K-SATURATED GRAPHS

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Abstract. We present some properties of k-existentially-closed and k-saturated graphs, as steps toward discovering a 4-saturated graph. We demonstrate a construction for 2-saturated graphs, prove the existence of k-e.c. graphs, and show that the chromatic number of k-saturated graphs increases with k. Finally, we detail an algorithm for finding k-existentially-closed and k-saturated graphs.

1. Introduction

Sometimes mathematicians have too much time on their hands so they invent with pretty pictures and pretend to be artists. Sadly, mathematicians make such poor artisits, they have in the past 30 years or so dedicated an entire field to the study of artistry, beginning with only dots and lines which define what most people know of math(read The Dot and The Line: A Romance in Lower Mathematics (1963)). From here, we see that the more triangles in these pretty pictures, the less clear it is (something about optical illusions), so here we try to make the prettiest picture we can without any triangles and without using too many dots.

1.1. Background.

We denote a graph G with vertex set $V(G)$ to be *k-existentially closed (k-e.c.)* if $|V(G)| \geq k$ and for every subset K of k vertices, every subset J of K of size j is adjoined to a common vertex in $V(G) - K$ which is not adjacent to any vertex in $K - J$. We refer to this vertex for some subset K and subset J as a (K, J) subset pair witness. Every vertex in the graph is a witness for at least one (K, J) subset pair, and more than one witness may, and usually will, exist for and (K, J) subset pair in $V(G)$.

Further, we say that a graph is *k-saturated* if it is *triangle-free*, meaning it contains no 3-cycles, and has the same witness conditions as a k-existentially closed graph, except that only subsets J which are independent are required to be witnessed. 1-, 2-, and 3-saturated graphs are known to exist, but no 4-saturated graph has been found or been proven to exist[**?**]. To check whether a graph is k-saturated is a mostly trivial excercise of computer programing, but the number of graphs which exist on n vertices is far too broad an area to check each graph.

1.2. Problem Statement and Outline.

Our ultimate research goal is to prove or disprove the existence of 4-saturated graphs. In this paper, we present a variety of algorithms and results about k-saturated and k-e.c. graphs that assist towards this goal.

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In Section 2, we present known examples and constructions of k-saturated graphs. In Section 3, we tighten the bounds on the smallest number of vertices for both k-saturated and k-e.c. graphs, and determine a probibalistic approach as to what number of vertices to begin the search for these graphs. In Section 4, we discuss the connection between k-saturation and chromatic number, and give an algorithm, given any positive integer n, for finding a k for which all k-saturated graphs must have chromatic number at least n. Finally, we present an algorithm to find k-saturated and k-e.c. graphs in Section **??** and some graphs found by running the algorithm in Section **??**.

2. Examples of k-Saturated Graphs

An example of a 1-saturated graph is 4-cycle. A 1-saturated graph implies that each vertex is adjacent to at least one vertex in the graph and not adjacent to at least one vertex in the graph. Each vertex in a 4-cycle is adjacent to two vertices and not adjacent to one.

A 2-saturated graph can be built using a construction called the *Mycielskian*. The Mycielskian construction, named after Jan Myscielski [**?**], proves that triangle free graphs exist of any given chromatic number. The construction is built by creating a creating a copy of each vertex in the graph, here after called a *shadow* of the vertex, and adjoining it to every vertex the original vertex is adjoined with, and creating a last vertex, called the *Central Vertex (CV)* which is not a shadow of any vertex, and attaching each shadow vertex to the CV. Note that the shadow vertices are not adjoined to their original vertex, and that this construction increases the number of vertices in a graph from n to $2n + 1$. Two properties about the Mycielskian construction of interest in this research are that if a graph has a chromatic number k, then the Mycielskian of this graph has a chromatic number $k + 1$, and that if a graph is triangle-free, the Mycielskian is also triangle-free.

