INTEGRAL OPERATORS WITH INFINITELY SMOOTH BI-CARLEMAN KERNELS OF MERCER TYPE

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Abstract: With the aim of applications to solving general integral equations, we introduce and study in this paper a special class of bi-Carleman kernels on $\mathbb{R} \times \mathbb{R}$, called K^∞ kernels of Mercer type, whose property of being infinitely smooth is stable under passage to certain left and right multiples of their associated integral operators. An expansion theorem in absolutely and uniformly convergent bilinear series concerning kernels of this class is proved extending to a general non-Hermitian setting both Mercer's and Kadota's Expansion Theorems for positive definite kernels. Another theorem proved in this paper identifies families of those bounded operators on a separable Hilbert space $\mathcal{H}$ that can be simultaneously transformed by the same unitary equivalence transformation into bi-Carleman integral operators on $L^2(\mathbb{R})$, whose kernels are K^∞ kernels of Mercer type; its singleton version implies in particular that any bi-integral operator is unitarily equivalent to an integral operator with such a kernel.

1. Introduction and the Main Results

In the theory of general linear integral equations in L^2 spaces, the equations with bounded infinitely differentiable bi-Carleman kernels (termed K^∞ kernels) should and do lend themselves well to solution by approximation and variational methods. The question of whether a second-kind integral equation with arbitrary kernel can be reduced to an equivalent one with a K^∞ kernel was positively answered using

a unitary-reduction method by the author [14]. In the present paper, we enrich the K^∞ kernels by imposing an extra condition (Definition 7) that guarantees the kernels and their partial and strong derivatives to be expandable in absolutely and uniformly convergent bilinear series which may also be relevant when solving integral equations. We then show that that condition can always be achieved by means of a unitary reduction which involves no loss of generality in the study of integral equations of the second kind.

Before we can write down our main results, we need to fix the terminology and notation and to give some definitions and preliminary material. Throughout this paper, $\mathcal{H}$ is a complex, separable, infinite-dimensional Hilbert space with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and the norm $\|\cdot\|_{\mathcal{H}}$. If $L \subset \mathcal{H}$, we write $\overline{L}$ for the norm closure of L in $\mathcal{H}$, and $\mathrm{Span}(L)$ for the norm closure of the set of all linear combinations of elements of L. Let $\mathfrak{R}(\mathcal{H})$ be the Banach algebra of bounded linear operators acting on $\mathcal{H}$. For an operator A of $\mathfrak{R}(\mathcal{H})$, A^* stands for the adjoint to A with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, $\mathrm{Ran}\, A = \{Af \mid f \in \mathcal{H}\}$ for the range of A. An operator $A \in \mathfrak{R}(\mathcal{H})$ is said to be *invertible* if it has an inverse which is also in $\mathfrak{R}(\mathcal{H})$, that is, if there is an operator $B \in \mathfrak{R}(\mathcal{H})$ for which $BA = AB = I_{\mathcal{H}}$ where $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$; B is denoted by A^{-1}. An operator $P \in \mathfrak{R}(\mathcal{H})$ is called *positive* and denoted by $P \geq 0$ if $\langle Px, x \rangle_{\mathcal{H}} \geq 0$ for all $x \in \mathcal{H}$. Also recall that a normal operator $A \in \mathfrak{R}(\mathcal{H})$ $(AA^* = A^*A)$ is called *diagonal(izable)* if $\mathcal{H}$ admits an orthonormal basis consisting of eigenvectors of A. If $T \in \mathfrak{R}(\mathcal{H})$, define the operator families $\mathcal{M}(T)$ and $\mathcal{M}^+(T)$ by

$$\mathcal{M}(T) = (T\mathfrak{R}(\mathcal{H}) \cup T^*\mathfrak{R}(\mathcal{H})) \cap (\mathfrak{R}(\mathcal{H})T \cup \mathfrak{R}(\mathcal{H})T^*)\,, \tag{1}$$

$$\mathcal{M}^+(T) = \{P \in \mathcal{M}(T) \mid P \geq 0\}, \tag{2}$$

where $S\mathfrak{R}(\mathcal{H}) = \{SV \mid V \in \mathfrak{R}(\mathcal{H})\}$, $\mathfrak{R}(\mathcal{H})S = \{VS \mid V \in \mathfrak{R}(\mathcal{H})\}$. The following are some simple remarks about families just defined.

Remark 1. First note from (1) that $\mathcal{M}(S) = \mathcal{M}(S^*)$, and that the set $\mathcal{M}(S)$ is alternatively defined as

$$\mathcal{M}(S) = (S\mathfrak{R}(\mathcal{H}) \cap \mathfrak{R}(\mathcal{H})S) \cup (S^*\mathfrak{R}(\mathcal{H}) \cap \mathfrak{R}(\mathcal{H})S^*)$$
$$\cup (S\mathfrak{R}(\mathcal{H}) \cap \mathfrak{R}(\mathcal{H})S^*) \cup (S^*\mathfrak{R}(\mathcal{H}) \cap \mathfrak{R}(\mathcal{H})S)\,,$$

which is the same as saying that an operator A belongs to $\mathcal{M}(S)$ if and only if there exist two operators $M, N \in \mathfrak{R}(\mathcal{H})$ such that at least one of the four relations

$$A = SM = NS, \quad A = S^*M = NS^*,$$
$$A = SM = NS^*, \quad \text{or } A = S^*N = MS, \tag{3}$$

holds. The family $\mathcal{M}^+(S)$ can then be characterized as the set of all positive operators $P \in \mathfrak{R}(\mathcal{H})$ that are expressible as $P = SB$, or as $P = BS$, where

$B \in \mathfrak{R}(\mathcal{H})$, that is to say, $\mathcal{M}^+(S) = \{P \in S\mathfrak{R}(\mathcal{H}) \cup \mathfrak{R}(\mathcal{H})S \mid P \geq 0\}$. If, finally, $S \in \mathcal{M}(T)$, then, by (3), there are four operators K, L, Q, $R \in \mathfrak{R}(\mathcal{H})$ such that $\mathcal{M}(S) = (TK\mathfrak{R}(\mathcal{H}) \cup T^*L\mathfrak{R}(\mathcal{H})) \cap (\mathfrak{R}(\mathcal{H})QT^* \cup \mathfrak{R}(\mathcal{H})RT))$, whence it follows via (1) that $\mathcal{M}(S) \subseteq \mathcal{M}(T)$, and, consequently, $\mathcal{M}^+(S) \subseteq \mathcal{M}^+(T)$.

Definition 2. A factorization of an operator $T \in \mathfrak{R}(\mathcal{H})$ into the product

$$T = WV^*, \tag{4}$$

provided that the operators V, $W \in \mathfrak{R}(\mathcal{H})$ are subject to the provisos

$$VV^* \in \mathcal{M}^+(T), \quad WW^* \in \mathcal{M}^+(T), \tag{5}$$

is called an $\mathcal{M}$ *factorization for* T.

Example 3. An ingenuous example of an $\mathcal{M}$ factorization for any $T \in \mathfrak{R}(\mathcal{H})$ is easy to come by. Just put $W = UP$ and $V = P$, where P is the positive square root of $|T| = (T^*T)^{\frac{1}{2}}$ and U is the partially isometric factor in the polar decomposition $T = U|T|$; since $T = WV^*$, $WW^* = U|T|U^* = TU^* = UT^* \in \mathcal{M}^+(T)$, and $VV^* = |T| = T^*U = U^*T \in \mathcal{M}^+(T)$, it follows that the requirements (4), (5) are satisfied.

The set of *regular values* for the operator $T \in \mathfrak{R}(\mathcal{H})$, denoted by $\Pi(T)$, is the set of complex numbers λ such that the operator $I_{\mathcal{H}} - \lambda T$ is invertible, that is, it has an inverse $R_\lambda(T) = (I_{\mathcal{H}} - \lambda T)^{-1} \in \mathfrak{R}(\mathcal{H})$ that satisfies

$$(I_{\mathcal{H}} - \lambda T) R_\lambda(T) = R_\lambda(T) (I_{\mathcal{H}} - \lambda T) = I_{\mathcal{H}}. \tag{6}$$

The operator

$$T_\lambda := TR_\lambda(T) \ (= R_\lambda(T)T) \tag{7}$$

of $\mathcal{M}(T)$ is then referred to as the *Fredholm resolvent* of T at λ. Remark that if λ is a regular value for T, then, for each fixed $g \in \mathcal{H}$, the (unique) solution $f \in \mathcal{H}$ to the second-kind equation $f - \lambda Tf = g$ may be written as

$$f = g + \lambda T_\lambda g. \tag{8}$$

Let $\mathbb{R}$ be the real line $(-\infty, +\infty)$ equipped with the Lebesgue measure, and let $L^2 = L^2(\mathbb{R})$ be the Hilbert space of (equivalence classes of) measurable complex-valued functions on $\mathbb{R}$ equipped with the inner product $\langle f, g \rangle_{L^2} = \int_{\mathbb{R}} f(s)\overline{g(s)} \, ds$ and the norm $\|f\|_{L^2} = \langle f, f \rangle^{\frac{1}{2}}$. A linear operator $T : L^2 \to L^2$ is *integral* if there is a complex-valued measurable function $\boldsymbol{T}$ (*kernel*) on the Cartesian product $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ such that

$$(Tf)(s) = \int_{\mathbb{R}} \boldsymbol{T}(s,t)f(t) \, dt$$

for every $f \in L^2$ and almost every $s \in \mathbb{R}$. Recall [5, Theorem 3.10] that integral operators are bounded, and need not be compact. A kernel $\boldsymbol{T}$ on $\mathbb{R}^2$ is said to be

Carleman if $\boldsymbol{T}(s,\cdot) \in L^2$ for almost every fixed s in $\mathbb{R}$. To each Carleman kernel $\boldsymbol{T}$ there corresponds a *Carleman function* $\boldsymbol{t} : \mathbb{R} \to L^2$ defined by $\boldsymbol{t}(s) = \overline{\boldsymbol{T}(s,\cdot)}$ for all s in $\mathbb{R}$ for which $\boldsymbol{T}(s,\cdot) \in L^2$. The Carleman kernel $\boldsymbol{T}$ is called *bi-Carleman* in case its conjugate transpose kernel $\boldsymbol{T}'$ $(\boldsymbol{T}'(s,t) = \overline{\boldsymbol{T}(t,s)})$ is also a Carleman kernel. Associated with the conjugate transpose $\boldsymbol{T}'$ of every bi-Carleman kernel $\boldsymbol{T}$ there is therefore a Carleman function $\boldsymbol{t}' : \mathbb{R} \to L^2$ defined by $\boldsymbol{t}'(s) = \overline{\boldsymbol{T}'(s,\cdot)} (= \boldsymbol{T}(\cdot,s))$ for all s in $\mathbb{R}$ for which $\boldsymbol{T}'(s,\cdot) \in L^2$. With each bi-Carleman kernel $\boldsymbol{T}$, we therefore associate the pair of Carleman functions $\boldsymbol{t}, \boldsymbol{t}' : \mathbb{R} \to L^2$, both defined, via $\boldsymbol{T}$, as above. An integral operator whose kernel is Carleman (resp., bi-Carleman) is referred to as the *Carleman* (resp., *bi-Carleman*) operator. The integral operator T is called *bi-integral* if its adjoint T^* is also an integral operator; in that case if $\boldsymbol{T^*}$ is the kernel of T^* then, in the above notation, $\boldsymbol{T^*}(s,t) = \boldsymbol{T}'(s,t)$ for almost all $(s,t) \in \mathbb{R}^2$ (see, e.g., [5, Theorem 7.5]). A bi-Carleman operator is always a bi-integral operator, but not conversely. Henceforth in this paper, kernels and Carleman functions will always be denoted by bold-face uppercase and bold-face lowercase letters, respectively.

Remark 4. From the viewpoint of the foundations of integral equation theory, the bad news about Fredholm resolvents is that the property of being an integral operator is not shared in general by Fredholm resolvents of integral operators; an example of an integral operator whose Fredholm resolvent at any non-zero regular value is not an integral operator can be found in [10], or in [11, Section 5, Theorem 8]. But, fortunately, this phenomenon can never be extended to Carleman operators due to an algebraic property of these operators, a property which is the content of the so called "Right-Multiplication Lemma" (see [5, Theorem 11.6] or [8, Corollary IV.2.8]):

Proposition 5. *Let T be a Carleman operator, let $\boldsymbol{t}$ be the Carleman function associated with the inducing Carleman kernel of T, and let $A \in \mathfrak{R}\left(L^2\right)$ be arbitrary. Then the product operator TA is also a Carleman operator, and the composition function $A^*(\boldsymbol{t}(\cdot)) : \mathbb{R} \to L^2$ is the Carleman function corresponding with its kernel.*

Thus, by (7), the Fredholm resolvent of a Carleman (resp., bi-Carleman) operator at its every regular point is always a Carleman (resp., bi-Carleman) operator. Furthermore, if the operator S is bi-Carleman, then it follows from the above proposition and (3) that the corresponding family $\mathcal{M}(S)$ must consist of bi-Carleman operators only. In the general theory of integral equations of the second kind in L^2, that is, equations of the form

$$f(s) - \lambda \int_{\mathbb{R}} \boldsymbol{T}(s,t)f(t)\,dt = g(s) \quad \text{for almost all } s \in \mathbb{R}, \tag{9}$$

it is customary to call the kernel $\boldsymbol{T}_\lambda$ (when it exists) of the Fredholm resolvent T_λ of a bi-integral operator T, induced on L^2 by the kernel $\boldsymbol{T}$, a *resolvent kernel for* $\boldsymbol{T}$ *at* λ. Once the resolvent kernel $\boldsymbol{T}_\lambda$ comes to be known, one can express the L^2

solution f to equation (9) in a direct fashion as

$$f(s) = g(s) + \lambda \int_{\mathbb{R}} \boldsymbol{T}_\lambda(s,t)g(t)\,dt,$$

regardless of the particular choice of the function g of L^2 (cf. (8)). Therefore, in the case when the kernel $\boldsymbol{T}$ is Carleman and $\lambda \in \Pi(T)$ is arbitrary, the problem of solving equation (9) may be reduced to the problem of explicitly constructing (in terms of $\boldsymbol{T}$) the resolvent kernel $\boldsymbol{T}_\lambda$ which is a priori known to exist. For a precise formulation of this latter problem (not solved here) and for comments to the solution of some of its special cases we refer to the works by Korotkov [9], [11] (in both the references, see Problem 4 in §5). Here we only recall that as long as a measurable kernel $\boldsymbol{T}$ of (9) is bi-Carleman but otherwise unrestricted, there seems to be as yet no analytic machinery for explicitly constructing its resolvent kernel $\boldsymbol{T}_\lambda$ at all $\lambda \in \Pi(T)$. With the objective of making problems like this more tractable, we propose in the present paper to treat some analytical restrictions which can be imposed on bi-Carleman kernels $\boldsymbol{T}$ without loss of generality as far as the solving of such equations as (9) is concerned. These restrictions will be introduced presently by means of Definitions 6 and 7, and the generality will be preserved using unitary reductions by means of Theorem 9 below.

