Kinematics in One Dimension

Introduction

Motion: change in position or orientation with respect to time.

Vectors have given us some basic ideas about how to describe the position of objects in the universe/ Now, we'll continue by extending those ideas to account for changes in that position. Of course the world would be awfully boring if the position of everything was constant.

Different Types of Motion We'll look at:

Linear motion involves the change in position of an object in one direction only. An example would be a train on a straight section of the track. The change in position is only in the horizontal direction.

Projectile motion occurs when objects are launched in the gravitational field near the earths surface. They experience motion in both the horizontal and the vertical directions.

Circular motion occurs in a few specific cases when an object travels in a perfect circle. Some special math can be used in these cases.

Rotational motion implies that the body in question is rotating around an axis. The axis doesn't necessary need to pass through the object.

... or a combination of them.

Dimensionality in physics

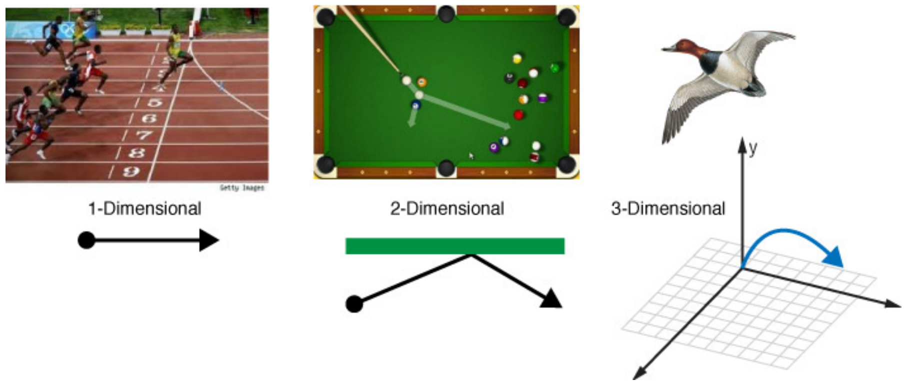


Fig. 1

Prelude to advanced physics and engineering: Later on, you'll have to expand your notion of dimensions a bit. It won't simply mean straight or curvy, but will instead be used to describe the *degrees of freedom* in a system. For example, an orbiting body, though it moves in a circle which requires x and y values to describe, can also be described by considering the radius and the angle of rotation instead. This is just another coordinate system: polar coordinates (usually: r and θ). If we describe the orbiting planet in this system, and say, it's going around in a perfect circle, then the r value doesn't change and the θ value become the only *dimension* of interest. Let's hold off on this approach for now, but when it comes back later on, welcome it with open arms because it allows for much more powerful and simple analysis of systems.

One dimensional kinematics

For the case of 1-dimensional motion, we'll only consider a change of position in one direction.

It could be any of the three coordinate axes.

Just a description of the motion, without attempting to analyze the cause. To describe motion we need:

1. Coordinate System (origin, orientation, scale)
2. the object which is moving

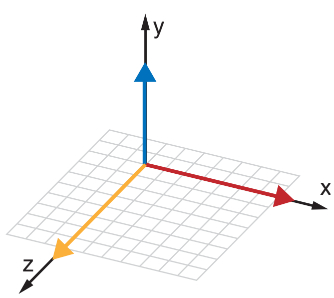


Fig. 2 A coordinate system

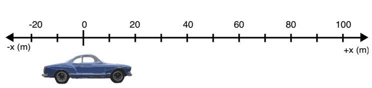


Fig. 3 A 1-d coordinate system

1d kinematics will be our starting point. It is the most straightforward and easiest mathematically to deal with since only one position variable will be changing with respect to time.

Particle model

We'll need to use an abstraction:

All real world objects take up space. We'll assume that they *don't*. In other words, things like cars, cats, and ducks are just point-like particles.

