrized by a function $\boldsymbol{r}$ satisfying Definition 5 in page 111. Consequently, there are unit vectors $N(p)$ that are normal to M at p such that $N(p)$ varies continuously as a function of p. Furthermore, suppose the boundary ∂M of M is a simple closed curve. We define a positive orientation for the curve ∂M using the orientation of M, see Figure 1. That is, using your right hand, let your thumb point to the direction of the unit vectors $N(p)$, and the direction along the curve ∂M as you curl your fingers is the **positive orientation** for ∂M.

In words, Stokes' Theorem states that the line integral of a differentiable vector field $\boldsymbol{F}$ in 3-space along a positively oriented boundary ∂M is the equal to the surface integral of the curl$\boldsymbol{F}$ on the surface M.

Theorem 2.12 Stokes' Theorem

Let M be an oriented surface such that its boundary ∂M is a simple closed curve. Suppose ∂M has the positive orientation. Let $\boldsymbol{F}$ be a vector field defined on M with values in $\mathbb{R}^3$. If $\boldsymbol{F}$ has continuous partial derivatives, then

$$\int_{\partial M} \boldsymbol{F} \cdot d\boldsymbol{r} = \int\int_M \text{curl} F \cdot d\mathbf{S}.$$

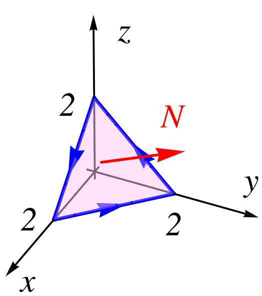


Figure 2
The boundary is oriented positively.

Example 1 Illustrating Stokes' Theorem

Verify Stokes' Theorem if $\boldsymbol{F}(x,y,z) = (y,-z,-x)$, M is the plane $x+y+z=2$ in the first octant, and M is oriented by unit vectors that point away from the origin. Notice, the boundary of M is oriented counter-clockwise, see Figure 2.

Solution Let $\boldsymbol{r}_1$ parametrize the straight path C_1 from $(2,0,0)$ to $(0,2,0)$.

$$\boldsymbol{r}_1(t) = (2-t, t, 0), \quad 0 \le t \le 2.$$

Likewise, let $\boldsymbol{r}_2$ parametrize the straight path C_2 from $(0,2,0)$ to $(0,0,2)$.

$$\boldsymbol{r}_2(t) = (0, 2-t, t), \quad 0 \le t \le 2.$$

Similarly, let $\boldsymbol{r}_3$ parametrize the straight path from $(0,0,2)$ to $(2,0,0)$.

$$\boldsymbol{r}_3(t) = (t, 0, 2-t), \quad 0 \le t \le 2.$$

The boundary of M is the union of the oriented lines, i.e., $\partial M = C_1 \cup C_2 \cup C_3$. Then the line integral of $\boldsymbol{F}$ along ∂M satisfies

$$\int_{\partial M} \boldsymbol{F} \cdot d\boldsymbol{r} \;=\; \int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_3} \boldsymbol{F} \cdot d\boldsymbol{r} \tag{23}$$

Since $\boldsymbol{F}(x, y, z) = (y, -z, -x)$, the line integral along $\boldsymbol{r}_1$ is given by

$$\begin{aligned} \int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_0^2 \boldsymbol{F}(\boldsymbol{r}_1(t)) \cdot \boldsymbol{r}_1'(t) dt \\ &= \int_0^2 (t, 0, t-2) \cdot (-1, 1, 0) dt \\ &= -\int_0^2 t dt \;=\; -2. \end{aligned}$$

Similarly, the line integral along $\boldsymbol{r}_2$ satisfies

$$\begin{aligned} \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_0^2 \boldsymbol{F}(\boldsymbol{r}_2(t)) \cdot \boldsymbol{r}_2'(t) dt \\ &= \int_0^2 (2-t, -t, 0) \cdot (0, -1, 1) dt \;=\; \int_0^2 t dt \;=\; 2. \end{aligned}$$

Likewise, we find $\int_{C_3} \boldsymbol{F} \cdot d\boldsymbol{r} = 2$. Applying (24), the sum of the line integrals equals

$$\int_{\partial M} \boldsymbol{F} \cdot d\boldsymbol{r} = 2. \tag{24}$$

Since the plane is $z = 2 - x - y$, we parametrize M by

$$\boldsymbol{r}(x, y) = (x, y, 2 - x - y)$$

where (x, y) lies in the xy-plane directly below M, see Figure 2a. Then

$$\frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y} \;=\; \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} \;=\; (1, 1, 1).$$

The curl of $\boldsymbol{F}$ satisfies

$$\text{curl}\boldsymbol{F} \;=\; \nabla \times \boldsymbol{F} \;=\; \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -z & -x \end{vmatrix} \;=\; (1, 1, -1).$$

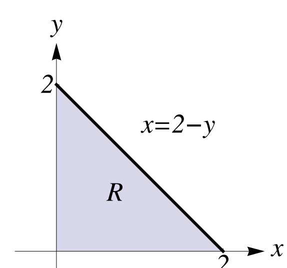


Figure 2a
The region R in the xy-plane below M.

Using the region of integration R in Figure 2a, the surface integral is given by

$$\begin{aligned} \int\int_M \text{curl}F \cdot d\mathbf{S} &= \int_0^2 \int_0^{2-y} (1, -1, -1) \cdot (1, 1, 1) dx dy \\ &= \int_0^2 \int_0^{2-y} dx dy \;=\; (\text{Area of } R) = 2. \end{aligned}$$

Hence, the surface integral agrees with the line integral (24), thereby, verifying Stokes' Theorem.

$\square$

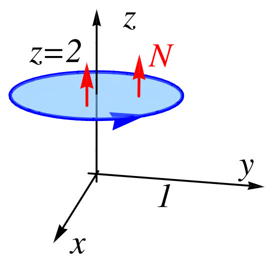


Figure 2b
A unit disk centered at $(0,0,2)$, and oriented by vector $N = \boldsymbol{k}$.

Try This 1

Let M be a circular region of radius 1 in the plane $z = 2$, and centered at $(0,0,2)$. Assume M is oriented by unit vectors N that point upward, as seen in Figure 2b. Verify Stokes' Theorem for the vector field $F(x,y,z) = (-y/2, x/2, 0)$.

Stokes' Theorem Applied to Surfaces that are Graphs of Functions

Let M be a surface that is the graph of a function, i.e., M is parametrized by

$$\boldsymbol{r}(x,y) = (x, y, f(x,y)) \tag{25}$$

where f is a function on an elementary region R in the xy-plane with continuous partial derivatives. Suppose the boundary ∂M of M is a simple closed curve.

Notice, M is oriented by unit vectors pointing away from the origin. Then the positive orientation on ∂M is the counter clockwise direction in the xy-plane.

In Section 2.4, we have seen

$$\frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y} = \text{Det}\begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{bmatrix} = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$$

Let $\boldsymbol{F}(x,y,z)$ be a vector field defined on M and with values in $\mathbb{R}^3$. Suppose $\text{curl}\boldsymbol{F}(\boldsymbol{r}(x,y)) = (L, N, P)$ where L, N, P are functions of (x,y).
Then the surface integral of $\text{curl}\boldsymbol{F}$ on M satisfies

$$\begin{aligned} \int\int_M \text{curl}\boldsymbol{F} \cdot d\mathbf{S} &= \int\int_R \text{curl}\boldsymbol{F}(\boldsymbol{r}(x,y)) \cdot \left(\frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y}\right) dA \\ &= \int\int_R \left(-L\frac{\partial f}{\partial x} - N\frac{\partial f}{\partial y} + P\right) dA \end{aligned} \tag{26}$$

where $dA = dxdy$ or $dA = dydx$. Thus, if M is a surface parametrized by (25), ∂M is a simple closed curve oriented in the counter clockwise motion, and $\text{curl}\boldsymbol{F} = (L, N, P)$, then we may rewrite Stokes' Theorem as follows:

$$\begin{aligned} \int_{\partial M} \boldsymbol{F} \cdot d\boldsymbol{s} &= \int\int_M \text{curl}\boldsymbol{F} \cdot d\mathbf{S} \\ &= \int\int_R \left(-L\frac{\partial f}{\partial x} - N\frac{\partial f}{\partial y} + P\right) dA \end{aligned} \tag{27}$$

Example 2 Surfaces Defined by Functions

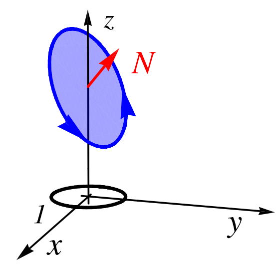


Figure 3
The region R in the xy-plane below M.

