JEE Advanced 2020 Maths | Paper-1 | Code-E

SECTION 1 (Maximum Marks: 18)

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options. **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:
  Full Marks : +3 If ONLY the correct option is chosen;
  Zero Marks : 0 If none of the options is chosen (i.e. the questions is unanswered);
  Negative Marks : −1 In all other cases.

1. Suppose a, b denote the distinct real roots of the quadratic polynomial $x^2 + 20x - 2020$ and suppose c, d denote the distinct complex roots of the quadratic polynomial $x^2 - 20x + 2020$. Then the value of

$$ac(a-c) + ad(a-d) + bc(b-c) + bd(b-d)$$

is

(A) 0

(B) 8000

(C) 8080

(D) 16000

Answer: D

Solution:

$x^2 + 20x - 2020 = 0$ → a, b

has two roots a, b $\in$ R.

$a + b = -20$ & $a.b = -2020$

& $x^2 - 20x + 2020 = 0$ → c, d

has two roots c, d $\in$ complex.

$c + d = 20$ & $c.d = 2020$

Now

$= ac(a-c) + ad(a-d) + bc(b-c) + bd(b-d)$

$= a^2c - ac^2 + a^2d - ad^2 + b^2c - bc^2 + b^2d - bd^2$

$= a^2(c+d) + b^2(c+d) - c^2(a+b) - d^2(a+b)$

$= (a^2 + b^2)(c+d) - (a+b)(c^2 + d^2)$

$= \{(a+b)^2 - 2ab\}(c+d) - \{(c+d)^2 - 2cd\}(a+b)$

Put value a, b, c & d then,

$= \{(-20)^2 - 2(-2020)\}(20) - \{(20)^2 - 2(2020)\}(-20)$

$= ((400 + 4040)(20) - (-20)((20)^2 - 4040)$

$= 20[4440 - 3640]$

$20[800] = 16000$

2. If the function f: R ⟶ R is defined by $(x) = |x|(x - \sin x)$, then which of the following statements is **TRUE**?

(A) f is one-one, but **NOT** onto

(B) f is onto, but **NOT** one-one

(C) f is **BOTH** one-one and onto

(D) f is **NEITHER** one-one **NOR** onto

Answer: C

Solution:

Given, $f(x) = |x|(x - \sin x)$

$f(-x) = -(|x|(x - \sin x))$

$f(-x) = -f(x) \Rightarrow f(x)$ is odd, non-periodic and continuous function.

Now

$$f(x) = \begin{bmatrix} x^2 - x\sin x, x \geq 0 \\ -x^2 + x\sin x, x < 0 \end{bmatrix}$$

$$\therefore f(\infty) = \lim_{x\to\infty}(x^2)\left(1 - \frac{\sin x}{x}\right) = \infty \qquad f(-\infty) = \lim_{x\to-\infty}(-x^2)\left(1 - \frac{\sin x}{x}\right) = -\infty$$

$\Rightarrow$ Range of $f(x) = R$

$\Rightarrow f(x)$ is an onto function .....(A)

$$f'(x) = \begin{bmatrix} 2x - \sin x - x\cos x & x \geq 0 \\ -2x + \sin x + x\cos x & x < 0 \end{bmatrix}$$

$$f'(x) = \underbrace{(x - \sin x)}_{\substack{\text{always}+\text{ive} \\ \text{or } 0}} + \underbrace{x(1 - \cos x)}_{\substack{\text{always}+\text{ive} \\ \text{or } 0}}$$

$\Rightarrow f'(x) > 0 \ \forall \ x \in (-\infty, \infty)$

$\Rightarrow$ f (x) is one - one function ....(B)

From equation (A) & (B), f(x) is both one – one & onto.

3. Let the functions: $R \longrightarrow R$ and $g : R \longrightarrow R$ be defined by

$$f(x) = e^{x-1} - e^{-|x-1|} \text{ and } g(x) = \frac{1}{2}\left(e^{x-1} + e^{1-x}\right)$$

Then, the area of the region in the first quadrant bounded by the curves $y = (x)$, $y = g(x)$ and $x = 0$ is

(A) $\left(2-\sqrt{3}\right)+\frac{1}{2}\left(e-e^{-1}\right)$

(B) $\left(2+\sqrt{3}\right)+\frac{1}{2}\left(e-e^{-1}\right)$

(C) $\left(2-\sqrt{3}\right)+\frac{1}{2}\left(e+e^{-1}\right)$

(D) $\left(2+\sqrt{3}\right)+\frac{1}{2}\left(e+e^{-1}\right)$

Answer: A

Solution:

