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Large deviations principle for the invariant measures of the 2D stochastic Navier–Stokes equations on a torus

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ABSTRACT

We prove here the validity of a large deviation principle for the family of invariant measures associated to a two dimensional Navier–Stokes equation on a torus, perturbed by a smooth additive noise.

Contents

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1. Introduction

In the present paper we are dealing with 2-D Navier–Stokes equations with periodic boundary conditions, perturbed by a small additive noise. These boundary conditions are usually realized by considering the problem on a two-dimensional torus $\mathbb{T}^2$, see Section 2 for more details. To fix readers attention, let us write down these equations in a functional form, as

$$du(t) + \mathrm{A}u(t)\,dt + \mathrm{B}(u(t), u(t))\,dt = \sqrt{\varepsilon}\,dw(t), \qquad u(0) = u_0, \tag{1.1}$$

for $0 < \varepsilon << 1$.

Full definitions of the symbols involved can be found later in Section 2, but, for the time being, let us recall that A is the Stokes operator, equal, roughly speaking, to the Laplace operator (acting on vector fields) composed with the Leray–Helmholtz projection P, defined on the space of zero mean and square integrable vector fields with values in the subspace H of divergence free vector fields, the convection $\mathrm{B}(u, u)$ is equal to $P(u\nabla u)$, $w(t)$ is a K-cylindrical Wiener process, for $\mathrm{K} = D(\mathrm{A}^{\frac{\alpha}{2}})$ with $\alpha > 1$, and $u_0 \in \mathrm{H}$. Of course, because P nullifies the gradients, the gradient of the pressure term ∇p disappears in such a formulation. Basic questions about such a problem are now well understood, and we simply refer to the papers [15] and [6] and to the chapter 15 of the monograph [11].

It is know that, for every fixed $\varepsilon > 0$, the Markov process on H generated by equation (1.1) has an invariant measure μ_ε (see [15]), which is also unique and ergodic (see [13] and also [17]). The objective of our paper is study of the validity of a large deviation principle (LDP) for the family of invariant measures $\{\mu_\varepsilon\}_{\varepsilon>0}$. To be more precise, our purpose is to show that the family of probability measure $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies a LDP, as $\varepsilon \downarrow 0$, with rate ε and action functional equal to the quasi-potential U associated to the controlled deterministic NSE, also known as the skeleton equation,

$$u'(t) + \mathrm{A}u(t) + \mathrm{B}(u(t), u(t)) = f(t), \qquad u(0) = u_0, \tag{1.2}$$

where $f \in L^2(0, \infty; D(\mathrm{A}^{\frac{\alpha}{2}}))$. The quasi-potential $\mathrm{U}(v)$, for $v \in \mathrm{H}$, can be defined as the infimum, over all $T > 0$, of the energy of the control f, with respect to the norm of the reproducing kernel Hilbert space K of the law $\mathcal{L}(W(1))$, i.e.

$$\frac{1}{2}\int_0^T |\mathrm{A}^{\frac{\alpha}{2}} f(t)|^2_{\mathrm{H}},$$

such that the solution u to the skeleton equation (1.2), with initial data $u(0) = 0$, reaches the state v at time T, i.e. $u(T) = v$. To this purpose, we refer to equation (4.12) for a version of the definition of U using both positive and negative times and (4.13) for a representation of U using the skeleton equation over the negative half-line $(-\infty, 0]$.

The quasi-potential U was an important object in our recent study [3] with M. Freidlin and in some sense our current paper is a natural continuation of that work. The two other works on which we depend a lot in our investigation is the paper [9] by the second named author and M. Röckner and [22] in which a similar question was investigated for reaction diffusion equation with polynomially bounded, resp. bounded, reaction term.

Let us make an important comment about the assumption $\alpha > 1$. In fact, the Markov process on H, generated by problem (1.1), for both periodic and Dirichlet boundary conditions, has a unique invariant measure μ_ε for $\alpha > 0$, (to this purpose, see [6, Corollary 9.1 and Remark 4.1 (c)]). However, an essential tool in proving the LDP is given by the exponential estimates for the invariant measures and we have been able to prove them only in the case of periodic boundary conditions and $\alpha > 1$ (see Theorem 5.1). As a matter of fact, we do not know if, even in the case of periodic boundary conditions, such exponential estimates are true without assuming that the covariance of the noise is a trace-class operator. This is the reason why, already from the very beginning, we assume that our problem is posed on a 2-D torus and that $\alpha > 1$.

The subject of this paper is closely related to recent research activity in mathematics and physics related to the so called rare events, see for instance [1,12,27].

Let us conclude this introduction by briefly describing the content of our paper. Section 2 is devoted to presenting basic notation and preliminaries. We try to explain the differences and similarities between the NSES with periodic and Dirichlet boundary conditions which lead us to consider only the latter case. In particular, we prove some estimates concerning the nonlinearity B with respect to norm in different fractional domains of the Stokes operator A, see Propositions 2.3 and 2.4.

In Section 3 we discuss the skeleton equation and, in addition to recalling some fundamental and useful results (also from our previous work [3]), we also discuss their generalizations to the general case $\alpha > 0$, valid however only for the case of NSEs on a 2-D torus.

In Section 4 we introduce the action functional, for the large deviation principle in $C([0,T];\mathrm{H})$ associated with the family of solutions $\{u_\varepsilon\}_{\varepsilon>0}$ of equation (1.1), and the corresponding quasi-potential. We formulate generalizations of the corresponding results from [3] to the general case $\alpha > 0$, again, valid only for the case of NSEs on a 2-D torus. Moreover, we state our main result, i.e. Theorem 4.5, about the LDP for the family of invariant measures $\{\mu_\varepsilon\}_{\varepsilon>0}$ for the stochastic NSEs on a 2-D torus. The remainder of the paper is devoted to the proof of that result.

So, in Section 5 we formulate and prove Theorem 5.1 about exponential estimates for the family of probability measures $\{\mu_\varepsilon\}_{\varepsilon>0}$. This result is based on the uniqueness and ergodicity of each invariant measure μ_ε. The basic ingredient in this proof is also Lemma 5.3, about uniform exponential estimates for the solutions u_ε of equation (1.1). Our proof is a simplification (and clarification) of a proof of a more general result from [16]. However, we should note that another proof of such a result is possible, which is based on an earlier paper [8] by the first named author and Peszat, see [2]. We believe that it is possible to obtain results similar to ours for stochastic NSEs with multiplicative noise. However, such a study is postponed till another publication.

Let us continue with the description of the content of our paper. In Section 6 we continue with the proof of Theorem 4.5 and show that the invariant measures μ_ε satisfy an appropriate lower bounds, see Theorem 6.1. In inequality (6.1) we already see the relationship between the invariant measures μ_ε and the quasi-potential U.

Sections 7 and 8 are devoted to the formulation and proof of appropriate lower bounds satisfied by the invariant measures μ_ε, see Theorem 7.1.

The paper is concluded with an appendix. It is devoted to a proof of precise behavior for large negative time of solutions to the skeleton equation (1.2) on the negative half-line $(-\infty, 0]$. Such results should be of independent interest.

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2. Notation and preliminaries

Our main results are formulated for the stochastic Navier–Stokes equations with periodic boundary conditions. Hence we begin with a brief introduction to the relevant notation in this case; all the mathematical background can be found in the small book [23] by Temam. Here we will not recall the notation in the case of the Dirichlet boundary conditions but only refer the reader to our earlier paper [3]. Some of our results are true also in this case. Proper generalization to this case, as well to the case of multiplicative noise, will be a subject of a forthcoming publication.

We denote here by $\mathbb{T}^2$ the two dimensional torus of fixed dimensions $L \times L$. The space H is equal to

$$\mathrm{H} = \{u \in L^2_0(\mathbb{T}^2, \mathbb{R}^2) : \mathrm{div}(u) = 0 \text{ and } \gamma_\nu(u)_{|\Gamma_{j+2}} = -\gamma_\nu(u)_{|\Gamma_j},\ j = 1, 2\},$$

where $L^2_0(\mathbb{T}^2,\mathbb{R}^2)$ is the Hilbert space consisting of those $u \in L^2(\mathbb{T}^2,\mathbb{R}^2)$ which satisfy the condition

$$\int_{\mathbb{T}^2} u(x)\,dx = 0, \tag{2.1}$$

γ_ν is the bounded linear map defined on divergence free vectors in $L^2(\mathbb{T}^2,\mathbb{R}^2)$ with values in the dual space of $H^{\frac{1}{2}}(\partial\mathbb{T}^2)$ (the image in $L^2(\partial\mathbb{T}^2)$ of the trace operator $H^1(\mathbb{T}^2) \to L^2(\partial\mathbb{T}^2)$), such that $\gamma_\nu(u)$ coincides with the restriction of $u\cdot\nu$ to $\partial\mathbb{T}^2$, if $u \in \overline{D(\mathbb{T}^2)}$, and Γ_j, $j = 1,\cdots,4$ are the four (not disjoint) parts of the boundary of $\partial(\mathbb{T}^2)$ defined by, for $j = 1,2$,

$$\Gamma_j = \{x = (x_1,x_2) \in [0,L]^2 : x_j = 0\},\ \Gamma_{j+2} = \{x = (x_1,x_2) \in [0,L]^2 : x_j = L\}.$$

We also define the vorticity space V by setting

$$\mathrm{V} \quad = \quad \Big\{u \in \mathrm{H} : D_j u \in L^2(\mathbb{T}^2,\mathbb{R}^2),\ u_{|\Gamma_{j+2}} \circ \tau_j = u_{|\Gamma_j},\ j = 1,2\Big\}, \tag{2.2}$$

where D_j, $j = 1,2$, are the 1st order weak derivatives in the interior of the torus.

Because of condition (2.1), the norm on the space V induced by the norm from the Sobolev spaces $\mathbb{H}^{1,2}$ is equivalent to the following one

$$\big(u,v\big)_{\mathrm{V}} = \sum_{i,j=1}^{2} \int_{\mathcal{O}} \frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}\,dx,\ \ u,v \in \mathrm{V}.$$

The Stokes operator A can be defined in a natural way as

$$\begin{cases} D(\mathrm{A}) & = \mathrm{V} \cap H^{2,2}(\mathbb{T}^2,\mathbb{R}^2) \\ \mathrm{A}u & = -P\Delta,\ u \in D(\mathrm{A}), \end{cases} \tag{2.3}$$

where

$$\mathrm{P} : \mathbb{L}^2(\mathcal{O}) \to \mathrm{H}$$

is the orthogonal projection, called usually the Leray–Helmholtz projection.

It is well known that A is a self-adjoint positive operator in H. In fact, its eigenvectors and eigenvalue can be explicitly found. In particular, A has bounded imaginary powers and thus by [26, Remark 2 in 1.15.2], the domains of the fractional powers of A are equal (with equivalent norms) to the complex interpolation spaces between $D(\mathrm{A})$ and H, i.e.

$$D(\mathrm{A}^\theta) = [\mathrm{H}, D(\mathrm{A})]_\theta,\ \ \theta \in (0,1). \tag{2.4}$$

This means that

$$D(\mathrm{A}^\theta) = \mathrm{H} \cap H^{2\theta,2}(\mathbb{T}^2, \mathbb{R}^2). \tag{2.5}$$

Moreover, it is well known, see for instance [24, p. 57], that $\mathrm{V} = D(\mathrm{A}^{1/2})$.

It follows from the above that, contrary to the Dirichlet boundary conditions case, compare with [3, Proposition 2.1], the Leray–Helmholtz projection

$$P : H^{\alpha,2}(\mathbb{T}^2, \mathbb{R}^2) \to D(\mathrm{A}^\alpha) \tag{2.6}$$

is a bounded linear map for every $\alpha \geq 0$. To this purpose, in the bounded domain case.

The Stokes operator A satisfies all the properties known in the bounded domain case, inclusive the strict positivity property, with $\lambda_1 = \frac{4\pi^2}{L^2}$,

$$\langle \mathrm{A}u, u\rangle_\mathrm{H} \geq \lambda_1 |u^2|_\mathrm{H}, \quad u \in D(\mathrm{A}). \tag{2.7}$$

Now, consider the trilinear form b on $V \times V \times V$ given by

$$b(u,v,w) = \sum_{i,j=1}^{2} \int_{\mathcal{O}} u_i \frac{\partial v_j}{\partial x_i} w_j \, dx, \quad u,v,w \in \mathrm{V}.$$

It is known that b is a continuous trilinear form such that

$$b(u,v,w) = -b(u,w,v), \quad u \in \mathrm{V}, v, w \in \mathbb{H}^1_0(\mathcal{O}), \tag{2.8}$$

and, for some constant $c > 0$ (see for instance [25, Lemma 1.3, p. 163] and [24]),

$$|b(u,v,w)| \leq c \begin{cases} |u|_\mathrm{H}^{1/2} |\nabla u|_\mathrm{H}^{1/2} |\nabla v|_\mathrm{H}^{1/2} |\mathrm{A}v|_\mathrm{H}^{1/2} |w|_\mathrm{H} & u \in \mathrm{V}, v \in D(\mathrm{A}), w \in \mathrm{H} \\ |u|_\mathrm{H}^{1/2} |\mathrm{A}u|_\mathrm{H}^{1/2} |\nabla v|_\mathrm{H} |w|_\mathrm{H} & u \in D(\mathrm{A}), v \in \mathrm{V}, w \in \mathrm{H} \\ |u|_\mathrm{H} |\nabla v|_\mathrm{H} |w|_\mathrm{H}^{1/2} |\mathrm{A}w|_\mathrm{H}^{1/2} & u \in \mathrm{H}, v \in \mathrm{V}, w \in D(\mathrm{A}) \\ |u|_\mathrm{H}^{1/2} |\nabla u|_\mathrm{H}^{1/2} |\nabla v|_\mathrm{H} |w|_\mathrm{H}^{1/2} |\nabla w|_\mathrm{H}^{1/2} & u, v, w \in \mathrm{V}. \end{cases} \tag{2.9}$$

Next, define the bilinear map $B : \mathrm{V} \times \mathrm{V} \to \mathrm{V}'$, by setting

$$\langle B(u,v), w\rangle = b(u,v,w), \quad u,v,w \in \mathrm{V},$$

and the homogeneous polynomial of second degree $B : \mathrm{V} \to \mathrm{V}'$ by

$$B(u) = B(u,u), \ u \in \mathrm{V}.$$

From the first inequality in (2.9), we have that if $v \in D(\mathrm{A})$, then $B(u,v) \in H$ and the following inequality follows directly

$$|\mathrm{B}(u,v)|_\mathrm{H}^2 \leq C |u|_\mathrm{H} |\nabla u|_\mathrm{H} |\nabla v|_\mathrm{H} |\mathrm{A}v|_\mathrm{H}, \ u \in \mathrm{V}, \ v \in D(\mathrm{A}). \tag{2.10}$$

Moreover, the following identity is a direct consequence of (2.8).

$$\langle \mathrm{B}(u,v),v\rangle = 0, \quad u,v\in \mathrm{V}. \tag{2.11}$$

Furthermore, we have the following property involving the nonlinear term B and the Stokes operator A

$$\langle \mathrm{A}u, B(u,u)\rangle_{\mathrm{H}} = 0, \quad u\in D(\mathrm{A}), \tag{2.12}$$

see [23, Lemma 3.1] for a proof.

Let us also recall the following facts (see [6, Lemma 4.2] and [25]).