Theorem 2.1. *For any graph G, if all (K, J) subset pairs, where $k = 2$, have witnesses except for the null set, then the Mycielski of the graph is 2-saturated.*

Proof. For use in this proof, we will define G to be the original set of vertices, and S to be the set of shadow vertices. Note that no edges exist between vertices in S because of the construction, thus there are four cases for (K, J) subset pairs, which need a witness. Only one case is drawn out in detail due to the similarity of the proofs.

case 1: For all 2 vertices in S

- A witness for the null set, a 0-witness, exists because any other vertex in S satisfies this condition
- Without loss of generality, a 1-witness for either of these shadow vertices is the 1-witness of its original vertex in G
- A 2-witness for this K set in S is the CV

case 2: For all 1 vertex in S and 1 vertex in G
case 3: For the CV and 1 vertex in S
case 4: For the CV and 1 vertex in G

□

The smallest known triangle free graph for which each (K, J) subset pair has at least one witness, where $k = 3$, consists of 16 vertices. This construction can be visualized by denoting four sets S_0, S_1, S_2, and S_3, each with four vertices $v_{S_i,0}$, $v_{S_i,1}$, $v_{S_i,2}$, $v_{S_i,3}$, where each $v_{S_i,j}$ is adjoined to each $v_{S_i+1,j}$ and each $v_{S_i,j}$ is adjoined to each $v \in S_i + 2 - v_{S_i+2,j}$. This is the graph $K(3, 3, 3)$.[?]

The Steiner system construction of the Higman-Sims graph bares much resemblance to the Mycielskian construction of triangle-free graphs. The smallest known graph in which each (K, J) subset pair is witnessed at least twice is known as the *Higman-Sims* graph, discovered by Donald G. Higman and Charles C. Sims. The graph is built on 100 vertices using a construction from Steiner systems based on block design. A *Steiner* system, S, with parameters t, k, and n is an n-element set S together with a set of k-element subsets of S (called blocks) with the property that each t-element subset of S is contained in exactly one block. As an example, a Steiner system where $t = 3$, $k = 6$, and $n = 22$, written $S(3, 6, 22)$ is the set of blocks of 6 vertices where no 3 vertices are in more than one block together. This particular system is the Higman-Sims graph [**?**]

Definition 2.2 (Regular)**.** *A graph G is called* ρ-regular *if every vertex in G has degree ρ*

Definition 2.3 (Strongly Regular)**.** *A graph G is called* strongly ρ-regular*, or simply strongly regular, if there exist positive integers κ, λ, and μ such that every vertex has κ neighbors (i.e., the graph is a regular graph), every adjacent pair of vertices has λ common neighbors, and every nonadjacent pair has μ common neighbors* [**?**]

Removing one vertex from the strongly regular Higman-Sims graph results in a 3-saturated graph which is not regular.

3. Proof of Main Results

We use the probabilistic method to determine an estimate of how many witnesses for each valid (K, J) subset pair are needed for a k-saturated graph to be $(k + 1)$-saturated.

Theorem 3.1. *For every n such that*

$$\binom{n}{k} 2^k \sum_{i=0}^{r-1} \binom{n-k}{i} 2^{-ki} (1 - 2^{-k})^{n-k-i} < 1 \tag{1}$$

there exists a graph on n vertices that is k-e.c. with r witnesses.

Proof. First we will demonstrate for $r = 1$.
We use the probability model $G_{n,(1/2)}$ where n is the number of vertices and the edges are chosen with independent, random probability of $(1/2)$. Then we can see that for each K set and each $J \subset K$, k edges determine a witness for $(K,\ J)$. So the probability that any vertex is a witness for $(K,\ J)$ is 2^{-k}. Then $(1 - 2^{-k})$ is the probability that any vertex is not a witness for $(K,\ J)$. Let $A_{K,J}$ be the event

that $(K,\, J)$ has no witness.

$$Pr(A_{K,J}) = (1-2^{-k})^{n-k}$$

Now let X equal the total number of unwitnessed sets. So we let $I_{K,J}$ be a random, independent variable such that

$$I_{K,J} = \begin{cases} 1 & \text{if event } A_{K,J} \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

Then

$$X = \sum_K \sum_J I_{K,J}$$

$$E(X) = \sum_K \sum_J E(I_{K,J})$$

$$= \binom{n}{k} 2^k (1-2^{-k})^{n-k}$$

Markov's Inequality 3.2. *If X takes only non-negative values, then $Pr(X \geq a) \leq E(X)/a$*