$f(x) = e^{x-1} - e^{-|x-1|}$

$$f(x) = \begin{bmatrix} e^{x-1} - e^{-(x-1)} & x \geq 1 \\ e^{x-1} - e^{(x-1)} = 0 & x < 1 \end{bmatrix}$$

$$f(x) = \begin{bmatrix} e^{x-1} - \frac{1}{e^{x-1}} & x \geq 1 \\ 0 & x < 1 \end{bmatrix}$$

$$\& \ g(x) = \frac{1}{2}\left(e^{x-1} + \frac{1}{e^{x-1}}\right) = \frac{1}{2}\left(e^{x-1} + e^{1-x}\right)$$

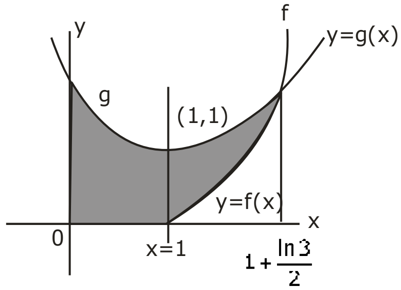
Now $f(x) = g(x)$

$$e^{x-1} - \frac{1}{e^{x-1}} = \frac{1}{2}\left(e^{x-1} + \frac{1}{e^{x-1}}\right)$$

$$2e^{x-1} - \frac{2}{e^{x-1}} = e^{x-1} + \frac{1}{e^{x-1}} c$$

$$e^{x-1} - \frac{3}{e^{x-1}} = 0 \Rightarrow e^{x-1} = \sqrt{3}$$

$$x = 1 + \frac{\ln 3}{2} = 1 + \ln\sqrt{3}$$



$$\text{Area} = \int_0^1 g(x)dx + \int_1^{1+\frac{\ln 3}{2}} \{g(x)-f(x)\}dx$$

$$= \int_0^1 \frac{1}{2}\left(e^{x-1}+\frac{1}{e^{x-1}}\right) + \int_1^{1+\frac{\ln 3}{2}} \frac{1}{2}\left(e^{x-1}+\frac{1}{e^{x-1}}\right)-\left(e^{x-1}-\frac{1}{e^{x-1}}\right)dx$$

$$= \frac{e-e^{-1}}{2} + 2-\sqrt{3}$$

4. Let a, b and λ be positive real numbers. Suppose P is an end point of the latus rectum of the parabola $y^2 = 4\lambda x$, and suppose the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through the point P. If the tangents to the parabola and the ellipse at the point P are perpendicular to each other, then the eccentricity of the ellipse is

(A) $\frac{1}{\sqrt{2}}$

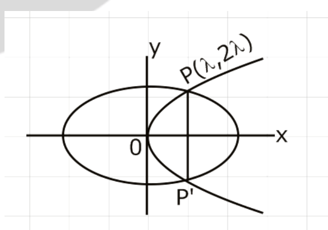
(B) $\frac{1}{2}$

(C) $\frac{1}{3}$

(D) $\frac{2}{5}$

Answer: A

Solution:



P(λ,2λ)

$$y^2 = 4\lambda x \Rightarrow \left(\frac{dy}{dx}\right)_A = 1 = m_1 \qquad \text{.......(1)}$$

Now E: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ Passes through P

$$\frac{\lambda^2}{a^2}+\frac{4\lambda^2}{b^2}=1 \Rightarrow \left(\frac{dy}{dx}\right)_A = \frac{-b^2}{2a^2}=m_2$$

$$\frac{2\lambda}{2\lambda}x-\frac{\lambda}{a^2}.\frac{b^2}{2\lambda} = -1 \qquad \text{.......(2)}$$

From eq. (1) and (2)

$m_1 . m_2 = -1 \Rightarrow b^2 = 2a^2$

$$\frac{a^2}{b^2}=\frac{1}{2}$$

for eccentricity of ellipse

$$e = \sqrt{1-\frac{a^2}{b^2}} = \sqrt{1-\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

5. Let C_1 and C_2 be two biased coins such that the probabilities of getting head in a single toss are $\frac{2}{3}$ and $\frac{1}{3}$, respectively. Suppose α is the number of heads that appear when C_1 is tossed twice, independently, and suppose β is the number of heads that appear when C_2 is tossed twice, independently. Then the probability that the roots of the quadratic polynomial $x^2 - \alpha x + \beta$ are real and equal, is

(A) $\frac{40}{81}$

(B) $\frac{20}{81}$

(C) $\frac{1}{2}$

(D) $\frac{1}{4}$

Answer: B

Solution:

C_1: P(H) = 2/3, P(T) = 1/3 ; C_2: P(H) = 1/3, P(T) = 2/3

Now roots of equation $x^2 - \alpha x + \beta = 0$ are real & equal

$\therefore$ D = 0

$\alpha^2 - 4\beta = 0$

$\alpha^2 = 4\beta$

$\Rightarrow (\alpha=0, \beta=0) \text{ or } (\alpha=2, \beta=1)$

$$P(E) = {}^2C_0\left(\frac{1}{3}\right)^2 . {}^2C_0\left(\frac{2}{3}\right)^2 + {}^2C_2\left(\frac{2}{3}\right)^2 . {}^2C_1\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)$$

$$\Rightarrow \frac{1}{9}\times\frac{4}{9}+\frac{4}{9}\times\frac{4}{9}=\frac{20}{81}$$