Throughout this paper, the symbols $\mathbb{C}$, $\mathbb{N}$, and $\mathbb{Z}$, refer to the complex plane, the set of all positive integers, and the set of all integers, respectively, and each of the letters i and j is reserved for all non-negative integers. $C(X, B)$, where B is a Banach space (with norm $\|\cdot\|_B$), denotes the Banach space (with the norm $\|f\|_{C(X,B)} = \sup_{x \in X} \|f(x)\|_B$) of continuous B-valued functions defined on a locally compact space X and *vanishing at infinity* (that is, given any $f \in C(X, B)$ and $\varepsilon > 0$, there exists a compact subset $X(\varepsilon, f) \subset X$ such that $\|f(x)\|_B < \varepsilon$ whenever $x \notin X(\varepsilon, f)$). In addition, we introduce the following notation: if an equivalence class $f \in L^2$ contains a function belonging to $C(\mathbb{R}, \mathbb{C})$, we write $[f]$ to mean that function, and we denote the i-th derivative of $[f]$, if exists, by $[f]^{(i)}$. We also say that the series $\sum_n f_n$ is B-*absolutely convergent in* $C(X, B)$ if $f_n \in C(X, B)$ $(n \in \mathbb{N})$ and the series $\sum_n \|f_n(x)\|_B$ converges in $C(X, \mathbb{R})$ (the sum notation $\sum_n$ will always be used instead of the more detailed symbol $\sum_{n=1}^{\infty}$).

Definition 6. A bi-Carleman kernel $\boldsymbol{T} : \mathbb{R}^2 \to \mathbb{C}$ is called a K^∞ *kernel* (see [13]) if it satisfies the three generally independent conditions:

(i) the function $\boldsymbol{T}$ and all its partial derivatives of all orders are in $C\left(\mathbb{R}^2, \mathbb{C}\right)$,

(ii) the Carleman function $\boldsymbol{t}$, $\boldsymbol{t}(s) = \overline{\boldsymbol{T}(s,\cdot)}$, and its (strong) derivatives, $\boldsymbol{t}^{(i)}$, of all orders are in $C\left(\mathbb{R}, L^2\right)$,

(iii) the Carleman function $\boldsymbol{t}'$, $\boldsymbol{t}'(s) = \overline{\boldsymbol{T}'(s,\cdot)} = \boldsymbol{T}(\cdot, s)$, and its (strong) derivatives, $(\boldsymbol{t}')^{(i)}$, of all orders are in $C\left(\mathbb{R}, L^2\right)$.

To deal with K^∞ kernels, we let D_r^i denote the i-th order partial derivative operator with respect to the r-th variable, and we let D_{r_1, r_2}^i denote the product

(in the operator sense) of i factors, each of which is the mixed second-order partial derivative operator $D_{r_2}^1 D_{r_1}^1$, so, for instance, $D_{r_1,r_2}^2 = D_{r_2}^1 D_{r_1}^1 D_{r_2}^1 D_{r_1}^1$.

Definition 7. Let $\boldsymbol{T}$ be a K^∞ kernel and let T be the integral operator it induces on L^2. We say that the K^∞ kernel $\boldsymbol{T}$ is of *Mercer type* if every operator belonging to $\mathcal{M}(T)$ is an integral operator with a K^∞ kernel.

The idea of a counter-example in [12, Section 2] may be used to show that there might be K^∞ kernels which are not of Mercer type. The (in a sense algebraic) condition on a K^∞ kernel for being of Mercer type is of course tailored with specific applications in mind. One of these is a theorem, the first main result of the present paper, which asserts, among other things, that any K^∞ kernel of Mercer type, along with all its partial and strong derivatives, is entirely recoverable from the knowledge of at least one $\mathcal{M}$ factorization for its associated integral operator, by means of bilinear series formulae universally applicable on arbitrary orthonormal bases of L^2:

Theorem 8. *Let* $T \in \mathfrak{R}\left(L^2\right)$ *be an integral operator, with a kernel* $\boldsymbol{T}_0$ *that is a* K^∞ *kernel of Mercer type. Then at each regular point* $\lambda \in \Pi(T)$

(a) *the Fredholm resolvent* T_λ *of* T *is also an integral operator and its kernel, the resolvent kernel* $\boldsymbol{T}_\lambda$ *for* $\boldsymbol{T}_0$, *is also a* K^∞ *kernel of Mercer type, and, moreover,*

(b) *for any* $\mathcal{M}$ *factorization* $T = WV^*$ *of* T *and for any orthonormal basis* $\{u_n\}$ *for* L^2, *the following formulae hold*

$$\left(D_2^j D_1^i \boldsymbol{T}_\lambda\right)(s,t) = \sum_n [R_\lambda(T)Wu_n]^{(i)}(s)\overline{[Vu_n]^{(j)}(t)}, \tag{10}$$

$$\boldsymbol{t}_\lambda^{(i)}(s) = \sum_n \overline{[R_\lambda(T)Wu_n]^{(i)}(s)}Vu_n,$$

$$\left(\boldsymbol{t}_\lambda'\right)^{(j)}(t) = \sum_n \overline{[Vu_n]^{(j)}(t)}R_\lambda(T)Wu_n, \tag{11}$$

$$[T_\lambda f]^{(i)}(s) = \sum_n \langle f, Vu_n\rangle_{L^2} [R_\lambda(T)Wu_n]^{(i)}(s), \tag{12}$$

for all i, j, *all* s, $t \in \mathbb{R}$, *and all* $f \in L^2$, *where the series of* (10) *converges* $\mathbb{C}$*-absolutely in* $C\left(\mathbb{R}^2, \mathbb{C}\right)$, *the two series of* (11), *in which* $\boldsymbol{t}_\lambda$, $\boldsymbol{t}_\lambda'$ *denote the associated Carleman functions of* $\boldsymbol{T}_\lambda$, *both converge in* $C\left(\mathbb{R}, L^2\right)$, *and the series of* (12) *converges* $\mathbb{C}$*-absolutely in* $C(\mathbb{R}, \mathbb{C})$.

For the case $\lambda = 0$, the results of the theorem have been announced without proofs in [15, Theorem 4]. In this particular case, $R_\lambda(T) = I_{L^2}$, $\boldsymbol{T}_\lambda = \boldsymbol{T}_0$, and the bilinear formula (10) reminds one of Mercer's (see [16, Theorem 4.24]) and Kadota's (see [7]) Theorems; the first theorem, recall, is about absoluteness and uniformity of convergence of bilinear (orthogonal) eigenfunction expansions for continuous compactly supported kernels of positive integral operators, and the second is about term-by-term differentiability of those expansions while retaining the absolute and

the uniform convergence. The similarity is closest when Theorem 8 is applied to a diagonal operator T by taking as $T = WV^*$ the $\mathcal{M}$ factorization of Example 3 and as $\{u_n\}$ any orthonormal basis with respect to which T is diagonalizable, because then formula (10) (with $\lambda = 0$) reduces, after a computation, to a bilinear eigenfunction expansion,

$$\left(D_2^j D_1^i \boldsymbol{T}_0 \right)(s,t) = \sum_n \lambda_n \left[u_n \right]^{(i)}(s) \overline{\left[u_n \right]^{(j)}(t)},$$

converging $\mathbb{C}$-absolutely in $C\left(\mathbb{R}^2, \mathbb{C} \right)$. Applied within the same setting, formula (12) looks like: $\left[Tf \right]^{(i)}(s) = \sum_n \lambda_n \langle f, u_n \rangle_{L^2} \left[u_n \right]^{(i)}(s)$ in the sense of $\mathbb{C}$-absolute convergence in $C\left(\mathbb{R}, \mathbb{C} \right)$, and gives something very akin to Schmidt's Theorem, see [16, Theorem 4.22]. In the general non-diagonalizable case, it is therefore natural to view formulae (10), (12) as a "necessary substitute" for the above classical diagonal ones, justifying the name chosen for the kernels defined in Definitions 6 and 7.

Formulae (10), (11) do not, of course, solve the problem (mentioned at the end of Remark 4) of explicitly constructing the resolvent kernel $\boldsymbol{T}_\lambda$ for $\boldsymbol{T}_0$, but succeed in reducing it to one of explicitly finding at most countably many functions $R_\lambda(T)Wu_n$, Vu_n ($n \in \mathbb{N}$), which we do not yet know how to resolve. Nevertheless, the formulae in Theorem 8 may (hopefully) prove quite interesting from a theoretical perspective in terms of developing a general scheme for bilinearly representing integral kernels via the use of the operators that they define but ignoring, if needed, the explicit knowledge about the spectra of those operators.

The proof of Theorem 8 is given in Section 2 below and actually proves a somewhat looser version of it, which will be formulated later, in Section 3, as Corollary 11. This corollary is of the same sort as Theorem 8, but, among others, we have relaxed the $\mathcal{M}$ factoring assumption (5) about the operators W, V being used in formulae (10)-(12). In Section 3, we also try to give a unified view of various bilinear expansion theorems which involve the use of canonical forms of operators, by deriving them as consequences of (the proof of) Theorem 8.

At first glance it may seem that the conditions defining K^∞ kernel of Mercer type are not only hardly verifiable, but also very artifical and rather contrived, and also that for most applications it is too restrictive to confine oneself to such kernels. It is somewhat surprising, therefore, to discover the fact that any bi-integral operator can be made to have as its kernel a K^∞ kernel of Mercer type. For the interpretation of the corresponding result, it is helpful to recall the notion of a unitary equivalence. A bounded linear operator $U : \mathcal{H} \to L^2$ is said to be *unitary* if $\operatorname{Ran} U = L^2$ and $\langle Uf, Ug \rangle_{L^2} = \langle f, g \rangle_{\mathcal{H}}$ for all $f, g \in \mathcal{H}$. An operator $S \in \mathfrak{R}(\mathcal{H})$ is said to be *unitarily equivalent* to an operator $T \in \mathfrak{R}\left(L^2 \right)$ if a unitary operator $U : \mathcal{H} \to L^2$ exists such that $T = USU^{-1}$. It is also relevant to mention the fact that a necessary and sufficient condition that an operator $S \in \mathfrak{R}(\mathcal{H})$ be unitarily equivalent to a (general) bi-Carleman integral operator is that there exist an orthonormal sequence $\{e_n\}$ in

$\mathcal{H}$ such that

$$\|Se_n\|_{\mathcal{H}} \to 0, \quad \|S^*e_n\|_{\mathcal{H}} \to 0 \quad \text{as } n \to \infty \tag{13}$$

(or, equivalently, that 0 belong to the essential spectrum of SS^*+S^*S). This fact was first stated by von Neumann in [17] for self-adjoint operators and was then extended by Korotkov to the general case (see [8, Theorem III.2.7], [5, Theorem 15.14]). Recall [5, Theorem 15.11] that the class of operators satisfying (13) includes all bi-integral operators when the Hilbert space $\mathcal{H}$ is L^2, or in general $L^2(Y, \mu)$ associated with a positive, σ-finite, separable, and not purely atomic, measure μ. The bi-integral operators, on the other hand, are generally involved in second-kind integral equations (like (9)) in $L^2(Y, \mu)$, as the adjoint equations to such equations are customarily required to be integral.

It is pleasant to know that the same condition as (13) proves necessary and sufficient for the operator S to be unitarily equivalent to a bi-Carleman operator generated by a K^∞ kernel of Mercer type. The second principal result of the present paper, Theorem 9 below, both states this fact and characterizes families incorporating those operators in $\mathfrak{R}(\mathcal{H})$ that can be simultaneously transformed by the same unitary equivalence transformation into bi-Carleman integral operators having as kernels K^∞ kernels of Mercer type.

Theorem 9. *Suppose that for an operator family $\mathcal{S} = \{S_\gamma\}_{\gamma \in \mathcal{G}} \subset \mathfrak{R}(\mathcal{H})$ with an index set of arbitrary cardinality there exists an orthonormal sequence $\{e_n\}$ in $\mathcal{H}$ such that*

$$\lim_{n \to \infty} \sup_{\gamma \in \mathcal{G}} \|S_\gamma e_n\|_{\mathcal{H}} = 0, \quad \lim_{n \to \infty} \sup_{\gamma \in \mathcal{G}} \|(S_\gamma)^* e_n\|_{\mathcal{H}} = 0. \tag{14}$$

Then there exists a unitary operator $U : \mathcal{H} \to L^2$ such that all the operators $T_\gamma = US_\gamma U^{-1}$ ($\gamma \in \mathcal{G}$) and their linear combinations are bi-Carleman operators on L^2, whose kernels are K^∞ kernels of Mercer type.

This result has recently been published without proof in [15, Theorem 3]. Section 4 of the present paper is entirely devoted to proving Theorem 9. The method of proof yields a technique for constructing that unitary operator $U : \mathcal{H} \to L^2$ whose existence the theorem asserts. The technique uses no spectral properties of the operators S_γ, other than their joint property imposed in (14), to determine the action of U by specifying two orthonormal bases, of $\mathcal{H}$ and of L^2, one of which is meant to be the image by U of the other, the basis for L^2 may be chosen to be an infinitely smooth wavelet basis.

2. Proof of Theorem 8

(a) Use (1), (7), and the invertibility of $R_\lambda(T)$, to see that

$$\mathcal{M}(T_\lambda) = \left(TR_\lambda(T)\mathfrak{R}\left(L^2\right) \cup T^*\left(R_\lambda(T)\right)^* \mathfrak{R}\left(L^2\right)\right)$$
$$\cap \left(\mathfrak{R}\left(L^2\right)R_\lambda(T)T \cup \mathfrak{R}\left(L^2\right)\left(R_\lambda(T)\right)^* T^*\right) = \mathcal{M}(T),$$

for each $\lambda \in \Pi(T)$. Now the first assertion in the theorem is immediate from Definition 7.

