5

Key understandings in
mathematics learning

Paper 5: Understanding space and its representation in mathematics

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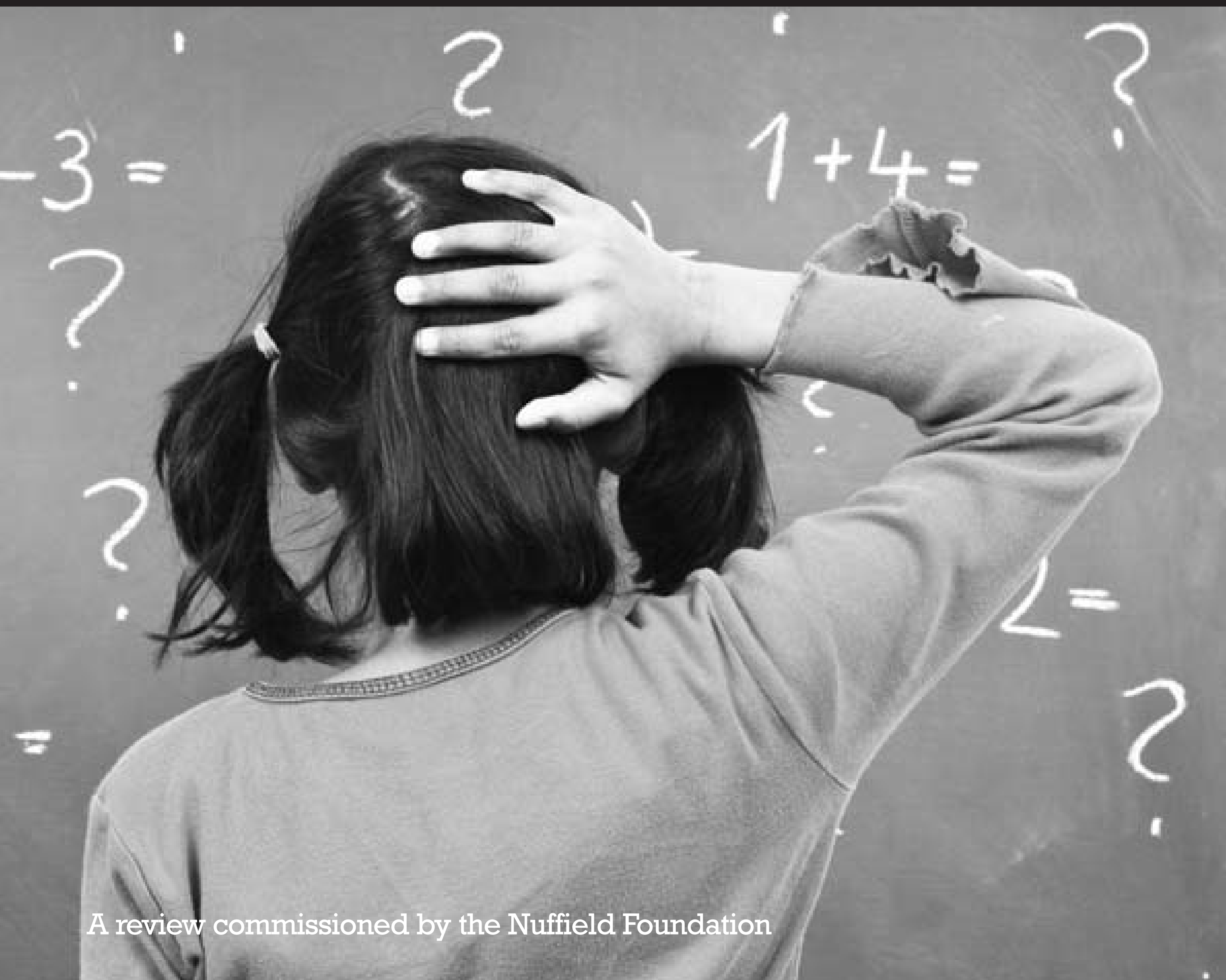

A review commissioned by the Nuffield Foundation

About this review

In 2007, the Nuffield Foundation commissioned a team from the University of Oxford to review the available research literature on how children learn mathematics. The resulting review is presented in a series of eight papers:

Paper 1: Overview
Paper 2: Understanding extensive quantities and whole numbers
Paper 3: Understanding rational numbers and intensive quantities
Paper 4: Understanding relations and their graphical representation
Paper 5: Understanding space and its representation in mathematics
Paper 6: Algebraic reasoning
Paper 7: Modelling, problem-solving and integrating concepts
Paper 8: Methodological appendix

Papers 2 to 5 focus mainly on mathematics relevant to primary schools (pupils to age 11 years), while papers 6 and 7 consider aspects of mathematics in secondary schools.

Paper 1 includes a summary of the review, which has been published separately as *Introduction and summary of findings*.

Summaries of papers 1-7 have been published together as *Summary papers*.

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About the author

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About the Nuffield Foundation

The Nuffield Foundation is an endowed charitable trust established in 1943 by William Morris (Lord Nuffield), the founder of Morris Motors, with the aim of advancing social well being. We fund research and practical experiment and the development of capacity to undertake them; working across education, science, social science and social policy. While most of the Foundation's expenditure is on responsive grant programmes we also undertake our own initiatives.

Summary of paper 5: Understanding space and its representation in mathematics

Headlines

- Children come to school with a great deal of knowledge about spatial relations. One of the most important challenges in mathematical education is how best to harness this implicit knowledge in lessons about space.

- Children's pre-school implicit knowledge of space is mainly relational. Teachers should be aware of kinds of relations that young children recognise and are familiar with, such as their use of stable background to remember the position and orientation of objects and lines.

- Measuring of length and area poses particular problems for children, even though they are able to understand the underlying logic of measurement. Their difficulties concern iteration of standard units, which is a new idea for them, and also the need to apply multiplicative reasoning to the measurement of area.

- From an early age children are able to extrapolate imaginary straight lines, which allows them to learn how to use Cartesian co-ordinates to plot specific positions in space with no difficulty. However, they need instruction about how to use co-ordinates to work out the relation between different positions.

- Learning how to represent angle mathematically is a hard task for young children, even though angles are an important part of their everyday life. There is evidence that children are more aware of angle in the context of movement (turns) than in other contexts and learn about the mathematics of angle relatively easily in this context. However, children need a great deal of help from to teachers to understand how to relate angles across different contexts.

- An important aspect of learning about geometry is to recognise the relation between transformed shapes (rotation, reflection, enlargement). This also can be difficult, since children's pre-school experiences lead them to recognise the same shapes as equivalent across such transformations, rather than to be aware of the nature of the transformation. However, there is very little research on this important question.

- Another aspect of the understanding of shape is the fact that one shape can be transformed into another, by addition and subtraction of its subcomponents. For example, a parallelogram can be transformed into a rectangle of the same base and height by the addition and subtraction of equivalent triangles and adding two equivalent triangles to a rectangle creates a parallelogram. Research demonstrates that there is a danger that children might learn about these transformations only as procedures without understanding their conceptual basis.

- There is a severe dearth of psychological research on children's geometrical learning. In particular we need long-term studies of the effects of intervention and a great deal more research on children's understanding of transformations of shape.

At school, children often learn formally about matters that they already know a great deal about in an informal and often quite implicit way. Sometimes their existing, informal understanding, which for the most part is based on experiences that they start to have long before going to school, fits well with what they are expected to learn in the classroom. At other times, what they know already, or what they think they know, clashes with the formal systems that they are taught at school and can even prevent them from grasping the significance of these formal systems.

Geometry is a good and an obvious example. Geometry lessons at school deal with the use of mathematics and logic to analyse spatial relations and the properties of shapes. The spatial relations and the shapes in question are certainly a common part of any child's environment, and psychological research has established that from a very early age children are aware of them and quite familiar with them. It has been shown that even very young babies not only discriminate regular geometric shapes but can recognise them when they see them at a tilt, thus co-ordinating information about the orientation of an object with information about the pattern of its contours.

Babies are also able to extrapolate imaginary straight lines (a key geometric skill) at any rate in social situations because they can work out what someone else is looking at and can thus construct that person's line of sight. Another major early achievement by young children is to master the logic that underlies much of the formal analysis of spatial relations that goes on in geometry. By the time they first go to school young children can make logical transitive inferences ($A > B$, $B > C$, therefore $A > C$; $A = B$, $B = C$, therefore $A = C$), which are the logical basis of all measurement. In their first few years at school they also become adept at the logic of inversion ($A + B – B$), which is a logical move that is an essential part of studying the relation between shapes.

Finally, there is strong evidence that most of the information about space that children use and remember in their everyday lives is relational in nature. One good index of this is that children's memory of the orientation of lines is largely based on the relation between these lines and the orientation of stable features in the background. For this reason children find it much easier to remember the orientation of horizontal and vertical lines than of diagonal lines, because horizontal and vertical features are quite common in the child's stable spatial environment. For the same reason, young children remember and reproduce right angles (perpendicular lines) better than acute or obtuse angles. The relational nature of children's spatial perception and memory is potentially a powerful resource for learning about geometry, since spatial relations are the basic subject matter of geometry.

With so much relevant informal knowledge about space and shape to draw on, one might think that children would have little difficulty in translating this knowledge into formal geometrical understanding. Yet, it is not always that easy. It is an unfortunate and well-documented fact that many children have persistent difficulties with many aspects of geometry.

One evidently successful link between young children's early spatial knowledge and their more formal experiences in the classroom is their learning how to use Cartesian co-ordinates to plot positions in two-dimensional space. This causes schoolchildren little difficulty, although it takes some time for them to understand how to work out the relation between two positions plotted in this way.

Other links between informal and formal knowledge are harder for young children. The apparently simple act of measuring a straight line, for example, causes them problems even though they are usually perfectly able to make the appropriate logical moves and understand the importance of one-to-one correspondence, which is an essential part of relating the units on a ruler to the line being measured. One problem here is that they find it hard to understand the idea of iteration: iteration is about repeated measurements, so that a ruler consists of a set of iterated (repeated) units like centimetres. Iteration is necessary when a particular length being measured is longer than the measuring instrument. Another problem is that the one-to-one correspondence involved in measuring a line with a ruler is asymmetrical. The units (centimetres, inches) are visible and clear in the ruler but have to be imagined on the line itself. It is less of a surprise that it also takes children a great deal of time to come to terms with the fact that measurement of area usually needs some form of multiplication, e.g. height × width with rectangles, rather than addition.

The formal concept of angle is another serious stumbling block for children even though they are familiar enough with angles in their everyday spatial environments. The main problem is that they find it hard to grasp that two angles in very different contexts are the same, e.g. themselves turning 90^{o} and the corner of a page in a book. Abstraction is an essential part of geometry but it has very little to do with children's ordinary spatial perception and knowledge.

For much the same reason, decomposing a relatively complex shape into several simpler component shapes – again an essential activity in geometry – is something that many children find hard to do. In their ordinary lives it is usually more important for them to see shapes as unities, rather than to be able to break them up into other shapes. This difficulty makes it hard for them to work out relationships between shapes.

For example, children who easily grasp that $a + b - b = a$, nevertheless often fail to understand completely the demonstration that a rectangle and a parallelogram with the same base and height are equal in area because you can transform the parallelogram into the rectangle by subtracting a triangle from one end of the parallelogram and adding an exactly equivalent triangle to the other end.

We know little about children's understanding of transformations of shape or of any difficulties that they might have when they are taught about these transformations. This is a serious gap in research on children's mathematical learning. It is well recognised, however, that children and some adults confuse scale enlargements with enlargements of area. They think that doubling the length of the contour of a geometric shape such as a square or a rectangle also doubles its area, which is a serious misconception. Teachers should be aware of this potential difficulty when they teach children about scale enlargements.

Researchers have been more successful in identifying these obstacles than in showing us how to help children to surmount them. There are some studies of ways of preparing children for geometry in the pre-school period or in the early years at school. This research, however, concentrated on short-term gains in children's geometric understanding and did not answer the question whether these early teaching programmes would actually help children when they begin to learn about geometry in the classroom.

There has also been research on teaching children about angle, mostly in the context of computer-based teaching programmes. One of the most interesting points to come out of this research is that teaching children about angle in terms of movements (turning) is successful, and there is some evidence that children taught this way are quite likely to transfer their new knowledge about angle to other contexts that do not involve movement.

However, there has been no concerted research on how teachers could take advantage of children's considerable spatial knowledge when teaching them geometry. We badly need long-term studies of interventions that take account of children's relational approach to the spatial environment and encourage them to grasp other relations, such as the relation between shapes and the relation between shapes and their subcomponent parts, which go beyond their informal spatial knowledge.

Recommendations

Research about mathematical learning	Recommendations for teaching and research
Children's pre-school knowledge of space is relational. They are skilled at using stable features of the spatial framework to perceive and remember the relative orientation and position of objects in the environment. There is, however, no research on the relation between this informal knowledge and how well children learn about geometry.	**Teaching** Teachers should be aware of the research on children's considerable spatial knowledge and skills and should relate their teaching of geometrical concepts to this knowledge. **Research** There is a serious need for longitudinal research on the possible connections between children's pre-school spatial abilities and how well they learn about geometry at school.
Children already understand the logic of measurement in their early school years. They can make and understand transitive inferences, they understand the inverse relation between addition and subtraction, and they can recognise and use one-to-one correspondence. These are three essential aspects of measurement.	**Teaching** The conceptual basis of measurement and not just the procedures should be an important part of the teaching. Teachers should emphasise transitive inferences, inversion of addition and subtraction and also one-to-one correspondence and should show children their importance. **Research** Psychologists should extend their research on transitive inference, inversion and one-to-one correspondence to geometrical problems, such as measurement of length and area.

Recommendations (continued)

Research about mathematical learning	Recommendations for teaching and research
Many children have difficulties with the idea of iteration of standard units in measurement.	**Teaching** Teachers should recognise this difficulty and construct exercises which involve iteration, not just with standard units but with familiar objects like cups and hands. **Research** Psychologists should study the exact cause of children's difficulties with iteration.
Many children wrongly apply additive reasoning, instead of multiplicative reasoning, to the task of measuring area. Children understand this multiplicative reasoning better when they first think of it as the number of tiles in a row times the number of rows than when they try to use a base times height formula.	**Teaching** In lessons on area measurement, teachers can promote children's use of the reasoning 'number in a row times number of rows' by giving children a number of tiles that is insufficient to cover the area. They should also contrast measurements which do, and measurements which do not, rest on multiplication.
Even very young children can easily extrapolate straight lines and schoolchildren have no difficulty in learning how to plot positions using Cartesian co-ordinates, but it is difficult for them to work out the relation between different positions plotted in this way.	**Teaching** Teachers, using concrete material, should relate teaching about spatial co-ordinates to children's everyday experiences of extrapolating imaginary straight lines. **Research** There is a need for intervention studies on methods of teaching children to work out the relation between different positions, using co-ordinates.
Research on pre-school intervention suggests that it is possible to prepare children for learning about geometry by enhancing their understanding of space and shapes. However, this research has not included long-term testing and therefore the suggestion is still tentative.	**Research** There will have to be long-term predictive and long-term intervention studies on this crucial, but neglected, question
Children often learn about the relation between shapes (e.g. between a parallelogram and a rectangle) as a procedure without understanding the conceptual basis for these transformations.	**Teaching** Children should be taught the conceptual reasons for adding and subtracting shape components when studying the relation between shapes. **Research** Existing research on this topic was done a very long time ago and was not very systematic. We need well-designed longitudinal and intervention studies on children's ability to make and understand such transformations.
There is hardly any research on children's understanding of the transformation of shapes, but there is evidence of confusion in many children about the effects of enlargement: they consider that doubling the length of the perimeter of a square, for example, doubles its area.	**Teaching** Teachers should be aware of the risk that children might confuse scale enlargements with area enlargements. **Research** Psychologists could easily study how children understand transformations like reflection and rotation but they have not done so. We need this kind of research.