Lemma 2.1. *The trilinear map $b : \mathrm{V}\times\mathrm{V}\times\mathrm{V}\to\mathbb{R}$ has a unique extension to a bounded trilinear map from $\mathbb{L}^4(\mathcal{O})\times(\mathbb{L}^4(\mathcal{O})\cap\mathrm{H})\times\mathrm{V}$ and from $\mathbb{L}^4(\mathcal{O})\times\mathrm{V}\times\mathbb{L}^4(\mathcal{O})$ into $\mathbb{R}$. Moreover, B maps $\mathbb{L}^4(\mathcal{O})\cap\mathrm{H}$ (and so V) into V' and*

$$|B(u)|_{\mathrm{V}'} \le C_1|u|^2_{\mathbb{L}^4(\mathcal{O})} \le 2^{1/2}C_1|u|_{\mathrm{H}}|\nabla u|_{\mathbb{L}^2(\mathcal{O})} \le C_2|u|_{\mathrm{H}}|u|_{\mathrm{V}} \le C_3|u|^2_{\mathrm{V}}, \quad u\in\mathrm{V}. \tag{2.13}$$

Lemma 2.2. *For any $T\in(0,\infty]$ and for any $u\in L^2(0,T;D(\mathrm{A}))$ with $u'\in L^2(0,T;\mathrm{H})$, we have*

$$\int_0^T |\mathrm{B}(u(t),u(t))|^2_{\mathrm{H}}\,dt < \infty.$$

The restriction of the map B to the space $D(\mathrm{A})\times D(\mathrm{A})$ has also the following representation

$$\mathrm{B}(u,v) = P(u\nabla v) = P(\sum_{j=1}^2 u^j D_j v), \quad u,v\in D(\mathrm{A}). \tag{2.14}$$

In view of (2.6), the above representation allows us to prove the following property of the map B (compare with a weaker result in [3, Proposition 2.5] for the Dirichlet boundary case).

Proposition 2.3. *Assume that $\alpha\in(0,1]$. Then for any $s\in(1,2]$ there exists a constant $c>0$ such that*

$$|\mathrm{B}(u,v)|_{D(\mathrm{A}^{\frac{\alpha}{2}})} \le c|u|_{D(\mathrm{A}^{\frac{s}{2}})}|v|_{D(\mathrm{A}^{\frac{\alpha+1}{2}})}, \quad u,v\in D(\mathrm{A}). \tag{2.15}$$

Proof. In view of equality (2.14), since by (2.6) the Leray–Helmholtz projection P is a well defined and continuous map from $\mathbb{H}^\alpha(\mathcal{O})$ into $D(\mathrm{A}^{\frac{\alpha}{2}})$ and since the norms in the spaces $D(\mathrm{A}^{\frac{s}{2}})$ are equivalent to norms in $\mathbb{H}^s(\mathcal{O})$, it is enough to show that

$$|u\nabla v|_{\mathbb{H}^\alpha} \leq c|u|_{\mathbb{H}^s}|v|_{\mathbb{H}^{\alpha+1}}, \quad u,v \in \mathbb{H}^2(\mathcal{O}).$$

Thus, it is sufficient to prove for scalar valued functions

$$|fg|_{H^\alpha} \leq c|f|_{H^s}|g|_{H^\alpha}, \quad f,g \in H^2. \tag{2.16}$$

First we consider the case $\alpha = 0$. In this case it is sufficient to assume that $s \in (1,2)$ and we have

$$|fg|_{L^2} \leq |f|_{L^\infty}|g|_{L^2} \leq |f|_{H^s}|g|_{L^2}$$

by the Gagliado–Nirenberg inequality, which implies that $H^s \hookrightarrow L^\infty$ continuously.

Secondly, we consider the case $\alpha = 1$. Also in this case it is sufficient to assume that $s \in (1,2)$. Then, by the Sobolev Gagliado–Nirenberg inequalities, we have

$$\begin{aligned}
|\nabla(fg)|_{L^2} &\leq |g\nabla f|_{L^2} + |f\nabla g|_{L^2} \\
&\leq |\nabla f|_{L^p}|g|_{L^q} + |f|_{L^\infty}|\nabla g|_{L^2} \\
&\leq |\nabla f|_{L^p}|g|_{H^1} + |f|_{H^s}|\nabla g|_{L^2} \\
&\leq |f|_{H^s}|g|_{H^1} + |f|_{H^s}|\nabla g|_{L^2}
\end{aligned}$$

where $p,q \in (2,\infty)$ are such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ and $\frac{1}{2} = \frac{1}{p} + \frac{s-1}{2}$, i.e. $\frac{1}{p} = 1 - \frac{s}{2}$. From the above two inequalities we trivially deduce that

$$|fg|_{H^1} \leq \sqrt{6}|f|_{H^s}|g|_{H^1}$$

what proves inequality (2.16) for $\alpha = 1$.

Finally, let us consider the case $\alpha \in (0,1)$. By a complex interpolation argument and the Marcinkiewicz Interpolation Theorem, we infer that for any $\alpha \in (0,1)$

$$|fg|_{H^\alpha} \leq 6^{\frac{\alpha}{2}}|f|_{H^s}|g|_{H^\alpha},$$

so that the proof of Proposition 2.3 is complete. □

Since the Sobolev space H^α is an algebra for $\alpha > 1$, we have the following result.

Proposition 2.4. *Assume that $\alpha \in (1,\infty)$. Then there exists a constant $c > 0$ such that*

$$|\mathrm{B}(u,v)|_{D(\mathrm{A}^{\frac{\alpha}{2}})} \leq c|u|_{D(\mathrm{A}^{\frac{\alpha}{2}})}|v|_{D(\mathrm{A}^{\frac{\alpha+1}{2}})}, \quad u \in D(\mathrm{A}^{\frac{\alpha}{2}}), v \in D(\mathrm{A}^{\frac{1+\alpha}{2}}). \tag{2.17}$$

Proof. Let us fix $\alpha \in (1, \infty)$. In view of equality (2.14), as in the proof of Proposition 2.3 it is enough to show that

$$|u\nabla v|_{\mathbb{H}^\alpha} \leq c|u|_{\mathbb{H}^\alpha}|v|_{\mathbb{H}^{\alpha+1}}, \quad u \in \mathbb{H}^\alpha(\mathcal{O}), \; v \in \mathbb{H}^{\alpha+1}(\mathcal{O}).$$

Since the Sobolev space H^α is an algebra and $|\nabla v|_{\mathbb{H}^\alpha} \leq c|v|_{\mathbb{H}^{\alpha+1}}$, the result follows. □

Remark 2.5. One consequence of Propositions 2.3 and 2.4 is that the 2-D NSEs with periodic boundary conditions are locally well posed in the space $D(\mathrm{A}^{\frac{\beta}{2}})$, for every $\beta \geq 0$. To be precise for every $u_0 \in D(\mathrm{A}^{\frac{\beta}{2}})$ and every $f \in L^2_{\mathrm{loc}}([0,\infty); D(\mathrm{A}^{\frac{\beta}{2}-\frac{1}{2}})$ there exists $T > 0$ and a strong solution u defined on $[0, T]$. This local existence result is well known for $\beta \in \{0, 1\}$ for 2-D NSEs with either Dirichlet or periodic boundary conditions. Moreover, it is rather a folk result for $\beta \in (0, 1)$. However, in [3] we proved it to be true also for $\beta \in (1, \frac{3}{2})$. The difference between the NSEs with general boundary conditions and NSEs on a torus stems from the fact that, while Propositions 2.3 and 2.4 hold in the latter case for any $\alpha > 0$, we have been able to establish a corresponding result in the former case for only $\alpha \in [0, \frac{1}{2})$. And the root for this difference lies in the properties of the Leray–Helmholtz projection P. Actually, while in the latter case, it is a bounded linear map from $H^{\alpha,2}(\mathbb{T}^2, \mathbb{R}^2)$ into $D(\mathrm{A}^\alpha)$, in the former case we have been able to prove an analogous result only for $\alpha \in (0, \frac{1}{2})$. As in [3], by using the global well-posedness in H, local well-posedness in $D(\mathrm{A}^{\frac{\beta}{2}})$ implies also global well-posedness. See Proposition 3.3 for precise formulations of these results.

Remark 2.6. Similar inequalities to those in Propositions 2.3 and 2.4 have also been studied in [4].

3. The skeleton equation

We consider here the following Navier–Stokes equation

$$\begin{cases} u'(t) + \mathrm{A}u(t) + \mathrm{B}(u(t), u(t)) = f(t), & t \in (a, b), \\ u(a) = u_0, \end{cases} \tag{3.1}$$

where $-\infty < a < b < \infty$.

Definition 3.1. Given any $f \in L^2(a, b; \mathrm{V}')$ and $u_0 \in \mathrm{H}$, a solution to problem (3.1) is a function $u \in L^2(a, b; \mathrm{V})$ such that $u' \in L^2(a, b; \mathrm{V}')$, $u(a) = u_0$ [2] and (3.1) is satisfied.

As shown in [25, Theorems III.3.1/2], for every $f \in L^2(a, b; \mathrm{V}')$ and $u_0 \in \mathrm{H}$ there exists exactly one solution u to problem (3.1).

[2] It is known (see for instance [25, Lemma III.1.2]) that these two properties of u imply that u is almost everywhere equal to a function $\bar{u} \in C([a, b], \mathrm{H})$. Thus, when we later write $u(a)$ we mean $\bar{u}(0)$.

Lemma 3.2. *For any $r > 0$ there exists $c_r > 0$ such that, if $T \in \mathbb{R}$ and $u \in C([T, +\infty); \mathrm{H})$ satisfies $f := u' + \mathrm{A}u + \mathrm{B}(u, u) \in L^2(T, +\infty; \mathrm{H})$, then*

$$|f|_{L^2(T,+\infty;\mathrm{H})} \leq r \Longrightarrow |u|_{C([T,+\infty);\mathrm{H})} \leq c_r + |u(T)|_{\mathrm{H}}. \tag{3.2}$$

More precisely, the following inequality holds:

$$|u(t)|_{\mathrm{H}} \leq \mathrm{e}^{-\lambda_1(t-T)}|u(T)|_{\mathrm{H}} + \frac{1}{\lambda_1}|f|^2_{L^2(T,+\infty;\mathrm{H})}, \qquad t \geq T. \tag{3.3}$$

Proof. By [25, Lemma III.1.2] and inequality (2.7) we have

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2_{\mathrm{H}} + |u(t)|^2_{\mathrm{V}} \leq \frac{1}{2}|u(t)|^2_{\mathrm{V}} + \frac{1}{2\lambda_1}|f(t)|^2_{\mathrm{H}}. \tag{3.4}$$

This implies that for all $b > a \geq T$,

$$|u|^2_{C([a,b);\mathrm{H})} + |u|^2_{L^2(a,b;\mathrm{V})} \leq |u(a)|^2_{\mathrm{H}} + \frac{1}{\lambda_1}|f|^2_{L^2(a,b;\mathrm{H})}, \tag{3.5}$$

what yields (3.2). Moreover, from (3.4) in view of inequality (2.7) we have

$$\frac{d}{dt}|u(t)|^2_{\mathrm{H}} + \lambda_1\,|u(t)|^2_{\mathrm{H}} \leq \frac{1}{\lambda_1}|f(t)|^2_{\mathrm{H}}, \quad t \geq T.$$

Hence, by the Gronwall lemma, for any $T \leq a \leq t < +\infty$ we have

$$|u(t)|^2_{\mathrm{H}} \leq |u(a)|^2_{\mathrm{H}}\mathrm{e}^{-\lambda_1(t-a)} + \frac{1}{\lambda_1}\int_a^t \mathrm{e}^{-\lambda_1(t-s)}|f(s)|^2_{\mathrm{H}}\,ds,$$

what implies (3.3). □

In [25, Theorem III.3.10] it is proven that, if $f \in L^2(a, b; \mathrm{H})$, the solution u of equation (3.1) has the following properties

$$\sqrt{(\cdot - a)}\,u \in L^2(a, b; D(\mathrm{A})) \cap L^\infty(a, b; \mathrm{V}), \quad \sqrt{(\cdot - a)}\,u' \in L^2(a, b; \mathrm{H}).$$

Moreover, there exists $c > 0$ such that for all $a < b$

$$\begin{aligned} &|\sqrt{(\cdot - a)}\,u|^2_{L^\infty(a,b;\mathrm{V})} + |\sqrt{(\cdot - a)}\,u|^2_{L^2(a,b;D(\mathrm{A}))} \\ &\quad \leq c\,\exp\left[c\big(|u_0|^4_{\mathrm{H}} + |f|^4_{L^2(a,b;\mathrm{V}')}\big)\right]\Big(|u_0|^2_{\mathrm{H}} + |f|^2_{L^2(a,b;\mathrm{V}')} + |b-a||f|^2_{L^2(a,b;\mathrm{H})}\Big). \end{aligned} \tag{3.6}$$

In [3, Proposition 3.3] we have also proved the following result for $\alpha \in (0, \frac{1}{2})$ for 2-D NSEs with both Dirichlet and periodic boundary conditions. It turns out that in the latter case it is true for any $\alpha \geq 0$.

Proposition 3.3. *Suppose that $\alpha \geq 0$. If $f \in L^2(a,b;D(\mathrm{A}^{\frac{\alpha}{2}})$ and $u_0 \in D(\mathrm{A}^{\frac{\alpha+1}{2}})$, for some $\alpha \geq 0$, then the unique solution u to the problem (3.1) satisfies*

$$u \in L^2(a,b;D(\mathrm{A}^{1+\frac{\alpha}{2}}) \cap C([a,b];D(\mathrm{A}^{\frac{\alpha+1}{2}})), \quad u'(\cdot) \in L^2(a,b;D(\mathrm{A}^{\frac{\alpha}{2}})). \tag{3.7}$$

Proof. As discussed in Remark 2.5, the above result follows from Propositions 2.3 and 2.4. The proof of the above result can be accomplished by following line by line the proof of [3, Proposition 3.3], which worked for both types of boundary conditions but only for $\alpha \in (0, \frac{1}{2})$. □

Now, for any $-\infty \leq a < b \leq \infty$ and for any two reflexive Banach spaces X and Y, such that $X \hookrightarrow Y$ continuously, we denote by $W^{1,2}(a,b;X,Y)$ the space of all $u \in L^2(a,b;X)$ which are weakly differentiable as Y-valued functions and their weak derivative belongs to $L^2(a,b;Y)$. The space $W^{1,2}(a,b;X,Y)$ is a separable Banach space (and Hilbert if both X and Y are Hilbert spaces), endowed with the natural norm

$$|u|^2_{W^{1,2}(a,b;X,Y)} = |u|^2_{L^2(a,b;X)} + |u'|^2_{L^2(a,b;Y)}, \quad u \in W^{1,2}(a,b;X,Y).$$

Later on, when no ambiguity is possible, we will use the shortcut notation

$$W^{1,2}(a,b) = W^{1,2}(a,b;D(\mathrm{A}),\mathrm{H}).$$

The following definition has first appeared in the paper [19] by Lions and Masmoudi as a natural tool in the investigation of the uniqueness questions for Navier–Stokes Equations in the Lebesgue spaces L^d.

Definition 3.4. Assume that $-\infty \leq a < b \leq \infty$ and $f \in L^2_{\mathrm{loc}}(a,b;\mathrm{H})$. A function $u \in C((a,b);\mathrm{H})$ is called a *very weak solution* to the Navier–Stokes equations (3.1) on the interval (a,b) if for all $\phi \in C^\infty((a,b) \times D)$, such that $\mathrm{div}\phi = 0$ on $(a,b) \times D$,

$$\begin{aligned}
\int_D u(t_1,\xi)\phi(t_1,\xi)d\xi &= \int_D u(t_0,\xi)\phi(t_0,\xi)d\xi \\
&+ \int_{[t_0,t_1]\times D} u(s,\xi)(\partial_s\phi(s,\xi) + \Delta\phi(s,\xi))\, dsd\xi \\
&+ \int_{t_0}^{t_1} b(u(s),u(s),\phi(s))\, ds + \int_{[t_0,t_1]\times D} f(s,\xi)\cdot\phi(s,\xi)\, dsd\xi,
\end{aligned} \tag{3.8}$$

for all $a < t_0 < t_1 < b$.

