Kapitza Pendulum: A Physically Transparent Simple Explanation

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Abstract. The phenomenon of dynamic stabilization of the inverted pendulum whose pivot is forced to oscillate with a high frequency in vertical direction is revisited. A simple physically meaningful explanation of the phenomenon is presented, followed by the derivation of an approximate quantitative criterion of stability. A computer program simulating the physical system is developed, which aids the analytical investigation of the phenomenon in a manner that is mutually reinforcing. Material is appropriate for undergraduate university students.

1. Introduction: The physical system

A fascinating feature in the behavior of a simple rigid pendulum whose suspension point is forced to vibrate with a high frequency along the vertical line is the dynamic stabilization of the inverted position. When the frequency and the amplitude of these vibrations are large enough, the inverted pendulum shows no tendency to turn down. Moreover, at moderate deviations from the vertical inverted position the pendulum tends to return to it. Being deviated, the pendulum executes relatively slow oscillations about the vertical line on the background of rapid oscillations of the suspension point.

Simple hand-made devices can be used for a classroom demonstration of this fascinating phenomenon of classical mechanics. A jig saw or an old electric shaver's mechanism can serve perfectly well to force the pivot of a light rigid pendulum vibrating with a high enough frequency and sufficient amplitude to make the inverted position stable (Figure 1).

The hand holds the shaver in the position which provides the vertical direction of the pivot oscillations. If the rod is turned into the inverted vertical position, it remains there as long as the axis is vibrating. When the rod is slightly deflected to one side and released, it oscillates slowly about the inverted position. Many videos illustrating this unexpected behavior of the pendulum can be found on the web.

This surprising phenomenon of dynamic stabilization was predicted originally by Stephenson [1] more than a century ago (in 1908). In 1951 such extraordinary behavior of the pendulum was rediscovered, explained physically and investigated experimentally in detail by Pjotr Kapitza [2]. The corresponding physical device is now widely known as "Kapitza's pendulum. Below is a citation from the paper of Kapitza [3] published in the Russian journal "Uspekhi":

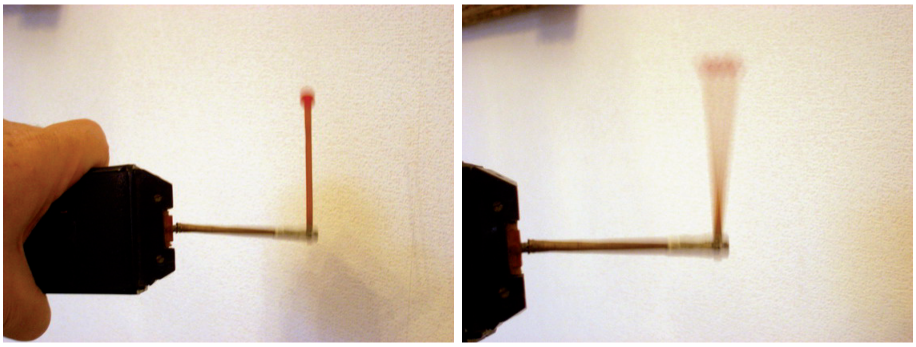

Figure 1. Demonstration of dynamic stabilization of the inverted pendulum.

> "Demonstration of oscillations of the inverted pendulum is very impressive. Our eyes cannot follow the fast small movements caused by vibrations of the pivot, so that behavior of the pendulum in the inverted position seems perplexing and even astonishing ... When we carefully touch the rod of the pendulum trying to deviate it from the vertical, the finger feels the resistance produced by the vibrational torque. After acquaintance with the experiment on dynamic stabilization of the inverted pendulum we reasonably conclude that this phenomenon is as much instructive as the dynamic stabilization of a gyroscope, and should be necessarily included in lecture demonstrations on classical mechanics."

Not surprisingly that after Kapitza this simple but very curious and intriguing physical system attracted attention of many researchers, and the theory of the phenomenon may seem to be well elaborated (see, for example, [4]). Nevertheless, more and more new features in the behavior of this really inexhaustible system are reported regularly. One can find many good texts and hundreds of papers on the subject. A vast list of references is provided in [5]. The author of this paper also contributed to investigation of the parametrically forced inverted pendulum (see [6]–[9]).

