Johns Hopkins Math Tournament 2020
Middle School Division

Proof Round: Graph Theory

February 22, 2020

Problem	Points	Score
1	5	
2	5	
3	15	
4	5	
5	15	
6	20	
7	10	
8	5	
9	5	
10	15	
Total	100	

Instructions

- The exam is worth 100 points.
- To receive full credit, the presentation must be legible, orderly, clear, and concise.
- If a problem says "list" or "compute," you need not justify your answer. If a problem says "determine," "find," or "show," then you must show your work or explain your reasoning to receive full credit, although such explanations do not have to be lengthy.
- Even if not proved, earlier numbered items may be used in solutions to later numbered items, but not vice versa.
- Pages submitted for credit should be **numbered in consecutive order at the top of each page** in what your team considers to be proper sequential order.
- **Please write on only one side of the answer papers.**
- Put the **team number** (NOT the team name) on the cover sheet used as the first page of the papers submitted. Do not identify the team in any other way.

1 What is a Graph?

Visually, a graph has a collection of vertices and edges connecting them. Vertices are points and the edges are lines connecting the vertices. Be careful, though, as the same graph can often be drawn in different ways, as long as the structure is the same. In both of the graphs below, each of the 4 vertices has an edge to each of the 3 other vertices, which makes the two graphs identical; they simply look different because the vertices are positioned differently.

Example 1.1. *Here is a graph drawn in two different ways:*

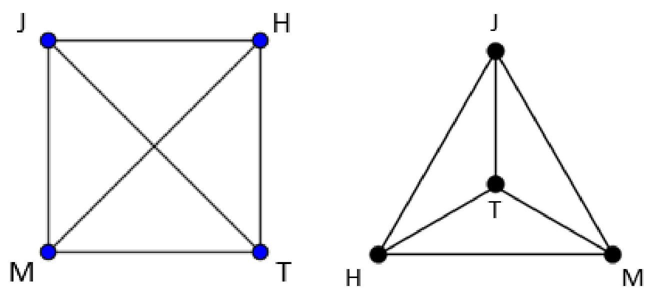


Figure 1: Different Visual Representations for the Same Graph

Definition 1.1. A graph is a ordered pair $G = (V, E)$, where

- V is a set of vertices of the graph
- E is a set of edges of the graph. Each edge is a pair of vertices, each in V

For the purposes of these problems, always assume non-trivial graphs — specifically meaning that every vertex has at least one edge, leaving no single points.

The graph G in Example 1.1 can be written as $G = (V, E)$, where $V = \{J, H, M, T\}$, $E = \{\{J, H\}, \{H, M\}, \{M, T\}, \{T, J\}, \{J, M\}, \{H, T\}\}$.

Problem 1:
(a) (2 points) Draw and label the graph $G = (V, E)$, where $V = \{1, 2, 3\}$, $E = \{\{1, 2\}\{2, 3\}\{3, 1\}\}$
(b) (3 points) Draw and label the graph $G = (V, E)$, where $V = \{1, 2, 3, 4, 5, 6\}$, $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{1, 4\}, \{2, 5\}, \{3, 6\}\}$

There are many equivalent solutions. One solution is:

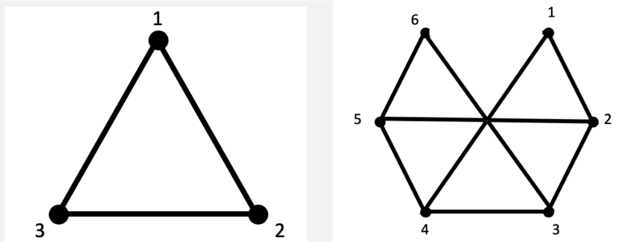


Figure 2: Solutions for 1(a) and 1(b)

Definition 1.2. A graph $g = (v, e)$ is a subgraph of $G = (V, E)$ if every vertex of g is a vertex of G and every edge of g is an edge of G.