One can observe that a solution to the Navier–Stokes equations (3.1) on the interval (a,b) is also a *very weak solution* to (3.1) on the interval (a,b). We need a notion of a *very weak solution* because a basic object in our study of large deviations is the space $\mathcal{X}$, see definition in equality (4.10), which consists only of H-valued continuous functions. It has been used in Proposition 3.6 which consequently was used in the proofs of Lemmata 3.8 and 3.9 in our previous paper [3].

By adapting some of the results from [19] to the 2-dimensional case, it is possible to prove the following result.

Proposition 3.5. *Assume that $-\infty \leq a < b \leq \infty$ and $f \in L^2_{loc}((a,b);\mathrm{H})$. Suppose that the functions $u, v \in C((a,b);\mathrm{H})$ are very weak solutions to the Navier–Stokes equations (3.1) on the interval (a,b), with $u(t_0) = v(t_0)$, for some $t_0 \in (a,b)$. Then $u(t) = v(t)$ for all $t \geq t_0$.*

Definition 3.6. Assume that $-\infty \leq a < b \leq \infty$. Given a function $u \in C((a,b);\mathrm{H})$ we say that

$$u' + \mathrm{A}u + \mathrm{B}(u,u) \in L^2(a,b;\mathrm{H}), \quad (\text{resp. } \in L^2_{\mathrm{loc}}((a,b);\mathrm{H}))$$

if there exists $f \in L^2(a,b;\mathrm{H})$, (resp. $f \in L^2_{\mathrm{loc}}((a,b);\mathrm{H})$) such that u is a very weak solution of the Navier–Stokes equations (3.1) on the interval (a,b).

Obviously, the corresponding function f is unique and we will denote it by $\mathcal{H}(u)$, i.e.

$$[\mathcal{H}(u)](t) := u'(t) + \mathrm{A}u(t) + \mathrm{B}(u(t),u(t)), \ \ t \in (a,b). \tag{3.9}$$

In [3, Proposition 10.2] we have proved the following result.

Lemma 3.7. *Assume that $\alpha \in (0, \frac{1}{2})$. Assume that $u \in C((-\infty,0];\mathrm{H})$ is such that $\mathcal{H}(u) := u' + \mathrm{A}u + \mathrm{B}(u,u) \in L^2(-\infty,0;D(\mathrm{A}^{\frac{\alpha}{2}}))$ and there exists $\{t_n\} \downarrow -\infty$, such that*

$$\lim_{n\to\infty} |u(t_n)|_{\mathrm{H}} = 0.$$

Then $u \in W^{1,2}(-\infty,0;D(\mathrm{A}^{1+\frac{\alpha}{2}}),D(\mathrm{A}^{\frac{\alpha}{2}}))$, $u(0) \in D(\mathrm{A}^{\frac{\alpha+1}{2}})$ and

$$\lim_{t\to-\infty} |u(t)|_{D(\mathrm{A}^{\frac{\alpha}{2}+\frac{1}{2}})} = 0.$$

In Appendix A, we generalize the above result, again only in the case of NSEs on a torus, to the case of any $\alpha > 0$.

4. LDP for stochastic NSEs on a 2-D torus

For any fixed $\varepsilon \in (0,1]$ and $x \in \mathrm{H}$, we consider the problem

$$du(t) + \mathrm{A}u(t) + \mathrm{B}(u(t),u(t)) = \sqrt{\varepsilon}\, dw(t), \qquad u(0) = x. \tag{4.1}$$

Here $w = \{w(t)\}_{t\geq 0}$ is an H-valued Wiener process with reproducing kernel Hilbert space denoted by K. In particular $\mathrm{K} \subset \mathrm{H}$ and the natural embedding $\mathrm{i} : \mathrm{K} \hookrightarrow \mathrm{H}$ is a

Hilbert–Schmidt operator. Let us fix an orthonormal basis $\{f_k\}_{k\in\mathbb{N}}$ of K and a sequence $\{\beta_k\}_{k=1}^{\infty}$ of independent Brownian motions defined on some filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$, where $\mathbb{F}=\{\mathcal{F}_t\}_{t\geq 0}$, such that the Wiener process w has the following representation

$$w(t)=\sum_{k=1}^{\infty}\beta_k(t)f_k,\qquad t\geq 0. \tag{4.2}$$

With the Wiener process w, one can associate a covariance operator $C\in\mathcal{L}(\mathrm{H})$ (usually denoted by Q), defined by

$$\langle Ch_1,h_2\rangle=\mathbb{E}\big[\langle h_1,w(1)\rangle_{\mathrm{H}}\,\langle w(1),h_2\rangle_{\mathrm{H}}\big],\quad h_1,h_2\in\mathrm{H}.$$

It is well known, see e.g. [10, Proposition 2.15], that C is a non-negative self-adjoint and trace class operator in H. Moreover, see for instance [5], $C=\mathrm{ii}^*$ and $\mathrm{K}=R(C^{\frac12})$. In this paper we **assume** that for some $\alpha>1$, the operator $Q:=C^{\frac12}$ is an isomorphism of H onto $D(\mathrm{A}^{\frac{\alpha}{2}})$, i.e. when $\mathrm{K}=D(\mathrm{A}^{\frac{\alpha}{2}})$.

Note that

$$\sum_{k=1}^{\infty}|\mathrm{i}f_k|_{\mathrm{H}}^2=\mathrm{tr}\,[C]<\infty.$$

It is now well known, see e.g. in [15], that for all $\varepsilon\in(0,1]$ and $x\in\mathrm{H}$, equation (4.1) has a unique solution u^x such that $u^x_\varepsilon\in L^p(\Omega;C([0,T];\mathrm{H}))$, for all $T>0$ and $p\geq 1$. Moreover, there exists an invariant measure ν_ε for the Markov process generated by this equation, i.e. a Borel probability measure on H such that for every $\varphi\in B_b(\mathrm{H})$, and every $t\geq 0$,

$$\int_{\mathrm{H}}\mathbb{E}\,\varphi(u^x_\varepsilon(t))\,\nu_\varepsilon(dx)=\int_{\mathrm{H}}\varphi(x)\,\nu_\varepsilon(dx), \tag{4.3}$$

where $B_b(\mathrm{H})$ denotes the set of bounded and continuous functions $\varphi:\mathrm{H}\to\mathbb{R}$. It is also known, see [13], that under our assumption, this invariant measure ν_ε is unique and ergodic. Thus is particular, for any bounded Borel measurable function $\varphi:\mathrm{H}\to\mathbb{R}$,

$$\int_{\mathrm{H}}\varphi(x)\,\nu_\varepsilon(dx)=\lim_{T\to\infty}\frac{1}{T}\int_0^T\mathbb{E}\,\varphi(u^0_\varepsilon(t))\,dt. \tag{4.4}$$

For a function[3] $u\in C([a,b];\mathrm{H})$, where $-\infty\leq a<b\leq+\infty$, such that

$$\mathcal{H}(u):=u'+\mathrm{A}u+\mathrm{B}(u,u)\in L^2_{\mathrm{loc}}((a,b);D(A^{\frac{\alpha}{2}})),$$

[3] If for instance $a=-\infty$), we assume that $u\in C((a,b];\mathrm{H})$.

we define the action functionals by

$$S_{t_0,t_1}(u) := \frac{1}{2}\int_{t_0}^{t_1} |Q^{-1}\mathcal{H}(u)(t)|_{\mathrm{H}}^2\,dt, \quad a \le t_0 < t_1 \le b. \tag{4.5}$$

Note that since Q is a bounded operator in H, we have the following useful inequality

$$\int_{t_0}^{t_1} |\mathcal{H}(u)(t)|_{\mathrm{H}}^2\,dt \le 2\|Q\|^2_{\mathcal{L}(\mathrm{H},\mathrm{H})} S_{t_0,t_1}(u). \tag{4.6}$$

If $\mathcal{H}(u) \notin L^2((t_0,t_1); D(A^{\frac{\alpha}{2}}))$, we put $S_{t_0,t_1}(u) = +\infty$. Moreover, we denote

$$S_{-T} := S_{-T,0}, \qquad S_T := S_{0,T}, \quad \text{for every } T > 0.$$

In particular, when $a = -\infty$ and $b = 0$, we set

$$S_{-\infty}(u) := \frac{1}{2}\int_{-\infty}^{0} |Q^{-1}\mathcal{H}(u)(t)|_{\mathrm{H}}^2\,dt. \tag{4.7}$$

An obvious sufficient condition for the finiteness of $S_{t_0,t_1}(u)$ is that u', Au and $B(u,u)$ all belong to $L^2(t_0,t_1; D(A^{\frac{\alpha}{2}})$. In fact, as we proved in [3, Lemma 3.9], in the case of 2-D NSEs with both Dirichlet and periodic boundary conditions, when $\alpha \in (0,\frac{1}{2})$, this is not so far from being a necessary condition. As earlier for Proposition 3.3, it turns out that in the latter case, [3, Lemma 3.9] holds true for any $\alpha \ge 0$.

Lemma 4.1. *Suppose that $\alpha \ge 0$ and $-\infty < a < b < \infty$.*

If a function $u \in C([a,b];\mathrm{H})$ satisfies

$$u' + \mathrm{A}u + \mathrm{B}(u,u) \in L^2(a,b; D(\mathrm{A}^{\frac{\alpha}{2}})),$$

then $u(b) \in D(\mathrm{A}^{\frac{\alpha+1}{2}})$ and $u \in W^{1,2}\big(t_0,b; D(\mathrm{A}^{\frac{\alpha}{2}+1}), D(\mathrm{A}^{\frac{\alpha}{2}})\big)$, for any $t_0 \in (a,b)$.

Moreover, if $u(a) \in D(\mathrm{A}^{\frac{\alpha+1}{2}})$, then $u \in W^{1,2}\big(a,b; D(\mathrm{A}^{\frac{\alpha}{2}+1}), D(\mathrm{A}^{\frac{\alpha}{2}})\big)$.

Proof. As discussed in Remark 2.5, the above result follows from Propositions 2.3 and 2.4. The proof of the above result can be accomplished by following the line of proof of [3, Lemma 3.9] which worked for both types of boundary conditions but only for $\alpha \in (0,\frac{1}{2})$. $\square$

As a consequence of the contraction principle and of certain continuity properties of the solution of equation (4.1) proven in [6], we infer the following result, see [3, Theorem 5.3].

Theorem 4.2. *For any $x \in$ H, the family $\{\mathcal{L}(u^x_\varepsilon)\}_{\varepsilon\in(0,1]}$ satisfies the large deviation principle in $C([0,T];\mathrm{H})$, with speed ε and action functional S_T, uniformly with respect to x in bounded subsets of* H.

We recall here (see e.g. [14]) that a family of probability measures $\{\mu_\varepsilon\}_{\varepsilon>0}$ on some complete metric space E satisfies a large deviation principle, with speed $\{\beta_\varepsilon\}_{\varepsilon>0}$ such that $\lim_{\varepsilon\searrow 0}\beta(\varepsilon) = 0$, and action functional I, iff $I : E \to [0,\infty]$ is a lower-semicontinuous[4] map such that

(1) For each $r > 0$, the level set

$$I_r := \{x \in E \ : \ I(x) \le r\},$$

is compact in E.

(2) *Lower bounds:* For every $\bar{x} \in E$ and for all $\delta,\gamma > 0$ there exists $\varepsilon_0 > 0$ such that[5]

$$\mu_\varepsilon\left(B_E(\bar{x},\delta)\right) \ge \exp\left(-\frac{I(\bar{x})+\gamma}{\beta_\varepsilon}\right), \qquad \varepsilon \le \varepsilon_0. \tag{4.8}$$

Here $B_E(\bar{x},\delta) = \{x \in E : |x-\bar{x}|_E < \delta\}$.

(3) *Upper bounds:* For every s,δ and $\gamma \in (0,s)$ there exists $\varepsilon_0 > 0$ such that

$$\mu_\varepsilon\left(\{x \in E : \mathrm{dist}_E\,(x, I_s) > \delta\}\right) \le \mathrm{e}^{-\frac{s-\gamma}{\beta_\varepsilon}}, \qquad \varepsilon \le \varepsilon_0. \tag{4.9}$$

Next, for any $x, y \in H$ and $a, b \in \mathbb{R}$, we introduce the following functional spaces

$$\begin{aligned}
&\mathcal{X} = \big\{u \in C((-\infty,0];\mathrm{H}) : \lim_{t\to-\infty}|u(t)|_\mathrm{H} = 0\big\},\\
&\mathcal{X}_x = \big\{u \in \mathcal{X} : u(0) = x\big\},\\
&C_{x,y}([a,b],\mathrm{H}) = \{u \in C([a,b];\mathrm{H}) : u(a) = x,\ u(b) = y\}.
\end{aligned} \tag{4.10}$$

We endow the space $\mathcal{X}$ with the topology of uniform convergence on compact intervals, i.e. the topology induced by the metric ρ defined by

$$\rho(u,v) := \sum_{n=1}^\infty 2^{-n}\left(\sup_{s\in[-n,0]}|u(s)-v(s)|_\mathrm{H} \wedge 1\right), \quad u, v \in \mathcal{X}.$$

The set $\mathcal{X}_x$ is a closed in $\mathcal{X}$ and we endow it with the trace topology induced by $\mathcal{X}$.

In [3, Propositions 5.4 and 5.5], we proved that the functional

$$S_{-\infty} : \mathcal{X} \to [0,+\infty], \tag{4.11}$$

[4] This condition is redundant if we also assume condition (1) below.

[5] Inequality (4.8) is trivially satisfied when $I(\bar{x}) = \infty$.

is lower-semicontinuous and has compact level sets. This result obviously holds for both types of boundary conditions, but only for $\alpha \in [0, \frac{1}{2})$. Its proof relied on [3, Propositions 10.1 and 10.2] which we generalize in Appendix A. Let us state it for the completeness sake.

Proposition 4.3. *Assume that $\alpha \geq 0$. Then the functional $S_{-\infty}$ defined by (4.7), is lower-semicontinuous on $\mathcal{X}$. Moreover, its level sets are compact in $\mathcal{X}$*

Next, we define the quasi-potential U associated with equation (4.1), by setting

$$\begin{aligned} \mathrm{U}(x) \quad &:= \quad \inf\left\{S_T(u) : T > 0,\, u \in C_{0,x}([0,T];\mathrm{H})\right\} \\ &= \quad \inf\left\{S_{-T}(u) : T > 0,\, u \in C_{0,x}([-T,0];\mathrm{H})\right\}, \quad x \in \mathrm{H}. \end{aligned} \tag{4.12}$$

In our previous paper [3], we thoroughly studied the functional U for the 2-D NSEs, for both Dirichelt and periodic boundary conditions, and we have shown that it satisfies the properties described below for $\alpha \in (0, \frac{1}{2})$. The following result generalizes [3, Theorem 6.2 and Propositions 6.1, 6.5 and 6.6] to $\alpha \geq 0$ but only, as all our generalizations, for the case of the 2-D NSEs on a torus.

Theorem 4.4. *Assume that $\alpha \geq 0$ and $x \in \mathrm{H}$. Then the following hold true.*

(1) $\mathrm{U}(x) < \infty \Longleftrightarrow x \in D(\mathrm{A}^{\frac{\alpha+1}{2}})$ *and*

$$\mathrm{U}(x) := \inf\left\{S_{-\infty}(u) : u \in \mathcal{X}_x\right\}. \tag{4.13}$$

The restriction of the map U *to the set $D(A^{\frac{\alpha+1}{2}})$, i.e. the map*

$$\mathrm{U} : D(A^{\frac{1+\alpha}{2}}) \to \mathbb{R}$$

is continuous.

(2) *For any $r > 0$, the level set*

$$K_r = \{x \in \mathrm{H} : \mathrm{U}(x) \leq r\} \tag{4.14}$$

is compact in H.