However, a great majority of papers and monographs on the subject are advanced texts written for experts and specialists, in which parametric excitation of the pendulum and associated phenomena are explained in terms of the theory of differential equations with periodic coefficients (Floquet theory, Hill and Mathieu equations). The nature of such texts is predominantly mathematical and actually gives very little insight into the phenomena, whose physical sense remains buried deeply in severe and nontransparent mathematics, which could turn out to be abstract and very complicated for physics students and their teachers.

In the abundant literature on the subject it is hardly possible to find a sufficiently simple and physically clear interpretation of the inverted pendulum dynamic stabilization. Understanding this interesting phenomenon is certainly a challenge to our intuition. The principal aim of this paper is to present a quite simple qualitative physical explanation of the phenomenon, and to find out the conditions at which it is possible to observe it. We focus also on an approximate quantitative analysis of the slow motion of the pendulum which can be developed on the basis of the suggested approach to the problem.

We consider for simplicity the rigid planar pendulum of length l with a point mass on its end assuming that all mass m of the pendulum is concentrated here. The force of gravity $m\mathrm{g}$ creates a restoring torque $-mgl\sin\varphi$ which is proportional to the sine of the angle φ of deflection from the equilibrium position. When the pivot is at rest, due to this this torque the pendulum swings about the lower stable equilibrium position.

When the pivot is forced to move with an acceleration, it is convenient to describe the motion of the pendulum using the non-inertial frame of reference associated with the pivot. To make the Newton's laws of motion applicable in this accelerated reference frame, we should add to all "real" forces the "pseudo" force of inertia. Due to translational acceleration $\mathbf{a}_{\mathrm{frame}}$ of the frame, an additional force, the force of inertia $\mathbf{F}_{\mathrm{in}} = -m\mathbf{a}_{\mathrm{frame}}$, is exerted on the pendulum. This force is directed oppositely to the acceleration of the frame.

We assume that the pivot is forced to execute a given harmonic oscillation along the vertical line with a frequency ω and an amplitude a, i. e., the motion of the axis is described by the following equation:

$$z(t) = a\cos\omega t \qquad \text{or} \qquad z(t) = a\sin\omega t. \tag{1}$$

Hence the pseudo force of inertia $F_{\mathrm{in}}(t)$ exerted on the bob in the non-inertial frame of reference associated with the pivot also has the same sinusoidal dependence on time:

$$F_{\mathrm{in}}(t) = -m\frac{d^2z(t)}{dt^2} = -m\ddot{z}(t) = m\omega^2 z(t). \tag{2}$$

This force is equivalent to a periodic modulation of the force of gravity. Indeed, $F_{\mathrm{in}}(t)$ is directed downward during the time intervals for which $z(t) < 0$, i.e., when the axis is below the middle point of its oscillations. We see this directly from equation for $F_{\mathrm{in}}(t)$, Eq. (2), whose right-hand side depends on time exactly as the z-coordinate of the axis (see Eq. (1)). Therefore during the corresponding half-period of the oscillation of the pivot this additional force is equivalent to some strengthening of the force of gravity. During the other half-period the axis is above its middle position, and the action of this additional force is equivalent to some weakening of the gravitational force. When the frequency and/or amplitude of the pivot are large enough (when $a\omega^2 > g$), for some part of the period the apparent gravity (the sum of real gravity and the force of inertia) is even directed upward.

2. Qualitative explanation of the dynamic stabilization

To explain physically the effect of dynamic stabilization of the inverted pendulum caused by fast vibrations of the pivot, we should take into account the influence of the force of inertia averaged over the period of these fast vibrations. According to eq. (2), the force of inertia depends on time sinusoidally and its mean value for a period is zero. However, the mean value of the *torque* of this force with respect to the axis of the pendulum is not zero. It is this mean torque of the force of inertia that is responsible for extraordinary, counterintuitive behavior of the pendulum.