Let C be the intersection of the surfaces $x^2+y^2=1$ and $z=3-x-y$, as in Figure 3. Assume curve C is oriented by the counterclockwise direction on the xy-plane. Apply Stokes' Theorem in evaluating the line integral

$$\int_C y^3dx + x^3dy + (y^3+z)dz.$$

Solution
The vector field in the line integral is

$$\boldsymbol{F}(x,y,z) = \left(y^3, x^3, y^3+z\right).$$

We evaluate the curl of $\boldsymbol{F}$:

$$\text{curl}\boldsymbol{F} = \nabla\times\boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3 & x^3 & y^3+z \end{vmatrix} = \left(3y^2, 0, 3x^2-3y^2\right).$$

Notice, the surface M that is enclosed by C is the graph of $f(x,y)=3-x-y$ where $x^2+y^2\le 1$. If $(L,N,P)=\text{curl}\boldsymbol{F}$, then

$$\left(-L\frac{\partial f}{\partial x} - N\frac{\partial f}{\partial y} + P\right) = \left(3y^2 - 0 + (3x^2-3y^2)\right) = 3x^2.$$

In applying Stokes' Theorem, we use the special case (27).

$$\begin{aligned}\int_C y^3dx + x^3dy + (y^3+z)dz &= \int\int_M \text{curl}\boldsymbol{F}\cdot d\mathbf{S} \\ &= \int\int_R \left(-L\frac{\partial f}{\partial x} - N\frac{\partial f}{\partial y} + P\right) dA \\ &= \int\int_R 3x^2 dA.\end{aligned}$$

The region of integration R is the unit disk of radius 1 centered at the origin. Using a change of variables to polar coordinates with $x=r\cos\theta$ and $dA=rdrd\theta$, we obtain

$$\begin{aligned}\int\int_R 3x^2dA &= \int_0^{2\pi}\int_0^1 3r^3\cos^2\theta\ drd\theta \\ &= \frac{3}{4}\int_0^{2\pi}\cos^2\theta\ d\theta \\ &= \frac{3\pi}{4}.\end{aligned}$$

Hence, by Stokes' Theorem we obtain

$$\int_C y^3dx + x^3dy + (y^3+z)dz = \frac{3\pi}{4}.$$

□

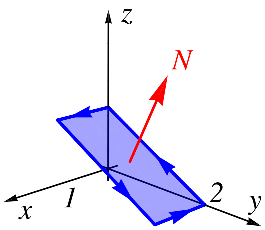


Figure 4
A 2 by 1 rectangular region and a normal vector N.

Try This 2

Let M be an oriented rectangular region with vertices $(0,0,1)$, $(1,0,1)$, $(1,2,0)$, and $(0,2,0)$, see Figure 4. Let the boundary ∂M of M be oriented positively. Then apply Stokes' Theorem in evaluating the line integral $\int_{\partial M}(xzdx + ydz)$.

Green's Theorem

We analyze a special case of Stokes' Theorem. In the parametrization (25), let $f(x,y)=0$. In particular, each point in M has the form $(x,y,0)$ where the set of of (x,y)'s lie in an elementary region R in the xy-plane, see Figure 5.

We assume M is oriented by the unit vector $\boldsymbol{k}$, and the boundary ∂M is a simple closed curve oriented with the counter clockwise direction.

Moreover, we consider vector fields of the form $\boldsymbol{F}(x,y,0) = (f_1(x,y), f_2(x,y), 0)$ where f_1, f_2 are real-valued functions defined on R. Then the curl is given by

$$\begin{aligned}\text{curl}\boldsymbol{F} &= \nabla \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1(x,y) & f_2(x,y) & 0 \end{vmatrix} \\ &= \left(0, 0, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right).\end{aligned}$$

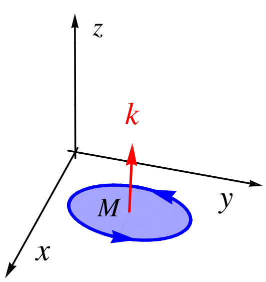


Figure 5
A surface M in the xy-plane oriented by the unit vector $\boldsymbol{k}$.

Applying a special case of Stokes' Theorem, i.e., (27), we obtain

$$\int_{\partial M}(f_1(x,y), f_2(x,y), 0)\cdot d\boldsymbol{s} = \int\int_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dA.$$

Since $z=0$, the vector field $\boldsymbol{F}$ may be realized as having values in $\mathbb{R}^2$.
Then Stokes' Theorem reduces to the following:

Theorem 2.13 Green's Theorem

Let M be an elementary region in the xy-plane whose boundary ∂M is a simple closed curve, oriented by the counter clockwise motion. If f_1, f_2 are real-valued functions defined on M with continuous partial derivatives, then

$$\int_{\partial M}(f_1(x,y)dx + f_2(x,y)dy) = \int\int_M \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dA.$$

Notice, the line integral $\int_{\partial M}(f_1(x,y)dx + f_2(x,y)dy)$ reduces to the area of M if $\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = 1$ for all $(x,y)\in M$.

Example 3 Applying Green's Theorem

Apply Green's Theorem in evaluating the line integral $\displaystyle\int_C y^2dx + 3xydy$ where C is a triangular path from the origin to point $(1,0)$ to point $(1,2)$ and to the origin.

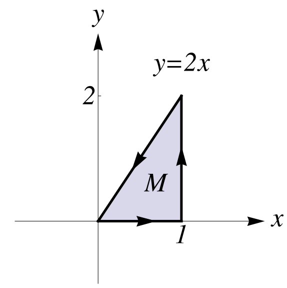


Figure 6
Triangular region with hypotenuse $y = 2x$.

Solution Let M be the triangular region enclosed by C, see Figure 6. Notice, $C = \partial M$, i.e, the boundary of M, is oriented in the counter clockwise direction. Applying Green's Theorem, we obtain

$$
\begin{aligned}
\int_C y^2dx + 3xydy &= \int\int_M \left(\frac{\partial(3xy)}{\partial x} - \frac{\partial(y^2)}{\partial y}\right) dA \\
&= \int\int_M (3y - 2y)\, dA \\
&= \int_0^1 \int_0^{2x} ydydx \\
&= \int_0^1 \frac{1}{2}y^2\bigg|_{y=0}^{y=2x} dx \\
&= \int_0^1 2x^2dx \\
\int_C y^2dx + 3xydy &= \frac{2}{3}.
\end{aligned}
$$

□

Try This 3

Apply Green's Theorem in evaluating the line integral $\displaystyle\int_C \sin x\ dx + 3xy^2dy$ where C is a triangular path from $(0,0)$ to $(2,1)$ to $(0,1)$ to $(0,0)$. See Figure 7.

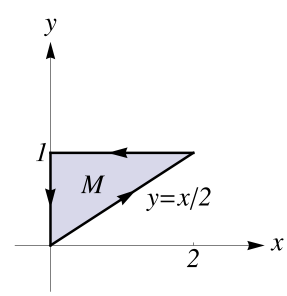


Figure 7
Triangular path with hypotenuse $y = x/2$.

Example 4 Illustrating Green's Theorem

Verify Green's Theorem if $\boldsymbol{F}(x,y) = \left(-\frac{1}{3}y^3, \frac{1}{3}x^3\right)$ and M is a circular region of radius 1 and centered at the origin. Suppose the unit circle ∂M, i.e., boundary of M, is oriented in the counter clockwise direction. See Figure 8.

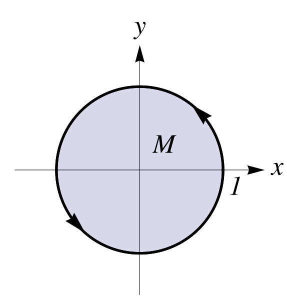


Figure 8
Counter clockwise circular path

Solution The unit circle ∂M is parametrized in the counter clockwise motion by

$$\boldsymbol{r}(t) = (\cos t, \sin t)$$

where $0 \le t \le 2\pi$. We evaluate the line integral of $\boldsymbol{F}$ along ∂M as follows:

$$\begin{aligned}
\int_{\partial M} \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_0^{2\pi} F\left(\boldsymbol{r}(t)\right) \cdot \boldsymbol{r}'(t)\ dt \\
&= \frac{1}{3}\int_0^{2\pi} \left(-\sin^3 t, \cos^3 t\right) \cdot \left(-\sin t, \cos t\right)\ dt \\
&= \frac{1}{3}\int_0^{2\pi} \left(\cos^4 t + \sin^4 t\right) dt \\
&= \frac{1}{3}\int_0^{2\pi} \left(\left(\frac{1+\cos 2t}{2}\right)^2 + \left(\frac{1-\cos 2t}{2}\right)^2\right) dt \\
&= \frac{1}{3}\int_0^{2\pi} \left(\frac{2+2\cos^2 2t}{4}\right) dt \\
&= \frac{1}{12}\int_0^{2\pi} (3+\cos 4t)\, dt \\
&= \frac{1}{12}\left(3t + \frac{\sin 4t}{4}\right|_{t=0}^{t=2\pi} \\
\int_{\partial M} \boldsymbol{F} \cdot d\boldsymbol{r} &= \frac{\pi}{2}.
\end{aligned}$$