Problem 2: (5 points) Is $g = (\{1, 2, 3, 4\}, \{\{1, 2\}, \{3, 4\}, \{1, 4\}, \{2, 5\}\})$ a subgraph of G, the graph from problem 1? Explain.

No. While g has both a subset of the vertices and edges of G, it is not a subgraph because namely it is not a graph! Notice that $\{2, 5\}$, an edge of g which contains 5, which is not a vertex of the g. By definition 1.1, this is not a graph and thus cannot be a subgraph.

2 Cycles and Trees

Definition 2.1. A trail in a graph $G = (V, E)$ is a sequence of at least two vertices $(v_1, v_2, v_3, \ldots, v_k)$ such that each edge $\{v_i, v_{i+1}\}$ is in E and that no edges are repeated.

In other words, it is a way of tracing through the vertices along edges in a specific order, making sure to avoid using the same edge twice. For any graph, there are generally many possible trails.

Example 2.1. *Below is an example of a trail in a graph.*

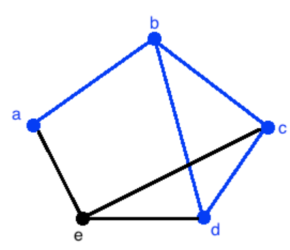


Figure 3: (a,b,c,d,b) is a trail in this graph

A specific type of trail is a cycle.

Definition 2.2. A cycle is a trail that ends at the same vertex where it began.

Example 2.2. *Note that there are multiple possible starting/ending vertices for any given cycle. If you have found one cycle, you have found many, differing only in their starting position.*

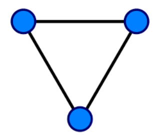


Figure 4: There are three different cycles in this graph.

Definition 2.3. The length of a trail (or cycle) is the number of edges it goes through.

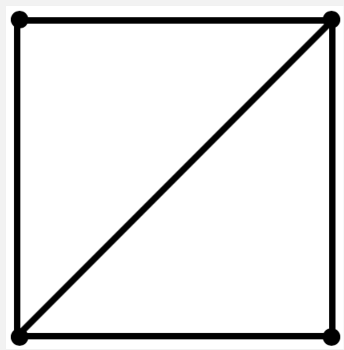
Problem 3:

(a) (4 points) Find the graph with the fewest edges such that it also contains 4 cycles of length 3.

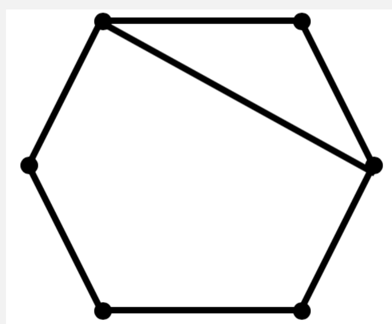
(b) (5 points) Draw a graph with a cycle of length 3 and a cycle of length 9 with at most 6 vertices and at most 9 edges.

(c) (6 points) Let G be a graph with an a cycle of length n. Explain why there must be at least $2n(n-1)$ trails in G.

(a) Look at example 2.2 for clarification for why this has 4 of length 3.



(b) There are many possible solutions. One such graph is:



(c) Within the cycle, we can create a trail between any two vertices. We first choose one of the n vertices to begin the trail, then choose one of the remaining n-1 vertices to end it. This gives us n(n-1) choices of endpoints for our trails. Because this is a cycle, there are two trails connecting this pair of vertices. This gives us 2n(n-1) trails composed of only the vertices and edges in the cycle. There can be other trails in the rest of the graph, so the entire graph has a minimum of 2n(n-1) trails.

Remark 2.1. *A cycle is said to be odd if it contains an odd number of edges. It is even if it contains an even number of edges. In Figure* 4*, the cycle on the left is even and the cycle on the right is odd.*

Definition 2.4. An Eulerian circuit is a cycle which includes all of the vertices in the graph.