(b) This part of the proof proves the second assertion in the theorem, and is divided into four steps. The first three steps are to establish formulae (10)-(12) for the case in which $\lambda = 0$. Step 4 takes care of the case of an arbitrary regular λ.

Throughout what follows let $\{u_n\}$ be an arbitrary but fixed orthonormal basis for L^2, and let W, V be arbitrary but likewise fixed operators of $\mathfrak{R}\left(L^2\right)$ such that both $F = VV^*$ and $G = WW^*$ are in $\mathcal{M}^+(T)$, and $T = WV^*$. We then let $\boldsymbol{F}$, $\boldsymbol{G}$ denote the K^∞ kernels of the integral positive operators F, G, respectively.

Step 1. A convenient way to begin the proof of assertion (b) for $\lambda = 0$ is to assume for the moment that the following properties of the function systems $\{Vu_n\}$, $\{Wu_n\}$ hold true:

(A) $[Vu_n]^{(i)}$, $[Wu_n]^{(i)} \in C(\mathbb{R}, \mathbb{C})$, for all $n \in \mathbb{N}$ and all i,

(B) the series $\sum_n \left|[Vu_n]^{(i)}\right|^2$, $\sum_n \left|[Wu_n]^{(i)}\right|^2$ converge in $C(\mathbb{R}, \mathbb{C})$, for all i.

Having made these assumptions, the first thing to do is to establish the existence of a K^∞ kernel $\boldsymbol{H}$, with associated Carleman functions $\boldsymbol{h}$ and $\boldsymbol{h}'$, such that, for all i, j,

$$\left(D_2^j D_1^i \boldsymbol{H}\right)(s, t) = \sum_n [Wu_n]^{(i)}(s)\overline{[Vu_n]^{(j)}(t)} \tag{15}$$

with the series converging $\mathbb{C}$-absolutely in $C\left(\mathbb{R}^2, \mathbb{C}\right)$, and

$$\boldsymbol{h}^{(i)}(s) = \sum_n \overline{[Wu_n]^{(i)}(s)}Vu_n, \quad \left(\boldsymbol{h}'\right)^{(j)}(t) = \sum_n \overline{[Vu_n]^{(j)}(t)}Wu_n, \tag{16}$$

in the sense of convergence in $C\left(\mathbb{R}, L^2\right)$. For this purpose, invoke the inequalities

$$\left(\sum_{n=p}^r \left|[Wu_n]^{(i)}(s)\overline{[Vu_n]^{(j)}(t)}\right|\right)^2 \leq \sum_{n=p}^r \left|[Wu_n]^{(i)}(s)\right|^2 \sum_{n=p}^r \left|[Vu_n]^{(j)}(t)\right|^2,$$

$$\left\|\sum_{n=p}^r \overline{[Wu_n]^{(i)}(s)}Vu_n\right\|_{L^2}^2 \leq \|V\|^2 \sum_{n=p}^r \left|[Wu_n]^{(i)}(s)\right|^2,$$

$$\left\|\sum_{n=p}^r \overline{[Vu_n]^{(j)}(t)}Wu_n\right\|_{L^2}^2 \leq \|W\|^2 \sum_{n=p}^r \left|[Vu_n]^{(j)}(t)\right|^2$$

to infer, via (A) and (B), that the series of (15) and of (16) do indeed converge in the senses above. Then apply the corresponding theorems on termwise differentiation of

series to conclude that functions $\boldsymbol{H}$ of $C\left(\mathbb{R}^2, \mathbb{C}\right)$ and $\boldsymbol{h}$, $\boldsymbol{h}'$ of $C\left(\mathbb{R}, L^2\right)$, defined as

$$\boldsymbol{H}(s,t) = \sum_n [Wu_n]\,(s)\overline{[Vu_n]\,(t)},$$

$$\boldsymbol{h}(s) = \overline{\boldsymbol{H}(s,\cdot)} = \sum_n \overline{[Wu_n]\,(s)}Vu_n, \tag{17}$$

$$\boldsymbol{h}'(t) = \boldsymbol{H}(\cdot,t) = \sum_n \overline{[Vu_n]\,(t)}Wu_n,$$

have the desired expansions (15), (16) for all i, j, and hence that

$$D_2^j D_1^i \boldsymbol{H} \in C\left(\mathbb{R}^2, \mathbb{C}\right), \quad \boldsymbol{h}^{(i)},\ \left(\boldsymbol{h}'\right)^{(j)} \in C\left(\mathbb{R}, L^2\right) \tag{18}$$

for all i, j. Since, moreover, each mixed partial derivative of $[Wu_n]\,(s)\overline{[Vu_n]\,(t)}$ is everywhere independent of the sequence in which the partial differentiations with respect to s and t are carried out, it follows that not only those of the form as in (18) but also all other partial derivatives of $\boldsymbol{H}$ belong to $C\left(\mathbb{R}^2, \mathbb{C}\right)$, hereby showing conclusively that $\boldsymbol{H}$ is a K^∞ kernel.

Now fix any $f \in L^2$, and observe then that

$$Tf = WV^*f = \sum_n \langle f, Vu_n\rangle_{L^2} Wu_n, \tag{19}$$

where the series converges to Tf in L^2. On the other hand, the convergence properties of the series of (17) make it possible to write, for each temporarily fixed $s \in \mathbb{R}$, the following chain of relations

$$\sum_n \langle f, Vu_n\rangle_{L^2}[Wu_n]\,(s) = \left\langle f, \sum_n \overline{[Wu_n]\,(s)}Vu_n \right\rangle_{L^2}$$

$$= \int_{\mathbb{R}}\left(\sum_n [Wu_n]\,(s)\overline{[Vu_n]\,(t)}\right)f(t)\,dt = \int_{\mathbb{R}} \boldsymbol{H}(s,t)f(t)\,dt. \quad (20)$$

Because of the assumptions (A), (B) made about the functions $[Wu_n]$, and also because of the inequality

$$\left(\sum_{n=p}^r \left|\langle f, Vu_n\rangle_{L^2}[Wu_n]^{(i)}\,(s)\right|\right)^2 \leq \sum_{n=p}^r |\langle V^*f, u_n\rangle_{L^2}|^2 \sum_{n=p}^r \left|[Wu_n]^{(i)}\,(s)\right|^2,$$

the first series of (20) is $\mathbb{C}$-absolutely convergent in $C(\mathbb{R}, \mathbb{C})$ and can be differentiated termwise any number of times while retaining this type of convergence. Comparison of (19) with (20) shows then that the series representation (12) holds with $\lambda = 0$ for all i, and that

$$(Tf)(s) = \int_{\mathbb{R}} \boldsymbol{H}(s,t)f(t)\,dt \quad \text{for almost every } s \in \mathbb{R}.$$

Since $f \in L^2$ was arbitrary, the latter equality means that the operator T (which is equal to the Fredholm resolvent T_λ at $\lambda = 0$) is an integral operator with the function $\boldsymbol{H}$ as its kernel, so, by the uniqueness of the kernel, $\boldsymbol{H} = \boldsymbol{T}_0$ in the $C\left(\mathbb{R}^2, \mathbb{C}\right)$ sense, and $\boldsymbol{h} = \boldsymbol{t}_0$, $\boldsymbol{h}' = \boldsymbol{t}_0'$ in the $C\left(\mathbb{R}, L^2\right)$ sense. The final conclusion is thus that, when $\lambda = 0$, assertion (b) in the theorem will follow once we show that the functions Vu_n, Wu_n ($n \in \mathbb{N}$) do in fact enjoy properties (A) and (B). This is the object of the next two steps.

Step 2. This step consists of proving that, under the assumptions made about V and W prior to Step 1, the series convergence properties stated in (B) always hold whenever the function properties stated in (A) are met, that is, that (A) implies (B). The proof is further given only for the first series of (B), as the proof for the second series, $\sum_n \left|[Wu_n]^{(i)}\right|^2$, is entirely similar.

If ℓ is a non-negative integer, m is a positive integer, and $Q = [a, b] \times [c, d]$ is a compact rectangle of $\mathbb{R}^2$, we define the three quantities

$$q_1(Q, m; \ell) = \iint\limits_Q \left(\left(D_{1,2}^\ell \boldsymbol{F}\right)(s, t) - \sum_{n \leq m} [Vu_n]^{(\ell)}(s)\overline{[Vu_n]^{(\ell)}(t)} \right) ds\, dt, \qquad (21)$$

$$q_2(Q, m; \ell) = \sum_{n > m} \int_a^b [Vu_n]^{(\ell)}(s)\, ds \int_c^d \overline{[Vu_n]^{(\ell)}(t)}\, dt, \qquad (22)$$

$$q_3(Q, m; \ell) = \iint\limits_Q \left(\sum_{n > m} [Vu_n]^{(\ell)}(s)\overline{[Vu_n]^{(\ell)}(t)} \right) ds\, dt, \qquad (23)$$

and prove that, for each ℓ,

$$q_1(Q, m; \ell) = q_2(Q, m; \ell) \quad \text{for all } Q \text{ and all } m. \qquad (24)$$

First, for this purpose, utilize the L^2 representation

$$Ff = VV^*f = \sum_n \langle f, Vu_n \rangle_{L^2} Vu_n \quad (f \in L^2)$$

in order to write, for each Q and m,

$$q_1(Q, m; 0)$$
$$= \sum_n \langle Vu_n, \chi_{[a,b]} \rangle_{L^2} \langle \chi_{[c,d]}, Vu_n \rangle_{L^2} - \sum_{n \leq m} \langle Vu_n, \chi_{[a,b]} \rangle_{L^2} \langle \chi_{[c,d]}, Vu_n \rangle_{L^2}$$
$$= \sum_{n > m} \langle Vu_n, \chi_{[a,b]} \rangle_{L^2} \langle \chi_{[c,d]}, Vu_n \rangle_{L^2} = q_2(Q, m; 0),$$

where χ_E denotes the characteristic function of a set E. This implies that the identity (24) holds with $\ell = 0$. Proceeding by induction over ℓ, suppose identity (24) to be satisfied for some fixed ℓ. The stage is now set for the induction step.

It is first necessary to remark that the integrand in (21) must be non-negative on the main diagonal of $\mathbb{R}^2$, that is, denoting the integrand by $\boldsymbol{F}_m^\ell(s,t)$:

$$\boldsymbol{F}_m^\ell(s,s) = \left(D_{1,2}^\ell \boldsymbol{F}\right)(s,s) - \sum_{n \leq m} \left|[V u_n]^{(\ell)}(s)\right|^2 \geq 0 \quad \text{for all } s \in \mathbb{R}. \qquad (25)$$

Indeed, if it were not so, there would exist a square Q', with centre at some point on the main diagonal of $\mathbb{R}^2$ and sides parallel to the coordinate axes, such that its corresponding quantity $q_1(Q', m; \ell)$ would be negative, contradicting the non-negativity of $q_2(Q', m; \ell)$ which in turn would be implied by coincidence of the integration intervals on the right side of (22). In more detail, if $\boldsymbol{F}_m^\ell(s_0, s_0) = -2\delta$ for some s_0 and $\delta > 0$, then by the continuity of $\boldsymbol{F}_m^\ell$ on $\mathbb{R}^2$, which follows from that of $D_{1,2}^\ell \boldsymbol{F}$ and from property (A), there is a square Q',

$$Q' = \left\{(s,t) \in \mathbb{R}^2 : |s - s_0| \leq \varepsilon/2, \ |t - s_0| \leq \varepsilon/2\right\} \quad (\varepsilon > 0)$$

in which $\operatorname{Re} \boldsymbol{F}_m^\ell(s,t) < -\delta$. By the induction hypothesis

$$\iint_{Q'} \boldsymbol{F}_m^\ell(s,t)\, ds\, dt = q_1(Q', m; \ell) = q_2(Q', m; \ell) \geq 0 \qquad (26)$$

so $0 \leq \iint_{Q'} \operatorname{Re} \boldsymbol{F}_m^\ell(s,t)\, ds\, dt < -\delta\varepsilon^2 < 0$, a contradiction. (Note: that the diagonal value, $\boldsymbol{F}_m^\ell(s,s)$, of the integrand in (21) at any point $s \in \mathbb{R}$ and the value of the integral in (26) taken over any square like $Q' = [a,b] \times [a,b]$ both have, for any ℓ and m, a zero imaginary part comes from the Hermiticity of $\boldsymbol{F}_m^\ell$, $\boldsymbol{F}_m^\ell(s,t) = \overline{\boldsymbol{F}_m^\ell(t,s)}$ for all $s, t \in \mathbb{R}$. This latter property is mainly inherited from that of $\boldsymbol{F}$ ($\boldsymbol{F}(s,t) = \overline{\boldsymbol{F}(t,s)}$ for all $s, t \in \mathbb{R}$ because of the self-adjointness of F) thanks to assumption (i) about $\boldsymbol{F}$, as follows: $\left(D_{1,2}^\ell \boldsymbol{F}\right)(s,t) = \overline{\left(D_{2,1}^\ell \boldsymbol{F}\right)(t,s)} = \overline{\left(D_{1,2}^\ell \boldsymbol{F}\right)(t,s)}$ for all $s, t \in \mathbb{R}$.)