SECTION 2 (Maximum Marks: 24)

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is(are) the correct answer(s).
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 If only (all) the correct option(s) is(are) chosen;

Partial Marks : +3 If all four options is correct but ONLY three options are chosen;

Partial Marks : +2 If three or more options are correct but ONLY two options are chosen; both of which are correct;

Partial Marks : +1 If two or more options are correct but ONLY one option is chosen and it is a correct option;

Zero Marks : 0 If none of the options is chosen (i.e. the questions is unanswered);

Negative Marks : –2 In all other cases.

6. Consider all rectangles lying in the region

$$\left\{(x,y)\in R\times R : 0\le x\le \frac{\pi}{2} \text{ and } 0\le y\le 2\sin(2x)\right\}$$

and having one side on the x-axis. The area of the rectangle which has the maximum perimeter among all such rectangles, is

(A) $\frac{3\pi}{2}$

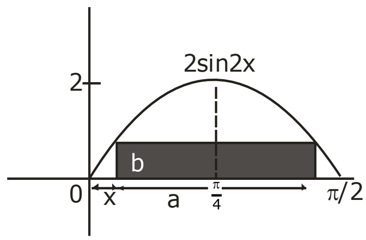
(B) π

(C) $\frac{\pi}{2\sqrt{3}}$

(D) $\frac{\pi\sqrt{3}}{2}$

Answer: C

Solution:



Let sides of rectangle are a & b

Then, perimeter = 2a + 2b

p = 2(a + b)

Now, b = 2sin2x & b = 2sin(2x + 2a) $\Rightarrow$ 2x + 2x + 2a = π

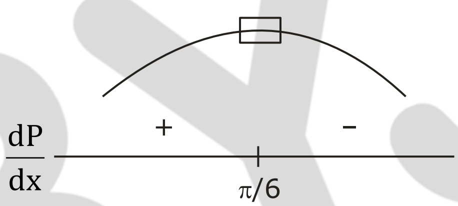
$$\left\{x = \frac{\pi}{4} - \frac{a}{2}\right\}$$

For perimeter maximum

P = 2a + 2b

P = π − 4x + 4sin2x

$$\frac{dP}{dx} = -4 + 8\cos 2x = 8\left\{\cos 2x - \frac{1}{2}\right\}$$



P_{max} at x = π/6

Now, Area at $x = \frac{\pi}{6}$

$$\text{Area} = \left(\frac{\pi}{2} - 2x\right).(2\sin 2x)$$

$$= \left(\frac{\pi}{2} - \frac{\pi}{3}\right)\left(2.\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}\sqrt{3} = \frac{\pi}{2\sqrt{3}}$$

7. Let the function $f: R \to R$ be defined by $(x) = x^3 - x^2 + (x - 1)\sin x$ and let $g: R \to R$ be an arbitrary function. Let $fg : R \to R$ be the product function defined by $(fg)(x) = f(x)g(x)$. Then which of the following statements is/are TRUE?

(A) If g is continuous at $x = 1$, then fg is differentiable at $x = 1$

(B) If fg is differentiable at $x = 1$, then g is continuous at $x = 1$

(C) If g is differentiable at $x = 1$, then fg is differentiable at $x = 1$

(D) If fg is differentiable at $x = 1$, then g is differentiable at $x = 1$

Answer: A, C

Solution:

$f : R \to R$

(A) $f(x) = x^3 - x^2 + (x - 1)\sin x$; $g : R \to R$

$h(x) = f(x).\, g(x) = \{x^3 - x^2 + (x - 1)\sin x\}.\, g(x)$

$$h'(1^+) = \lim_{h\to 0}\frac{\left\{(1+h)^3-(1+h)^2+h.\sin(1+h)\right\}g(1+h)}{h}$$

$$= \lim_{h\to 0}\frac{\left(1+h^3+3h+3h^2-1-h^2-2h+h\sin(1+h)\right)g(1+h)}{h}$$

$$= \lim_{h\to 0}\frac{\left(h^3+2h^2+h+h\sin(1+h)\right)g(1+h)}{h}$$

$$= \lim_{h\to 0}\left(1+\sin(1+h)\right)g(1+h)$$

$$h'(1^-) = \lim_{h\to 0}\frac{\left((1-h)^3-(1-h)^2+(-h)\sin(1-h)\right)g(1-h)}{-h}$$

$$= \lim_{h\to 0}\frac{\left(-h^3-3h+3h^2-h^2+2h-h\sin(1-h)\right)g(1-h)}{-h}$$

$$= \lim_{h\to 0}\left(1+\sin(1-h)\right)g(1-h)$$

as g(x) is constant at x =1

$\therefore$ $g(1+h) = g(1 - h) = g(1)$

$h'(1^+) = h'(1^-) = (1 + \sin 1)\, g(1)$

'A' is Correct.