To better understand the influence of the force of inertia upon the system, we first forget for a while about the force of gravity. Without oscillations of the pivot, in the absence of

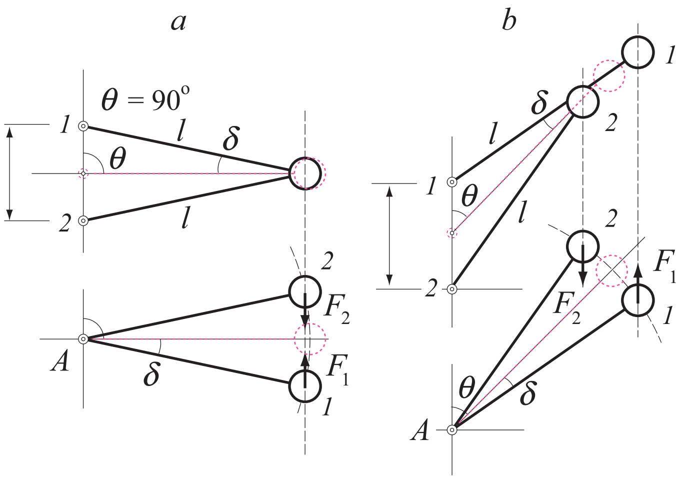

Figure 2. The forces of inertia F_1 and F_2 exerted on the pendulum in the noninertial reference frame at the extreme positions *1* and *2* of the oscillating axis A.

gravity the pendulum is in the neutral state of equilibrium at any orientation of its rod. Let us begin with the case in which the rod of the pendulum is oriented horizontally, that is, at the right angle $\theta = \pi/2$ with respect to direction of the pivot oscillations (see Figure 2, *a*). If the massive bob has zero initial velocity, in the absence of gravity it remains practically at the same level with respect to the laboratory inertial reference frame while the axis A oscillates between the extreme points *1* and *2*. The rod simply turns down and up through a small angle δ, as shown in the upper panel of Figure 2, *a*.

In the non-inertial frame of reference associated with the oscillating axis, the same motion of the rod is shown in the lower panel of Figure 2, *a*: The bob of the pendulum moves up and down along an arc of a circle and occurs in positions *1* and *2* at the instants at which the oscillating axis reaches its extreme positions *1* and *2*, respectively (the upper panel of Figure 2, *a*). Indeed, at any time moment the rod has the same simultaneous orientations in both reference frames.

In position *1* the force of inertia $\mathbf{F}_1$ exerted on the bob, according to Eq. (2), is directed upward, and in position *2* the force $\mathbf{F}_2$ of the same magnitude is directed downward. The arm of the force in positions *1* and *2* is the same. It is evident that the torque of this force of inertia, averaged over the period of oscillations, is zero. Hence in the absence of gravity this orientation of the pendulum (perpendicularly to the direction of the axis' oscillations) corresponds to a dynamic equilibrium position (an unstable one, as we shall see later).

Now let us consider the case in which on average the rod is deflected through an arbitrary angle θ from the direction of oscillations, and the axis oscillates between extreme points *1* and *2*, as shown in the upper panel of Figure 2, *b*. By virtue of these vertical oscillations of the axis the rod turns periodically up and down from its middle position through some small angle δ. In the non-inertial frame of reference associated with the oscillating axis, the bob moves at these oscillations between points *1* and *2* (the lower panel of Figure 2, *b*) along an arc of a circle whose center coincides with the axis A of the pendulum.

We note again that at any time moment the rod has the same simultaneous orientations in both reference frames. This is true at moment *1* as well as at moment *2.* When the axis is displaced upward (to position *1* from its midpoint), the force of inertia F_1 exerted on the bob is also directed upward. In the other extreme position *2* the force of inertia F_2 has an equal magnitude and is directed downward. However, now the *torque* of the force of inertia in position *1* is greater than in position *2* because the *arm* of the force in this position is greater.

Therefore on average the force of inertia creates a torque about the axis that tends to turn the pendulum upward, into the vertical inverted position, in which the rod is parallel to the direction of oscillations. Certainly, if the pendulum makes an acute angle with respect to the downward vertical position, the mean torque of the force of inertia tends to turn the pendulum downward.

Thus, the torque of the force of inertia, averaged over a period of oscillations, tends to align the pendulum along the direction of forced oscillations of the axis. Figure 2, *b* presents an utterly simple and clear explanation to the origin of this torque. Since this torque is induced by vibrations of the axis, Kapitza (see [2]–[3]) called it "vibrational," but we can also call it "inertial," because its origin is related to the force of inertia that arises in the reference frame of the axis due to the fast forced vibrations of the axis. For given values of the driving frequency and amplitude, this mean torque depends only on the angle of the pendulum's deflection from the direction of the pivot's vibration.