Understanding space and its representation in mathematics

From informal understanding to formal misunderstanding of space

This paper is about children's informal knowledge of space and spatial relations and about their formal learning of geometry. It also deals with the connection between these two kinds of knowledge. This connection is much the same as the one between knowledge about quantitative relations on the one hand and about number on the other hand, which we described in Papers 1 and 2. We shall show how young children build up a large and impressive, but often implicit, understanding of spatial relations before they go to school and how this knowledge sometimes matches the relations that they learn in geometry very well and sometimes does not.

There is a rich vein of research on children's spatial knowledge – knowledge which they acquire informally and, for the most part, long before they go to school – and this research is obviously relevant to the successes and the difficulties that they have when they are taught about geometry at school. Yet, with a few honourable exceptions, the most remarkable of which is a recent thorough review by Clements and Sarama (2007 *b*), there have been very few attempts indeed to link research on children's informal, and often implicit, knowledge about spatial relations to their ability to carry out the explicit analyses of space that are required in geometry classes.

The reason for this gap is probably the striking imbalance in the contribution made by psychologists and by maths educators to research on geometrical learning. Although psychologists have studied children's informal understanding of space in detail and with great success, they have virtually ignored children's learning about geometry, at any rate in recent years. Despite Wertheimer's (1945) and Piaget, Inhelder and Szeminska's (1960) impressive pioneering work on children's understanding of geometry, which we shall describe later, psychologists have virtually ignored this aspect of children's education since then. In contrast, mathematics educators have made steady progress in studying children's geometry with measures of what children find difficult and studies of the effects of different kinds of teaching and classroom experience.

One effect of this imbalance in the contribution of the two disciplines to research on learning about geometry is that the existing research tells us more about educational methods than about the underlying difficulties that children have in learning about geometry. Another result is that some excellent ideas about enhancing children's geometrical understanding have been proposed by educationalists but are still waiting for the kind of empirical test that psychologists are good at designing and carrying out.

The central problem for anyone trying to make the link between children's informal spatial knowledge and their understanding of geometry is easy to state. It is the stark contrast between children's impressive everyday understanding of their spatial environment and the difficulties that they have in learning how to analyse space mathematically. We shall start our review with an account of the basic spatial knowledge that children acquire informally long before they go to school.

Early spatial knowledge: perception

Shape, size, position and extrapolation of imagined straight lines

Spatial achievements begin early. Over the last 30 years, experimental work with young babies has clearly shown that they are born with, or acquire very early on in their life, many robust and effective perceptual abilities. They can discriminate objects by their shape, by their size and by their orientation and they can perceive depth and distinguish differences in distance (Slater, 1999; Slater and Lewis 2002; Slater, Field and Hernadez-Reif, 2002; Bremner, Bryant and Mareschal, 2006).

They can even co-ordinate information about size and distance (Slater, Mattock and Brown, 1990), and they can also co-ordinate information about an object's shape and its orientation (Slater and Morrison, 1985). The first co-ordination makes it possible for them to recognise a particular object, which they first see close up, as the same object when they see it again in the distance, even though the size of the visual impression that it now makes is much smaller than it was before. With the help of the second kind of co-ordination, babies can recognise particular shapes even when they see them from completely different angles: the shape of the impression that these objects make on the visual receptors varies, but babies can still recognise them as the same by taking the change in orientation into account.

We do not yet know how children so young are capable of these impressive feats, but it is quite likely that the answer lies in the relational nature of the way that they deal with size (and, as we shall see later, with orientation), as Rock (1970) suggested many years ago. A person nearby makes a larger visual impression on your visual system than a person in the distance but, if these two people are roughly the same size as each other, the relation between their size and that of familiar objects near each of them, such as cars and bus-stops and wheelie-bins, will be much the same.

The idea that children judge an object's size in terms of its relation to the size of other objects at the same distance receives some support from work on children's learning about relations. When four-year-old children are asked to discriminate and remember a particular object on the basis of its size, they do far better when it is possible to solve the problem on the basis of size relations (e.g. it is always the smaller one) than when they have to remember its absolute size (e.g. it is always exactly so large) (Lawrenson and Bryant, 1972).

Another remarkable early spatial achievement by infants, which is also relational and is highly relevant to much of what they later have to learn in geometry lessons, is their ability to extrapolate imaginary straight lines in three dimensional space (Butterworth, 2002). Extrapolation of imagined straight lines is, of course, essential for the use of Cartesian co-ordinates to plot positions in graphs and in maps, but it also is a basic ingredient of very young children's social communication (Butterworth and Cochrane, 1980; Butterworth and Grover, 1988). Butterworth and Jarrett (1991) showed this in a study in which they asked a mother to sit opposite her baby and then to stare at some predetermined object which was either in front and in full view of the child or was behind the child, so that he had to turn his head in order to see it. The question was whether the baby would then look at the same object, and to do this he would have to extrapolate a straight line that represented his mother's line of sight. Butterworth and Jarrett found that babies younger than 12 months manage to do this most of the time when the object in question was in front of them. They usually did not also turn their heads to look at objects behind them when these apparently caught their mothers' attention. But 15-month-old children did even that: they followed their mother's line of sight whether it led them to objects already in full view or to ones behind them. A slightly later development that also involves extrapolating imaginary straight lines is the ability to point and to look in the direction of an object that someone else is pointing at, which infants manage do with great proficiency (Butterworth and Morisette, 1996; Butterworth and Itakura, 2000).

Orientation and position

The orientation of objects and surfaces are a significant and highly regular and predictable part of our everyday spatial environments. Walls usually are, and usually have to be, vertical: objects stay on horizontal surfaces but tend to slide off sloping surfaces. The surface of still liquid is horizontal: the opposite edges of many familiar manufactured objects (doors, windows, television sets, pictures, book pages) are parallel: we ourselves are vertical when we walk, horizontal when we swim. Yet, children seem to have more difficulty distinguishing and remembering information about orientation than information about other familiar spatial variables.

Horizontals and verticals are not the problem. Five-year-old children take in and remember the orientation of horizontal and vertical lines extremely well (Bryant 1969, 1974; Bryant and Squire, 2001). In contrast, they have a lot of difficulty in remembering either the direction or slope of obliquely oriented lines. There is, however, an effective way of helping them over this difficulty with oblique lines. If there are other obliquely oriented lines in the background that are parallel to an oblique line that they are asked to remember, their memory of the slope and direction for this oblique line improves dramatically (see Figure 5.1). The children use the parallel relation between the line that they have to remember and stable features in the background framework to store and recognise information about the oblique line.

This result suggests a reason for the initial radical difference in how good their memory is for vertical and horizontal features and how poor it is for obliquely oriented ones. The reason, again, is about relations. It is that that there are usually ample stable horizontal and vertical features in the background to relate these lines to. Stable, background features that parallel particular lines which are not either vertical or horizontal are much less common. If this idea is right, young children are already relying on spatial relations that are at the heart of Euclidean geometry to store information about the spatial environment by the time that they begin to be taught formally about geometry.

However, children do not always adopt this excellent strategy of relating the orientation of lines to permanent features of the spatial environment. Piaget and Inhelder's (1963) deservedly famous and often-repeated experiment about children drawing the level of water in a tilted container is the best example. They showed the children tilted glass containers (glasses, bottles) with liquid in them (though the containers were tilted, the laws of nature dictated that the level of the liquid in them was horizontal). They also gave the children a picture of an empty, tilted container depicted as just above a table top which was an obvious horizontal background feature. The children's task was to draw in the level of the liquid in the drawing so that it was

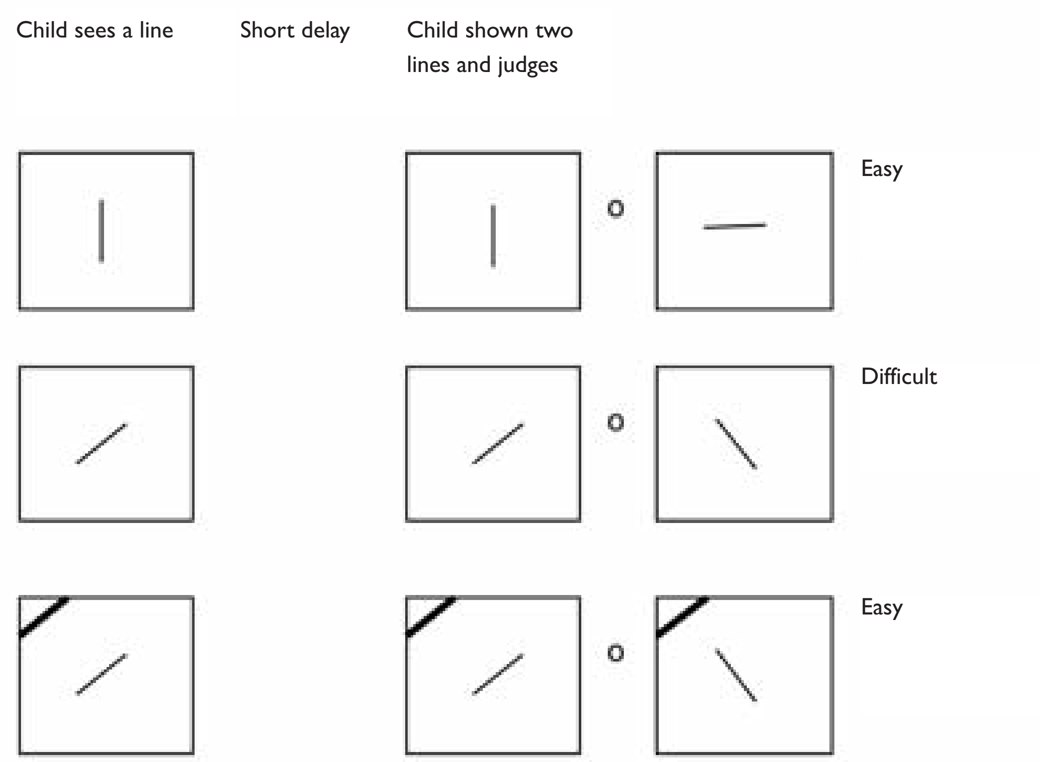

Figure 5.1: Children easily remember horizontal or vertical lines but not oblique lines unless they can relate oblique lies to a stable background feature.

exactly like the liquid in the experimenter's hand. The question that Piaget and Inhelder asked was whether they would draw the liquid as parallel to the table top or, in other words, as horizontal.

Children below the age of roughly eight years did not manage to do this. Many of them drew the liquid as perpendicular to the sides (when the sides were straight) and parallel to the bottom of the container. It seems that the children could not take advantage of the parallel relation between the liquid and the table top, probably because they were preoccupied with the glass itself and did not manage to shift their attention to an external feature.

Piaget and Inhelder treated the young child's difficulties in this drawing task as a failure on the child's part to notice and take advantage of a basic Euclidean relation, the parallel relation between two horizontal lines. They argued that a child who makes this mistake does not have any idea about horizontality: he or she is unaware that horizontal lines and surfaces are an important part of the environment and that some surfaces, such as still liquid, are constantly horizontal.

Piaget and Inhelder then extended their argument to verticality. They asked children to copy pictures of objects that are usually vertical, such as trees and chimneys. In the pictures that the children had to copy, these objects were positioned on obliquely oriented surfaces: the trees stood vertically on the side of a steeply sloping hill and vertical chimneys were placed on sloping roofs. In their copies of these pictures, children younger than about eight years usually drew the trees and chimneys as perpendicular to their baselines (the side of the hill or the sloping roof) and therefore with an oblique orientation. Piaget and Inhelder concluded that children of this age have not yet realised that the space around them is full of stable vertical and horizontal features.

There is something of a conflict between the two sets of results that we have just presented. One (Bryant 1969, 1974; Bryant and Squire, 2001) suggests that young children detect, and indeed rely on, parallel relations between objects in their immediate perception and stable background features. The other (Piaget and Inhelder, 1963) leads to the conclusion that children completely disregard these relations. However, this is not a serious problem. In the first set of experiments the use that children made of parallel relations was probably implicit. The second set of experiments involved drawing tasks, in which the children had to make explicit judgements about such

When children see 2-line figure A and are asked to copy in the missing line on B either by placing or drawing a straight wire, they represent the line as nearer to the perpendicular than it is

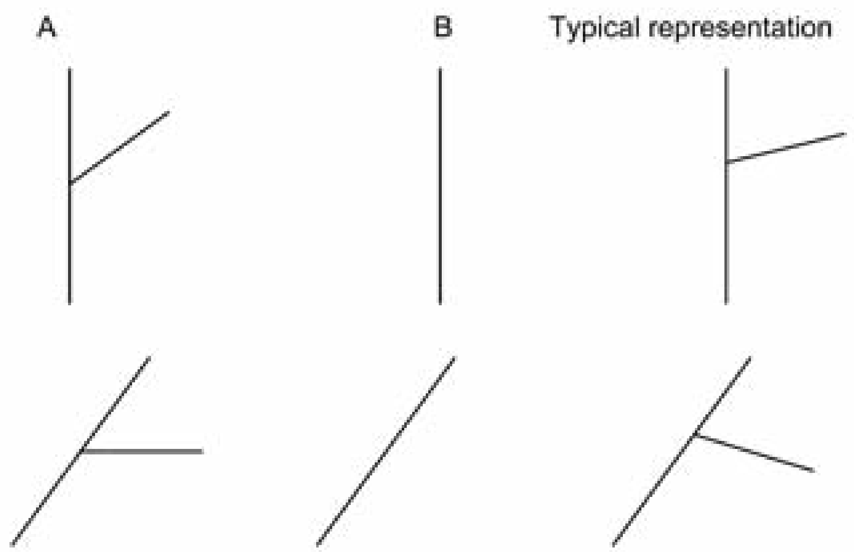

Figure 5.2: The perpendicular bias

relations. Children probably perceive and make use of parallel relations without being aware of doing so. The implication for teaching children is an interesting one. It is that one important task for the teacher of geometry is to transform their implicit knowledge into explicit knowledge.