Using Markov's Inequality, we will let $a = 1$ and the random variable X. Then $Pr(X \geq 1) \leq E(X)$. So $Pr(X = 0) > 1 - E(X)$. Hence when $E(X) < 1$ we are guaranteed the existence of a k-e.c. graph. So choose n such that

$$E(X) = \binom{n}{k} 2^k (1-2^{-k})^{n-k} < 1$$

Thus for any k there exists a graph with 1 witness for every J set.
Now consider when $r = 2$. The probability that there exists exactly one witness for $(K,\, J)$ (i.e. v is a witness for $(K,\, J)$ for some vertex, v, in the graph and no other vertex is a witness) is $2^{-k}(1-2^{-k})^{n-k-1}$. Then the probability that $(K,\, J)$ has less than 2 witnesses is:

$$(1-2^{-k})^{n-k} + (n-k)2^{-k}(1-2^{-k})^{n-k-1}$$

Similarly for $r = 3$, the probability that $(K,\, J)$ has less than 3 witnesses is:

$$(1-2^{-k})^{n-k} + (n-k)2^{-k}(1-2^{-k})^{n-k-1} + \binom{n-k}{2} 2^{-2k}(1-2^{-k})^{n-k-2}$$

In general, let $R_{K,J}$ be the event that $(K,\, J)$ has less than r witnesses. Then

$$Pr(R_{K,J}) = \sum_{i=0}^{r-1} \binom{n-k}{i} 2^{-ki}(1-2^{-k})^{n-k-i}$$

In this case we let X equal the total number of sets with less than r witnesses and we let $I_{K,J}$ be a random variable such that

$$I_{K,J} = \begin{cases} 1 & \text{if event } R_{K,J} \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

Applying Markov's Inequality, we see that we can choose n such that

$$E(X) = \binom{n}{k} 2^k \sum_{i=0}^{r-1} \binom{n-k}{i} 2^{-ki}(1-2^{-k})^{n-k-i} < 1$$

□

When $r = 1$, we examine the asymptotics of the expected value and find an expression for n as k increases,

$$n \sim k^2 2^k \ln 2 (1 + \frac{\ln k}{k \ln 2} + \frac{\ln 2 + 1 + \ln\ln 2}{k \ln 2} + \frac{\ln(1 + \frac{\ln k}{k\ln 2} + \frac{\ln 2+1+\ln\ln 2}{k \ln 2})}{k \ln 2})$$

We notice that $\binom{n}{k} = An^k/k!$ where $A = \prod_k (1 - \frac{i}{n})$.
Then

$$\log A = \sum \log(1 - \frac{i}{n})$$
$$= \sum \frac{-i}{n}(1 - \frac{x}{2} + \frac{x^2}{3} - \ldots)$$
$$\approx -\sum_{i=1}^{k} \frac{i}{n} = \frac{-k^2 - k}{2n}$$

As $k << \sqrt{n}$ then $\frac{-k^2-k}{2n}$ goes to 0 and $e^{\frac{-k^2-k}{2n}}$ goes to 1. So $\binom{n}{k} \approx n^k/k!$
We also notice that from Stirling's Approximation, $k! \approx \frac{k^k}{e^k}$.
Hence when $k << \sqrt{n}$,

$$E(X) \approx \frac{n^k e^k}{k^k} 2^k (1 - 2^{-k})^n$$

.
We use our approximation for n in this equation to get

$$E(X) \approx (\frac{k^2 2^k \ln 2(1 + \frac{\ln k}{k \ln 2} + \frac{\ln 2+1+\ln\ln 2}{k\ln 2} + \frac{\ln(1+\frac{\ln k}{k \ln 2}+\frac{\ln 2+1+\ln\ln 2}{k\ln 2})}{k \ln 2})2e}{k})^k$$
$$*(1-2^{-k})^{k^2 2^k \ln 2(1+\frac{\ln k}{k\ln 2}+\frac{\ln 2+1+\ln\ln 2}{k \ln 2}+\frac{\ln(1+\frac{\ln k}{k\ln 2}+\frac{\ln 2+1+\ln\ln 2}{k\ln 2})}{k\ln 2})}$$