This mean inertial torque does not depend on time explicitly, and its influence on the pendulum can be considered exactly in the same way as the influence of other ordinary external torques, such as the torque of the gravitational force. The inertial torque gives the desired explanation for the physical reason of existence (in the absence of gravity) of the two stable equilibrium positions that correspond to the two preferable orientations of the pendulums rod along the direction of the pivots vibration.

With gravity, the inverted pendulum is stable with respect to small deviations from the inverted vertical position provided the mean torque of the force of inertia is greater than the torque of the force of gravity that tends to tip the pendulum down.

3. An approximate quantitative analysis

On the basis of the above-described physical considerations, we can calculate the criterion of dynamic stabilization, that is, determine the quantitative conditions, which provide the stability of the inverted pendulum. Rapid vertical vibrations of the axis make the inverted position stable if at small deflections from this position the torque of the force of inertia, averaged over the period of rapid oscillations, is greater in magnitude than the torque of the gravitational force that tends to turn the pendulum down.

Due to the forced vertical vibrations of the axis, the force of inertia $F_{\rm in}(t)$ oscillates with a high frequency ω of these vibrations. The momentary arm of this force (the horizontal distance between the axis and the pendulum's bob) also varies with the same frequency ω. As we have seen in the previous section, the fast variations of this arm together with synchronous fast variations of the force of inertia are responsible for the effect of dynamic stabilization.

What we need to calculate now is the momentary torque of this oscillating force, and the non-zero mean value of this torque.

We can consider, after Kapitza [2]–[3], the motion of the pendulum whose axis is vibrating with a high frequency as a superposition of two components: a "slow" or "smooth" component, whose variation during a period of forced axis' vibrations is small, and a "fast" (or "vibrational") component. Let's imagine an observer who does not notice (or does not want to notice) the vibrational component of this compound motion. The observer, which uses, for example, a stroboscopic illumination with a short interval between the flashes that equals the period of forced vibrations of the pendulum's axis, can see only the slow component of the motion. Our principal interest is to determine this slow component.

In other words, we can represent the instantaneous value $\varphi(t)$ of the pendulum's deflection angle from the vertical (see Figure 2) as the sum of a slowly varying function $\theta(t)$ and a small fast term $\delta(t)$: $\varphi(t) = \theta(t) + \delta(t)$. This additional angle $\delta(t)$ oscillates with the high frequency ω, and its mean value is zero. At time moment t the axis is displaced from its mid-point through $z(t)$. If $\theta = 90^\circ$ (see Figure 2, *a*), the momentary value of $\delta(t)$ equals $z(t)/l$. When θ is non-zero (see Figure 2, *b*), $\delta(t) \approx (z(t)/l)\sin\theta$.

At time moment t the bob in its oscillating motion along the arc between the utmost points *1* and *2* is displaced from its mid-point through the distance $l\delta(t)$. This displacement adds to the arm of force $F_{\text{in}}(t)$ the value $l\delta(t)\cos\theta$, as can be seen from Figure 2, *b*. Because $\delta(t) = (z(t)/l)\sin\theta$, this additional arm $z(t)\sin\theta\cos\theta$ varies with time sinusoidally, in the same way as $z(t)$. It is just this additional variable arm that is responsible for the effect of dynamic stabilization, because it varies with time in the same way as does itself the force of inertia: $F_{\text{in}}(t) = m\omega^2 z(t)$. Hence the magnitude of additional torque of the force of inertia associated with this additional variable arm at any time moment t is proportional to $z^2(t)$:

$$F_{\text{in}}(t)z(t)\sin\theta\cos\theta = m\omega^2 z^2(t)\sin\theta\cos\theta. \tag{3}$$

In order to calculate the mean torque of the force of inertia, we can average expression (3) over the period $T = 2\pi/\omega$ of the fast oscillations, assuming the slow varying angle θ to be constant ("frozen") during this short period. Taking into account that at sinusoidal vibration of the pivot $\langle z^2(t)\rangle = a^2/2$, where a is the amplitude of the pivot fast vibration, we find the desired mean value $\langle T_{\text{in}}(t)\rangle$ of the torque:

$$\langle T_{\text{in}}(t)\rangle = -\frac{1}{2}ma^2\omega^2\sin\theta\cos\theta = -\frac{1}{4}ma^2\omega^2\sin 2\theta. \tag{4}$$