There is another point to be made about the children's mistakes in Piaget and Inhelder's studies. One possible reason, or partial reason, for these mistakes might have been that in every case (the liquid in a tilted container, trees on the hillside, chimneys on the sloping roofs) the task was to draw the crucial feature as non-perpendicular in relation to its baseline. There is plenty of evidence (Ibbotson and Bryant, 1976) that, in copying, children find it quite difficult to draw one straight line that meets another straight line, the baseline, when the line that they have to draw is obliquely oriented to that baseline (see Figure 5.2).

They tend to misrepresent the line that they are drawing either as perpendicular to the baseline or as closer to the perpendicular than it should be. There are various possible reasons for this 'perpendicular error', but at the very least it shows that children have some difficulty in representing non-perpendicular lines. The work by Piaget *et al.* establishes that the presence of stable, background features of the spatial environment, like the table top, does not help children surmount this bias.

Early spatial knowledge: logic and measurement

Inferences about space and measurement

The early spatial achievements that we have described so far are, broadly speaking, perceptual ones. Our next task is to consider how young children reason about space. We must consider whether young children are able to make logical inferences about space and can understand other people's inferential reasoning about space by the age when they first go to school.

We can start with spatial measurements. These depend on logical inferences about space. Measurement allows us to make comparisons between quantities that we cannot compare directly. We can work out whether a washing line is long enough to stretch between posts by measuring the line that we have and the distance between the posts. We compare the two lengths, the length of the line and the distance from one post to the other indirectly, by comparing both directly to the same measuring instrument – a tape measure or ruler. We combine two direct comparisons to make an indirect comparison.

When we put two pieces of information together in this way in order to produce a new conclusion, we are making a logical inference. Inferences about continua, like length, are called *transitive inferences*. We, adults, know that if A = B in length and B = C, then A is necessarily the same length as C, even though we have never seen A and C together and therefore have not been able to compare them directly. We also know, of course, that if A > B and B > C (in length), then A > C, again without making a direct comparison between A and C. In these inferences B is the independent measure through which A and C can be compared.

Piaget, Inhelder and Szeminska (1960) were the first to discuss this link between understanding logic and being able to measure in their well-known book on geometry. They argued that the main cause of the difficulties that children have in learning about measurement is that they do not understand transitive inferences. These authors' claim about the importance of transitive inferences in learning about measurement is indisputable and an extremely important one. However, their idea that young children cannot make or understand transitive inferences has always been a controversial one, and it is now clear that we must make a fundamental distinction between being able to make the inference and knowing when this inference is needed and how to put it into effect.

There are usually two consecutive parts to a transitive inference task. In the first, the child is given two premises (A = B, B = C) and in the second he or she has to try to draw an inference from these premises. For example, in Piaget's first study of transitive inferences, which was not about length but about the behaviour of some fictional people, he first told the children that 'Mary is naughtier than Sarah, and Sarah is naughtier than Jane' and then asked then 'Who was the naughtier, Mary or Jane?' Most children below the age of roughly nine years found, and still do find, this an extremely difficult question and often say that they cannot tell. The failure is a dramatic one, but there are at least two possible reasons for it.

One, favoured by Piaget himself, that the failure is a logical one – that children of this age simply cannot put two premises about quantity together logically. It is worth noting that Piaget thought that the reason that young children did not make this logical move was that they could not conceive that the middle term (B when the premises are A > B and B > C) could simultaneously have one relation to A and another, different, relation to C.

The second possible reason for children not making the transitive inference is about memory. The children may be unable to make the inference simply because they have forgotten, or because they did not bother to commit to memory in the first place, one or both of the premises. The implication here is that they would be able to make the inference if they could remember both premises at the time that they were given the inferential question.

One way to test the second hypothesis is to make sure that the children in the study do remember the premises, and also to take the precaution of measuring how well they remember these premises at the same time as testing their ability to draw a transitive inference. Bryant and Trabasso (1971) did this by repeating the information about the premises in the first part of the task until the children had learned it thoroughly, and then in the second part checking how well they remembered this information and testing how well they could answer the inferential questions at the same time. In this study even the four-year-olds were able remember the premises and they managed to put them together successfully to make the correct transitive inference on 80% of the trials. The equivalent figure for the five-year-olds was 89%.

Young children's success in this inferential task suggests that they have the ability to make the inference that underlies measurement, but we still have to find out how well they apply this ability to measurement itself. Here, the research of Piaget *et al.* (1960) on measurement provides some interesting suggestions. These researchers showed children a tower made of bricks of different sizes. The tower was placed on a small table and each child was
asked to build another tower of the same height on another lower table that was usually, though not always, on the other side of the other side of a partition, so that the child had to create the replica without being able compare it directly to the original tower. Piaget *et al.* also provided the child with various possible measurement instruments, such as strips of paper and a straight stick, to help her with the task, and the main question that they asked was whether the child would use any of these as measures to compare the two towers.

Children under the age of (roughly) eight years did not take advantage of the measuring instruments. Either they tried to do the task by remembering the original while creating the replica, which did not work at all well, or they used their hands or their body as a measuring instrument. For example, some children put one hand at the bottom and the other at the top of the original tower and then walked to the other tower trying at the same time to keep their hands at a constant distance from each other. This strategy, which Piaget *et al.* called 'manual transfer', tended not to be successful either, for the practical reason that the children also had to use and move their hands to add and subtract bricks to their own tower. Older children, in contrast, were happy to use the strips of paper or the dowel rod as a makeshift ruler to compare the two towers. Piaget *et al.* claimed that the children who did not use the measuring instruments failed the task because they were unable to reason about it logically. They also argued that children's initial use of their own body was a transitional step on the way to true measurement using an 'independent middle term'.

This might be too pessimistic a conclusion. There is an alternative explanation for the reactions of the children who did not attempt to use a measure at all. It is that children not only have to be able to make an inference to do well in any measuring task: they also have to realise that a direct comparison will not do, and thus that instead they should make an indirect, inferential, comparison with the help of a reliable intervening measure.

There is some evidence to support this idea. If it is right, children should be ready to measure in a task in which it is made completely obvious that direct comparisons would not work. Bryant and Kopytynska (1976) devised a task of this sort. First, they gave a group of five- and six-year-old children a version of Piaget *et al.*'s two towers task, and all of them failed. Then, in a new task, they gave the children two blocks of wood, each with a hole sunk in the middle in such a way that it was impossible to see how deep either hole was. They asked the children to find out whether the two holes were as deep as each other or not. The children were also given a rod with coloured markings. The question was whether the children, who did not measure in Piaget *et al.*'s task, would start to use a measure in this new task in which it was clear that a direct comparison would be useless.

Nearly all the children used the rod to measure both holes in the blocks of wood at least once (they were each given four problems) and over half the children measured and produced the right answer in all four problems. It seems that children of five years or older are ready to use an intervening measure to make an indirect comparison of two quantities. Their difficulty is in knowing when to distrust direct comparisons enough to resort to measurement.

Iteration and measurement

One interesting variation in the study of measurement by Piaget *et al.* (1960) was in the length of the straight dowel rod, which was the main measuring tool in this task. The rod's length equalled the height of the original tower (R = T) in some problems but in others the rod was longer (R > T) and in others still it was shorter (R < T) than the tower.

The older children who used the rod as a measure were most successful when R = T. They were slightly less successful when R > T and they had to mark a point on the rod which coincided with the summit of the tower. In contrast, the R < T problems were particularly difficult, even for the children who tried to use the rod as a measure. The solution to such problems is iteration which, in this case, is to apply the rod more than once to the tower: the child has to mark a point to represent the length of the ruler and then to start measuring again from this point.

It is worth noting that iteration also involves a great deal of care in its execution. You must cover all the surface that you are measuring, all its length in these examples, but you must never overlap – never measure any part of the surface twice.

Iteration in measurement is interesting because the people who do it successfully are actually constructing their own measure and therefore certainly have a strong and effective understanding of measurement. Piaget *et al.* (1960) also argued that children's eventual realisation that iteration is the solution to some measuring problems is the basis for their eventual understanding of the role of standardised units such as centimetres and metres. We use these units, they argued, in an iterative way: 1 metre is made up of 100 iterations of 1 centimetre, and one kilometre consists of 1000 iterations of 1 metre. Children's first insight into this iterative system, according to Piaget *et al.*, comes from their initial experiences with R < T problems. This is an interesting causal hypothesis that has some serious educational implications. It should be tested.

Conclusions about children's early spatial knowledge

- Children have a well-developed and effective relational knowledge of shape, position, distance, spatial orientation and direction long before they go to school. This knowledge may be implicit and non-numerical for the most part, but it is certainly knowledge that is related to geometry.

- The mistakes that children make in drawing horizontal and vertical lines are probably due to preferring to concentrate on relations between lines close to each other (liquid in a glass is perpendicular to the sides of the glass) rather than to separated lines (liquid in a glass is parallel to horizontal surfaces like table tops). This is a mistake not in relational perception, but in picking the right relation.

- Children are also able to understand and to make transitive inferences, which are the basic logical move that underlies measurement, several years before being taught about geometry.

- We do not yet know how well they can cope with the notion of iteration in the school years.

- There is no research on the possible causal links between these impressive early perceptual and logical abilities and the successes and difficulties that children have when they first learn about geometry. This is a serious gap in our knowledge about geometrical learning.

The connections between children's knowledge of space before being taught geometry and how well they learn when they are taught about geometry

To what extent does children's early spatial development predict their success in geometry later on? The question is simple, clear and overwhelmingly important. If we were dealing with some other school subject – say learning to read – we would have no difficulty in finding an answer, perhaps more than one answer, about the importance of early, informal learning and experience, because of the very large amount of work done on the subject. With geometry, however, it is different. Having established that young children do have a rich and in many ways sophisticated understanding of their spatial environment, psychologists seem to have made their excuses and left the room. Literally hundreds of longitudinal and intervention studies exist on what children already know about language and how they learn to read and spell. Yet, as far as we know, no one has made a systematic attempt, in longitudinal or intervention research, to link what children know about space to how they learn the mathematics of spatial relations, even though there are some extremely interesting and highly specific questions to research.

To take one example, what connections are there between children's knowledge of measurement before they learn about it and how well they learn to use and understand the use of rulers? To take another, we know that children have a bias towards representing angles as more perpendicular than they are: what connection is there between the extent of this bias and the success that children have in learning about angles, and is the relation a positive or a negative one? These are practicable and immensely interesting questions that could easily be answered in longitudinal studies. It is no longer a matter of what is to be done. The question that baffles us is: why are the right longitudinal and intervention studies not being done?

How can we intervene to prepare young children in the pre-school period for geometry? If there is a connection between the remarkable spatial knowledge that we find in quite young children and their successes and failures in learning about geometry later on, it should be possible to work on these early skills and enhance them in various ways that will help them learn about geometry when the time comes.

Here the situation is rather different. Educators have produced systematic programmes to prepare children for formal instruction in geometry. Some of these are ingenious and convincing, and they deserve attention. The problem in some cases is a lack of empirical evaluation.

One notable programme comes from the highly respected Freudenthal Institute in the Netherlands. A team of educational researchers there (van den Heuven-Panhuizen and Buys, 2008) have produced an ingenious and original plan for enhancing children's geometric skills before the age when they would normally be taught in a formal way about the subject. We shall concentrate here on the recommendations that van den Heuven-Panhuizen and Buys make for introducing kindergarten children to some basic geometrical concepts. However, we shall begin with the remark that, though their recommendations deserve our serious attention, the Freudenthal team offer us no empirical evidence at all that they really do work. Neither intervention studies with pre-tests and post-tests and randomly selected treatment groups, nor longitudinal predictive projects, seem to have played any part in this particular initiative.

The basic theoretical idea behind the Freudenthal team's programme for preparing children for geometry is that children's everyday life includes experiences and activities which are relevant to geometry but that the geometric knowledge that kindergarten children glean from these experiences is implicit and unsystematic. The solution that the team offers is to give these young children a systematic set of enjoyable game-like activities with familiar material and after each activity is finished to discuss and to encourage the children to reflect on what they have just done.

Some of these activities are about measurement (Buys and Veltman, 2008). In one interesting example, a teacher encourages the children to find out how many cups of liquid would fill a particular bottle and, when they have done that, to work out how many cups of liquid the bottle would provide when the bottle is not completely full. This leads to the idea of putting marks on the bottle to indicate when it contains one or more cups' worth of liquid. Thus, the children experience measurement units and also iteration. In another measurement activity children use conventional measures. They are given three rods each a metre long and are asked to measure the width of the room. Typically the children start well by forming the rods into a straight 3-metre line, but then hit the problem of measuring the remaining space: their first reaction is to ask for more rods, but the teacher then provides the suggestion that instead they try moving the first rod ahead of the third and then to move the second rod: Buys and Veltman report that the children readily follow this suggestion and apparently understand the iteration involved perfectly well.

Other exercises, equally ingenious, are about constructing and operating on shapes (van den Heuvel-Panhuizen, Veltman, Janssen and Hochstenbach, 2008). The Freudenthal team use the device of folding paper and then cutting out shapes to encourage children to think about the relationship between shapes: cutting an isosceles triangle across the fold, for example, creates a regular parallelogram when the paper is unfolded. The children also play games that take the form of four children creating a four-part figure between them with many symmetries: each child produces the mirror-image of the figure that the previous child had made (see Figure 5.3). The aim of such games is to give children systematic experience of the transformations, rotation and reflection, and to encourage them to reflect on these transformations.

If this group of researchers is right, children's early knowledge of geometric relationships and comparisons, though implicit and unsystematic, plays an important part in their eventual learning about geometry. It is a resource that can be enhanced by sensitive teaching of the kind that the Freudenthal group has pioneered. They may be right, but someone has to establish, through empirical research, how right they are.

There are a few empirical studies of ways of improving spatial skills in pre-school children. In these the children are given pre-tests which assess how well they do in spatial tasks which are suitable for children of that age, then go through intervention sessions which are designed to increase some of these skills and finally, soon after the end of this teaching, they are given post-tests to measure improvement in the same skills.

Two well designed studies carried out by Casey and her colleagues take this form (Casey, Erkut, Ceder and Young, 2008; Casey, Andrews, Schindler, Kersh and Young, 2008). In both studies the researchers were interested in how well five- and six-year-old children can learn to compose geometric shapes by combining other geometric shapes and how well they decompose shapes into component shapes, and also whether it is easier to improve this particular skill when it is couched in the context of a story than when the context is a more formal and abstract one.