We now apply the approximation, $(1 - \frac{a}{n})^n \approx e^{-a}$ to $(1-2^{-k})^{2^k}$. So

$$E(X) \approx (\frac{k 2^k \ln 2(1 + \frac{\ln k}{k \ln 2} + \frac{\ln 2+1+\ln\ln 2}{k\ln 2} + \frac{\ln(1+\frac{\ln k}{k \ln 2}+\frac{\ln 2+1+\ln\ln 2}{k\ln 2})}{k \ln 2})2e}{e^{k \ln 2(1+\frac{\ln k}{k\ln 2}+\frac{\ln 2+1+\ln\ln 2}{k \ln 2}+\frac{\ln(1+\frac{\ln k}{k\ln 2}+\frac{\ln 2+1+\ln\ln 2}{k\ln 2})}{k\ln 2})}})^k$$
$$\approx (\frac{(1 + \frac{\ln k}{k \ln 2} + \frac{\ln 2+1+\ln\ln 2}{k\ln 2} + \frac{\ln(1+\frac{\ln k}{k \ln 2}+\frac{\ln 2+1+\ln\ln 2}{k\ln 2})}{k \ln 2})}{e^{k \ln 2(\frac{\ln(1+\frac{\ln k}{k\ln 2}+\frac{\ln 2+1+\ln\ln 2}{k\ln 2})}{k\ln 2})}})^k$$
$$\approx (\frac{\frac{\ln(1+\frac{\ln k}{k \ln 2}+\frac{\ln 2+1+\ln\ln 2}{k\ln 2})}{k \ln 2}}{1 + \frac{\ln k}{k\ln 2} + \frac{\ln 2+1+\ln\ln 2}{k \ln 2}})^k$$

For our approximation to be accurate

$$\lim_{k\to\infty} (\frac{\frac{\ln(1+\frac{\ln k}{k \ln 2}+\frac{\ln 2+1+\ln\ln 2}{k\ln 2})}{k \ln 2}}{1 + \frac{\ln k}{k\ln 2} + \frac{\ln 2+1+\ln\ln 2}{k \ln 2}})^k \longrightarrow 1$$

Or

$$\lim_{k\to\infty} k\ln\left(\left(\frac{\frac{\ln(1+\frac{\ln k}{k\ln 2}+\frac{\ln 2+1+\ln\ln 2}{k\ln 2})}{k\ln 2}}{1+\frac{\ln k}{k\ln 2}+\frac{\ln 2+1+\ln\ln 2}{k\ln 2}}\right)^k\right) \longrightarrow 0$$

Using L'Hopital's Rule, it can be verified that this holds true.

In general for $r << \sqrt{n}$,

$$n \sim (k+r-1)2^k k\ln 2(1+a+b+c+d)$$

where

$$a = \frac{-k\ln k-(r-1)\ln(r-1)}{(k+r-1)k\ln 2}$$
$$b = \frac{(k+r-1)+k\ln 2}{(k+r-1)k\ln 2}$$
$$c = \frac{\ln(k+r-1)+\ln k+\ln\ln 2}{k\ln 2}$$
$$d = \frac{\ln(1+a+b+c)}{(k+r-1)k\ln 2}$$

4. k-Saturated and Chromatic Number

By examining chromatic number, we find that k-saturated graphs are very rare. The first thing to note is then any k-saturated graph is also j-saturated for all $0 < j < k$.

Definition 4.1. *A graph has chromatic number ℓ if the vertices can be colored with ℓ colors in such a way that every edge has endpoints of different colors and it cannot be colored in such a way with fewer than ℓ colors. We say that a graph is ℓ-colorable if it has chromatic number $\leq n$.*

We show in this section that as k increases, the chromatic number a graph must have to be k-saturated increases as well. Erdos, Kleitman, and Rothschild[**?**] showed that

$$|T(n)| = |B(n)|(1+o(1))$$

where $T(n)$ is the set of triangle-free graphs on n vertices and $B(n)$ is the set of bipartite graphs on n vertices. Promel, Schickinger, and Steger[**?**] extended this result to show that

$$|T(n)\setminus B(n)| = |B_1(n)|(1+o(1))$$

where $B_1(n)$ is the set of *quasibipartite graphs*, or graphs which become bipartite after the removal of a single vertex.

Note 4.2. *A bipartite graph has chromatic number 2 and a quasibipartite graph has chromatic number 3.*

Then asymptotically, a random triangle-free graph is almost surely bipartite, and a random non-bipartite triangle-free graph almost surely has chromatic number larger than 3.