For $\theta < \pi/2$, that is, if the pendulum makes an acute angle with the upward vertical direction, the average torque of the force of inertia tends to turn the pendulum up to the vertical. Otherwise, this mean torque tends to turn the pendulum downward. Hence in the absence of gravity, instead of a neutral equilibrium at an arbitrary angle, the pendulum has two equivalent dynamically stabilized equilibrium positions pointing (up and down) along both directions of the forced fast oscillations of the axis.

The other (non-vibrating) part of the arm equals $l\sin\theta$, so it is nearly constant during the period T of the fast oscillations. Therefore the torque of the oscillating force of inertia $F_{\text{in}}(t) = m\omega^2 z(t)$ associated with this arm, $F_{\text{in}}(t)l\sin\theta = m\omega^2 z(t)l\sin\theta$, has zero mean value, because at sinusoidal vibrations of the pivot $\langle z(t)\rangle = 0$.

With gravity, the mean torque of the force of inertia is added to the mean torque created by the force of gravity, $\langle T_{\text{grav}}(t)\rangle = mgl\sin\theta$, which is tending to tip the pendulum down.

At small deviations from the vertical, when $\theta \ll 1$, we can replace $\sin 2\theta$ in Eq. (4) by its argument 2θ. Hence for small deviations from the upper vertical both the mean torque of the force of inertia and the mean torque of the force of gravity are proportional to the angle θ:

$$\langle T_{\text{in}}(t)\rangle \approx -\frac{1}{2}ma^2\omega^2\theta, \quad \langle T_{\text{grav}}(t)\rangle \approx mgl\theta. \tag{5}$$

Comparing these torques, we see that the mean torque of the force of inertia $\langle T_{\text{in}}(t)\rangle$ can exceed in magnitude the torque of the gravitational force (at small deviations θ from the vertical), when the following condition is fulfilled:

$$a^2\omega^2 > 2gl. \tag{6}$$

This is the desired approximate criterion of dynamic stabilization of the pendulum in the inverted position. Thus, the inverted position of the pendulum is stable if the maximal velocity $a\omega$ of the vibrating axis is greater than the velocity $\sqrt{2gl}$ attained by a body during a free fall from the height that equals the pendulum length l. We can write this approximate criterion of stability in another form, using the expression $\omega_0^2 = g/l$ for the frequency of small natural oscillations of the pendulum in the absence of forced vibrations of the axis. Substituting $g = l\omega_0^2$ in Eq. (6), we get

$$\frac{a}{l}\cdot\frac{\omega}{\omega_0} > \sqrt{2}. \tag{7}$$

According to Eq. (7), for stabilization of the inverted pendulum the product of the dimensionless normalized amplitude of forced oscillations of the axis a/l and the dimensionless (normalized) frequency of these oscillations ω/ω_0 must exceed $\sqrt{2}$. For instance, for the pendulum whose length $l = 20$ cm and the frequency of forced oscillations of the axis $f = \omega/2\pi$ = 100 Hz, the amplitude a must be greater than 3.2 mm. For a physical pendulum, the condition of dynamic stability in the inverted position is expressed by the same equation (6) or (7) provided we imply by the quantity l the equivalent length of the physical pendulum I/md, where I is the moment of inertia with respect to the axis of rotation, m is the mass, and d is the distance between the axis and the center of mass. We note that the criterion (6) or (7) is independent of friction.

The above-developed approach is not restricted to small deviations of the pendulum from the vertical. In particular, for given values of the frequency ω and amplitude a of forced oscillations of the pivot at which criterion (6) or (7) is fulfilled, we can find the maximal admissible angular deflection from the inverted vertical position $\theta_{\max}$ for which the pendulum will return to this position. To do this, we should equate the average torque of the force of inertia $\langle T_{\text{in}}(t)\rangle$ given by Eq. (4), which tends to return the pendulum to the inverted position, and the torque $mgl\sin\theta$ of the gravitational force, which tends to tip the pendulum down. This yields the following maximal deviation:

$$\cos\theta_{\max} = \frac{2gl}{a^2\omega^2} = 2\left(\frac{\omega_0}{\omega}\frac{l}{a}\right)^2. \tag{8}$$

This expression for an admissible angular excursion from the inverted equilibrium position is valid for arbitrarily large values of θ. The greater the product ωa of the frequency and the amplitude of forced vibrations of the axis, the closer the angle $\theta_{\rm max}$ to $\pi/2$.