The results of these two studies showed that the special instruction did, on the whole, help children to compose and decompose shapes and did have an effect on related spatial skills in the children who were taught in this way. They also showed that the narrative context added to the effect of teaching children at this age. Recently, Clements and Sarama (2007 *a*) reported a very different study of slightly younger, nursery children. These researchers were interested in the effects of a pre-school programme, called Building Blocks, the aim of which is to prepare children for mathematics in general including geometry. This programme is based on a theory about children's mathematical development: as far as

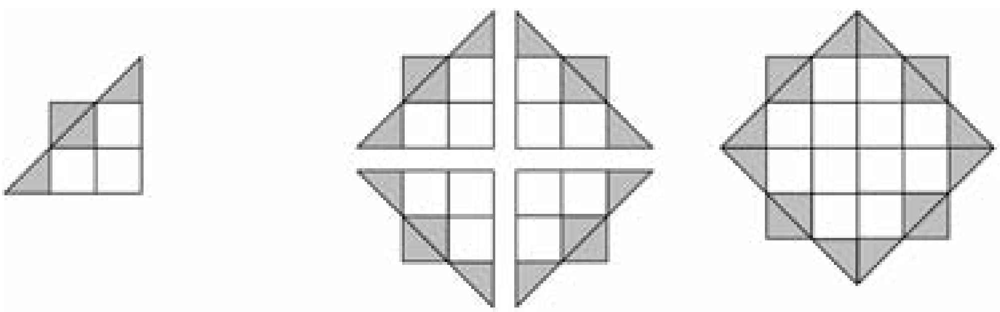

Figure 5.3: An activity devised by van den Heuvel-Panhuizen, Veltman, Janssen and Hochstenbach: four children devise a four-part shape by forming mirror-images.

geometry is concerned Clement and Sarama's strongest interest is in children's awareness of the composition of shapes and the relationship between different shapes. They also believe that the actual teaching given to individual children should be determined by their developmental levels. Thus the day-to-day instruction in their programme depends on the children's developmental trajectories. Clements and Sarama report that the young children taught in the Building Blocks programmes improved from pre-test to post-test more rapidly than children taught in other ways in tasks that involved constructing or relating shapes.

These are interesting conclusions and a good start. However, research on the question of the effects of intervention programmes designed to prepare children for geometry need to go further than this. We need studies of the effects of pre-school interventions on the progress that children make when they are eventually taught geometry at school a few years later on. We cannot be sure that the changes in the children's skills that were detected in these studies would have anything to do with their successes and failures later on in geometry.

Summary

1 Children have a well-developed and effective relational knowledge of shape, position, distance, spatial orientation and direction long before they go to school. This knowledge may be implicit and non-numerical for the most part, but it is certainly knowledge that is related to geometry.

2 The mistakes that children make in drawing horizontal and vertical lines are probably due to them preferring to concentrate on relations between lines close to each other (liquid in a glass is perpendicular to the sides of the glass) rather than to separated lines (liquid in a glass is parallel to horizontal surfaces like table tops). This is a mistake not in relational perception, but in picking the right relation.

3 Children are also able to understand and to make transitive inferences, which are the basic logical move that underlies measurement, several years before being taught about geometry.

4 We do not yet know how well they can cope with the notion of iteration in the school years.

5 There is little research on the possible causal links between these impressive early perceptual and logical abilities and the successes and difficulties that children have when they first learn about geometry. This is a serious gap in our knowledge about geometrical learning.

Learning about geometry

The aim of teaching children geometry is to show them how to reason logically and mathematically about space, shapes and the relation between shapes, using as tools conventional mathematical measures for size, angle, direction, orientation and position. In geometry classes children learn to analyse familiar spatial experiences in entirely new ways, and the experience of this novel and explicit kind of analysis should allow them to perceive and understand spatial relationships that they knew nothing about before.

In our view the aspects of analysing space geometrically that are new to children coming to the subject for the first time are:
- representing spatial relations which are already familiar to them, like length, area and position, in numbers
- learning about relations that are new to them, at any rate in terms of explicit knowledge, such as angle
- forming new categories for shapes, such as triangles, and understanding that the properties of a figure depends on its geometric shape
- understanding that there are systematic relations between shapes, for instance between rectangles and parallelograms
- understanding the relation between shapes across transformations, such as rotation, enlargement and changes in position.

Applying numbers to familiar spatial relations and forming relations between different shapes

Length measurement

Young children are clearly aware of length. They know that they grow taller as they grow older, and that some people live closer to the school than others. However, putting numbers on these changes and differences, which is one of their first geometric feats, is something new to them.

Standard units of measurement are equal subdivisions of the measuring instrument, and this means that children have to understand that this instrument, a ruler or tape measure or protractor, is not just a continuous quantity but is also subdivided into units that are exactly the same as each other. The child has to understand, for example, that by using a ruler, she can represent an object's length through an iteration of measurement units, like the centimetre.

When children measure, for example, the length of a straight line, they must relate the units on the ruler to the length that they are measuring, which is a one-to-one correspondence, but of a relatively demanding form. In order to see that the measured length is, for example, 10 cm long, they have to understand that the length that they are measuring can also be divided into the same unit and that ten of the units on the ruler are in one-to-one correspondence with ten imaginary but exactly similar units on what is being measured. This is an active form of one-to-one correspondence, since it depends on the children understanding that they are converting a continuous into a discontinuous quantity by dividing it into imaginary units. Here, is a good example of how even the simplest of mathematical analysis of space makes demands on children's imagination: they must imagine and impose divisions on undivided quantities in order to create the one-to-one correspondence which is basic to all measurement of length.

Measuring a straight line with a ruler is probably the simplest form of measurement of all, but children even make mistakes with this task and their mistakes suggest that they do not at first grasp that measuring the line takes the form of imposing one-to-one correspondence of the units on the measure with imagined units on the line. This was certainly suggested by the answers that a large number of children who were in their first three years of secondary school (11-, 12-, 13- and 14-year-olds) gave to a question about the length of a straight line, which was part of a test devised by Hart, Brown, Kerslake, Küchemann and Ruddock (1985). The children were shown a picture of straight line beside a ruler that was marked in centimetres. One end of the line was aligned with the 1 cm mark on the ruler and the other end with the 7 cm mark. The children were asked how long the line was and, in the youngest group, almost as many of them (46%) gave the answer 7 cm as gave the right answer (49%) which was 6 cm. Thus even at the comparatively late age of 11 years, after several years experience of using rulers, many children seemed not to understand, or at any rate not to understand perfectly, that it is the number of units on the ruler that the line corresponded to that decided its length.

A study by Nunes, Light and Mason (1993) gives us some insight into this apparently persistent difficulty. These experimenters asked pairs of six- to eight-year-old children to work together in a measuring task. They gave both children in each pair a piece of paper with a straight line on it, and the pair's task was to find out whether their lines were the same length or, if not whose was longer and whose shorter. Neither child could see the other's line because the two children did the task in separate rooms and could only talk to each other over a telephone. Both children in each pair were given a measure to help them compare these lines and the only difference between the pairs was in the measures that the experimenters gave them.

The pairs of children were assigned to three groups. In one group, both children in each pair were given a string with no markings: this therefore was a measure without units. In a second group, each child in every pair was given a standard ruler, marked in centimetres. In the third group the children were also given rulers marked in centimetres, but, while one of the children had a standard ruler, the other child in the pair was given a 'broken' ruler: it started at four centimetres. The child with the broken ruler could not produce the right answer just by reading out the number from the ruler which coincided with the end of the line. If the line was, for example, 7 cm long, that number would be 11 cm. The children in these pairs had to pay particular attention to the units in the ruler.

The pairs in the second group did well. They came up with the correct solution 84% of the time. The few mistakes that they made were mostly about placing the ruler or counting the units. Some children aligned one endpoint with the 1 cm point on the ruler rather than the 0 cm point and thus overestimated the length by 1 cm. This suggests that they were wrongly concentrating on the boundaries between units rather than the units themselves. Teachers should be aware that some children have this misconception. Nevertheless the ruler did, on the whole, help the children in this study since those who worked with complete rulers did much better than the children who were just given a string to measure with.

However, the broken ruler task was more difficult. The children in the standard ruler group were right 84%, and those in the broken ruler group 63%, of the time. Those who got it right despite having a broken ruler either counted the units or read off the last number (e.g. 11 cm for a 7 cm line) and then subtracted 4, and since they managed to do this more often than not, their performance established that these young children have a considerable amount of understanding of how to use the units in a measuring instrument and of what the units mean. However, on 30% of the trials the children in this group seemed not to understand the significance of the missing first four centimetres in the ruler. Either they simply read off the number that matched the line's endpoint (11 cm for 7 cm) or they did not subtract the right amount from it.

A large-scale American study (Kloosterman, Warfield, Wearne, Koc, Martin and Strutchens, 2004; Sowder, Wearne, Martin and Strutchens, 2004) later confirmed this striking difficulty. In this study, the children had to judge the length of an object which was pictured just above a ruler, though neither of its endpoints was aligned with the zero endpoint on that ruler. Less than 25% of the 4th graders (nine-year-olds) solved the problem correctly and only about 60% of the 8th graders managed to find the correct answer. This strong result, combined with those reported by Nunes *et al.*, suggests that many children may know how to use a standard ruler, but do not fully understand the nature or structure of the measurement units that they are dealing with when they do measure. Their mistake, we suggest, is not a misunderstanding of the function of a ruler: it is a failure in an active form of one-to-one correspondence – in imagining the same units on the line as on the ruler and then counting these units.

Summary

1 Measuring a straight line with a ruler is a procedure and it is also a considerable intellectual feat.

2 The procedure is to place the zero point of the ruler at one end of the straight line and to read off the number of standardised units on the ruler that corresponds to the other end of this line. There is no evidence that this procedure causes young children any consistent difficulty.

3 The intellectual feat is to understand that the ruler iterates a standardised unit (e.g. the centimetre) and that the length of the line being measured is the number of units in the part of the ruler that is in correspondence to the line. Thus measurement of length is a one-to-one correspondence problem, and the correspondence is between units that are displayed on the ruler but have to be imagined on the line itself. This act of imagination seem obvious and easy to adults but may not be so for young children.

4 Tests and experiments in which the line being measured is not aligned with zero show that initially children do not completely understand how measurement is based on imagining one-to-one correspondence of iterated units.

Measurement of area: learning about the relationship between the areas of different shapes

There is a striking contrast between young children's apparently effortless informal discriminations of size and area and the difficulties that they have in learning how to analyse and measure area geometrically. Earlier in this chapter we reported that babies are able to recognise objects by their size and can do so even when they see these objects at different distance on different occasions. Yet, many children find it difficult at first to measure or to understand the area of even the simplest and most regular of shapes.

All the intellectual requirements for understanding how to measure length, such as knowing about transitivity, iteration, and standardised units, apply as well to measuring area. The differences are that:

- area is necessarily a more complex quantity to measure than length because now children have to learn to consider and measure two dimensions and to co-ordinate these different measurements. The co-ordination is always a multiplicative one (e.g. base x height for rectangles; πr^2 for circles etc.).

- the standardised units of area – square centimetres and square metres or square inches etc. – are new to the children and need a great deal of explanation. This additional step is usually quite a hard one for children to take.

Rectangles

Youngsters are usually introduced to the measurement of area by being told about the base x height rule for rectangles. Thus, rectangles provide them with their first experience of square centimetres. The large-scale study of 11- to 14-year-old children by Hart *et al.* (1985), which we have mentioned already, demonstrates the difficulties that many children have even with this simplest of area measurements. In one question the children were shown a rectangle, drawn on squared paper, which measured 4 squares (base) by 2½ squares (height) and then were asked to draw another rectangle of the same area with a base of 5 squares. Only 44% of the 11-year-old children got this right: many judged that it was impossible to solve this problem.

We have to consider the reason for this difficulty. One reason might be that children find it hard to come to terms with a new kind of measuring unit, the square. In order to explain these new units teachers often give children 'covering' exercises. The children cover a rectangle with squares, usually 1 cm squares, arranged in columns and rows and the teacher explains that the total number of squares is a measure of the rectangle's area. The arrangement of columns and rows also provides a way of introducing children to the idea of multiplying height by width to calculate a rectangle's area. If the rectangle has five rows and four columns of squares, which means that its height is 5 cm and its width 4 cm, it is covered by 20 squares.

This might seem like an easy transition, but it has its pitfalls. These two kinds of computation are based on completely different reasoning: counting is about finding out the number that represents a quantity and involves additive reasoning whereas multiplying the base by the height involves understanding that there is a multiplicative relation between each of these measures and the area. Therefore, practice on one (counting) will not necessarily encourage the child to adopt the other formula (multiplying). Another radical difference is that the covering exercise provides the unit, the square centimetre, from the start but when the child uses a ruler to measure the sides and then to multiply height by width, she is measuring with one unit, the centimetre, but creating a new unit, the square centimetre (for further discussion, see Paper 3).

This could be an obstacle. The French psychologist, Gerard Vergnaud (1983), rightly distinguishes problems in which the question and the answer are about the same units ('A plant is 5 cm high at the beginning of the week and by the end of the week it is 2 cm higher. How high is it at the end of the week?') and those in which the question is couched in one unit and the answer in another ('The page on your book is 15 cm high and 5 cm wide. What is the area of this page?'). The answer to the second question must be in square centimetres even though the question itself is couched only in terms of centimetres. Vergnaud categorised the first kind of problem as 'isomorphism of measures' and the second as 'product of measures'. His point was that product of measures problems are intrinsically the more difficult of the two because, in order to solve such problems, the child has to understand how one kind of unit can be used to create another.

At first, even covering tasks are difficult for many young children. Outhred and Mitchelmore (2000) gave young children a rectangle to measure and just one 1 cm^2 square tile to help them to do this. The children also had pencils and were encouraged to draw on the rectangle itself. Since the children had one tile only to work with, they could only 'cover' the area by moving that tile about. Many of the younger children adopted this strategy but carried it out rather unsuccessfully. They left gaps between their different placements of the tile and there were also gaps between the squares in the drawings that they made to represent the different positions of the tiles.

These mistakes deserve attention, but they are hard to interpret because there are two quite different ways of accounting for them. One is that these particular children made a genuinely conceptual mistake about the iteration of the measuring unit. They may not have realised that gaps are not allowed – that the whole area must be covered by these standardised units. The alternative account is that this was an executive, not a conceptual, failure. The children may have known about the need for complete covering, and yet may have been unable to carry it out. Moving a tile around the rectangle, so that the tile covers every part of it without any overlap, is a complicated task, and children need a great deal of dexterity and a highly organised memory to carry it out, even if they know exactly what they have to do. These 'executive' demands may have been the source of the children's problems. Thus, we cannot say for sure what bearing this study has on Vergnaud's distinction between isomorphism and product of measures until we know whether the mistakes that children made in applying the measure were conceptual or executive ones.

Vergnaud's analysis, however, fits other data that we have on children's measurement of area quite well.

