Just Tuning and the Unavoidable Discrepancies

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I. *Introduction*

Although in the past just tuning has been more of a theoretical than a practical music concept, the advent of computerized instruments and synthesizers makes it quite possible that a piece of music could be programmed to be performed always in just tuning, perhaps with the tuning scheme modulating in concordance with modulations in the composition. In any event, a keyboard-driven synthesizer could easily be programmed to perform nonchromatic music in just tuning, without the necessity of manual retuning for different keys. For these reasons, I have thought it worthwhile to study the just-tuning scheme in some detail, with a view to improving it. I have chosen the scheme described in Thomas D. Rossing's book, Table 9.3, as the paradigm with which to begin my study.[1]

To review briefly what is meant by the term "just tuning," the I, IV, and V chords are all required to obey the canonical frequency ratios of 1:5/4:3/2 = 4:5:6. (For convenience, the canonical—or

[1]Thomas D. Rossing, *The Science of Sound*, 2d ed. (Reading, MA: Addison-Wesley Pub. Co., 1990), 181. This table gives the frequency ratios of the principal twenty-two chromatic notes in a single octave relative to the lowest note, denoted C. The octave is denoted C'. To retune in another key, simply replace C with the relevant tonic note.

just—ratios for various intervals are given in Appendix C, Table 1.) Serendipitously, the iii and vi chords also come out justly, in the ratios 10:12:15. The frequency ratios for the white keys are presented in Table 2, assuming that the scheme has been applied in the key of C. For any other key, it is simply necessary to transpose appropriately, C→Do, D→Re, etc. The necessity for retuning with each transposition is what renders the scheme essentially useless for conventional keyboard instruments such as the piano, but quite feasible for computer-driven or even computer-coupled instruments, as described above.

During the Middle Ages, keyboard instruments were commonly tuned by the Pythagorean method.[2] This involved tuning eleven consecutive beatless intervals around the circle of fifths. (Beatless intervals are easy to tune, explaining the popularity of the method.) Deficiencies of the scheme included the facts that the twelfth beatless fifth did not close the circle (the "ditonic comma," described below) and that the major thirds were badly mistuned (the "syntonic comma," also described below). In fact, in Medieval music the third was considered a dissonance, and so polyphonic compositions were generally written in organum—parallel fourths, fifths and octaves.

The polyphony introduced in the Renaissance was still compatible with the Pythagorean scheme, since chords were not stressed. It was the introduction of homophonic music (monody) at the beginning of the Baroque era (to accommodate opera—the words had to be understood!) and the related use of the basso continuo chordal accompaniment which required an improvement. The first and most obvious improvement was a just scheme as described above, with the major deficiency as noted, namely that it was only "just" in a single (major) key. Like the Pythagorean tuning, just schemes were relatively easy to tune because beatless chords were set.

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[2]A rather nice historical survey of tunings and temperament from a scientist's point of view is given by Stillman Drake, "Renaissance Music and Experimental Science," *Journal of the History of Ideas* XXXI (1970): 483-500. For a musicologist's approach, see J. Murray Barbour, *Tuning and Temperament: A Historical Survey* (East Lansing: Michigan State University Press, 1951), especially chapter 1.

The eventual necessity to "circulate"—to play in different keys without continual retuning (think of the *Well-Tempered Clavier*)—and chromaticism led to meantone and, ultimately, equal temperament.[3] The assumption on which this article is based is that the just-tuning scheme is the best in terms of musical quality, if only its deficiencies could be overcome, as evidently they might be by computers.[4]

The frequency ratios given in Table 1 represent only one possible just-tuning scheme. Many (but not all) of these agree insofar as the diatonic notes are concerned, but there is no general agreement with regard to accidentals (let alone the double accidentals). Examples may be found in various references.[5] Of these only Rossing and Helmholtz present a scheme which distinguishes between enharmonic notes. Rossing's scheme, which is similar to "Marpurg's First Just Monochord,"[6] is to introduce the lesser diesis (defined below) as the interval between all pairs of enharmonic notes, such as B♭–A♯, F–E♯,

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[3]Rossing, 76-80. Or, better, see Donald E. Hall, *Musical Acoustics*, 2d ed. (Pacific Grove, CA: Brooks/Cole Pub. Co., 1990), 412-420.

[4]John Fonville, "Ben Johnston's Extended Just Intonation: A Guide for Interpreters," *Perspectives of New Music* 29, no. 2 (1991): 106-37. Fonville (107), quoting Ben Johnston, is of the opinion that "Just intonation . . . results in greatly heightened purity and clarity of sound." Johnston's 21-note "Just Enharmonic Scale" turns out to be identical with my just scale (Table 4, Column Z) with the double accidentals deleted. It is interesting that Johnston arrived at this scale in a different fashion, i.e., by enforcing consonant relations on higher partials, whereas I arrived at it from the simple Axiom of Chromatic Invariance (Section III). Johnston did not use this 21-note scale in composition; rather he used it as the basis for a microtonal scale. Its application in composition has been studied by Steven Eliter, "A Harmonic and Serial Analysis of Ben Johnston's String Quartet No. 6," *Perspectives of New Music* 29, no. 2 (1991): 138-65.

[5]Rossing, 411; Barbour, 89-102; Hermann L. F. Helmholtz, *On the Sensations of Tone as a Physiological Basis for the Theory of Music*, trans. Alexander J. Ellis, 2d ed. (New York: Dover Pub., 1954), 440; J. Murray Barbour and Fritz A. Kuttner, "The Theory and Practice of Just Tuning," Notes accompanying *Theory Series A* (New York: Musurgia Records, 1955), Record A-2.