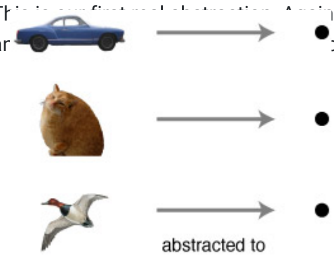


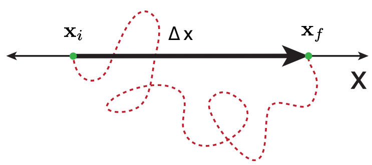
Fig. 4 The particle model

Th… , since we are trying to predict everything, we would like to figure out the rules that describe how ar… or example. If we asked a question like "when does the C train enter 59th street station?", a natural follow-up would be "well, do you mean the front of the train, or the middle of the train, or the end of the train? Each of these answers might be different by a few seconds.

How do we deal with this? By considering the train to be a 'point', we can neglect the actual length of the train and focus on what's more interesting: how the train moves.

The goal is to find the underlying physics that describes *all* trains. Once we do that, then we can improve our model by including information about the length of the individual train we are interested in.

Displacement Vector



To quantify the motion, we'll start by defining the displacement vector.

$$\Delta \mathrm{x} = \mathrm{x} - \mathrm{x}_0$$

In the case of our wandering bug, this would be the difference between the final position and the initial position.

This figure shows the displacement vector $\Delta \mathrm{x}$. This might be different than the distance traveled by the bug (shown in the dotted line).

Note on Notation!

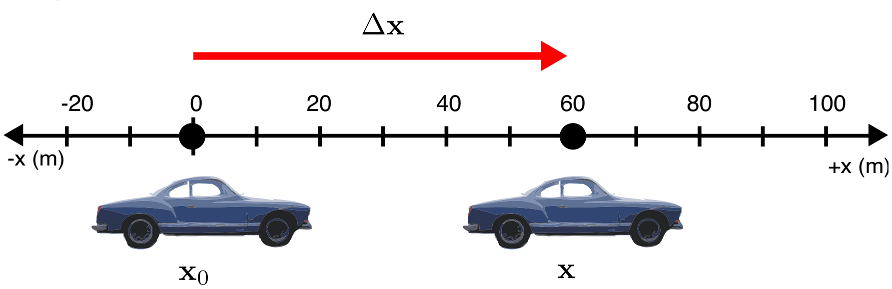
x_f is the same thing as x

x_i is the same thing as x_0

When describing motions, we usually have an initial position and a final position. We can call these x_i and x_f respectively, when we do our algebra.

Or another way of writing these quantities is to say our initial position is x_0 and our final position is just x. This is a slightly more general way of writing things.

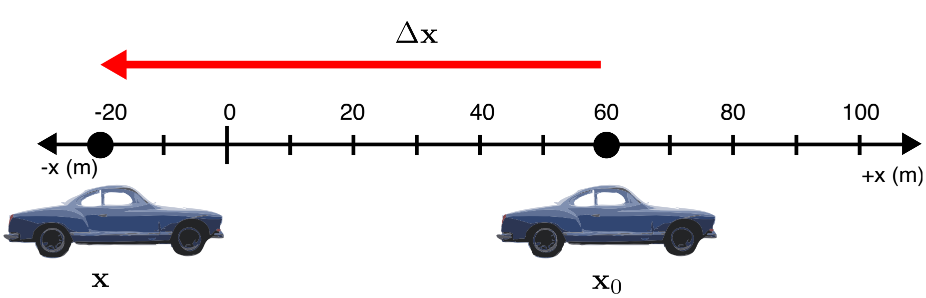
Displacement in 1-D



Here's a car that moves from x_0 to x creating a displacement vector of:

$$\Delta \mathrm{x} = \mathrm{x} - \mathrm{x}_0 = 60 \mathrm{~m} - 0 \mathrm{~m} = 60 \mathrm{~m}$$

The car then reverses to $x = -20$.



The leads to a displacement vector of $\Delta \mathrm{x} = -80 \mathrm{m}$.

About notation. Δx ("delta x") refers to the change in x. That is, difference between a final and initial value:

$$\Delta x = x - x_0$$

Or, in words, the final x position minus the original x position is equal to the change in x.

Distance Traveled

To get the *distance traveled*, we just neet to take the magnitude of the displacement during a certain motion.

$$|\Delta \mathrm{x}| = \text{Distance Traveled}$$

This equation will only be true if the displacement is always in the same direction. If however, the displacement vector were to change direction during a trip, the the distance traveled might not be equal to the total displacement. For example, if you walk 100 feet forward, then turn around and walk 50 backwards. You displacement from the initial to final position will only be 50 feet, but you will have walked a total of 150 feet.