To verify Green's Theorem, we evaluate a double integral as in Theorem 2.13. We apply a change of variables by changing from Cartesian to polar coordinates. Notice, M is the circular region centered at the origin and radius 1.

$$\begin{aligned}
\int\int_M \left(\frac{\partial}{\partial x}\left[\frac{1}{3}x^3\right] - \frac{\partial}{\partial y}\left[-\frac{1}{3}y^3\right]\right) dA &= \int\int_M \left(x^2+y^2\right) dA \\
&= \int_0^{2\pi}\int_0^1 r^3 dr d\theta \\
&= \int_0^{2\pi} \frac{1}{4} d\theta \ = \ \frac{\pi}{2}.
\end{aligned}$$

since $x^2 + y^2 = r^2$ and $dA = r dr d\theta$. Then the double integral and line integral have the same values. This completes the verification of Green's Theorem.

□

Try This 4

Verify Green's Theorem if $\boldsymbol{F}(x, y) = \left(-\frac{y}{2}, \frac{x}{2}\right)$ and M is the circular region of radius 1 and centered at $(1, 1)$. Assume circle ∂M, i.e., the boundary of M is oriented in the counter clockwise direction.

Gauss' Divergence Theorem

We consider a solid M in $\mathbb{R}^3$ whose boundary ∂M consists of finitely many surfaces that are oriented by unit vectors pointing outward from the solid. An example of such a solid is a cube, and its boundary consists of six planar faces. Another example of a solid is the ball $x^2+y^2+z^2 \le r^2$, and its boundary is the sphere $x^2+y^2+z^2=r^2$ where $r>0$. See Figures 9 and 10.

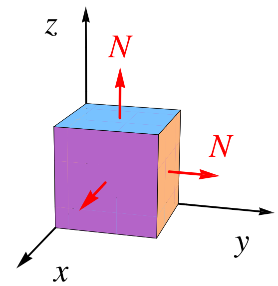


Figure 9
A cube oriented by unit vectors pointing outward.

Theorem 2.14 Gauss' Divergence Theorem

Let M be a solid region in $\mathbb{R}^3$ such that its boundary ∂M consists of finitely many surfaces oriented by unit vectors pointing outward from M. Let $\boldsymbol{F}$ be a vector field defined on M and with values in $\mathbb{R}^3$. If $\boldsymbol{F}$ has continuous partial derivatives, then

$$\int\int_{\partial M} \boldsymbol{F}\cdot d\boldsymbol{S} = \int\int\int_M \operatorname{div}(\boldsymbol{F})\ dV.$$

Example 5 Illustrating Gauss' Divergence Theorem

Let M be the ball $x^2+y^2+z^2 \le r^2$ of radius $r>0$. Verify the Divergence Theorem for $\boldsymbol{F}(x,y,z)=(0,0,z)$. Assume the boundary ∂M of M is oriented by unit vectors pointing outward from the ball, see Figure 10.

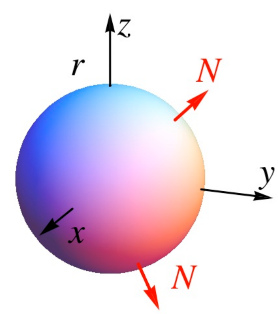


Figure 10
Sphere of radius r and outward pointing vectors.

Solution We parametrize ∂M using spherical coordinates, i.e.,

$$\boldsymbol{r}(\phi,\theta) = (r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)$$

where $0\le\phi\le\pi$ and $0\le\theta<2\pi$, see Section 2.2. Furthermore, we find

$$\begin{aligned}
\frac{\partial \boldsymbol{r}}{\partial\phi}\times\frac{\partial \boldsymbol{r}}{\partial\theta} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ r\cos\phi\cos\theta & r\cos\phi\sin\theta & -r\sin\phi \\ -r\sin\phi\sin\theta & r\sin\phi\cos\theta & 0 \end{vmatrix} \\
&= \left(r^2\sin^2\phi\cos\theta, r^2\sin^2\phi\sin\theta, r^2\sin\phi\cos\phi\right)
\end{aligned}$$

Notice, the above cross product is a vector that points away from the ball.

Applying Definition 6 in Section 2.4, the surface integral of $\boldsymbol{F}$ on ∂M satisfies

$$\begin{aligned}
\int\int_{\partial M} \boldsymbol{F}\cdot d\boldsymbol{S} &= \int_0^{\pi}\int_0^{2\pi} F\left(\boldsymbol{r}(\phi,\theta)\right)\cdot\left(\frac{\partial \boldsymbol{r}}{\partial\phi}\times\frac{\partial \boldsymbol{r}}{\partial\theta}\right) d\theta d\phi \\
&= \int_0^{\pi}\int_0^{2\pi} (0,0,r\cos\phi)\cdot\left(\frac{\partial \boldsymbol{r}}{\partial\phi}\times\frac{\partial \boldsymbol{r}}{\partial\theta}\right) d\theta d\phi \\
&= \int_0^{\pi}\int_0^{2\pi} r^3\cos^2\phi\sin\phi\ d\theta d\phi \\
&= 2\pi r^3\int_0^{\pi}\cos^2\phi\sin\phi\ d\phi \\
\int\int_{\partial M} \boldsymbol{F}\cdot d\boldsymbol{S} &= \frac{4\pi r^3}{3}.
\end{aligned}$$

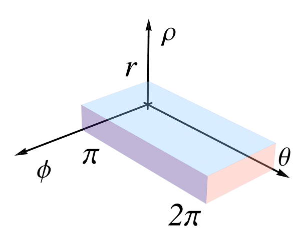


Figure 11
Rectangular solid T

Next, we evaluate the triple integral as in Theorem 2.14. Since $\operatorname{div}(\boldsymbol{F}) = 1$, we find

$$\int\int\int_M \operatorname{div}(\boldsymbol{F})\ dV \quad = \quad \int\int\int_M dV.$$

We apply a change of variables and use the spherical coordinates $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho\cos\phi$ where $0 \le \rho \le r$, $0 \le \phi \le \pi$, and $0 \le \theta < 2\pi$.

The ball M is mapped into a rectangular solid T, see Figure 11, by the transformation that sends $(x, y, z) \in M$ into $(\phi, \theta, \rho) \in T$. In page 83, the Jacobian determinant of the transformation is given by

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} \quad = \quad \operatorname{Det}\begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{bmatrix} \quad = \quad \rho^2 \sin\phi.$$

Applying the change of variables, we obtain

$$\begin{aligned} \int\int\int_M dV \quad &= \quad \int\int\int_T \left| \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} \right| d\rho d\phi d\theta \\ &= \quad \int_0^{2\pi} \int_0^{\pi} \int_0^{r} \rho^2 \sin\phi\ d\rho d\phi d\theta \\ &= \quad \frac{2\pi r^3}{3} \int_0^{\pi} \sin\phi\ d\phi \quad = \quad \frac{4\pi r^3}{3} \end{aligned}$$

Hence, in the case of Example 5, we have verified

$$\int_{\partial M} \boldsymbol{F} \cdot d\boldsymbol{S} = \int\int\int_M \operatorname{div}(\boldsymbol{F})\ dV.$$

□

Try This 5

Let M be the cube in the first octant bounded by $x = s$, $y = s$, $z = s$ where $s > 0$ as in Figure 9. Let ∂M be oriented by unit vectors that point away from M. If $\boldsymbol{F}(x, y, z) = (2x, 0, 0)$, evaluate $\int\int_{\partial M} \boldsymbol{F} \cdot d\boldsymbol{S}$ by Gauss' Divergence Theorem.

Example 6 Applying Gauss' Divergence Theorem

Let M be the solid circular cylinder bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 1$. Let the boundary ∂M of M be oriented by unit vectors that point away from M, see Figure 12. If $\boldsymbol{F}(x, y, z) = \left(x^3, 2x^2 y, x^2 z\right)$, apply Gauss' Divergence Theorem in evaluating the surface integral

$$\int\int_{\partial M} \boldsymbol{F} \cdot d\boldsymbol{S}.$$

Solution The divergence of $\boldsymbol{F}$ is given by

$$\begin{aligned}\operatorname{div}\left(x^3, 2x^2y, x^2z\right) &= \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(2x^2y) + \frac{\partial}{\partial z}(x^2z) \\ &= 3x^2 + 2x^2 + x^2 = 6x^2.\end{aligned}$$

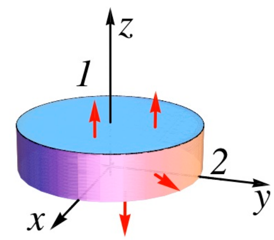


Figure 12
A disk oriented by unit vectors pointing outward.