[6]Barbour and Kuttner.

etc. He does not consider doubly accidental enharmonics (D♭♭–C, G–F♯♯, etc.).[7] Helmholtz's work is of such historical importance that I have devoted Section IV to a brief analysis of it. The other authors' schemes basically differ only in which enharmonic note is chosen for the chromatic scale, i.e., some choose A♭, others G♯, etc.

To understand how the various accidentals are introduced, it is necessary to consider the so-called "four unavoidable tuning discrepancies."[8] These are described musically and mathematically in Table 3. In words:

(1) The syntonic comma (SC) is the ratio by which four perfect fifths exceed a major third (after the fifths have been brought down twice to stay within the octave). Musically, the sequence of notes would be, for example, C→G→D'→D→A→E'→E, which is then compared to the note whose frequency is 5/4 that of C. The E arrived at by the above sequence is sharp, in the ratio 81/80.

(2) The ditonic comma (DC) is the ratio by which a complete circuit of the circle of fifths (twelve fifths) exceeds seven octaves. The circuit runs C→G→D→A→E→B→F♯→C♯→G♯→D♯→A♯→E♯→B♯. Thus, it might be thought that the DC is the ratio by which B♯ is sharp with respect to C. This, however, is not the case, as will become clear in Section 2.

(3) The lesser diesis (LD) is the ratio of the octave to three major thirds. Starting with C, the octave is C', whereas three major thirds bring us to B♯. C' is higher in pitch than B♯ by the LD. The discrepancy between the B♯ defined here and that defined in

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[7]Just tuning of the double accidentals is not simply an academic exercise, since such notes are required for the proper spelling of various important chords. For example, the diminished seventh chord in C minor is C–E♭–G♭–B♭♭. Augmented chords involve double sharps, i.e., G♯–B♯–D♯♯, F♯–A♯–C♯♯.

[8]Hall, 410-11. In the future, I shall use the standard notation X' for the note an octave above X (see, for example, Christine Ammer, "Chord," in *Harper's Dictionary of Music* [New York: Barnes and Noble, 1972], 67).

(2) above illustrates the ambiguity which may arise and the care with which one must define the pitches of the sharps and flats. From these definitions, the discrepancy between the two is seen to be LD + DC.

(4) The greater diesis (GD) is the ratio of four minor thirds to the octave—the relevant sequence is C—E♭—G♭—B♭♭—D♭♭.

It is worth noting that the definition of the ditonic comma stated in (2) above could be expressed in terms of the circle of fourths: C—F—B♭—E♭—A♭—D♭—G♭—C♭—F♭—B♭♭—E♭♭—A♭♭—D♭♭. Musicians tend to think of this sequence as the *descending* circle of fifths, but I believe it is better to think of it as the *ascending* circle of fourths. Then the DC is the amount by which twelve fourths is *flat* with respect to five octaves, i.e., $2^5 \div (4/3)^{12}$. Similarly, the syntonic comma is the amount by which a minor third exceeds three perfect fourths (brought down one octave):

$$SC = 6/5 \div 1/2(4/3)^3$$

It is interesting that these alternate and perfectly valid definitions of the commas do not appear in most texts. The D♭♭ obtained by following the circle of fourths twelve steps is flat of the D♭♭ of definition (4) by the sum of the GD and the DC, or about a semitone.

In Section 2, I study these four discrepancies and prove that there are two relations between them. (Mathematicians would say that they are "linearly dependent.") These relations will explain, for example, the difference between the B♯'s defined in (2) and (3) above, and allow us to decide that the B♯ of paragraph (3) is the right one to adopt in a just-tuning scheme. They will also provide insight into the structure of scales and especially into the frequency ratios of enharmonic notes, as discussed in Section 3 where I present my proposed just-tuning scheme. As indicated above, Section 4 is denoted as a short discussion of Helmholtz's work on tunings.

I have also provided three appendices. In Appendix A, I explain and illustrate "interval arithmetic," the mathematical procedure for dealing rigorously with musical intervals. (The reader should refer to this appendix before reading further.) One of the illustrations is the tritonal discrepancy G♭/F♯ which I use to help argue that the Rossing just-tuning scheme is deficient. In Appendix B, *I take out a lifetime of frustration with choral conductors who have for fifty years accused me of singing flat*. In Appendix C, I provide tables which the reader is referred to throughout the course of the paper. I would like to acknowledge fruitful discussions with Professors Andrew Schloss and Reinhard Illner, of the University of Victoria, B.C., and especially Professor Donald Hall of California State University.

II. *The Relations*

Using the ratio definitions of the two commas and the two dieses (Table 3), I wish to prove the following two relations:

Relation 1. SC = GD - LD
Relation 2. LD + DC = 3SC

As explained in Appendix A, a proof cannot be constructed using the values of the discrepancies in terms of cents. Thus we use "interval arithmetic" as explained in that Appendix. However, it is easy to check, again using Table 3, that the two relations are indeed satisfied in cents to three places; cents are defined in Appendix A. It was this discovery which led me to suspect that the relations were rigorous and led me to construct the proofs which I now give. I shall present both arithmetical and musical proofs, the latter to cast some insight into the origin and meaning of the two relations.

Proof of Relation 1. Arithmetical:

Referring to Table 3, we see that it is sufficient to prove that

Equation (1) $$81/80 = (648/625) \div (128/125).$$

It is better not to carry this calculation out on a pocket calculator, as round-off error might occur in the decimal representations of these fractions. Rather, express all six numbers above in terms of prime factors, and then cancel:

$$81 = 3^4$$
$$80 = 5*2^4$$
$$648 = 2^3*3^4$$
$$625 = 5^4$$
$$125 = 5^3$$
$$128 = 2^7$$

If the reader will take the trouble to substitute these values into Equation (1), he or she will find that it reduces to an arithmetic identity.