Speed and Velocity

$$\boxed{\text{Average Speed} \equiv \frac{\text{Distance in a given time}}{\text{Elapsed time}}}$$

The 'elapsed time' is determined in the same way as the distance: $\Delta t = t - t_0$.

Again, t_0 is the starting time, and t is the final time.

Example Problem #1:

Taking the A train between 59th and 125th takes about 8 minutes. The C, which is a local, takes 12 minutes (on a good day). Find the average speed for both of these trips.

...with a direction

Calculating the average speed didn't tell us anything about the direction of travel. For this, we'll need average velocity.

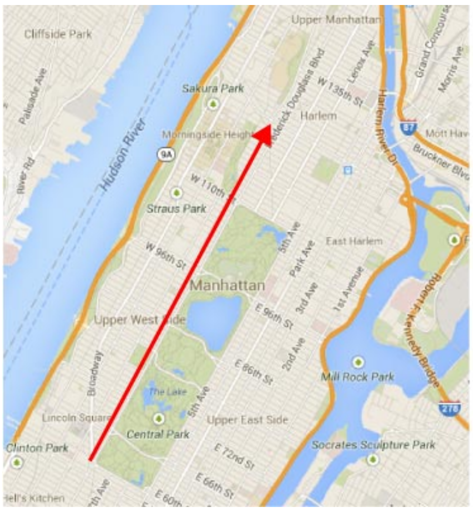
$$\boxed{\text{Average Velocity} \equiv \frac{\text{Displacement}}{\text{Elapsed time}}}$$

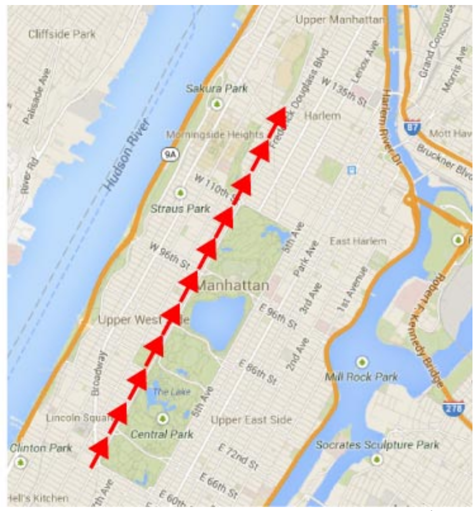
In mathematical terms:

$$\boxed{\bar{\mathrm{v}} \equiv \frac{\mathrm{x}-\mathrm{x}_0}{t-t_0} = \frac{\Delta \mathrm{x}}{\Delta t}}$$

(SI units of average velocity are m/s)
In one-dimension, velocity can either be in the positive or negative direction.

Thinking about the A train, it's clear that its speed and velocity stayed essentially constant between 59th and 125th ideally). However, the C train had to start and stop at 7 stations. To quantify, this difference in motion, we'll need to introduce the concept of *instantaneous velocity*.





If we imagine making many measurements of the velocity over the course of the travel, by reducing the Δx we are considering, then we can begin to see how we can more accurately assess the motion of the train.

The concept of instantaneous velocity involves considering an *infinitesimally small* section of the motion:

$$\mathrm{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathrm{x}}{\Delta t} = \frac{dx}{dt}$$

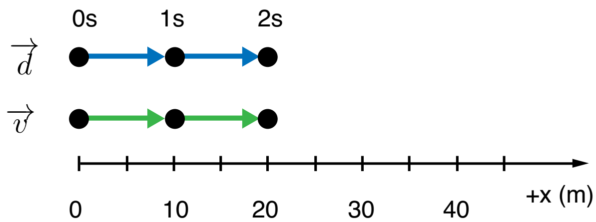
This will enable us to talk about the velocity at a particle's specific position or time rather than for an entire trip.

In general, this is what we'll mean when we say 'velocity' or 'speed'.