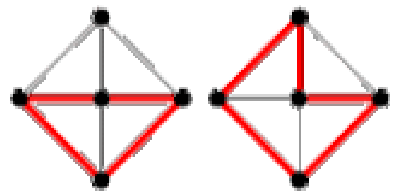


Figure 5: Two of the many possible cycles in this graph

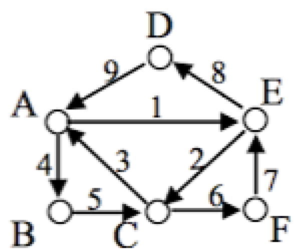
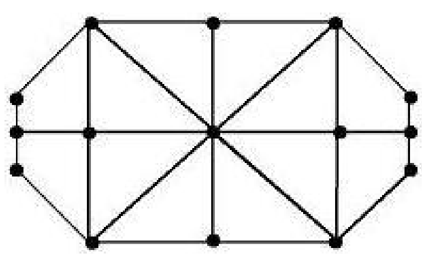
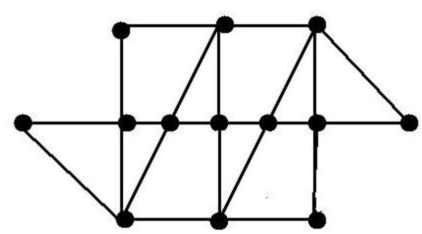


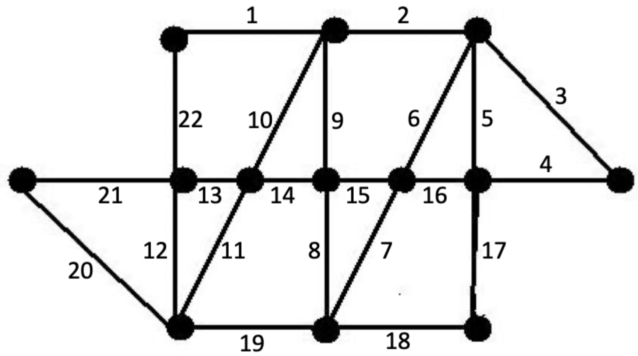
Figure 6: The cycle starting and ending at A and following the order of the numbered edges is an Eulerian circuit.

Problem 4: (5 points) One of the following graphs has an Eulerian circuit. Determine which graph has an Eulerian circuit and identify the circuit by numbering the edges as in Figure 4.





The graph on the right has an Eulerian circuit. One possible ordering of the edges is as follows:



Definition 2.5. A tree is a graph that contains no cycles.

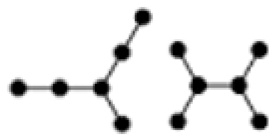


Figure 8: Trees with 6 vertices

Definition 2.6. A spanning tree of G is a subgraph of G that contains every vertex of G and is also a tree.

Problem 5:
(a) (5 points) Give an example of a graph with at least 6 vertices and no odd cycles.
(b) (5 points) Construct a spanning tree for your graph.
(c) (5 points) Show that any graph with no odd cycles contains a spanning tree.

(a) There are many possible solutions.Three such graphs are:

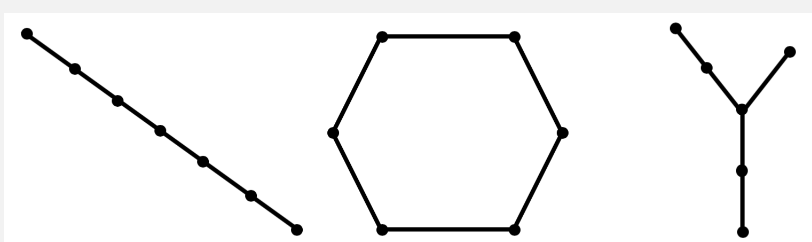
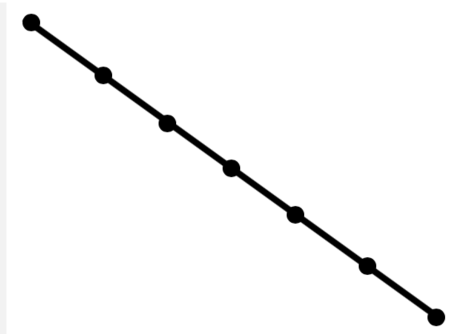