If the angle θ equals $\pm\pi/2$, that is, if the pendulum is oriented perpendicularly to the direction of pivot's oscillations, the mean torque of the force of inertia, according to Eq. (4), is zero: in the absence of gravity the pendulum at such orientations is in equilibrium. However, these equilibria are unstable: at a slightest deviation from such orientation to one or to the other side the mean torque of the force of inertia becomes non-zero and, according to Eq. (4), tends to increase the deviation, turning the pendulum towards the nearest stable equilibrium, in which the pendulum is oriented along the direction of forced vibrations of its pivot. With gravity, deviations from the upper vertical through the angle $\pm\theta_{\rm max}$ given by Eq. (8) also correspond to unstable equilibrium positions.

4. Oscillations about the equilibrium positions

Being deviated from the vertical position through an angle that does not exceed $\theta_{\rm max}$, the pendulum will execute relatively slow oscillations about this inverted position. This slow motion occurs both under the mean torque of the force of inertia and the force of gravity. Fast oscillations with the frequency of forced vibrations of the axis superimpose on this slow motion of the pendulum. With friction, the slow motion gradually damps, and the pendulum wobbles up settling eventually in the inverted position.

The simulation program "Pendulum with the vertically driven pivot" [10] demonstrates clearly slow oscillations of the pendulum about the inverted position, distorted by high frequency vibrations of the pivot. The program allows us to change parameters of the system in wide ranges and to vary the time scale in order to make visible subtle details of such counterintuitive behavior.

Similar behavior of the pendulum with vibrating pivot can be observed when it is deflected from the lower vertical position. But in this case the frequency $\omega_{\rm down}$ of slow oscillations is greater than the frequency $\omega_{\rm up}$ for the inverted pendulum. Indeed, for the hanging down pendulum both the averaged torque of the force of inertia and the torque of the gravitational force tend to return the pendulum to the lower vertical position. Therefore the frequency $\omega_{\rm down}$ of these slow oscillations is greater than the frequency $\omega_{\rm slow}$ of slow oscillations in the absence of gravity. The frequency $\omega_{\rm down}$ is also greater than the frequency ω_0 of natural oscillations of the same pendulum under the gravitational force in the absence of forced vibrations of the axis. Regarding the latter conclusion, Kapitza noted that the clock with a pendulum subjected to a fast vertical vibration will be always ahead of time.

The approximate differential equations for the slow motion of the pendulum $\theta(t)$ can be written under the assumption that the angular acceleration $\ddot{\theta}(t)$ in this slow motion is determined both by the mean torque of the force of gravity $mg\sin\theta$ and the torque of the force of inertia $\langle T_{\rm in}(t)\rangle$ given by Eq. (4). For small oscillations $\sin\theta\approx\theta$, and we can write

$$\ddot{\theta} = (\omega_0^2 - \frac{1}{2}\frac{a^2}{l^2}\omega^2)\theta, \qquad \ddot{\theta} = (-\omega_0^2 - \frac{1}{2}\frac{a^2}{l^2}\omega^2)\theta \tag{9}$$

for oscillations about the inverted and hanging down positions, respectively. The mean torque on the right-hand side of Eqs. (9) is calculated approximately under the assumption that the slowly varying angular coordinate $\theta(t)$ is "frozen."

It follows from (9) that frequencies $\omega_{\rm down}$ and $\omega_{\rm up}$ of small slow oscillations about the lower ($\theta = 0$) and upper ($\theta = \pm\pi$) equilibrium positions are given by the following expressions:

$$\omega_{\rm down}^2 = \frac{a^2\omega^2}{2l^2} + \omega_0^2, \qquad \omega_{\rm up}^2 = \frac{a^2\omega^2}{2l^2} - \omega_0^2. \tag{10}$$

If we put $\omega_0 = 0$ into Eqs. (10), they yield for the frequency $\omega_{\rm slow}$ of small slow oscillations of the pendulum with vibrating axis in the absence of the gravitational force the following approximate expression:

$$\omega_{\rm slow} = \omega\frac{a}{\sqrt{2}\,l}. \tag{11}$$

These oscillations can occur about either of the two equivalent stable equilibrium positions located oppositely one another along the direction of forced vibrations of the axis.