We show first two specific results: that no 2-saturated graph can be bipartite and no 4-saturated graph can be 3-colorable. When $k \geq 4$, k-saturated graphs are 4-saturated, meaning that they are not bipartite or quasibipartite and are hence rare. We use $v \parallel u$ to denote "v is adjacent to u" and $v \Vdash u$ to denote "v is not adjacent to u".

Theorem 4.3. *No k-saturated graph for $k \geq 2$ is bipartite.*

Proof. It is sufficient to show that no 2-saturated graphs are bipartite, since all k-saturated for $k \geq 2$ are also 2-saturated.

Let G be bipartite, and let the vertices of G be split into two non-empty sets V_1 and V_2 by color, so that the vertices in each set are colored a single color. We prove that G cannot be 2-saturated from examining the following cases.

Case 1: G is complete bipartite, in other words, there is an edge between every pair of vertices (v_1, v_2) where $v_1 \in V_1$ and $v_2 \in V_2$.

Let v_1 be a vertex in V_1 and v_2 be a vertex in V_2. Then there is no witness for $K = \{v_1, v_2\}$ and $J = \emptyset$.

Case 2: G is not complete bipartite.

Then there exists some pair of vertices (v_1, v_2) with $v_1 \in V_1$ and $v_2 \in V_2$ such that there is no edge between v_1 and v_2. There is no witness for $K = \{v_1, v_2\}$ and $J = \{v_1, v_2\}$. □

To show a more general result, we need the following lemma.

Lemma 4.4. *Given a k-saturated graph on n vertices divided into ℓ sets by color where $k > \ell$ and $k \geq 4$, there is at least one set with at least k vertices.*

Proof. First we show that there is an independent set of k vertices. Choose any k-set K and name one vertex v_1. Let v_2 be the witness for the J subset $\emptyset$. Now choose any K subset containing v_1 and v_2 and let v_3 be a witness for the J subset $\emptyset$. Next choose any K subset containing v_1, v_2, and v_3 and let v_4 be a witness for the J subset $\emptyset$. Continue this process until there is a set of k independent vertices $\{v_1, \ldots, v_k\}$.

Now there must be 2^k witnesses for the independent set of k vertices, divided into ℓ sets by color. By pidgeon-hole principle, there must be at least one color-set with $\frac{2^k}{\ell}$ vertices in it. Now $\frac{2^k}{\ell} > \frac{2^k}{k}$ and for $k \geq 4$, $\frac{2^k}{k} \geq k$. So for $k \geq 4$, there is at least one color-set with at least k vertices. □

We prove that no k-saturated graph for $k \geq 4$ is 3-colorable as a specific example and then generalize the proof to find a k for any ℓ such that no k-saturated is ℓ-colorable.

Theorem 4.5. *No k-saturated graph for $k \geq 4$ is 3-colorable.*

Proof. It is sufficient to show that no 4-saturated graphs are 3-colorable.

Let G be 3-colorable, assume G is 4-saturated, and let the vertices of G be split into three non-empty sets V_1, V_2, and V_3 by color. There must be a set with at least 4 vertices, let's call it V_1. Choose 4 vertices from V_1 as a K subset and call them v_1, v_2, v_3, and v_4.

Now consider the J subsets which are subsets of $\{v_1, v_2\}$. There are $2^2 = 4$ such subsets: $\{v_1, v_2\}$, $\{v_1\}$, $\{v_2\}$, and $\emptyset$, and $4 - 1 = 3$ of these are not the empty set. Thus there are 3 witnesses for non-empty J subsets contained in $\{v_1, v_2\}$. Each of these witnesses is not in V_1 and is not adjacent to v_3 or v_4. Pidgeon-hole principle shows that either V_2 or V_3 has at least two of the three witnesses. We can assume without loss of generality that it is V_2 and call the two witnesses v_5 and v_6.

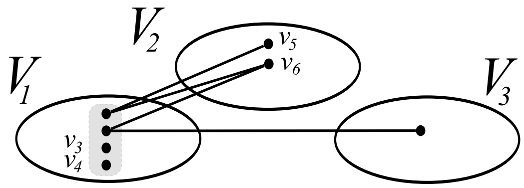

FIGURE 1. Finding vertices v_5 and v_6 not adjacent to v_3 or v_4.