Proof of Relation 2. Arithmetical:

Again, using Table 3, it is seen that it is necessary to verify that

$$(128/125) * (3^{12}/2^{19}) = (81/80)^3.$$

Using the prime factors provided above and cancelling like terms between numerator and denominator, the reader can easily convince himself or herself that this also is an identity.

Proof of Relation 1. Musical:

From Table 3,

Equation (2) $\quad SC = (4P5 - 2P8) - M3.$

Now, it is a well-known fact that a perfect fifth is the sum of a major third plus a minor third:

Equation (3) $\quad P5 = M3 + m3.$

Substituting Equation (3) into Equation (2) gives

$$SC = (4M3 + 4m3 - 2P8) - M3;$$

rearranging,

$$\begin{aligned} SC &= (4m3 - P8) - (P8 - 3M3) \\ &= (GD - LD) \end{aligned}$$

again using the definitions in Table 3.

Similarly,

Proof of Relation 2. Musical:

Begin with Equation (2) and multiply by 3 to obtain

$$12P5 - 6P8 - 3M3 = 3SC.$$

This is the same as

$$(12P5 - 7P8) + (1P8 - 3M3) = 3SC,$$

or, again referring to Table 3

$$DC + LD = 3SC, \text{ qed.}$$

I should like to present a couple of applications of these results. First, refer to the just-tuning scheme of Table 2, and observe that there are two types of whole tones: a major whole tone of 9/8 (D/C, G/F, B/A) and a minor whole tone of 10/9 (E/D, A/G). The ratio of these, $9/8 \div 10/9 = 81/80$, is the syntonic comma.[9] The syntonic comma

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[9]The notation Y/X for two notes X and Y means the ratio of the frequencies of Y to X. The notation |X - Y| will be used for the cents on the interval starting with X and ending with Y, $Y > X$.

should be thought of as the amount by which E and A are flatted in forming a just scale rather than in terms of definition 1 of Section 1. This makes it clear why the E produced by the sequence of four perfect fifths is sharp; it is in fact the major third which is slightly flat, by the SC. Similarly, the major sixth is flat by the SC when compared with a sequence of three fifths. That the magnitude of this discrepancy is precisely the same as the discrepancy of E has not always been appreciated in the literature.[10] Incidentally, this latter discrepancy can create problems with string quartets who tune to A in perfect fifths and is one reason why they eschew open strings. The inability to vibrate is another.

We can now see that the B♯'s of definitions 2 and 3 of Section 1 differ because of Relation 2; i.e., the B♯ which follows after twelve perfect fifths, starting with C, has involved three cycles of four perfect fifths (3 * 4 = 12), each introducing a syntonic comma. Relation 2 tells us that if we reduced the pitch by the SC after each cycle—thus arriving at the just M3—the end note would be flat of C' by the LD rather than sharp by the DC. Thus, the B# of definition 3 (three major thirds above C) is the correct one for chordal integrity! Similarly, I have shown in Appendix A that the tritonic discrepancy G♭/F♯ = GD.

Now it is generally accepted[11] that

$$A\flat/G\sharp = LD.$$

The argument is that

$$A\flat = C' - M3,$$

implying

$$A\flat/C = 2 \div 5/4 = 8/5,$$

[10]Frank Budden, *The Fascination of Groups* (London: Cambridge University Press, 1972), 433.

[11]Hall, 411.

while

$$G\sharp = C + 2M3,$$

so that

$$G\sharp/C = (5/4)^2 = 25/16.$$

The ratio of these two intervals

$$8/5 \div 25/16 = 128/125,$$

is the LD. So the question is, why is

$$A\flat/G\sharp = \text{LD, but } G\flat/F\sharp = \text{GD?}$$

The answer is in Relation 1. In the transition from A♭→G♭, we are dropping a minor whole tone, while the transition G♯→F♯ is a *major* whole tone. Thus, the interval G♭/F♯ stretches by the ratio of the two whole tones, i.e., by the SC. Adding this to the LD yields the GD (by Relation 1). The above verbal argument can be expressed algebraically:

Equation (4) $$G\flat/F\sharp = G\sharp/F\sharp * A\flat/G\sharp \div A\flat/G\flat$$

$$= \text{MWT} + \text{LD} - \text{mwt}$$
$$= \text{LD} + \text{SC}$$
$$= \text{GD}.$$

The last set of equations illustrates how elegantly musical structure can be described algebraically! (To verify Equation (4), treat the symbols algebraically and cancel.) The axiom of chromatic invariance, introduced in the next section, has been assumed in obtaining this result, i.e., I have set G♯/F♯ = G/F.

It might be noted that the two tritones embedded in the white keys, i.e., the diminished fifth B–F and the augmented fourth F–B, differ exactly by LD - SC. (The reader is invited to prove this fact, using tables 2 and 3.) It is important to remember that the tritonic discrepancy I have referred to previously is an *enharmonic* tritonic discrepancy, i.e., the amount by which the octave exceeds the sum of the two augmented fourths | C' - F♯ | + | G♭ - C | (or by which the sum of the diminished fifths | C' - G♭ | + | F♯ - C | exceeds the octave). Another application of Relations 1 and 2 is presented in Appendix C.

III. *The Just Scale*

The "white notes" of the scale of just intonation are well accepted, as given in Table 2. The problem is to fill in the accidentals. The scheme used in Reference 2, leading to Rossing's Table 9.3, is to assume that all enharmonic pairs involving at most one sharp and/or one flat (i.e., diminished seconds) differ by the lesser diesis. This portion of musical folklore is actually viewed with skepticism in some circles.[12] Although Rossing does not include double sharps and double flats in his scheme, I have extrapolated the above rule to assume that all enharmonic pairs which *do* involve double flats and/or double sharps differ by the greater diesis (since D♭♭/C = GD—see definition 4, Section 1).