Figure 9

(b) The spanning tree depends on the graph provided in part (a). For the graphs given in this solution, the left and right graphs are already trees. For the graph in the middle, the following is a spanning tree:



(c) We can construct a spanning tree of any graph by removing one edge from each cycle, thus breaking that cycle. When no cycles remain, the resulting graph is a tree, and since we have not removed any vertices, it is a spanning tree.

3 Graceful Graphs

Definition 3.1. Given a graph with x edges, suppose we label each vertex with a unique integer between 0 and x inclusive. Notice that since most graphs have more edges than vertices, you won't necessarily use up all the integers from 0 to x. Then, label each edge with the absolute difference of the integer label of its two vertices. If the resulting edge labels cover every integer from 1 to x, the labeling is said to be graceful.

Example 3.1. *Below is an example of a graceful graph. Since there are* 6 *edges, the vertices can be labeled with any integer between* 0 *and* 6 *inclusive, as long as all vertex labels are unique. Then, the edge labels are calculated by taking the absolute difference of the vertex labels. Since the edge labels cover every integer from* 1 *to* 6*, this labeling is graceful.*

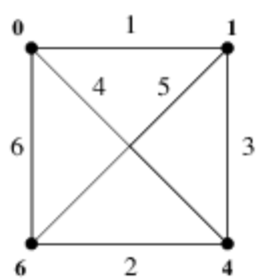
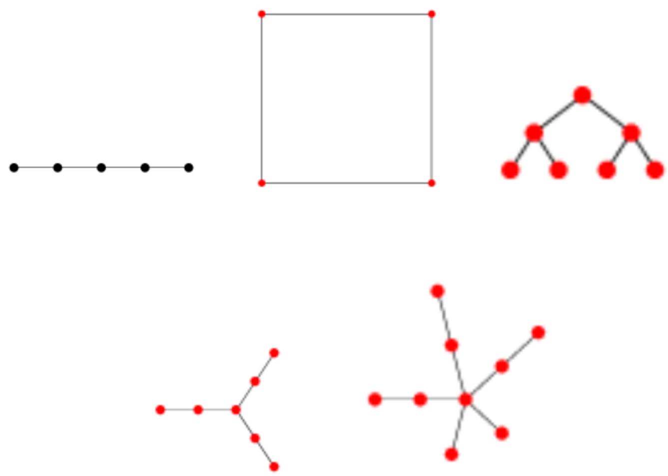
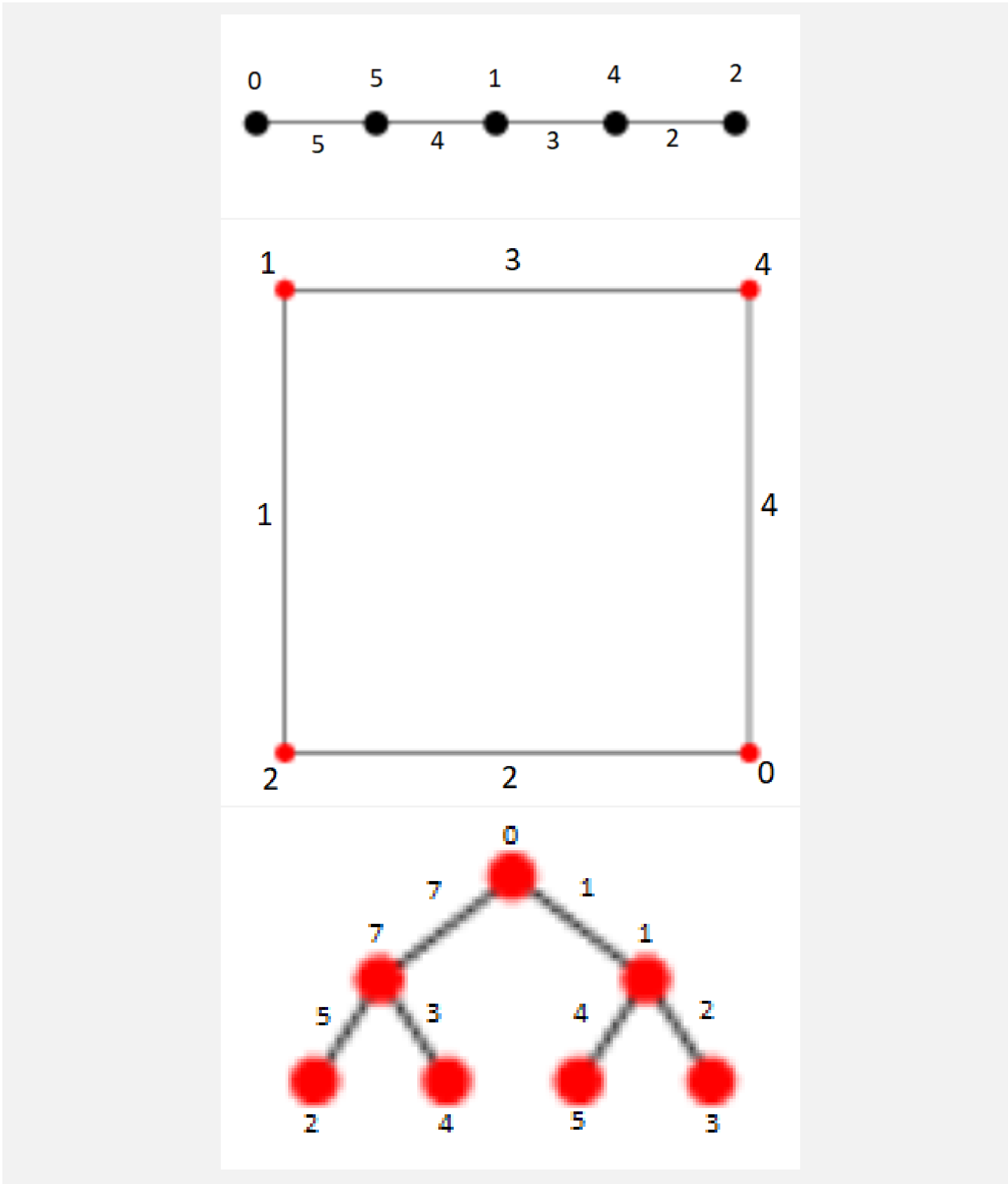


Figure 10: A graceful graph with 4 vertices and 6 edges

Problem 6: (20 points) Create a graceful labeling for the following graphs. Each of the five parts is worth 4 points.