Further, it is seen from (25) that, for each m, there is the inequality

$$\sum_{n \leq m} \left|[V u_n]^{(\ell)}(s)\right|^2 \leq \left(D_{1,2}^\ell \boldsymbol{F}\right)(s,s) \leq C_\ell = \sup_{s \in \mathbb{R}} \left(D_{1,2}^\ell \boldsymbol{F}\right)(s,s) \quad (s \in \mathbb{R}), \qquad (27)$$

from which it follows in particular that the series in the integrand of (23) is termwise integrable, implying (via (22) and (24)) that $q_1(Q, m; \ell) = q_3(Q, m; \ell)$ for all m and all Q. In turn, this new identity implies that

$$\left(D_{1,2}^\ell \boldsymbol{F}\right)(s,t) = \sum_n [V u_n]^{(\ell)}(s)\overline{[V u_n]^{(\ell)}(t)} \qquad (28)$$

almost everywhere in $\mathbb{R}^2$. The series here converges by (27) uniformly in each variable separately for all values of the other, and its sum-function, denoted by $\boldsymbol{S}_\ell$, is therefore a continuous function of either argument. One can now appeal directly to a subtle Example 2 from [18, Chapter 14, pp. 545-546], as applied to functions $f_1 = D_{1,2}^\ell \boldsymbol{F}$ and $f_2 = \boldsymbol{S}_\ell$, in order to be sure that

$$\left(D_{1,2}^\ell \boldsymbol{F}\right)(s,s) = \sum_n \left|[Vu_n]^{(\ell)}(s)\right|^2 \quad \text{for all } s \in \mathbb{R}. \tag{29}$$

Since $\left(D_{1,2}^\ell \boldsymbol{F}\right)(s,s) \to 0$ as $|s| \to \infty$, Dini's Monotone Convergence Theorem may now be applied to the 1-point compactification of $\mathbb{R}$, to yield the conclusion that the series of (29) does converge in $C(\mathbb{R}, \mathbb{C})$ (compare this result with property (B)). In particular, it follows that the series of (28) is converging in $C\left(\mathbb{R}^2, \mathbb{C}\right)$ to $D_{1,2}^\ell \boldsymbol{F}$. That, in turn, justifies the following computation

$$
\begin{aligned}
&q_1(Q, m; \ell+1) \\
&= \left(D_{1,2}^\ell \boldsymbol{F}\right)(b,d) - \left(D_{1,2}^\ell \boldsymbol{F}\right)(b,c) - \left(D_{1,2}^\ell \boldsymbol{F}\right)(a,d) + \left(D_{1,2}^\ell \boldsymbol{F}\right)(a,c) \\
&\quad - \sum_{n \leq m} \left([Vu_n]^{(\ell)}(b) - [Vu_n]^{(\ell)}(a)\right)\left(\overline{[Vu_n]^{(\ell)}(d) - [Vu_n]^{(\ell)}(c)}\right) \\
&\qquad = \sum_n \left([Vu_n]^{(\ell)}(b)\overline{[Vu_n]^{(\ell)}(d)} - [Vu_n]^{(\ell)}(b)\overline{[Vu_n]^{(\ell)}(c)}\right. \\
&\qquad\qquad \left. - [Vu_n]^{(\ell)}(a)\overline{[Vu_n]^{(\ell)}(d)} + [Vu_n]^{(\ell)}(a)\overline{[Vu_n]^{(\ell)}(c)}\right) \\
&\qquad\quad - \sum_{n \leq m} \left([Vu_n]^{(\ell)}(b) - [Vu_n]^{(\ell)}(a)\right)\left(\overline{[Vu_n]^{(\ell)}(d) - [Vu_n]^{(\ell)}(c)}\right) \\
&\qquad = \sum_{n > m} \left([Vu_n]^{(\ell)}(b) - [Vu_n]^{(\ell)}(a)\right)\left(\overline{[Vu_n]^{(\ell)}(d) - [Vu_n]^{(\ell)}(c)}\right) \\
&\hspace{9cm} = q_2(Q, m; \ell+1),
\end{aligned}
$$

which proves that (24) with $\ell+1$ instead of ℓ holds true for all m and $Q = [a,b] \times [c,d]$. Therefore, by induction, the identity (24) is true for every non-negative integer ℓ. Hence, as (24) implies the $C(\mathbb{R}, \mathbb{C})$ convergence of the series of (29) by what has just been seen in the course of the induction step, the first series of (B) converges in $C(\mathbb{R}, \mathbb{C})$ for each fixed i.

Step 3. In this step, the proof of assertion (b) will be completed for $\lambda = 0$, by showing that, under the conditions laid on V and W at the beginning of the proof, property (A) always holds. We shall restrict ourselves to dealing only with the functions Vu_n ($n \in \mathbb{N}$), because the proof of (A) for Wu_n ($n \in \mathbb{N}$) can be obtained in a similar way, but using respectively $\operatorname{Ran} W^*$ and $\boldsymbol{G}$ instead of $\operatorname{Ran} V^*$ and $\boldsymbol{F}$.

Choose $\{v_k\}$ to be an orthonormal basis for the subspace $\overline{\operatorname{Ran} V^*}$, with the property that $\{v_k\} \subset \operatorname{Ran} V^*$, and let $\{\widetilde{u}_n\}$ be any orthonormal basis for L^2 such

that $\{v_k\} \subset \{\widetilde{u}_n\}$. Observe that $v_k \in \operatorname{Ran} V^*$ implies $Vv_k = F f_k$ for some $f_k \in L^2$. Therefore, by property (ii) for the Carleman function $\boldsymbol{f}(s) = \overline{\boldsymbol{F}(s, \cdot)}$, there is in the equivalence class Vv_k a function of $C(\mathbb{R}, \mathbb{C})$, namely $[Vv_k](s) = \langle f_k, \boldsymbol{f}(s) \rangle_{L^2}$ ($s \in \mathbb{R}$), such that its every derivative, $[Vv_k]^{(i)}(\cdot) = \left\langle f_k, \boldsymbol{f}^{(i)}(\cdot) \right\rangle_{L^2}$, is also in $C(\mathbb{R}, \mathbb{C})$. Then, since every $V\widetilde{u}_n$ is equal either to Vv_k, for some k, or to the zero function and therefore all $V\widetilde{u}_n$ satisfy (A), the reasoning of Step 2 can be applied to the function system $\{V\widetilde{u}_n\}$ in place of $\{Vu_n\}$ in order to arrive at the conclusion that, for each i, the series $\sum_n \left| [V\widetilde{u}_n]^{(i)} \right|^2$ converges in $C(\mathbb{R}, \mathbb{C})$ (cf. (B)). After that, one can conclude immediately that each series of the form $\sum_n \langle f, \widetilde{u}_n \rangle_{L^2} [V\widetilde{u}_n]^{(i)}$ ($f \in L^2$) also converges in $C(\mathbb{R}, \mathbb{C})$, and hence defines a $C(\mathbb{R}, \mathbb{C})$ function, which is nothing else than $[Vf]^{(i)}$ because $Vf = \sum_n \langle f, \widetilde{u}_n \rangle_{L^2} V\widetilde{u}_n$ in the sense of convergence in L^2. In particular, $[Vu_n]^{(i)} \in C(\mathbb{R}, \mathbb{C})$, for all $n \in \mathbb{N}$ and all i.

Step 4. Now let λ be an arbitrary non-zero regular value of T, and factorize the Fredholm resolvent T_λ of T at λ in the form $T_\lambda = W_\lambda V^*$ where $W_\lambda = R_\lambda(T)W$ (see (7), (4)). This factorization need not be an $\mathcal{M}$ factorization for T_λ. If, however, the operator $G_\lambda = W_\lambda (W_\lambda)^*$ is known to be integral with a K^∞ kernel, then the previous three steps of the proof may easily be adapted, with W_λ written instead of W, to show that the formulae (10)-(12) all hold exactly as stated in the theorem.

Let us therefore focus attention on the operator G_λ. That this operator is Carleman follows form the representation

$$G_\lambda = W_\lambda (W_\lambda)^* = G + \lambda T R_\lambda(T) G + \bar{\lambda} G (R_\lambda(T))^* T^*$$
$$+ |\lambda|^2 T R_\lambda(T) G (R_\lambda(T))^* T^* \quad (30)$$

which may be established with the aid of the equality $R_\lambda(T) = I_{L^2} + \lambda T R_\lambda(T)$ (cf. (6)), and in which each term is a Carleman operator by the right-multiplication lemma. In addition, if $\boldsymbol{g}$ is the Carleman function corresponding with the K^∞ kernel $\boldsymbol{G}$ of the integral operator $G = WW^*$, then the function $\boldsymbol{g}_\lambda : \mathbb{R} \to L^2$ defined by

$$\boldsymbol{g}_\lambda(\cdot) = \boldsymbol{g}(\cdot) + \bar{\lambda} (R_\lambda(T)G)^* (\boldsymbol{t}_0(\cdot)) + \lambda ((R_\lambda(T))^* T^*)^* (\boldsymbol{g}(\cdot))$$
$$+ |\lambda|^2 (R_\lambda(T) G (R_\lambda(T))^* T^*)^* (\boldsymbol{t}_0(\cdot))$$

can be regarded as a Carleman function associated with the Carleman kernel of G_λ, by Proposition 5 again. Since, for each i,

$$\boldsymbol{g}^{(i)}, \boldsymbol{t}_0^{(i)}, \left(\boldsymbol{t}_0'\right)^{(i)} \in C\left(\mathbb{R}, L^2\right), \quad (31)$$

it follows that

$$\boldsymbol{g}_\lambda^{(i)}(\cdot) = \boldsymbol{g}^{(i)}(\cdot) + \bar{\lambda} (R_\lambda(T)G)^* \left(\boldsymbol{t}_0^{(i)}(\cdot)\right) + \lambda ((R_\lambda(T))^* T^*)^* \left(\boldsymbol{g}^{(i)}(\cdot)\right)$$

$$+ |\lambda|^2 \left(R_\lambda(T) G \left(R_\lambda(T) \right)^* T^* \right)^* \left(\boldsymbol{t}_0^{(i)}(\cdot) \right) \in C \left(\mathbb{R}, L^2 \right), \quad (32)$$

because the operators $R_\lambda(T)$, G, and T, are bounded. Analogously, for each i,

$$\left(\left(R_\lambda(T) \right)^* \left(\boldsymbol{t}_0(\cdot) \right) \right)^{(i)} = \left(R_\lambda(T) \right)^* \left(\boldsymbol{t}_0^{(i)}(\cdot) \right) \in C \left(\mathbb{R}, L^2 \right),$$

$$\left(R_\lambda(T)(\boldsymbol{g}(\cdot)) \right)^{(i)} = R_\lambda(T) \left(\boldsymbol{g}^{(i)}(\cdot) \right) \in C \left(\mathbb{R}, L^2 \right), \quad (33)$$

$$R_\lambda(T) G \left(R_\lambda(T) \right)^* \left(\boldsymbol{t}_0(\cdot) \right) \right)^{(i)} = R_\lambda(T) G \left(R_\lambda(T) \right)^* \left(\boldsymbol{t}_0^{(i)}(\cdot) \right) \in C \left(\mathbb{R}, L^2 \right).$$

Note also that each of the last three terms in (30) is a product of two Carleman operators. From this it becomes possible to express the inducing Carleman kernel $\boldsymbol{G}_\lambda$ of G_λ by means of convolutions as follows:

$$\boldsymbol{G}_\lambda(s,t) = \boldsymbol{G}(s,t) + \lambda \left\langle \boldsymbol{g}(t), \left(R_\lambda(T) \right)^* \left(\boldsymbol{t}_0(s) \right) \right\rangle_{L^2} + \bar{\lambda} \left\langle \boldsymbol{t}_0'(t), R_\lambda(T)(\boldsymbol{g}(s)) \right\rangle_{L^2}$$
$$+ |\lambda|^2 \left\langle \boldsymbol{t}_0'(t), R_\lambda(T) G \left(R_\lambda(T) \right)^* \left(\boldsymbol{t}_0(s) \right) \right\rangle_{L^2}.$$

Then, for each fixed i, j, a partial differentiation of $\boldsymbol{G}_\lambda$ yields, after the equalities of (33) are taken into account, the following equality on $\mathbb{R}^2$:

$$\left(D_2^j D_1^i \boldsymbol{G}_\lambda \right)(s,t) = \left(D_2^j D_1^i \boldsymbol{G} \right)(s,t) + \lambda \left\langle \boldsymbol{g}^{(j)}(t), \left(R_\lambda(T) \right)^* \left(\boldsymbol{t}_0^{(i)}(s) \right) \right\rangle_{L^2}$$
$$+ \bar{\lambda} \left\langle \left(\boldsymbol{t}_0' \right)^{(j)}(t), R_\lambda(T) \left(\boldsymbol{g}^{(i)}(s) \right) \right\rangle_{L^2}$$
$$+ |\lambda|^2 \left\langle \left(\boldsymbol{t}_0' \right)^{(j)}(t), R_\lambda(T) G \left(R_\lambda(T) \right)^* \left(\boldsymbol{t}_0^{(i)}(s) \right) \right\rangle_{L^2};$$

clearly we are free to compute each term on the right-hand side here by alternatively applying D_2^1 and D_1^1 in any order we please. Hence, according to (31), (33), and by the continuity of the inner product, the partial derivatives of $\boldsymbol{G}_\lambda$ are all in $C \left(\mathbb{R}^2, \mathbb{C} \right)$. This acquisition in conjunction with (32) implies that $\boldsymbol{G}_\lambda$ is a K^∞ kernel, thereby completing the proof of assertion (b) of the theorem.

3. Corollaries and Applications

Remark 10. An easier way to see that $G_\lambda = W_\lambda \left(W_\lambda \right)^*$ in (30) is an integral operator with a K^∞ kernel is by making more use of the assumption in Theorem 8 that the K^∞ kernel $\boldsymbol{T}_0$ of T is of Mercer type, that is, that the operator family $\mathcal{M}(T)$ consists only of integral operators with K^∞ kernels. The argument might be as follows: the first and last terms in the right-hand side of (30) clearly both belong to $\mathcal{M}^+(T)$ (see (5) and Remark 1). The second term $\lambda T R_\lambda(T) G$, and hence its adjoint $\bar{\lambda} G \left(R_\lambda(T) \right)^* T^*$ (which is just the third one), does belong to $\mathcal{M}(T)$, because $\lambda T R_\lambda(T) G = \lambda T R_\lambda(T) B^* T^* \in T \mathfrak{R} \left(L^2 \right) \cap \mathfrak{R} \left(L^2 \right) T^* \subset \mathcal{M}(T)$ or $\lambda T R_\lambda(T) G =$

$\lambda T R_\lambda(T) B T \in T\mathfrak{R}\left(L^2\right) \cap \mathfrak{R}\left(L^2\right) T \subset \mathcal{M}(T)$ according as $G = TB$ or $G = BT$ (see again Remark 1). Each summand on the right of (30) is thus an integral operator with a K^∞ kernel and hence so is G_λ itself.

In addition to Remark 10 let us confess that also in the course of the whole proof of assertion (b) in Theorem 8 we have not used the full strength of the assumption on the K^∞ kernel $\boldsymbol{T}_0$ of being of Mercer type, but only a consequence of it, namely the existence of inducing K^∞ kernels for $F = VV^*$ and $G = WW^*$, whenever $T = WV^*$ is an $\mathcal{M}$ factorization for T. So what that proof really proves is the following, more general, result intended for the case where $T = WV^*$ is not necessarily an $\mathcal{M}$ factorization for, at first, not necessarily integral T.