This leads to the scheme of Table 4, column *R*, where I have now switched to cents (since I have already worked out the proofs). Slight discrepancies, by ~1 cent, are due to round-off error and should be ignored. The scheme I propose is based on the following axiom, which I call the

Axiom of chromatic Invariance: The interval between any pair of notes is not affected by chromatic alteration.

From this axiom we propose

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[12]Donald E. Hall, Private Communication (1986).

Theorem: All augmented unisons have the value 71 cents.

Proof: First, it is easy to calculate $|B - B\sharp| = 71$ cents. For (Table 2) $|B| = 1088$ cents. Also $|B\sharp - C'| = LD = 41$ cents, so $|B\sharp| = 1200$ cents $- 41$ cents $= 1159$ cents $= 1088$ cents $+ 71$ cents.

Next, the axiom implies that for any two notes X and Y,

$$X/Y = X\sharp/Y\sharp;$$

so that

$$\begin{aligned} X/X\sharp &= Y/Y\sharp \\ &= B/B\sharp; \end{aligned}$$

thus

$$|X - X\sharp| = 71 \text{ cents.}$$

From the axiom again,

$$X/X\sharp = X\flat/X;$$

so

$$|X\flat - X| = 71 \text{ cents also.}$$

Similarly,

$$\begin{aligned} |X\flat\flat - X\flat| &= |X\sharp - X\sharp\sharp| \\ &= 71 \text{ cents,} \end{aligned}$$

and the theorem is proved.

Before I present my proposed just-tuning scheme based on the

"71-cent rule," I want to show that some previously determined pitches obey it. For example, we know that

$$| D\flat\flat - C | = GD = 63 \text{ cents.}$$

From the theorem,

$$\begin{aligned} | D\flat | &= | D | - 71 \text{ cents} \\ &= 204 \text{ cents} - 71 \text{ cents} \\ &= 133 \text{ cents} \end{aligned}$$

so that

$$\begin{aligned} | D\flat\flat - D\flat | &= 133 \text{ cents} - 63 \text{ cents} \\ &= 70 \text{ cents} \end{aligned}$$

as the Theorem avers (recall that discrepancies of 1 cent are supposed to be ignored due to the round-off error).

Similarly, one computes

$$A\flat/G\sharp = A\flat/A * A/G * G/G\sharp,$$

implying

$$\begin{aligned} | A\flat - G\sharp | &= - | A\flat - A | + | G - A | - | G\sharp - G | \\ &= -71 \text{ cents} + 182 \text{ cents} - 71 \text{ cents} \\ &= 40 \text{ cents.} \end{aligned}$$

This is the classic example of the lesser diesis,[13] so again there is agreement.

Using this theorem, it is possible (and easy) to compute all of the pitches of the accidentals and double accidentals. These are given in Table 4, Column Z. The pitches in my scheme which differ from Rossing's are underlined.

[13]See note 11 above.

The two just-tuning schemes have been compared in Tables 5 and 6. First Table 5: all enharmonic pairs are listed—note that they occur in threes, except for the specific pair G♯—A♭. The intervals between the pairs are presented according to my scheme, by reference to Table 4, Column Z (I have written "L" and "G" for "LD" and "GD" to decrease clutter). According to Rossing's scheme, all intervals with single sharps and/or flats would be marked "L" and all others "G." Perhaps it is not obvious which scheme should be preferred on the basis of this comparison, but I point out that at least the tritonic discrepancy, G♭/F♯, is the GD in my scheme, whereas in the Rossing scheme G♭/F♯ is the LD. One could also argue that | C - D♭♭ | should be the same as | C♯ - D♭ |; it is in mine, but not in Rossing's.

The structure of Table 5 can be understood in another way, combining the 70-cent rule with Relation 1 of Section 2. Consider a major whole tone, for example, F—G = 204 cents. Then | F♯ | = | F | + 70 cents, | G♭ | = | G | - 70 cents so that | G♭ - F♯ | = | G - F | - 140 cents = 64 cents = GD. If this calculation is repeated for a minor whole tone, say G — A, the interval | A - G | is less than the interval | G - F | by the SC so | A♭ - G♯ | is smaller by the SC, i.e., GD - SC = LD.

Let us turn now to Table 6, where various intervals are compared according to the two schemes. We observe:

(1) The Rossing scheme is seen to have eleven mistuned fifths, compared to my nine (mistuned fifths not including double sharps or double flats are marked with asterisks; I have five, Rossing six).

(2) Rossing has eight mistuned major thirds compared with my four, although none of his are asterisked while two of mine are.

(3) Rossing is slightly better in minor thirds (five to five total, but two to three in asterisks).

In the major second category, Rossing's scheme seems really to fail in that intervals involving major whole tones evolve to minor whole tones (and vice versa) once accidentals are introduced. (Mine are guaranteed to be correct by the Axiom of Chromatic Invariance.) One interval (E♭♭ – F♭) is even a horrendous 160 cents. Minor seconds are also better according to my scheme, 12 to 16 (although I lose 6 to 5 in asterisks) with two of Rossing's being a dreadful 155 cents. (In fact, the Rossing scheme does satisfy the principle of coherence[14] but only barely, in that these minor seconds are within 5 cents of the 160 cents major second cited above.) Finally, Rossing has fourteen mistuned augmented unisons (three asterisks); eleven of them are a completely unacceptable 48 cents wide.[15]

It must be admitted that in tuning chords, the Rossing scheme is comparable to the one which I suggest. Indeed, it is better in some respects. For example, of the 21 possible pitches listed in Table 4 which do not involve double sharps or flats, there are 63 chords (diminished, minor, major). The chords may involve double sharps or flats but not as the tonic note. In my scheme, 45 of these chords are tuned justly (vs. 40 in Rossing's scheme). If we eliminate the diminished chords from the statistics, there are 42 major and minor chords; of these, 30 are tuned justly according to my scheme, 29 according to Rossing's.