Applying the Gauss' Divergence Theorem, we rewrite the surface integral.

$$\int\int_{\partial M} \boldsymbol{F}\cdot d\boldsymbol{S} = \int\int\int_M \operatorname{div}(\boldsymbol{F})\ dV = \int\int\int_M 6x^2\ dV.$$

To evaluate the above triple integral, we use cylindrical coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

as discussed in Section 2.2. Notice, $0 \le r \le 2$, $0 \le \theta \le 2\pi$, and $0 \le z \le 1$.

The Jacobian determinant of the transformation from cartesian to cylindrical coordinates is given by

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \operatorname{Det}\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = r$$

Consequently, we obtain

$$\begin{aligned}\int\int\int_M 6x^2\ dV &= \int_0^1\int_0^{2\pi}\int_0^2 6(r\cos\theta)^2\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| drd\theta dz \\ &= \int_0^1\int_0^{2\pi}\int_0^2 6r^3\cos^2\theta\ drd\theta dz \\ &= \int_0^1\int_0^{2\pi} \frac{3r^4}{2}\bigg|_{r=0}^{r=2}\cos^2\theta\ d\theta dz \\ &= 24\int_0^1\int_0^{2\pi}\cos^2\theta\ d\theta dz \\ &= 12\int_0^1\int_0^{2\pi}(1+\cos 2\theta)\ d\theta dz \\ &= 12\int_0^1\left(\theta + \frac{1}{2}\sin 2\theta\right|_{\theta=0}^{\theta=2\pi} dz \\ &= 12\int_0^1 2\pi\ dz \\ \int\int_M 6x^2\ dV &= 24\pi.\end{aligned}$$

Hence, we have

$$\int\int_{\partial M}\boldsymbol{F}\cdot d\boldsymbol{S} = 24\pi.$$

$\square$

Try This 6

Let M be the solid bounded by $x^2+y^2=1$, $z=-2$ and $z=2$. Suppose the boundary ∂M is oriented by unit vectors pointing away from M. If $\boldsymbol{F}(x,y,z)=\big(z,y^3,x\big)$, apply Gauss' Divergence Theorem in evaluating the surface integral

$$\int\int_{\partial M}\boldsymbol{F}\cdot d\boldsymbol{S}.$$

Proof of Green's Theorem

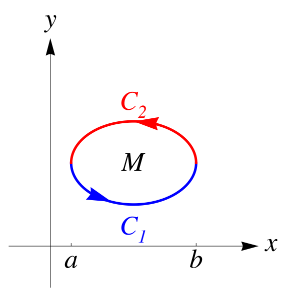


Figure 13a
$g(x)\le f(x)$ for $a\le x\le b$

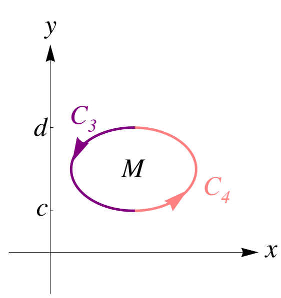


Figure 13b
$m(y)\le n(y)$ for $c\le y\le d$

We prove Green's Theorem for the case when $M\subset\mathbb{R}^2$ is an elementary region that is both of type R_x and R_y, see page 61. That is, we may express M as

$$\begin{aligned} M &= \{(x,y): a\le x\le b,\ g(x)\le y\le f(x)\} \qquad \text{Type } R_x \\ &= \{(c,d): c\le y\le d,\ m(y)\le x\le n(y)\} \qquad \text{Type } R_y \end{aligned}$$

where f,g and m,n are continuous real-valued functions on $[a,b]$ and $[c,d]$, respectively. Assume the boundary ∂M is a simple closed curve oriented by the counter clockwise motion. In Figure 13a, let $\partial M=C_1\cup C_2$ where C_1 and $-C_2$ are parametrized by

$$\boldsymbol{r}_1(x,y)=(x,g(x))$$
$$\boldsymbol{r}_2^*(x,y)=(x,f(x))$$

and $a\le x\le b$. If f_1 is a real-valued function defined on M, the line integral of the vector field $(f_1,0)$ along ∂M satisfies

$$\begin{aligned} \int_{\partial M} f_1dx &= \int_{C_1} f_1dx-\int_{-C_2} f_1dx \\ &= \int_a^b f_1(x,g(x))dx-\int_a^b f_1(x,f(x))dx \\ &= \int_a^b\int_{g(x)}^{f(x)}-\frac{\partial f_1}{\partial y}dydx \\ \int_{\partial M} f_1dx &= \int\int_M-\frac{\partial f_1}{\partial y}dA. \end{aligned}$$

Similarly, if f_2 is defined on M, the line integral of $(0,f_2)$ along ∂M satisfies

$$\int_{\partial M} f_2dy \;=\; \int\int_M\frac{\partial f_2}{\partial x}dA.$$

Adding the last two line integrals, we obtain

$$\int_{\partial M} f_1dx+f_2dy=\int\int_M\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)dA.$$

This proves Green's Theorem when M is an elementary region that is of both type R_x and R_y.

Proof of Stokes' Theorem

We prove Stokes' Theorem for surfaces M that are graphs of functions. Following (25), we parametrize M by

$$\boldsymbol{r}(x, y) = (x, y, f(x, y))$$

where f is a function with continuous partial derivatives defined on an elementary region R in the xy-plane . In Section 2.4, we proved

$$\frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y} = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right). \tag{28}$$

Notice, M is oriented by unit vectors pointing away from the origin. Assume the boundary ∂M of M is a simple closed curve. Then the positive orientation on ∂M is the counter clockwise direction in the xy-plane.

Let $\boldsymbol{F}$ be a vector field defined on M such that the values of $\boldsymbol{F}$ lie in $\mathbb{R}^3$. We write $\boldsymbol{F} = (P, Q, R)$ where P, Q, R are real-valued functions with continuous partial derivatives defined on R. Then the curl is given by

$$\mathrm{curl}\boldsymbol{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\boldsymbol{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\boldsymbol{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\boldsymbol{k} \tag{29}$$

see (30) in Section 1.3.

On the other hand, the boundary ∂M of M is parametrized by

$$\boldsymbol{c}(t) = (x(t), y(t), f(x(t), y(t)))$$

where $x(t), y(t)$ are certain real-valued function defined on $[a, b]$. Then the line integral of $\boldsymbol{F}$ along the boundary ∂M satisfies

$$\begin{aligned}
\int_{\partial M} \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_a^b (\boldsymbol{F} \circ \boldsymbol{c})(t) \cdot \boldsymbol{c}'(t) dt \\
&= \int_a^b ((P, Q, R) \circ \boldsymbol{c})(t) \cdot \left(x'(t), y'(t), \left.\frac{\partial f}{\partial x}\right|_{\boldsymbol{c}(t)} x'(t) + \left.\frac{\partial f}{\partial y}\right|_{\boldsymbol{c}(t)} y'(t) \right) dt
\end{aligned}$$

where the chain rule is used in the last line. Evaluating the dot product, we find

$$\begin{aligned}
\int_{\partial M} \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_a^b \left(P(\boldsymbol{c}(t))x'(t) + R(\boldsymbol{c}(t)) \left.\frac{\partial f}{\partial x}\right|_{\boldsymbol{c}(t)} x'(t) \right) dt \;+ \\
&\qquad \int_a^b \left(Q(\boldsymbol{c}(t))y'(t) + R(\boldsymbol{c}(t)) \left.\frac{\partial f}{\partial y}\right|_{\boldsymbol{c}(t)} y'(t) \right) dt \\
&= \int_{\partial M} \left(P + R\frac{\partial f}{\partial x}\right) dx + \left(Q + R\frac{\partial f}{\partial y}\right) dy
\end{aligned}$$

where we used the definition of the line integral in the previous line.