Corollary 11. *Suppose that an operator $T \in \mathfrak{R}\left(L^2\right)$ has a factorization into a product $T = WV^*$ $(V, W \in \mathfrak{R}\left(L^2\right))$ such that both $F = VV^*$ and $G = WW^*$ are integral operators with K^∞ kernels. Then at each regular point $\lambda \in \Pi(T)$ the Fredholm resolvent T_λ of T is an integral operator and its kernel $\boldsymbol{T}_\lambda$ is a K^∞ kernel. Moreover, for any orthonormal basis $\{u_n\}$ for L^2, the following formulae hold*

$$\left(D_2^j D_1^i \boldsymbol{T}_\lambda\right)(s,t) = \sum_n [R_\lambda(T)Wu_n]^{(i)}(s)\overline{[Vu_n]^{(j)}(t)}, \qquad (34)$$

$$\boldsymbol{t}_\lambda^{(i)}(s) = \sum_n \overline{[R_\lambda(T)Wu_n]^{(i)}(s)}Vu_n,$$

$$\left(\boldsymbol{t}_\lambda'\right)^{(j)}(t) = \sum_n \overline{[Vu_n]^{(j)}(t)}R_\lambda(T)Wu_n, \qquad (35)$$

$$[T_\lambda f]^{(i)}(s) = \sum_n \langle f, Vu_n\rangle_{L^2}[R_\lambda(T)Wu_n]^{(i)}(s), \qquad (36)$$

for all i, j, all s, $t \in \mathbb{R}$, and all $f \in L^2$, where the series of (34) converges $\mathbb{C}$-absolutely in $C\left(\mathbb{R}^2, \mathbb{C}\right)$, the two series of (35) both converge in $C\left(\mathbb{R}, L^2\right)$, and the series of (36) converges $\mathbb{C}$-absolutely in $C(\mathbb{R}, \mathbb{C})$.

The proof is exactly the same as the proof given in the previous section for assertion (b) of Theorem 8. Corollary 11 also opens a slightly different, but equivalent, way to define a K^∞ kernel of Mercer type, involving families $\mathcal{M}^+(\cdot)$ in place of $\mathcal{M}(\cdot)$ (cf. Definition 7).

Corollary 12. *An operator $T \in \mathfrak{R}\left(L^2\right)$ is an integral operator with a K^∞ kernel of Mercer type if and only if the family $\mathcal{M}^+(T)$ consists only of integral operators with K^∞ kernels.*

Proof. The "only if" is immediate from the inclusion $\mathcal{M}^+(T) \subset \mathcal{M}(T)$. For the proof of the "if", assume that every operator $A \in \mathcal{M}^+(T)$ is an integral operator with a K^∞ kernel. If $S \in \mathcal{M}(T)$, it is to be proved that S is an integral operator with a K^∞ kernel. For this, let $S = WV^*$ be an $\mathcal{M}$ factorization for S, where, according to (5) and Remark 1, VV^*, $WW^* \in \mathcal{M}^+(S) \subseteq \mathcal{M}^+(T)$. Then it follows

by assumption that both $F = VV^*$ and $G = WW^*$ are integral operators with K^∞ kernels. Apply Corollary 11 to the operator $S = WV^*$ and when $\lambda = 0$ conclude that S is an integral operator with a K^∞ kernel. The corollary is proved. $\square$

Remark 13. The next result, Dostanić's [1]-[2] extension of Mercer's Theorem to a class of continuous non-Hermitian kernels on $[0,1]^2$, deserves mention because it turns out to also fit into the general bilinear expansion scheme proved in Section 2.

Proposition 14. If $T = H\left(I_{L^2(0,1)} + S\right)$, where $H \geq 0$, $S = S^*$ are integral operators induced on $L^2(0,1)$ by continuous kernels on $[0,1]^2$, and if either (I) $I_{L^2(0,1)} + S$ is positive and invertible, or (II) $I_{L^2(0,1)} + S$ is invertible and H is one-to-one, then the kernel $\boldsymbol{T}_0$ of the integral operator T is represented on $[0,1]^2$ by the absolutely and uniformly convergent series

$$\boldsymbol{T}_0(s,t) = \sum_n \lambda_n \psi_n(s)\overline{\varphi_n(t)}, \tag{37}$$

where $\{\psi_n\}$ and $\{\varphi_n\}$ are biorthogonal systems of eigenfunctions for T and for T^*, respectively: $T\psi_n = \lambda_n\psi_n$, $T^*\varphi_m = \lambda_m\varphi_m$, $\langle\psi_n,\varphi_m\rangle_{L^2(0,1)} = \delta_{nm}$ (Kronecker delta).

The proof of this result rests on an eigenvalue-eigenfunction analysis (see, e.g., [4, Chapter 5, §8]) of compact operators that are self-adjoint with respect to the definite or indefinite inner product $[f,g]_{L^2(0,1)} = \langle(I_{L^2(0,1)} + S)f,g\rangle_{L_2(0,1)}$ according as case (I) or case (II) is in question. Alternatively, the result may be proved without direct recourse to that analysis by using a factorization argument similar to that used for Theorem 8(b); a possible outline of a proof may be roughly sketched as follows. First note that the integral operator T in Proposition 14 has an $\mathcal{M}$ factorization $T = WV^*$ such that both VV^* and WW^* are integral operators with continuous kernels on $[0,1]^2$. Indeed, define the factors by $V = (I_{L^2(0,1)} + S)^{\frac{1}{2}}\Lambda^{\frac{1}{2}}$, $W = (I_{L^2(0,1)} + S)^{-\frac{1}{2}}\Lambda^{\frac{1}{2}}$, where $\Lambda = (I_{L^2(0,1)} + S)^{\frac{1}{2}}H(I_{L^2(0,1)} + S)^{\frac{1}{2}}$, in case (I), or by $V = (I_{L^2(0,1)} + S)H^{\frac{1}{2}}$, $W = H^{\frac{1}{2}}$ in case (II); in both cases, then, $VV^* = (I_{L^2(0,1)} + S)T$, $WW^* = T(I_{L^2(0,1)} + S)^{-1} = H$, and $T = WV^*$, where the integral operators H, S, and T, are known to have continuous kernels. Having thus factorized the integral operator T, it can be proved by adapting arguments in Section 2 that any series of the form $\sum_n Wu_n(s)\overline{Vu_n(t)}$, where $\{u_n\}$ is an orthonormal basis in $L^2(0,1)$, converges absolutely and uniformly to the kernel $\boldsymbol{T}_0$ of T. Thus, in order to prove the desired convergence behavior of representation (37) in Proposition 14, it suffices to prove that $\lambda_n\psi_n(s)\overline{\varphi_n(t)} = Wu_n(s)\overline{Vu_n(t)}$ $(n \in \mathbb{N})$ for some orthonormal basis $\{u_n\}$. It is a straightforward calculation to show that in case (I) (resp., (II)) such a $\{u_n\}$ can be chosen to be an orthonormal basis in $L^2(0,1)$ with respect to which the compact, self-adjoint operator $(I_{L^2(0,1)} + S)^{\frac{1}{2}}H(I_{L^2(0,1)} + S)^{\frac{1}{2}}$ (resp., $H^{\frac{1}{2}}(I + S)H^{\frac{1}{2}}$) diagonalizes (see proof of Corollary 15 below for more details).

The following generalization of Proposition 14 is included as an application of Corollary 11.

Corollary 15. *Let $T = H(I_{L^2} + S)$, where $0 \leq H \in \mathfrak{R}(L^2)$, $S \in \mathfrak{R}(L^2)$, and both H and $(I_{L^2} + S^*)H(I_{L^2} + S)$ are integral operators with K^∞ kernels. Then at each regular point $\lambda \in \Pi(T)$ the Fredholm resolvent T_λ of T is an integral operator whose kernel $\boldsymbol{T}_\lambda$ is a K^∞ kernel. If, in addition, $\Lambda = H^{\frac{1}{2}}(I + S)H^{\frac{1}{2}}$ is a diagonal operator with diagonal entries $\lambda_1, \lambda_2, \lambda_3, \ldots$, then there are functions ψ_n, φ_n ($n \in \mathbb{N}$) in L^2 for which the following formulae hold: $T\psi_n = \lambda_n \psi_n$, $T^*\varphi_m = \bar{\lambda}_m \varphi_m$, $\langle \psi_n, \varphi_m \rangle_{L^2} = \lambda_n \delta_{nm}$ for all $n, m \in \mathbb{N}$, and*

$$\left(D_2^j D_1^i \boldsymbol{T}_\lambda \right)(s,t) = \sum_n \frac{1}{1 - \lambda\lambda_n} [\psi_n]^{(i)}(s)\overline{[\varphi_n]^{(j)}(t)}, \tag{38}$$

$$\boldsymbol{t}_\lambda^{(i)}(s) = \sum_n \frac{1}{1 - \overline{\lambda\lambda_n}}\overline{[\psi_n]^{(i)}(s)}\varphi_n,$$

$$\left(\boldsymbol{t}_\lambda'\right)^{(j)}(t) = \sum_n \frac{1}{1 - \lambda\lambda_n}\overline{[\varphi_n]^{(j)}(t)}\psi_n, \tag{39}$$

$$[T_\lambda f]^{(i)}(s) = \sum_n \frac{1}{1 - \lambda\lambda_n}\langle f, \varphi_n \rangle_{L^2}[\psi_n]^{(i)}(s), \tag{40}$$

for all $\lambda \in \Pi(T)$, all i, j, and all s, $t \in \mathbb{R}$, and all $f \in L^2$, where the series of (38) converges $\mathbb{C}$-absolutely in $C\left(\mathbb{R}^2, \mathbb{C}\right)$, the two series of (39) both converge in $C\left(\mathbb{R}, L^2\right)$, and the series of (40) converges $\mathbb{C}$-absolutely in $C(\mathbb{R}, \mathbb{C})$.

Proof. Put $W = H^{\frac{1}{2}}$, $V = (I_{L^2} + S^*)H^{\frac{1}{2}}$, and let the orthonormal basis $\{u_n\}$ for L^2 allow the diagonal operator $\Lambda = H^{\frac{1}{2}}(I_{L^2} + S)H^{\frac{1}{2}}$ to be written as

$$\Lambda = \sum_n \lambda_n \langle \cdot, u_n \rangle_{L^2} u_n. \tag{41}$$

Since, by assumption, both operators $WW^* = H$ and $VV^* = (I_{L^2} + S^*)H(I_{L^2} + S)$ are integral and are defined by K^∞ kernels, Corollary 11 can be applied with respect to factorization $T = H(I_{L^2} + S) = WV^*$ and basis $\{u_n\}$ to conclude that the Fredholm resolvent T_λ of T is an integral operator, with a kernel $\boldsymbol{T}_\lambda$ that is a K^∞ kernel for which the expansion formulae (34)-(36) hold. By the change of notation: $\psi_n = Wu_n$, $\varphi_n = Vu_n$ ($n \in \mathbb{N}$), these formulae can be rewritten in forms (38)-(40), respectively, because then $T\psi_n = H(I_{L^2} + S)H^{\frac{1}{2}}u_n = H^{\frac{1}{2}}\Lambda u_n = \lambda_n \psi_n$ by (41), and therefore $R_\lambda(T)\psi_n = \frac{1}{1-\lambda\lambda_n}\psi_n$. Moreover, $T^*\varphi_n = (I_{L^2} + S^*)HVu_n = (I_{L^2} + S^*)H(I_{L^2} + S^*)H^{\frac{1}{2}}u_n = (I_{L^2} + S^*)H^{\frac{1}{2}}\Lambda^*u_n = \bar{\lambda}_n\varphi_n$ and $\langle \psi_n, \varphi_m \rangle_{L^2} = \left\langle H^{\frac{1}{2}}u_n, (I_{L^2} + S^*)H^{\frac{1}{2}}u_m \right\rangle_{L^2} = \langle \Lambda u_n, u_m \rangle_{L^2} = \lambda_n \langle u_n, u_m \rangle_{L^2} = \lambda_n\delta_{nm}$ for all m, $n \in \mathbb{N}$. The corollary is proved. $\qquad\square$

For the next corollary, we recall that a family $\{\varphi_n\}$ of functions in L^2 is a *Riesz basis* (see [4]) for L^2 if there exist an invertible operator $A \in \mathfrak{R}(L^2)$ and an orthonormal basis $\{u_n\}$ for L^2 such that $\varphi_n = Au_n$ for all $n \in \mathbb{N}$.

Corollary 16. *Let $T \in \mathfrak{R}\left(L^2\right)$ be an integral operator induced by a K^∞ kernel of Mercer type $\boldsymbol{T}_0$ and suppose that T is representable as*

$$T = \sum_n \lambda_n \left\langle \cdot, \varphi_n \right\rangle_{L^2} \psi_n, \quad \{\varphi_n\} \subset \{\varphi_n'\}, \quad \{\psi_n\} \subset \{\psi_n'\}, \tag{42}$$

where $\{\varphi_n'\}$ and $\{\psi_n'\}$ are Riesz bases for L^2 and $\{\lambda_n\}$ is a family of complex numbers. Then

$$\left(D_2^j D_1^i \boldsymbol{T}_0\right)(s,t) = \sum_n \lambda_n \, [\psi_n]^{(i)}\,(s)\overline{[\varphi_n]^{(j)}\,(t)}, \tag{43}$$

$$\boldsymbol{t}_0^{(i)}(s) = \sum_n \overline{\lambda_n \, [\psi_n]^{(i)}\,(s)}\varphi_n, \quad \left(\boldsymbol{t}_0'\right)^{(j)}(t) = \sum_n \lambda_n \overline{[\varphi_n]^{(j)}\,(t)}\psi_n, \tag{44}$$

$$[Tf]^{(i)}\,(s) = \sum_n \lambda_n \left\langle f, \varphi_n \right\rangle_{L^2} [\psi_n]^{(i)}\,(s), \tag{45}$$

for all i, j, and all s, $t \in \mathbb{R}$, and all $f \in L^2$, where the series of (43) converges $\mathbb{C}$-absolutely in $C\left(\mathbb{R}^2, \mathbb{C}\right)$, the two series of (44) both converge in $C\left(\mathbb{R}, L^2\right)$, and the series of (45) converges $\mathbb{C}$-absolutely in $C(\mathbb{R}, \mathbb{C})$.