If we consider a keyboard instrument with fixed tuning, and assume that the black keys are split, there are 13 major and 13 minor chords which can be played. These are listed in Table 7. According to my scheme, 18 of the 26 chords are tuned justly, while Rossing has 20 just tuned.

Even without split black keys, one can obtain more justly tuned chords than are generally supposed. If the black keys are tuned to C♯, D♯, F♯, G♯, and A♯ or D♭, E♭, G♭, A♭, and B♭, one can play 16 chords; both my tuning scheme and Rossing's tune 12 of these justly (the other 8 chords are enharmonically altered, and involve a diesis discrepancy). However, both Hall and Kuttner suggest tuning C♯, E♭,

[14]Gerald J. Balzano, "The Group-Theoretic Description of the 12-Fold and Microtonal Pitch Systems," *Computer Music Journal* 4 (1980): 66-84.

[15]Recall that in my scheme all augmented unions are 70 cents wide.

F♯, G♯, and B♭, which allows a marginal improvement.[16] In this case, there are actually 17 chords which can be played without enharmonic alteration and 13 which can be tuned justly (see Table 8).

I will leave it up to the reader to decide the merits of the two schemes, although I will admit to being biased in favor of my own. Ultimately, computers will be programmed as described in Section 1 and we shall be able to hear the two schemes performed, after which the final judgement should be possible.

IV. *Connection with Helmholtz*

In Helmholtz's book, there are extensive discussions of tunings and scales—just and tempered—as well as the consonance and dissonance of intervals and chords.[17] Included are references to and descriptions of musical scales and tunings from antiquity to the present and from all parts of the world. It is a book which no serious musical acoustician should be without, even though its 19th-century Teutonic organization and somewhat formal style make it difficult to read. There are also numerous addenda by the translator, Alexander J. Ellis.

As complete as Helmholtz's book is, it does have some gaps. For example, the greater diesis, GD, is never introduced. The lesser diesis, LD, is given but is referred to as the "Greater Diesis."[18] It is possible that Ellis or Helmholtz was aware of Relation 1, although it is never mentioned in either text or appendices as far as I can determine. Then the specification of the GD would be superfluous, as it could be inferred as LD + SC, both of which are given on page 453 (although the latter is referred to as the "Comma of Didymus!").

A very useful idea introduced by Helmholtz is that of revising the

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[16]Hall, 414; Barbour and Kuttner (see note 5 above).

[17]Helmholtz, 250-79 and 421-513.

[18]Ibid., 453.

tuning scheme to fit each mode,[19] a subject I intend to pursue in a future paper (thus in the minor mode, the i, iv, and v chords would be tuned up—to 10:12:15). To motivate these tunings he presents a discussion of harmonization of the various modes.[20] His names for the modes are more descriptive than our present Dorian, Phrygian, etc. The mode is named for the interval from the modal tonic up to the relative major. Thus, the Mixolydian mode would be called the "Mode of the Fourth" and the Dorian the "Mode of the Minor Seventh," etc.

Helmholtz also introduced some useful notation which is no longer seen. A note symbol without a subscript or a superscript such as A, B, C, etc., refers to a note obtained from C by following the circle of fifths—or fourths. A symbol with a numerical subscript, E_n, represents a note which is n syntonic commas flat of E while a superscript, E^n, represents a note sharp by the same amount. Using this notation, he describes an instrument designed to play in just intonation in all keys;[21] these ideas are extended further,[22] but I have found it impossible to figure out a full "just" tuning in any given key, such as Rossing's or mine presented in Table 5. (It is gratifying to think that the computer may now serve to fulfill Helmholtz's dream of playing justly in all keys without retuning.) By referring to Helmholtz,[23] I have written out the names of the notes of my suggested just scale using Helmholtz's notation (see Table 9). Note that the flats are always superscripted and the sharps and naturals subscripted. (Double sharps and flats are omitted.) In the second column of this table, I have indicated that the sum of the subscripts and superscripts of enharmonic pairs adds to three when the pairs differ by the LD and by four when they differ by the GD. This is a consequence of both Relation 1 and

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[19]Ibid., 277.

[20]Ibid., 290-309.

[21]Ibid., 312.

[22]Ibid., Appendix XVII. This appendix was written by Alexander J. Ellis.

[23]Ibid., 403.

Relation 2. Consider, for example, the pair $F_2\sharp$ and $G^2\flat$: $G\flat - F\sharp =$ -DC (by definition). Since $F_2\sharp = F\sharp - 2SC$ and $G^2\flat = G\flat + 2SC$, then $| G^2\flat - F_2\sharp | = | G\flat - F\sharp + 4SC | = -DC + 4SC = GD$ by Relations 1 and 2. Rossing's just scale is obtained from Table 9 by replacing $D^2\flat$ with $D^1\flat$, $F_2\sharp$ by $F_1\sharp$, and $A_3\sharp$ by $A_2\sharp$.