Now consider a witness v_7 for the K subset $\{v_3, v_4, v_5, v_6\}$ and the J subset $\{v_4, v_6\}$. Then $v_7 \Vdash v_5, v_5 \Vdash v_3$, and $v_3 \Vdash v_7$.

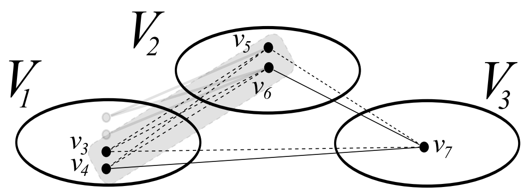

FIGURE 2. Creating a triangle with v_3, v_5, and v_7. (non-adjacencies denoted by dashed lines)

Consider any 4-set as a K subset which contains $\{v_3, v_5, v_7\}$. There is no witness for the J subset $\{v_3, v_5, v_7\}$. So G is not 4-saturated. □

Corollary 4.6. *No k-saturated graph for $k \geq 4$ is quasibipartite.*

We now show that in general, as k increases, the chromatic number of a k-saturated graph must increase as well. To do this, we present a construction, given ℓ, to find k such that all k-saturated graphs cannot be ℓ-colorable. The algorithm is as follows: given ℓ, construct a sequence $\{a_1, \ldots, a_{\ell-2}\}$ and $\{a'_1, \ldots, a'_{\ell-1}\}$ such that $a'_{i+1} = \lfloor \frac{(2^{a_i}-1)^i}{\ell-i} \rfloor$ for all i and $a'_i = a_i + a'_{i+1}$ for all $i < \ell - 1$ with $a'_{\ell-1} = 2$. Then $k = \max_{1 \leq i \leq \ell-1}(ia'_i)$.

Proposition 4.7. *Using the above construction to find k, a k-saturated graph is not ℓ-colorable.*

Proof. Let G be a k-saturated ℓ-colorable graph, and let $V_1, V_2, \ldots V_\ell$ be the sets of vertices of each color of an ℓ-coloring. One color-set contains at least k vertices by Lemma **??**. Assume it is V_1 and designate a k-set $K_0 \subset V_1$. Notice that $k \geq a'_1$ and designate a subset $B_1 \subset K_0$ of size a'_1.

Now let K_0 be a K subset, and choose any a_1-set $A_1 \subset B_1$. Every subset of A_1 needs a witness as a J-set, and all $2^{a_1} - 1$ witnesses to non-empty subsets are not in V_1. By the pidgeon-hole principle, we can find a color-set, say V_2, which contains at least a'_2 witnesses. Denote the set of a'_2 witnesses in V_2 by B_2. By construction, each element of B_2 is not adjacent to any vertex in $B_1 \setminus A_1$.

Since $|B_1 \setminus A_1| = a'_2$ and $2a'_2 \leq k$, we can choose a k-set containing $(B_1 \setminus A_1) \cup B_2$ and consider a set A_2 consisting of a_2 vertices from $B_1 \setminus A_1$ and a_2 vertices from B_2. By pidgeon-hole, we can find a different color-set, say V_3, which contains at least a'_3 witnesses of subsets of A_2. Denote the set of a'_3 witnesses by B_3 and notice that no element of B_3 is adjacent to any element of $(B_1 \setminus A_1) \setminus A_2$ or $B_2 \setminus A_2$ by construction. Furthermore, $|(B_1 \setminus A_1) \setminus A_2| = |B_2 \setminus A_2| = a'_3$ and $3a'_3 \leq k$. So we can choose a k-set containing the independent set $((B_1 \setminus A_1) \setminus A_2) \cup (B_2 \setminus A_2) \cup B_3$ and continue the process.