In particular, the function $\mathrm{U} : \mathrm{H} \to [0, \infty]$ *is lower semi-continuous.*

Proof. The proofs of the above results follow the proofs of [3, Theorem 6.2 and Propositions 6.1, 6.5 and 6.6], while taking into account Propositions 2.3 and 2.4. □

As mentioned in the introduction, in the present paper we want to prove the following theorem.

Theorem 4.5. *The family $\{\nu_\varepsilon\}_{\varepsilon>0}$ of the invariant measures for equation (4.1) satisfies a large deviation principle in* H*, with speed $\beta_\varepsilon = \varepsilon$ and action functional* U *(defined in formula (4.12)).*

In Theorem 4.4, we have seen that U has compact level sets. Thus, in order to prove Theorem 4.5, in what follows we have only to prove the validity of the lower and upper bounds.

5. Exponential estimates

In the proof of lower bounds for the large deviation principle we need to prove that there exists some $\bar{R} > 0$ such that

$$\lim_{\varepsilon\to 0} \nu_\varepsilon\big(B^{\mathrm{c}}_{\mathrm{H}}(0, \bar{R})\big) = 0. \tag{5.1}$$

On the other hand, in the proof of upper bounds we need something stronger. Actually we need that the convergence to zero in (5.1) is exponential.

Theorem 5.1. *For any $s > 0$, there exist $\epsilon_s > 0$ and $R_s > 0$ such that*

$$\nu_\varepsilon\big(B^{\mathrm{c}}_{\mathrm{H}}(0, R_s)\big) \leq \exp\left(-\frac{s}{\varepsilon}\right), \qquad \varepsilon \leq \varepsilon_s. \tag{5.2}$$

This fundamental result will be used in the proof of Theorem 7.1. Let us note that the proof uses the ergodicity of the invariant measure.

Remark 5.2. An essential part of the proof of the above result is given by the following exponential estimates. Their proof can in fact be traced to the paper [8], but we present here an independent one based on the use of a suitable Lyapunov function. This proof goes back to the paper [16], but that paper tried to treat so many cases simultaneously that we decided to write down an independent statement and proof.

Lemma 5.3. *In the framework of Theorem 5.1, for any arbitrary $\varepsilon > 0$ there exists $\gamma > 0$ such that*

$$\mathbb{E}\, \mathrm{e}^{\frac{\gamma}{\varepsilon}|u^x_\varepsilon(t)|^2_{\mathrm{H}}} \leq \mathrm{e}^{-\frac{\lambda_1}{2}t}\mathrm{e}^{\frac{\gamma}{\varepsilon}|x|^2_{\mathrm{H}}} + 2, \quad t > 0.$$

Remark 5.4. The result from Theorem 5.1 is also true for the stochastic Navier–Stokes equations with multiplicative noise

$$du(t) + \mathrm{A}u(t) + \mathrm{B}(u(t), u(t)) = \sqrt{\varepsilon} g(u)\, dw(t), \qquad u(0) = u_0, \tag{5.3}$$

where $w(t)$ is a cylindrical Wiener process on some separable Hilbert space K, provided the map $g : \mathrm{H} \to R(\mathrm{K}, \mathrm{H})$ is a continuous and bounded and there exists unique ergodic invariant measure ν_ε of the corresponding Markov process.

The result from Lemma 5.3 is also true for the stochastic Navier–Stokes equations with multiplicative noise (5.3) provided w is a cylindrical Wiener process on some separable Hilbert space K and a continuous and bounded map $g : \mathrm{H} \to R(\mathrm{K}, \mathrm{H})$.

Proof of Lemma 5.3. Let us fix the initial data $x \in \mathrm{H}$. Take arbitrary $\varepsilon, \lambda, \gamma > 0$, that will be specified later on. Let us denote the solution u_ε^x by u. Let us recall the Itô's formula due to Pardoux [21] applied to a $C^{1,2}$-class function $\varphi : \mathbb{R}_+ \times \mathrm{H} \to \mathbb{R}$ and the process u:

$$d\varphi(t, u(t)) = \frac{\partial \varphi(t, u(t))}{\partial t}\, dt + \langle \frac{\partial \varphi(t, u(t))}{\partial u}, du(t)\rangle + \frac{\varepsilon}{2} \sum_k \langle \frac{\partial^2 \varphi(t, u(t))}{\partial u^2}(\mathrm{i} f_k), \mathrm{i} f_k\rangle.$$

We apply this formula to the following function

$$\varphi : \mathbb{R}_+ \times \mathrm{H} \ni (t, u) \mapsto \mathrm{e}^{\lambda t}\, \mathrm{e}^{\frac{\gamma}{\varepsilon}|u|^2_{\mathrm{H}}} \in \mathbb{R}.$$

Since $\frac{\partial \varphi(t,u)}{\partial t} = \lambda \varphi(t, u)$, $\frac{\partial \varphi(t,u)}{\partial u} = \frac{2\gamma}{\varepsilon} \varphi(t, u)u \in \mathrm{H}$ and, for $y, z \in \mathrm{H}$, $\langle \frac{\partial^2 \varphi(t,u)}{\partial u^2} y, z\rangle = \varphi(t, u)\big[\frac{4\gamma^2}{\varepsilon^2}\langle u, y\rangle\langle u, z\rangle + \frac{2\gamma}{\varepsilon}\langle y, z\rangle\big]$, by (2.11) and (2.7), we infer that

$$\begin{aligned}
\langle \frac{\partial \varphi(t, u(t))}{\partial u}, du(t)\rangle &= \frac{2\gamma}{\varepsilon}\varphi(t, u)\Big[\langle u, -\mathrm{A}u\rangle + \langle u, \mathrm{B}(u, u) + \langle u\sqrt{\varepsilon}\, dw(t)\rangle\Big] \\
&\le \varphi(t, u)\Big[-\lambda_1 \frac{2\gamma}{\varepsilon}|u|^2 + \frac{2\gamma}{\sqrt{\varepsilon}}\langle u, dw(t)\rangle\Big],
\end{aligned}$$

$$\begin{aligned}
\frac{\varepsilon}{2} \sum_k \langle \frac{\partial^2 \varphi(t, u(t))}{\partial u^2}(\mathrm{i} f_k), \mathrm{i} f_k\rangle &= \frac{\varepsilon}{2}\varphi(t, u)\Big[\frac{4\gamma^2}{\varepsilon^2}\sum_k \langle u, \mathrm{i} f_k\rangle\langle u, \mathrm{i} f_k\rangle + \frac{2\gamma}{\varepsilon}\sum_k \langle \mathrm{i} f_k, \mathrm{i} f_k\rangle\Big] \\
&= \varphi(t, u)\big[\frac{2\gamma^2}{\varepsilon}|\mathrm{i}^* u|^2_{\mathrm{K}} + \gamma\ \mathrm{tr}(C)\big] \\
&\le \varphi(t, u)\big[\frac{2\gamma^2}{\varepsilon}|\mathrm{i}^*|^2_{\mathcal{L}(\mathrm{H},\mathrm{K})}|u|^2 + \gamma\ \mathrm{tr}(C)\big].
\end{aligned}$$

Therefore, we infer that

$$d\varphi(t, u(t)) \le \varphi(t, u)\Big[-\frac{2\gamma}{\varepsilon}\big(\lambda_1 - \gamma|\mathrm{i}^*|^2_{\mathcal{L}(\mathrm{H},\mathrm{K})}\big)|u(t)|^2 + \lambda + \gamma\, \mathrm{tr}(C)\Big] + \varphi(t, u)\frac{2\gamma}{\sqrt{\varepsilon}}\langle u, dw(t)\rangle.$$

Now, if we put $\lambda = \lambda_1/2$ and choose (small enough) $\gamma > 0$ such that

$$\lambda_1 - \gamma\, |\mathrm{i}^*|^2_{\mathcal{L}(\mathrm{H},\mathrm{K})} \ge \frac{\lambda_1}{2}, \qquad \gamma\ \mathrm{tr}(C) \le \frac{\lambda_1}{2},$$

we get

$$d\varphi(t, u(t)) \le \varphi(t, u)\Big[-\frac{\gamma}{\varepsilon}\lambda_1|u(t)|^2 + \lambda_1\Big] + \varphi(t, u)\frac{2\gamma}{\sqrt{\varepsilon}}\langle u, dw(t)\rangle.$$

By taking expectation (and considering stopping times as for instance in [7]) we infer that

$$\mathbb{E}\left[\mathrm{e}^{\frac{\lambda_1}{2}t}\,\mathrm{e}^{\frac{\gamma}{\varepsilon}|u(t)|^2}\right] \leq \mathrm{e}^{\frac{\bar{\gamma}}{\varepsilon}|x|^2} + \lambda_1\,\mathbb{E}\int_0^t \mathrm{e}^{\frac{\lambda_1}{2}s}\,\mathrm{e}^{\frac{\gamma}{\varepsilon}|u(s)|^2}\left[-\frac{\gamma}{\varepsilon}|u(s)|_{\mathrm{H}}^2+1\right]ds.$$

Since $e^r(-r+1)\leq 1$, for any $r\geq 0$, and $\lambda_1\int_0^t \mathrm{e}^{\frac{\lambda_1}{2}s}\,ds = 2\big(\mathrm{e}^{\frac{\lambda_1}{2}t}-1\big)$ this yields

$$\mathbb{E}\mathrm{e}^{\frac{\gamma}{\varepsilon}|u(t)|_{\mathrm{H}}^2} \leq \mathrm{e}^{-\frac{\lambda_1}{2}t}\mathrm{e}^{\frac{\gamma}{\varepsilon}|x|_{\mathrm{H}}^2}+2.$$

The proof is now complete. □

Now, we continue with the proof of the main result in this section.

Proof of Theorem 5.1. We use the notation introduced in the proof of Lemma 5.3. Let us fix $R>0$ and $t>0$. By the previous lemma and Chebyshev's inequality, we have

$$\begin{aligned}&\mathbb{P}\big(u_\varepsilon^x(t)\in B_{\mathrm{H}}^{\mathrm{c}}(0,R)\big) = \mathbb{P}\left(|u_\varepsilon^x(t)|_{\mathrm{H}}>R\right) = \mathbb{P}\left(\mathrm{e}^{\frac{\gamma}{\varepsilon}|u_\varepsilon^x(t)|_{\mathrm{H}}^2} > \mathrm{e}^{\frac{R^2\gamma}{\varepsilon}}\right)\\ &\leq \mathrm{e}^{-\frac{R^2\gamma}{\varepsilon}}\mathbb{E}\left(\mathrm{e}^{\frac{\gamma}{\varepsilon}|u_\varepsilon^x(t)|_{\mathrm{H}}^2}\right) \leq \mathrm{e}^{-\frac{R^2\gamma}{\varepsilon}}\left[\mathrm{e}^{-\frac{\lambda_1}{2}t}\mathrm{e}^{\frac{\gamma}{\varepsilon}|x|_{\mathrm{H}}^2}+2\right].\end{aligned}$$

Now, due to the ergocity of the invariant measure ν_ε, for any function $\varphi:\mathrm{H}\to\mathbb{R}$, Borel and bounded,

$$\int_{\mathrm{H}}\varphi(x)\,\nu_\varepsilon(dx) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbb{E}\,\varphi(u_\varepsilon^0(s))\,ds.$$

This implies that for any $R>0$

$$\begin{aligned}\nu_\varepsilon\big(B_{\mathrm{H}}^{\mathrm{c}}(0,R)\big) &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbb{P}\left(u_\varepsilon^0(s)\in B_{\mathrm{H}}^{\mathrm{c}}(0,R)\right)\,ds\\ &\leq \mathrm{e}^{-\frac{R^2\gamma}{\varepsilon}}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\left(\mathrm{e}^{-\frac{\lambda_1}{2}s}+2\right)\,ds = 2\,\mathrm{e}^{-\frac{R^2\gamma}{\varepsilon}}.\end{aligned}$$

Hence, if we fix $s>0$ and put

$$R_s := \sqrt{\frac{2s}{\gamma}},\qquad \varepsilon_s := \frac{R_s^2\gamma}{2\log 2},$$

we have that

$$\nu_\varepsilon\big(B_{\mathrm{H}}^{\mathrm{c}}(0,R_s)\big) \leq \mathrm{e}^{-\frac{R_s^2\gamma}{2\varepsilon}} = \mathrm{e}^{-\frac{s}{\varepsilon}},\qquad \varepsilon\leq\varepsilon_s,$$

and this concludes proof of Theorem 5.1. □

6. Lower bounds

Our purpose here is proving the following upper bound.

Theorem 6.1. *For any $\delta, \gamma > 0$ and $\bar{x} \in \mathrm{H}$, there exists $\varepsilon_0 > 0$ such that*

$$\nu_\varepsilon\big(B_{\mathrm{H}}(\bar{x}, \delta)\big) \geq \mathrm{e}^{-\frac{\mathrm{U}(\bar{x})+\gamma}{\varepsilon}}, \qquad \varepsilon \leq \varepsilon_0. \tag{6.1}$$

Let us point out that in the proof of the result we will use that fact that ν_ε is an invariant measure of the Markov process corresponding to the stochastic Navier–Stokes

Before proceeding with the proof of Theorem 6.1, we need to prove a preliminary result.

Lemma 6.2. *Suppose that $\gamma, \bar{T} > 0$, $\bar{x} \in \mathrm{H}$ and $\bar{\varphi} \in L^2(0, \bar{T}; \mathrm{H})$ satisfy*

$$\frac{1}{2}|\bar{\varphi}|^2_{L^2(0,\bar{T};\mathrm{H})} \leq \mathrm{U}(\bar{x}) + \frac{\gamma}{4}. \tag{6.2}$$

Moreover, assume there exists a solution $\bar{z} \in C([0, \bar{T}]; \mathrm{H})$ to the problem

$$\bar{z}\,'(t) + \mathrm{A}\bar{z}(t) + B(\bar{z}(t), \bar{z}(t)) = Q\bar{\varphi}(t), \quad \bar{z}(0) = 0, \quad \bar{z}(\bar{T}) = \bar{x}. \tag{6.3}$$

Then, for all δ and $R > 0$ there exists $T_0 > 0$ and $\varphi_0 \in L^2(0, T_0 + \bar{T}; \mathrm{H})$ such that

$$\frac{1}{2}\,|\varphi_0|^2_{L^2(0,T_0+\bar{T};\mathrm{H})} \leq \mathrm{U}(\bar{x}) + \frac{\gamma}{4} \tag{6.4}$$

and

$$\sup_{|x|_{\mathrm{H}} \leq R} \big|z_x(\varphi_0)(T_0 + \bar{T}) - \bar{x}\big|_{\mathrm{H}} \leq \frac{\delta}{2}, \tag{6.5}$$

where $z_x(\varphi_0) \in C([0, T_0 + \bar{T}]; \mathrm{H})$ is the (unique) solution of the control problem

$$z'(t) + \mathrm{A}z(t) + B(z(t), z(t)) = Q\varphi_0(t), \quad t \in [0, T_0 + \bar{T}], \qquad z(0) = x. \tag{6.6}$$

Proof. Let us assume that $\gamma, \bar{T} > 0$, $\bar{x} \in \mathrm{H}$, $\bar{\varphi} \in L^2(0, \bar{T}; \mathrm{H})$ and $\bar{z} \in C([0, \bar{T}]; \mathrm{H})$ satisfy the assumptions of our Lemma. Let us fix $\delta > 0$ and $R > 0$.