8. Let M be a 3 × 3 invertible matrix with real entries and let I denote the 3 × 3 identity matrix. If M^{-1} = adj (adj M), then which of the following statements is/are ALWAYS TRUE?

(A) $M = I$

(B) det $M = 1$

(C) $M^2 = I$

(D) $(\text{adj } M)^2 = I$

Answer: B, C, D

Solution:

$M^{-1} = \text{adj}(\text{adj}(M))$

$(\text{adj } M)M^{-1} = (\text{adj}M)(\text{adj}(\text{adj}(M)))$

$(\text{adj } M)M^{-1} = N.\,\text{adj}(N)$ { Let adj(M) = N }

$(\text{adj } M)M^{-1} = |N|I$

$(\text{adj}M)M^{-1} = |\text{adj}(M)|I_3$

$(\text{adj}M) = |M|^2\,.M$ ....(1)

$|\text{adj } M| = ||M|^2.M|$

$|M|^2 = |M^6|.|M|$

$|M|=1,\ |M| \neq 0$

From equation (1)

adj.M =M ....(2)

Multiply by matrix M

$M.\text{adj } M = M^2$

$|M|I_3 = M^2$

$M^2 = I$

From (2) adj M =M

$(\text{adj } M)^2 = M^2 = I$

9. Let S be the set of all complex numbers z satisfying $|z^2 + z + 1| = 1$. Then which of the following statements is/are TRUE?

(A) $\left|z+\frac{1}{2}\right| \le \frac{1}{2}$ for all $z \in S$

(B) $|z| \le 2$ for all $z \in S$

(C) $\left|z+\frac{1}{2}\right| \ge \frac{1}{2}$ for all $z \in S$

(D) The set S has exactly four elements

Answer: B, C

Solution:

$$|z^2 + z + 1| = 1$$

$$\Rightarrow \left|\left(z+\frac{1}{2}\right)^2 + \frac{3}{4}\right| = 1$$

$$\Rightarrow \left|\left(z+\frac{1}{2}\right)^2 + \frac{3}{4}\right| \le \left|z+\frac{1}{2}\right|^2 + \frac{3}{4}$$

$$\Rightarrow 1 \le \left|z+\frac{1}{2}\right|^2 + \frac{3}{4} \Rightarrow \left|\left(z+\frac{1}{2}\right)\right|^2 \ge \frac{1}{4}$$

$\Rightarrow \left|z+\frac{1}{2}\right| \ge \frac{1}{2}$ Option (C) is correct

Also $|(z^2+z)+1| = 1 \ge \left||z^2+z|-1\right|$

$$\Rightarrow |z^2+z|-1 \le 1$$

$$\Rightarrow |z^2+z| \le 2$$

$$\Rightarrow \left||z^2|-|z|\right| \le |z^2+z| \le 2$$

$$\Rightarrow |r^2-r| \le 2$$

$$\Rightarrow r = |z| \le 2; \forall z \in S$$

Also we can always fine root of the equation $z^2 + z + 1 = e^{i\theta}$; $\forall \theta \in R$

Hence set 'S' is infinite.

10. Let x, y and z be positive real numbers. Suppose x, y and z are the lengths of the sides of a triangle opposite to its angles X, Y and Z, respectively. If

$$\tan\frac{X}{2}+\tan\frac{Z}{2}=\frac{2y}{x+y+z},$$

then which of the following statements is/are TRUE?

(A) $2Y = X + Z$

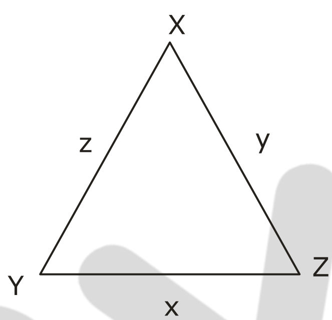
(B) $Y = X + Z$

(C) $\tan\frac{X}{2}=\frac{x}{y+z}$

(D) $x^2 + z^2 - y^2 = xz$

Answer: B, C

Solution:



$$\tan\frac{X}{2}+\tan\frac{Z}{2}=\frac{2y}{x+y+z}$$

$$\Rightarrow \frac{\Delta}{S(S-x)}+\frac{\Delta}{S(S-z)}=\frac{2y}{2S}$$

$$\Rightarrow \frac{\Delta}{S}\left(\frac{2S-(x+z)}{(S-x)(S-z)}\right)=\frac{y}{S}$$