Nunes *et al.* (1993) asked pairs of eight- and nine-year-old children to work out whether two rectangles had the same area or not. The dimensions of the two rectangles were always different, even when their areas were the same (e.g. 5 x 8 and 10 x 4 cm). The experimenters gave all the children standard rulers, and also 1 cm^3 bricks to help them solve the problem.

The experimenters allowed the pairs of children to make several attempts to solve each problem until they agreed with each other about the solution. Most pairs started by using their rulers, as they had been taught to at school, but many of them then decided to use the bricks instead. Overall the children who measured with bricks were much more successful than those who relied entirely on their rulers. This clear difference is a demonstration of how difficult it is, at first, for children to use one measurement unit (centimetres) to create another (square centimetres). At this age they are happier and more successful when working just with direct representation of the measurement units that they have to calculate than when they have to use a ruler to create these units.

The success of the children who used the bricks was not due to them just counting these bricks. They hardly ever covered the area and then laboriously counted all the bricks. Much more often, they counted the rows and the columns of bricks and then either multiplied the two figures or used repeated addition or a mixture of the two to come up with the correct solution (A: *'Eight bricks in a row. And 5 rows. What's five eights?'* B: *'Two eights is 16 and 16 is 32. Four eights is 32. 32. 40'*). In fact, the children who used bricks multiplied in order to calculate the area more than three times as often as the children who used the ruler. Those who used rulers often concentrated on the perimeter: they measured the length of the sides and added lengths instead.

This confusion of area and perimeter is a serious obstacle. It can be traced back in time to a systematic bias in judgements that young children make about area long before they are taught the principles of area measurement. This bias is towards judging the area of a figure by its perimeter.

The bias was discovered independently in studies by Wilkening (1979) in Germany and Anderson and Cuneo (1978) and Cuneo (1983) in America. Both groups of researchers asked the same two questions (Wilkening and Anderson, 1982).

1 If you ask people to judge the area of different rectangles that vary both in height and in width, will their judgement be affected by both these dimensions? In other words, if you hold the width of two rectangles constant will they judge the higher of the two as larger, and if you hold their height constant will they judge the wider one as the larger? It is quite possible that young children might attend to one dimension only, and indeed Piaget's theory about spatial reasoning implies that this could happen.

2 If people take both dimensions into account, do they do so in an *additive* or a *multiplicative* way? The correct approach is the multiplicative one, because the area of a rectangle is its height multiplied by its width. This means that the difference that an increase in the rectangle's height makes to the area of the rectangle depends on its width, and vice versa. An increase of 3 cm in the height of a 6 cm wide rectangle adds another 18 cm^2 to its area, but the same increase in height to an 8 cm wide rectangle adds another 24 cm^2. The additive approach, which is wrong, would be to judge that an equal change in height to two rectangles has exactly the same effect on their areas, even if their widths differ. This is not true of area, but it is true of perimeter. To increase the height of a rectangle by 3 cm has exactly the same effect on the perimeter of a 6 cm and an 8 cm wide rectangle (and increase of 6 cm) and the same goes for increases to the width of rectangles with different heights. Also, the same increase in width has exactly the same effect on the two rectangles' perimeters, but very different effects on their area. It follows that anyone who persistently makes additive judgements about area is probably confusing area with perimeter.

The tasks that these two teams of experimenters gave to children and adults in their studies were remarkably similar, and so we will describe only Wilkening's (1979) experiment. He showed 5-, 8- and 11-year-old children and a group of adults a series of rectangles that varied both in height (6, 12 and 18 cm) and in width (again 6, 12 and 18 cm). He told the participants that these could be broken into pieces of a particular size, which he illustrated by showing them also the size of one of these pieces. The children's and adults' task was to imagine what would happen if each rectangle was broken up and the pieces were arranged in a row. How long would this row be?

The most striking contrast in the pattern of these judgments was between the five-year-old children and the adults. To put it briefly, five-year-old children made additive judgements and adults made multiplicative judgements.

The five-year-olds plainly did take both height and width into account, since they routinely judged rectangles of the same height but different widths as having different areas and they did the same with rectangles of the same widths but different heights. This is an important result, and it must be reassuring to anyone who has to teach young schoolchildren about how to measure area. They are apparently ready to take both dimensions into account.

However, the results suggest that young children often co-ordinate information about height and width in the wrong way. The typical five-year-old judged, for example, that a 6 cm difference in height would have the same effect on 12 cm and 18 cm wide rectangles. In contrast, the adults' judgements showed that they recognised that the effect would be far greater on the 18 cm than on the 12 cm wide figures. This is evidence that young children rely on the figures' perimeters, presumably implicitly, in order to judge their area. As have already seen, when children begin to use rulers many of them fall into the trap of measuring a figure's perimeter in order to work out its area, (Nunes, Light and Mason, 1993). Their habit of concentrating on the perimeter when making informal judgements about area may well be the basis for this later mistake. The existence among schoolchildren of serious confusion between area and perimeter was confirmed in later research by Dembo, Levin and Siegler (1997).

We can end this section with an interesting question. One obvious possible cause of the radical difference in the patterns of 5-year-olds' and adults' judgements might be that the 5-year-olds had not learned how to measure area while the adults had. In other words, mathematical learning could alter this aspect of people's spatial cognition. The suggestion does not seem far-fetched, especially when one also considers the performance of the older children in Wilkening's interesting study. The 5-year-olds had not been taught about measurement at all: the 8-year-olds had had some instruction, but not a great deal: the 12-year-olds were well-versed in measurement, but probably still made mistakes. Wilkening found some signs of a multiplicative pattern in the responses of the 8-year-olds, but this was slight: he found stronger signs of this pattern among the 12-year-olds, though not as pronounced as in the adult group. These changes do not prove that being taught how to measure and then becoming increasingly experienced with measuring led to this difference between the age groups, but they are certainly consistent with that idea. There is an alternative explanation, which is that adults and older children have more informal experience than 8-year-olds do of judging and comparing areas, as for example when they have to judge how much paint they need to cover different walls. Here is a significant and interesting question for research: do teachers alone change our spatial understanding of area or does informal experience play a part as well?

Summary

1 Measuring area is a multiplicative process: we usually multiply two simple measurements (e.g. base by height for rectangles) to produce a total measure of an area. The process also produces a different unit (i.e. product of measures): measuring the base and height in centimetres and then multiplying them produces a measure in terms of square centimetres.

2 Producing a new measure is a difficult step for children to make. They find it easier to measure a rectangle when they measure with units which directly instantiate square centimetres than when they use a ruler to measure its base and height in centimetres.

3 The multiplicative aspect of area measurement is also a problem for young children who show a definite bias to judge the area of a rectangle by adding its base and height rather than by multiplying them. They confuse, therefore, perimeter and area.

Parallelograms: forming relations between rectangles and parallelograms

The measurement of parallelograms takes us into one of the most exciting aspects of learning about geometry. The base-by-height rule applies to these figures as well as to rectangles. One way of justifying the base by height rule for parallelograms is that any parallelogram can be transformed into a rectangle with the same base and height measurements by adding and subtracting congruent areas to the parallelogram.

Figure 5.4 presents this justification which is a commonplace in geometry classes. It is based on the inversion principle (see Paper 2). Typically the teacher shows children a parallelogram and then creates two congruent triangles (*A* and *S*) by dropping vertical lines from the top two corners of the parallelogram and then extending the baseline to reach the new vertical that is external to the parallelogram. Triangle *A* falls outside the original parallelogram, and therefore is an addition to the figure. Triangle *S* falls inside the original parallelogram, and the nub of the teacher's demonstration is to point out that the effect of adding Triangle *A* and subtracting Triangle *S* would be to transform the figure into a rectangle with the same base and height as the original parallelogram. Triangles *A* and *S* are congruent and so their areas are equal. Therefore, adding one and subtracting the other triangle must produce a new figure (the rectangle) of exactly the same size as the original one (the parallelogram).

This is a neat demonstration, and it is an important one from our point of view, because it is our first example of the importance in geometry of understanding that there are systematic relations between shapes. Rectangles and parallelograms are different shapes but they are measured by the same base-by-height rule because one can transform any rectangle into any parallelogram, or vice versa, with the same base and height without changing the figure's area.

Some classic research by the well-known Austrian psychologist Max Wertheimer (1945) suggests that many children learn the procedure for transforming parallelograms into rectangles quite easily, but apply it inflexibly. Wertheimer witnessed a group of 11-year-old children learning from their teacher why the same base-by-height rule applied to parallelograms as well. The teacher used the justification that we have already described, which the pupils appeared to understand. However, Wertheimer was not certain whether these children really had understood the underlying idea. So, he gave them another parallelogram whose height was longer than its base (diagram 3 in Figure 5.4). When a parallelogram is oriented in this way, dropping two vertical lines from its top two corners does not create two congruent triangles. Wertheimer found that most of the children tried putting in the two vertical lines, but were at a loss when they saw the results of doing so. A few, however, did manage to solve the problem by rotating the new figure so that the base was longer than its height, which made it possible for them to repeat the teacher's demonstration.

The fact that most of the pupils did not cope with Wertheimer's new figure was a clear demonstration that they had learned more about the teacher's procedure than about the underlying idea about transformation that he had hoped to convey. Wertheimer argued that this was probably because the teaching itself concentrated too much on the procedural sequence and too little on the idea of

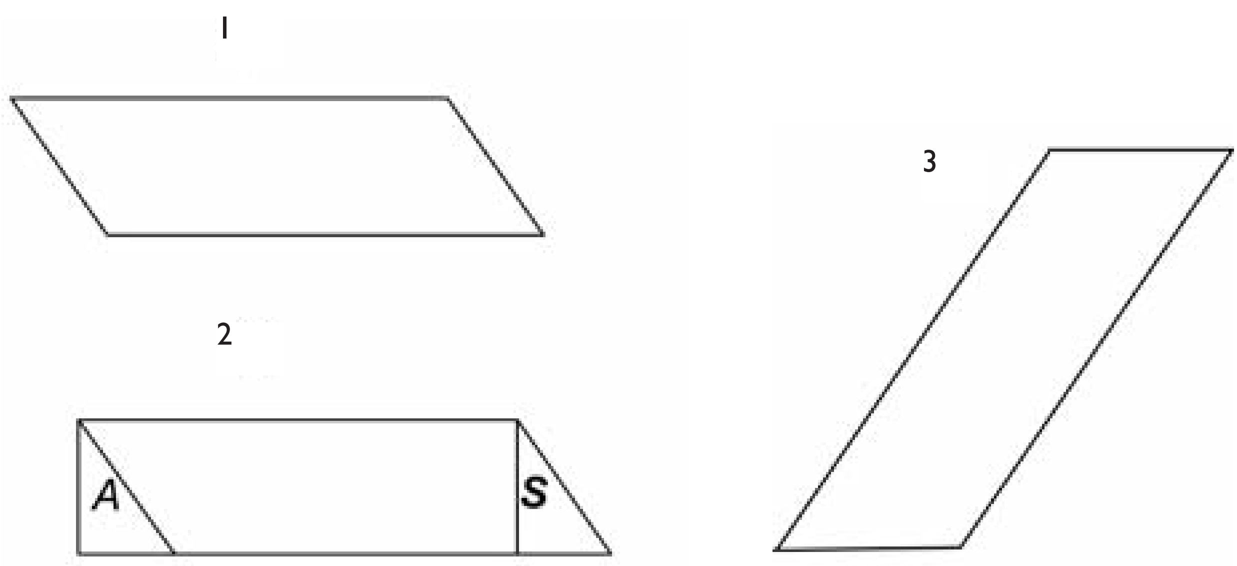

Figure 5.4: Demonstrating by transforming a parallelogram into a rectangle that the base-by-height rule applies to parallelograms as well as to rectangles.

transformation. In later work, that is only reported rather informally (Luchins and Luchins, 1970), Wertheimer showed children two figures at a time, one of which could be easily transformed into a measurable rectangle while the other could not (for example, the *A* and *B* figures in Figure 5.5). Wertheimer reported that this is an effective way of preparing children for understanding the relation between the area of parallelograms and rectangles.

It is a regrettable irony that this extraordinarily interesting and ingenious research by a leading psychologist was done so long ago and is so widely known, and yet few researchers since then have studied children's knowledge of how to transform one geometric shape into another to find its area.

In fact, Piaget *et al.* (1960) did do a relevant study, also a long time ago. They asked children to measure the area of an irregular polygon (Figure 5.6). One good way to solve this difficult problem is to partition the figure by imaging the divisions represented by the dotted lines in the right hand figure. This creates the Triangle *x* and also a rectangle which includes another Triangle *y*. Since the two triangles are congruent and Triangle *x* is part of the original polygon while Triangle *y* is not, the area of the polygon must add up to the area of the rectangle (plus Triangle *x* minus Triangle *y*).

Piaget *et al.* report that the problem flummoxed most of the children in their study, but report that some 10-year-olds did come up with the solution that we have just described. They also tell us that many children made no attempt to break up the figure but that others, more advanced, were ready to decompose the figure into smaller shapes, but did not have the idea of in effect adding to the figure by

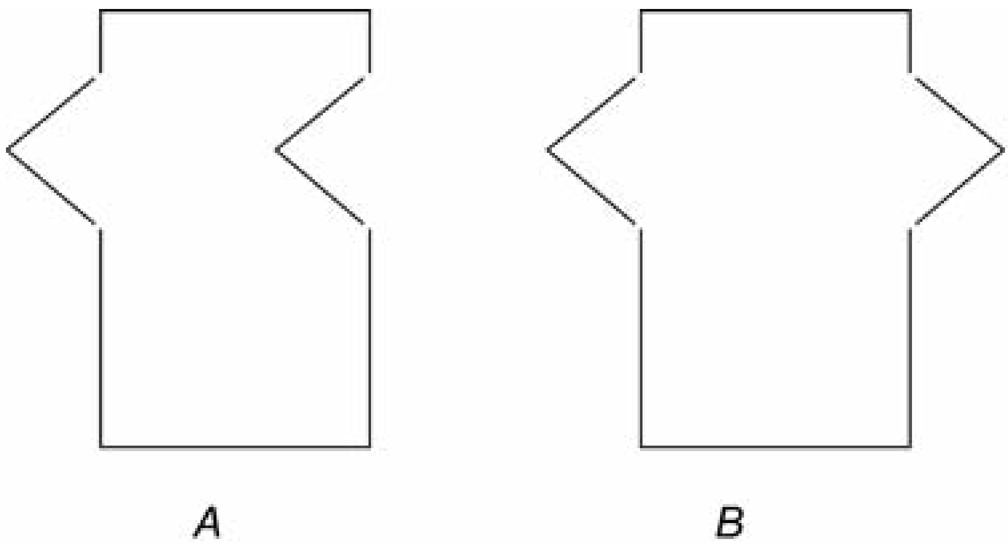

Figure 5.5: An example of Wertheimer's A and B figures. A figures could be easily transformed into a simple rectangle. B figures could not.