Since, as it is well known,[6] the solution of problem (6.3) depends continuously on the initial condition in $C([0, \bar{T}]; \mathrm{H})$, we infer there exists $\rho > 0$ such that

$$|y_0|_{\mathrm{H}} \leq \rho \Longrightarrow |y_{y_0}(\bar{\varphi})(\bar{T}) - \bar{x}|_{\mathrm{H}} \leq \frac{\delta}{2}, \tag{6.7}$$

[6] For a proof, see e.g. [6, Theorem 4.6].

where $y_{y_0}(\bar{\varphi}) \in C([0,\bar{T}];\mathrm{H})$ is the solution of the problem

$$y'(t) + \mathrm{A}y(t) + B(y(t),y(t)) = Q\bar{\varphi}(t), \quad t \in [0,\bar{T}], \qquad y(0) = y_0.$$

Now, let us consider a solution $u_x \in C([0,T_0];\mathrm{H})$ of the homogeneous Navier–Stokes equation

$$u'(t) + \mathrm{A}u(t) + \mathrm{B}(u(t),u(t)) = 0, \qquad u(0) = x. \tag{6.8}$$

According to (3.3), we have

$$|u_x(t)|_{\mathrm{H}} \le \mathrm{e}^{-\lambda_1 t}|x|_{\mathrm{H}}, \qquad t \ge 0.$$

Hence, if we choose $T_0 > 0$ such that $R\mathrm{e}^{-\lambda_1 T_0} \le \rho$, we have

$$\sup_{|x|_{\mathrm{H}} \le R} |u_x(T_0)|_{\mathrm{H}} \le \rho. \tag{6.9}$$

Next, let us define a control $\varphi_0 \in L^2(0,T_0+\bar{T};\mathrm{H})$ by setting

$$\varphi_0(t) = \begin{cases} 0, & t \in [0,T_0], \\ \bar{\varphi}(t-T_0), & t \in [T_0,T_0+\bar{T}], \end{cases}$$

and next let us fix $x \in \mathrm{H}$ such that $|x|_{\mathrm{H}} \le R$. Then, the function $z \in C([0,T_0+\bar{T}];\mathrm{H})$ defined by

$$z(t) = \begin{cases} u_x(t), & t \in [0,T_0], \\ y_{u_x(T_0)}(\bar{\varphi})(t-T_0), & t \in [T_0,T_0+\bar{T}], \end{cases}$$

is the unique solution to problem

$$z'(t) + \mathrm{A}z(t) + \mathrm{B}(z(t),z(t)) = Q\varphi_0(t), \qquad t \in [0,T_0+\bar{T}], \qquad z(0) = x.$$

In particular we infer that $z = z_x(\varphi_0)$.

Since moreover $z_x(\varphi_0)(T_0+\bar{T}) = y_{u_x(T_0)}(\bar{\varphi})(\bar{T})$ and $|x|_{\mathrm{H}} \le R$, due to (6.9) and (6.7), we infer that

$$|z_x(\varphi_0)(T_0+\bar{T}) - \bar{x}|_{\mathrm{H}} \le \frac{\delta}{2}.$$

This proves condition (6.5). It remains to prove that φ_0 satisfies (6.4). This however follows directly from the definition of φ_0 and assumption (6.2). $\square$

Now, we are ready to prove Theorem 6.1. As we have proved Lemma 6.2, its proof is analogous to the proof of [22, C.2] and [9, Theorem 6.1].

Proof of Theorem 6.1. Let us fix $\delta, \gamma > 0$ and $\bar{x} \in \mathrm{H}$. Without loss of generality, we may assume that $\mathrm{U}(\bar{x}) < \infty$. Note, that in view of Theorem 4.4, this implies that $\bar{x} \in D(\mathrm{A}^{\frac{1+\beta}{2}})$. Moreover, by the definitions (4.12) for the quasi-potential U and (4.5) for the energy, we infer that there exists $\bar{T} > 0$, a control $\bar{\varphi} \in L^2(0, \bar{T}; \mathrm{H})$ and a function $\bar{z} \in C([0, \bar{T}; \mathrm{H})$ such that

$$\frac{1}{2}|\bar{\varphi}|^2_{L^2(0,\bar{T};\mathrm{H})} \leq \mathrm{U}(\bar{x}) + \frac{\gamma}{4}$$

and $\bar{z}$ is a solution to the problem

$$z'(t) + \mathrm{A}z(t) + B(z(t), z(t)) = Q\varphi(t), \quad z(0) = 0, \quad z(T) = \bar{x}.$$

By (5.1) we can find $\bar{R} > 0$, sufficiently large, and $\varepsilon_1 > 0$ such that

$$\nu_\varepsilon(B_{\mathrm{H}}(0, \bar{R}) \geq 1 - (1 - \mathrm{e}^{-\frac{\gamma}{2}}) = \mathrm{e}^{-\frac{\gamma}{2}}, \quad \varepsilon \in (0, \varepsilon_1]. \tag{6.10}$$

Note that trivially, the above implies that

$$\nu_\varepsilon(B_{\mathrm{H}}(0, \bar{R}) \geq \mathrm{e}^{-\frac{\gamma}{2\varepsilon}}, \quad \varepsilon \in (0, 1 \wedge \varepsilon_1]. \tag{6.11}$$

With all the data given and constructed, we can apply Lemma 6.2 and we can find $T_0 > 0$ and $\varphi_0 \in L^2(0, T_0 + \bar{T}; \mathrm{H})$ such that

$$\frac{1}{2}\,|\varphi_0|^2_{L^2(0,T_0+\bar{T};\mathrm{H})} \leq \mathrm{U}(\bar{x}) + \frac{\gamma}{4}$$

and

$$\sup_{|x|_{\mathrm{H}} \leq R} \left|z_x(\varphi_0)(T_0 + \bar{T}) - \bar{x}\right|_{\mathrm{H}} \leq \frac{\delta}{2},$$

where $z_x(\varphi_0) \in C([0, T_0 + \bar{T}]; \mathrm{H})$ is the solution of the control problem (6.6). Let us recall that for $x \in \mathrm{H}$ and $\varepsilon > 0$, the unique solution to the stochastic problem (4.1) is denoted by u^x_ε.

Now, since by Theorem 4.2, the family $\{u^x_\varepsilon\}_{\varepsilon>0}$ satisfies the uniform large deviation principle in $C([0, T_0 + \bar{T}]; \mathrm{H})$, there exists $\varepsilon_2 > 0$ such that for $|x|_{\mathrm{H}} \leq \bar{R}$ and all $\varepsilon \in (0, \varepsilon_2]$,

$$\mathbb{P}\left(|u^x_\varepsilon - z_x(\varphi_0)|_{C([0,T_0+\bar{T}];\mathrm{H})} < \frac{\delta}{2}\right) \geq \mathrm{e}^{-\frac{|\varphi_0|^2_{L^2(0,T_0+\bar{T};\mathrm{H})} + \frac{\gamma}{2}}{2\varepsilon}} \geq \mathrm{e}^{-\frac{\mathrm{U}(\bar{x}) + \frac{\gamma}{2}}{\varepsilon}}. \tag{6.12}$$

Let us fix $x \in \mathrm{H}$ such that $|x|_{\mathrm{H}} \leq \bar{R}$. Then by inequality (6.5) we have

$$\begin{aligned} |u^x_\varepsilon(T_0 + \bar{T}) - \bar{x}|_{\mathrm{H}} &\leq |u^x_\varepsilon(T_0 + \bar{T}) - z_x(\varphi_0)(T_0 + \bar{T})|_{\mathrm{H}} + |z_x(\varphi_0)(T_0 + \bar{T}) - \bar{x}|_{\mathrm{H}} \\ &\leq |u^x_\varepsilon(T_0 + \bar{T}) - z_x(\varphi_0)(T_0 + \bar{T})|_{\mathrm{H}} + \frac{\delta}{2}. \end{aligned}$$

This implies that

$$|u_\varepsilon^x(T_0+\bar{T})-z_x(\varphi_0)(T_0+\bar{T})|_{\mathrm{H}}<\frac{\delta}{2}\Longrightarrow |u_\varepsilon^x(T_0+\bar{T})-\bar{x}|_{\mathrm{H}}<\delta.$$

Therefore, since ν_ε is an invariant measure for the Markov process u_x^ε, we infer that

$$\nu_\varepsilon\big(B_{\mathrm{H}}(\bar{x},\delta)\big)=\nu_\varepsilon\left(|x-\bar{x}|_{\mathrm{H}}<\delta\right)=\int_{\mathrm{H}}\mathbb{P}\left(|u_\varepsilon^x(T_0+\bar{T})-\bar{x}|_{\mathrm{H}}<\delta\right)\,\nu_\varepsilon(dx)$$

$$\geq\int_{\mathrm{H}}\mathbb{P}\left(|u_\varepsilon^x(T_0+\bar{T})-z_x(\varphi_0)(T_0+\bar{T})|_{\mathrm{H}}<\frac{\delta}{2}\right)\,\nu_\varepsilon(dx)$$

$$\geq\int_{\mathrm{H}}\mathbb{P}\left(|u_\varepsilon^x-z_x(\varphi_0)|_{C([0,T_0+\bar{T}];\mathrm{H})}<\frac{\delta}{2}\right)\,\nu_\varepsilon(dx)$$

$$\geq\int_{B_{\mathrm{H}}(0,\bar{R})}\mathbb{P}\left(|u_\varepsilon^x-z_x(\varphi_0)|_{C([0,T_0+\bar{T}];\mathrm{H})}<\frac{\delta}{2}\right)\,\nu_\varepsilon(dx).$$

Applying (6.12) we infer that for $\varepsilon\in(0,\varepsilon_2]$,

$$\nu_\varepsilon\big(B_{\mathrm{H}}(\bar{x},\delta)\big)\geq\nu_\varepsilon\big(B_{\mathrm{H}}\big(0,\bar{R}\big)\big)\mathrm{e}^{-\frac{\mathrm{U}(\bar{x})+\frac{\gamma}{2}}{\varepsilon}}.$$

To conclude the proof, let us take $\varepsilon_0:=\min\{1,\varepsilon_1,\varepsilon_2\}$. Then, by (6.11), we infer that for $\varepsilon\in(0,\varepsilon_0]$,

$$\nu_\varepsilon\big(B_{\mathrm{H}}(\bar{x},\delta)\big)\geq\mathrm{e}^{-\frac{\gamma}{2\varepsilon}}\mathrm{e}^{-\frac{\mathrm{U}(\bar{x})+\frac{\gamma}{2}}{\varepsilon}}=\mathrm{e}^{-\frac{\mathrm{U}(\bar{x})+\gamma}{\varepsilon}}.$$

This completes the proof of Theorem 6.1. □

7. Upper bounds

Let us recall here that K_s is the level set of the quasipotential U, as defined in (4.14), that is

$$K_s:=\{x\in\mathrm{H}:\mathrm{U}(x)\leq s\}.$$

Theorem 7.1. *For all $\delta,\gamma>0$ and $s\geq 0$, there exists $\varepsilon_0>0$ such that*

$$\nu_\varepsilon\left(\{x\in\mathrm{H}:\ dist_{\mathrm{H}}(x,K_s)\geq\delta\}\right)\leq\mathrm{e}^{-\frac{s-\gamma}{\varepsilon}},\qquad\varepsilon\leq\varepsilon_0.\tag{7.1}$$

Before proceeding with the proof of Theorem 7.1, we state two auxiliary results, whose proofs are postponed till next section.

Lemma 7.2. *For all $\delta > 0$ and $s > 0$, there exist $\lambda = \lambda(\delta, s) > 0$ and $\bar{T} = \bar{T}(\delta, s) > 0$ such that for every $t \geq \bar{T}$ and $z \in C([-t, 0]; \mathrm{H})$*

$$|z(-t)|_{\mathrm{H}} < \lambda, \quad S_{-t}(z) \leq s \Longrightarrow dist_{\mathrm{H}}(z(0), K_s) < \delta. \tag{7.2}$$

Lemma 7.3. *For all $s, \delta, r > 0$, there exists $\bar{n} \in \mathbb{N}$ such that*

$$\beta_{\bar{n}} := \inf \{S_{\bar{n}}(u) \ : \ u \in H_{r,s,\delta}(\bar{n})\} > s,$$

where for each $n \in \mathbb{N}$, $s > 0$, $\delta > 0$ and $r > 0$, the set $H_{r,s,\delta}(n)$ is defined by

$$H_{r,s,\delta}(n) := \{u \in C([0, n]; \mathrm{H}), \ |u(0)|_{\mathrm{H}} \leq r, \ |u(j)|_{\mathrm{H}} \geq \lambda, \quad j = 1, \dots, n\}, \tag{7.3}$$

and λ is the constant depending on s and δ, obtained in Lemma 7.2.

Assuming Lemmata 7.2 and 7.3, the proof of Theorem 7.1 follows the same line of the proofs of [22, C.3] and [9, Theorem 7.3]. We give here the proof, with some additional details, for the reader's convenience.

Proof of Theorem 7.1. Let us fix $\delta > 0$, $\gamma > 0$ and $s \geq 0$ and let us choose positive constants R_s and ε_s, as in Theorem 5.1.

Because ν_ε is an invariant measure for the Markov process generated by equation (4.1) and the set $\{x \in \mathrm{H} \ : \ \mathrm{dist}(x, K_s) \geq \delta\}$ is closed and hence a Borel subset of H, we infer that

$$\begin{aligned}
\nu_\varepsilon\left(\{x \in \mathrm{H} \ : \ \mathrm{dist}(x, K_s) \geq \delta\}\right) &= \int_H \mathbb{P}\left(\mathrm{dist}\left(u_\varepsilon^y(t), K_s\right) \geq \delta\right) \nu_\varepsilon(dy) \\
&= \int_{B_H^c(0,R_s)} \mathbb{P}\left(\mathrm{dist}\left(u_\varepsilon^y(t), K_s\right) \geq \delta\right) \nu_\varepsilon(dy) \\
&\quad + \int_{B_H(0,R_s)} \mathbb{P}\left(\mathrm{dist}\left(u_\varepsilon^y(t), K_s\right) \geq \delta\right) \nu_\varepsilon(dy).
\end{aligned} \tag{7.4}$$

Thanks to Theorem 5.1, for any $\varepsilon \leq \varepsilon_s$ we have

$$\int_{B_H^c(0,R_s)} \mathbb{P}\left(\mathrm{dist}\left(u_\varepsilon^y(t), K_s\right) \geq \delta\right) \nu_\varepsilon(dy) \leq \mathrm{e}^{-\frac{s}{\varepsilon}}. \tag{7.5}$$

Now, in view of Lemma 7.3, we can pick $\bar{n} \in \mathbb{N}$ such that

$$u \in H_{R_s,s,\delta}(\bar{n}) \Longrightarrow S_{\bar{n}}(u) \geq s.$$

Since the set $H_{R_s,s,\delta}(\bar{n})$ is closed in $C([0,n];\mathrm{H})$, and since by Theorem 4.2 the family $\{u_\varepsilon^y\}_{\varepsilon>0}$ satisfies the large deviation principle in $C([0,n];\mathrm{H})$ uniformly with respect to $y \in B_{\mathrm{H}}(0,R_s)$, we infer that there exists $\varepsilon_1 > 0$ such that

$$\sup_{y\in B_{\mathrm{H}}(0,R_s)} \mathbb{P}\left(u_\varepsilon^y \in H_{R_s,s,\delta}(\bar{n})\right) \leq \mathrm{e}^{-\frac{s-\gamma/2}{\varepsilon}}, \qquad \varepsilon \leq \varepsilon_1. \tag{7.6}$$

This implies that for $\varepsilon \in (0,\varepsilon_1)$,

$$\int_{B_H(0,R_s)} \mathbb{P}\left(\mathrm{dist}\left(u_\varepsilon^y(t),K_s\right) \geq \delta,\ u_\varepsilon^y \in H_{R_s,s,\delta}(\bar{n})\right) \nu_\varepsilon(dy) \leq \mathrm{e}^{-\frac{s-\gamma/2}{\varepsilon}}.$$

Thus, for $\varepsilon \in (0,\varepsilon_1)$,

$$\begin{aligned}\int_{B_H(0,R_s)} \mathbb{P}\left(\mathrm{dist}\left(u_\varepsilon^y(t),K_s\right) \geq \delta\right) &\leq \mathrm{e}^{-\frac{s-\gamma/2}{\varepsilon}} \\ &+ \int_{B_H(0,R_s)} \mathbb{P}\left(\mathrm{dist}\left(u_\varepsilon^y(t),K_s\right) \geq \delta,\ u_\varepsilon^y \notin H_{R_s,s,\delta}(\bar{n})\right) \nu_\varepsilon(dy).\end{aligned} \tag{7.7}$$

Thus we only have to deal with the second integral on the RHS of (7.7).