$$\Rightarrow \frac{\Delta y}{S(S-x)(S-z)}=\frac{y}{S}$$

$$\Rightarrow \Delta^2=(S-x)^2(S-z)^2$$

$$\Rightarrow S(S-y) = (S-x)(S-z)$$

$$\Rightarrow (x + y + z)(x + z - y) = (y + z - x)(x + y - z)$$

$$\Rightarrow (x+z)^2-y^2=y^2-(z-x)^2$$

$\Rightarrow (x+z)^2 + (x-z)^2 = 2y^2$

$\Rightarrow x^2 + z^2 = y^2$ $\Rightarrow \angle Y = \frac{\pi}{2}$

$\Rightarrow \angle Y = \angle X + \angle Z$

$$\tan\frac{X}{2} = \frac{\Delta}{S(S-x)}$$

$$\tan\frac{X}{2} = \frac{\frac{1}{2}xz}{\frac{(y+z)^2 - x^2}{4}}$$

$$\tan\frac{X}{2} = \frac{2xz}{y^2 + z^2 + 2yz - x^2}$$

$$\tan\frac{X}{2} = \frac{2xz}{2z^2 + 2yz} \quad \text{(using } y^2 = x^2 + z^2\text{)}$$

$$\tan\frac{X}{2} = \frac{x}{y+z}$$

11. Let L_1 and L_2 be the following straight lines.

$L_1 : \frac{x-1}{1} = \frac{y}{-1} = \frac{z-1}{3}$ and $L_2 : \frac{x-1}{-3} = \frac{y}{-1} = \frac{z-1}{1}$

Suppose the straight line

$L : \frac{x-\alpha}{l} = \frac{y-1}{m} = \frac{z-\gamma}{-2}$

Lies in the plane containing L_1 and L_2, and passes through the point of intersection of L_1 and L_2. If the line L bisects the acute angle between the lines L_1 and L_2, then which of the following statements is/are TRUE?

(A) $\alpha - \gamma = 3$

(B) $l + m = 2$

(C) $\alpha - \gamma = 1$

(D) $l + m = 0$

Answer: A, B

Solution:

$L_1: \frac{x-1}{1} = \frac{y}{-1} = \frac{z-1}{3} = \lambda$ (let)

$L_2: \frac{x-1}{-3} = \frac{y}{-1} = \frac{z-1}{1} = \mu$ (let)

$P(\lambda+1, -\lambda, 3\lambda+1)$, $Q(-3\mu+1, -\mu, \mu+1)$

For point of Intersection

$\lambda + 1 = -3\mu + 1$ $(\lambda = \mu)$

$\lambda = \mu = 0$

Point of Intersection (1,0,1)

$\therefore \frac{x-\alpha}{l} = \frac{y-1}{m} = \frac{z-\gamma}{-2}$ passes through (1,0,1)

$\frac{1-\alpha}{l} = \frac{-1}{m} = \frac{1-\gamma}{-2}$ ...(1)

Direction ratio of $L_1(1, -1, 3)$,direction ratio of $L_2(-3, -1, 1)$

$\vec{V}_1$ = direction cosine of $L_1\left(\frac{1}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{3}{\sqrt{11}}\right)$;

$\vec{V}_2$ = direction cosine of $L_2\left(\frac{-3}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right)$

$\vec{V}_1 . \vec{V}_2 > 0$

$\therefore$ direction ratio of $\angle$bisector of L_1 and L_2

$= \left(\frac{-2}{\sqrt{11}}, \frac{-2}{\sqrt{11}}, \frac{4}{\sqrt{11}}\right)$

Or $\frac{l}{-2} = \frac{m}{-2} = \frac{n}{4}$

$\frac{l}{-1} = \frac{m}{-1} = \frac{n}{2} \Rightarrow l = m = 1$

From eq.(1)

$\frac{1-\alpha}{1} = -1 \Rightarrow \alpha = 2$

$\& \frac{1-\gamma}{-2} = -1 \Rightarrow \gamma = -1$

12. Which of the following inequalities is/are TRUE?

(A) $\int_0^1 x\cos x\,dx \geq \frac{3}{8}$

(B) $\int_0^1 x\sin x\,dx \geq \frac{3}{10}$

(C) $\int_0^1 x^2\cos x\,dx \geq \frac{1}{2}$

(D) $\int_0^1 x^2\sin x\,dx \geq \frac{2}{9}$

Answer: A, B, D

Solution:

(A)