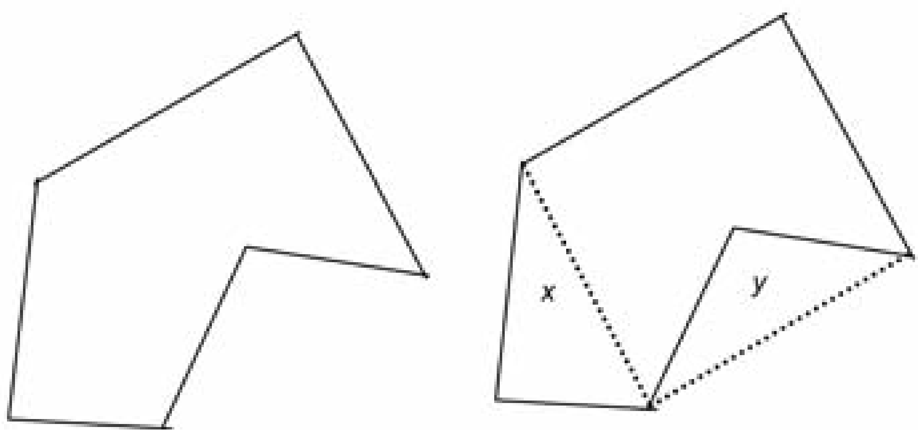

Figure 5.6: Piaget et al.'s irregular polygon whose area could be measured by decomposition.

imagining the BD line which was external to the figure. Thus, the stumbling block for these relatively advanced children was in adding to the figure. This valuable line of research, long abandoned, needs to be restarted.

Triangles: forming relations between triangles and parallelograms

Other transformations from one shape to another are equally important. Triangles can be transformed into parallelograms, or into rectangles if they are right-angle triangles, simply by being doubled (see Figure 5.7). Thus the area of a triangle is half that of a parallelogram with the same base and height.

Hart *et al.*'s (1985) study shows us that 11- to 14-year-old children's knowledge of the (base × height)/2 rule for measuring triangles is distinctly sketchy. Asked to calculate the area of a right-angle triangle with a base of 3 cm and a height of 4 cm, only 48% of the children in their third year in secondary school (13- and 14-year-olds) gave the right answer. Only 31% of the first year secondary school children (11-year-olds) succeeded, while almost an equal number of them – 29% – gave the answer 12, which means that they correctly multiplied base by height but forgot to halve the product of that multiplication.

Children are usually taught about the relationship between triangles, rectangles and parallelograms quite early on in their geometry lessons. However, we know of no direct research on how well children understand the relationships between different shapes or on the best way to teach them about these relations.

Summary

1 Learning about the measurement of the area of different shapes is a cumulative affair which is based not just on formulas for measuring particular shapes but on grasping the relationships, through transformations, of different shapes to each other. Parallelograms can be transformed into rectangles by adding and subtracting congruent triangles: triangles can be transformed into parallelograms by being doubled.

2 There is little direct research on children's understanding of the importance of the relationships between shapes in their measurement. Wertheimer's observations suggest very plausibly that their understanding depends greatly on the quality of the teaching.

3 We need more research on what are the most effective ways to teach children about these relationships.

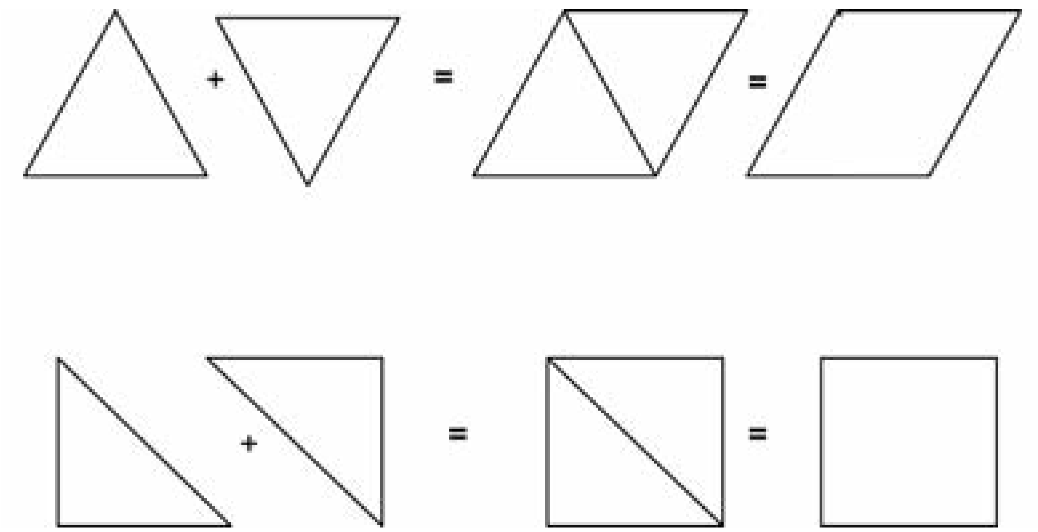

Figure 5.7: A demonstration that any two congruent triangles add up to a measurable parallelogram and any two congruent right angle triangles add up to a measurable rectangle.

Understanding new relationships: the case of angle

Angle is an abstract relation. Sometimes it is the difference between the orientations of two lines or rays, sometimes the change in your orientation from the beginning of a turn that you are making to its end, and sometimes the relation between a figure or a movement to permanent aspects of the environment: the angle at which an aeroplane rises after take-off is the relation between the slope of its path and the spatial horizontal. Understanding that angles are a way of describing such a variety of contexts is a basic part of learning plane geometry. Yet, research by psychologists on this important and fascinating topic is remarkably thin on the ground.

Most of the relatively recent studies of children's and adults' learning about angles are about the effectiveness of computer-based methods of teaching. This is estimable and valuable work, but we also need a great deal more information about children's basic knowledge about angles and about the obstacles, which undoubtedly do exist, to forming an abstract idea of what angles are.

In our everyday lives we experience angles in many different contexts, and it may not at first be easy for children to connect information about angles encountered in different ways. The obvious distinction here is between perceiving angles as configurations, such as the difference between perpendicular and non-perpendicular lines in pictures and diagrams, and as changes in movement, such as changing direction by making a turn. These forms of experiencing angles can themselves be subdivided: it may not be obvious to school children that we make the same angular change in our movements when we walk along a path with a right-angle bend as when we turn a door-knob by 90° (Mitchelmore, 1998).

Another point that children might at first find hard to grasp is that angles are relational measures. When we say that the angles in some of the figures in Figure 5.8a are 90° ones and in others 45°, we are making a statement about the relation between the orientations of the two lines in each figure and not about the absolute orientation of any of the individual lines, which vary from each other. Also, angles affect the distance between lines, but only in relation to the distance along the lines: in Figure 5.8b the distance between the lines in the figure with the larger angle is greater than in the other figure when

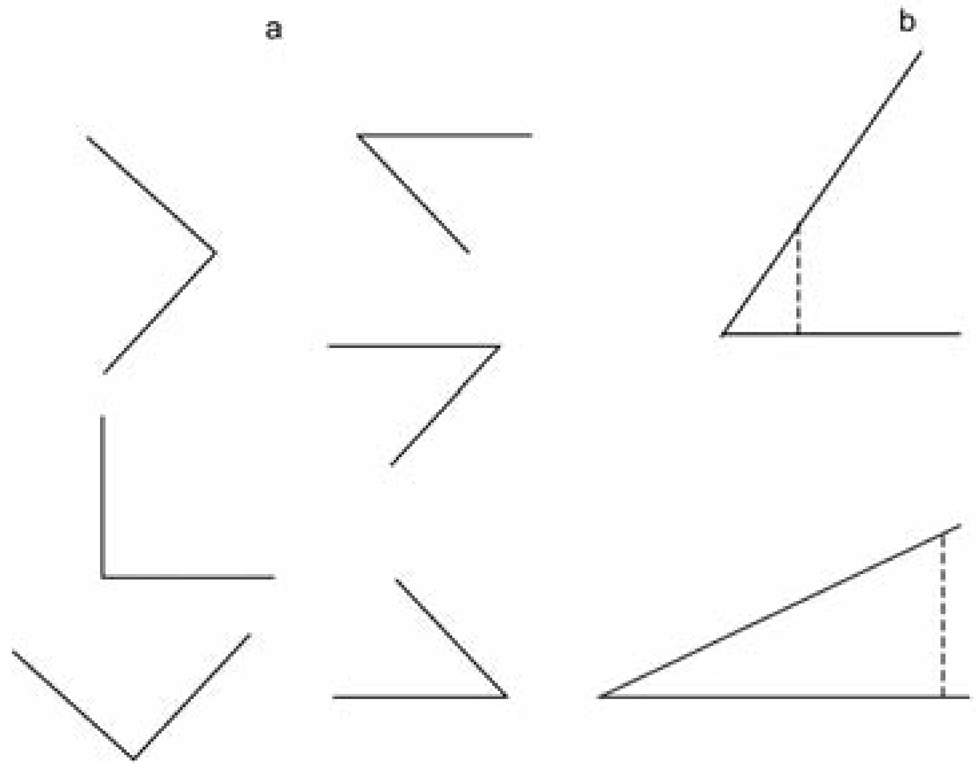

Figure 5.8: Angles as relations

the distance is measured at equivalent points along lines in the two figures, but not necessarily otherwise.

A third possible obstacle is that children might find some representations of angle more understandable than others. Angles are sometimes formed by the meeting of two clear lines, like the peak of a roof or the corner of a table. With other angles, such as the inclination of a hill, the angle is clearly represented by one line – the hill itself, but the other, a notional horizontal, is not so clear. In still other cases, such as the context of turning, there are no clearly defined lines at all: the angle is the amount of turning. Children may not see the connection between these very different perceptual situations, and they may find some much easier than others.

There are few theories of how children learn about angles, despite the importance of the topic. The most comprehensive and in many ways the most convincing of the theories that do exist was produced by Mitchelmore and White (2000). The problem that these researchers tried to solve is how children learn to abstract and classify angles despite the large variety of situations in which they experience them. They suggest that children's knowledge of angles develops in three steps:

1 **Situated angle concepts** Children first register angles in completely specific ways, according to Mitchelmore and White. They may realise that a pair of scissors, for example, can be more open or less open, and that some playground slides have steeper slopes than others but they make no link between the angles of scissors and of slides, and would not even recognise that a slide and a roof could have the same slope.

2 **Contextual angle concepts** The next step that children take is to realise that there are similarities in angles across different situations, but the connections that they do make are always restricted to particular, fairly broad, contexts. Slope, which we have mentioned already, is one of these contexts; children begin to be able to compare the slopes of hills, roofs and slides, but they do not manage to make any connection between these and the angles of, for instance, turns in a road. They begin to see the connection between angles in very different kinds of turns – in roads and in a bent nail, for example – but they do not link these to objects turning round a fixed point, like a door or a door-knob.

3 **Abstract angle concepts** Mitchelmore and White's third step is itself a series of steps. They claim that children begin to compare angles across contexts, for example, between slopes and turns, but that initially these connections across contexts are limited in scope; for example, these researchers report that even at the age of 11 years many children cannot connect angles in bends with angles in turns. So, children form one or more restricted abstract angle 'domains' (e.g. a domain that links intersections, bends, slopes and turns) before they finally develop this into a completely abstract concept of standard angles.

To test this theoretical framework, Mitchelmore and White gave children of 7, 9 and 11 years pictures of a wide range of situations (doors, scissors, bends in roads etc.) and asked them to represent the angles in these, using a bent pipe cleaner to do so, and also to compare angles in pairs of different situations. The study certainly showed different degrees of abstraction among these children and provided some evidence that abstraction about angles increases with age. This is a valuable contribution, but we certainly need more evidence about this developmental change for at least two reasons.

One is methodological. The research that we have just described was cross-sectional: the children in the different age groups were different children. A much better way of testing any hypothesis about a series of developmental cognitive changes is to do a longitudinal study of the ideas that the same children hold and then change over time as they get older. If the hypothesis is also about what makes the changes happen, one should combine this longitudinal research with an intervention study to see what provokes the development in question. We commented on the need for combining longitudinal and intervention studies in Paper 2. Once again, we commend this all-too-rarely adopted design to anyone planning to do research on children's mathematics.

The second gap in this theory is its concentration on children learning what is irrelevant rather than what is relevant to angle. The main claim is that children eventually learn, for example, that the same angles are defined by two clear lines in some cases but not in others, that some angular information is about static relations and some about movement, but that it is still exactly the same kind of information. This claim is almost certainly right, but it does not tell us what children learn instead.

One possibility, suggested by Piaget, Inhelder and Szeminska (1960), is that children need to be able to relate angles to the surrounding Euclidean framework in order to reproduce them and compare them to other angles. In one study they asked children to copy a triangle like the one in Figure 5.9.

They gave the children some rulers, strips of paper and sticks (but apparently no protractor) to help them do this. Piaget *et al.* wanted to find out how the children set about reproducing the angles CAB, ABC and ACB. In the experimenters' view the best solution was to measure all three lines, and also introduce an additional vertical line (KB in Figure 5.9) or to extend the horizontal line and then introduce a new vertical line (CK' in Figure 5.9). It is not at all surprising that children below the age of roughly ten years did not think of this Euclidean solution. Some tried to copy the triangle perceptually. Others used the rulers to measure the length of the lines but did not take any other measures. Both strategies tended to lead to inaccurate copies.

The study is interesting, but it does not establish that children have to think of angles in terms of their relationship to horizontal and vertical lines in order to be able to compare and reproduce particular angles. The fact that the children were not given the chance to use the usual conventional measure for angles – the protractor – either in this study or in Mitchelmore and White's study needs to be noted. This measure, despite being quite hard to use, may play a significant and possibly even an essential part in children's understanding of angle.

Another way of approaching children's understanding of angles is through what Mitchelmore and White called their situated angle concepts. Magina and Hoyles (1997) attempted to do this by investigating children's understanding of angle in the context of clocks and watches. They asked Brazilian children, whose ages ranged from 6 to 14 years, to show them where the hands on a clock would be in half an hour's time and also half an hour before the time it registered at that moment. Their aim was to find out if the children could judge the correct degree of turn. Magina and Hoyles report that the younger children's responses tended to be either quite unsystematic or to depend on the initial position of the minute hand: these latter children, mostly 8- to 11-year-olds, could move the minute hand to represent half an hour's difference well enough when the hand's initial position was at 6 (half past) on the clock face, but not when it was at 3 (quarter past). Thus, even in this highly familiar situation, many children seem to have an incomplete understanding of the angle as the degree of a turn. This rather disappointing result suggests that the origins of children's understanding may not lie in their informal spatial experiences.