Let now fix $t \geq \bar{n}$, $\varepsilon \in (0,\varepsilon_s \wedge \varepsilon_1)$ and $y \in B_{\mathrm{H}}(0,R_s)$. In view of the definition of $H_{r,s,\delta}(\bar{n})$, we have that

$$\{u \in C([0,n];\mathrm{H}), |u(0)|_{\mathrm{H}} \leq r\} \setminus H_{r,s,\delta}(n) = \bigcup_{j=1}^{n} \{u \in C([0,n];\mathrm{H}), |u(j)|_{\mathrm{H}} < \lambda\}. \tag{7.8}$$

Therefore, because $|u_\varepsilon^y(0)|_{\mathrm{H}} = |y|_{\mathrm{H}} \leq R_s$, we infer that

$$\begin{aligned}&\Big\{\omega \in \Omega : \mathrm{dist}\big(u_\varepsilon^y(t),K_s\big) \geq \delta,\ u_\varepsilon^y \notin H_{R_s,s,\delta}(\bar{n})\Big\} \\ &\quad= \bigcup_{j=1}^{n} \Big\{\omega \in \Omega : \mathrm{dist}\big(u_\varepsilon^y(t),K_s\big) \geq \delta, |u_\varepsilon^y(j)|_{\mathrm{H}} < \lambda\Big\}.\end{aligned}$$

Moreover, by the Markov property of the process u_ε^y, we infer that if $P(\tau,t,dz)$, $0 \leq \tau \leq t$, is the transition probability function corresponding to the Markov process $u_\varepsilon^y(t)$, $t \geq 0$ and $y \in \mathrm{H}$, then

$$\begin{aligned}&\mathbb{P}\Big(\mathrm{dist}\big(u_\varepsilon^y(t),K_s\big) \geq \delta, |u_\varepsilon^y(j)|_{\mathrm{H}} < \lambda\Big) \\ &\quad= \int_{\{|u_\varepsilon^y(j)|_{\mathrm{H}}<\lambda\}} P(j,y,dz)\mathbb{P}\big(\mathrm{dist}(u_\varepsilon^z(t-j),K_s) \geq \delta\big) &(7.9) \\ &\quad\leq \sup_{|z|_{\mathrm{H}}<\lambda} \mathbb{P}\big(\mathrm{dist}(u_\varepsilon^z(t-j),K_s) \geq \delta\big). &(7.10)\end{aligned}$$

Therefore,

$$\begin{aligned}&\int\limits_{\{|y|_{\mathrm{H}}\leq R_s\}} \mathbb{P}\left(\mathrm{dist}\left(u^y_\varepsilon(t),K_s\right)\geq\delta,\ u^y_\varepsilon\notin H_{R_s,s,\delta}(\bar n)\right)\nu_\varepsilon(dy)\\&\qquad\leq\sum_{j=1}^{\bar n}\sup_{z\in B_{\mathrm{H}}(0,\lambda)}\mathbb{P}\left(\mathrm{dist}\left(u^z_\varepsilon(t-j),K_s\right)\geq\delta\right).\end{aligned} \tag{7.11}$$

Next, in order to estimate the RHS of the last equality, we fix $z\in B_{\mathrm{H}}(0,\lambda)$ and define two auxiliary sets $K_s(\lambda,t)$ and $\tilde K_s(z,t)$ by

$$K_s(\lambda,t):=\left\{u\in C([0,t];\mathrm{H})\ :\ S_t(u)\leq s,\ |u(0)|_{\mathrm{H}}\leq\lambda\right\},$$

and

$$\tilde K_s(z,t):=\left\{u\in C([0,t];\mathrm{H})\ :\ S_t(u)\leq s,\ u(0)=z\right\}.$$

Since, $|z|_{\mathrm{H}}\leq\lambda$, we observe that

$$\tilde K_s(z,t)\subset K_s(\lambda,t).$$

Moreover, according to Lemma 7.2, there exists $\bar T>0$ such that for any $T\geq\bar T$

$$\varphi\in K_s(\lambda,T)\Longrightarrow\mathrm{dist}(\varphi(T),K_s)\leq\frac{\delta}{2}.$$

In what follows we fix $t\geq\max\{\bar T,\bar n\}$, and we prove that for any $u\in C([0,t];\mathrm{H})$ such that

$$\mathrm{dist}_{C([0,t];\mathrm{H})}(u,K_s(\lambda,t))<\frac{\delta}{2}, \tag{7.12}$$

we have

$$\mathrm{dist}(u(t),K_s)<\delta.$$

Indeed, if (7.12) holds, then there exists $\varphi\in K_s(\lambda,t)$ such that

$$\mathrm{dist}_{C([0,t];\mathrm{H})}(u,\varphi)<\frac{\delta}{2},$$

so that $|u(t)-\varphi(t)|_{\mathrm{H}}<\frac{\delta}{2}$. Hence, by the triangle inequality, we infer that

$$\mathrm{dist}(u(t),K_s)\leq|u(t)-\varphi(t)|_{\mathrm{H}}+\mathrm{dist}(u(t),K_s)<\frac{\delta}{2}+\frac{\delta}{2}=\delta.$$

Since, $\tilde{K}_s(z,t) \subset K_s(\lambda,t)$ we deduce that

$$\mathbb{P}\left(\mathrm{dist}\left(u_\varepsilon^z(t), K_s\right) \geq \delta\right) \leq \mathbb{P}\left(\mathrm{dist}_{C([0,t];\mathrm{H})}\left(u_\varepsilon^y, K_s(\lambda,t)\right) > \frac{\delta}{2}\right)$$
$$\leq \mathbb{P}\left(\mathrm{dist}_{C([0,t];\mathrm{H})}\left(u_\varepsilon^z, \tilde{K}_s(z,t)\right) > \frac{\delta}{2}\right).$$

Next, as the set

$$\left\{u \in C([0,t];\mathrm{H}) : \mathrm{dist}_{C([0,t];\mathrm{H})}\left(u_\varepsilon, K_s(y,t)\right) \geq \frac{\delta}{2}\right\}$$

is closed in $C([0,t];\mathrm{H})$, and, by Theorem 4.2, the family $\{u_\varepsilon^z\}_{\varepsilon>0}$ satisfies the large deviation principle in $C([0,t];\mathrm{H})$ uniformly with respect to $z \in B_{\mathrm{H}}(0,\lambda)$, we infer that there exists $\varepsilon_2(t) > 0$ such that

$$\sup_{z\in B_{\mathrm{H}}(0,\lambda)} \mathbb{P}\left(\mathrm{dist}_{C([0,t];\mathrm{H})}\left(u_\varepsilon^z, K_s(y,t)\right) > \frac{\delta}{2}\right) \leq \mathrm{e}^{-\frac{s-\gamma/2}{\varepsilon}}, \qquad \varepsilon \leq \varepsilon(t).$$

Therefore, if we define

$$\varepsilon_3 := \min\{\varepsilon_2(t-1), \ldots, \varepsilon_2(t-\bar{n}), \varepsilon_s, \varepsilon_1\},$$

due to (7.4), (7.5), (7.7) and (7.11), we deduce that for $\varepsilon \leq \varepsilon_3$,

$$\nu_\varepsilon\left(x \in \mathrm{H} \ : \ \mathrm{dist}(x, K(s)) \geq \delta\right)$$
$$\leq \mathrm{e}^{-\frac{s}{\varepsilon}} + (1+\bar{n})\mathrm{e}^{-\frac{s-\gamma/2}{\varepsilon}} = \mathrm{e}^{-\frac{s}{\varepsilon}}\left(1 + (1+\bar{n})\mathrm{e}^{-\frac{\gamma/2}{\varepsilon}}\right).$$

This clearly implies (7.1), if we take ε_0 sufficiently small. □

8. Proof of Lemmata 7.2 and 7.3

Proof of Lemma 7.2. Suppose that there exist $\delta > 0$ and $s > 0$ such that for every $n \in \mathbb{N}$ there exists a function $z_n \in C([-n,0];H)$ with

$$S_{-n}(z_n) \leq s, \qquad \mathrm{dist}_{\mathrm{H}}(z_n(0), K_s) \geq \delta, \tag{8.1}$$

and

$$\beta_n := |z_n(-n)|_H^2 \searrow 0, \quad \text{as } n \to \infty. \tag{8.2}$$

We will show that this leads to a contradiction.

Note that for every $n \in \mathbb{N}$, the function z_n satisfies the following a priori inequality

$$\sup_{s\in[-n,0]} |z_n(s)|_H^2 + \int_{-n}^{0} |z_n(s)|_V^2\, ds \leq |z_n(-n)|_H^2 + \int_{-n}^{0} |f_n(s)|_{V'}^2\, ds, \tag{8.3}$$

where

$$f_n(s) := \mathcal{H}(z_n)(s) = z_n'(s) + \mathrm{A}z_n(s) + \mathrm{B}(z_n(s), z_n(s)), \quad s \in (-n, 0).$$

Therefore, in view of inequality (4.6), by conditions (8.1) and (8.2), we infer that there exists $c > 0$ such that

$$\sup_{s\in[-n,0]} |z_n(s)|_H^2 + \int_{-n}^{0} |z_n(s)|_V^2 \, ds \leq \beta_n + c\, s.$$

Moreover, by inequality (3.6), there exists a constant $c > 0$ such that

$$\begin{aligned} &|\sqrt{\cdot + n}\, z_n(\cdot)|^2_{L^\infty(-n,0;\mathrm{V})} + |\sqrt{\cdot + n}\, z_n(\cdot)|^2_{L^2(-n,0;D(\mathrm{A}))} \\ &\leq c \exp\left[c\Big(|z_n(-n)|^4_{\mathrm{H}} + |f_n|^4_{L^2(-n,0;\mathrm{V}')}\Big)\right] \\ &\times \Big(|z_n(-n)|^2_{\mathrm{H}} + |f_n|^2_{L^2(-n,0;\mathrm{V}')} + n|f_n|^2_{L^2(-n,0;\mathrm{H})}\Big). \end{aligned} \tag{8.4}$$

If $s \in [-\frac{n}{2}, 0]$, we have $s + n \geq \frac{n}{2}$ and therefore, from (8.4), we get

$$\begin{aligned} &\frac{n}{2}\Big(|z_n|^2_{L^\infty(-\frac{n}{2},0;\mathrm{V})} + |\, z_n|^2_{L^2(-\frac{n}{2},0;D(\mathrm{A}))}\Big) \\ &\leq c \exp\left[c\Big(|z_n(-n)|^4_{\mathrm{H}} + |f_n|^4_{L^2(-n,0;\mathrm{V}')}\Big)\right] \\ &\times \Big(|z_n(-n)|^2_{\mathrm{H}} + |f_n|^2_{L^2(-n,0;\mathrm{V}')} + n|f_n|^2_{L^2(-n,0;\mathrm{H})}\Big). \end{aligned}$$

This implies that there exists a constant $c_2 = c_2(s, |z_1(-1)|_H^2)$ such that

$$|z_n|_{L^\infty(-\frac{n}{2},0;\mathrm{V})} + |\, z_n|_{L^2(-\frac{n}{2},0;D(\mathrm{A}))} \leq c_2, \quad n \in \mathbb{N}. \tag{8.5}$$

Moreover, this implies that there exists a constant $c_3 > 0$ such that

$$|z_n'(\cdot)|_{L^2(-\frac{n}{2},0;\mathrm{H})} \leq c_3. \tag{8.6}$$

Indeed, since for $n \in \mathbb{N}$,

$$z_n'(t) = f_n(t) - \mathrm{A}z_n(t) - B(z_n(t), z_n(t)), \qquad t \in (-\frac{n}{2}, 0),$$

we infer that

$$\begin{aligned} &|z_n'|_{L^2(-\frac{n}{2},0;\mathrm{H})} \\ &\leq |f_n|_{L^2(-\frac{n}{2},0;\mathrm{H})} + |\mathrm{A}z_n|_{L^2(-\frac{n}{2},0;\mathrm{H})} + |B(z_n, z_n)|_{L^2(-\frac{n}{2},0;\mathrm{H})} \\ &\leq c\, f_n|^2_{L^2(-\frac{n}{2},0;\mathrm{H})} + c\,|z_n|^2_{L^2(-\frac{n}{2},0;D(\mathrm{A}))} + |B(z_n, z_n)|_{L^2(-\frac{n}{2},0;\mathrm{H})} + 1. \end{aligned} \tag{8.7}$$

Next, from inequality (2.10), we deduce that

$$|\mathrm{B}(z_n,z_n)|^2_{L^2(-\frac{n}{2},0;\mathrm{H})} \le |z_n|_{L^\infty(-\frac{n}{2},0;\mathrm{H})}|z_n|_{L^\infty(-\frac{n}{2},0;\mathrm{V})}|z_n|_{L^2(-\frac{n}{2},0;\mathrm{V})}|z_n|_{L^2(a,b;D(\mathrm{A}))}.$$

Thanks to (8.3) and (8.5), this implies that for some constant $c > 0$

$$|\mathrm{B}(z_n,z_n)|_{L^2(-\frac{n}{2},0;\mathrm{H})} \le c, \qquad n \in \mathbb{N}.$$

Hence inequality (8.6) follows, due to (8.7) and (8.5).

Now, let us fix $k \in \mathbb{N}$. Notice that if $n \ge 2k$ then $[-k,0] \subset [-\frac{n}{2},0]$. We can consider the sequence $\{z_n\}_{n=2k}^\infty$, or more precisely, the sequence of restrictions of that sequence to the time interval $[-k,0]$. According to (8.3), (8.5) and (8.6), this sequence satisfies

$$\sup_{r\in[-k,0]} |z_n(r)|_H^2 + \int_{-k}^0 |z_n(r)|_V^2\,dr \le \beta_n + c\,s$$

and

$$|z_n|^2_{L^\infty(-k,0;\mathrm{V})} + |\,z_n|^2_{L^2(-k,0;D(\mathrm{A}))} + |z_n'|_{L^2(-k,0;\mathrm{H})} \le c.$$

Moreover, as $S_{-n}(z_n) \le s$, for any $n \in \mathbb{N}$ we have

$$|f_n|_{L^2(-k,0;D(\mathrm{A}^{\frac{\alpha}{2}}))} \le \sqrt{2s}.$$

Hence, by a standard compactness argument, (compare with the first method of proof of Theorem 6.1 in [18, page 71 onwards] and the proof of Theorem III.3.10 from [25]), for each fixed $k \in \mathbb{N}$ there exist two subsequences

$$\{z^k_{n_j}\}_{j=1}^\infty \subseteq \{z_n\}_{n=2k}^\infty \text{ and } \{f^k_{n_j}\}_{j=1}^\infty \subseteq \{f_n\}_{n=2k}^\infty,$$

and two functions $f^k \in L^2(-k,0,\mathrm{H})$ and $u^k \in C([-k,0];\mathrm{V}) \cap L^2(-k,0;D(\mathrm{A}))$, with $D_t u^k \in L^2(-k,0;\mathrm{H})$, such that, as $j \to \infty$,

$$z^k_{n_j} \to u^k, \text{ weakly in } L^2(-k,0;D(\mathrm{A})) \text{ and strongly in } L^2(-k,0;\mathrm{V}) \cap C([-k,0];\mathrm{H}),$$

and

$$f^k_{n_j} \to f^k, \text{ weakly in } L^2(-k,0;D(\mathrm{A}^{\frac{\alpha}{2}})),$$

u^k satisfies

$$D_t u^k + \mathrm{A}u^k + \mathrm{B}(u^k,u^k) = f_k \text{ on } (-k,0),$$

and

$$|u^k|^2_{C([-k,0];\mathrm{H})} \leq c\, s, \qquad |f^k|^2_{L^2(-k,0;D(\mathrm{A}^{\frac{\alpha}{2}}))} \leq 2s.$$

Moreover, by an inductive argument, the sequences $\{z^k_{n_j}\}_{j=1}^\infty$ and $\{f^k_{n_j}\}_{j=1}^\infty$ can be chosen in such a way that, for any $k \in \mathbb{N}$, the restrictions of u^{k+1} and f^{k+1} to $(-k, 0)$ are equal to u^k and f^k, respectively.