In the end, we have a set $B_{\ell-1}$ of consisting of $a'_{\ell-1} = 2$ vertices, and an independent set $S = ((B_1 \setminus A_1) \setminus A_2) \setminus \cdots \setminus A_{\ell-2}) \cup ((B_2 \setminus A_2) \setminus \cdots \setminus A_{\ell-2}) \cup \cdots \cup B_{\ell-1}$ consisting of 2 vertices in every one of $\ell - 1$ color-sets, say $V_1, \ldots, V_{\ell-1}$. We can choose a superset of S as a K subset because $|S| \leq k$. Now we can chose 1 vertex from S in each color-set V_j, $1 \leq j \leq \ell - 1$ as a J subset, and call it T. Then the witness v for T is not adjacent to any vertex in $S \setminus T$. But $(S \setminus T) \cup \{v\}$ has cardinality less than k and contains a vertex from every color-set $V_1, \ldots, V_\ell$. Let $(S \setminus T) \cup \{v\}$ be a J-subset for any K subset containing it. There is no witness for $(S \setminus T) \cup \{v\}$. So G cannot be both k-saturated and ℓ-colorable. □

It is clear by construction that as ℓ increases, the k needed to show a k-saturated graph is not ℓ-colorable increases as well. Similarly, as k increases, all k-saturated graphs must not be ℓ-colorable for increasingly many ℓ.

5. Description of Algorithm

To find k-saturated and k-existentially-closed graphs, we use a simulated annealing algorithm.

Definition 5.1. *A simulated annealing algorithm is an algorithm governing a walk from an initial state to a global minimum or maximum avoiding local minima and maxima that works as follows:*

1. *The potential next step s is chosen randomly from a set of possible steps*
2. *If s improves the current state, the step s is taken*
3. *If s does not improve the current state, s taken with some probability p*
4. *The probability p of accepting an edge fluctuates throughout the process and decreases globally as the algorithm continues to run*

The idea behind such an algorithm is to "cool" from an initial state to a global minimum while "heating" temporarily to escape from local minima along the way.

Our algorithm for finding graphs uses a seed graph as the initial state. A *step* is the addition or deletion of an edge or set of edges. The overall global minimum we would like to reach is zero unwitnessed statements, and we say that the state is *improved* by a step if the addition or deletion of the chosen edges decreases the number of unwitnessed statements.

More specifically, our algorithm works as follows to find an n-vertex, k-saturated or k-existentially-closed graph. We use as a seed a graph that is far from bipartite to avoid the issues raised in Section 4. We then choose two vertices randomly. If the addition/deletion of the edge between these two vertices decreases the number of unwitnessed statements, the edge is added/deleted. Otherwise, the edge is added/deleted with probability $p(\Delta) = e^{-\Delta/b}$ where b is a constant initialized to 10 and Δ is the change in the number of unwitnessed statements due to the addition/deletion of the edge. Hence the worse the edge is, the less likely it is to be added/deleted. If the edge improves the graph, b is set to 2, thereby decreasing the probability of keeping a "bad" edge since there are "good" choices around. If the edge does not improve the graph, the constant b is increased by $(5\sqrt{1000 + i/30})^{-1}$ where i is the total number of iterations of a random edge choice since the start of the algorithm. Adding to b increases the probability of keeping a bad edge, but as the program continues to run, the increase becomes progressively smaller. Thus locally the algorithm is "heating," but it is globally "cooling."

If the number of unwitnessed statements remains the same for $100\binom{n}{2}$ iterations, we determine that the program is. "stuck." We systematically try every possible edge one at a time to determine which will improve the current state. If at least one edge does improve the current state, we choose one such edge randomly and continue the algorithm as before, setting b to 2. If no edges improve the graph, we choose a vertex v_0 and a random number j from 1 to $\frac{n}{3}+1$. We randomly select j distinct vertices from $V(G) - \{v_0\}$ and determine if the addition/deletion of all of these edges at once improves the state. Just as in the case of adding an edge at a time, we add/delete this set of edges if doing so improves the state and we add/delete this set of edges with the same probability function $p(\Delta)$ if doing so does not improve the state. We then continue adding/deleting an edge at a time. The pseudo-code is given in Algorithm 1.