V. *Conclusion*

A number of novel ideas have been introduced in this paper. They include the following:

(1) Symbolic representation and algebraic manipulation of musical intervals

(2) "Interval arithmetic" (Presented in Appendix A)

(3) Reduction of the "four unavoidable tuning discrepancies" to two

(4) A suggested new scale of just intonation

(5) A consistent assignment of the appropriate diesis to each possible enharmonic pair

It seems remarkable that item 3 above has not been discovered previously, but it appears that it has not. Modern books still routinely refer to the "four unavoidable discrepancies."[24] Hopefully, at least some of these ideas will be useful to musicians and music theorists. Eventually, computers may be programmed to implement the just-tuning scheme. One (certainly solvable) problem on a keyboard synthesizer would be the assignment of the appropriate member of an

[24]Hall, 414.

enharmonic trio to a particular key for a given note.[25] However, complete electronic programming, note by note, would not suffer from this problem. In any event, the use of a just tuning-scheme should make for a consonant harmonic structure seldom heard today in keyboard instruments.

Appendix A: Interval Arithmetic

Musicians often think of intervals in terms of cents. Using the notation described in Footnote 9 from Section 2, for two notes X and Y,

Equation (A-1) $$| X - Y | = 1200 \log_2 Y/X.$$

Since logarithms to base 2 are not readily calculated, it is convenient to rewrite Equation (A-1) in the following form which is easy to evaluate on a pocket calculator:

Equation (A-1') $$| X - Y | = 3986 \log Y/X.$$

In Equation (A-1'), the logarithm is the ordinary log, i.e., base 10.

From Equation (A-1), we note (recalling Table 1) that for the interval of the perfect octave Y/X = 2 and that the octave contains 1200 cents ($\log_2 2 = 1$). Thus, 100 cents corresponds, roughly speaking, to a semitone. We should keep this in mind to have a musical feeling for the meaning of the cent unit. Other helpful concepts:

(1) Hall suggests that professional musicians frequently play 12–20

[25]See Douglas Leedy, "A Venerable Temperament Rediscovered," *Perspectives of New Music* 29, no. 2 (1991): 202-11. Leedy (210) quotes Wendy Carlos: "I suggest that we go back to meantone, using computers to keep recentering the pitch so [that] you avoid the wolf tones." This is essentially my own suggestion, except for going back to just rather than meantone; cf., note 4 above.

cents out of tune.[26]

(2) Many professional musicians have claimed (and proved to me) that they can distinguish between A440 and A441. This corresponds to 4 cents and might be considered the ''JND'' (''Just Noticeable Difference'').[27]

Having belabored this point, I shall now explain why sometimes we cannot use cents in our work. To prove that two numbers are equal, it would be sufficient to prove that their logarithms are equal. But in general, rational numbers have irrational logarithms—i.e., numbers whose decimal representations are infinite and nonrepeating. Clearly, no finite algorithm can be constructed to prove that two such representations are equal. Since the frequency ratios that we deal with are all rational (3/2, 6/5, 648/625), we cannot use cents, which involve logarithms, but we must deal directly with the ratios according to the following rules:

Rule 1: Let I_1 be an interval corresponding to frequency ratio r_1, and I_2 an interval with frequency ratio r_2. Then the interval $I_1 + I_2$ has the ratio $r_1 \times r_2$.

Example 1: From Table 1, we see that the ratio of the M3 is 5/4 and of the m3, 6/5. Then, since P5 = M3 + m3, the ratio of P5 = 5/4 x 6/5 = 3/2, which agrees with the ratio given in Table 1.

Example 2: Referring to Table 1, we see that M6 + m3 has the ratio 5/3 x 6/5 = 2. This simply states that the M6 and m3 are inversions, i.e., their sum is the octave.

Rule 2: Tuning discrepancies are also computed by ratio.

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[26]Hall, 402.

[27]Hall, 97; Rossing, 111.

Example 3: Consider LD (Table 3)

$$\begin{aligned} LD &= P8 - 3M3 \\ &= 2 \div (5/4)^3 \\ &= 2 \times 64/125 \\ &= 128/125. \end{aligned}$$

Caution: $3M3 = (5/4)^3$, *not* 3 x 5/4. Converting 128/125 to cents by Equation (A-1') gives the well-known result of 41 cents.[28] The other three discrepancies can be calculated in the same way using their definitions. The ratios are presented in Table 3.

Example 4: The tritones.

(1) The "harmonic tritone," or diminished fifth, consists of two minor thirds.

$$2m3 = (6/5)^2 = 36/25.$$

(2) The "melodic tritone," or augmented fourth, is the inversion of the diminished fifth.

$$P8 - 2m3 = 2 \div 36/25 = 25/18.$$

The ratio of these two intervals is

$$36/25 \div 25/18 = 648/625,$$

the greater diesis. In the key of C, this represents G♭/F♯. From its definition (Section 1), we have already seen that the GD is also D♭♭/C. Incidentally, this also indicates that the interval F♯−E, i.e., the third step in the whole-tone scale, should be a minor whole tone.

[28]Hall, 410.

Appendix B: Why Do We Sing Flat?

Most music theorists are aware of the fact that somehow it is the tuning discrepancies which cause *a capella* choirs to sing flat. (A job only slightly more difficult than squaring the circle would be to convince any choral director of this fact!) Hall gave an example of a chord progression which inevitably goes flat,[29] while Pierce provides a somewhat contrived example of a melodic line which cannot be sung in tune.[30] The idea behind Pierce's example is that if the intervals are sung in tune, the final note will differ from the starting note—by the syntonic comma.

In Figure B-1, I present an example similar to Pierce's, except that it is more likely to be sung since it does not involve large intervals.

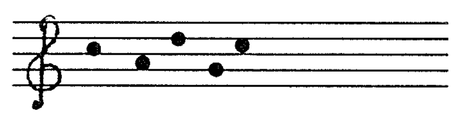

Fig. B-1. (*It cannot be sung in tune!*)

Since the intervals are down a minor third, up a fourth, down a fifth, and up a fourth, it is only necessary to apply the canonical ratios of Table 1 to the first note, C, to compute the frequency of the last note, presumably also a C:

$$C_{final} = C_{initial} \times 5/6 \times 4/3 \times 2/3 \times 4/3$$

$$= C_{initial} \times 80/81$$

[29]Donald E. Hall, "Quantitative Evaluation of Musical Scale Tunings," *American Journal of Physics* 42 (1974): 543-52.