This allows us to define two functions u and f on the interval $(-\infty, 0)$ such that for every k, the restrictions of u and f to $(-k, 0)$ are equal to u^k and f^k, respectively. These functions u and f satisfy, for every $k \in \mathbb{N}$,

$$D_t u + \mathrm{A}u + \mathrm{B}(u, u) = f \text{ on } (-k, 0),$$

and

$$|u|^2_{C([-k,0];\mathrm{H})} \leq c\, s, \qquad |f|^2_{L^2(-k,0;D(\mathrm{A}^{\frac{\alpha}{2}}))} \leq 2s.$$

The last of these properties implies that $f \in L^2(-\infty, 0; D(\mathrm{A}^{\frac{\beta}{2}}))$ and

$$|f|^2_{L^2(-\infty,0;D(\mathrm{A}^{\frac{\beta}{2}}))} \leq 2s.$$

Moreover

$$D_t u + \mathrm{A}u + \mathrm{B}(u, u) = f \text{ on } (-\infty, 0),$$

so that $S_{-\infty}(u) \leq s$. Finally, due to (8.2), there exists a sequence $\{t_n\} \downarrow -\infty$ such that

$$\lim_{n\to\infty} |u(t_n)|_{\mathrm{H}} = 0.$$

Therefore, by Lemma 3.7 we infer that $u \in \mathcal{X}$. Thus, thanks to the characterization of U given in equality (4.13) in Theorem 4.4, we can conclude that $\mathrm{U}(u(0)) \leq s$, so that $u(0) \in K_s$.

On the other hand,

$$\lim_{j\to\infty} z^1_{n_j} = u, \qquad \text{in } C([-1, 0]; \mathrm{H}),$$

and, by our assumptions, $\mathrm{dist}_{\mathrm{H}}(z_n(0), K_s) \geq \delta$. Hence

$$\mathrm{dist}_{\mathrm{H}}(u(0), K_s) \geq \delta$$

which contradicts the fact that $u(0) \in K_s$. $\square$

Proof of Lemma 7.3. Let us assume that there exist $s > 0$, $\delta > 0$ and $r > 0$ such that for every $n \in \mathbb{N}$ there exists $u_n \in H_{r,s,\delta}(n)$ such that

$$S_n(u_n) \leq s + 1.$$

In particular, due to (3.2), we have

$$|u_n|_{C([0,n];\mathrm{H})} \leq c_{\sqrt{2(s+1)}}(1 + r) =: c_{s,r}, \quad n \in \mathbb{N}. \tag{8.8}$$

Now, for any $k \in \mathbb{N}$, we define

$$\gamma_k := \inf \{S_k(u) \,;\, u \in C([0,k];\mathrm{H}),\ |u(0)|_{\mathrm{H}} \leq c_{s,r} \wedge r,\ |u(k)|_{\mathrm{H}} \geq \lambda\}\,.$$

If we show that there exists $\bar{k} \in \mathbb{N}$ such that $\gamma_{\bar{k}} > 0$, then, due to (8.8), we have

$$S_{n\bar{k}}(u_{n\bar{k}}) \geq n\,\gamma_{\bar{k}}, \qquad n \in \mathbb{N},$$

which contradicts the fact that $S_{n\bar{k}}(u_{n\bar{k}}) \leq s + 1$. Therefore, in order to conclude our proof, we show that there exists some $\bar{k} \in \mathbb{N}$ such that $\gamma_{\bar{k}} > 0$.

For any $x \in \mathrm{H}$, we denote by $z_x(t)$ the solution of the problem

$$z_x'(t) + \mathrm{A}z_x(t) + B(z_x(t), z_x(t)) = 0, \qquad z_x(1) = x.$$

According to (3.3), there exists some integer $\bar{k} \geq 1$ such that

$$|x|_{\mathrm{H}} \leq c_{s,r} \Longrightarrow |z_x(t)|_{\mathrm{H}} \leq \frac{\lambda}{2}, \qquad t \geq \bar{k}. \tag{8.9}$$

We show that, for such $\bar{k}$, it holds $\gamma_{\bar{k}} > 0$. Actually, if $\gamma_{\bar{k}} = 0$, then there exists a sequence

$$\{v_n\}_{n\in\mathbb{N}} \subset \left\{u \in C([0,\bar{k}];\mathrm{H})\,;\, |u(0)|_{\mathrm{H}} \leq c_{s,r} \wedge r,\ |u(k)|_{\mathrm{H}} \geq \lambda\right\},$$

such that

$$\lim_{n\to\infty} S_{\bar{k}}(v_n) = 0. \tag{8.10}$$

Thus, there exists $\bar{n} \in \mathbb{N}$ such that $S_{\bar{k}}(v_n) \leq s + 1$, for any $n \geq \bar{n}$ and hence, according to (8.8), $|v_n(1)|_{\mathrm{H}} \leq c_{s,r}$, for any $n \geq \bar{n}$. Moreover, thanks to (3.6), there exists a constant $\tilde{c}_{s,r,\bar{k}}$ such that

$$|v_n|_{L^\infty(1,\bar{k};\mathrm{V})} \leq \tilde{c}_{s,r,\bar{k}}, \qquad n \geq \bar{n}. \tag{8.11}$$

This means, in particular, that there exists a subsequence $\{v_{n_j}\}_{j\in\mathbb{N}} \subset \{v_n\}_{n\in\mathbb{N}}$ and $\bar{x} \in \mathrm{H}$ such that

$$\lim_{j\to\infty} |v_{n_j}(1) - \bar{x}|_{\mathrm{H}} = 0. \tag{8.12}$$

Since $|v_n(1)|_{\mathrm{H}} \leq c_{s,r}$, it follows that $|\bar{x}|_{\mathrm{H}} \leq c_{s,r}$, and then, due to (8.9), it follows that

$$|z_{\bar{x}}(\bar{k})|_{\mathrm{H}} \leq \frac{\lambda}{2}. \tag{8.13}$$

Now, as a consequence of (8.10), for every $n \in \mathbb{N}$ there exists $f_n \in L^2(0, \bar{k}; \mathrm{H})$ such that

$$v_n'(t) + \mathrm{A}v_n(t) + \mathrm{B}(v_n(t), v_n(t)) = f_n(t),$$

and

$$\lim_{n\to\infty} |f_n|_{L^2(0,\bar{k};\mathrm{H})} = 0.$$

According to (8.12), this implies that

$$\lim_{j\to\infty} |v_{n_j} - z_{\bar{x}}|_{C([1,\bar{k}];\mathrm{H})} = 0,$$

so that $|z_{\bar{x}}(\bar{k})|_{\mathrm{H}} \geq \lambda$, which contradicts (8.13). □

Appendix A. Behavior of the solutions of the Navier–Stokes equations for large negative times

In our paper with Mark Freidlin [3] we proved the following two results, see Propositions A.1 and A.2, for the general 2-D Navier–Stokes Equations. We formulate them in way that does not need to use special notation used by us.

Proposition A.1. *Assume that $z \in C((-\infty, 0]; \mathrm{H})$ is such that*

$$\lim_{t\to-\infty} |z(t)|_{\mathrm{H}} = 0 \tag{A.1}$$

and $S_{-\infty}(z) < \infty$, i.e.

$$\int_{-\infty}^{0} |z'(t) + \mathrm{A}z(t) + \mathrm{B}(z(t), z(t))|_{\mathrm{H}}^2 \, dt < \infty. \tag{A.2}$$

Then, we have $z(0) \in V$,

$$\lim_{t\to-\infty} |z(t)|_{\mathrm{V}} = 0, \tag{A.3}$$

and

$$\int_{-\infty}^{0} |Az(t)|_{\mathrm{H}}^2\,dt + \int_{-\infty}^{0} |z'(t)|_{\mathrm{H}}^2\,dt < \infty. \tag{A.4}$$

Moreover, there exists a continuous and strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ *such that* $\varphi(0) = 0$ *and, if* z *satisfying condition (A.1) is a solution to the problem*

$$z'(t) + \mathrm{A}z(t) + \mathrm{B}(z(t), z(t)) = f(t), \quad t \le 0, \tag{A.5}$$

with f *being an element of* $L^2(-\infty, 0; \mathrm{H})$*, then*

$$|z(0)|_{\mathrm{V}}^2 + \int_{-\infty}^{0} |Az(t)|_{\mathrm{H}}^2\,dt + \int_{-\infty}^{0} |z'(t)|_{\mathrm{H}}^2\,dt \le \varphi\Big(\int_{-\infty}^{0} |f(t)|_{\mathrm{H}}^2\,dt\Big). \tag{A.6}$$

Proposition A.2. *Assume that* $\alpha \in (0, 1/2)$*. If a function* $z \in C((-\infty, 0]; \mathrm{H})$*, satisfying condition (A.1), satisfies also*

$$\int_{-\infty}^{0} |z'(t) + \mathrm{A}z(t) + \mathrm{B}(z(t), z(t))\,|_{D(\mathrm{A}^{\frac{\alpha}{2}})}^2\,dt < \infty, \tag{A.7}$$

we have

$$z(0) \in D(\mathrm{A}^{\frac{\alpha}{2}+\frac{1}{2}}), \tag{A.8}$$

$$\lim_{t\to-\infty} |z(t)|_{D(\mathrm{A}^{\frac{\alpha}{2}+\frac{1}{2}})} = 0, \tag{A.9}$$

and

$$\int_{-\infty}^{0} |A^{\frac{\alpha}{2}+1} z(t)|_{\mathrm{H}}^2\,dt + \int_{-\infty}^{0} |A^{\frac{\alpha}{2}} z'(t)|_{\mathrm{H}}^2\,dt < \infty. \tag{A.10}$$

Moreover, there exists a continuous and strictly increasing function $\varphi_\alpha : [0,\infty) \to [0,\infty)$ *such that* $\varphi_\alpha(0) = 0$ *and if* z*, satisfying condition (A.1), is a solution to problem (A.5) with* $f \in L^2(-\infty, 0; D(\mathrm{A}^{\frac{\alpha}{2}}))$*, then*

$$|z(0)|_{D(\mathrm{A}^{\frac{\alpha}{2}+\frac{1}{2}})}^2 + \int_{-\infty}^{0} |A^{\frac{\alpha}{2}+1} z(t)|_{\mathrm{H}}^2\,dt + \int_{-\infty}^{0} |A^{\frac{\alpha}{2}} z'(t)|_{\mathrm{H}}^2\,dt \tag{A.11}$$

$$\le \varphi_\alpha(|f|_{L^2(-\infty,0);D(\mathrm{A}^{\frac{\alpha}{2}})}^2). \tag{A.12}$$

The reason for the restriction $\alpha \in (0, 1/2)$ in Proposition A.2 lies in the fact that we have used continuity of the Leray–Helmholtz projection P from $\mathrm{H}^\alpha(\mathcal{O}, \mathbb{R}^2)$ into $D(\mathrm{A}^{\alpha/2})$, see Proposition 2.1 in [3] (and (2.6) in the current paper).

The aim of this section is show that in the case of the 2-D NSEs with periodic boundary conditions, i.e. NSEs on a 2-dimensional torus, Proposition A.2 holds true for any $\alpha > 0$. Of course we will only need to consider the case $\alpha \geq \frac{1}{2}$. The main result in this section is as follows.

Proposition A.3. *Assume that $\alpha > 0$. If z satisfies conditions (A.1) and (A.7), then it satisfies (A.8), (A.9), and (A.10) as well.*

Moreover, there exists a continuous and strictly increasing function $\varphi_\alpha : [0, \infty) \to [0, \infty)$ such that $\varphi_\alpha(0) = 0$ and if z, satisfying condition (A.1), is a solution to problem (A.5), with $f \in L^2(-\infty, 0; D(\mathrm{A}^{\frac{\alpha}{2}}))$, then inequality (A.11) holds as well.

The following proof is an adaptation of the proof of Proposition 2.2 from [3]. In fact, we follow the lines quite literary. As mentioned earlier, we only need to consider the case $\alpha \in [1/2, \infty)$. Since the estimates for the nonlinear term B given in Propositions 2.3 and 2.4 are different for $\alpha \leq 1$ and $\alpha > 1$ we will have to consider two cases: $\alpha \in [1/2, 1]$ and $\alpha \in (1, \infty)$. If $\alpha \in [1/2, 1]$ then we can use inequality (2.15) with $s = 2$. In this case the proof from [3] is virtually the same. Note however, that if $\alpha > 1$ and inequality (2.17) holds, this does not imply that inequality (2.15) with $s = 2$ holds.

Proof of Proposition A.3. In the whole proof all the norms and scalar products are in H.

For the readers convenience and the completeness of the results we will prove our result in the special case

$$\alpha = 1$$

So, let us fix $\phi \in D(\mathrm{A})$ and a function z satisfying conditions (A.1) and (A.7). Following the methods from the proof of Proposition 2.1 from [3] it is sufficient to prove (A.8), (A.9), and (A.10).

Since the function z satisfies inequality (A.6), we can find a decreasing sequence $\{s_n\}$ such that $s_n \searrow -\infty$, $z(s_n) \in D(\mathrm{A})$, $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} |\mathrm{A}z(s_n)|_{\mathrm{H}} = 0. \tag{A.13}$$

Arguing as in the proof of [3, Proposition 3.3], we infer that the function $|\mathrm{A}z(\cdot)|^2_{\mathrm{H}}$ is absolutely continuous and satisfies the following identity on $(-\infty, 0]$

$$\frac{1}{2}\frac{d}{dt}|\mathrm{A}z(t)|^2_{\mathrm{H}} + |\mathrm{A}^{\frac{3}{2}}z(t)|^2_{\mathrm{H}} = -(B(z(t), z(t)), \mathrm{A}^2 z(t))_{\mathrm{H}} + (f(t), \mathrm{A}^2 z(t))_{\mathrm{H}}. \tag{A.14}$$

In view of inequality (2.15), with $s = 2$ and $\alpha = 1$, we infer that there exists $c > 0$ such the following inequality is satisfied for $f \in D(\mathrm{A}^{\frac{1}{2}})$

$$-(B(z,z),\mathrm{A}^2z)+(\mathrm{A}f,\mathrm{A}^2z)=-(\mathrm{A}^{\frac12}B(z,z),\mathrm{A}^{\frac32}z)+(\mathrm{A}^{\frac12}f,\mathrm{A}^{\frac32}z)$$
$$\leq\frac12|\mathrm{A}^{\frac32}z|^2+c\,|\mathrm{A}z|^4+|\mathrm{A}^{\frac12}f|^2,\quad z\in D(\mathrm{A}).$$

Hence, we infer that on $(-\infty,0]$, we have

$$\frac{d}{dt}|\mathrm{A}z(t)|^2_{\mathrm{H}}+|\mathrm{A}^{\frac32}z(t)|^2_{\mathrm{H}}\leq c\,|\mathrm{A}z(t)|^2_{\mathrm{H}}|\mathrm{A}z(t)|^2_{\mathrm{H}}+2|\mathrm{A}^{\frac12}f(t)|^2_{\mathrm{H}}. \tag{A.15}$$

Therefore, by the Gronwall Lemma, we get

$$|\mathrm{A}z(t)|^2_{\mathrm{H}}\leq|\mathrm{A}z(s)|^2_{\mathrm{H}}\mathrm{e}^{c\int_s^t|\mathrm{A}z(r)|^2_{\mathrm{H}}\,dr} \tag{A.16}$$
$$+2\int_s^t|\mathrm{A}^{\frac12}f(r)|^2_{\mathrm{H}}\mathrm{e}^{c\int_r^t|\mathrm{A}z(\rho)|^2_{\mathrm{H}}\,d\rho}\,dr,\quad -\infty<s\leq t\leq 0.$$

Note that by inequality (A.6) we have

$$\sup_{n\geq1}\int_{s_n}^t|\mathrm{A}z(r)|^2_{\mathrm{H}}\,dr\leq\varphi(|f|^2),\quad t\leq 0,$$

where for the sake of brevity, we set $|f|_{L^2(-\infty,0;\mathrm{H})}=|f|$.