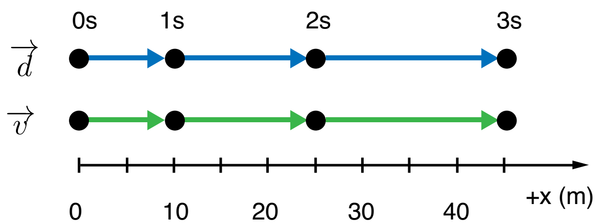
Change in velocity.

Naturally, in order to begin moving, an object must *change its velocity*.

Here's a graph of a bicyclist riding at a constant velocity. (In this case it's 10 m/s)



Now, here's a graph of the same bicyclist riding and changing his velocity during the motion



In the upper motion graph, notice how the length of the displacement vector $\vec{d}$ is the same at each interval in time. Meaning, that after 1 second has passed, the displacement is 10 m, after another second passes, another 10 meters displacement has occurred, making the total displacement equal to 20 m. This is motion at a constant velocity. This also apparent in the length of the velocity vectors at each point. They are always the same.

In the bottom graph, the displacement, and velocity vectors, change each time they are measured. This is representative of motion with non-constant velocity. The velocity is changing as time moves on.

Acceleration

This change in velocity we'll call acceleration, and we can define it in a very similar way to our definition of velocity:

$$\bar{\mathrm{a}} = \frac{\mathrm{v} - \mathrm{v}_0}{t - t_0} = \frac{\Delta \mathrm{v}}{\Delta t}$$

Again, in this case we're talking about *average* acceleration.

> **Example Problem #2:**
>
> At $t = 0$, the A train is at rest at 59th street. 5 seconds later, it's traveling north at 19 meters per second. What is the average acceleration during this time interval?

If we considered the same very small change in time, the infinitesimal change, then we could talk about instantaneous acceleration

$$\mathrm{a} = \lim_{\Delta t \to 0} \frac{\Delta \mathrm{v}}{\Delta t} = \frac{dv}{dt}$$

The SI units of acceleration are meters per second per second, or ms^{-2}. That's probably a little bit of a weird unit, but, it makes sense to think about like this:

$$\frac{\left(\frac{m}{s}\right)}{s} \text{ or } \frac{vel}{s}$$

Acceleration, the math.

To quantify to the acceleration of a moving body, say this car, we'll need to know its initial and final velocities

The car has a build in speedometer, so we can look at that to get the speed, and if we don't change direction, then the velocity will be always pointed in the same direction.

For this case of a car starting from rest, and then increasing velocity, the acceleration will be a positive quantity.

$$\bar{\mathrm{a}} = \frac{\mathrm{v} - \mathrm{v}_0}{t - t_0} = \frac{20\mathrm{mph} - 0\mathrm{mph}}{2\mathrm{s} - 0\mathrm{s}} = \frac{20\mathrm{mph}}{2\mathrm{s}}$$

$$\bar{\mathrm{a}} = \frac{9\mathrm{m/s}}{2\mathrm{s}} = +4.5\mathrm{ms}^{-2}$$

Slowing down

What if we ask about a car slowing down. Now, our $\mathrm{v}_0 = +9m/s$ while $\mathrm{v} = 0$.

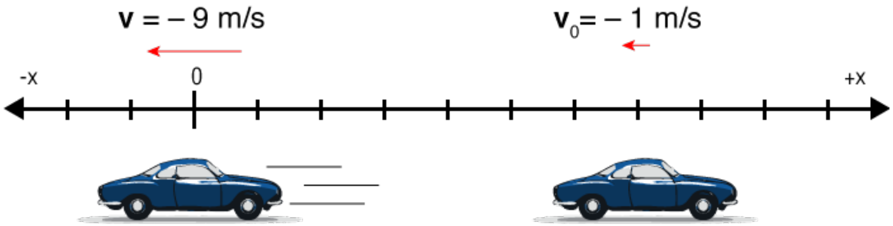
Now the math looks like this:

$$\bar{\mathrm{a}} = \frac{\mathrm{v} - \mathrm{v}_0}{t - t_0} = \frac{0\mathrm{m/s} - 9\mathrm{m/s}}{2\mathrm{s} - 0\mathrm{s}} = -\frac{9\mathrm{m/s}}{2\mathrm{s}} = -4.5\mathrm{m/s}^2$$

We notice that the acceleration is *negative*.