Expansion of cosx = $1-\frac{x^2}{2!}+\frac{x^4}{4!}$

$\therefore\ \cos x \geq 1-\frac{x^2}{2!}$

Multiply x both side

$x\cos x \geq x-\frac{x^3}{2!}$

Integration both side

$$\int_0^1 x\cos x\,dx \geq \int_0^1 \left(x-\frac{x^3}{2}\right)dx$$

$$\int_0^1 x\cos x\,dx \geq \left(\frac{x^2}{2}-\frac{x^4}{8}\right)_0^1$$

$$\int_0^1 x\cos x\,dx \geq \frac{1}{2}-\frac{1}{8}$$

$$\int_0^1 x\cos x\,dx \geq \frac{3}{8}$$

Similarly

(B) expansion of sinx = $x-\frac{x^3}{3!}+\frac{x^5}{5!}\ldots$

$\sin x \geq\ x - \frac{x^3}{3!}$

Multiply x both side

$x\sin x \ge x^2 - \frac{x^4}{6}$

Integration both side

$$\int_0^1 x\sin x\,dx \ge \int_0^1 \left(x^2 - \frac{x^4}{6}\right)dx$$

$$\int_0^1 x\sin x dx \ge \left(\frac{x^3}{3} - \frac{x^5}{6\cdot 5}\right)_0^1$$

$$\int_0^1 x\sin x\,dx \ge \left(\frac{x^3}{3} - \frac{x^5}{30}\right)_0^1$$

$$\int_0^1 x\sin x\,dx \ge \frac{1}{3} - \frac{1}{30}$$

$$\int_0^1 x\sin x\,dx \ge \frac{3}{10}$$

(C) $\cos x < 1$

Multiply x^2 both side

$x^2 \cos x < x^2$

Integration both side

$$\int_0^1 x^2 \cos x\,dx < \int_0^1 x^2 dx$$

$$\int_0^1 x^2 \cos x\,dx < \frac{1}{3}$$

(D) $$\int_0^1 x^2 \sin x\,dx \ge \int_0^1 x^2\left(x - \frac{x^3}{6}\right)dx$$

$$\int_0^1 x^2 \sin x\,dx \ge \left(\frac{x^4}{4} - \frac{x^6}{36}\right)_0^1$$

$$\int_0^1 x^2 \sin x\,dx \ge \frac{1}{4} - \frac{1}{36}$$

$$\int_0^1 x^2 \sin x\,dx \ge \frac{2}{9}$$

SECTION 3 (Maximum Marks: 24)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE.**
- For Each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numerical keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:
  Full Marks : +4 If ONLY the correct numerical value is entered;
  Zero Marks : 0 In all other cases.

13. Let m be the minimum possible value of $\log_3\left(3^{y_1}+3^{y_2}+3^{y_3}\right)$, where y_1, y_2, y_3 are real numbers for which $y_1 + y_2 + y_3 = 9$. Let M be the maximum possible value of $(\log_3 x_1 + \log_3 x_2 + \log_3 x_3)$, where x_1, x_2, x_3 are positive real numbers for which $x_1 + x_2 + x_3 = 9$. Then the value of $\log_2(m^3) + \log_3(M^2)$ is _____

Answer: 8.00

Solution:

Using A.M$\geq$G.M.

$$\frac{3^{y_1}+3^{y_2}+3^{y_3}}{3}\geq\left(3^{y_1}.3^{y_2}.3^{y_3}\right)^{\frac{1}{3}}$$

$$3^{y_1}+3^{y_2}+3^{y_3}\geq 3.\left(3^{y_1+y_2+y_3}\right)^{\frac{1}{3}} \qquad \{\therefore y_1+y_2+y_3=9\}$$

$$3^{y_1}+3^{y_2}+3^{y_3}\geq 3.\left(3^{9}\right)^{\frac{1}{3}}$$

$$3^{y_1}+3^{y_2}+3^{y_3}\geq 81$$

$\Rightarrow m = \log_3 81 = \log_3 3^4 = 4\log_3 3 = 4$

Again, using A.M$\geq$G.M.

$$\frac{x_1+x_2+x_3}{3}\geq\left(x_1.x_2.x_3\right)^{\frac{1}{3}}$$

$$= \frac{9}{3}\geq\left(x_1.x_2.x_3\right)^{\frac{1}{3}} \qquad \{\because x_1+x_2+x_3=9\}$$

$\Rightarrow\ 27\geq x_1x_2x_3$

$M = \log_3 x_1 + \log_3 x_2 + \log_3 x_3$

$M = \log_3(x_1x_2x_3) = \log_3(27) = 3$

$\therefore \log_2(m)^3 + \log_3(M)^2 \Rightarrow \log_2(2^6) + \log_3(3^2) = 6 + 2 = 8$

14. Let a_1, a_2, a_3,… be a sequence of positive integers in arithmetic progression with common difference 2. Also, let b_1, b_2, b_3,… be a sequence of positive integers in geometric progression with common ratio 2. If $a_1 = b_1 = c$, then the number of all possible values of c, for which the equality

$$2(a_1 + a_2 + \cdots + a_n) = b_1 + b_2 + \cdots + b_n$$

holds for some positive integer n, is _____

Answer: 1.00

Solution:

$2(a_1+a_2+\ldots\ldots+a_n) = b_1 + b_2 + \ldots + b_n$

$\Rightarrow 2\left[\frac{n}{2}\left(2a_1+(n-1)2\right)\right] = \frac{b_1\left(2^n-1\right)}{2-1}$

$\Rightarrow 2n[a_1 + (n-1)] = b_1(2^n - 1)$

$\Rightarrow 2na_1 + 2n^2 - 2n = a_1(2^n - 1)$ $\{\because a_1 = b_1\}$

$\Rightarrow 2n^2 - 2n = a_1(2 - 1 - 2n)$

$a_1 = \frac{2\left(n^2-n\right)}{\left(2^n-1-2n\right)} = c$ $\{\because a_1 = c\}$

$\because c \geq 1$

$\Rightarrow \frac{2\left(n^2-n\right)}{2^n-1-2n} \geq 1$

$2(n^2 - n) \geq 2^n - 1 - 2n$ $\{\because n^2 - n \geq 0 \text{ for } n \geq 1\}$

$= 2n^2 + 1 \geq 2^n$

Therefore, n =1, 2,3,4,5,6

$n = 1 \Rightarrow c = 0$ (rejected)

$n = 2 \Rightarrow c < 0$ (rejected)

$n = 3 \Rightarrow c = 12$ (correct)

$n = 4 \Rightarrow c =$ not Integer

$n = 5 \Rightarrow c =$ not Integer

$n = 6 \Rightarrow c =$ not Integer

$\therefore c = 12$ for $n = 3$

Hence, no. of such c = 1

15. Let f: [0, 2] ⟶ R be the function defined by

$$f(x)=(3-\sin(2\pi x))\sin\left(\pi x-\frac{\pi}{4}\right)-\sin\left(3\pi x+\frac{\pi}{4}\right).$$

If $\alpha, \beta \in [0,2]$ are such that $\{x \in [0, 2] : f(x) \geq 0\} = [\alpha, \beta]$, then the value of $\beta - \alpha$ is _____

Answer: 1.00

Solution:

$$f(x)=(3-\sin(2\pi x))\sin\left(\pi x-\frac{\pi}{4}\right)-\sin\left(3\pi x+\frac{\pi}{4}\right)$$

Let $\pi x - \pi/4 = \theta$

$f(x) \geq 0$

$\Rightarrow (3 - \sin 2(\theta + \pi/4)) \sin\theta - \sin\left[3\left(\frac{\pi}{4}+\theta\right)+\frac{\pi}{4}\right] \geq 0$

$\Rightarrow 3\sin\theta - \sin\theta.\sin\left(\frac{\pi}{2}+2\theta\right)-\sin(\pi+3\theta)\geq 0$

$\Rightarrow$ $3\sin\theta - \sin\theta\cos 2\theta + \sin 3\theta \geq 0$

$\Rightarrow \left[3\sin\theta-\sin\theta\left(1-2\sin^2\theta\right)+\left(3\sin\theta-4\sin^3\theta\right)\right]\geq 0$

$\Rightarrow \sin\theta\,[3 - (1 - 2\sin^2\theta) + 3 - 4\sin^2\theta] \geq 0$

$\Rightarrow \sin\theta\left[5-2\sin^2\theta\right]\geq 0$

$\sin\theta\,[4+\cos 2\theta]\geq 0$

$\therefore \sin\theta \geq 0$

$\Rightarrow \theta \in [0,\pi]$

$0\leq\theta\leq\pi$

$0\leq \pi x-\frac{\pi}{4}\leq\pi$

$\frac{1}{4}\leq x\leq\frac{5}{4}$

$x\in\left[\frac{1}{4},\frac{5}{4}\right]$

$\alpha = 1/4$; $\beta = 5/4$

$\beta - \alpha = 1$

16. In a triangle PQR, let $\vec{a}=\overrightarrow{QR}$, $\vec{b}=\overrightarrow{RP}$ and $\vec{c}=\overrightarrow{PQ}$. If

$$|\vec{a}|=3, \qquad |\vec{b}|=4 \text{ and } \frac{\vec{a}.\left(\vec{c}-\vec{b}\right)}{\vec{c}.\left(\vec{a}-\vec{b}\right)}=\frac{|\vec{a}|}{|\vec{a}|+|\vec{b}|}$$

Then the value of $\left|\vec{a}\times\vec{b}\right|^2$ is ____

Answer: 108.00

Solution:

$\vec{a}+\vec{b}+\vec{c}=0$ ....(1)

$\Rightarrow \vec{c}=-\vec{a}-\vec{b}$ ....(2)

Now, $\dfrac{\vec{a}.(\vec{c}-\vec{b})}{\vec{c}.(\vec{a}-\vec{b})}=\dfrac{|\vec{a}|}{|\vec{a}|+|\vec{b}|}$