Proof. Assume, with no loss of generality, that $\lambda_n = |\lambda_n|\,e^{\imath\theta_n} \neq 0$ for all $n \in \mathbb{N}$. Since both $\{\varphi_n'\}$ and $\{\psi_n'\}$ are Riesz bases, it follows that there exist two invertible operators A, $B \in \mathfrak{R}(L^2)$ and an orthonormal basis $\{u_n\}$ for L^2 such that $T = A\Lambda B$, where

$$\Lambda = \sum_n \lambda_n \left\langle \cdot, u_{k_n} \right\rangle_{L^2} u_{m_n}, \quad u_{m_n} = A^{-1}\psi_n, \quad u_{k_n} = (B^*)^{-1}\,\varphi_n \ (n \in \mathbb{N}), \tag{46}$$

for some subsequences $\{m_n\}$, $\{k_n\}$ of the sequence $\{n\}_{n=1}^\infty$. Introduce operators U, $|\Lambda| \in \mathfrak{R}\left(L^2\right)$, defined by

$$U = \sum_n e^{\imath\theta_n} \left\langle \cdot, u_{k_n} \right\rangle_{L^2} u_{m_n}, \quad |\Lambda| = \sum_n |\lambda_n| \left\langle \cdot, u_{k_n} \right\rangle_{L^2} u_{k_n},$$

and it is clear that $\Lambda = U\,|\Lambda|$ and $|\Lambda| \geq 0$. If $W = AU\,|\Lambda|^{\frac{1}{2}}$, $V = B^*\,|\Lambda|^{\frac{1}{2}}$, then $WW^* = AU\,|\Lambda|\,U^*A^* = TB^{-1}U^*A^* \in \mathcal{M}^+(T)$, $VV^* = B^*\,|\Lambda|\,B = B^*U^*A^{-1}T \in \mathcal{M}^+(T)$, and $T = A\Lambda B = AU\,|\Lambda|^{\frac{1}{2}}\,|\Lambda|^{\frac{1}{2}}\,B = WV^*$, that is to say $T = WV^*$ is an $\mathcal{M}$ factorization for T. In addition to this, for each $r \in \mathbb{N}$, $Wu_r = \delta_{rk_n}\,|\lambda_n|^{\frac{1}{2}}\,AUu_{k_n} = \delta_{rk_n}e^{\imath\theta_n}\,|\lambda_n|^{\frac{1}{2}}\,Au_{m_n} = \delta_{rk_n}e^{\imath\theta_n}\,|\lambda_n|^{\frac{1}{2}}\,\psi_n$, and $Vu_r = B^*\,|\Lambda|^{\frac{1}{2}}\,u_r = \delta_{rk_n}\,|\lambda_n|^{\frac{1}{2}}\,B^*u_{k_n} = \delta_{rk_n}\,|\lambda_n|^{\frac{1}{2}}\,\varphi_n$, by (46). Then $[Wu_r]\,(s)\overline{[Vu_r]\,(t)} = \delta_{rk_n}\lambda_n[\psi_n](s)\overline{[\varphi_n](t)}$ for all s, $t \in \mathbb{R}$, so that formulae (43)-(45) stated in the theorem are already implied by corresponding formulae (10)-(12) in Theorem 8(b) when the latter is applied, with $\lambda = 0$, to the above $\mathcal{M}$ factorization $T = WV^*$ and basis $\{u_n\}$. The corollary is proved. $\qquad\square$

Remark 17. Since, at each fixed regular point $\lambda \in \Pi(T)$, the set $\{R_\lambda(T)\psi'_n\}$ remains a Riesz basis for L^2 and, by Theorem 8(a), the resolvent kernel $\boldsymbol{T}_\lambda$ for $\boldsymbol{T}_0$ is a K^∞ kernel of Mercer type, formulae (43)-(45) continue to hold with 0 and ψ_n replaced respectively by λ and $R_\lambda(T)\psi_n$, due to the same Corollary 16.

Remark 18. When the Riesz bases $\{\varphi'_n\}$ and $\{\psi'_n\}$ are both orthonormal and when $\lambda_n \to 0$ as $n \to \infty$, the assumed representation (42) for T is strongly reminiscent of Schmidt's decomposition for compact operators (see [4], and (63) below). For any $T \in \mathfrak{R}(\mathcal{H})$, meanwhile, there is a generalized Schmidt's decomposition due to I. Fenyö [3]:

Proposition 19. *For every operator $T \in \mathfrak{R}(\mathcal{H})$ there exist an orthonormal basis $\{x_n\} \subset \operatorname{Ran} T$ for $\overline{\operatorname{Ran} T}$, an orthonormal basis $\{y_n\} \subset \operatorname{Ran} T^*$ for $\overline{\operatorname{Ran} T^*}$, and bounded number sequences $\{\varkappa_n\}$, $\{\mu_n\}$ in $\mathbb{C}$, such that, for each $f \in \mathcal{H}$,*

$$Tf = \sum_n \alpha_n \langle f, v_n \rangle x_n, \quad Tf = \sum_n \beta_n \langle f, y_n \rangle w_n \qquad (47)$$

in the sense of convergence in $\mathcal{H}$, where

$$\alpha_n = \sqrt{|\varkappa_n|^2 + |\mu_{n-1}|^2}, \quad v_n = \frac{1}{\alpha_n}\left(\overline{\varkappa}_n y_n + \overline{\mu}_{n-1} y_{n-1}\right) \ (\mu_0 = 0),$$

$$\beta_n = \sqrt{|\varkappa_n|^2 + |\mu_n|^2}, \quad w_n = \frac{1}{\beta_n}\left(\varkappa_n x_n + \mu_n x_{n+1}\right).$$

The elements x_n, $y_n \in \mathcal{H}$ ($n \in \mathbb{N}$) and the numbers $\varkappa_n$, $\mu_n \in \mathbb{C}$ ($n \in \mathbb{N}$), whose existence the proposition guarantees, can be determined simultaneously in a recursive way, as follows. Let $\varkappa_0 = 0$, and let $x_0 = y_0 = 0$. If, for some positive integer n, the subset $\{x_j\}_{j=0}^{n-1}$ of $\operatorname{Ran} T$, the element $y_{n-1} \in \mathcal{H}$, and the number $\varkappa_{n-1} \in \mathbb{C}$, are already defined, and if

$$\operatorname{Span}\left(\{x_j\}_{j=0}^{n-1}\right) \neq \overline{\operatorname{Ran} T}, \qquad (48)$$

then let the number μ_{n-1} and the element x_n of $\operatorname{Ran} T$ be chosen to satisfy

$$\mu_{n-1} x_n = Ty_{n-1} - \varkappa_{n-1} x_{n-1} \qquad (49)$$

subject to the restrictions $\sum_{j=0}^{n-1} |\langle x_n, x_j \rangle_{\mathcal{H}}|^2 = 0$, $\|x_n\|_{\mathcal{H}} = 1$. Next, having made the proper choice of μ_{n-1} and x_n, let $\varkappa_n \in \mathbb{C}$ and $y_n \in \mathcal{H}$ be defined from $\overline{\varkappa}_n y_n = T^* x_n - \overline{\mu}_{n-1} y_{n-1}$ provided that $\|y_n\|_{\mathcal{H}} = 1$. Either this process of determining x_n, y_n, μ_{n-1}, and $\varkappa_n$, from the previously defined $\{x_j\}_{j=0}^{n-1}$, y_{n-1}, and $\varkappa_{n-1}$, is repeated indefinitely with unending growth of n, or it is terminated whenever inequality (48) fails to hold. In any event, the process may generate different sequences $\{x_n\}$, $\{y_n\}$, $\{\varkappa_n\}$, and $\{\mu_n\}$, because of a freedom in choosing x_n when the right side of (49) happens to be zero. *We do not yet know whether, given a bi-integral operator T*

on $\mathcal{H} = L^2$, there always exist operators U_1, $U_2 \in \mathfrak{R}(L^2)$ such that $U_1 v_n = x_n$ and $U_2 w_n = y_n$ for all $n \in \mathbb{N}$, with the notation of (47). If so, then an application of Theorem 8(b) might yield the following result.

Corollary 20. If $T \in \mathfrak{R}(L^2)$ is an integral operator with a K^∞ kernel $\boldsymbol{T}_0$ of Mercer type, then, with the notations of Proposition 19,

$$
\left(D_2^j D_1^i \boldsymbol{T}_0 \right)(s,t) = \sum_n \alpha_n \left[x_n \right]^{(i)}(s) \overline{\left[v_n \right]^{(j)}(t)},
$$

$$
\left(D_2^j D_1^i \boldsymbol{T}_0 \right)(s,t) = \sum_n \beta_n \left[w_n \right]^{(i)}(s) \overline{\left[y_n \right]^{(j)}(t)},
$$

(50)

$$
\boldsymbol{t}_0^{(i)}(s) = \sum_n \overline{\alpha_n \left[x_n \right]^{(i)}(s)} v_n, \quad \left(\boldsymbol{t}_0' \right)^{(j)}(t) = \sum_n \alpha_n \overline{\left[v_n \right]^{(j)}(t)} x_n,
$$

$$
\boldsymbol{t}_0^{(i)}(s) = \sum_n \overline{\beta_n \left[w_n \right]^{(i)}(s)} y_n, \quad \left(\boldsymbol{t}_0' \right)^{(j)}(t) = \sum_n \beta_n \overline{\left[y_n \right]^{(j)}(t)} w_n,
$$

(51)

$$
\left[Tf \right]^{(i)}(s) = \sum_n \alpha_n \left\langle f, v_n \right\rangle_{L^2} \left[x_n \right]^{(i)}(s),
$$

$$
\left[Tf \right]^{(i)}(s) = \sum_n \beta_n \left\langle f, y_n \right\rangle_{L^2} \left[w_n \right]^{(i)}(s),
$$

(52)

for all i, j, all s, $t \in \mathbb{R}$, and all $f \in L^2$, where the series of (50) converge $\mathbb{C}$-absolutely in $C\left(\mathbb{R}^2, \mathbb{C} \right)$, the series of (51) converge in $C\left(\mathbb{R}, L^2 \right)$, and the series of (52) converge $\mathbb{C}$-absolutely in $C(\mathbb{R}, \mathbb{C})$.

4. Proof of Theorem 9

The proof is broken up into three steps. The first step is to specify a pair of orthonormal bases, $\{f_n\}$ for $\mathcal{H}$ and $\{u_n\}$ for L^2. The second step is to define a unitary operator from $\mathcal{H}$ onto L_2 by sending in a suitable manner the basis $\{f_n\}$ onto the basis $\{u_n\}$. This operator is suggested as U in the theorem, and the third step of the proof is a straightforward verification that it is indeed as desired.

Step 1. Let $\mathcal{S} = \{S_\gamma\}_{\gamma \in \mathcal{G}} \subset \mathfrak{R}(\mathcal{H})$ be a family satisfying (14) for some orthonormal sequence $\{e_n\}_{n=1}^\infty$ in $\mathcal{H}$. Let us suppose that we have a pair of orthonormal bases: $\{f_n\}$ for $\mathcal{H}$ and $\{u_n\}$ for L^2, where the latter has the property that, for each i, the i-th derivative, $[u_n]^{(i)}$, of $[u_n]$ belongs to $C(\mathbb{R}, \mathbb{C})$:

$$
\left[u_n \right]^{(i)} \in C(\mathbb{R}, \mathbb{C}) \quad (n \in \mathbb{N}).
$$

(53)

Let us suppose further that each of these bases can be subdivided into two infinite subsequences: $\{f_n\}$ into $\{x_k\}_{k=1}^\infty$ and $\{y_k\}_{k=1}^\infty$, while $\{u_n\}$ into $\{g_k\}_{k=1}^\infty$ and $\{h_k\}_{k=1}^\infty$, such that

$$
\sum_k d(x_k) \le 2
$$

(54)

and, for each i,

$$\sum_k \left\| [h_k]^{(i)} \right\|_{C(\mathbb{R},\mathbb{C})} < \infty, \tag{55}$$

$$\sum_k d(x_k) \left\| [g_k]^{(i)} \right\|_{C(\mathbb{R},\mathbb{C})} < \infty, \tag{56}$$

where

$$d(x_k) := 2 \left(\sup_{\gamma \in \mathcal{G}} \|S_\gamma x_k\|_{\mathcal{H}}^{1/4} + \sup_{\gamma \in \mathcal{G}} \|(S_\gamma)^* x_k\|_{\mathcal{H}}^{1/4} \right) \quad (k \in \mathbb{N}). \tag{57}$$

The proof will make use of the orthonormal bases $\{f_n\}$, $\{u_n\}$ just described to define the action of the desired unitary operator $U \colon \mathcal{H} \to L^2$ in the theorem. By using the assumption (14) and some facts from the theory of wavelets, the following example affirmatively answers the existence question for such a pair of orthonormal bases.

Example 21. Let $\mathbb{N}$ be decomposed into two infinite subsequences $\{l(k)\}_{k=1}^{\infty}$ and $\{m(k)\}_{k=1}^{\infty}$, and let $\{u_n\}$ be an orthonormal basis for L^2 that has property (53), and such that, for each i,

$$\left\| [u_n]^{(i)} \right\|_{C(\mathbb{R},\mathbb{C})} \le N_n D_i \quad (n \in \mathbb{N}), \tag{58}$$

where $\{N_n\}_{n=1}^{\infty}$, $\{D_i\}_{i=0}^{\infty}$ are two sequences of positive reals, the first of which is subject to the restriction that

$$\sum_k N_{l(k)} < \infty. \tag{59}$$

Then the orthonormal basis $\{u_n\}$ can be paired (in the sense above) with an orthonormal basis $\{f_n\}$ for $\mathcal{H}$. Indeed, from conditions (58), (59) it follows that the requirement (55) may be satisfied for all i by taking $h_k = u_{l(k)}$ $(k \in \mathbb{N})$. At the same time, on account of (14) and of (58), the sequence $\{e_k\}_{k=1}^{\infty}$ does always have an infinite subsequence $\{x_k\}_{k=1}^{\infty}$ that satisfies both the requirement (54) and the requirement (56) for each i with $g_k = u_{m(k)}$ $(k \in \mathbb{N})$. Once such a subsequence $\{x_k\}_{k=1}^{\infty}$ of $\{e_k\}_{k=1}^{\infty}$ has been fixed, the remaining task is simply to complete the x_k's to an orthonormal basis, and to let y_k $(k \in \mathbb{N})$ denote the new elements of that basis. Now $\{f_n\} = \{x_k\}_{k=1}^{\infty} \cup \{y_k\}_{k=1}^{\infty}$ and $\{u_n\} = \{g_k\}_{k=1}^{\infty} \cup \{h_k\}_{k=1}^{\infty}$ constitute a pair of orthonormal bases satisfying (53)-(57).