By Green's Theorem, chain rule, and the product rule, we obtain

$$\begin{aligned}
\int_{\partial M} \boldsymbol{F} \cdot d\boldsymbol{r} &= \int\int_R \left(\frac{\partial}{\partial x}\left[Q + R\frac{\partial f}{\partial y}\right] - \frac{\partial}{\partial y}\left[P + R\frac{\partial f}{\partial x}\right]\right) dA \\
&= \int\int_R \left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial f}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial f}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + R\frac{\partial^2 f}{\partial x \partial y}\right) dA \\
&\quad - \int\int_R \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial f}{\partial y} + \frac{\partial R}{\partial y}\frac{\partial f}{\partial x} + \frac{\partial R}{\partial z}\frac{\partial f}{\partial y}\frac{\partial f}{\partial x} + R\frac{\partial^2 f}{\partial y \partial x}\right) dA \\
&= \int\int_R \left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial f}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial f}{\partial y}\right) dA \\
&\quad - \int\int_R \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial f}{\partial y} + \frac{\partial R}{\partial y}\frac{\partial f}{\partial x}\right) dA.
\end{aligned}$$

Grouping and factoring $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, we find

$$\begin{aligned}
\int_{\partial M} \boldsymbol{F} \cdot d\boldsymbol{r} &= \int\int_R \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\left(-\frac{\partial f}{\partial x}\right) + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\left(-\frac{\partial f}{\partial y}\right) + \right. \\
&\qquad \left. \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right] dA \\
&= \int\int_R \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \cdot \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right) dA \\
&= \int\int_R (\mathrm{curl}\boldsymbol{F}) \cdot \left(\frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y}\right) dA
\end{aligned}$$

where in the last line we used the curl of $\boldsymbol{F}$ in (29), and the cross product in (28). Note, the right side of the last line is the definition of the surface integral of curl$\boldsymbol{F}$ on surface M. Finally, we obtain

$$\int_{\partial M} \boldsymbol{F} \cdot d\boldsymbol{r} = \int\int_M \mathrm{curl}\boldsymbol{F} \cdot d\boldsymbol{S}.$$

This completes the proof of Stokes' Theorem for the case when the surface M is the graph of a function $z = f(x, y)$.

Proof of Gauss' Divergence Theorem

Let M be an elementary solid in 3-space, see page 66 in Section 2.1. Assume M is oriented by unit vectors pointing away from M. Let $\boldsymbol{F} = (P, Q, R)$ be a vector field where P, Q, R are real-valued functions on M. We express the surface integral of $\boldsymbol{F}$ on the boundary ∂M of M as a sum:

$$\int\int_{\partial M} \boldsymbol{F} \cdot d\boldsymbol{S} = \int\int_{\partial M} (P\boldsymbol{i}) \cdot d\boldsymbol{S} + \int\int_{\partial M} (Q\boldsymbol{j}) \cdot d\boldsymbol{S} + \int\int_{\partial M} (R\boldsymbol{k}) \cdot d\boldsymbol{S}.$$

Using the definition of the divergence, we obtain

$$\int\int\int_M \operatorname{div}\boldsymbol{F} dV = \int\int\int_M \frac{\partial P}{\partial x} dV + \int\int\int_M \frac{\partial Q}{\partial y} dV + \int\int\int_M \frac{\partial R}{\partial z} dV.$$

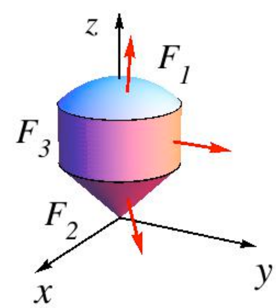


Figure 14
Solid with three faces F_1, F_2, F_3, and outward pointing vectors.

The Gauss' Divergence Theorem will follow if the three statements below are true.

$$\int\int_{\partial M} (P\boldsymbol{i}) \cdot d\boldsymbol{S} = \int\int\int_M \frac{\partial P}{\partial x} dV \tag{30}$$

$$\int\int_{\partial M} (Q\boldsymbol{j}) \cdot d\boldsymbol{S} = \int\int\int_M \frac{\partial Q}{\partial y} dV \tag{31}$$

$$\int\int_{\partial M} (R\boldsymbol{k}) \cdot d\boldsymbol{S} = \int\int\int_M \frac{\partial R}{\partial z} dV \tag{32}$$

We only prove (32), and in a special case. Suppose ∂M consists of three faces F_i for $i = 1, 2, 3$ such that F_3 is a lateral surface, and $\boldsymbol{k}$ is perpendicular to the normal vectors to F_3, see Figure 14.

If $\boldsymbol{r_3}$ is an orientation preserving parametrization of F_3 defined on $D_3 \subseteq \mathbb{R}^2$, then

$$\boldsymbol{k} \cdot \left(\frac{\partial \boldsymbol{r_3}}{\partial x} \times \frac{\partial \boldsymbol{r_3}}{\partial y} \right) = 0.$$

Consequently, by the definition of the surface integral we obtain

$$\int\int_{F_3} R\boldsymbol{k} \cdot d\boldsymbol{S} = \int\int_{D_3} R\boldsymbol{k} \cdot \left(\frac{\partial \boldsymbol{r_3}}{\partial x} \times \frac{\partial \boldsymbol{r_3}}{\partial y} \right) dA = 0. \tag{33}$$

Also, we know the surface integral of $R\boldsymbol{k}$ on ∂M is the sum of the surface integrals on F_1, F_2, F_3. Combined with (33), we obtain

$$\int\int_{\partial M} R\boldsymbol{k} \cdot d\boldsymbol{S} = \int\int_{F_1} R\boldsymbol{k} \cdot d\boldsymbol{S} + \int\int_{F_2} R\boldsymbol{k} \cdot d\boldsymbol{S}. \tag{34}$$

Let D be an elementary region in the xy-plane, and let Φ_1, Φ_2 be continuous real-valued functions on D satisfying

$$M = \{(x, y, z) : (x, y) \in D,\ \Phi_1(x, y) \le z \le \Phi_2(x, y)\}.$$

For $i = 1, 2$, we parametrize F_i as follows:

$$\boldsymbol{r_i}(x, y) = (x, y, \Phi_i(x, y)),\ (x, y) \in D.$$

Since F_1 is oriented by upward pointing unit vectors, we have

$$\begin{aligned}
\int\int_{F_1} R\boldsymbol{k} \cdot d\boldsymbol{S} &= \int\int_D R\boldsymbol{k} \cdot \left(\frac{\partial \boldsymbol{r_1}}{\partial x} \times \frac{\partial \boldsymbol{r_1}}{\partial y} \right) dA \\
&= \int\int_D R\boldsymbol{k} \cdot \left(-\frac{\partial \Phi_1}{\partial x}, -\frac{\partial \Phi_1}{\partial y}, 1 \right) dA \\
&= \int\int_D R(x, y, \Phi_1(x, y)) dA.
\end{aligned}$$

We have oriented F_2 by downward pointing unit vectors. Notice, the cross product

$$-\frac{\partial \boldsymbol{r_2}}{\partial x} \times \frac{\partial \boldsymbol{r_2}}{\partial y} = \left(\frac{\partial \Phi_2}{\partial x}, \frac{\partial \Phi_2}{\partial y}, -1 \right)$$

is downward pointing and normal to F_2. Then we obtain

$$\begin{aligned} \int \int_{F_2} R\boldsymbol{k} \cdot d\boldsymbol{S} &= \int \int_D R\boldsymbol{k} \cdot \left(-\frac{\partial \boldsymbol{r_2}}{\partial x} \times \frac{\partial \boldsymbol{r_2}}{\partial y} \right) dA \\ &= -\int \int_D R(x, y, \Phi_2(x, y)) dA. \end{aligned}$$

Then we rewrite (34) as follows:

$$\begin{aligned} \int \int_{\partial M} R\boldsymbol{k} \cdot d\boldsymbol{S} &= \int \int_D \frac{\partial R(x, y, z)}{\partial z} \Bigg|_{z=\Phi_2(x,y))}^{z=\Phi_1(x,y))} dA \\ &= \int \int \int_M \frac{\partial R}{\partial z} dV. \end{aligned}$$

Thus, we have proved (32). The proofs of (30)-(31) are similar when we assume M has a symmetry in its parametrizations:

$$\begin{aligned} M &= \{(x, y, z) : (x, z) \in R',\ \Phi_1'(x, z) \leq y \leq \Phi_2'(x, z)\} \\ &= \{(x, y, z) : (y, z) \in R'',\ \Phi_1''(y, z) \leq x \leq \Phi_2''(y, z)\} \end{aligned}$$

where Φ_1', Φ_2' and Φ_1'', Φ_2'' are defined on subsets of the xz-plane and yz-plane, respectively. This completes a sketch of the proof of Gauss' Divergence Theorem.

Theorem 2.15 Path Independence and Conservative Vector Fields

Let $\boldsymbol{F}$ be a vector field from $\mathbb{R}^3$ into $\mathbb{R}^3$ with continuous partial derivatives except at finitely many points. Then $\boldsymbol{F}$ is a conservative vector field if and only if $\int_C \boldsymbol{F} \cdot d\boldsymbol{s} = 0$ for all simple closed curves C in $\mathbb{R}^3$.