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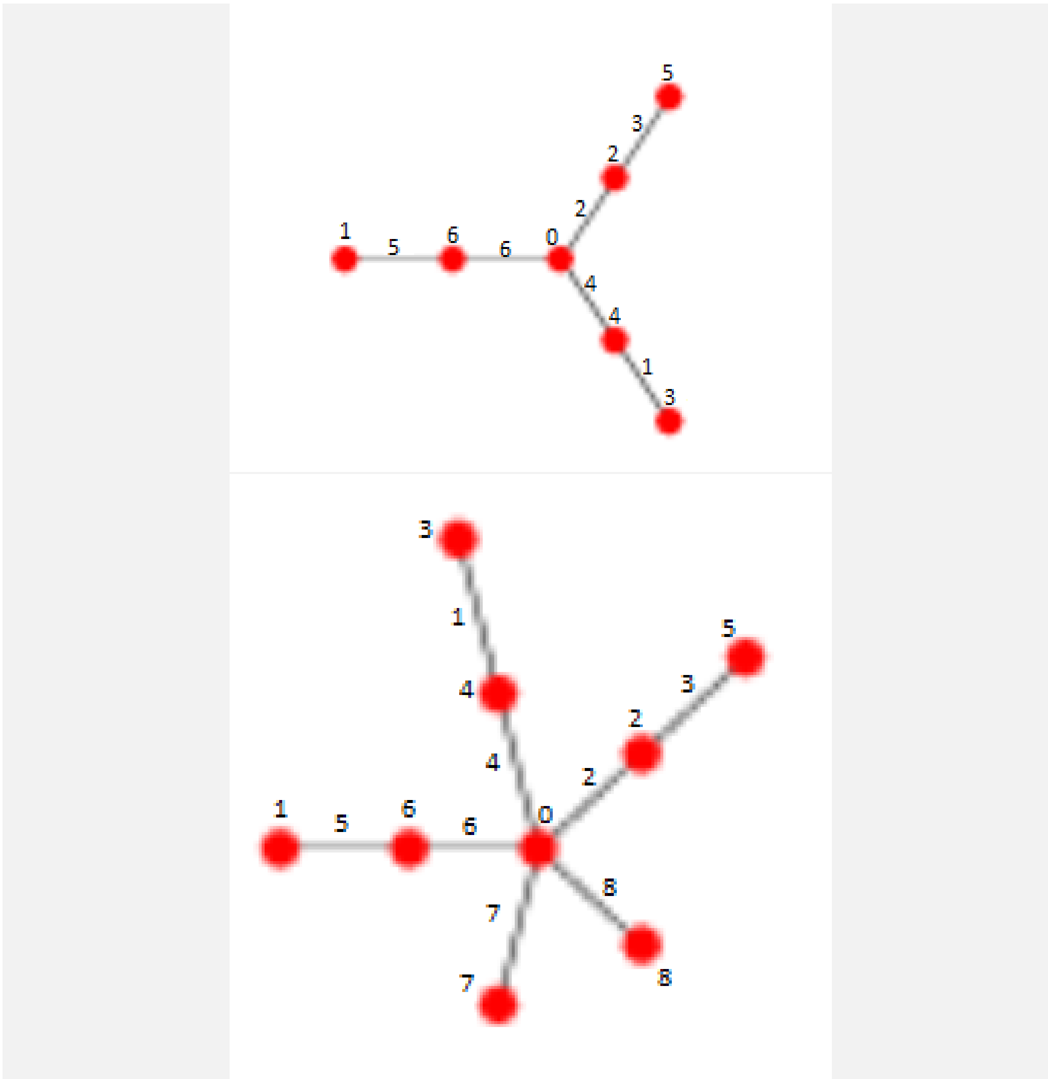

4 Graph Coloring and Bipartite Graphs

Definition 4.1. A coloring of a graph is when each of the vertices is assigned a unique marking, called the color. A proper coloring is a coloring where all vertices connected by an edge are colored a different color.

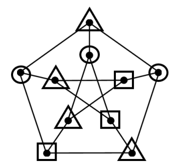

Figure 15: A proper coloring of the Petersen Graph

Problem 7: (10 points) Give a proper coloring of the following graph using the fewest possible colors, using shapes to denote colors as used in the example above.

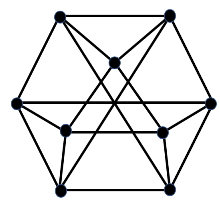

Figure 16

There are many possible correct colorings. However, the minimum number of colors needed is 4, and the coloring should obey the definition of a proper coloring. Here is one correct proper coloring:

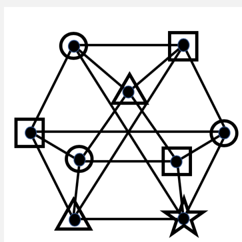

Figure 17: Caption

Remark 4.1. *A graph is n-colorable if n is the minimum number of colors required for a proper coloring.*

Definition 4.2. A graph is bipartite if its vertices can be divided into two groups such that there is no edge between vertices in the same group. (See graph below.)