[30]John R. Pierce, *The Science of Musical Sound* (New York: Scientific American Library, 1983), 67.

The final C is flat by the syntonic comma! The difficulty can be traced to the fact that the second note, A, is "too low" by the syntonic comma, i.e., the interval G–A is a minor whole tone rather than a major whole tone. One would think that any time an A or E entered the melodic line, there might be trouble, but this would be simplistic (see Figure B-2).

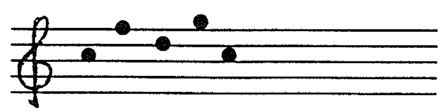

Fig. B-2. (*It cannot be sung in tune either!*)

We see that

$$C_{\text{final}} = C_{\text{initial}} \times 4/3 \times 5/6 \times 4/3 \times 2/3$$

$$= C_{\text{initial}} \times 80/81.$$

Again, we are flat by the syntonic comma. In principle, a choir, or even a solo musician–not necessarily a singer–would go flat every time either of the above melodies was iterated. After five iterations the discrepancy would amount to over 100 cents, that is, in excess of a tempered semitone; or put the two melodies together, as in Figure B-3.

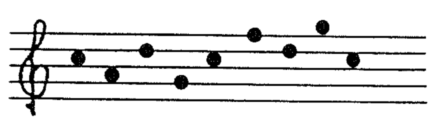

Fig. B-3. (Combination of Figs. B-1 and B-2)

In principle, if each interval was sung justly, the singer would end 2SC below the starting pitch, or about a quarter-tone flat! The only saving grace might be that choirmaster's ideal, the innate sense of tonic. This theory has some influential supporters (for example, Goldovsky[31] uses it to explain the "doctrine of affections," —the theory that different musical keys have different emotional content)[32] and it is certainly a real effect, or else the phenomenon of resolution would not exist in music. Still, it remains to be seen to what extent the innate sense of the tonic is able to overcome the problem of the syntonic comma.

Appendix C: Tables

Table 1. Canonical Ratios for Intervals

Interval	Canonical Ratio	Cents
P8	2	1200
M6	5/3	884
m6	8/5	814
P5	3/2	702
P4	4/3	498
M3	5/4	386
m3	6/5	316
Major Whole Tone* MWT	9/8	204
Minor Whole Tone* mwt	10/9	182
Just Semitone*	16/15	112

*See Table 2

[31]Boris Goldovsky, Private communication.

[32]Hall, 419.

Table 2. Just Tuning for C Major

0	204	386	498	702	884	1088	1200
C	D	E*	F	G	A*	B	C
1	9/8	5/4	4/3	3/2	5/3	15/8	2
9/8	10/9	16/15	9/8	10/9	9/8	16/15	

*Indicates notes lying a minor whole tone above the previous note. Observe the five perfectly tuned chords: the major chords FAC, CEG, GBD; and the minor chords EGB, ACE. The minor chord DFA and the diminished chord BDF are mistuned because the interval D−F is a syntonic comma narrower than a minor third.

Table 3. Tuning Discrepancies

Discrepancy	Abbreviation	Definition	Ratio	Cents
Syntonic Comma	SC	(4P5-2P8)-M3	81/80	21.5
Ditonic Comma	DC	12P5-7P8	$3^{12}/2^{19}$	23.5
Lesser Diesis	LD	P8-3M3	128/125	41.1
Greater Diesis	GD	4m3-P8	648/625	62.6

Table 4. Just Tuning in C: Column "Z," as suggested here; Column "R" Rossing extended.

		Z			R
C'		1200		1200	
B♯		1159		1159	
C♭		1130		1129	
B		1088		1088	
	C♭♭		1059		1081
	A♯♯		1025		1025
B♭		1018		1018	
A♯		954		977	
	B♭♭		947		947
A		884		884	
	G♯♯		843		821
A♭		814		814	
G♯		773		773	
	A♭♭		743		765
G		702		702	
	F♯♯		639		639
G♭		632		631	
F♯		568		590	
	G♭♭		561		561
	E♯♯		527		527
F		498		498	
E♯		456		457	
F♭		428		427	
E		386		386	
	F♭♭		357		379
	D♯♯		345		323
E♭		316		316	
D♯		274		275	
	E♭♭		245		267
D		204		204	
	C♯♯		141		141
D♭		134		112	

Table 4. (continued)

		Z			R
C♯		70		71	
	D♭♭		63		63
	B♯♯		<u>29</u>		8
C		0		0	

Table 5. Enharmonic notes. Notes in the same column are played by the same piano key.

B♯	B♯♯	C♯♯	D♯	D♯♯	E♯	E♯♯	F♯♯		G♯♯	A♯	A♯♯
L	L	G	L	L	L	L	G		L	G	G
C	C♯	D	E♭	E	F	F♯	G	G♯	A	B♭	B
G	G	L	L	L	G	G	L	L	G	L	L
D♭♭	D♭	E♭♭	F♭♭	F♭	G♭♭	G♭	A♭♭	A♭	B♭♭	C♭♭	C♭

Table 6. Mistunings: Column "Z" refers to the tuning scheme suggested in this work, Column "R" to that of Rossing extended. (Asterisks indicate mistunings in intervals not including ♯♯'s or ♭♭'s.)