In turn, an explicit example of a basis $\{u_n\}$ that obeys the above three conditions (53), (58), and (59), can be adopted from the wavelet theory, as follows. Let ψ be the Lemarié-Meyer wavelet,

$$[\psi](s) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{\imath \xi (\frac{1}{2}+s)} \mathrm{sgn}\, \xi\, b(|\xi|)\, d\xi \quad (s \in \mathbb{R})$$

with the bell function b being infinitely smooth and compactly supported on $[0, +\infty)$ (see [6, Example D, p. 62] for details). Then $[\psi]$ is of the Schwartz class $\mathcal{S}(\mathbb{R})$, so its every derivative $[\psi]^{(i)}$ is in $C(\mathbb{R}, \mathbb{C})$. In addition, the "mother wavelet" ψ generates an orthonormal basis $\{\psi_{\alpha\beta}\}_{\alpha, \beta \in \mathbb{Z}}$ for L^2 by

$$\psi_{\alpha\beta} = 2^{\frac{\alpha}{2}} \psi(2^{\alpha} \cdot -\beta) \quad (\alpha, \beta \in \mathbb{Z}).$$

In any manner, rearrange the two-indexed set $\{\psi_{\alpha\beta}\}_{\alpha, \beta \in \mathbb{Z}}$ into a simple sequence, so that it looks like $\{u_n\}_{n=1}^{\infty}$; clearly each term here, u_n, has property (53) for each i. Besides, it is easily verified that a norm estimate like (58) holds for each $[u_n]^{(i)}$: the factors in the right-hand side of (58) may be taken as

$$N_n = \begin{cases} 2^{\alpha_n^2} & \text{if } \alpha_n > 0, \\ 2^{\alpha_n/2} & \text{if } \alpha_n \leq 0, \end{cases} \qquad D_i = 2^{(i+1/2)^2} \left\| [\psi]^{(i)} \right\|_{C(\mathbb{R}, \mathbb{C})},$$

with the convention that $u_n = \psi_{\alpha_n \beta_n}$ ($n \in \mathbb{N}$) in conformity with that rearrangement. This choice of N_n also gives $\sum_k N_{l(k)} < \infty$ (cf. (59)) whenever $\{l(k)\}_{k=1}^{\infty}$ is a subsequence of $\{l\}_{l=1}^{\infty}$ satisfying $\alpha_{l(k)} \to -\infty$ as $k \to \infty$.

Step 2. In this step our intention is to construct a candidate for the desired unitary operator $U \colon \mathcal{H} \to L^2$ in the theorem. Recalling that

$$\begin{aligned} \{f_1, f_2, f_3, \dots\} &= \{x_1, x_2, x_3, \dots\} \cup \{y_1, y_2, y_3, \dots\}, \\ \{u_1, u_2, u_3, \dots\} &= \{g_1, g_2, g_3, \dots\} \cup \{h_1, h_2, h_3, \dots\}, \end{aligned} \tag{60}$$

where the unions are disjoint, define the unitary operator U from $\mathcal{H}$ onto L^2 by setting

$$Ux_k = g_k, \quad Uy_k = h_k \quad \text{for all } k \in \mathbb{N}, \tag{61}$$

in the harmless assumption that $Uf_n = u_n$ for all $n \in \mathbb{N}$.

Step 3. This step of the proof is to show that, in fact, if the unitary operator $U \colon \mathcal{H} \to L^2$ is defined as in (61), then the operators $US_\gamma U^{-1} \colon L^2 \to L^2$ ($\gamma \in \mathcal{G}$) are all simultaneously bi-Carleman operators with K^∞ kernels of Mercer type.

Fix, to begin with, an arbitrary index $\gamma \in \mathcal{G}$, abbreviate S_γ to S, and let $T = USU^{-1}$. It is to be proved that T is an integral operator with a K^∞ kernel of Mercer type. The idea is first to conveniently split the operators S, S^*, each into two parts as follows. If E is the orthogonal projection of $\mathcal{H}$ onto $\text{Span}\,(\{x_k\}_{k=1}^{\infty})$, write

$$S = (1 - E)S + ES, \quad S^* = (1 - E)S^* + ES^*. \tag{62}$$

It is an immediate consequence of (54) and (60) that the operators $J = SE$ and $\widetilde{J} = S^*E$ are Hilbert-Schmidt. Write down their Schmidt's decompositions

$$J = \sum_n s_n \langle \cdot, p_n \rangle_{\mathcal{H}} q_n, \quad \widetilde{J} = \sum_n \widetilde{s}_n \langle \cdot, \widetilde{p}_n \rangle_{\mathcal{H}} \widetilde{q}_n, \tag{63}$$

and introduce compact operators B and $\widetilde{B}$, which are defined by

$$B = \sum_n s_n^{\frac{1}{4}} \langle \cdot, p_n \rangle_{\mathcal{H}} \, q_n, \quad \widetilde{B} = \sum_n \widetilde{s}_n^{\frac{1}{4}} \langle \cdot, \widetilde{p}_n \rangle_{\mathcal{H}} \, \widetilde{q}_n; \tag{64}$$

here the s_n are the singular values of J (eigenvalues of $(JJ^*)^{\frac{1}{2}}$), $\{p_n\}$, $\{q_n\}$ are orthonormal sets in $\mathcal{H}$ (the p_n are eigenvectors for J^*J and q_n are eigenvectors for JJ^*). The explanation of the notation for $\widetilde{J}$ is similar.

For each $f \in \mathcal{H}$, let

$$c(f) := \|Bf\|_{\mathcal{H}} + \|B^*f\|_{\mathcal{H}} + \left\|\widetilde{B}f\right\|_{\mathcal{H}} + \left\|\left(\widetilde{B}\right)^* f\right\|_{\mathcal{H}}. \tag{65}$$

Applying Schwarz's inequality yields

$$
\begin{aligned}
c(x_k) &= \sqrt{\sum_n s_n^{\frac{1}{2}} |\langle x_k, p_n \rangle_{\mathcal{H}}|^2} + \sqrt{\sum_n s_n^{\frac{1}{2}} |\langle x_k, q_n \rangle_{\mathcal{H}}|^2} \\
&\quad + \sqrt{\sum_n \widetilde{s}_n^{\frac{1}{2}} |\langle x_k, \widetilde{p}_n \rangle_{\mathcal{H}}|^2} + \sqrt{\sum_n \widetilde{s}_n^{\frac{1}{2}} |\langle x_k, \widetilde{q}_n \rangle_{\mathcal{H}}|^2} \\
&= \left\|(J^*J)^{\frac{1}{8}} x_k\right\|_{\mathcal{H}} + \left\|(JJ^*)^{\frac{1}{8}} x_k\right\|_{\mathcal{H}} \\
&\quad + \left\|\left(\left(\widetilde{J}\right)^* \widetilde{J}\right)^{\frac{1}{8}} x_k\right\|_{\mathcal{H}} + \left\|\left(\widetilde{J}\left(\widetilde{J}\right)^*\right)^{\frac{1}{8}} x_k\right\|_{\mathcal{H}} \\
&\leq \|Jx_k\|_{\mathcal{H}}^{\frac{1}{4}} + \|J^*x_k\|_{\mathcal{H}}^{\frac{1}{4}} + \left\|\widetilde{J}x_k\right\|_{\mathcal{H}}^{\frac{1}{4}} + \left\|\left(\widetilde{J}\right)^* x_k\right\|_{\mathcal{H}}^{\frac{1}{4}} \\
&\leq 2\left(\|Sx_k\|^{\frac{1}{4}} + \|S^*x_k\|^{\frac{1}{4}}\right) \leq d(x_k) \quad (k \in \mathbb{N}),
\end{aligned}
\tag{66}
$$

whence it follows again from (54) and (60) that B and $\widetilde{B}$ are both Hilbert-Schmidt operators, implying

$$\sum_n s_n^{\frac{1}{2}} < \infty, \quad \sum_n \widetilde{s}_n^{\frac{1}{2}} < \infty. \tag{67}$$

Now define $Q = (1-E)S^*$, $\widetilde{Q} = (1-E)S$. Since $\{y_k\}$ is, by construction, an orthonormal basis for the subspace $(1-E)\mathcal{H}$, it follows that

$$
\begin{aligned}
Qf &= \sum_k \langle Qf, y_k \rangle_{\mathcal{H}} \, y_k = \sum_k \langle f, Sy_k \rangle_{\mathcal{H}} \, y_k, \\
\widetilde{Q}f &= \sum_k \left\langle \widetilde{Q}f, y_k \right\rangle_{\mathcal{H}} \, y_k = \sum_k \langle f, S^*y_k \rangle_{\mathcal{H}} \, y_k,
\end{aligned}
\tag{68}
$$

for all $f \in \mathcal{H}$, with the series converging in $\mathcal{H}$.

Having considered the splittings (62), which now look like

$$S = \widetilde{Q} + \left(\widetilde{J}\right)^*, \quad S^* = Q + J^*, \tag{69}$$

our next task is to verify that each of the four operators UQU^{-1}, $U\widetilde{Q}U^{-1}$, UJ^*U^{-1}, and $U\left(\widetilde{J}\right)^* U^{-1}$, is a Carleman operator with a kernel satisfying conditions (i), (ii) in Definition 6. The checking is straightforward, and goes by representing all pertinent kernels and Carleman functions as infinitely smooth sums of termwise differentiable series of infinitely smooth functions, without any use of the method of Section 2. Theorems on termwise differentiation of series will however be used repeatedly (and usually tacitly) in the subsequent analysis.

If $P = UQU^{-1}$, $\widetilde{P} = U\widetilde{Q}U^{-1}$, then from (68) and (61) it follows that, for each $f \in L^2$,

$$Pf = \sum_k \langle f, Th_k \rangle_{L^2} h_k, \quad \widetilde{P}f = \sum_k \langle f, T^*h_k \rangle_{L^2} h_k, \tag{70}$$

in the sense of convergence in L^2. Represent the equivalence classes $Th_k, T^*h_k \in L^2$ ($k \in \mathbb{N}$) by the Fourier expansions with respect to the orthonormal basis $\{u_n\}$:

$$Th_k = \sum_n \langle y_k, S^* f_n \rangle_{\mathcal{H}} u_n, \quad T^*h_k = \sum_n \langle y_k, Sf_n \rangle_{\mathcal{H}} u_n,$$

where the convergence is in the L^2 norm. But somewhat more than that can be said about convergence, namely that, for each fixed i, the series

$$\sum_n \langle y_k, S^* f_n \rangle_{\mathcal{H}} [u_n]^{(i)}(s), \quad \sum_n \langle y_k, Sf_n \rangle_{\mathcal{H}} [u_n]^{(i)}(s) \quad (k \in \mathbb{N}) \tag{71}$$

converge in the norm of $C(\mathbb{R}, \mathbb{C})$. Indeed, all the series are dominated everywhere on $\mathbb{R}$ by one series

$$\sum_n \left(\|S^* f_n\|_{\mathcal{H}} + \|Sf_n\|_{\mathcal{H}} \right) \left| [u_n]^{(i)}(s) \right|,$$

which is uniformly convergent on $\mathbb{R}$ for the following reason: its component subseries (cf. (61))

$$\sum_k \left(\|Sx_k\|_{\mathcal{H}} + \|S^* x_k\|_{\mathcal{H}} \right) \left| [g_k]^{(i)}(s) \right|, \quad \sum_k \left(\|Sy_k\|_{\mathcal{H}} + \|S^* y_k\|_{\mathcal{H}} \right) \left| [h_k]^{(i)}(s) \right|$$

are uniformly convergent on $\mathbb{R}$ because they are in turn dominated by the convergent series

$$\sum_k d(x_k) \left\| [g_k]^{(i)} \right\|_{C(\mathbb{R}, \mathbb{C})}, \quad \sum_k 2\|S\| \left\| [h_k]^{(i)} \right\|_{C(\mathbb{R}, \mathbb{C})}, \tag{72}$$

respectively (see (57), (54), (56), and (55)).

It is now evident that the pointwise sums of the series of (71) define functions that belong to $C(\mathbb{R}, \mathbb{C})$ and that are none other than $[Th_k]^{(i)}$, $[T^*h_k]^{(i)}$ ($k \in \mathbb{N}$), respectively. Moreover, the above arguments prove that given any i, there exists a positive constant C_i such that

$$\left\| [Th_k]^{(i)} \right\|_{C(\mathbb{R}, \mathbb{C})} < C_i, \quad \left\| [T^*h_k]^{(i)} \right\|_{C(\mathbb{R}, \mathbb{C})} < C_i,$$

for all k. Hence, by (55), it is possible to infer that, for all i, j, the series

$$\sum_k [h_k]^{(i)}(s)\overline{[Th_k]^{(j)}(t)}, \quad \sum_k [h_k]^{(i)}(s)\overline{[T^*h_k]^{(j)}(t)}$$

converge and even absolutely in the norm of $C\left(\mathbb{R}^2, \mathbb{C}\right)$. This makes it clear that functions $\boldsymbol{P}$, $\widetilde{\boldsymbol{P}} \colon \mathbb{R}^2 \to \mathbb{C}$, defined by

$$\boldsymbol{P}(s, t) = \sum_k [h_k](s)\overline{[Th_k](t)}, \quad \widetilde{\boldsymbol{P}}(s, t) = \sum_k [h_k](s)\overline{[T^*h_k](t)}, \qquad (73)$$

satisfy condition (i) in Definition 6.