Algorithm 1 Finding k-saturated and k-e.c. graphs

Input: The number of vertices n and saturation number k.
Output: A k-saturated or k-e.c. graph

Let G be a graph with n vertices and edges between vertices v_i and v_j, $j > i$ whenever $v_j = v_i + 1 \mod n$ and whenever $v_j = v_i + 4 \mod n$.
Let $b = 10$.
Let $u(G)$ be the number of unwitnessed statements of G
while $u(G) > 0$ **do**
  if not stuck **then**
    choose an edge e
    Let G' be the result of adding/deleting e.
    if $u(G') < u(G)$ **then**
      Set $b = 2$ and $G = G'$
    else
      Set $b = b + \frac{1}{5\sqrt{1000+\frac{i}{30}}}$
      With probability $e^{(u(G)-u(G')/b}$ set $G = G'$.
    end if
  else
    Let E be the set of edges that improve $w(G)$.
    if $E \neq \{\}$ **then**
      add/delete a randomly chosen element of E to G
    else
      Chose a vertex v_0 and a random number j, $1 \leq j \leq \frac{n}{3} + 1$.
      Chose j edges with an endpoint at v_0 and add/delete them all. Let the resulting graph be G"
      if $u(G') < u(G)$ **then**
        Set $b = 2$ and $G = G'$
      else
        Set $b = b + \frac{1}{5\sqrt{1000+\frac{i}{30}}}$
        With probability $e^{(u(G)-u(G')/b}$ set $G = G'$.
      end if
    end if
  end if
end while
return G

There are two differences in the implementation of the algorithm between k-saturated and k-existentially closed. The first is in the adding of an edge. Any edge is added if we are looking for a k-e.c. graph. If we are looking for a k-saturated graph, an edge is only added if the resulting graph remains triangle free. Secondly, the value $u(G)$ is the total number of unwitnessed statements in the k-e.c. case. In the k-saturated case, $u(G)$ is the number of unwitnessed statements minus the number of statements that do not need witnesses due to non-independent J-sets.

The implementation of the algorithm uses two data structures. The first is an $n \times n$ bit array A called the adjacency matrix. The entries a_{ij} of A are defined as follows:

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge between } v_i \text{ and } v_j \\ 0 & \text{otherwise} \end{cases}$$

The second is a matrix M with rows indexed by the $\binom{n}{k}$ K-sets and columns indexed by the 2^k J-sets such that

$$m_{ij} = \text{the number of witnesses for the } i\text{th } K\text{-set and } j\text{th } J\text{-set}$$

Additionally, when we look for k-saturated graphs, we use another array Q indexed exactly like M where

$$q_{ij} = \begin{cases} 1 & \text{if the } j\text{th } J\text{-set of the } i\text{th } K\text{-set is not independent} \\ 0 & \text{otherwise} \end{cases}$$

6. Figures

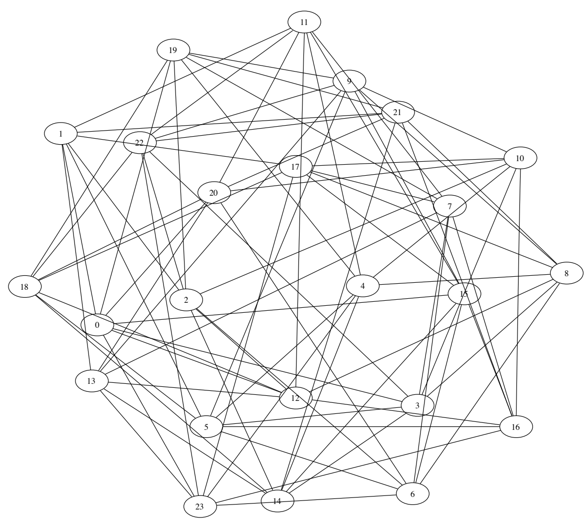

Figure 3. 3-saturated graph on 24 vertices

k \ r	1	2	3	4	5	6	7	8	9	10
3	111	139	162	183	202	220	238	255	271	287
4	363	425	477	524	568	609	649	687	724	760
5	1061	1195	1309	1412	1507	1598	1684	1767	1847	1925
6	2888	3174	3418	3638	3843	4037	4223	4401	4573	4740
7	7502	8103	8617	9083	9518	9929	10322	10700	11065	11420
8	18835	20086	21160	22138	23051	23915	24741	25536	26305	27051
9	46085	48673	50906	52943	54847	56651	58377	60039	61647	63208
10	110502	115834	120453	124676	128627	132376	135964	139420	142765	146015

Table 1. Values of n for each k and r witnesses

7. Conclusion/Discussion/Future Work

As noted in the introduction, the similarities in graph design between the Steiner system and the Mycielskian have been noticed and we would like to inspect these graphs further.

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