Proof Let M be an oriented surface in $\mathbb{R}^3$ such that $C = \partial M$ has the positive orientation. By definition, if $\boldsymbol{F}$ is a conservative vector field then curl$\boldsymbol{F} = \boldsymbol{0}$, see page 37. Applying Stokes' Theorem, we obtain

$$\begin{aligned} \int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= \int \int_M \text{curl} F \cdot d\boldsymbol{S} \\ &= \int \int_M \boldsymbol{0} \cdot d\boldsymbol{S} \ = \ 0. \end{aligned}$$

To prove the converse, suppose $\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0$ for any simple closed curve C. Let C_1 and C_2 be two paths from points P to Q such that $C = C_1 \cup (-C_2)$ is a simple closed path. Since $\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} - \int_{C2} \boldsymbol{F} \cdot d\boldsymbol{r}$, we find

$$\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{C2} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

That is, the line integral of $\boldsymbol{F}$ is independent of the path from P to Q. Let $f(x, y, z)$ denote the value of the line integral of $\boldsymbol{F}$ from $(0, 0, 0)$ to point (x, y, z). For simplicity, suppose $x, y, z > 0$. Let $\boldsymbol{r_1}(t_1) = (t_1, 0, 0)$, $\boldsymbol{r_2}(t_2) = (x, t_2, 0)$, and $\boldsymbol{r_3}(t_2) = (x, y, t_3)$.

Notice, the union or the sum $\boldsymbol{r} = \boldsymbol{r_1} + \boldsymbol{r_2} + \boldsymbol{r_3}$ is a path from the origin to (x, y, z). If $\boldsymbol{F} = (F_1, F_2, F_3)$, then

$$\begin{aligned} f(x,y,z) &= \int_{\boldsymbol{r}} \boldsymbol{F} \cdot d\boldsymbol{r} \\ &= \int_{\boldsymbol{r_1}} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{\boldsymbol{r_2}} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{\boldsymbol{r_3}} \boldsymbol{F} \cdot d\boldsymbol{r} \\ &= \int_0^x F_1(t_1, 0, 0) dt_1 + \int_0^y F_2(x, t_2, 0) dt_2 + \int_0^z F_3(x, y, t_3) dt_3. \end{aligned}$$

Applying the Fundamental Theorem of Calculus, we obtain

$$\frac{\partial f}{\partial z} = F_3.$$

Similarly, by choosing different paths, we find $\frac{\partial f}{\partial x} = F_1$ and $\frac{\partial f}{\partial y} = F_2$. Thus, $\nabla f = \boldsymbol{F}$. From the identity $\mathrm{curl}(\nabla f) = \boldsymbol{0}$, we find $\boldsymbol{F}$ is a conservative vector field.

$\square$

2.5 Check-It Out

1. Apply Stokes' Theorem in evaluating the line integral $\displaystyle\int_{\partial M} \boldsymbol{F} \cdot d\boldsymbol{r}$ where M is the part of the plane $2x+y+z = 2$ in the first octant, and $\boldsymbol{F}(x, y, z) = (z, y, 0)$.

2. Apply Green's Theorem in evaluating the line integral $\displaystyle\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$. where C is the boundary of the rectangular region $0 \le x \le 2$, $0 \le y \le 1$, and $\boldsymbol{F} = (e^y, e^x)$.

3. Apply Gauss' Divergence Theorem in evaluating the surface integral $\displaystyle\int\int_{\partial M} \boldsymbol{F} \cdot d\boldsymbol{S}$ where $\boldsymbol{F}(x, y, z) = (x, -y, z)$, and M is the solid cube bounded by the planes $x = 1$, $y = 1$, $z = 1$, and the coordinate planes.

True or False. *If false, revise the statement to make it true or explain.*

1. The line integral of a vector field $\boldsymbol{F}$ along the boundary ∂M of surface M is equal to the surface integral of the curl$\boldsymbol{F}$ on M.

2. The surface integral of a vector field $\boldsymbol{F}$ along the boundary ∂M of a solid M is equal to the triple integral of div$\boldsymbol{F}$ on M.

3. Let R be a region that is enclosed by a simple closed curve C that is oriented in the counter clockwise direction. If $\boldsymbol{F}(x,y) = (-x^2, xy)$, then $\displaystyle\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int\int_R (y-2x)dA$.

4. Let M be parametrized by $\boldsymbol{r}(x,y) = (x, y, x^2+y^2)$ where (x,y) lies in a region R. Assume the boundary ∂M is oriented in the counter clockwise motion as seen from above the z-axis. If curl$\boldsymbol{F}(x,y,z) = (x,-y,-1)$, then $\displaystyle\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int\int_R (x,-y,-1)\cdot(2x,2y,1)dA$.

5. Let $\boldsymbol{F}(x,y,z) = (x,y,z)$, and let M be the ball defined $x^2+y^2+z^2 \le r^2$ and oriented by unit vectors that point away from the origin. Applying Gauss' Divergence Theorem, we find $\displaystyle\int\int_{\partial M} \boldsymbol{F}\cdot d\boldsymbol{S}$ is equal to the volume of M.

Exercises for Section 2.5

In Exercises 1-8, apply Stokes' Theorem in evaluating a line integral of a vector field $\boldsymbol{F}$ along the boundary ∂M of a surface M. Assume M is oriented by upward pointing unit normal vectors, and ∂M is oriented positively by the counter clockwise direction.

1. $\displaystyle\int_{\partial M} \boldsymbol{F}\cdot d\boldsymbol{r}$ where $\boldsymbol{F}(x,y,z) = (0,z,x)$, M is the part of the surface $z = y^2$ in the first octant above the rectangular region in the xy-plane with vertices at the origin, $(2,0,0)$, $(2,1,0)$ and $(0,1,0)$.

2. $\displaystyle\int_{\partial M} (e^z - 2y)dx + e^y dy + e^x dz$ where M is the part of the plane $z = 2$ that lies above the rectangular region in the xy-plane satisfying $0 \le x \le 2$, $0 \le y \le 3$.

3. $\displaystyle\int_{\partial M} \boldsymbol{F}\cdot d\boldsymbol{r}$ where $\boldsymbol{F}(x,y,z) = (y,z,-4x)$, and M is the part of the plane $2x+3y+3z = 6$ in the first octant.

4. $\displaystyle\int_{\partial M} \boldsymbol{F}\cdot d\boldsymbol{r}$ where M is the plane $2x+2y+z = 2$ in the first octant, and $\boldsymbol{F}(x,y,z) = (-x,-z^2,-y)$.

5. $\displaystyle\int_{\partial M} \boldsymbol{F}\cdot d\boldsymbol{r}$ where $\boldsymbol{F}(x,y,z)=(0,e^x,e^y)$, and M is the part of the plane $x+y+z=1$ in the first octant.

6. $\displaystyle\int_{\partial M} \langle -xy, z-y, z^2\rangle\cdot d\boldsymbol{r}$ where M is the part of the plane $z=4$ in the first octant that lies above the interior of the quarter of the unit circle in the xy-plane centered at the origin.

7. $\displaystyle\int_{\partial M} \boldsymbol{F}\cdot d\boldsymbol{r}$ where M is the upper hemisphere $z=\sqrt{1-x^2-y^2}$ and $\boldsymbol{F}(x,y,z)=(-x^2y,0,0)$.

8. $\displaystyle\int_{\partial M} \boldsymbol{F}\cdot d\boldsymbol{r}$ where $\boldsymbol{F}(x,y,z)=(1,y+x^3,z)$, and M is the part of the hemisphere $z=\sqrt{4-x^2-y^2}$ that is bounded by the cylinder $x^2+y^2=1$.

In Exercises 9-16, apply Green's Theorem in evaluating the line integral of $\boldsymbol{F}$ along the indicated curve C oriented in the counter clockwise direction.

9. $\displaystyle\int_C y^2dx+4xydy$ where C is a triangular path from the origin to point $(1,0)$ to point $(1,4)$ and to the origin.

10. $\displaystyle\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ where $\boldsymbol{F}(x,y)=(x-y^2,2x+y)$, C is a triangular path from the origin to point $(1,1)$ to point $(0,2)$ and to the origin.

11. $\displaystyle\int_C \left(-y^3dx+x^3dy\right)$ where C is the circle of radius 2, and centered at the origin.

12. $\displaystyle\int_C \left(x^3+y\right)dy$ where C is the unit circle centered at the origin.

13. $\displaystyle\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ where $\boldsymbol{F}(x,y)=\left(x^2,20xy\right)$, and C is a closed curve formed by $y=\sqrt{x}$ and $y=x^2$.