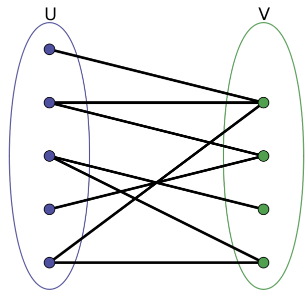

Figure 18: Bipartite graph and division of vertices

Hint: For the proofs below, first convince yourself that the statements are true before giving a formal answer. Use the graphs given throughout the test or create your own graphs as examples for each statement.

Problem 8: (5 points) Show that a graph is bipartite if and only if it is 2-colorable. (That is, show bipartite implies 2-colorable and that 2-colorable implies bipartite)

If it is 2-colorable, then there exists a partition where each group shares no edges with like-colored vertices. By considering these two sets as the division described in the definition of bipartite, this graph is bipartite.

If it is bipartite, then there exists a partition where each vertex in one of the two groups does not share an edge with its fellow group members. Color each group a different color —this is a two coloring. Since this group is nontrivial, it is not 1-colorable, so the minimum number of colors is two, making it 2-colorable.

Problem 9: (5 points) Show that a bipartite graph cannot contain any odd cycles.
Hint: Notice that in a bipartite graph, every edge must go from one group of vertices, to the other, or else the graph isn't bipartite. Now, think about why that means odd cycles can't exist.

Consider a bipartite graph with vertices divided into sets A and B. Every edge must contain one vertex from each set since the graph is bipartite. For an odd cycle, we need three edges that start and end at the same vertex. The first two edges of our cycle go from a vertex in A to B and from B to A. To complete our cycle, the last edge must connect our first and last vertex. But these vertices are both in A, so there is no edge between them. Therefore a bipartite graph can contain no odd cycles.

Problem 10: (15 points)
(a) Show that the spanning tree for a graph is 2-colorable.
(b) Show that a 2-coloring for the spanning tree of a graph G is also a 2-coloring of G if G contains no odd cycles.
(c) Show that a graph is bipartite if and only if it contains no odd cycles.

(a) Consider only the spanning tree of the graph. Therefore there are no cycles, and there is a unique trail between each pair of vertices. Color one vertex red. Then color each of its neighbors blue. Continue alternating colors for the neighbors of each vertex. In general, vertices that are an odd number of edges from the initial vertex are colored blue, and vertices that are an even number of edges from the initial vertex are colored red. Every vertex can be reached by a unique trail from the initial vertex, so it is always either and odd or an even number of edges away. So, we can color every vertex of the spanning tree with two colors in this way. Finally, note that no even or odd vertices will share an edge, meeting the conditions for a bipartite graph.

(b) We showed earlier that every graph with no odd cycles contains a spanning tree. Consider the coloring of that spanning tree as described above. When we add the remaining edges of the graph back, we need to show that no vertices that were colored the same in the spanning tree now share an edge. If two vertices were an even number of edges away from each other in the spanning graph (and thus were colored the same color), then adding an edge between them would create an odd cycle (because an even number plus one is odd). Since there are no odd cycles, there are no such issues, and so the 2-coloring of the spanning tree is also a valid 2-coloring for the graph.

(c) We showed one direction of this statement in Problem 9. The other direction is proven as follows: If a graph has no odd cycles, then we can give a proper 2-coloring of the graph using the 2-coloring of the spanning tree described in (a). From Problem 8, a 2-colorable graph is bipartite. Thus, since we can give a 2-coloring of any graph with no odd cycles, these graphs are all bipartite.