Acceleration in the negative

What if the car starts accelerating in the negative direction?

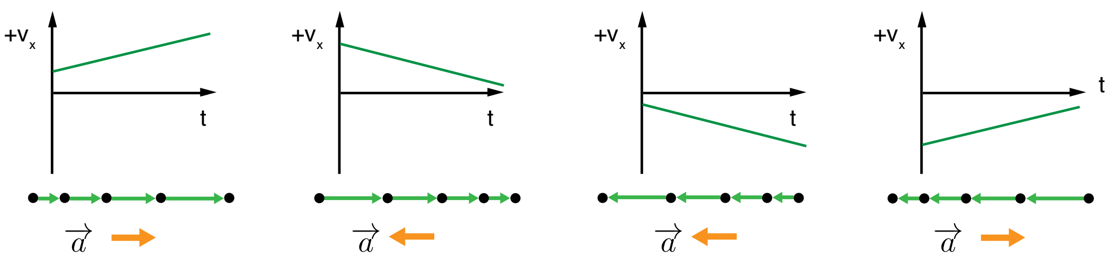


Now, even the speed is *increasing*, the velocity is getting *more negative*.

If we do the math, we'll see that the acceleration vector points in the negative direction.

Summary of acceleration signage.

When the signs of an object’s velocity and acceleration are the same (in same direction), the object is speeding up
When the signs of an object’s velocity and acceleration are opposite (in opposite directions), the object is slowing down and speed decreases



Kinematic equations

$$1.\ \bar{a} = a = \frac{v - v_0}{t} \quad \Rightarrow \quad v = v_0 + at$$

$$2.\ \bar{v} = \frac{x - x_0}{t - t_0} \quad \Rightarrow \quad x - x_0 = \bar{v}t = \frac{1}{2}(v_0 + v)t$$

We can do a lot by rearranging these equations.

Putting v from (1) into (2) will give us:

$$3.\ x - x_0 = v_0 t + \frac{1}{2}at^2$$

or, solving (1) for t, then inserting that into (2) will give us:

$$4.\ v^2 = v_0^2 + 2a(x - x_0)$$

1. $v = v_0 + at$
2. $x = \bar{v}t = \frac{1}{2}(v_0 + v)t$
3. $x = x_0 + v_0 t + \frac{1}{2}at^2$
4. $v^2 = v_0^2 + 2ax$

$$\boxed{v = v_0 + at}$$

Here we have an equation for *velocity* which is changing due to an acceleration, a.

It tells us how fast something will be going (and the direction) if has been accelerated for a time, t.

- It can determine an object's velocity at any time t when we know its initial velocity and its acceleration
- Does not require *or give* any information about position
- Ex: "How fast was the car going after 10 seconds while accelerating from rest at 10 m/s^2"
- Ex: "How long did it take to reach 20 miles per hour"

$$\boxed{x = \bar{v}t = \frac{(v + v_0)t}{2}}$$

This equation will tell us the position of an object based on the initial and final velocities, and the time elapsed.

It does not require knowing, nor will it give you, the acceleration of the object.

- Ex: How far did the duck walk if it took 10 seconds to reach 50 miles per hour under constant acceleration.

$$\boxed{x = x_0 + v_0 t + \frac{at^2}{2}}$$

Gives position at time t in terms of initial velocity and acceleration

- Doesn't require or give final velocity.
- Ex: "How far up did the rocket go?"

$$\boxed{v^2 = v_0^2 + 2ax}$$

Gives velocity at time t in terms of acceleration and position

- Does not require or give any information about the time.
- Ex: "How fast was penny going when it reached the bottom of the well?"

Equations of Motion (1-D)

Things to be aware of:

1. They are *only* for situations where the acceleration is constant.
2. The way we have written them is really just for 1-D motion.

Equation	Missing Variable	Good for finding
$v = v_0 + at$	x	a,t,v
$x = \frac{(v+v_0)t}{2}$	a	x,t,v
$x = x_0 + v_0 t + \frac{at^2}{2}$	v	x,a,t
$v^2 = v_0^2 + 2ax$	t	a,x,v

Solving Problems

1. Diagram: draw a picture
2. Characters: Consider the problem a story. Who are the characters?
3. Find: clearly list symbolically what we're looking for.
4. Solve: state the basic idea behind solution, in a few words (physical principles used, etc.)
5. Assess: does answer make sense?