Hence, using inequality (A.16) with $s=s_n$ from (A.13) and then taking the limit as $n\to\infty$, we infer that

$$|\mathrm{A}z(t)|^2_{\mathrm{H}}\leq 2\int_{-\infty}^t|\mathrm{A}^{\frac12}f(r)|^2_{\mathrm{H}}\mathrm{e}^{c\int_r^t|\mathrm{A}z(\rho)|^2_{\mathrm{H}}\,d\rho}\,dr,\quad t\leq 0. \tag{A.17}$$

Therefore, we conclude that

$$\sup_{t\leq 0}|\mathrm{A}z(t)|^2\leq 2\mathrm{e}^{C\varphi(|f|^2)})\int_{-\infty}^0|\mathrm{A}^{\frac12}f(r)|^2_{\mathrm{H}}\,dr. \tag{A.18}$$

Moreover, by assumption (A.7), definition (A.5) of the function f and inequality (A.6) we have

$$\int_{-\infty}^0|\mathrm{A}^{\frac12}f(r)|^2\mathrm{e}^{c\int_r^0|\mathrm{A}z(\rho)|^2_{\mathrm{H}}\,d\rho}\,dr\leq\mathrm{e}^{c\int_{-\infty}^0|\mathrm{A}z(\rho)|^2_{\mathrm{H}}\,d\rho}\int_{-\infty}^0|\mathrm{A}^{\frac12}f(r)|^2\,dr<\infty,$$

we infer that

$$\lim_{t\to-\infty}\int_{-\infty}^t|\mathrm{A}^{\frac12}f(r)|^2_{\mathrm{H}}\exp\mathrm{e}^{c\int_r^t|\mathrm{A}z(\rho)|^2_{\mathrm{H}}\,d\rho}\,dr=0.$$

Hence, due to (A.17), we have that (A.9) holds for $\alpha = 1$, i.e.

$$\lim_{t\to-\infty} |z(t)|^2_{D(\mathrm{A})} = 0.$$

Now, let us prove the first one of inequalities (A.10), with $\alpha = 1$. For this aim, let us observe that from (A.15) we deduce that

$$|\mathrm{A}z(0)|^2_{\mathrm{H}} + \int_{-\infty}^{0} |\mathrm{A}^{\frac{3}{2}} z(t)|^2_{\mathrm{H}}\, dt \le c \int_{-\infty}^{0} |\mathrm{A}z(t)|^4_{\mathrm{H}}\, dt + 2\int_{-\infty}^{0} |\mathrm{A}^{\frac{1}{2}} f(t)|^2_{\mathrm{H}}\, dt$$
$$\le c \sup_{t\le 0} |\mathrm{A}z(t)|^2_{\mathrm{H}} \int_{-\infty}^{0} |\mathrm{A}z(t)|^2_{\mathrm{H}}\, dt + 2\int_{-\infty}^{0} |\mathrm{A}^{\frac{1}{2}} f(t)|^2_{\mathrm{H}}\, dt.$$

Taking into account inequalities (A.18) and (A.6), we infer that

$$|\mathrm{A}z(0)|^2_{\mathrm{H}} + \int_{-\infty}^{0} |\mathrm{A}^{\frac{3}{2}} z(t)|^2_{\mathrm{H}}\, dt \le 2\int_{-\infty}^{0} |\mathrm{A}^{\frac{1}{2}} f(t)|^2_{\mathrm{H}}\, dt \tag{A.19}$$
$$+ 2c\varphi(|f|^2)\mathrm{e}^{c\varphi(|f|^2)} \int_{-\infty}^{0} |\mathrm{A}^{\frac{1}{2}} f(r)|^2_{\mathrm{H}}\, dr,$$

and this concludes the proof of the first part of inequalities (A.10).

In order to prove the second of inequalities (A.10) (with $\alpha = 1$), i.e.

$$\int_{-\infty}^{0} |\mathrm{A}^{\frac{1}{2}} z'(t)|^2_{\mathrm{H}}\, dt < \infty$$

by the maximal regularity of the linear Stokes problem, it is enough to show that

$$\int_{-\infty}^{0} |\mathrm{A}^{\frac{1}{2}} B(z(t), z(t))|^2_{\mathrm{H}}\, dt < \infty.$$

According to inequality (2.15) (with $s = 2$ and $\alpha = 1$), we get, similarly to (A.19), the following estimate

$$\int_{-\infty}^{0} |\mathrm{A}^{\frac{1}{2}} B(z(t), z(t))|^2_{\mathrm{H}}\, dt \le C \int_{-\infty}^{0} |\mathrm{A}z(t)|^4_{\mathrm{H}}\, dt \tag{A.20}$$
$$\le 2c\varphi(|f|^2)\mathrm{e}^{c\varphi(|f|^2)} \int_{-\infty}^{0} |\mathrm{A}^{\frac{1}{2}} f(r)|^2_{\mathrm{H}}\, dr.$$

The proof, for $\alpha = 1$, is now complete.

The case $\alpha > 1$ has to be treated very carefully. To this purpose, we first consider the case $\alpha \in (1,2]$. We fix $\phi \in D(\mathrm{A}^{\frac{\alpha+1}{2}})$ and a function $z \in C((-\infty,0];\mathrm{H})$, such that $z(0)=\phi$, satisfying conditions (A.1) and (A.7), i.e.

$$\lim_{t\to-\infty} |z(t)|_{\mathrm{H}} = 0,$$

and

$$\int_{-\infty}^{0} |f(t)|^2_{D(\mathrm{A}^{\frac{\alpha}{2}})}\,dt := \int_{-\infty}^{0} |z'(t) + \mathrm{A}z(t) + \mathrm{B}(z(t),z(t))\,|^2_{D(\mathrm{A}^{\frac{\alpha}{2}})}\,dt < \infty.$$

Since the assumptions in the present proposition are stronger than the assumptions of Propositions A.1 and A.2, we can freely use the results from their proofs, see [3].

As before, it is sufficient to prove that z satisfies conditions (A.8), (A.9), and (A.10). We notice that, due to inequality (A.10) with $\alpha = 1$, we can find a decreasing sequence $\{s_n\}$ such that $s_n \searrow -\infty$ and

$$\lim_{n\to\infty} |\mathrm{A}^{\frac{3}{2}} z(s_n)|_{\mathrm{H}} = 0. \tag{A.21}$$

Hence, as $\alpha \le 2$, we get

$$\lim_{n\to\infty} |\mathrm{A}^{\frac{\alpha+1}{2}} z(s_n)|_{\mathrm{H}} = 0.$$

Therefore we can deduce that the function $|\mathrm{A}^{\frac{\alpha+1}{2}} u(t)|^2_{\mathrm{H}}$ is absolutely continuous and satisfies the following identity on $(-\infty,0]$

$$\frac{1}{2}\frac{d}{dt}|\mathrm{A}^{\frac{\alpha+1}{2}} z(t)|^2_{\mathrm{H}} + |\mathrm{A}^{\frac{\alpha}{2}+1} z(t)|^2_{\mathrm{H}} = -(B(z(t),z(t)),\mathrm{A}^{\alpha+1}z(t)) + (f(t),\mathrm{A}^{\alpha+1}z(t)) \tag{A.22}$$

By inequality (2.17), since $\frac{\alpha}{2} \le \frac{3}{2}$, we infer that

$$\begin{aligned} &-(B(z,z),\mathrm{A}^{\alpha+1}z) = (\mathrm{A}^{\frac{\alpha}{2}} B(z,z),\mathrm{A}^{1+\frac{\alpha}{2}}z) \le c|\mathrm{A}^{\frac{\alpha}{2}}z|_{\mathrm{H}}|\mathrm{A}^{\frac{\alpha+1}{2}}z|_{\mathrm{H}}|\mathrm{A}^{1+\frac{\alpha}{2}}z|_{\mathrm{H}} \\ &\le \frac{1}{4}|\mathrm{A}^{1+\frac{\alpha}{2}}z|^2_{\mathrm{H}} + c|\mathrm{A}^{\frac{\alpha}{2}}z|^2_{\mathrm{H}}|\mathrm{A}^{\frac{\alpha+1}{2}}z|^2_{\mathrm{H}} \le \frac{1}{4}|\mathrm{A}^{1+\frac{\alpha}{2}}z|^2_{\mathrm{H}} + c|\mathrm{A}^{\frac{3}{2}}z|^2_{\mathrm{H}}|\mathrm{A}^{\frac{\alpha+1}{2}}z|^2_{\mathrm{H}}. \end{aligned}$$

Due to (A.22), this implies that

$$\frac{d}{dt}|\mathrm{A}^{\frac{\alpha+1}{2}}z(t)|^2_{\mathrm{H}} + |\mathrm{A}^{\frac{\alpha}{2}+1}z(t)|^2_{\mathrm{H}} \le c|\mathrm{A}^{\frac{3}{2}}z(t)|^2_{\mathrm{H}}|\mathrm{A}^{\frac{\alpha+1}{2}}z(t)|^2_{\mathrm{H}} + 2|\mathrm{A}^{\frac{\alpha}{2}}f(t)|^2_{\mathrm{H}}.$$

Therefore, by the Gronwall Lemma, for any $-\infty < s \le t \le 0$ we get

$$|A^{\frac{\alpha+1}{2}} z(t)|_H^2 \le |A^{\frac{\alpha+1}{2}} z(s)|_H^2 \exp\left(c \int_s^t |A^{\frac{3}{2}} z(r)|_H^2 \, dr \right) \qquad (A.23)$$
$$+ 2 \int_s^t |A^{\frac{\alpha}{2}} f(r)|_H^2 \exp\left(c \int_r^t |A^{\frac{3}{2}} z(\rho)|_H^2 \, d\rho \right) dr.$$

Note that by inequality (A.12) with $\alpha = 1$ we have

$$\sup_{n\ge 1} \int_{s_n}^t |A^{\frac{3}{2}} z(r)|_H^2 \, dr \le \varphi_1(|f|_{\frac{1}{2}}^2),$$

where we use notation shortcut $|f|_{\alpha/2} = |f|_{L^2(-\infty,0);D(A^{\frac{\alpha}{2}})}$. Hence, using inequality (A.23) with $s = s_n$ from (A.21) and then taking the limit as $n \to \infty$, we infer that

$$|A^{\frac{\alpha+1}{2}} z(t)|_H^2 \le 2 \int_{-\infty}^t |A^{\frac{\alpha}{2}} f(r)|_H^2 e^{c \int_r^t |A^{\frac{3}{2}} z(\rho)|_H^2 \, d\rho} \, dr, \quad t \le 0, \qquad (A.24)$$

so that

$$\sup_{t\le 0} |A^{\frac{\alpha+1}{2}} z(t)| \le 2 \int_{-\infty}^0 |A^{\frac{\alpha}{2}} f(r)|_H^2 \, dr e^{\varphi_1(|f|_{\frac{1}{2}}^2)} = 2|f|_{\frac{\alpha}{2}}^2 e^{\varphi_1(|f|_{\frac{1}{2}}^2)}. \qquad (A.25)$$

Moreover, as

$$\int_{-\infty}^0 |A^{\frac{\alpha}{2}} f(r)|_H^2 e^{c \int_r^t |A^{\frac{3}{2}} z(\rho)|_H^2 \, d\rho} \, dr < \infty,$$

we have that

$$\lim_{t\to -\infty} \int_{-\infty}^t |A^{\frac{\alpha}{2}} f(r)|_H^2 e^{c \int_r^t |A^{\frac{3}{2}} z(\rho)|_H^2 \, d\rho} \, dr = 0,$$

and (A.9) follows from (A.24).

Next, we observe that, due to (A.15),

$$|A^{\frac{\alpha+1}{2}} z(0)|_H^2 + \int_{-\infty}^0 |A^{\frac{\alpha+2}{2}} z(t)|_H^2 \, dt \le c \int_{-\infty}^0 |A^{\frac{3}{2}} z(t)|^2 |A^{\frac{\alpha+1}{2}} z(t)|_H^2 \, dt$$
$$+2 \int_{-\infty}^0 |A^{\frac{\alpha}{2}} f(t)|_H^2 \, dt \le c \sup_{t\le 0} |A^{\frac{\alpha+1}{2}} z(t)|_H^2 \int_{-\infty}^0 |A^{\frac{3}{2}} z(t)|_H^2 \, dt + 2|f|_{\frac{\alpha}{2}}^2.$$

Taking into account inequalities (A.25) and (A.10) with $\alpha = 1$, we infer that

$$|\mathrm{A}^{\frac{\alpha+1}{2}} z(0)|_{\mathrm{H}}^2 + \int_{-\infty}^{0} |\mathrm{A}^{\frac{\alpha+2}{2}} z(t)|_{\mathrm{H}}^2\, dt \leq 2\,|f|_{\frac{\alpha}{2}}^2 + c|f|_{\frac{\alpha}{2}}^2 \mathrm{e}^{\varphi_1(|f|_{\frac{1}{2}}^2)} \varphi_1(|f|_{\frac{1}{2}}^2) \tag{A.26}$$

and this concludes the proof of the first part of inequality (A.10).

Invoking the maximal regularity of the Stokes evolution equation, in order to prove the second inequality in (A.10), it is enough to show that

$$\int_{-\infty}^{0} |\mathrm{A}^{\frac{\alpha}{2}} B(z(t), z(t))|_{\mathrm{H}}^2\, dt < \infty.$$

According to inequalities (2.17), (A.25) and (A.10) with $\alpha = 1$, we have

$$\begin{aligned} \int_{-\infty}^{0} |\mathrm{A}^{\frac{\alpha}{2}} B(z(t), z(t))|_{\mathrm{H}}^2\, dt &\leq c \int_{-\infty}^{0} |\mathrm{A}^{\frac{\alpha}{2}} z(t)|_{\mathrm{H}}^2 |\mathrm{A}^{\frac{\alpha+1}{2}} z(t)|_{\mathrm{H}}^2\, dt \\ &\leq c \sup_{t\leq 0} |\mathrm{A}^{\frac{\alpha+1}{2}} z(t)|_{\mathrm{H}}^2 \int_{-\infty}^{0} |\mathrm{A}^{\frac{3}{2}} z(t)|_{\mathrm{H}}^2\, dt \\ &\leq |f|_{\frac{\alpha}{2}}^2 \mathrm{e}^{\varphi(|f|_{\alpha/2}^2)} \varphi_1(|f|_{\frac{1}{2}}^2). \end{aligned} \tag{A.27}$$

The proof in the case $\alpha \in (1, 2]$ is now complete. A simple extension of the last argument and mathematical induction with respect to the integer part of α can provide a complete proof for all $\alpha \geq 0$. □

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