14. $\displaystyle\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ where $\boldsymbol{F}(x,y)=(x-y,x+y)$, and C is a closed curve formed by $y=4$ and $y=x^2$.

15. $\displaystyle\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ where $\boldsymbol{F}(x,y)=\frac{1}{2}\left(y^2,x^2\right)$, C is the parallelogram with vertices $(0,0)$, $(2,0)$, $(3,1)$, and $(1,1)$.

16. $\displaystyle\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ where $\boldsymbol{F}(x,y)=\left(x^2,e^{x-y}\right)$, and C is a parallelogram with vertices $(0,0)$, $(1,0)$, $(2,1)$, and $(1,1)$.

In Exercises 17-24, apply Gauss' Divergence Theorem in evaluating the surface integral of $\boldsymbol{F}$ *on the boundary* ∂M *of a solid* M *that is oriented by unit vectors that point away from* M*.*

17. $\displaystyle\int\int_{\partial M} \boldsymbol{F}\cdot d\boldsymbol{S}$ where $\boldsymbol{F}(x,y,z) = (x, y^2, z^3)$ and M is the solid bounded by the planes $x = 3$, $y = 2$, $z = 1$, and the coordinate planes.

18. $\displaystyle\int\int_{\partial M} \boldsymbol{F}\cdot d\boldsymbol{S}$ where $\boldsymbol{F}(x,y,z) = (z, y, x)$ and M is the solid bounded by the cylinder $x^2 + y^2 = 9$, the planes $z = 0$, and $z = 2$.

19. $\displaystyle\int\int_{\partial M} \boldsymbol{F}\cdot d\boldsymbol{S}$ where $\boldsymbol{F}(x,y,z) = (x, x^2y, y^2z)$ and M is the solid bounded by the cylinder $x^2 + y^2 = 1$, the planes $z = 1$, and $z = 2$.

20. $\displaystyle\int\int_{\partial M} \boldsymbol{F}\cdot d\boldsymbol{S}$ where $\boldsymbol{F}(x,y,z) = (xz^2, x^2z, xy^2)$, and M is the sphere $x^2 + y^2 + z^2 = 1$.

21. $\displaystyle\int\int_{\partial M} \boldsymbol{F}\cdot d\boldsymbol{S}$ where $\boldsymbol{F}(x,y,z) = (x^3, y^3, z^3)$, and M is the sphere $x^2 + y^2 + z^2 = r^2$ with $r > 0$.

22. $\displaystyle\int\int_{\partial M} \boldsymbol{F}\cdot d\boldsymbol{S}$ where $\boldsymbol{F}(x,y,z) = (y, x, z^2)$, and M is the solid bounded by the cylinder $x^2 + y^2 = 4$, the hemisphere $z = \sqrt{16 - x^2 - y^2}$, and the plane $z = 0$.

23. $\displaystyle\int\int_{\partial M} \boldsymbol{F}\cdot d\boldsymbol{S}$ where $\boldsymbol{F}(x,y,z) = (y, x^2, z^3)$, and M is the solid bounded by the ellipsoid $x^2 + y^2 + 4z^2 = 4$.

24. $\displaystyle\int\int_{\partial M} \boldsymbol{F}\cdot d\boldsymbol{S}$ where $\boldsymbol{F}(x,y,z) = \left(z, \frac{y^3}{18}, x\right)$, and M is the solid bounded by the ellipsoid $x^2 + 9y^2 + 4z^2 = 36$.

25. Let $\boldsymbol{r}(x,y,z) = (x,y,z)$ be the position vector field, and let
$$\boldsymbol{F}(x,y,z) = \frac{\boldsymbol{r}(x,y,z)}{\|\boldsymbol{r}(x,y,z)\|^3}.$$
If M is a solid that does not contain $(0,0,0)$ and for which Gauss' Divergence Theorem applies, verify $\displaystyle\int\int_{\partial M} \boldsymbol{F}\cdot d\boldsymbol{S} = 0$.

26. *One of Gauss' Several Lemmas* Let $\boldsymbol{F}$ be the vector field in Exercise 25.

    Let M be a sphere centered at $(0,0,0)$, and of any radius $\varepsilon > 0$. Prove

$$\int\int_M \boldsymbol{F} \cdot d\boldsymbol{S} = 4\pi.$$

27. If $\boldsymbol{F}$ is the vector field in Exercise 25, show $\boldsymbol{F} \neq \text{curl}\boldsymbol{G}$ for any vector field $\boldsymbol{G}$.

28. Let $\boldsymbol{F}$ be a vector field from $\mathbb{R}^3$ to $\mathbb{R}^3$ with continuous first partial derivatives. If $\text{div}\boldsymbol{F} = 0$, prove $\boldsymbol{F} = \text{curl}\boldsymbol{G}$ for some vector field $\boldsymbol{G}$. Notice, $\boldsymbol{F}$ should be defined on all of $\mathbb{R}^3$, see Exercise 27.

Chapter 2 Multiple Choice Test

1. A unit tangent vector to the curve $r(t) = \left(t, t^2, -t^2\right)$ at the origin is given by

   A. $\boldsymbol{i} - \boldsymbol{k}$ **B.** $\boldsymbol{i} + \boldsymbol{k}$ **C.** $\boldsymbol{j} - \boldsymbol{k}$ **D.** $-\boldsymbol{i}$

2. The curvature to $r(t) = (\cos t, \sin t, 2t)$ at point $P(1, 0, 0)$ is equal to

   A. $\frac{1}{5}$ **B.** $\frac{\sqrt{5}}{5}$ **C.** $\sqrt{5}$ **D.** 5

3. Let C be a triangular path from the origin to point $(1, 0)$ to point $(1, 2)$ and to the origin. Then the line integral $\int_C y dx + 3x dy$ equals

   A. 1 **B.** 2 **C.** 4 **D.** 8

4. If $u = x + y$ and $v = x - y$, the absolute value of the Jacobian determinant $\frac{\partial(x,y)}{\partial(u,v)}$ is equal to

   A. 4 **B.** 2 **C.** $\frac{1}{4}$ **D.** $\frac{1}{2}$

5. Let C be directed line segment from the origin to point $(1, 3)$. Then the line integral $\int_C dx + dy$ equals

   A. 2 **B.** 3 **C.** 4 **D.** 5

6. The iterated integral $\int_0^2 \int_0^3 (2x + y^2) dy dx$ equals

   A. 30 **B.** 21 **C.** 13 **D.** 7

7. The iterated integral $\int_0^4 \int_{x/2}^{\sqrt{x}} f(x, y) dy dx$ equals

   A. $\int_0^4 \int_{\sqrt{y}}^{2y} f(x, y) dx dy$ **B.** $\int_0^2 \int_{2y}^{\sqrt{y}} f(x, y) dx dy$

   C. $\int_0^2 \int_{y^2}^{2y} f(x, y) dx dy$ **D.** $\int_0^4 \int_{\sqrt{y}}^{y^2} f(x, y) dx dy$

8. Let C be a rectangular path that joins the points from $(0,0)$ to $(2,0)$ to $(2,1)$ to $(0,1)$, and to $(0,0)$. Applying Green's Theorem, the line integral $\int_C (xy, x^2) \cdot d\boldsymbol{r}$ is equal to

   A. $\int_0^2 \int_0^1 (2x + y) dy dx$ **B.** $\int_0^2 \int_0^1 (2x - y) dy dx$

   C. $\int_0^2 \int_0^1 (y + 2x) dy dx$ **D.** $\int_0^2 \int_0^1 (y - 2x) dy dx$

9. Let C be a path from point $(0,1)$ to $(\frac{\pi}{4}, 2)$. Applying the Fundamental Theorem of Line Integrals, $\int_C 6y\cos(2x)dx + 3\sin(2x)dy$ is equal to

   A. 12 **B.** 9 **C.** 6 **D.** 3

10. If $R = \{(x,y) : x^2 + y^2 \leq 4\}$, the double integral $\int\int_R \sqrt{x^2+y^2}\ dA$ equals

    A. $\int_0^{2\pi} \int_0^4 r^3 dr d\theta$ **B.** $\int_0^{2\pi} \int_0^4 r^2 dr d\theta$

    C. $\int_0^{2\pi} \int_0^2 r dr d\theta$ **D.** $\int_0^{2\pi} \int_0^2 r^2 dr d\theta$

11. Let M be the portion of the plane $x + y + z = 2$ that lies in the fist octant. If $\boldsymbol{F}(x,y,z) = (x,y,z)$, then the surface integral $\int\int_M \boldsymbol{F}(x,y,z) \cdot d\boldsymbol{S}$ equals

    A. 2 **B.** 4 **C.** 8 **D.** None of the given

12. Let S be the solid in the first octant that is bounded by the surfaces $z = 1 - y^2$, $y = 1$, $x = 0$, $z = 0$, and $y = x$. The volume of S equals