Example Problem #3:

A taxi is sitting at a red light. The light turns green and the taxi accelerates at 2.5 m/s^2 for 3 seconds. How far does it travel during this time?

Example Problem #4:

A particle is at rest. What acceleration value should we give it so that it will be 2 meters away from its starting position after 0.4 seconds?

Example Problem #5:

A subway train accelerates starting at x = 200 m uniformly until it reaches x = 350 m, at a uniform acceleration value of 0.5 m/s^2.

a. If it had an initial velocity of 0 m/s, what will the duration of this acceleration be?
b. If it had an initial velocity of 8 m/s, what will the duration of this acceleration be?

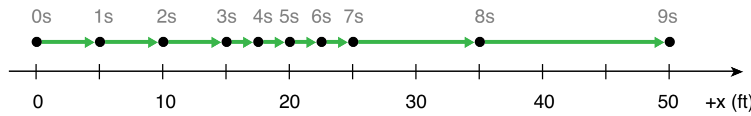
Example Problem #6:

If $x(t) = 4 - 27t + t^3$, find $v(t)$ and $a(t)$. Also, find the time when the velocity is zero.

Example Problem #7:

Traian Vuia, a Romanian Inventor, wanted to reach 17 m/s in order to take off in his flying machine. His plane could accelerate at 2m/s^2. The only runway he had access to was 80 meters long. Will he reach the necessary speed?

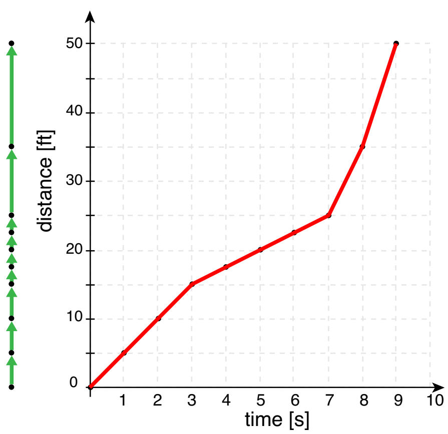
Plotting



Let's look at the motion of a honey badger.

After each second, we note where the honey badger is along the x axis.

t[s]	x[ft]
0	0.00
1	5.00
2	10.0
3	15.0
4	17.5
5	20.0
6	22.5
7	25.0
8	35.0
9	50.0



Derive kinematics using calculus.

We can *derive* nearly all of kinematics (for casses with constant acceleration) by considering the relationships between derivatives and integrals. Let's begin with the definition of acceleration:

$$a = \frac{\Delta v}{\Delta t}$$

If we make the Δv and Δv infinitesimally small, dv and dt, then we can rewrite this as:

$$a = \frac{dv}{dt} \Rightarrow dv = a\ dt$$

Now, we can take the indefinite integral of both sides:

$$\int dv = \int a\ dt$$

Since a is assumed to be constant, we can remove from the integrand. Performing the indefinite integrals:

$$v = at + C_1$$

where C_1 is the constant of integration. To determine the constant C, consider the equation when $t = 0$. This is the 'initial condition', thus the velocity at this point will be the initial velocity: v_0. We therefore obtain:

$$v = v_0 + at$$

by considering just the definition of acceleration and the concept of integration.

We can likewise consider the definition of instantaneous velocity:

$$v = \frac{dx}{dt}$$

A similar operation leads to:

$$\int dx = \int v\ dt$$

Now, we cannot remove v from this integrand since it is not a constant value. However, we just figured out a relation between velocity and time above, so:

$$\int dx = \int (v_0 + at)\ dt$$

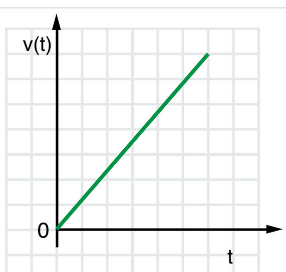
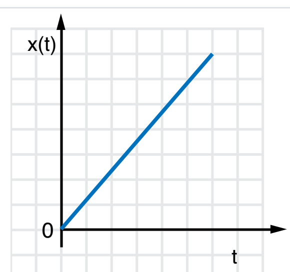
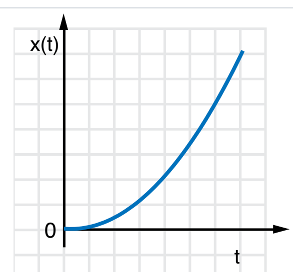
In this case, v_0 and a are both constants. So the indefinite integral can be solved:

$$x = v_0 t + \frac{1}{2}at^2 + C_2$$

Again, we have a constant of integration to solve for: C_2. Let's again consider $t = 0$, i.e. the initial condition. When $t = 0$, the object will be located at the initial x position, x_0. Thus $C_2 = x_0$. Finally, we have an equation for x as a function of time given all the initial conditions of position and velocity:

$$x = x_0 + v_0 t + \frac{1}{2}at^2$$

This is our fundamental quadratic equation that describes the motion of a particle undergoing translation with constant acceleration.

		
$v = v_0 + at$	$x = vt$	$x = v_0 t + \frac{at^2}{2}$
velocity as a function of time: $v(t)$ Acceleration is constant	position as a function of time $x(t)$. (vel. constant, accel = 0)	position as a function of time

Example Problem #8:

A turtle and a rabbit are to have a race. The turtle's average speed is 0.9 m/s. The rabbit's average speed is 9 m/s. The distance from the starting line to the finish line is 1500 m. The rabbit decides to let the turtle run before he starts running to give the turtle a head start. What, approximately, is the maximum time the rabbit can wait before starting to run and still win the race?

Example Problem #9:

A car and a motorcycle are at $x_0 = 0$ at $t = 0$. The car moves at a constant velocity v_0. The motorcycle starts at rest and accelerates with constant acceleration a.

a. Find the t where they meet.
b. Find the position x where they meet.
c. Find the velocity of the motorcycle when they meet.

This problem is asking us to describe the kinematics of the situation in the most general terms possible. There are no numbers given, so we must do everything using symbolic algebra. First, let's make sure we understand the setup. There are two vehicles: a car and a motorcycle. They can be considered particles meaning they are point like. The action starts at t = 0. At this time, both vehicles are located at the origin. The motorcycle is stationary, but the car has a velocity, v_0. (*v_0 is just a symbol that could be a number, like 10 m/s or 34.3 mph. But we leave it as a symbol so that we can solve this problem in a general way, applicable to any car!) Now the car will move farther than the motorcycle at first. However, the motorcycle will catch up and overtake the car because it is accelerating.

a) Find out when, i.e. at what time, they are at the same position. So, we need functions that tell us where each vehicle is located at a given time. We can start with the basic kinematic equation of motion:

$$x = x_0 + v_0 t + \frac{1}{2}at^2$$

For the car, since there is no acceleration, $a = 0$, and $x_0 = 0$, this equation simplifies to:

$$x_{\text{car}} = v_0 t$$

For the motorcycle, it has no intitial velocity, $v_0 = 0$, but it does has an acceleration a. It also starts from the origin:

$$x_{\text{moto}} = \frac{1}{2}at^2$$

The question ask when the objects meet? That is, when are the x values the same. So, we can just set the two equations equal to each other.

$$x_{\text{car}} = x_{\text{moto}}$$

$$v_0 t = \frac{1}{2}at^2$$

and solve this for t.

$$t = \frac{2v_0}{a}$$

. Now we have an equation for t that we can use given any acceleration and initial velocity.

b) Where does this occur? We can use the time expression in one of the previous position equations.

$$x_{\text{car}} = v_0 \frac{2v_0}{a} = 2\frac{v_0^2}{a}$$

It should also be the same if we put in the time in the motorcycle's position equation:

$$x_{\text{moto}} = \frac{1}{2}at^2 = \frac{1}{2}a\left(\frac{2v_0}{a}\right)^2 = 2\frac{v_0^2}{a}$$