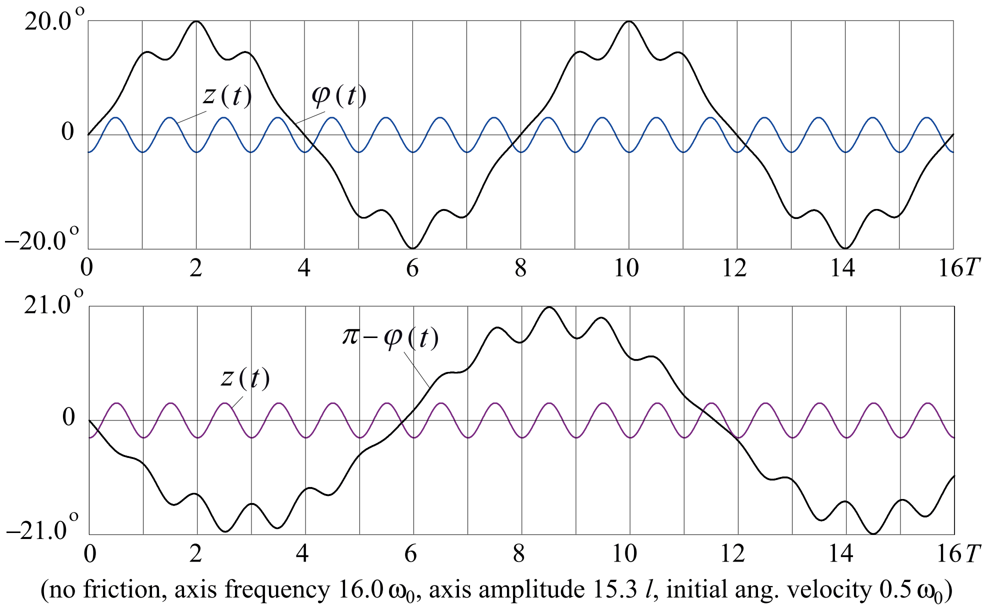

Figure 3. The graphs of $\varphi(t)$ for oscillations of the pendulum about the lower and upper equilibrium positions, respectively, and the graph $z(t) = -a\cos\omega t$ of the pivot motion. The graphs are obtained by a numerical integration of the exact differential equation (12) for the momentary angular deflection.

Expressions (10) for the frequencies $\omega_{\rm up}$ and $\omega_{\rm down}$ of slow small oscillations are illustrated by the graphs in Figure 3, obtained by a numerical integration of the exact differential equation for the momentary angular deflection $\varphi(t) = \theta(t) + \delta(t)$:

$$\ddot{\varphi} + (\frac{g}{l} - \frac{a}{l}\omega^2\cos\omega t)\sin\varphi = 0. \tag{12}$$

The graphs in Figure 12 are plotted with the help of the simulation program [10].

It is assumed in Eq. (12) that φ is measured from the lover position. We note that oscillations about the inverted position can be formally described by the same differential equation, Eq. (12), with negative values of g. In other words, we can treat the acceleration of free fall g in (12) as a control parameter whose variation is physically equivalent to variation of the force of gravity exerted on the pendulum. When parameter g is reduced to zero and further on to negative values, the time-independent torque of the force of gravity turns to zero and then reverses its sign. Such reversed force of "gravity" tends to bring the pendulum to the inverted position $\varphi = \pi$, making this position stable (in the absence of the pivot vibration), and making position $\varphi = 0$ – unstable. That is, at $g < 0$ the upper equilibrium position in (12) is equivalent to the lover position at positive values of parameter g.