One way in which children may learn about angles is through movement. The idea of children learning about spatial relations by monitoring their own actions in space fits well with Piaget's framework, and it is the basis for Logo, the name that Papert (1980) gave to his well-known computer-system that has often been used for teaching aspects of geometry. In Logo, children learn to write programmes to move a 'turtle' around a spatial environment. These programmes consist of a series of instructions that determine a succession of movements by the turtle. The instructions are about the length and direction of each movement, and the instructions about direction

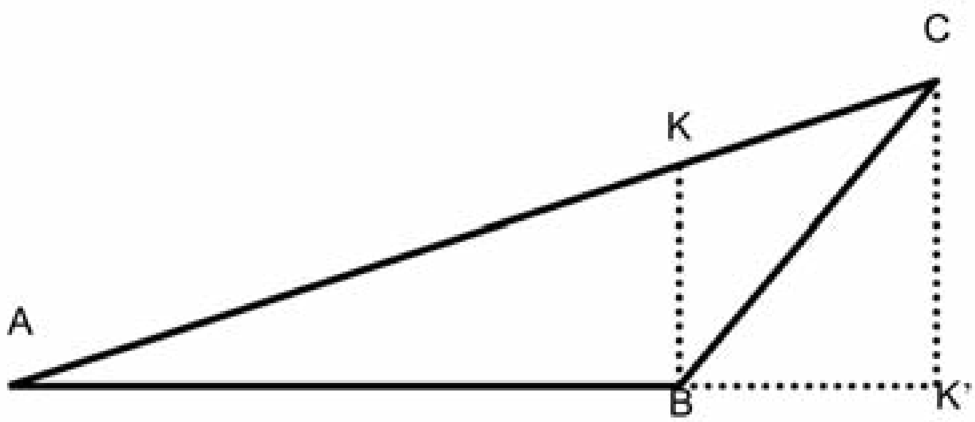

Figure 5.9: The triangle (represented by the continuous lines) that Piaget, Inhelder and Szeminska asked children to copy, with the vertical and horizontal (dotted) lines which some of the children created to help them to solve this problem.

take the form of angular changes e.g. L90 is an instruction for the turtle to make a 90° turn to the left. Since the turtle's movements leave a trace, children effectively draw shapes by writing these programmes.

There is evidence that experience with Logo does have an effect on children's learning about angles, and this in turn supports the idea that representations of movement might be one effective way of teaching this aspect of geometry. Noss (1988) gave a group of 8- to 11-year-old children, some of whom had attended Logo classes over a whole school year, a series of problems involving angles. On the whole the children who had been to the Logo classes solved these angular problems more successfully than those who had not. The relative success of the Logo group was particularly marked in a task in which the children had to compare the size of the turn that people would have to make at different points along a path, and this is not surprising since this specific task resonated with the instructions that children make when determining the direction of the turtle's movements (see Figure 5.10).

However, the Logo group also did better than the comparison in more static angular problems. This is an interesting result because it suggests that the children may have generalised what they learned about angles and movement to other angular tasks which involve no movement at all. We need more research to be sure of this conclusion and, as far as studies of the effects of Logo and other computer-based programmes are concerned, we need studies in which pre-tests are given before the children go through these programmes as well as post-tests that follow these classes.

Children do not just learn about single angles in isolation from each other. In fact, to us, the most interesting question in this area is about their learning of the relations between different angles. These relations are a basic part of geometry lessons: pupils learn quite early on in these lessons that, for example, when two straight lines intersect opposite angles are equal and that alternate angles in a Z-shape figure are equal also, but how easily this knowledge comes to them and how effectively they use it to solve geometric problems are interesting but unanswered questions (at any rate, unanswered by psychologists). Some interesting educational research by Gal and Vinner (1997) on 14-year-old students' reaction to perpendicular lines suggests that they have some difficulties in understanding the relation between the angles made by intersecting lines when the lines are perpendicular to each other. Many of the students did not realise at first that if one of the four angles made by two intersecting lines is a right angle the other three must be so as well. The underlying reason for this difficulty needs investigation.

You are walking along this path. You start at point A and you finish at point G.

- At which point would you have to turn most?
- At which point would you have to turn least?

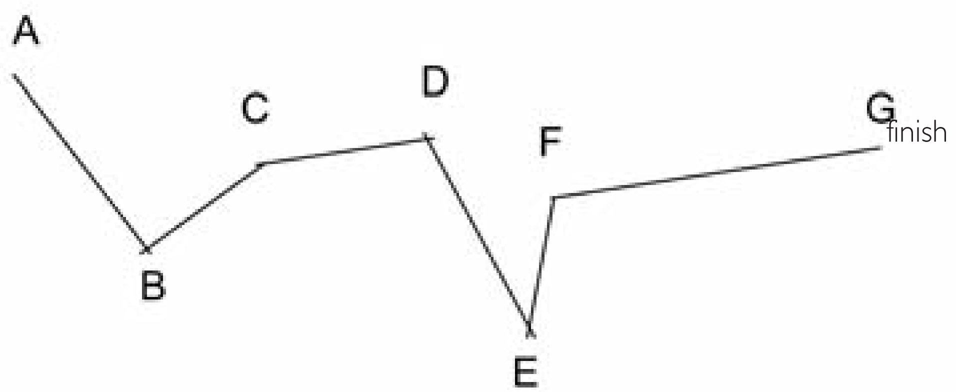

Figure 5.10: The judgement about relative amount of turns in Noss's study.

Summary

1 Although young children are aware of orientation, they seem to know little about angle (the relation between orientations) when they start on geometry.

2 The concept of angle is an abstract one that cuts across very different contexts. This is difficult for children to understand at first.

3 There is some evidence, mainly from work on Logo, that children can learn about angle through movement.

4 Children's understanding of the relation between angles within figures (e.g. when straight lines intersect, opposite angles are equivalent) is a basic part of geometry lessons, but there seems to be no research on their understanding of this kind of relation.

Spatial frameworks

Horizontal and vertical lines

Most children's formal introduction to geometry is a Euclidean one. Children are taught about straight lines, perpendicular lines, and parallel lines, and they learn how a quite complex system of geometry can be derived from a set of simple, comprehensible, axioms. It is an exercise in logic, and it must, for most children, be their first experience of a formal and explicit account of two-dimensional space. The principal feature of this account is the relation between lines such as parallel and perpendicular and intersecting lines.

These fundamental spatial relations are probably quite familiar, but in an implicit way, to the seven- and eight-year-old children when they start classes in geometry. In spatial environments, and particularly in 'carpentered' environments, there are obvious horizontal and vertical lines and surfaces, and these are at right angles (perpendicular) to each other. We reviewed the evidence on children's awareness of these spatial relations in an earlier part of this paper, when we reached the following two conclusions.

1 Although quite young children can relate the orientation of lines to stable background features and often rely on this relation to remember orientations, they do not always do this when it would help them to do so. Thus, many young children and some adults too (Howard, 1978) do not recognise that the level of liquid is parallel to horizontal features of the environment like a table top.

2 Part of the difficulty that they have in Piaget's horizontality and verticality tasks is that these depend on children being able to represent acute and obtuse angles (i.e. non-perpendicular lines). They tend to do this inaccurately, representing the line that they draw as closer to perpendicular than it actually is. This bias towards the perpendicular may get in the way of children's representation of angle.

The role of horizontal and vertical axes in the Cartesian system

The Euclidean framework makes it possible to pinpoint any position in a two-dimensional plane. We owe this insight to René Descartes, the 17th century French mathematician and philosopher, who was interested in linking Euclid's notions with algebra. Descartes devised an elegant way of plotting positions by representing them in terms of their position along two axes in a two dimensional plane. In his system one axis was vertical and the other horizontal, and so the two axes were perpendicular to each other. Descartes pointed out that all that you need to know in order to find a particular point in two-dimensional space is its position along each of these two axes. With this information you can plot the point by extrapolating an imaginary straight perpendicular line from each axis. The point at which these two lines intersect is the position in question. Figure 5.11 shows two axes, *x* and *y*, and points which are expressed as positions on these axes.

This simple idea has had a huge impact on science and technology and on all our daily lives: for example, we rely on Cartesian co-ordinates to interpret maps, graphs and block diagrams. The Cartesian co-ordinate system is a good example of a cultural tool (Vygotsky, 1978) that has transformed all our intellectual lives.

To understand and to use the Cartesian system to plot positions in two-dimensional space, one has to be able to extrapolate two imaginary perpendicular straight lines, and to co-ordinate the two in order to work out where they intersect. Is this a difficult or even an impossible barrier for young children? Teachers certainly need the answer to this question because children are introduced to graphs and block diagrams in primary school, and as we have noted these mathematical representations depend on the use of Cartesian co-ordinates.

Earlier in this section we mentioned that, in social contexts, very young children do extrapolate imaginary straight lines. They follow their mother's line of sight in order, apparently, to look at whatever it is that is attracting her attention at the time (Butterworth, 1990). If children can extrapolate straight lines in three-dimensional space, we can quite reasonably expect them to be able to do so in two-dimensional space as well. The Cartesian requirement that these extrapolated lines are perpendicular to their baselines should not be a problem either, since, as we have seen already, children usually find it easier to create perpendicular than non-perpendicular lines. The only requirement that this leaves is the ability to work out where the paths of the two imaginary straight lines intersect.

A study by Somerville and Bryant (1985) established that children as young as six years usually have this ability. In the most complex task in this study, young children were shown a fairly large square space on a screen and 16 positions were clearly marked within this space, sometimes arranged in a regular grid and sometimes less regularly than that. On the edge of the square waited two characters, each just about to set off across the square. One of these characters was standing on a vertical edge (the right or left side of the square) and the other on a horizontal edge of the square. The children were told that both characters could only walk in the direction they were facing (each character had a rather prominent nose to mark this direction, which was perpendicular to the departure line, clear), and that the two would eventually meet at one particular position in the square. It was the child's task to say which position that would be.

The task was slightly easier when the choices were arranged in a grid than when the arrangement was irregular, but in both tasks all the children chose the right position most of the time. The individual children's choices were compared to chance (if a child followed just one extrapolated line instead of co-ordinating both, he or she would be right by chance 25% of the time) and it turned out that the number of correct decisions made by every individual child was significantly above chance. Thus, all of these 6-year-old children were able to plot the intersection of two extrapolated, imaginary straight lines, which means that they were well equipped to understand Cartesian co-ordinates.

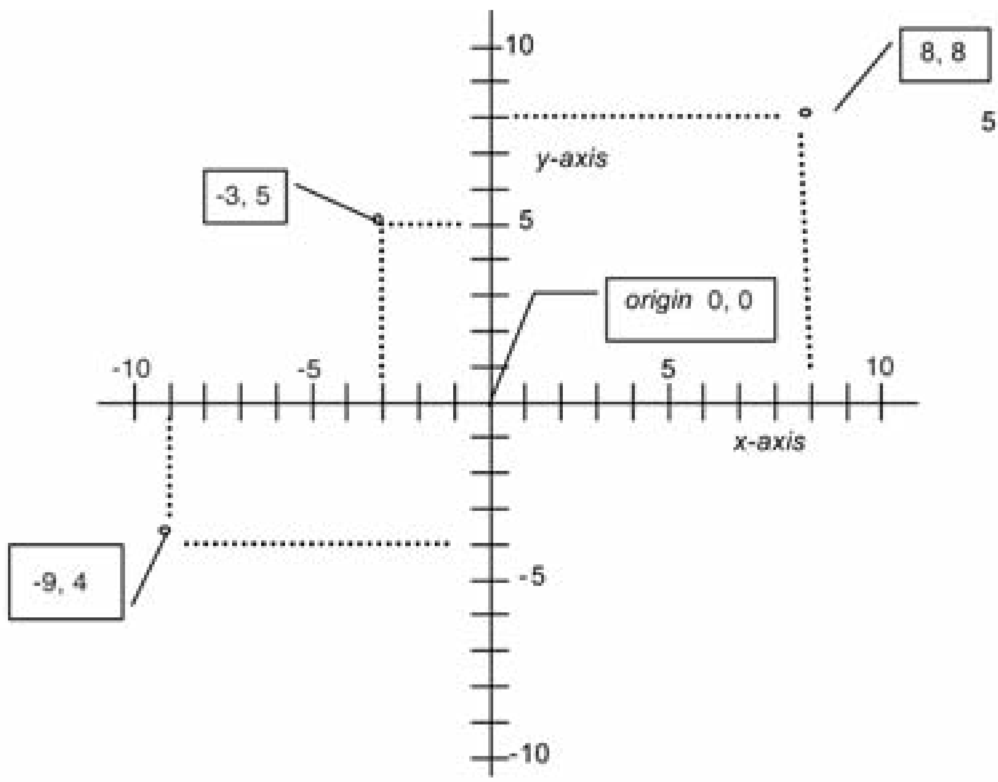

Figure 5.11: Descartes' co-ordinates: three points (8, 8), (-3, 5) and (-9, -4) are plotted by their positions on the x- and y-axes

Piaget *et al.* (1960) were less optimistic about children's grasp of co-ordinates, which they tested with a copying task. They gave each child two rectangular sheets of paper, one with a small circle on it and the other a complete blank. They also provided the children with a pencil and a ruler and strips of paper, and then they asked them to put a circle on the blank sheet in exactly the same position as the circle on the other sheet. Piaget *et al.*'s question was whether any of the children would use a co-ordinate system to plot the position of the existing circle and would then use the co-ordinates to position the circle that they had to draw on the other sheet. This was a difficult task, which children of six and seven years tended to fail. Most of them tried to put the new circle in the right place simply by looking from one piece of paper to the other, and this was a most unsuccessful strategy.

There is no conflict between the success of all the children in the Somerville and Bryant study, and the grave difficulties of children of the same age in the Piaget *et al.* task. In the Piaget *et al.* task the children had to decide that co-ordinates were needed and then had to measure in order to establish the appropriate position on each axis. In the Somerville and Bryant study, the co-ordinates were given and all that the child had to do was to use them in order to find the point where the two extrapolated lines met. So, six- and seven-year-old children can establish a position given the co-ordinates but often cannot set up these co-ordinates in the first place.

Some older children in Piaget *et al.*'s study (all the successful children given as examples in the book were eight- or-nine-years-old) did apparently spontaneously use co-ordinates. It seems unlikely to us that these children managed to invent the Cartesian system for themselves. How could eight- and nine-year-old children come up, in one experimental session, with an idea for which mankind had had to wait till Descartes had his brilliant insight in the middle of the 17th century?
A more plausible reason for these children's success is that, being among the older children in Piaget *et al.*'s sample, they had been taught about the use of Cartesian co-ordinates in maps or graphs already.

It appears that this success is not universal. Many children who have been taught about Cartesian co-ordinates fail to take advantage of them or to use them properly. Sarama, Clements, Swaminathan, McMillen and Gomez (2003) studied a group of nine-year-old children while they were being given intensive instruction, which the researchers themselves designed. In the tasks that they gave to the children Sarama *et al.* represented the x and y co-ordinates as numbers, which was a good thing to include because it is a fundamental part of the Cartesian system, and they also imposed a rectangular grid on many of the spaces that they gave the children to work with. Thus, one set of materials was a rectangular grid, presented as a map of a grid-plan city with 'streets' as the vertical and 'avenues' as the horizontal lines.