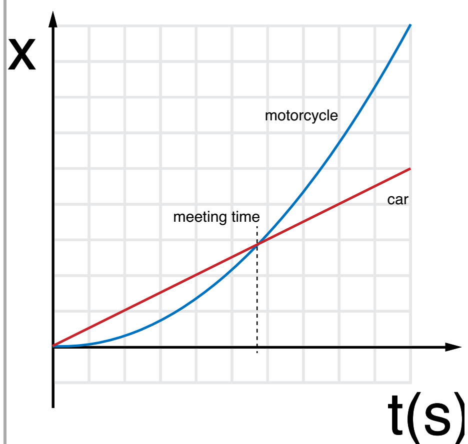
c) What is the speed of the motorcycle? We first need to find an equation for speed of the motorcycles. Let's the relationship between position and velocity:

$$v = \frac{dx}{dt} = at$$

So, when time is $t = \frac{2v_0}{a}$, the speed of the motorcycle will be:

$$v = at = a\left(\frac{2v_0}{a}\right) = 2v_0$$

Notice how the acceleration term is gone. The speed of the motorcycle when the two object meet is independent of its acceleration. That's an interesting bit of information that would have been lost if we did this problem using numbers instead of letters.



This plot shows graphically the situation. We can compare the slopes that the intersection and see that the slope of the motorcycle is roughly twice that of the car.

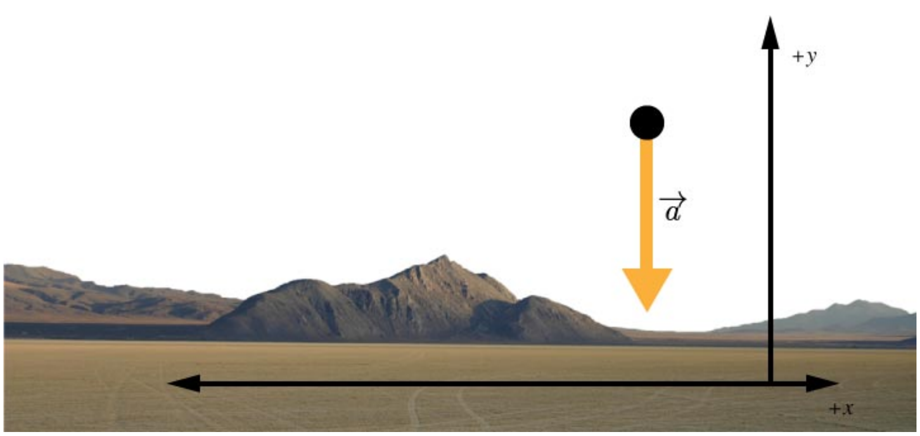
Free Fall

A freely falling object is any object moving freely under the influence of gravity alone.

Object could be:

1. Dropped = released from rest
2. Thrown downward
3. Thrown upward

It does not depend upon the initial motion of the object.



1. The acceleration of an object in free fall is directed downward (negative direction), regardless of the initial motion.
2. The magnitude of free fall acceleration is $\boxed{9.8\mathrm{m/s^2} = g}$.
3. We can neglect air resistance.
4. We'll choose our y axis to be positive upward.
5. Consider motion near Earth's surface for now.

Kinematic equation in the case of free fall:

1. $v = v_0 - gt$
2. $y = \bar{v}t = \frac{1}{2}(v_0 + v)t$
3. $y = y_0 + v_0 t - \frac{1}{2}gt^2$
4. $v^2 = v_0^2 - 2gy$

They are the same. We just replaced $x \rightarrow y$ and $a \rightarrow -g$.

Example Problem #10:

An object is thrown upward at 20 m/s:

a. How long will it take to reach the top
b. How high is the top?
c. How long to reach the bottom?
d. How fast will it be going when it reaches the bottom?

Example Problem #11:

If an object is thrown upward from a height y_0 with a speed v_0, when will it hit the ground?

Example Problem #12:

Drop a wrench

A worker drops a wrench down the elevator shaft of a tall building.

a. Where is the wrench 1.5 seconds later?
b. How fast is the wrench falling at that time?

Example Problem #13:

A rock is thrown upward with a velocity of 49 m/s from a point 15 m above the ground.

a. When does the rock reach its maximum height?
b. What is the maximum height reached?
c. When does the rock hit the ground?

How high was this?

Example Problem #14:

Draw position, velocity, and acceleration graphs as a functions of time, for an object that is let go from rest off the side of a cliff.