    A. $\frac{2}{3}$ **B.** $\frac{1}{3}$ **C.** $\frac{5}{12}$ **D.** None of the given

13. Let M be the planar surface $x+y+z=1$ in the first octant. Assume M is oriented by unit vectors that point away from the origin, and the boundary ∂M has the positive orientation. By Stokes' Theorem, $\int_{\partial M} \boldsymbol{F}\cdot d\boldsymbol{r}$ is equal to

A. $\int_0^1 \int_0^{1-x} \mathrm{curl}\boldsymbol{F}\cdot(-1,-1,1)dydx$ **B.** $\int_0^1 \int_0^1 \mathrm{curl}\boldsymbol{F}\cdot(-1,-1,1)dxdy$

C. $\int_0^1 \int_0^1 \mathrm{curl}\boldsymbol{F}\cdot(1,1,1)dxdy$ **D.** $\int_0^1 \int_0^{1-x} \mathrm{curl}\boldsymbol{F}\cdot(1,1,1)dydx$

14. Let $\boldsymbol{F}(x,y,z)=(x,y,z)$, let M be the disk $x^2+y^2+z^2\le 1$, and let ∂M be the sphere $x^2+y^2+z^2=1$. Assume ∂M is oriented by unit vectors that point away from the origin. By Gauss' Divergence Theorem, the surface integral $\int\int_{\partial M} \boldsymbol{F}\cdot d\boldsymbol{S}$ is equal to

A. $\int_0^1 \int_0^{2\pi} \int_0^{\pi} 3\rho^2 \sin\phi \; d\phi d\theta d\rho$ **B.** $\int_0^1 \int_0^{2\pi} \int_0^{\pi} 3\rho \sin\phi \; d\phi d\theta d\rho$

C. $\int_0^1 \int_0^{2\pi} \int_0^{\pi} 2\rho^2 \sin\phi \; d\phi d\theta d\rho$ **D.** $\int_0^1 \int_0^{2\pi} \int_0^{\pi} 2\rho \sin\phi \; d\phi d\theta d\rho$

15. A point $(x,y,z)=(\sqrt{3},-1,2)$ is given in Cartesian coordinates. The cylindrical coordinates of the point are equal to

A. $(2,\frac{7\pi}{6},2)$ **B.** $(2,-\frac{\pi}{6},2)$ **C.** $(4,\frac{11\pi}{6},2)$ **D.** $(4,-\frac{\pi}{6},2)$

16. A point $(\rho,\theta,\phi)=(4,\frac{\pi}{3},\frac{\pi}{6})$ is given in spherical coordinates. The Cartesian coordinates (x,y,z) of the point are equal to

A. $(1,\sqrt{3},2\sqrt{3})$ **B.** $(4,4\sqrt{3},8\sqrt{3})$ **C.** $(12,4\sqrt{3},8)$ **D.** $(3,\sqrt{3},2)$

Investigation Projects

Maxwell's Equations

Let $\boldsymbol{E}$, $\boldsymbol{H}$, and $\boldsymbol{J}$ be vector-valued functions of (x, y, z, t) such that the values of $\boldsymbol{E}$, $\boldsymbol{H}$, and $\boldsymbol{J}$ lie in $\mathbb{R}^3$. Let ρ be a real-valued function of (x, y, z, t). In [1], a set of Maxwell's equations is given as follows:

$$\begin{aligned} \mathrm{div}\boldsymbol{E} &= \rho \quad \text{(Gauss' law)} && (35) \\ \mathrm{div}\boldsymbol{H} &= 0 && (36) \\ \mathrm{curl}\boldsymbol{E} + \frac{\partial \boldsymbol{H}}{\partial t} &= \boldsymbol{0} \quad \text{(Faraday's law)} && (37) \\ \mathrm{curl}\boldsymbol{H} - \frac{\partial \boldsymbol{E}}{\partial t} &= \boldsymbol{J} \quad \text{(Ampere's law)} && (38) \end{aligned}$$

The divergence and curl of a vector field is evaluated by fixing t. If in component form, we write $\boldsymbol{E}(x, y, z, t) = (E_1(x, y, z, t), E_2(x, y, z, t), E_3(x, y, z, t))$, the divergence is defined in the usual way:

$$\mathrm{div}\boldsymbol{E}(x, y, z, t) = \frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} + \frac{\partial E_3}{\partial z}.$$

For each t, the curl of $\boldsymbol{E}$ is

$$\mathrm{curl}\boldsymbol{E}(x, y, z, t) = \mathrm{Det}\begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_1 & E_2 & E_3 \end{pmatrix}.$$

If ϕ is a real-valued function of (x, y, z, t), the *Laplacian* of ϕ is denoted $\nabla^2\phi$, and defined by

$$\nabla^2\phi = \mathrm{div}(\nabla\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}.$$

Likewise, if $\boldsymbol{A} = (A_1, A_2, A_3)$ is a vector-valued function of (x, y, z, t) with values in $\mathbb{R}^3$, the Laplacian of $\boldsymbol{A}$ is defined by

$$\nabla^2\boldsymbol{A} = \left(\nabla^2 A_1, \nabla^2 A_2, \nabla^2 A_3\right).$$

Let $\boldsymbol{J}$ and ρ be given, and suppose $\boldsymbol{A}$ and ϕ satisfy

$$\begin{aligned} \mathrm{div}\boldsymbol{A} + \frac{\partial\phi}{\partial t} &= 0 && (39) \\ \nabla^2\phi &= \frac{\partial^2\phi}{\partial t^2} - \rho && (40) \\ \nabla^2\boldsymbol{A} &= \frac{\partial^2\boldsymbol{A}}{\partial t^2} - \boldsymbol{J}. && (41) \end{aligned}$$

We assume t lies in an open interval I, and $\boldsymbol{A}$, $\boldsymbol{H}$, $\boldsymbol{J}$ are defined for all (x, y, z) in $\mathbb{R}^3$. Also, for each $t \in I$, assume $\boldsymbol{E}$, ρ, ϕ are everywhere in $\mathbb{R}^3$ except for finitely many (x, y, z)'s. We assume $\boldsymbol{A}$ and ϕ have continuous second partial derivatives.

In Exercises 1-5, we outline a proof that the vector-valued functions defined by

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} \quad \text{and} \quad \boldsymbol{H} = \text{curl}\boldsymbol{A} \tag{42}$$

satisfy Maxwell's equations (35)-(38).

1. Verify div$\boldsymbol{E} = \rho$. *Hint:* Apply (39) and div $\left(\frac{\partial}{\partial t}\boldsymbol{A}\right) = \frac{\partial}{\partial t}\,(\text{div}\boldsymbol{A})$.

2. Prove div$\boldsymbol{H} = 0$. *Hint:* div(curl$A$) $= 0$.

3. Show curl$\boldsymbol{E} + \frac{\partial \boldsymbol{H}}{\partial t} = \boldsymbol{0}$. *Hint:* curl$(\nabla\phi) = \boldsymbol{0}$ and curl $\left(\frac{\partial}{\partial t}\boldsymbol{A}\right) = \frac{\partial}{\partial t}\,(\text{curl}\boldsymbol{A})$.

4. Prove curl(curl$\boldsymbol{A}$) $= \nabla(\text{div}\boldsymbol{A}) - \nabla^2\boldsymbol{A}$.

5. Verify curl$H - \frac{\partial}{\partial t}\boldsymbol{E} = \boldsymbol{J}$. *Hint:* Apply Exercise 4.

In Exercises 6-7, let $f_i(x)$ and $g_j(x)$ be functions that are selected from $\{\sin x, \cos x\}$ for $i, j = 1, 2, 3$. We consider the following functions:

$$\phi(x, y, z, t) = f_1(nx)g_1(nt) \tag{43}$$

$$\boldsymbol{A}(x, y, z, t) = \big(f_1'(nx)g_1'(nt), f_2(mz)g_2(mt), f_3(py)g_3(pt)\big) \tag{44}$$

where $n, m, p \in \mathbb{R}$ are constants.

6. If $\boldsymbol{J} \equiv \boldsymbol{0}$ and $\rho \equiv 0$ are the zero functions, prove ϕ and $\boldsymbol{A}$ as defined by (43)-(44) satisfy (39), (40), and (41).

7. Let $\boldsymbol{J} \equiv \boldsymbol{0}$ and $\rho \equiv 0$. If ϕ and $\boldsymbol{A}$ are defined by (43)-(44), evaluate the functions $\boldsymbol{E}$ and $\boldsymbol{H}$ in (42). Notice, $\boldsymbol{E}$ and $\boldsymbol{H}$ satisfy Maxwell's equations because of Exercises 1-5.