To make the verification of our approximate expressions (10) for the frequencies of slow oscillations with the simulation program [10] easier, the following values of the system parameters were chosen for numerical integration: the amplitude of the pivot vibration $a = 0.153\,l$, its frequency $\omega = 16\,\omega_0$, so that $(a^2/2l^2)\omega^2 = 3.0\,\omega_0^2$. In this case Eqs. (10) give for the frequency about the lower position the value $\omega_{\rm down} = 2\omega_0$, which is twice the natural frequency. This means that the period of slow oscillations $T_{\rm down}$ must equal one half of the period T_0 of natural oscillations in the absence of pivot vibrations. Figure 3 shows that the pendulum executes, as expected, exactly two slow oscillations about the lower equilibrium position during one period T_0, which in this case (at $\omega = 16\,\omega_0$) equals 16 periods $T = 2\pi/\omega$ of pivot vibrations. (The units T are used for the time scale.)

For the frequency of slow oscillations about the inverted position, Eq. (10) gives $\omega_{\rm up} = \sqrt{2}\omega_0$, so that their period should equal $T_{\rm up} = T_0/\sqrt{2}$. This value of the period is also in good agreement with the lower graph in Figure 3.

The graphs in Figure 3 show that the slow motion is distorted by fast oscillations most of all near the utmost deflections of the pendulum, while the distortions of $\varphi(t)$ graphs are rather insignificant when the pendulum crosses the equilibrium positions. This peculiarity is also consistent with the above developed approach. Indeed, the angular amplitude of the fast oscillations $\delta(t)$ is proportional to the sine of the mean deflection angle θ that describes the slow component of pendulum's oscillations: $\delta(t) = (z(t)/l)\sin\theta$.

5. Concluding discussion

A simple physical explanation is suggested in this paper for the phenomenon of dynamic stabilization of the inverted pendulum whose pivot is forced to oscillate with a high frequency. The approximate criterion (6) or (7) obtained here on the basis of the suggested approach agrees with the well-known lower boundary of stability of the inverted pendulum obtained by approximating the exact nonlinear equation of motion, Eq. (12), with the linear Mathieu equation, the solutions of which are widely documented in the extensive literature concerning the problem (see, e. g., [11]–[12]). However, the investigation based on the Mathieu equation and infinite Hill's determinants gives little physical insight into the problem and, more importantly, is restricted to motion within small deviations from the vertical. On the contrary, the above explanation in this paper shows clearly the physical reason for the dynamic stabilization of the inverted pendulum and is free from the restriction of small angles.

Criterion of stabilization (6) or (7) is obtained by a decomposition of pendulum's motion on slow oscillations and fast vibrations with the driving frequency. Hence these results, being physically clear and transparent, are approximate and valid when the amplitude of the forced vibration of the axis is small compared to the pendulum's length ($a \ll l$), and when their frequency is high enough ($\omega \gg \omega_0$). An enhanced and more exact analytical criterion of dynamic stabilization of the inverted pendulum, valid in a wider region of the system parameters, is obtained in Ref. [9].

For some intervals of the pivot frequency the lower equilibrium position becomes unstable due to the phenomenon of parametric resonance at which small initial oscillations increase progressively. This conclusion does not follow from the investigation based on the decomposition of motion on slow and rapid components. This is by no means surprising because parametric resonance occurs at such driving frequencies (for the principal parametric resonance $\omega \approx 2\omega_0$) for which this decomposition is not applicable.

The inverted (dynamically stabilized) position also becomes unstable at large enough amplitudes of the pivot oscillations: the pendulum is involved in so-called "flutter" oscillations about the inverted position. With friction, such oscillations eventually become stationary (limit cycle). Their period covers two cycles of excitation. The "flutter" mode of oscillations is closely related to ordinary parametric resonance of the hanging down pendulum. This relationship is shown in Ref. [9], in which also the upper boundary of stability of the inverted pendulum is obtained. We emphasize that parametric resonance, flutter mode and other complicated regular and chaotic regimes occur at such frequencies and amplitudes of the pivot, for which the decomposition of motion on the slow and fast components is not applicable. Various complicated modes of behavior of the parametrically forced pendulum are described in Refs. [7]–[9]. These modes are illustrated by the simulation program [10], which contains a wealth of predefined examples of extraordinary, counterintuitive motions of the pendulum. To observe these motions, there is no need in defining the required parameters: desired examples can be launched by simply choosing them from the list.

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