1. <u>680¢ Fifths</u>

<u>Z</u>	<u>R</u>
*B – F♯	C♯♯ – G♯♯
B♯ – F♯♯	C♭♭ – G♭♭
*B♭ – F	*B♯ – F♯
B♭♭ – F♭	*B♭ – F
*D – A	B♭♭ – F♭

Table 6. (continued)

Z	R
*D♯ – A♯	*A♯ – E♯
D♯♯ – A♯♯	*G♭ – D♭
*D♭ – A♭	*F♯ – C♯
D♭♭ – A♭♭	E♯♯ – B♯♯
	E♭♭ – B♭♭
	*D – A

2. 364¢ Major Thirds

Z	R
*D – F♯	D♯ – F♯♯
D♯ – F♯♯	D♭♭ – F♭
*D♭ – F	E♯ – G♯♯
D♭♭ – F♭	E♭♭ – G♭
	A♯ – C♯♯
	A♭♭ – C♭
	B♯ – D♯♯
	B♭♭ – D♭

3. Mistuned Minor Thirds

Z		R	
*D – F	(294¢)	C♭ – E♭♭	(338¢)
*D♯ – F♯	(294¢)	*D – F	(294¢)
D♯♯ – F♯♯	(294¢)	E♯♯ – G♯♯	(294¢)
*D♭ – F♭	(294¢)	E♭♭ – G♭♭	(294¢)
D♭♭ – F♭♭	(294¢)	*F♯ – A	(294¢)

Table 6. (continued)

4. Mistuned Major Seconds

R	
*C♭ – D♭	(182)
C♯ – D♯♯	(182)
C♭♭ – D♭♭	(182)
*D♭ – E♭	(204)
D♯♯ – E♯♯	(204)
D♭♭ – E♭♭	(204)
*E – F♯	(204)
E♭♭ – F♭	(160) !!
*F♯ – G♯	(183)
F♯♯ – G♯♯	(183)
F♭♭ – G♭♭	(183)
*G♯ – A♯	(204)
G♯♯ – A♯♯	(204)
G♭♭ – A♭♭	(204)
*A♯ – B♯	(182)
A♯♯ – B♯♯	(182)
A♭♭ – B♭♭	(182)

5. Mistuned Minor Seconds

Z		R	
*C – D♭	(134)	C♭ – D♭♭	(134)
C♭ – D♭♭	(134)	*C♯ – D	(134)
*C♯ – D	(134)	C♯♯ – D♯	(134)
C♯♯ – D♯	(134)	D♭ – E♭♭	(155) !!
*F – G♭	(134)	*D♯ – E♯	(134)
F♭ – G♭♭	(134)	*E♯ – F♯	(134)
*F♯ – G	(134)	*F – G♭	(134)
F♯♯ – G♯	(134)	F♭ – G♭♭	(134)

Table 6. (continued)

Z		R	
*A – B♭	(134)	F♯♯ – G♯	(134)
A♭ – B♭♭	(134)	G♭ – A♭♭	(134)
*A♯ – B	(134)	G♯♯ – A♯	(156) !!
A♯♯ – B♯	(134)	*A – B♭	(134)
		A♭ – B♭♭	(134)
		A♯♯ – B♯	(134)
		B♭♭ – C♭♭	(134)
		B♯♯ – C♯♯	(134)

6. Mistuned Augmented Unisons

R	
C♭♭ – C♭	(48¢)
C♯ – C♯♯	(48¢)
D♭♭ – D♭	(48¢)
*D♭ – D	(92¢)
D♯ – D♯♯	(48¢)
E♭♭ – E♭	(48¢)
F♭♭ – F♭	(48¢)
*F – F♯	(92¢)
F♯ – F♯♯	(48¢)
G♯ – G♯♯	(48¢)
A♭♭ – A♭	(48¢)
*A – A♯	(93¢)
A♯ – A♯♯	(48¢)
B♯ – B♯♯	(48¢)

Table 7. The thirteen major and thirteen minor chords which can be played on a keyboard with split black keys. X's mark mistuned chords. Z=Zweifel, R=Rossing.

Major	Z	R	Minor	Z	R
CEG			CE♭G		
D♭FA♭	X		C♯EG♯		
DF♯A	X	X	DFA	X	X
E♭GB♭			D♯F♯A♯	X	
EG♯B			E♭G♭B♭		
FAC			EGB		
F♯A♯C♯		X	FA♭C		
G♭B♭D♭			F♯AC♯		X
GBD			GB♭D		
A♭CE♭			G♯BD♯		
AC♯E			ACE		
B♭DF	X	X	B♭D♭F	X	X
BD♯F♯	X		BDF♯	X	

Table 8. Chords in the scheme suggested by Hall. X's indicate mistuned chords according to Zweifel and Rossing.

Major	Z	R	Minor	Z	R
CEG			CE♭G		
DF♯A	X	X	C♯EG♯		
E♭GB♭			DFA	X	X
EG♯B			EGB		
FAC			F♯AC♯		X
GBD			GB♭D		
AC♯E			G♯BD♯		
B♭DF	X	X	ACE		
			BDF♯	X	

Table 9. The just scale proposed in this paper in Helmholtz notation, double flats and sharps omitted.

$C^2\flat$	C	$C_2\sharp$
$D^2\flat$	D	$D_2\sharp$
$E^1\flat$	E_1	$E_3\sharp$
$F^1\flat$	F	$F_2\sharp$
$G^2\flat$	G	$G_2\sharp$
$A^1\flat$	A_1	$A_3\sharp$
$B^1\flat$	B_1	$B_3\sharp$

B♯–C	3	L
B–C♭	3	L
C♯–D♭	4	G
D♯–E♭	3	L
E–F♭	3	L
E♯–F	3	L
F♯–G♭	4	G
G♯–A♭	3	L
A♯–B♭	4	G