$\Rightarrow \dfrac{\vec{a}.(-\vec{a}-2\vec{b})}{(-\vec{a}-\vec{b})(\vec{a}-\vec{b})}=\dfrac{3}{3+4}$

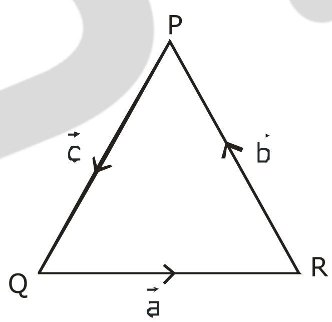
$\Rightarrow \dfrac{\vec{a}.(\vec{a}+2\vec{b})}{(\vec{a}+\vec{b})(\vec{a}-\vec{b})}=\dfrac{3}{7}$

$\Rightarrow \dfrac{|\vec{a}|^2+2\vec{a}\cdot\vec{b}}{|\vec{a}|^2-|\vec{b}|^2}=\dfrac{3}{7}$

$\Rightarrow \dfrac{(3)^2+2\vec{a}.\vec{b}}{(3)^2-(4)^2}=\dfrac{3}{7}$

$\Rightarrow 2\vec{a}\cdot\vec{b}=-12$

$\Rightarrow \vec{a}\cdot\vec{b}=-6$



Now, $|\vec{a}\times\vec{b}|^2=a^2b^2-(\vec{a}\cdot\vec{b})^2$

$= 9\times16-(-6)^2$

$= 108$

17. For a polynomial $g(x)$ with real coefficients, let m_g denote the number of distinct real roots of $g(x)$. Suppose S is the set of polynomials with real coefficients defined by

$S = \{(x^2 - 1)^2(a_0 + a_1x + a_2x^2 + a_3x^3) : a_0, a_1, a_2, a_3 \in R\}$

For a polynomial f, let f' and f'' denote its first and second order derivatives, respectively. Then the minimum possible value of $(m_{f'} + m_{f''})$, where $f \in S$, is _____

Answer: 5.00

Solution:

$f(x) = (x^2 - 1)^2h(x)$; $h(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

Now, $f(1) = f(-1) = 0$

$\Rightarrow f'(\alpha) = 0, \alpha \in (-1,1)$ [Rolle's Theorem]

Also, $f'(1) = f'(-1) = 0 \Rightarrow f'(x) = 0$ has atleast3 root $-1, \alpha, 1$ with $-1 < \alpha < 1$

$\Rightarrow f''(x) = 0$ will have at least 2 root, say β, γ such that

$-1 < \beta < \alpha < \gamma < 1$ [Rolle's Theorem]

So, $\min\left(m_{f''}\right) = 2$

and we find $(m_{f'} + m_{f''}) = 5$ for $f(x) = (x^2 - 1)^2$

Thus, Ans = 5

18. Let e denote the base of the natural logarithm. The value of the real number a for which the right hand limit

$$\lim_{x\to 0^+}\frac{(1-x)^{\frac{1}{x}}-e^{-1}}{x^a}$$

is equal to a nonzero real number, is _____

Answer: 1.00

Solution:

$$L=\lim_{x\to 0^+}\frac{(1-x)^{\frac{1}{x}}-e^{-1}}{x^a}$$

$$L=\lim_{x\to 0^+}\frac{e^{\ln(1-x)^{1/x}}-e^{-1}}{x^a}$$

$$L=\lim_{x\to 0^+}\frac{e^{\frac{1}{x}\ln(1-x)}-e^{-1}}{x^a}$$

$$L=\lim_{x\to 0^+}\frac{e^{\frac{1}{x}\left(-x-\frac{x^2}{2}-\frac{x^3}{3}\dots\right)}-e^{-1}}{x^a}$$

$$L=\lim_{x\to 0^+}\frac{e^{-1}.e^{-\left(\frac{x}{2}+\frac{x^2}{3}\dots\right)}-e^{-1}}{x^a}$$

$$L=\lim_{x\to 0^+}\frac{e^{-1}\left[e^{-\left(\frac{x}{2}+\frac{x^2}{3}\dots\right)}-1\right]}{x^a}$$

$$L=\lim_{x\to 0^+}\frac{e^{-1}\left[\left(1+\left(-\frac{x}{2}-\frac{x^2}{3}\right)+\frac{\left(\frac{x}{2}+\frac{x^2}{3}\right)^2}{2!}\dots\right)-1\right]}{x^a}$$

$$=\lim_{x\to 0^+}\frac{e^{-1}\left(\left(-\frac{1}{2}-\frac{x}{3}\dots\right)+\frac{x\left(\frac{1}{2}+\frac{x}{3}\right)^2}{2!}\dots\right)}{x^{a-1}}$$

For Non - Zero limit a – 1 = 0 $\Rightarrow$ a = 1