Now we prove that (Carleman) functions $\boldsymbol{p}$, $\widetilde{\boldsymbol{p}} \colon \mathbb{R} \to L^2$, defined by

$$\boldsymbol{p}(s) = \overline{\boldsymbol{P}(s, \cdot)} = \sum_k \overline{[h_k](s)}Th_k, \quad \widetilde{\boldsymbol{p}}(s) = \overline{\widetilde{\boldsymbol{P}}(s, \cdot)} = \sum_k \overline{[h_k](s)}T^*h_k, \qquad (74)$$

are both subject to requirement (ii) in Definition 6. Indeed, the series displayed converge absolutely in the $C\left(\mathbb{R}, L^2\right)$ norm, because those two series whose terms are respectively $|[h_k](s)| \|Th_k\|_{L^2}$ ($k \in \mathbb{N}$) and $|[h_k](s)| \|T^*h_k\|_{L^2}$ ($k \in \mathbb{N}$) are dominated by the second series of (72) with $i = 0$. For the remaining i, a similar reasoning implies the same convergence behavior of the series $\sum_k [h_k]^{(i)}(s)Th_k$, $\sum_k \overline{[h_k]^{(i)}(s)}T^*h_k$. The asserted property of both Carleman functions $\boldsymbol{p}$ and $\widetilde{\boldsymbol{p}}$ to satisfy (ii) then follows by the corresponding theorem on termwise differentiation of series.

From (55) it follows that the series of (70), viewed as series in $C(\mathbb{R}, \mathbb{C})$, converge (and even absolutely) in $C(\mathbb{R}, \mathbb{C})$ norm, and therefore that their pointwise sums are none other than $[Pf]$ and $\left[\widetilde{P}f\right]$, respectively. On the other hand, the established properties of the series of (74) and of (73) make it possible to write, for each temporarily fixed $s \in \mathbb{R}$, the following chains of relations

$$\sum_k \langle f, Th_k \rangle_{L^2} [h_k](s) = \left\langle f, \sum_k \overline{[h_k](s)}Th_k \right\rangle_{L^2}$$

$$= \int_{\mathbb{R}} \left(\sum_k [h_k](s)\overline{[Th_k](t)} \right) f(t)\,dt = \int_{\mathbb{R}} \boldsymbol{P}(s, t)f(t)\,dt,$$

$$\sum_k \langle f, T^* h_k \rangle_{L^2} [h_k](s) = \left\langle f, \sum_k \overline{[h_k](s)} T^* h_k \right\rangle_{L^2}$$

$$= \int_{\mathbb{R}} \left(\sum_k [h_k](s) \overline{[T^* h_k](t)} \right) f(t)\, dt = \int_{\mathbb{R}} \widetilde{\boldsymbol{P}}(s,t) f(t)\, dt,$$

whenever f is in L^2. These imply that P and $\widetilde{P}$ are Carleman integral operators, the kernels of which are $\boldsymbol{P}$ and $\widetilde{\boldsymbol{P}}$ respectively, both are subject to requirements (i), (ii) in Definition 6 by what precedes.

Now we consider the Hilbert-Schmidt integral operators $F = U J^* U^{-1}$ and $\widetilde{F} = U \left(\widetilde{J} \right)^* U^{-1}$ on L^2, and prove that the kernels of these operators are K^∞ kernels. Associate with Schmidt's decompositions (63) for J, $\widetilde{J}$ two functions $\boldsymbol{F}, \widetilde{\boldsymbol{F}} \colon \mathbb{R}^2 \to \mathbb{C}$, defined by

$$\boldsymbol{F}(s,t) = \sum_n s_n^{\frac{1}{2}} [U B^* q_n](s) \overline{[U B p_n](t)}$$

$$\left(= \sum_n s_n [U p_n](s) \overline{[U q_n](t)} \right),$$

$$\widetilde{\boldsymbol{F}}(s,t) = \sum_n \widetilde{s}_n^{\frac{1}{2}} \left[U \left(\widetilde{B} \right)^* \widetilde{q}_n \right](s) \overline{\left[U \widetilde{B} \widetilde{p}_n \right](t)}$$

$$\left(= \sum_n \widetilde{s}_n [U \widetilde{p}_n](s) \overline{[U \widetilde{q}_n](t)} \right), \tag{75}$$

whenever s, t are in $\mathbb{R}$; for the auxiliary operators B, $\widetilde{B}$ here used, see (64). It is to be noted that without the square brackets the bilinear series just written do converge almost everywhere on $\mathbb{R}^2$ to Hilbert-Schmidt kernels that induce respectively F and $\widetilde{F}$ (see (67)). Hence, and again because of (67), the conclusion that the above-defined functions $\boldsymbol{F}$ and $\widetilde{\boldsymbol{F}}$ are the kernels of F and $\widetilde{F}$, respectively, and are subject to condition (i) of Definition 6 can be inferred as soon as it is known that for each fixed i the terms of the sequences

$$\left\{ [U B p_k]^{(i)} \right\}, \quad \left\{ [U B^* q_k]^{(i)} \right\}, \quad \left\{ \left[U \widetilde{B} \widetilde{p}_k \right]^{(i)} \right\}, \quad \left\{ \left[U \left(\widetilde{B} \right)^* \widetilde{q}_k \right]^{(i)} \right\} \tag{76}$$

make sense, are all in $C(\mathbb{R}, \mathbb{C})$, and their $C(\mathbb{R}, \mathbb{C})$ norms are bounded, regardless of k. To see that the conditions just listed are all fulfilled, it suffices to observe that, once i is fixed, all the series

$$\sum_n \langle p_k, B^* f_n \rangle_{\mathcal{H}} [u_n]^{(i)}(s), \quad \sum_n \langle q_k, B f_n \rangle_{\mathcal{H}} [u_n]^{(i)}(s),$$

$$\sum_n \left\langle \widetilde{p}_k, \left(\widetilde{B} \right)^* f_n \right\rangle_{\mathcal{H}} [u_n]^{(i)}(s), \quad \sum_n \left\langle \widetilde{q}_k, \widetilde{B} f_n \right\rangle_{\mathcal{H}} [u_n]^{(i)}(s) \quad (k \in \mathbb{N}) \tag{77}$$

(which, with $i = 0$, are merely the L^2-convergent Fourier expansions, with respect to the orthonormal basis $\{u_n\}$, for UBp_k, UB^*q_k, $U\widetilde{B}\widetilde{p}_k$, and $U(\widetilde{B})^*\widetilde{q}_k$, respectively) are dominated by one series

$$\sum_n c(f_n) \left| [u_n]^{(i)}(s) \right|,$$

where $c(f)$ is defined in (65) above. This last series is uniformly convergent on $\mathbb{R}$, because it is composed of the two subseries

$$\sum_n c(x_k) \left| [g_k]^{(i)}(s) \right|, \quad \sum_n c(y_k) \left| [h_k]^{(i)}(s) \right|,$$

that converge uniformly in $\mathbb{R}$, having as their dominant series the convergent series

$$\sum_k d(x_k) \left\| [g_k]^{(i)} \right\|_{C(\mathbb{R},\mathbb{C})}, \quad \sum_k 2 \left(\|B\| + \left\| \widetilde{B} \right\| \right) \left\| [h_k]^{(i)} \right\|_{C(\mathbb{R},\mathbb{C})}$$

(see (66), (56), (55)). Thus, for each i, all the series in (77) converge (and even absolutely) in the $C(\mathbb{R},\mathbb{C})$ norm, and their sums are none other than, respectively, $[UBp_k]^{(i)}$, $[UB^*q_k]^{(i)}$, $\left[U\widetilde{B}\widetilde{p}_k \right]^{(i)}$, and $\left[U(\widetilde{B})^*\widetilde{q}_k \right]^{(i)}$ ($k \in \mathbb{N}$). Then, in virtue of (67) and the above-established boundedness of the sequences of (76) in $C(\mathbb{R},\mathbb{C})$, the series

$$\sum_n s_n^{\frac{1}{2}} [UB^*q_n]^{(i)}(s) \overline{[UBp_n]^{(j)}(t)}, \quad \sum_n \widetilde{s}_n^{\frac{1}{2}} \left[U\left(\widetilde{B}\right)^*\widetilde{q}_n \right]^{(i)}(s) \overline{\left[U\widetilde{B}\widetilde{p}_n \right]^{(j)}(t)},$$

converge (and even absolutely) in $C(\mathbb{R}^2,\mathbb{C})$, for all i, j. This in conjunction with (75) is sufficient to conclude that both the functions $\boldsymbol{F}$ and $\widetilde{\boldsymbol{F}}$ satisfy condition (i) of Definition 6, and are the Hilbert-Schmidt kernels of F and of $\widetilde{F}$, respectively.

Again by the properties of the sequences of (76) and by (67), the series

$$\sum_n s_n^{\frac{1}{2}} \overline{[UB^*q_n]^{(i)}(s)} UBp_n, \quad \sum_n s_n^{\frac{1}{2}} UB^*q_n \overline{[UBp_n]^{(i)}(s)},$$

$$\sum_n \widetilde{s}_n^{\frac{1}{2}} \overline{\left[U\left(\widetilde{B}\right)^*\widetilde{q}_n \right]^{(i)}(s)} U\widetilde{B}\widetilde{p}_n, \quad \sum_n \widetilde{s}_n^{\frac{1}{2}} U\left(\widetilde{B}\right)^*\widetilde{q}_n \overline{\left[U\widetilde{B}\widetilde{p}_n \right]^{(i)}(s)}$$

converge and even absolutely in the $C\left(\mathbb{R}, L^2\right)$ norm, for each i. Observe, via (75), that four of these series, namely those with $i = 0$, represent the Carleman functions $\boldsymbol{f}(s) = \overline{\boldsymbol{F}(s,\cdot)}$, $\boldsymbol{f}'(s) = \boldsymbol{F}(\cdot,s)$, $\widetilde{\boldsymbol{f}}(s) = \overline{\widetilde{\boldsymbol{F}}(s,\cdot)}$, and $\widetilde{\boldsymbol{f}}'(s) = \widetilde{\boldsymbol{F}}(\cdot,s)$, respectively, which therefore do satisfy conditions (ii), (iii) in Definition 6. This finally implies that the Hilbert-Schmidt kernels $\boldsymbol{F}$ and $\widetilde{\boldsymbol{F}}$ are K^∞ kernels of F and of $\widetilde{F}$, respectively.

In accordance with (69), the operator T (which, recall, is the transform by U of S) and its adjoint decompose as $T = \widetilde{P} + \widetilde{F}$, $T^* = P + F$ where all the terms are already known to be Carleman operators, with kernels satisfying (i), (ii). Hence, both T and T^* are Carleman operators, and their kernels, say $\boldsymbol{T}$ and $\boldsymbol{T^*}$, defined as

$$\boldsymbol{T}(s,t) = \widetilde{\boldsymbol{P}}(s,t) + \widetilde{\boldsymbol{F}}(s,t), \quad \boldsymbol{T^*}(s,t) = \boldsymbol{P}(s,t) + \boldsymbol{F}(s,t), \tag{78}$$

for all s, $t \in \mathbb{R}$, inherit the smoothness properties (i), (ii) from their terms. But then, since the operator T is bi-integral, $\boldsymbol{T}(s,t) = \overline{\boldsymbol{T^*}(t,s)}$ for all s, $t \in \mathbb{R}$; hence $\boldsymbol{T}(\cdot,t) = \overline{\boldsymbol{T^*}(t,\cdot)}$ in the L^2 sense for each fixed $t \in \mathbb{R}$. That implies that the kernel $\boldsymbol{T}$ also satisfies condition (iii) in Definition 6, and is thus a K^∞ kernel that induces the bi-Carleman operator T.

This K^∞ kernel $\boldsymbol{T}$ is moreover of Mercer type. A main ingredient in proving the claim is the following remark: if $d(x_k)$ are defined as in (57), if U is defined as in (61), and if an operator $A \in \mathfrak{R}(\mathcal{H})$ fulfils

$$2\left(\|Ax_k\|_{\mathcal{H}}^{\frac{1}{4}} + \|A^*x_k\|_{\mathcal{H}}^{\frac{1}{4}}\right) \le d(x_k) \quad \text{for each } k \in \mathbb{N},$$

then, like $T = USU^{-1}$, the operator UAU^{-1} is a bi-Carleman operator with a K^∞ kernel, no matter whether A is in the initial family $\mathcal{S}$ or not. This additional feature of U may be checked by directly applying the above verification procedure, leading from (62) to (78), to the operator A in place of S.

If now a nonzero operator A is restricted to lie in the set $\mathcal{M}(S)$, then whatever its representation from among those in (3) is, the scalar multiple

$$A_1 = (16 \max\{\|M\|, \|N\|\})^{-1} \cdot A$$

of A obeys the following easily verifiable inequality valid for all $k \in \mathbb{N}$:

$$2\left(\|A_1x_k\|_{\mathcal{H}}^{\frac{1}{4}} + \|A_1^*x_k\|_{\mathcal{H}}^{\frac{1}{4}}\right) \le 2\left(\|Sx_k\|_{\mathcal{H}}^{\frac{1}{4}} + \|S^*x_k\|_{\mathcal{H}}^{\frac{1}{4}}\right) \le d(x_k).$$

By the above remark, this implies that the unitary operator U defined in (61) carries, besides S, every other member, A, of $\mathcal{M}(S)$ onto an integral operator, UAU^{-1}, with a K^∞ kernel. Then, since $\mathcal{M}(T) = U\mathcal{M}(S)U^{-1}$, the K^∞ kernel $\boldsymbol{T}$ of T, constructed in (78) above, is of Mercer type by virtue of the definition. Moreover, since $T = US_\gamma U^{-1}$ where the index γ was arbitrarily fixed, it follows that all the operators $T_\gamma = US_\gamma U^{-1}$ ($\gamma \in \mathcal{G}$) are bi-Carleman operators whose kernels are K^∞ kernels of Mercer type.

The only thing that remains to be proved is that those K^∞ kernels which induce finite linear combinations of $US_\gamma U^{-1}$ ($\gamma \in \mathcal{G}$) are also of Mercer type. Indeed, consider any finite linear combination $G = \sum z_\gamma S_\gamma$ with $\sum |z_\gamma| \le 1$. It is seen easily that, for each n,

$$\left\|\sum z_\gamma S_\gamma e_n\right\|_{\mathcal{H}} \le \sup_{\gamma \in \mathcal{G}} \|S_\gamma e_n\|_{\mathcal{H}}, \quad \left\|\sum \overline{z}_\gamma \left(S_\gamma\right)^* e_n\right\|_{\mathcal{H}} \le \sup_{\gamma \in \mathcal{G}} \|(S_\gamma)^* e_n\|_{\mathcal{H}}.$$

There is, therefore, no barrier to assuming that G was, from the start, in the initial family $\mathcal{S}$. The proof of the theorem is now complete.

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