The children were given various tasks before, during and after the instruction. Some of these involved relating locations to x and y co-ordinates: the children had to locate positions given the co-ordinates and also to work out the co-ordinates of particular positions. Sarama *et al.* reported that most of the children learned about this relation quickly and well, as one might expect given their evident ability to co-ordinate extrapolated lines in a rectangular context (Somerville and Bryant, 1985). However, when they had to co-ordinate information about two or more locations, they were in greater difficulty. For example, some children found it hard to work out the distance between two locations in the grid-like city, because they thought that the number of turns in a path affected its distance, and some did not realise that the numerical differences in the x and y co-ordinate addresses between the two locations represents the distance between them.

Thus, the children understood how to find two locations, given their coordinates, but struggled with the idea that a comparison between the two pairs of co-ordinates told them about the spatial relation between these locations. Further observations showed that the problem that some of the children had in working out the relations between two co-ordinate pairs was created by a certain tension between absolute and relative information. The two co-ordinate pairs 10,30 and 5,0 represent two absolute positions: however, some children, who were given first 10,30 and then 5,0 and asked to work out a path between the two, decided that 5,0 represented the difference between the first and the second location and plotted a location five blocks to the right of 10,30. They treated absolute information about the second position as relative information about the difference between the two positions. However, most of the children in this ingenious and important study seem to have overcome this difficulty during the period of instruction, and to have learned reasonably well that co-ordinate pairs represent the relation between positions as well as the absolute positions themselves.

Finally, we should consider children's understanding of the use of Cartesian co-ordinates in graphs. Here, research seems to lead to much the same conclusion as we reached in our discussion of the use of co-ordinates to plot spatial positions. With graphs, too, children find it easy to locate single positions, but often fail to take advantage of the information that graphs provide that is based on the relation between different positions. Bryant and Somerville (1986) gave six- and nine-year-old children a graph that represented a simple linear function. Using much the same technique as in their previous research on spatial co-ordinates, these researchers measured the children's ability to plot a position on *x*-axis given the position on the *y*-axis and vice versa. This was quite an easy task for the children in both age groups, and so the main contribution of the study was to establish that children can co-ordinate extrapolated straight lines in a graph-like task fairly well even before they have had any systematic instruction about graphs.

The function line in a graph is formed from a series of positions, each of which is determined by Cartesian co-ordinates. As in the Sarama *et al.* study, it seems to be hard for children to grasp what the relation between these different positions means. An interesting study by Knuth (2000) showed that American students 'enrolled in 1st-year algebra' (Knuth does not say how old these students were) are much more likely to express linear functions as equations than graphically. We do not yet know the reasons for this preference

Summary

1 Cartesian co-ordinates seem to pose no basic intellectual difficulty for young children. They are able to extrapolate imaginary straight lines that are perpendicular to horizontal and vertical axes and to work out where these imaginary lines would meet in maps and in graphs.

2 However, it is harder for children to work out the relation between two or more positions that are plotted in this way, either in an map-like or in a graph-like task.

3 Students prefer expressing functions as equations to representing them graphically.

4 Thus, although children have the basic abilities to understand and use co-ordinates well, there seem to be obstacles that prevent them using these abilities in tasks which involve two or more plotted positions. We need research on how to teach children to surmount these obstacles.

Categorising, composing and decomposing shapes

We have chosen to end this chapter on learning about space and geometry with the question of children's ability to analyse and categorise shapes, but we could just as easily have started the section with this topic, because children are in many ways experts on shape from a very early age. They are born, apparently, with the ability to distinguish and remember abstract, geometric shapes, like squares, triangles, and circles, and with the capacity to recognise such shapes as constant even when they see them from different angles on different occasions so that the shape of the retinal image that they make varies quite radically over time. We left shape to the end because much of the learning that we have discussed already, about measurement and angle and spatial co-ordinates, undoubtedly affects and changes schoolchildren's understanding of shape.

There is nearly complete agreement among those who study mathematics education that children's knowledge about shapes undergoes a series of radical changes during their time at school. Different theories propose different changes but many of these apparent disagreements are really only semantic ones. Most claim, though in different terms, that school-children start by being able to distinguish and classify shapes in a perceptual and implicit way and eventually acquire the ability to analyse the properties of shapes conceptually and explicitly.

The model developed by the Dutch educationalist van Hiele (1986) is currently the best known theoretical account of this kind of learning. This is a good example of what we called a 'pragmatic theory' in our opening paper. Van Hiele claimed that children have to take a sequence of steps in a fixed order in their geometric learning about shape. There are five such steps in van Hiele's scheme, but he agreed that not all children get to the end of this sequence:

Level 1: Visualization/recognition Students recognise and learn to name certain geometric shapes but are usually only aware of shapes as a wholes, and not of their properties or of their components.

Level 2: Descriptive/analytic Students begin to recognise shapes by their properties.

Level 3: Abstract/relational Students begin to form definitions of shapes based on their common properties, and to understand some proofs.

Level 4: Formal deduction Students understand the significance of deduction as a way of establishing geometric theory *within* an axiomatic system, and comprehend the interrelationships and roles of axioms, definitions, theorems, and formal proof.

Level 5: Rigour Students can themselves reason formally about different geometric systems.

There have been several attempts to elaborate and refine this system. For example, Clements and Sarama (2007 b) argue that it would better to rename the Visual/recognition stage as Syncretic, given its limitations. Another development was Guttierez's (1992) sustained attempt to extend the system to 3-D figures as well as 2-D ones. However, although van Hiele's steps provide us with a useful and interesting way of assessing improvements in children's understanding of geometry, they are descriptive. The theory tells us about changes in what children do and do not understand, but not about the underlying cognitive basis for this understanding, nor about the reasons that cause children to move from one level to the next. We shall turn now to what is known and what needs to be known about these cognitive bases.

Composing and decomposing

If van Hiele is right, one of the most basic changes in children's analysis of shape is the realisation that shapes, and particularly complex shapes, can be decomposed into smaller shapes. We have already discussed one of the reasons why children need to be able to compose and decompose shapes, which is that it is an essential part of understanding the measurement of the area of different shapes. Children, as we have seen, must learn, for example, that you can compose a parallelogram by putting together two identical triangles (and thus that you can decompose any parallelogram into two identical triangles) in order to understand how to measure the area of triangles. As far as we know, there has been no direct research on the relationship between children's ability to compose and decompose shapes and their understanding of the rules for measuring simple geometric figures, though such research would be easy to do.

The Hart *et al.* (1985) study included two items which dealt with shapes that were decomposed into two parts and these parts were then re-arranged. In one item the re-arranged parts were a rectangle and two triangles, which are simple and familiar geometric shapes, and in the other they were unfamiliar and more complex shapes. In both cases the children were asked about the effect of the re-arrangement on the figure's total area: was the new figure's area bigger or smaller or the same as the area of the original one? A large proportion of the 11- to 14-year-old students in the study (over 80% in each group) gave the correct answer to the first of these two questions but the second was far harder: only 60% of the 11-year-old group understood that the area was the same after the rearrangement of parts as before it. There are two reasons for being surprised at this last result. The first is that it is hard to see why there was such a large difference in the difficulty of the two problems when the logic for solving both was exactly the same. The second is that the mistakes which the children made in such abundance with the harder problem are in effect conservation failures, and yet these children are well beyond the age when conservation of area should pose any difficulty for them. This needs further study.

The insight that understanding composition and decomposition may be a basic part of children's learning about shapes has been investigated in another way. Clements, Wilson and Sarama (2004) looked at a group of three- to seven-year-old children's ability to assemble target patterns, like the figure of a man, by assembling the right component wooden shapes. This interesting study produced evidence of some sharp developmental changes: the younger children tended to create the figures bit by bit whereas the older children tended to create units made out of several bits (an arm unit for example) and the oldest dealt in units made out of other units. The next step in this research should be to find out whether there is a link between this development and the eventual progress that children make in learning about geometry. Once again we have to make a plea for longitudinal studies (which are far too rare in research on children's geometry) and intervention studies as well.

Summary

1 Although young school children are already very familiar with shapes, they have some difficulty with the idea of decomposing these into parts, e.g. a parallelogram decomposed into two congruent triangles or an isosceles triangle decomposed into two right-angle triangles, and also with the inverse process of composing new shapes by combining two or more shapes to make a different shape.

2 The barrier here may be that these are unusual tasks for children who might learn how to carry them out easily given the right experience. This is a subject for future research.

Transforming shapes: enlargement, rotation and reflection

We have already stressed the demands that measurement of length and angle make on children's imagination, and the same holds for their learning about the basic transformations of shapes – translation, enlargement, rotation and reflection. Children have to learn to imagine how shapes would change as a result of each of these transformations and we know that this is not always easy. The work by Hart (1981) and her colleagues on children's solutions to reflection and rotation problems suggests that these transformations are not always easy for children to work out. They report that there is a great deal of change in between the ages of 11 and 16 years in students' understanding of what changes and what stays constant as a result of these two kinds of transformation.

One striking pattern reported by this research group was that the younger children in the group being studied were much more successful with rotation and reflection problems that involved horizontal and vertical figures than with sloping figures: this result may be related to the evidence, mentioned earlier, that much younger children discriminate and remember horizontal and vertical lines much better than sloping ones.

Psychologists, in contrast to educationalists, have not thrown a great deal light on children's learning about transformations, even though some research on perceptual development has come close to doing so. They have shown that children remember symmetrical figures better than asymmetrical figures (Bornstein, Ferdinandsen and Gross, 1981), and there are observations of pre-school children spontaneously constructing symmetrical figures in informal play (Seo and Ginsburg, 2004). However, the bulk of the psychological work on rotation and reflection has treated these transformations in a negative sense. The researchers (Bomba, 1984; Quinn, Siqueland and Bomba, 1985; Bryant, 1969,1974) were concerned with children's confusions between symmetrical, mirror-image figures (usually reflections around a vertical axis): they studied the development of children's ability to tell symmetrical figures apart, not to understand the relation between them. Here is another bridge still to be crossed between psychology and education.

Enlargement raises some interesting issues about children's geometric understanding. We know of no direct research on teaching children or on children learning about this transformation, at any rate in the geometrical sense of shapes being enlarged by a designated scale factor. These scale factors of course directly affect the perimeter of the shapes: the lengths of the sides of, for example, a right-angle triangle enlarged by a factor of 2 are twice those of the original triangle. But, of course, the relation between the areas of the two triangles is different: the area of the larger triangle is 4 times that of the smaller one.

There is a danger that some children, and even some adults, might confuse these different kinds of relation between two shapes, one of which is an enlargement of the other. Piaget *et al.* (1960) showed children a 3 cm x 3 cm square which they said represented a field with just enough grass for one cow, and then they asked each child to draw a larger field of the same shape which would produce enough for two cows. Since this area measured 9 cm^2 the new square would have to have an area of 18 $cm.^2$ and therefore sides of roughly 4.24 cm. since 4.24 is very nearly the square root of 18. In this study most of the children under the age of 10 years either acted quite unsystematically or made the mistake of doubling the sides of the original square in the new figure that they drew, which meant that their new square (an enlargement of the original square by a scale factor of two) had 6 cm sides and an area of around 36 cm^2 which is actually 4 times the area of the first square. Older children, however, recognised the problem as a multiplicative one and calculated each of the two squares' areas my multiplying its height by its width. The younger children's difficulties echoed those of the slave who, in Plato's Meno, was lucky enough to be instructed about measuring area by Socrates himself.

The widespread existence of this apparently prevailing belief that doubling the length of the sides of a shape will double its area as well was recently confirmed by a team of psychologists in Belgium ((De Bock, Verschaffel and Janssens, 1998, 2002; De Bock, Van Dooren, Janssens and Verschaffel, 2002; De Bock, Verschaffel, Janssens, Van Dooren and Claes, 2003). There could be an educational conflict here between teaching children about scale factors on the one hand and about proportional changes in area on the other. It is possible that the misconceptions expressed by students in the studies by Piaget *et al.* (1960) and also by the Belgian team may actually have been the result of confusion between the use of scale factors in drawing and effect of doubling the sides of figures. In Paper 4 we discussed in detail the difficulties of the studies carried out by the Belgian team but we still need to find out, by research, whether scale drawing does provoke this confusion. It would be easy to do such research.

Summary

1 Understanding, and being able to work out, the familiar transformations of reflection, rotation and enlargement are a basic part of the geometry that children learn at school. They are another instance of the importance of grasping the relations between shapes in learning geometry.

2 Psychology tells us little about children's understanding of these relations, though it would be easy enough for psychologists to do empirical research on this basic topic. The reason for psychologists' neglect of transformations is that they have concentrated on children distinguishing between shapes rather than on their ability to work out the relations between them.

3 There is the possibility of a clash between learning about scale factors in enlargement and about the measurement of area. This should be investigated.

General conclusions on learning geometry

1 Geometry is about spatial relations.

2 Children have become highly familiar with some of these relations, long before they learn about them formally in geometry classes: others are new to them.

3 In the case of the spatial relations that they know about already, like length, orientation and position relations, the new thing that children have to learn is to represent them numerically. The process of making these numerical representations is not always straightforward.

4 Representing length in standard units depends on children using one-to-one correspondence between the units on the ruler and imagined units on the line being measured. This may seem to be easy to do to adults, but some children find it difficult to understand.

5 Representing the area of rectangles in standard units depends on children understanding two things: (a) why they have to multiply the base with the height in centimetres (b) why this multiplication produces a measure in different units, square centimetres. Both ideas are difficult ones for young children.

6 Understanding how to measure parallelograms and triangles depends on children learning about the relation between these shapes and rectangles. Learning about the relations between shapes is a significant part of learning about geometry and deserves attention in research done by psychologists.

7 The idea of angle seems to be new to most children at the time that they begin to learn about geometry. Research suggests that it takes children some time to a form an abstract concept of angle that cuts across different contexts. More research is needed on children's understanding of the relations between angles in particular figures.

8 Children seem well-placed to learn about the system of Cartesian co-ordinates since they are, on the whole, able to extrapolate imaginary perpendicular lines from horizontal and vertical co-ordinates and to work out where they intersect. They do, however, often find co-ordinate tasks which involve plotting and working out the relationship between two or more positions quite difficult. We need research on the reasons for this particular difficulty.

9 There is a serious problem about the quality of the research that psychologists have done on children learning geometry. Although psychologists have carried out good work on children's spatial understanding, they have done very little to extend this work to deal with formal learning about the mathematics of space. There is a special need for longitudinal studies, combined with intervention studies, of the link between informal spatial knowledge and success in learning geometry.

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